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ESTIMATES FOR THE RATE OF APPROXIMATION OF FUNCTIONS OF BOUNDED VARIATION BY POSITIVE LINEAR OPERATORS

The Ohio State University

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300 N. Zeeb Road, Ann Arbor, MI 48106
ESTIMATES FOR THE RATE OF APPROXIMATION OF FUNCTIONS OF BOUNDED VARIATION BY POSITIVE LINEAR OPERATORS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By
Fuhua Cheng

* * * * *

The Ohio State University
1982

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INTRODUCTION

The classical Dirichlet-Jordan's theorem states that if 

\( f \) is a \( 2\pi \)-periodic function of bounded variation on \([-\pi, \pi]\)

and \( S_n(f, x) \) the nth partial sum of the Fourier series of \( f \)

then \( S_n(f, x) \to \frac{1}{2}(f(x^+) + f(x^-)) \) as \( n \to \infty \), where \( f(x^+) \) and \( f(x^-) \)

are the right-hand and left-hand limits of \( f \) at the point \( x \).

An estimate for the rate of convergence of this theorem was given by R. Bojanic [1] in 1979 showing that

\[
|S_n(f, x) - \frac{1}{2}(f(x^+) + f(x^-))| \leq \frac{3}{n} \sum_{k=1}^{\pi/k} V_0(g_x),
\]

where

\[
g_x(t) = f(x+t) + f(x-t) - f(x^+) - f(x^-)
\]

and \( V^b_a(g) \) is the total variation of \( g \) on \([a, b]\). If \( f \) is a continuous \( 2\pi \)-periodic function of bounded variation on \([-\pi, \pi]\),

the preceding inequality becomes

\[
|S_n(f, x) - f(x)| \leq \frac{3}{n} \sum_{k=1}^{x+\pi/k} V_{x-\pi/k}(f).
\]

In 1908, E. W. Hobson [2] proved that if \( f \) is a function of bounded variation on \([-1,1]\) and

\[
L_n(f, x) = \sum_{k=0}^{n} a_k(f)p_k(x)
\]

the nth partial sum of the Fourier-Legendre series of \( f \), where

\( p_k(x) \) is the Legendre polynomial of degree \( k \) normalized so that \( p_n(1) = 1 \) and
then \( \ln(f,x) - \frac{1}{2}(f(x^+)+f(x^-)) \) as \( n \to \infty \) at every point \( x \in (-1,1) \).

As for the rate of convergence, R. Bojanic and M. Vuilleumier [3] proved in 1981 that

\[
|L_n(f,x) - \frac{1}{2}(f(x^+)+f(x^-))| < \frac{2^a(1-x^2)^{-3/2}}{n} \frac{x+(1-x)/k}{\sum_{k=1}^{n} x-(1+x)/k} (g_x)
\]

\[+ \frac{(1-x^2)^{-1}}{\pi n} |f(x^+)-f(x^-)|,
\]

where

\[
g_x(t) = \begin{cases} 
f(t)-f(x^+), & x < t < 1 \\
0, & t = x \\
f(t)-f(x^-), & -1 < t < x.
\end{cases}
\]

In addition, if \( f \) is continuous on \([-1,1]\) then the inequality (0.1) becomes

\[
|L_n(f,x)-f(x)| < \frac{2^a(1-x^2)^{-3/2}}{n} \frac{x+(1-x)/k}{\sum_{k=1}^{n} x-(1+x)/k} (f).
\]

It was proved in the same paper that inequality (0.2) is essentially the best possible for functions continuous and of bounded variation on \([-1,1]\).

In this thesis, we shall study similar estimates for the rate of convergence of Szász-Mirakyan operator, Bernstein polynomials and Hermite-Fejér interpolatory polynomials of functions of bounded variation. Our work will be divided into three chapters, each of independent interest. A more
detailed introduction is given in the beginning of each chapter.

In Chapter I, the behavior of the Szász-Mirakyan operator is studied. If \( f \) is a function defined on the infinite interval \([0,\infty)\), the Szász-Mirakyan operator \( S_n(f) \) is defined by

\[
S_n(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) p_k(nx); \quad p_k(t) = e^{-t} \frac{t^k}{k!}.
\]

In order to insure the convergence of the infinite series it is necessary to restrict the growth of \( f \) at \( \infty \). The simplest such hypothesis is that \( f(t) = O(t^\alpha) \) \( (t \to \infty) \) for some \( \alpha > 0 \). Assuming this condition and, further, that \( f \) is bounded on every finite interval of \([0,\infty)\) and continuous at \( t = x, 0 \). Szász proved in 1950 that

\[
\lim_{n \to \infty} S_n(f,x) = f(x).
\]

Estimates for the rate of convergence of \( S_n(f,x) \) were obtained later for continuous functions on \([0,\infty)\) under more general hypothesis about the growth of \( f \) at \( \infty \). In Chapter I we prove that if \( f \) is of bounded variation on every finite interval of \([0,\infty)\) and \( f(t) = O(t^\alpha) \) \( (t \to \infty) \) for some \( \alpha > 0 \) as \( t \to \infty \); if \( x \in (0,\infty) \) and \( x \neq k/n \) for any \( k,n \) then for \( n \) sufficiently large we have

\[
|S_n(f,x) - \frac{1}{2}(f(x^+)+f(x^-))| \leq \frac{(3+x)x^{-1}}{n} \sum_{k=1}^{n} \frac{x+x/\sqrt{k}}{x-x/\sqrt{k}}(g_x) + \frac{O(x^{-\frac{1}{2}})}{n^{\frac{1}{2}}} |f(x^+)-f(x^-)|
\]

\[
+ \frac{O(1)(4x)^{\frac{1}{2}}(nx)}{e^{\frac{nx}{\pi}}},
\]
where $g_x$ is defined as in (0.2).

If, in addition, $f$ is continuous at $x$ and not constant in any neighborhood of $x$ then (0.4) becomes

$$\left|S_n(f,x) - f(x)\right| < \frac{(4+x)x^{-1}}{n} \sum_{k=1}^{\infty} \frac{x+x/\sqrt{k}}{x-x/\sqrt{k}} \omega_f(f).$$

Estimate (0.5) is essentially the best possible.

In Chapter II we study the rate of convergence of Bernstein polynomials $B_n(f,x)$ of a function $f$ of bounded variation on $[0,1]$. Bernstein polynomial of $f$ is defined by

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f^k.$$

It is well known that

$$\lim_{n \to \infty} B_n(f,x) = f(x)$$

uniformly on $[0,1]$ whenever $f$ is a continuous function on $[0,1]$. More precisely,

$$\left|B_n(f,x) - f(x)\right| \leq \frac{3}{2} \omega_f(\frac{1}{n})$$

where $\omega_f$ is the modulus of continuity of $f$ defined by

$$\omega_f(\delta) = \max\{|f(x)-f(y)|: x,y \in [0,1], |x-y| \leq \delta\}.$$

In 1946, F. Herzog and D. J. Hill ([4], see also [5]) proved that

$$\lim_{n \to \infty} B_n(f,x) = \frac{1}{2}(f(x^+) + f(x^-))$$
whenever \( f \) is a function of bounded variation on \([0,1]\).

In Chapter II, we obtain the following estimate for the rate of convergence of (0.6):

\[
|B_n(f,x)-\frac{1}{2}(f(x^+)+f(x^-))| \leq \frac{3(x(1-x))^{-1}}{n} \sum_{k=1}^{n} \frac{x+(1-x)/\sqrt{n}}{x-x/\sqrt{n}} (g_x)
+ \frac{18(x(1-x))^{-5/2}}{n^{1/6}} |f(x^+)-f(x^-)|.
\]

where \( g_x \) is defined as in (0.2).

If, in addition, \( f \) is continuous at \( x \) then (0.7) becomes

\[
|B_n(f,x)-f(x)| \leq \frac{3(x(1-x))^{-1}}{n} \sum_{k=1}^{n} \frac{x+(1-x)/\sqrt{n}}{x-x/\sqrt{n}} (f).
\]

This inequality can not be improved asymptotically.

Chapter III is based on a joint work of Professor Ranko Bojanic and the author [29]. The Hermite-Fejér interpolatory polynomial \( H_n(f,x) \) based on the Chebyshev nodes \( x_{kn} = \cos \left( \frac{2k-1}{2n} \pi \right), \quad k = 1,2,\ldots,n \) is defined by

\[
H_n(f,x) = \sum_{k=1}^{n} f(x_{kn})(1-x_{kn}x)\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2.
\]

The approximation-theoretical properties of the sequence \( (H_n(f,x)) \) are well known when \( f \) is a continuous function on \([-1,1]\). In 1916, L. Fejér [18] proved that

\[
\lim_{n \to \infty} H_n(f,x) = f(x)
\]

uniformly on \([-1,1]\) whenever \( f \) is a continuous function on
Various estimates of the rate of convergence are known. The most recent result is the following inequality of Goodenough and Mills [26]:

\[ |H_n(f,x) - f(x)| \leq \frac{c_1}{n} T_n(x) \sum_{k=1}^{n} \left( \omega_f\left(\frac{\sqrt{1-x^2}}{k}\right) + \omega_f\left(\frac{1}{k^2}\right)\right) + c_2 \omega_f\left(\frac{T_n(x)}{n}\right), \]

where \( c_1 \) and \( c_2 \) are positive constants.

Here we study the corresponding properties when \( f \) is a function of bounded variation on \([-1,1]\). We obtain an estimate for the rate of convergence of \( H_n(f,x) \) of functions of bounded variation at points of continuity as follows:

\[ |H_n(f,x) - f(x)| \leq 2 \sqrt{\frac{x+\pi}{x-\pi}} |T_n(x)|/2n \sum_{k=1}^{n} \left( \omega_f\left(\frac{x+\pi/k}{2n}\right) + \omega_f\left(\frac{x-\pi/k}{2n}\right)\right), \]

and prove that this estimate is asymptotically the best possible. On the other hand, we prove that, at points of discontinuity where \( f(x+) \neq f(x-) \), the sequence \( (H_n(f,x)) \) diverges by showing that

\[ \lim_{n \to \infty} \sup_n H_n(f,x) = \frac{1}{2} (f(x+) + f(x-)) + \frac{1}{2} |f(x+) - f(x-)| \beta(x) \]

where \( \beta(x) = 1 \) if \( x = \cos \alpha \pi \) and \( \alpha \) is irrational, and

\[ \beta(x) = \left(\frac{\sin(\pi/2q)}{\pi/2q}\right)^2 \left(1 - \sum_{k=1}^{\infty} \frac{8qk}{(4q^2k^2-1)^2}\right) \]

if \( x = \cos(p\pi/q) \).
CHAPTER I
ON THE RATE OF CONVERGENCE OF SZÁSZ-MIRAKYAN OPERATOR
FOR FUNCTIONS OF BOUNDED VARIATION

1. INTRODUCTION. Let $f$ be a function defined on the infinite interval $[0, \infty)$. The Szász-Mirakyan operator $S_n(f)$ for $f$ is defined by

$$S_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) P_k(nx)$$

where

$$P_k(t) = e^{-t} \frac{t^k}{k!}.$$

O. Szász [6] proved in 1950 the following theorem:

**THEOREM A.** If $f(t)$ is bounded in every finite interval of $[0, \infty)$; if $f(t) = O(t^k)$ for some $k > 0$ as $x \to \infty$ and if $f(t)$ is continuous at the point $t = x$, then $S_n(f, t)$ converges uniformly to $f(t)$ at $t = x$.

Later on, in 1971 J. Gróf [8] gave the following estimate of the rate of convergence of $S_n(f, x)$ when $f(t)$ is continuous on $[0, \infty)$:

---

1 G. Mirakyan [7] considered the partial sum $\psi_{m,n}(f,x)$ of $S_n(f,x)$

$$\psi_{m,n}(f,x) = \sum_{k=0}^{m} f\left(\frac{k}{n}\right) P_k(nx)$$

and proved that $\lim_{n \to \infty} \psi_{m,n}(f,x) = f(x)$ uniformly in $[0, r']$, if $\lim_{n \to \infty} \frac{m}{n} = r > r' > 0$. We don't consider this case in this paper.
THEOREM B. If \( f \) is continuous on \([0,\infty)\) and \( f(x) = 0(e^{\alpha x}) \) for some \( \alpha > 0 \) as \( x \to \infty \) then for all \( A > 0 \)

\[ |S_n(f,x) - f(x)| \leq 0(\omega_2A(f,n^{-1})) \quad \forall x \in [0,A] \]

where

\[ \omega_A(f,\delta) = \sup\{|f(x+t)-f(x)| : |t| \leq \delta, x \in [0,A]|. \]

It is easy to see, by considering the function \( f(t) = |t-x| \)
at the point \( t = x \) (\( x > 0 \)), that this result is essentially the best possible.

This result was then improved by T. Hermann [9] in 1977 by showing that (1.2) holds if \( f(t) = 0(t^{\alpha t}) \) (\( \alpha > 0 \)). He proved in the same paper that \( S_n(f,x) \) does not exist if \( f(t) \geq t^{\phi(t) \cdot t} \) where \( \phi(t) \) is any monotonically increasing function such that \( \lim_{t \to \infty} \phi(t) = \infty \). Therefore \( f(t) = 0(t^{\alpha t}) \) (\( \alpha > 0 \)) is the strongest condition which can be imposed on the magnitude of \( f \).

In this paper, we shall study \( S_n(f,x) \) for functions of bounded variation on every finite interval of \([0,\infty)\) and prove that \( S_n(f,x) \) converges to \( \frac{1}{2}(f(x+0)+f(x-0)) \) under the Hermann's condition on the magnitude of \( f \) by giving quantitative estimates of the rate of convergence. We shall also prove that our estimates are essentially the best possible.

2. RESULTS and REMARKS. Our main result may be stated as follows:
THEOREM. Let $f$ be a function of bounded variation on every finite interval of $[0,\infty)$ and $f(t) = 0(t^{\alpha})$ for some $\alpha > 0$ as $t \to \infty$. If $x \in (0,\infty)$ and $x \neq k/n$ for any $k,n$ then for $n$ sufficiently large we have

$$|S_n(f,x) - \frac{1}{2}(f(x^+) + f(x^-))| \leq \sum_{k=1}^{n} \frac{\sqrt{x + x/\sqrt{k}}}{n} \mathcal{V}_{x/\sqrt{k}}(g_x)$$

$$+ \frac{O(x^{-\frac{1}{2}})}{n^{\frac{1}{2}}} |f(x^+) - f(x^-)|$$

$$+ O(1)\frac{4^\alpha x}{(n^{\alpha} x^{\alpha})} \frac{1}{n} \mathcal{V}(g_x)$$

where $\mathcal{V}(g)$ is the total variation of $g$ on $[a,b]$ and

$$g_x(t) = \begin{cases} f(t) - f(x), & x < t < \infty \\ 0, & t = x \\ f(t) - f(x^-), & 0 < t < x. \end{cases}$$

The right hand side of (2.1) converges to zero as $n \to \infty$ since continuity of $g_x(t)$ at $t = x$ implies that

$$\mathcal{V}_{x+\delta}(g_x) \to 0 \quad (\delta \to 0+).$$

REMARKS. 1. If $f$ is not constant in any neighborhood of $x$ then

$$|S_n(f,x) - \frac{1}{2}(f(x^+) + f(x^-))| \leq \sum_{k=1}^{n} \frac{\sqrt{x + x/\sqrt{k}}}{n} \mathcal{V}_{x/\sqrt{k}}(g_x)$$

for $n$ sufficiently large. So in that case (2.1) becomes

$$|S_n(f,x) - \frac{1}{2}(f(x^+) + f(x^-))| \leq \frac{(4^\alpha x)}{n} \frac{1}{nx} \mathcal{V}(g_x)$$
If, in addition, \( f \) is continuous at \( x \) then (2.3) can be further simplified as

\[
\left| S_n(f,x) - f(x) \right| \leq \frac{(4+x)x^{-1}}{n^2} \sum_{k=1}^{n} \frac{x+x/\sqrt{k}}{x-x/\sqrt{k}}(f)
\]

for sufficiently large \( n \).

2. If, however, \( f \) is constant in some neighborhood of \( x \) then, since \( V_{x-x/\sqrt{k}}(f) = 0 \) for all except finite number of \( k \)'s implies that

\[
\sum_{k=1}^{n} \frac{x+x/\sqrt{k}}{x-x/\sqrt{k}}(f) = c
\]

for some positive constant \( c \) (depending on \( x \)), with (2.2) we find that (2.1) becomes

\[
\left| S_n(f,x)-f(x) \right| \leq \frac{c(4+x)}{nx}
\]

when \( n \) is sufficiently large.

As far as the precision of the above estimates is concerned, we can prove that (2.4) can not be improved asymptotically.

Consider the function \( f(t) = |t-x| \) \( (x > 0) \) at \( t = x \). From (2.4) we have

\[
\left| S_n(f,x)-f(x) \right| \leq \frac{4+x}{nx} \sum_{k=1}^{n} \frac{x+x/\sqrt{k}}{x-x/\sqrt{k}}(f).
\]

Since \( V_{x-\delta}(f) = 2\delta \), it follows that

\[
\left| S_n(f,x)-f(x) \right| \leq \frac{2(4+x)}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq \frac{4(4+x)}{\sqrt{n}}.
\]
On the other hand, by a result of O. Szász ([6], pp.240),

\[ |S_n(f, x) - f(x)| = \sum_{k=0}^{\infty} \left| \frac{k}{n} - x \right| p_k(nx) \]
\[ = 2xe^{-nx} \frac{\binom{nx}{[nx]}}{[nx]!} \]
\[ > \frac{(2x/\pi e^4)^{\frac{1}{2}}}{n^{\frac{1}{2}}} . \]

Hence, for the function \( f(t) = |t-x| (x > 0) \), we have

\[ \frac{(2x/\pi e^4)^{\frac{1}{2}}}{n^{\frac{1}{2}}} \leq |S_n(f, x) - f(x)| \leq \frac{\ln(n+x)}{n^{\frac{1}{2}}} . \]

Therefore (2.4) cannot be improved asymptotically.

3. LEMMAS. The proof of our theorems is based on a number of lemmas. The first lemma is one of the questions posed by S. Ramanujan in a letter of January 16, 1913, to G. H. Hardy. A complete proof of this theorem was given by G. N. Watson [10], G. Szegö [11] and J. Karamata [12].

LEMMA 1. If \( x \) is a positive integer then

\[ 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^x}{x!} \theta(x) = \frac{1}{2} e^x \]

where \( \theta(x) \) lies between 1/2 and 1/3.

For arbitrary positive number \( x \) we have the following

LEMMA 2. If \( x \) is a positive number then

\[ e^{-x} \sum_{k=0}^{[x]} \frac{x^k}{k!} = \frac{1}{2} + O(1/\sqrt{x}). \]
PROOF OF LEMMA 2. Let \( n \leq x < n+1 \). That is, \([x] = n\).

Define a function \( \psi(t) \) on \([n,n+1)\) as follows:

\[
\psi(t) = e^{-t} \sum_{k=0}^{n} \frac{tk}{k!}, \quad t \in [n,n+1).
\]

Since

\[
\psi'(t) = -e^{-t} \frac{tn}{n!} < 0,
\]

so if \( n \leq t < n+1 \) we have

\[
\psi((n+1)-) \leq \psi(t) \leq \psi(n).
\]

In particular

\[
\psi((n+1)-) \leq e^{-x} \sum_{k=0}^{n} \frac{xk}{k!} \leq \psi(n).
\]

By Lemma 1,

\[
\psi(n) = e^{-n} \sum_{k=0}^{n} \frac{nk}{k!} = e^{-[x]} \sum_{k=0}^{[x]} \frac{[x]k}{k!}
\]

\[
= \frac{1}{2} + e^{-[x]} \frac{[x][x]}{[x]!} (1-\theta([x]));
\]

\[
\psi((n+1)-) = e^{-x} \sum_{k=0}^{n} \frac{(n+1)k}{k!} = e^{-[x]+1} \sum_{k=0}^{[x]+1} \frac{([x]+1)k}{k!}
\]

\[
= \frac{1}{2} - e^{-(x+1)} \frac{([x]+1)[x]+1}{([x]+1)!} \theta([x]+1),
\]

and the Stirling's formula

\[
e^{-k} \frac{k^k}{k!} < \frac{1}{\sqrt{2\pi}k}
\]

we then have

\[
\frac{1}{2} - O\left(\frac{1}{\sqrt{x}}\theta([x]+1)\right) \leq e^{-x} \sum_{k=0}^{[x]} \frac{xk}{k!} \leq \frac{1}{2} + O\left(\frac{1}{\sqrt{x}}(1-\theta([x]))\right).
\]
Lemma 2 now follows immediately from the fact that \( \theta(x) \) is uniformly bounded.

Lemma 2 has evident application in Probability. As we can see that \( S_n(f,x) \) corresponds to the Poisson distribution in the terminology of Probability Theory; the distribution function is

\[
P(X < x) = \sum_{k=0}^{[x]} \frac{e^{-\alpha} \alpha^k}{k!}
\]

with parameter \( \alpha > 0 \). If \( \alpha = nx \) for some \( x > 0 \) then by Lemma 2

\[
P(X < \alpha) = \sum_{k=0}^{[nx]} \frac{e^{-nx}(nx)^k}{k!} = \frac{1}{2} + O\left(\frac{1}{\sqrt{nx}}\right).
\]

This result can not be proved directly by applying the Central Limit Theorem though we have a similar result using the Liapounov Theorem when \( x \) is a positive integer (see, e.g. [13], pp.302).

The following lemma was proved by O. Szász (see [6], pp.239).

**Lemma 3.** If \( x \) is a positive number then

\[
e^{-x} \sum_{|k-x| \geq \delta} \frac{x^k}{k!} \leq \frac{x}{\delta^2}.
\]

Lemma 4 is a Ramanujan type problem. The second part will not be needed in the proof of our theorems. The reason we put it here is because of its own interest.

**Lemma 4.** (i) If \( 2x \) is a positive integer then

\[
\sum_{k=2x+1}^{\infty} \frac{x^k}{k!} = \delta(x) \frac{x^{2x}}{(2x)!}
\]
where $\delta(x)$ lies between $2\sqrt{e} - 3$ and 1.

(i) If $x$ is a positive integer then

$$
\alpha(x) = \frac{x^x}{x!} + \frac{x^{x+1}}{(x+1)!} + \cdots + \frac{x^{2x-1}}{(2x-1)!} + \frac{x^{2x}}{(2x)!}
$$

$$
\beta(x) = \frac{1}{2} e^x
$$

where $\alpha(x)$ lies between $\frac{1}{2}$ and $2/3$; $\beta(x)$ lies between $2(\sqrt{e} - 1)$ and 2.

**PROOF OF LEMMA 4.** It is easy to see that, when $2x$ is a positive integer,

$$
\delta(x) = \frac{e^{x-2} \cdot \sum_{k=0}^{2x} \frac{x^k}{k!}}{x^{2x}(2x)!} = \frac{x}{2x+1} + \frac{x^2}{(2x+2)(2x+1)} + \frac{x^3}{(2x+3)(2x+2)(2x+1)} + \cdots
$$

We shall adopt the last expression as the definition of $\delta(x)$ for positive number $x \geq \frac{1}{2}$. It is obvious that $\delta(\frac{1}{2}) = 2\sqrt{e} - 3$ and $\lim_{x \to \infty} \delta(x) = 1$. Therefore (i) will be proved if we can show that $\delta(x)$ is an increasing function. However this follows immediately from the fact that

$$
\frac{x^j}{\prod_{i=1}^{j} (2x+1)} \leq \frac{y^j}{\prod_{i=1}^{j} (2y+1)}, \quad j = 1, 2, 3, \ldots,
$$

if $x \leq y$.

(ii) is a direct consequence of Lemma 1 and (i).

As for positive number $x$ we have the following

**LEMMA 5.** If $x$ is a positive number $\geq \frac{1}{2}$, then
\[ c_1 \frac{1}{\sqrt{x}} \left( \frac{e}{\pi} \right)^x \leq e^{-x} \sum_{k=2x}^{\infty} \frac{x^k}{k!} \leq c_2 \frac{1}{\sqrt{x}} \left( \frac{e}{4} \right)^x \]

where \( c_1 = (2\sqrt{e} - 3)/2\sqrt{\pi e} \) and \( c_2 = \sqrt{\frac{e}{4\pi}} \).

**Proof of Lemma 5.** First, let us assume that \( n < x < n + \frac{1}{2} \) where \( n \) is a positive integer. Define a function \( \psi(t) \) on \([n, n + \frac{1}{2}]\) as follows:

\[
\psi(t) = e^{-t} \sum_{k=2n+1}^{\infty} \frac{t^k}{k!}, \quad t \in [n, n + \frac{1}{2}].
\]

Since

\[
\psi'(t) = e^{-t} \frac{t^{2n}}{(2n)!} > 0,
\]

so if \( n \leq t < n + \frac{1}{2} \), we have

\[
\psi(n) \leq \psi(t) \leq \psi((n + \frac{1}{2})^-).
\]

In particular

\[
\psi(n) \leq e^{-x} \sum_{k=2x}^{\infty} \frac{x^k}{k!} \leq \psi((n + \frac{1}{2})^-).
\]

By the fact that \( n = [x] \), Lemma 4 and Stirling's formula

\[
\psi(n) = e^{-n} \sum_{k=2n+1}^{\infty} \frac{n^k}{k!} = e^{-[x]} \sum_{k=2[x]+1}^{\infty} \frac{[x]^k}{k!} = e^{-[x]} \delta([x]) \frac{[x]}{(2[x])!}
\]

\[
\geq \frac{1}{\sqrt{[x]}} \left( \frac{e}{\pi} \right)^{[x]} \frac{1}{4\sqrt{\pi}} \delta([x])
\]

\[
> \frac{1}{\sqrt{x}} \left( \frac{e}{\pi} \right)^x \frac{1}{2\sqrt{\pi e}} \delta([x]);
\]
\[
\psi((n+\frac{1}{2})-) = e^{-n-\frac{1}{2}} \sum_{k=2n+1}^{\infty} \frac{(n+\frac{1}{2})^k}{k!} = e^{-\lfloor x \rfloor -\frac{1}{2}} \sum_{k=2\lfloor x \rfloor +1}^{\infty} \frac{(\lfloor x \rfloor +\frac{1}{2})^k}{k}
\]
\[
= e^{-\lfloor x \rfloor -\frac{1}{2}} \left(1+\delta([x]+\frac{1}{2})\right) \frac{2[x]+1}{(2[x]+1)!}
\]
\[
\leq \frac{1}{\sqrt{x}(\frac{e}{4})^x} \frac{1}{2\sqrt{\pi}} \frac{1}{(1+\delta([x]+\frac{1}{2}))}
\]
\[
\leq \frac{1}{\sqrt{x}} \left(\frac{e}{4}\right)^x \frac{\sqrt{e}}{4\sqrt{\pi}} (1+\delta([x]+\frac{1}{2})).
\]

Therefore when \( n \leq x < n+\frac{1}{2} \) according to the fact that \( \delta(x) \) lies between \( 2\sqrt{e} - 3 \) and 1, we have

\[
(3.1) \quad \frac{2\sqrt{e} - 3}{2\sqrt{\pi e}} \frac{1}{\sqrt{x}} \left(\frac{e}{4}\right)^x \leq e^{-x} \sum_{k>2x} \frac{x^k}{k!} \leq \frac{\sqrt{e}}{4\sqrt{\pi}} \cdot \frac{1}{\sqrt{x}} \left(\frac{e}{4}\right)^x .
\]

Next, if \( n+\frac{1}{2} \leq x < n+1 \) for some nonnegative integer \( n \), we define \( \psi(t) \) on \([n+\frac{1}{2}, n+1)\) as follows:

\[
\psi(t) = e^{-t} \sum_{k=2n+2}^{\infty} \frac{t^k}{k!}, \quad t \in [n+\frac{1}{2}, n+1).
\]

Along the same line, we can prove that, when \( n+\frac{1}{2} \leq x < n+1 \) for some nonnegative integer \( n \), \( e^{-x} \sum_{k>2x} x^k/k! \) satisfies the same inequality as (3.1). And this completes the proof.

The last lemma of this section is similar to a result in T. Hermann's paper, but with a more precise estimate.

**Lemma 6.** For any fixed positive numbers \( \alpha \) and \( x \)

\[
\sum_{k>2x} \binom{k}{n} \frac{\alpha}{n} p_k(x) \leq \frac{3}{2} \left(\frac{2x+1}{n}\right)^\alpha \frac{2x+1}{n} \frac{1}{\sqrt{\pi x}} \left(\frac{e}{4}\right)^x
\]

if \( n \) is sufficiently large.
PROOF OF LEMMA 6. Let

\[ b_k = \left( \frac{k}{n} \right)^{\alpha_k} p_k(x), \quad k > 2x. \]

By a simple calculation we can show that

\[ \frac{b_{k+1}}{b_k} < \frac{2}{3} \]

if \( n \) is sufficiently large. Hence, if \( k > 2x \)

\[ \sum_{k > 2x} \left( \frac{k}{n} \right)^{\alpha_k} p_k(x) \leq 3 \frac{(2x+1)}{[2x]+1} \left( \frac{2x+1}{n} \right)^{\alpha} p_{[2x]+1}(x). \]

By Stirling's formula

\[ p_{[2x]+1}(x) \leq \frac{1}{\sqrt{4\pi x}} \left( e^{\frac{x}{[2x]+1}} + \log \frac{x}{[2x]+1} \right) [2x]+1. \]

Since the function \( g(y) = 1-y+\log y \) defined on \((0, \frac{1}{2})\) is increasing and \( g\left(\frac{1}{2}\right) = \frac{1}{2} - \log 2 < 0 \). Therefore

\[ p_{[2x]+1}(x) \leq \frac{1}{\sqrt{4\pi x}} \left( e^{\frac{1}{2} - \log 2} [2x]+1 \right) \]

\[ \leq \frac{1}{\sqrt{4\pi x}} \left( e^{\frac{1}{2} - \log 2} 2x \right) \]

\[ = \frac{1}{\sqrt{4\pi x}} \left( \frac{e}{n} \right)^x. \]

Lemma 6 follows immediately from (3.2) and (3.3).

If, in Lemma 6, we replace \( x \) by \( nx \) and when \( n \) is sufficiently large, we get the following inequality
which is what we really need in the proof of our theorems.

4. PROOF. For any fixed \( x \in (0, \infty) \) define \( g_x \) as follows:

\[
(4.1) \quad g_x(t) = \begin{cases} 
  f(t) - f(x), & x < t < \infty \\
  0, & t = x \\
  f(t) - f(x^-), & 0 < t < x.
\end{cases}
\]

\( g_x \) is continuous at \( t = x \) and inherits all the properties of \( f \). By using (4.1), (1.1) can be written as

\[
S_n(f, x) = S_n(g_x, x) + \frac{f(x^+) + f(x^-)}{2} + \frac{f(x^+) - f(x^-)}{2} (A_n(x) - B_n(x))
\]

where

\[
A_n(x) = \sum_{k \geq nx} p_k(nx) = e^{-nx} \sum_{k \geq nx} \frac{(nx)^k}{k!};
\]

\[
B_n(x) = \sum_{k < nx} p_k(nx) = e^{-nx} \sum_{k < nx} \frac{(nx)^k}{k!}.
\]

Hence

\[
(4.2) \quad |S_n(f, x) - \frac{1}{2}(f(x^+) + f(x^-))| \leq |S_n(g_x, x)| + \frac{1}{2} \cdot |f(x^+) - f(x^-)| \cdot |A_n(x) - B_n(x)|.
\]

By Lemma 2

\[
B_n(x) = \frac{1}{2} + O\left(\frac{1}{\sqrt{nx}}\right)
\]

and

\[
A_n(x) = 1 - B_n(x) = \frac{1}{2} + O\left(\frac{1}{\sqrt{nx}}\right),
\]

therefore, for the second term of the right hand side of (4.2) we have
\[(4.3)\] \( \frac{1}{2} |f(x^+)-f(x^-)| \cdot |A_n(x)-B_n(x)| = O\left(\frac{1}{\sqrt{n}x}\right) |f(x^+)-f(x^-)| \)

and our Theorem will be proved if we establish that

\[(4.4)\] \(|S_n(g_x, x)| \leq \frac{(3+x)x-1}{n} \sum_{k=1}^{x+x/\sqrt{k}} (g_x) \left(\frac{x+x/\sqrt{k}}{k}\right) + O(1)(4x)^{\frac{1}{4}} \left(\frac{x}{1}\right)^{\frac{1}{2}} (g_x) nx \)

for sufficiently large \( n \).

To do this, first, we observe that \( S_n(g_x, x) \) can be written as a Lebesgue-Stieltjes integral

\[(4.5)\] \( S_n(g_x, x) = \int_{0}^{\infty} g_x(t) \, d_t K_n(x, t) \)

where the kernel \( K_n(x, t) \) is defined by

\[
K_n(x, t) = \begin{cases} 
\sum_{k < nt} p_k(nx), & 0 < t < \infty \\
0, & t = 0,
\end{cases}
\]

the so called Poisson distribution in Probability. We decompose the integral on the right hand side of (4.5) into three parts, as follows.

\[(4.6)\] \( \int_{0}^{\infty} g_x(t) d_t K_n(x, t) = L_n(f, x) + M_n(f, x) + R_n(f, x) \)

where

\[
L_n(f, x) = \int_{0}^{x-x/\sqrt{n}} g_x(t) \, d_t K_n(x, t),
\]

\[
M_n(f, x) = \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} g_x(t) \, d_t K_n(x, t),
\]

\[
R_n(f, x) = \int_{x+x/\sqrt{n}}^{\infty} g_x(t) \, d_t K_n(x, t).
\]
We shall evaluate $M_n(f,x)$, $L_n(f,x)$ and $R_n(f,x)$ in sequence.

For $t \in \left[\frac{x-x}{\sqrt{n}}, \frac{x+x}{\sqrt{n}}\right]$, 

$$|g_x(t)| = |g_x(t) - g_x(x)| \leq \frac{x+x}{\sqrt{n}} V_{x-x/\sqrt{n}}(g_x).$$

Hence 

$$|M_n(f,x)| \leq \frac{x+x}{\sqrt{n}} \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} d_t K_n(x,t).$$

Since 

$$\int_{a}^{b} d_t K_n(x,t) \leq 1 \text{ for any } [a,b] \subseteq [0,\infty),$$

therefore 

$$|M_n(f,x)| \leq \frac{x+x}{\sqrt{n}} \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} d_t K_n(x,t).$$

(4.7) 

Next, we evaluate $L_n(f,x)$. The method used here is similar to the approach used by R. Bojanic and M. Vuilleumier [3].

Using partial integration with $y = \frac{x-x}{\sqrt{n}}$ we have 

$$L_n(f,x) = g_x(y+K_n(x,y+) - \int_{0}^{y} \hat{K}_n(x,t)d_t g_x(t)$$

where $\hat{K}_n(x,t)$ is the normalized form of $K_n(x,t)$. Since 

$$K_n(x,y+) = K_n(x,y) \text{ if } 0 < y < \infty$$

and 

$$|g_x(y+)| = |g_x(y+) - g_x(x)| \leq \frac{x}{y+}(g_x)$$

where $V^{x}_{y+}(g_x) = \lim_{\epsilon \to 0^+} V^{x}_{y+}(g_x)$. Therefore 

$$|L_n(f,x)| \leq V^{x}_{y+}(g_x)K_n(x,y) + \int_{0}^{y} \hat{K}_n(x,t)d_t (-V^{x}_{y+}(g_x)).$$

Since $\hat{K}_n(x,t) \leq K_n(x,t)$ on $(0,\infty)$ and by Lemma 3, 

$$K_n(x,t) = \sum_{k \leq nt} p_k(nx) \leq \frac{x}{n(t-x)^2}, \ 0 \leq t < x.$$
Hence

\[ |L_n(f,x)| \leq \nu_{y+}(g_x)K_n(x,y) + \frac{x}{n} \int_0^y \frac{1}{(t-x)^2} dt (-V_t^x(g_x)) + \frac{1}{2} \cdot e^{-nx} \int_0^y \frac{1}{(t-x)^2} dt (-V_t^x(g_x)) \]

As we can see that

\[ \frac{1}{2} \cdot e^{-nx} \int_0^y \frac{1}{(t-x)^2} dt (-V_t^x(g_x)) \]

for any real number \( nx \) and

\[ \frac{x}{n} \int_0^y \frac{1}{(t-x)^2} dt (-V_t^x(g_x)) + \frac{1}{nx} \int_0^y \frac{1}{(t-x)^2} dt (-V_t^x(g_x)) = \frac{x}{n} \int_0^y \frac{1}{(t-x)^2} dt (-V_t^x(g_x)) \]

Consequently

\[ |L_n(f,x)| \leq \nu_{y+}(g_x)K_n(x,y) + \frac{x}{n} \int_0^y \frac{1}{(t-x)^2} dt (-V_t^x(g_x)) \]

Using partial integration again, we have

\[ \int_0^y \frac{1}{(t-x)^2} dt (-V_t^x(g_x)) = \frac{1}{x-y^2} + \frac{1}{x^2} \int_0^y \frac{1}{(x-t)^3} \]

Hence

\[ |L_n(f,x)| \leq \frac{x}{n} \left( \frac{1}{x^2} + 2 \int_0^y \frac{1}{(x-t)^3} \right) \]

Replacing the variable \( t \) in the last integral by \( x-x/\sqrt{\tau} \), we find that

\[ \int_0^{x-x/\sqrt{x}} \nu_t^x(g_x) \frac{dt}{(x-t)^3} = \frac{1}{2x^2} \int_1^{x} \frac{x}{x-x/\sqrt{x}}(g_x)dt \]

\[ \leq \frac{1}{2x^2} \sum_{k=1}^n \nu_{x-x/\sqrt{x}}(g_x). \]
Thus

\[(4.8)\quad |L_n(f,x)| \leq \frac{1}{nx} \left( V_0(g_x) + \sum_{k=1}^{n} V^{x}_{x-x/\sqrt{k}}(g_x) \right) \]

\[\leq \frac{2}{nx} \sum_{k=1}^{n} V^{x}_{x-x/\sqrt{k}}(g_x).\]

Finally, we evaluate $R_n(f,x)$. Let $z = x + x/\sqrt{n}$ and define $Q_n(x,t)$ on $[0,2x]$ as follows:

\[Q_n(x,t) = \begin{cases} 
1 - P_n(x,t-), & 0 \leq t < 2x \\
0, & t = 2x. 
\end{cases}\]

Then

\[(4.9)\quad R_n(f,x) = -\int_{z}^{2x} g_x(t) d_t Q_n(x,t) - g_x(2x) \sum_{k>2nx} p_k(nx) + \int_{2x}^{\infty} g_x(t) d_t K_n(x,t) \]

\[= R_{1n} + R_{2n} + R_{3n}.\]

Using partial integration for the first term on the right hand side of (4.9), we get

\[R_{1n} = g_x(z-)Q_n(x,z-) + \int_{z}^{2x} \hat{Q}_n(x,t) d_t g_x(t)\]

where $\hat{Q}_n(x,t)$ is the normalized form of $Q_n(x,t)$. Since $Q_n(x,z-) = Q_n(x,z), 0 \leq z < 1$ and $|g_x(z-)| \leq V_x^z(g_x)$, so that

\[|R_{1n}| \leq V_x^z(g_x)Q_n(x,z) + \int_{z}^{2x} \hat{Q}_n(x,t) d_t V_x(g_x).\]

By Lemma 3
\[ Q_n(x,t) = \sum_{k \geq nt} p_k(nx) \leq \frac{x}{n(t-x)^2}, \quad x < t < 2x \]

and the fact that \( \hat{Q}_n(x,t) \leq Q_n(x,t) \) on \([0,2x)\) we then have

\[
|R_{1n}| \leq \int_{2x}^{-} v_x^z(g_x) \frac{x}{n(z-x)^2} + \frac{x}{n} \int_{2x}^{x} \frac{1}{(t-x)^2} d_t v_x^t(g_x) \]

\[ + \frac{1}{2} \left( \sum_{k>2nx} p_k(nx) \right) \int_{2x}^{x} v_{2x}^z(g_x). \]

Since, by Lemma 5,

\[
\frac{1}{2} \left( \sum_{k>2nx} p_k(nx) \right) \int_{2x}^{x} v_{2x}^z(g_x) \leq \frac{1}{\sqrt{nx\pi}} \left( \frac{e}{4} \right)^{nx} \int_{2x}^{x} v_{2x}^z(g_x) \]

\[ \leq \frac{1}{nx} v_{2x}^z(g_x) \]

and

\[
\frac{x}{n} \int_{2x}^{x} \frac{1}{(t-x)^2} d_t v_x^t(g_x) + \frac{1}{n} \int_{2x}^{x} v_{2x}^z(g_x) = \int_{2x}^{x} \frac{1}{(t-x)^2} d_t v_x^t(g_x). \]

Therefore

\[
|R_{1n}| \leq \int_{2x}^{x} v_x^z(g_x) \frac{x}{n(z-x)^2} + \frac{x}{n} \int_{2x}^{x} \frac{1}{(t-x)^2} d_t v_x^t(g_x). \]

For the integral on the right hand side of the preceding inequality we integrate by parts, get

\[
\int_{2x}^{x} \frac{1}{(t-x)^2} d_t v_x^t(g_x) = \frac{v_x^z(g_x)}{x^2} - \frac{v_x^z(g_x)}{(z-x)^2} + 2 \int_{z}^{x} V_x^t(g_x) \frac{dt}{(t-x)^3}. \]

Hence

\[
|R_{1n}| \leq \frac{1}{n} \left( \frac{V_x^z(g_x)}{x^2} + 2 \int_{z}^{x} V_x^t(g_x) \frac{dt}{(t-x)^3} \right). \]

Replacing the variable in the last integral by \( x + x/\sqrt{t} \), we find that
\[ \int_{z}^{2x} \frac{v_{x}(g_{x})}{(t-x)^{3}} \, dt = \frac{1}{2x^{2}} \int_{1}^{n} \frac{x+x/\sqrt{k}}{v_{x}(g_{x})} \, dt \leq \frac{1}{2x} \sum_{k=1}^{n} \frac{x+x/\sqrt{k}}{v_{x}(g_{x})}. \]

Therefore

\[ (4.10) \quad |R_{1n}| \leq \frac{1}{nx} \left( v_{x}(g_{x}) + \sum_{k=1}^{n} \frac{x+x/\sqrt{k}}{v_{x}(g_{x})} \right) \leq \frac{2}{nx} \sum_{k=1}^{n} \frac{x+x/\sqrt{k}}{v_{x}(g_{x})}. \]

The evaluation of \( R_{2n} \) is relatively easy. By Lemma 5, we have

\[ |R_{2n}| \leq |g_{x}(2x)| \cdot \frac{1}{\sqrt{nx\pi} \left( \frac{4}{e} \right)^{nx}}. \]

But

\[ |g_{x}(2x)| \leq \sum_{k=1}^{n} \frac{x+x/\sqrt{k}}{v_{x}(g_{x})} \]

and

\[ \frac{1}{\sqrt{nx\pi} \left( \frac{4}{e} \right)^{nx}} \leq \frac{1}{nx}. \]

Consequently

\[ (4.11) \quad |R_{2n}| \leq \frac{1}{nx} \sum_{k=1}^{n} \frac{x+x/\sqrt{k}}{v_{x}(g_{x})}. \]

As for \( R_{3n} \), by Lemma 6 and the assumption that \( f(t) = 0(t^{\alpha})(\alpha > 0) \) as \( t \to \infty \), we see that when \( n \) is sufficiently large

\[ (4.12) \quad |R_{3n}| \leq M \sum_{k=2nx}^{\infty} \left( \frac{k}{n} \right)^{\alpha k} p_{k}(nx) \]
for some positive constant $M$.

Hence, from (4.10), (4.11) and (4.12) when $n$ is sufficiently large,

\[
|R_n(f,x)| \leq \frac{3}{n^x} \sum_{k=1}^{n} V_{x-k}^{x+k/\sqrt{n}} (g_x) + O(1)(\frac{\log x}{n^x})^{1/2} \left(\frac{\epsilon}{\log x}\right)^{nx}.
\]

(4.13) now follows from (4.5), (4.9), (4.7), (4.8), (4.13) and the fact that

\[
V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} (g_x) \leq \frac{1}{n^x} \sum_{k=1}^{n} V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} (g_x).
\]
CHAPTER II
ON THE RATE OF CONVERGENCE OF BERNSTEIN POLYNOMIALS
OF FUNCTIONS OF BOUNDED VARIATION

1. INTRODUCTION. If $f$ is a function defined on $[0,1]$ then the Bernstein polynomial $B_n(f)$ of $f$

$$B_n(f,x) = \sum_{k=0}^{n} f(\frac{k}{n}) p_{n,k}(x); \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},$$

converges to $f(x)$ uniformly on $[0,1]$ if $f$ is continuous on $[0,1][14]$. A precise estimate for the rate of convergence of this result was given by T. Popoviciu [15] showing that

$$|B_n(f,x) - f(x)| \leq k \omega_f(n^{-\frac{1}{2}}),$$

where $k$ is a positive constant and $\omega_f$ is the modulus of continuity of $f$. It is known that (1.2) can not be asymptotically improved.¹

As for discontinuous function, Herzog and Hill, et al. ([4], see also [5]) proved that if $f$ is bounded on $[0,1]$ and $x$ is a point of discontinuity of the first kind then

$$\lim_{n \to \infty} B_n(f,x) = \frac{1}{2}(f(x^+)+f(x^-)).$$

In particular, if $f$ is of bounded variation on $[0,1]$ then (1.3) holds for every $x$ in $(0,1)$.

¹ P.C. Sikkema [29] proved in 1961 that the best value of the constant $k$ for which (1.2) is true for each $f \in C[0,1]$ and each $n$ is $(8306 + 837\sqrt{5})/5932$. 26
We shall give here an estimate for the rate of convergence of (1.3) for functions of bounded variation in terms of the arithmetic means of the sequence of total variations and prove that our estimate is essentially the best possible at points of continuity. Results of this type for Fourier series of $2\pi$-periodic functions of bounded variation and for Fourier-Legendre series of functions of bounded variation were proved in [1] and [3].

2. RESULTS. Let $f$ be a function defined on $[0,1]$. For any fixed $x \in (0,1)$, define $g_x$ as follows if both $f(x^+)$ and $f(x^-)$ exist:

$$g_x(t) = \begin{cases} f(t) - f(x^+), & x < t \leq 1 \\ 0, & t = x \\ f(t) - f(x^-), & 0 < t < x. \end{cases}$$

$g_x$ is continuous at the point $t = x$. With this definition of $g_x$ and a simple algebra (1.1) can be expressed as

$$B_n(f,x) = \frac{1}{2}(f(x^+) + f(x^-))$$

$$= B_n(g_x,x) + \frac{1}{2}(f(x^+) - f(x^-))(\sum_{k>n x} p_{kn}(x) - \sum_{k<n x} p_{kn}(x)).$$

Furthermore, if we let $\sigma_c(t) = \text{sign}(t-c)$ then

$$B_n(\sigma_x,x) = \sum_{k=0}^{n} \sigma_x^{(k)} p_{kn}(x)$$

$$= \sum_{k>n x} p_{kn}(x) - \sum_{k<n x} p_{kn}(x)$$

and so
(2.1) \[ B_n(f,x) - \frac{1}{2}(f(x^+) + f(x^-)) = B_n(g_x,x) + \frac{1}{2}(f(x^+) - f(x^-))B_n(\sigma_x,x). \]

It shows that to estimate \( |B_n(f,x) - \frac{1}{2}(f(x^+) + f(x^-)) | \) we only have to evaluate \( B_n(g_x,x) \) and \( B_n(\sigma_x,x) \).

Our main result may be stated as follows:

**Theorem.** Let \( f \) be of bounded variation on \([0,1]\) and \( v^b_a(g_x) \) be the total variation of \( g_x \) on \([a,b]\). Then for every \( x \in (0,1) \) and \( n > (3/\varepsilon (1-x)) \) we have

\[
(2.2) \quad |B_n(f,x) - \frac{1}{2}(f(x^+) + f(x^-))| \leq \frac{3(1-x)-1}{n} \sum_{k=1}^{n} \frac{x+(1-x)/\sqrt{k}}{x-x/\sqrt{k}} (g_x) + \frac{18(x(1-x))^{-5/2}}{n^{1/6}} |f(x^+)-f(x^-)|.
\]

The right hand side of (2.2) converges to zero as \( n \to \infty \) since continuity of \( g_x \) at \( x \) implies that

\[
v^a_x (g_x) \to 0 \quad (\alpha, \beta \to 0+).
\]

If \( f \) is of bounded variation on \([0,1]\) and continuous at \( x \) then the inequality (2.2) becomes

\[
(2.3) \quad |B_n(f,x) - f(x)| \leq \frac{3(1-x)-1}{n} \sum_{k=1}^{n} \frac{x+(1-x)/\sqrt{k}}{x-x/\sqrt{k}} (f).
\]

Let us now consider the function \( f(t) = |t-x| (0 < x < 1) \) on \([0,1]\). We have, for any small \( \delta \),

\[
(2.4) \quad \sum_{k=0}^{n} \frac{k}{n} - x |p_{kn}(x)| \leq \left( \sum_{k=0}^{\varepsilon} + \sum_{k=\varepsilon}^{\delta} \right) \frac{k}{n} - x |p_{kn}(x)| \leq \delta + \frac{1}{\delta} \sum_{k=0}^{n} \frac{k}{n} - x)^2 |p_{kn}(x)|
\]
\[ \leq \delta + \frac{x(1-x)}{n\delta} \]

and
\[
\sum_{k=0}^{n} \left| \frac{k}{n} - x \right| p_{kn}(x) \geq \sum_{|k/n-x| \leq \delta} \left| \frac{k}{n} - x \right| p_{kn}(x)
\]
\[ \geq \frac{1}{\delta} \sum_{|k/n-x| \leq \delta} \left( \frac{k}{n} - x \right)^2 p_{kn}(x) \]
\[ \geq \frac{x(1-x)}{n^{\delta}} - \frac{1}{\delta} \sum_{|k/n-x| > \delta} \left( \frac{k}{n} - x \right)^2 p_{kn}(x). \]

Since
\[
\sum_{|k/n-x| > \delta} \left( \frac{k}{n} - x \right)^2 p_{kn}(x) \leq \frac{1}{\delta^2} \sum_{k=0}^{n} \left( \frac{k}{n} - x \right)^4 p_{kn}(x)
\]
\[ \leq \frac{1}{\delta^2} \left( \frac{3x^2(1-x)^2}{n^2} + \frac{1}{n^3} (x(1-x) - 6x^2(1-x)^2) \right) \]
\[ \leq \frac{x^2(1-x)^2}{n^2\delta^2} (3 + \frac{1}{nx(1-x)}), \]

it follows that
\[
(2.5) \quad \sum_{k=0}^{n} \left| \frac{k}{n} - x \right| p_{kn}(x) \geq \frac{x(1-x)}{n\delta} - \frac{7}{2} \frac{x^2(1-x)^2}{n^2\delta^2},
\]

if \( n > \frac{2}{x(1-x)} \). Choose \( \delta = 2 \left( \frac{x(1-x)}{n} \right)^{\frac{1}{2}} \), we obtain from (2.4) that
\[
\sum_{k=0}^{n} \left| \frac{k}{n} - x \right| p_{kn}(x) \leq \frac{5}{2} \frac{(x(1-x))^{\frac{3}{2}}}{n^{\frac{3}{2}}}
\]
and from (2.5) that
\[
\sum_{k=0}^{n} \left| \frac{k}{n} - x \right| p_{kn}(x) \geq \frac{1}{2} \frac{(x(1-x))^{\frac{3}{2}}}{n^{\frac{3}{2}}} - \frac{7}{8} \frac{x^2(1-x)^2}{n^2(x(1-x)/n)^{3/2}} \]
\[ \geq \frac{1}{16} \frac{(x(1-x))^{\frac{3}{2}}}{n^{\frac{3}{2}}}. \]
Therefore, if \( n > 2x(1-x) \) then we have

\[
(2.6) \quad \frac{1}{16} \left( \frac{x(1-x)}{n^{1/2}} \right)^{1/2} \leq \sum_{k=0}^{n} \left| \frac{k}{n} - x \right| \frac{p_{kn}(x)}{n^{1/2}} \leq \frac{5}{2} \left( \frac{x(1-x)}{n^{1/2}} \right)^{1/2}.
\]

On the other hand, from (2.3) since \( V_{x}^{x+\alpha}(f) = \alpha - \beta \), it follows that

\[
(2.7) \quad |B_n(f,x) - f(x)| = \sum_{k=0}^{n} \left| \frac{k}{n} - x \right| p_{kn}(x) \leq \frac{3}{n} \frac{(x(1-x))^{1/2}}{\sqrt{k} x^{1/2}} \leq \frac{3}{n} \frac{(x(1-x))^{1/2}}{\sqrt{k} x^{1/2}} \leq \frac{3}{n} \frac{(x(1-x))^{1/2}}{x^{1/2}}.
\]

Hence by comparing (2.6) and (2.7) we see that (2.3) cannot be asymptotically improved for functions of bounded variation at points of continuity as we have mentioned before.

A more precise version of (2.6)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} \left| \frac{k}{n} - x \right| p_{kn}(x) = (2x(1-x))^{1/2}
\]

was proved in [16].

3. PROOF. EVALUATION OF \( B_n(\sigma_x,x) \). The convergence of the sequence \( B_n(\sigma_x,x) \) to zero as \( n \to \infty \) follows immediately from the well known central limit theorem of probability. However, what we are interested here is finding an estimate for the rate of convergence of this result. To do so, we first decompose \( B_n(\sigma_x,x) \) into three parts as follows:
\[(3.1) \quad B_n(x, x) = A_n(x) - B_n(x) + C_n(x)\]

with

\[A_n(x) = \sum_{x<k/n<x+n} -\alpha P_{kn}(x),\]

\[B_n(x) = \sum_{x-n}^{-\alpha} P_{kn}(x),\]

\[C_n(x) = \left(-\sum_{0<k/n<x-n} -\alpha + \sum_{x+n}^{-\alpha} P_{kn}(x)\right) \frac{1}{n},\]

where \(0 < \alpha < 1\).

The evaluation of \(C_n(x)\) is relatively easy. Observe that

\[\left|\sum_{k=0}^{n} \frac{k}{n} - x\right| \leq n^{2\alpha} \cdot \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 P_{kn}(x).\]

Since

\[\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 P_{kn}(x) = x(1-x)/n,\]

it follows that

\[|C_n(x)| \leq \frac{x(1-x)}{n^{1-2\alpha}} \leq \frac{1/4}{n^{1-2\alpha}}.\]

To evaluate \(A_n(x)\) and \(B_n(x)\), we need a convenient asymptotic form for those \(P_{kn}(x)\)'s satisfying the inequality \(|k/n-x| \leq n^{-\alpha}\). Using Stirling's formula

\[n! = (2\pi n)^{\frac{1}{2}} n^n e^{-n} H_n,\]

\[H_n = e^{\theta_n/12n}, \quad 0 < \theta_n < 1,\]

\[A\ more\ precise\ version\ of\ (3.2)\ can\ be\ found\ in\ [17],\ pp.15.\]

However, the role played by \(C_n(x)\) in our proof is not essential, (3.2) is all we need.
we obtain

\[ p_{kn}(x) = \left( \frac{n}{2\pi k(n-k)} \right)^{\frac{3}{2}} w_{kn}(x) H_{kn}, \]

where

\[ w_{kn}(x) = \frac{n^n}{k^k(n-k)^{n-k}} x^k(1-x)^{n-k}; \]

\[ H_{kn}(x) = \frac{H_n}{H_k H_{n-k}}. \]

It is easy to see that if \( n > (2/x(1-x))^{1/\alpha} \) and \( \left| \frac{k}{n} - x \right| \leq n^{-\alpha} \) then

\[ |H_{kn} - 1| \leq \frac{2}{3nx(1-x)} \]

and

\[
\left| \left( \frac{n}{2\pi k(n-k)} \right)^{\frac{3}{2}} - \left( \frac{1}{2\pi nx(1-x)} \right)^{\frac{3}{2}} \right| \leq \frac{4}{3\sqrt{2\pi} n^{\alpha+\frac{3}{2}}(x(1-x))^{3/2}}.
\]

On the other hand since \( \frac{n}{2\pi k(n-k)} \right)^{\frac{3}{2}}, w_{kn}(x) = p_{kn}(x)/H_{kn} \) is uniformly bounded by 2, it follows that

\[
(3.3) \quad |p_{kn}(x) - \left( \frac{1}{2\pi nx(1-x)} \right)^{\frac{3}{2}} w_{kn}(x)|
\]

\[
\leq \left| \left( \frac{n}{2\pi k(n-k)} \right)^{\frac{3}{2}} w_{kn}(x)(H_{kn} - 1) \right| + \left| w_{kn}(x) \right| \left| \left( \frac{n}{2\pi k(n-k)} \right)^{\frac{3}{2}} - \left( \frac{1}{2\pi nx(1-x)} \right)^{\frac{3}{2}} \right|
\]

\[
\leq \frac{4}{3nx(1-x)} + \frac{4}{3\sqrt{2\pi} n^{\alpha+\frac{3}{2}}(x(1-x))^{3/2}}
\]

if \( n > \left( \frac{2}{x(1-x)} \right)^{1/\alpha} \) and \( \left| \frac{k}{n} - x \right| \leq n^{-\alpha} \).

The following lemma which gives us a precise estimation of \( w_{kn}(x) \) is the key to the evaluation of \( B_n(\sigma_x, x) \).

**LEMMA (Laplace's formula of Probability).** If \( \frac{1}{3} < \alpha < 1 \) and \( n \geq (3/x(1-x))^{2/(3\alpha-1)} \) then
uniformly for all $k$ satisfying the inequality $|\frac{k}{n} - x| \leq n^{-\alpha}$.

In particular, $W_{kn}(x)$ is then bounded by 2.

**Proof of the Lemma.** By Taylor's formula for $|u| < 1$

$$\log(1+u) = u - \frac{1}{2} u^2 + \frac{1}{3} u^3 (1 + tu)^{-3}$$

$$= u - \frac{1}{2} u^2 [1 - \frac{2}{3} u(1 + tu)^{-3}]$$

$$= u - \frac{1}{2} u^2 \rho,$$

where $0 < t < 1$, $\rho = 1 - \frac{2}{3} u(1+tu)^{-3} = 1 + \varepsilon u$, where $\varepsilon = -\frac{2}{3}(1+tu)^{-3}$. If $|u| \leq \frac{1}{2}$ then $|\varepsilon| \leq 16/3$ and $|\rho| \leq 11/3$. Similarly we can express $\log(1 - u)$ as

$$\log(1 - u) = -u - \frac{1}{2} u^2 \rho_1$$

with $\rho_1 = 1 + \varepsilon_1 u$ for some $\varepsilon_1$ such that $|\varepsilon_1| \leq 16/3$ and $|\rho_1| \leq 11/3$ if $|u| \leq \frac{1}{2}$.

Since

$$-\log W_{kn}(x) = k \log(1+x^{-1}(\frac{k}{n} - x)) + (n-k)\log(1-(1-x)^{-1}(\frac{k}{n} - x))$$

and $|x^{-1}(\frac{k}{n} - x)| \leq \frac{1}{2}$, $|(1-x)^{-1}(\frac{k}{n} - x)| \leq \frac{1}{2}$ if $n \geq (\frac{3}{x(1-x)})^{2/3\alpha-1}$. Therefore

$$-\log W_{kn}(x) = k(x^{-1}(\frac{k}{n} - x) - \frac{1}{2} x^{-2}(\frac{k}{n} - x)^2 \rho)$$

$$- (n-k)((1-x)^{-1}(\frac{k}{n} - x) + \frac{1}{2}(1-x)^{-2}(\frac{k}{n} - x)^2 \rho_1)$$
\[
= (n x + n \left( \frac{k}{n} - x \right))(x^{-1}(\frac{k}{n} - x) - \frac{1}{2}x^{-2}(\frac{k}{n} - x)^2 \rho)
\]
\[
- (n(1-x) - n (\frac{k}{n} - x))( (1-x)^{-\frac{1}{2}} \left( \frac{k}{n} - x \right) + \frac{1}{2}(1-x)^{-2} (\frac{k}{n} - x)^2 \rho_1)
\]
\[
= n (\frac{k}{n} - x)^2 (x^{-1}(1 - \frac{1}{2}x - \frac{1}{2}x^{-1} \rho (\frac{k}{n} - x))
\]
\[
+ (1-x)^{-1}(1 - \frac{1}{2} \rho_1 + \frac{1}{2}(1-x)^{-1}(\frac{k}{n} - x) \rho_1)
\]
\[
= (2x(1-x))^{-1} n (\frac{k}{n} - x)^2 + n (\frac{k}{n} - x)^2 (x^{-1}(1 - \frac{1}{2}x - \frac{1}{2}x^{-1} \rho (\frac{k}{n} - x))
\]
\[
+ (1-x)^{-1}(\frac{1}{2} - \frac{1}{2} \rho_1 + \frac{1}{2}(1-x)^{-1} \rho_1 (\frac{k}{n} - x))
\]
\[
= (2x(1-x))^{-1} n (\frac{k}{n} - x)^2 + \frac{1}{2}n (\frac{k}{n} - x)^3 (-x^{-2}(\varepsilon + \rho)
\]
\[
+ (1-x)^{-2}(\varepsilon_1 + \rho_1))
\]

and so

\[
| \log W_{kn}(x) + (2x(1-x))^{-1} n (\frac{k}{n} - x)^2 | \leq \frac{9}{2n^{3\alpha-1}(x(1-x))^2}.
\]

But if \( \frac{1}{3} < \alpha < 1 \) and \( n^2 (3/x(1-x))^2 / (3\alpha-1) \) then we have

\[
| \exp \left( \frac{9}{2n^{3\alpha-1}(x(1-x))^2} \right) - 1 | \leq \frac{9}{n^{3\alpha-1}(x(1-x))^2}.
\]

Hence

\[
| W_{kn}(x) - \exp \left( -(2x(1-x))^{-1} n (\frac{k}{n} - x)^2 \right) |
\]
\[
\leq \exp \left( -(2x(1-x))^{-1} n (\frac{k}{n} - x)^2 \right) \exp \left( \log W_{kn}(x) + (2x(1-x))^{-1} n (\frac{k}{n} - x)^2 - 1 \right)
\]
\[
\leq \frac{9}{n^{3\alpha-1}(x(1-x))^2}.
\]

The boundedness of \( W_{kn}(x) \) follows from the fact that

\[
\frac{9}{n^{3\alpha-1}(x(1-x))^2} \leq 1.
\]
if \( n > \frac{(3/x(1-x))^{2/(3\alpha-1)}}{2} \). This completes the proof of the lemma.

Consequently, if \( \frac{1}{3} < \alpha < 1 \) and \( n > \frac{(3/x(1-x))^{2/(3\alpha-1)}}{2} \), by Laplace's formula and (3.3) we obtain

\[
(3.4) \quad |p_{kn}(x) - (2\pi n x(1-x))^{-\frac{1}{2}} \exp\left(-\left(2x(1-x)\right)^{-1} n\left(\frac{k}{n} - x\right)^2\right)|
\]

\[
\leq |p_{kn}(x) - (2\pi n x(1-x))^{-\frac{1}{2}} \mathcal{W}_{kn}(x)| + \left| (2\pi n x(1-x))^{-\frac{1}{2}} \mathcal{W}_{kn}(x) - (2\pi n x(1-x))^{-\frac{1}{2}} \exp\left(-\left(2x(1-x)\right)^{-1} n\left(\frac{k}{n} - x\right)^2\right) \right|
\]

\[
\leq \frac{\alpha}{3n x(1-x)} + \frac{\alpha + \frac{1}{2}}{\sqrt{2\pi} n^{\frac{3}{2}} (x(1-x))^3} + \frac{9}{\sqrt{2\pi} n^{\frac{3}{2}} (x(1-x))^{5/2}}
\]

for all \( k \) satisfying the inequality \( \left| \frac{k}{n} - x \right| \leq n^{-\alpha} \).

However, to estimate the sums of \( p_{kn}(x) \) we need a more convenient form. With a simple algebra we can show that

\[
(2\pi n x(1-x))^{-\frac{1}{2}} \exp\left(-\left(2x(1-x)\right)^{-1} n\left(\frac{k}{n} - x\right)^2\right)
\]

\[
= \left(\frac{n}{2\pi x(1-x)}\right)^{\frac{1}{2}} \int_{k/n}^{(k+1)/n} \exp\left(-\frac{n}{2x(1-x)} (u-x)^2\right) du
\]

\[
+ \left(\frac{n}{2\pi x(1-x)}\right)^{\frac{1}{2}} \int_{k/n}^{(k+1)/n} \exp\left(-\frac{n}{2x(1-x)} \left(\frac{k}{n} - x\right)^2\right)
\]

\[
\cdot (1 - \exp\left(-\frac{n}{2x(1-x)} (u - \frac{k}{n})(u + \frac{k}{n} - 2x)\right)) du.
\]

If \( n > \frac{3}{x(1-x)^{2/(3\alpha-1)}} \) then absolute value of the second term on the right hand side of the last equation

\[
\leq (2\pi n x(1-x))^{-\frac{1}{2}} \max_{\frac{k}{n} \leq u \leq \frac{k+1}{n}} |1 - \exp\left(-\frac{n}{2x(1-x)} (u - \frac{k}{n})(u + \frac{k}{n} - 2x)\right)|
\]
\[
\leq (2\pi n x(1-x))^{-\frac{1}{2}} \cdot \max_{\frac{k}{n} < u < \frac{k+1}{n}} \left| -\frac{n}{2x(1-x)} (u - \frac{k}{n})(u + \frac{k}{n} - 2x) \right|
\]
\[
\leq \frac{1}{\sqrt{2\pi}} \frac{1}{n^{\frac{1}{2}}(x(1-x))^{3/2}} \cdot \max_{\frac{k}{n} < u < \frac{k+1}{n}} |u + \frac{k}{n} - 2x|
\]
\[
\leq \frac{3}{\sqrt{2\pi}} \frac{1}{n^{\alpha + \frac{1}{2}}(x(1-x))^{3/2}}
\]

Hence from (3.4) and the above inequality we see that

\[
(3.5) \quad |p_{kn}(x) - (\frac{n}{2\pi x(1-x)})^{\frac{1}{2}} \int_{k/n}^{(k+1)/n} \exp(-\frac{n}{2x(1-x)} (u-x)^2) du| \leq \frac{4}{3nx(1-x)} + \frac{17}{3\sqrt{2\pi}} \frac{\alpha + 1/2}{n^{\alpha + 1/2}(x(1-x))^{3/2}} + \frac{9}{\sqrt{2\pi}} \frac{3\alpha - 1/2}{n^{3\alpha - 1/2}(x(1-x))^{5/2}}
\]

if \(n > \frac{3}{x(1-x)} \) and \(|\frac{k}{n} - x| \leq n^{-\alpha} (\frac{1}{3} < \alpha < 1)\).

We now apply (3.5) to estimate the sums

\[
A_n(x) = \sum_{x < k/n < x+n} a p_{kn}(x).
\]

Assuming that \(3/\alpha < \alpha < \frac{1}{2}\).

Let \(k'\) and \(k''\) be the smallest and largest of the \(k\) resp. which satisfy the inequality \(x < \frac{k}{n} < x+n^{-\alpha}\). Since the number of \(k\)'s between \(k'\) and \(k''\) is at most \([n^{1-\alpha}]\), by (3.5) it follows that

\[
|A_n(x) - (\frac{n}{2\pi x(1-x)})^{\frac{1}{2}} \int_{x}^{x+n^{-\alpha}} \exp(-\frac{n}{2x(1-x)} (u-x)^2) du| \leq \left| (\frac{n}{2\pi x(1-x)})^{\frac{1}{2}} \left( \int_{x+n^{-\alpha}}^{(k''+1)/n} - \int_{x}^{k'/n} \right) \exp(-\frac{n}{2x(1-x)} (u-x)^2) du \right|
\]
\[\begin{align*}
&+ \frac{4}{3n^2 x(1-x)} + \frac{17}{3\sqrt{2\pi} n^{2\alpha-\frac{1}{2}}(x(1-x))^{3/2}} + \frac{9}{\sqrt{2\pi} n^{4\alpha-3/2}(x(1-x))^{5/2}} \\
&\leq \frac{2}{\sqrt{2\pi} n^{\frac{1}{2}}(x(1-x))^{3/2}} + \frac{4}{3n^2 x(1-x)} + \frac{17}{3\sqrt{2\pi} n^{2\alpha-\frac{1}{2}}(x(1-x))^{3/2}} \\
&+ \frac{9}{\sqrt{2\pi} n^{4\alpha-3/2}(x(1-x))^{5/2}} .
\end{align*}\]

Moreover, since \((x(1-x))^{-a} \geq (x(1-x))^{-b}\) if \(a > b > 0\) and \(n^{-\alpha} < n^{-(2\alpha-\frac{1}{2})} < n^{-(\frac{\alpha}{2}-\frac{3}{2})}\) if \(3/\beta < \alpha < 1\), we find that

\[|A_n(x) - \left(\frac{n}{3\pi x(1-x)}\right)^{3/2} \int_{x}^{x+n^{-\alpha}} \exp(-\frac{n}{2x(1-x)}(u-x)^2)du|\]

\[\leq \frac{1}{n^{\frac{\alpha}{2}-\frac{3}{2}}(x(1-x))^{5/2}} \left(\frac{2}{\sqrt{2\pi}} + \frac{4}{3} + \frac{17}{3\sqrt{2\pi}} + \frac{9}{\sqrt{2\pi}}\right)\]

\[\leq \frac{8}{n^{\frac{\alpha}{2}-\frac{3}{2}}(x(1-x))^{5/2}} ,\]

or

\[(3.6) \quad |A_n(x) - \frac{1}{\sqrt{\pi}} \int_{0}^{M_n} e^{-v^2} dv| \leq \frac{8}{n^{\frac{\alpha}{2}-\frac{3}{2}}(x(1-x))^{5/2}} ,\]

where \(M_n = n^{\frac{1}{2} - \alpha}(2x(1-x))^{\frac{1}{2}}\).

With an easy calculation we can show that

\[\frac{\sqrt{\pi}}{2} \sqrt{1-e^{-t^2}} \leq \int_{0}^{t} e^{-v^2} dv, \quad t > 0.\]

Therefore

\[\frac{1}{\sqrt{\pi}} \int_{t}^{\infty} e^{-v^2} dv \leq \frac{1}{2}(1 - \sqrt{1 - e^{-t^2}}) .\]

On the other hand, since \(1-(1-y)^{\frac{1}{2}} \leq y/2\) if \(0 \leq y < 1\) and
\( e^{-z} \leq (1+z)^{-1} \) if \( z \geq 0 \), it follows that
\[
\frac{1}{\sqrt{\pi}} \int_{M_n} e^{-v^2} dv \leq \frac{1}{4} \frac{1}{1+M_n^2} < \frac{1/8}{n^{1-2\alpha}}.
\]
Hence, from (3.6),
\[
\left| A_n(x) - \frac{1}{2} \right| \leq \frac{1/8}{n^{1-2\alpha}} + \frac{8}{n^{4\alpha-3/2}(x(1-x))^{5/2}}
\]
\[
\leq \frac{1/8}{n^{1-2\alpha}(x(1-x))^{5/2}} + \frac{8}{n^{4\alpha-3/2}(x(1-x))^{5/2}}.
\]
However, it is easy to see that, on \((\frac{3}{8}, \frac{1}{2})\), the right hand side of the last inequality asymptotically drops most rapidly when \( \alpha = 5/12 \). Therefore, by choosing \( \alpha = 5/12 \), we get the best estimate for \( A_n(x) \), namely,
\[
(3.7) \quad \left| A_n(x) - \frac{1}{2} \right| \leq \frac{(65/8)(x(1-x))^{-5/2}}{n^{1/6}}
\]
if \( n \geq (3/x(1-x))^{2/(3\alpha-1)} = (3/x(1-x))^8 \).

The evaluation of \( B_n(x) \) is similar to that of \( A_n(x) \). Repeat the same process we can prove that
\[
(3.8) \quad \left| B_n(x) - \frac{1}{2} \right| \leq \frac{(65/8)(x(1-x))^{-5/2}}{n^{1/6}}
\]
if \( n \geq (3/x(1-x))^8 \).

Then by (3.1), (3.2), (3.7) and (3.8) with \( \alpha = 5/12 \) in (3.2) follows that
\[
(3.9) \quad \left| B_n(c, x) \right| \leq \frac{18(x(1-x))^{-5/2}}{n^{1/6}}, \quad n \geq (3/x(1-x))^8.
\]
EVALUATION OF $B_n(g(x,x))$. As we know, $B_n(g(x,x)) = \sum_{k=0}^{n} \frac{g(x,k)}{n}$. 

$p_{kn}(x)$ may be written in the form of a Lebesgue-Stieltjes integral in the variable $t$

\[(3.10) \quad \sum_{k=0}^{n} \frac{g(x,k)}{n} p_{kn}(x) = \int_{0}^{1} g(x,t) d_{t}k_{n}(x,t)\]

with the kernel

\[k_{n}(x,t) = \begin{cases} \sum_{k<tn} p_{kn}(x), & 0 < t \leq 1 \\ 0, & t = 0. \end{cases}\]

To estimate $\int_{0}^{1} g(x,t) d_{t}k_{n}(x,t)$, we decompose it into three parts, as follows.

\[(3.11) \quad \int_{0}^{1} g(x,t) d_{t}k_{n}(x,t) = L_{n}(f,x) + M_{n}(f,x) + R_{n}(f,x)\]

with

\[L_{n}(f,x) = \int_{0}^{x-x/\sqrt{n}} g(x,t) d_{t}k_{n}(x,t),\]

\[M_{n}(f,x) = \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} g(x,t) d_{t}k_{n}(x,t),\]

\[R_{n}(f,x) = \int_{x+(1-x)/\sqrt{n}}^{1} g(x,t) d_{t}k_{n}(x,t).\]

First, we evaluate $M_{n}(f,x)$. For $t \in [x - \frac{x}{\sqrt{n}}, x + \frac{1-x}{\sqrt{n}}]$, we have

\[|g(x,t)| = |g(x(t)) - g(x(x))| \leq \frac{x+(1-x)/\sqrt{n}}{x-x/\sqrt{n}} (g(x)),\]

and so

\[|M_{n}(f,x)| \leq \frac{x+(1-x)/\sqrt{n}}{x-x/\sqrt{n}} (g(x)) \cdot \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} d_{t}k_{n}(x,t).\]
Since
\[ \int_a^b d_t k_n(x,t) \leq 1 \] for any \([a,b] \subseteq [0,1]\),
therefore

\[ |M_n(f,x)| \leq \frac{x+(1-x)/\sqrt{n}}{|x-x/\sqrt{n}|} (g_x). \] (3.12)

To estimate \( L_n(f,x) \), let \( y = x-x/\sqrt{n} \) and note that \( g_x \) is of bounded variation. Using partial integration for Lebesgue-Stieltjes integral we find that

\[ L_n(f,x) = g_x(y+)k_n(x,y) - \int_0^y \hat{k}_n(x,t) d_t g_x(t), \]

where \( \hat{k}_n(x,t) \) is the normalized form of \( k_n(x,t) \). Since

\[ k_n(x,y+) = k_n(x,y), \quad 0 < y \leq 1 \]

and

\[ |g_x(y+)| = |g_x(y+) - g_x(x)| \leq V_x^y (g_x), \]

where \( V_x^y (g_x) = \lim_{\varepsilon \to 0^+} V_x^{y+\varepsilon} (g_x) \), it follows that

\[ |L_n(f,x)| \leq V_x^y (g_x)k_n(x,y) + \int_0^y k_n(x,t) d_t (-v_x^t (g_x)). \]

By the well known inequality

\[ k_n(x,t) \leq \frac{x(1-x)}{n(x-t)^2}, \quad 0 \leq t < x \]

(see, e.g., [17], pp.6) and the fact \( \hat{k}_n(x,t) \leq k_n(x,t) \) on \((0,1]\), we obtain
\begin{align*}
|L_n(f,x)| &\leq \int_0^n x(y)(g_x)\frac{x(1-x)}{y(n-y)^2} + \frac{x(1-x)}{n} \int_0^n 1 \frac{1}{(x-t)^2} dt(-V_t^x(g_x)) \\
&+ \frac{(1-x)^n}{2} \int_0^n 1 \frac{1}{(x-t)^2} dt(-V_t^x(g_x)).
\end{align*}

Actually, since \((1-x)^n/2 \leq x(1-x)/nx^2\) and

\[
\frac{x(1-x)}{n} \int_0^n 1 \frac{1}{(x-t)^2} dt(-V_t^x(g_x)) + \frac{x(1-x)}{nx^2} \int_0^n 1 \frac{1}{(x-t)^2} dt(-V_t^x(g_x))
\]

it follows that

\[
|L_n(f,x)| \leq \int_0^n x(y)(g_x)\frac{x(1-x)}{y(n-y)^2} + \frac{x(1-x)}{n} \int_0^n 1 \frac{1}{(x-t)^2} dt(-V_t^x(g_x)).
\]

Furthermore, since

\[
\int_0^n 1 \frac{1}{(x-t)^2} dt(-V_t^x(g_x)) = -\frac{V^x_{y+}(g_x)}{(x-y)^2} + \frac{V^x_0(g_x)}{x^2} + 2 \int_0^n V^x_t(g_x)\frac{dt}{(x-t)^3}
\]

and \(y = x-x/\sqrt{n}\), we have

\[
|L_n(f,x)| \leq \frac{x(1-x)}{n} \frac{V^x_0(g_x)}{x^2} + 2 \int_0^{x-x/\sqrt{n}} V^x_t(g_x)\frac{dt}{(x-t)^3}.
\]

Replacing the variable \(t\) in the last integral by \(x-x/\sqrt{t}\), we find that

\[
\int_0^{x-x/\sqrt{n}} V^x_t(g_x)\frac{dt}{(x-t)^3} = \frac{1}{2x^2} \int_1^n V^x_{x-x/\sqrt{k}}(g_x)dx
\]

Hence
To estimate \( R_n(f,x) \), let \( z = x + (1-x)/\sqrt{n} \) and define \( H_n(x,t) \) on \([0,1] \) as follows:

\[
H_n(x,t) = 1 - k_n(x,t-), \quad 0 \leq t < 1 \\
H_n(x,1) = 0.
\]

Then

\[
R_n(f,x) = - \int_z^1 g_x(t) d_t H_n(x,t). 
\]

Using partial integration for Lebesgue-Stieltjes integral

\[
R_n(f,x) = g_x(z-)[H_n(x,z-)] + \int_z^1 \hat{H}_n(x,t)d_t g_x(t),
\]

where \( \hat{H}_n(x,t) \) is the normalized form of \( H_n(x,t) \). Since

\[
H_n(x,z-) = H_n(x,z), \quad 0 \leq z < 1
\]

and

\[
|g_x(z-)| = |g_x(z-) - g_x(x)| \leq V^z_x(g_x),
\]

so that

\[
|R_n(f,x)| \leq V^z_x(g_x)H_n(x,z) + \int_z^1 \hat{H}_n(x,t)d_t V^t_x(g_x).
\]

By inequality

\[
H_n(x,t) = \sum_{k\geq nt} p_{kn}(x) \leq \frac{x(1-x)}{n(x-t)^2}, \quad x \leq t < 1
\]
and the fact that \( \hat{H}_n(x,t) \leq H_n(x,t) \) on \([0,1)\), we have then

\[
|R_n(f,x)| \leq V^z_x(g_x) \frac{x(1-x)}{n(x-1)^2} + \frac{x(1-x)}{n} \int_0^1 \frac{1}{(x-t)^2} \, dt \cdot V_x(g_x) + \frac{x}{2} V^1_x(g_x). 
\]

But as we did for \( L_n(f,x) \), since \( x/2 < x(1-x)/n(1-x)^2 \) and

\[
\frac{x(1-x)}{n} \int_0^1 \frac{1}{(x-t)^2} \, dt \cdot V_x(g_x) + \frac{x(1-x)}{n(1-x)^2} V^1_x(g_x)
\]

\[
= \frac{x(1-x)}{n} \int_0^1 \frac{1}{(x-t)^2} \, dt (V_x(g_x)),
\]

We actually have

\[
|R_n(f,x)| \leq \frac{x(1-x)}{n} \left( \frac{V_x(g_x)}{(x-z)^2} + \int_0^1 \frac{1}{(x-t)^2} \, dt (V_x(g_x)) \right).
\]

Furthermore, since

\[
\int_0^1 \frac{1}{(x-t)^2} \, dt \cdot V_x(g_x) = \frac{1}{V_x(g_x)} - \frac{V_x(g_x)}{(z-x)^2} + 2 \int_0^1 \frac{V_x(g_x)}{(t-x)^3} \, dt
\]

and \( z = x + (1-x)/\sqrt{n} \), the preceding inequality becomes

\[
|R_n(f,x)| \leq \frac{x(1-x)}{n} \left( \frac{V_x(g_x)}{(1-x)^2} + 2 \int_{x+(1-x)/\sqrt{n}}^1 V_x(g_x) \frac{dt}{(t-x)^3} \right).
\]

Replacing the variable \( t \) by \( x+(1-x)/\sqrt{n} \),

\[
\int_{x+(1-x)/\sqrt{n}}^1 \frac{V_x(g_x)}{(t-x)^3} \, dt = \frac{1}{2(1-x)^2} \int_0^1 \frac{V_x(g_x)(g_x)}{(t-x)^3} \, dt
\]

\[
\leq \frac{1}{2(1-x)^2} \sum_{k=1}^{n-1} V_x(g_x).
\]

Therefore
From (3.10), (3.11), (3.12), (3.13) and (3.14), it follows that

\begin{equation}
|R_n(f, x)| \leq \frac{x}{n(1-x)} \left( V_x^n(g_x) + \sum_{k=1}^{n-1} \frac{x+(1-x)\sqrt{k}}{V_x} (g_x) \right) \\
\leq \frac{2}{nx(1-x)} \sum_{k=1}^{n} \frac{x+(1-x)/\sqrt{k}}{V_x} (g_x). 
\end{equation}

Our theorem now follows from (2.1), (3.9) and (3.15).
CHAPTER III
ON HERMITE-FEJÉR INTERPOLATION FOR
FUNCTIONS OF BOUNDED VARIATION

1. **INTRODUCTION.** Let $f$ be a function on $[-1,1]$. The polynomial $H_n(f,x)$ of Hermite-Fejér interpolation based on the zeros $x_{kn} = \cos\left(\frac{2k-1}{2n}\pi\right)$, $k = 1, 2, \ldots, n$, of the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$ is defined by

\[ H_n(f,x) = \sum_{k=1}^{n} f(x_{kn})H_{kn}(x); \]

(1.1)

\[ H_{kn}(x) = (1-x \cdot x_{kn}) \left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2. \]

It was proved by L. Fejér [18] that for a continuous function $f$ on $[-1,1]$ \( \lim_{n \to \infty} H_n(f,x) = f(x) \) uniformly on $[-1,1]$. Since then, this subject has been studied extensively and various quantitative versions of Fejér's result have been given (see, e.g., [20, 21, 22, 23]). Among them, R. Bojanic [19] proved that

\[ |H_n(f,x) - f(x)| < \frac{c}{n} \sum_{k=1}^{n} \omega_f \left(\frac{1}{k}\right), \]

(1.2)

where $c$ is a positive constant and $\omega_f$ is the modulus of continuity of $f$. This estimate improved all earlier results. Later on, P. O. Vertesi [24] and R. B. Saxena [25] improved further the inequality (1.2) by showing that

\[ |H_n(f,x) - f(x)| \leq \frac{c}{n} \sum_{k=1}^{n} \left( \omega_f \left(\frac{1}{k}\right) + \omega_f \left(\frac{\sqrt{1-x^2}}{k}\right) + \omega_f \left(\frac{1}{k^2}\right) \right). \]

(1.3)

(1.3) shows that the approximation is considerably better at the end points than it may be at the center of the interval.
Most recent result in this direction was given by S. J. Goodenough and T. M. Mills [26]. They proved that

\[
(1.4) \quad |H_n(f, x) - f(x)| \leq \frac{c_1}{n} T_n^2(x) \sum_{k=1}^{n} \left( \omega_f\left(\frac{1-x^2}{k}\right) + \omega_f\left(\frac{1}{k^2}\right) \right)
\]

\[+ c_2 \omega_f\left(\frac{|T_n(x)|}{n}\right),\]

where \(c_1\) and \(c_2\) are positive constants. (1.4) reflects the fact that if \(x\) is a node of interpolation then \(|H_n(f, x) - f(x)| = 0\).

In this paper, we shall study the behavior of Hermite-Fejér interpolatory polynomials for functions of bounded variation. More precisely, we shall give an estimate for the rate of convergence of \(H_n(f, x)\) for functions of bounded variation on \([-1, 1]\) at points of continuity and prove that our estimate is essentially the best possible. In addition, we shall prove that at points of discontinuity where \(f(x^+) \neq f(x^-)\) Hermite-Fejér interpolatory polynomials of functions of bounded variation do not converge by finding explicitly expressions of \(\limsup H_n(f, x)\) and \(\liminf H_n(f, x)\).

2. RESULTS and REMARKS. Our first result is as follows:

**THEOREM 1.** If \(f\) is a function of bounded variation on \([-1, 1]\) and continuous at \(x \in (-1, 1)\) then for all \(n\) sufficiently large

\[
(2.1) \quad |H_n(f, x) - f(x)| \leq \frac{96 T_n^2(x)}{n} \sum_{k=1}^{n} V_{x-\pi/k}(f) + 2 V_{x-\pi/2 |T_n(x)|/2n(f)}\]
Here $V_a^b(f)$ is the total variation of $f$ on $[a,b]$.  \(^1\)

The right hand side of (2.1) converges to zero as $n \to \infty$ since continuity of $f$ at $x$ implies that

$$V_{x-\delta}^{x+\delta}(f) \to 0 \quad (\delta \to 0^+).$$

As far as the precision of (2.1) is concerned, (2.1) gives clearly the exact estimate at nodes of interpolation. As for the other points, consider the Hermite-Fejér interpolatory polynomials of the function $f(x) = x^2$ at $x = 0$, for even $n$. Since $T_n(0) = 1$ if $n$ is an even integer, we have

$$H_n(f,0)-f(0) = \sum_{k=1}^{n} \frac{T_n^2(0)}{n^2} = \frac{1}{n}.$$  \(^2\)

On the other hand, from (2.1) it follows that

$$|H_n(f,0)-f(0)| \leq \frac{192}{n} \sum_{k=1}^{n} V_{-\pi/k}(f) + 4 V_{-\pi/2n}(f)$$

$$\leq \frac{192}{n} \sum_{k=1}^{n} \frac{\pi}{k} V_{\delta}(f) + 4 V_{\delta}(f).$$

Since $V_{\delta}(f) = \delta^2$, we have

\[^1\] We assume here and in the rest of the paper that $f$ is extended to $[-1-\pi, 1+\pi]$ by assuming the value $f(1)$ on $(1,1+\pi]$ and $f(-1)$ on $[-1-\pi, -1)$. Hence the total variation of $f$ on any part of $[1,1+\pi]$ and $[-1-\pi, -1]$ is zero. i.e., $V_a^b(f)$ is actually the total variation of $f$ on $[a,b] \cap [-1,1]$ even if $1 < b \leq 1+\pi$ or $-1-\pi \leq a < -1$.  

\[^2\] We have $2 \leq \frac{\pi}{k}$ for all $k$. Thus, if $k$ is an integer, we get

$$\sum_{k=1}^{n} \frac{\pi}{k} \leq \int_{1}^{n} \frac{\pi}{x} \, dx = \pi \ln n.$$
for some $c > 1$. Hence for the function $f(x) = x^2$ when $n$ is an even integer we have

$$\frac{1}{n} < |H_n(f,0) - f(0)| \leq \frac{c}{n}$$

for some positive constant $c$. Therefore (2.1) can not be improved asymptotically.

However, if $x$ is a point of discontinuity of $f$ where $f(x^+) \neq f(x^-)$, the sequence $(H_n(f,x))$ is no longer convergent. This follows from the following theorem.

**THEOREM 2.** If $f$ is a function of bounded variation on $[-1,1]$ and $x \in (-1,1)$ then

$$\lim_{n \to \infty} \sup_{n} \inf_{h} H_n(f,x) = \frac{1}{2}(f(x^+)+f(x^-)) + \frac{1}{2}|f(x^+)-f(x^-)|\beta(x)$$

where $\beta(x) = 1$ if $x = \cos(\alpha \pi)$ and $\alpha$ is irrational, and

$$\beta(x) = \left(\frac{\sin(\pi/2q)}{\pi/2q}\right)^2 (1 - \sum_{k=1}^{8qk} \frac{8qk}{(4q^2k^2-1)^2})$$

if $x = \cos(p\pi/q)$.

Unlike Fourier series of $2\pi$-periodic functions of bounded
variation or Bernstein polynomials of functions of bounded variation which all converge to \( \frac{1}{2}(f(x^+)+f(x^-)) \), Theorem 2 implies that Hermite-Fejér interpolatory polynomial of a function of bounded variation on \([-1,1]\) converges only if \( f(x^+)=f(x^-) \).

3. PROOFS. PROOF OF THEOREM 1. We have, for any \( x \in (-1,1) \),

\[
|H_n(f,x)-f(x)| \leq \sum_{k=1}^{n} |f(x_{kn})-f(x)|H_{kn}(x)
\]

where \( t_{kn} = |x-x_{kn}| \). Let \( x = \cos \theta, 0 < \theta < \pi, x_{kn} = \cos \theta_{kn} \), \( \theta_{kn} = (2k-1)\pi/2n, k = 1,2,\ldots,n \), and define

\[
E_r(n, \theta) = \{ k: \frac{r\pi}{2n} < |\theta-\theta_{kn}| \leq \frac{(r+1)\pi}{2n} \},
\]

\( r = 0,1,2,\ldots,2n-1 \). We have then

\[
|H_n(f,x)-f(x)| \leq \sum_{r=0}^{2n-1} \sum_{k \in E_r(n, \theta)} V x+t_{kn}(f)H_{kn}(x),
\]

Since \( t_{kn} = |x-x_{kn}| = |\cos \theta - \cos \theta_{kn}| \leq |\theta - \theta_{kn}| \leq \frac{\pi}{2n}|T_n(x)| \) if \( k \in E_0(n, \theta) \) (see [26], pp.257), it follows that

\[
\sum_{k \in E_0(n, \theta)} V x+t_{kn}(f)H_{kn}(x) \leq 2 V x+\pi|T_n(x)|/2n (f).
\]

\[
\sum_{k \in E_0(n, \theta)} V x-t_{kn} \leq 2 V x-\pi|T_n(x)|/2n (f).
\]
On the other hand, for \( r = 1, 2, \ldots, 2n-1 \), since

\[
H_{kn}(x) \leq \frac{4T_n^2(x)}{r^2}; \quad t_{kn} \leq \frac{(r+1)\pi}{2n}
\]

if \( k \in E \), we find that

\[
\sum_{k \in E \cap (n, \theta)} V_{x-t_{kn}}(f) H_{kn}(x) \leq \frac{8T_n^2(x)}{r^2} \frac{x+(r+1)\pi/2n}{2n} V_{x-(r+1)\pi/2n}(f),
\]

\( r = 1, 2, \ldots, 2n-1 \). Therefore

\[
(3.1) \quad |H_{n}(f, x) - f(x)| \leq 2 V_{x-\pi} |T_n(x)|/2n(f)
\]

\[
+ 8 T_n(x) \sum_{r=1}^{2n-1} \frac{1}{r^2} V_{x-(r+1)\pi/2n}(f)
\]

\[
\leq 2 V_{x-\pi} |T_n(x)|/2n(f)
\]

\[
+ 32 T_n(x) \sum_{r=2}^{2n} \frac{1}{r^2} V_{x-r\pi/2n}(f).
\]

Next, let \( p(t) = V_{x-t}(f), t \geq 0 \). Since \( p(t) \) is increasing and \( 1/t^2 \) is continuous on \([r\pi/2n, (r+1)\pi/2n]\), we have

\[
\int_{r\pi/2n}^{(r+1)\pi/2n} \frac{p(t)}{t^2} dt \geq p(r\pi/2n) \int_{r\pi/2n}^{(r+1)\pi/2n} \frac{dt}{t^2}
\]

\[
\geq p(r\pi/2n) \frac{2n}{\pi(r+1)^2}
\]

or

\[
\frac{1}{r^2} p(r\pi/2n) \leq 2\pi/n \int_{r\pi/2n}^{(r+1)\pi/2n} \frac{p(t)}{t^2} dt.
\]

Hence,
\[ \sum_{r=2}^{2n} \frac{1}{r^2} \frac{p(r\pi/2n)}{2n} \leq 2\pi/n \int_{\pi/n}^{(2n+1)\pi/2n} \frac{p(t)}{t^2} \, dt \]

\[ = \frac{2}{n} \int_{2n/2n+1}^{n} p(\pi/t) \, dt \]

\[ \leq \frac{3}{n} \sum_{k=1}^{n} p(\pi/k). \]

Therefore

\[ \sum_{r=2}^{2n} \frac{1}{r^2} \frac{V(x+r\pi/2n)}{2n} \leq \frac{3}{n} \sum_{k=1}^{n} V(x+\pi/k). \]

(3.2)

And Theorem 1 follows from (3.1) and (3.2).

**Proof of Theorem 2.** For any \( x \in (-1,1) \), \( x \neq x_{kn} \), \( k = 1, 2, \ldots, n \), define \( g_x \) as follows:

\[ g_x(t) = \begin{cases} 
  f(t) - f(x+), & x < t \leq 1 \\
  0, & t = x \\
  f(t) - f(x-), & -1 < t < x.
\end{cases} \]

It is easy to see that \( g_x \) is continuous at \( t = x \) and of bounded variation on \([-1,1]\). By a simple calculation we obtain from (1.1) that

\[ H_n(f, x) = H_n(g_x, x) + \frac{1}{2}(f(x+) + f(x-)) + \frac{1}{2}(f(x+) - f(x-))(\sum_{x_{kn} < x} H_{kn}(x) + \sum_{x_{kn} > x} H_{kn}(x)). \]

Let

\[ \sigma_c(t) = \text{sign}(t - c). \]

We have, for \( c \neq x_{kn} \), \( k = 1, 2, \ldots, n \),
\[ H_n(\sigma_c, x) = \sum_{k=1}^{n} \sigma_c(x_{kn}) H_{kn}(x) \]

\[ = - \sum_{x_{kn} < x} H_{kn}(x) + \sum_{x_{kn} > x} H_{kn}(x). \]

Therefore

\[ (3.3) \quad H_n(f, x) = H_n(g_x, x) + \frac{1}{2}(f(x^+) + f(x^-)) \]

\[ + \frac{1}{2}(f(x^+) - f(x^-)) H_n(\sigma_x, x). \]

Since \( g_x \) is continuous at \( t = x \), \( H_n(g_x, x) \rightarrow g_x(x) = 0 \) (\( n \to \infty \)).

Hence, to prove Theorem 2, it is sufficient to show that the following result is true.

**Theorem 2'.** For any \( x = \cos(\alpha \pi) \in (-1, 1) \), the sequence

\[ H_n(\sigma_x, x) = - \sum_{x_{kn} < x} H_{kn}(x) + \sum_{x_{kn} > x} H_{kn}(x) \]

satisfies the following relations:

\[ \limsup_{n \to \infty} H_n(\sigma_x, x) = \beta(x) \]

where \( \beta(x) \) is defined as in Theorem 2.

**Proof of Theorem 2'.** The proof of Theorem 2' is based on two lemmas which deal with asymptotic properties of \( H_n(\sigma_x, x) \).

**Lemma 1.** For \( 0 < \alpha < 1 \) and \( \theta \in (0, \pi) \) we have

\[ \limsup_{n \to \infty} n^{1-\alpha} \frac{1}{|\theta - \theta_{kn}|} H_{kn}(\cos \theta) \leq \infty. \]
PROOF OF LEMMA 1. Obviously

\[ (3.4) \quad \left| \sum_{\theta_k \leq \theta - n^{-\alpha}} H_{\theta_k}(\cos \theta) \right| \leq \frac{2}{n^2} \sum_{\theta_k \leq \theta - n^{-\alpha}} \frac{1}{(\cos \theta - \cos \theta_{kn})^2}. \]

Let \( \ell \) be the smallest value of \( k \) such that \( \theta - n^{-\alpha} < \theta_{kn} < \theta \). Then

\[ |\cos \theta - \cos \theta(\ell-1),n| \geq \cos(\theta - n^{-\alpha}) - \cos \theta = 2 \sin(\theta - \frac{n^{-\alpha}}{2}) \sin\left(\frac{n^{-\alpha}}{2}\right) \]

\[ \geq \frac{2}{\pi} n^{-\alpha} \sin(\theta - \frac{n^{-\alpha}}{2}), \]

and

\[ |\cos \theta - \cos \theta(\ell-2),n| \geq \cos(\theta - n^{-\alpha} - \frac{\pi}{n}) - \cos \theta \]

\[ = 2 \sin(\theta - \frac{n^{-\alpha} + \pi/n}{2}) \sin\left(\frac{n^{-\alpha} + \pi/n}{2}\right) \]

\[ \geq \frac{2}{\pi} (n^{-\alpha} + \frac{\pi}{n}) \sin(\theta - \frac{n^{-\alpha} + \pi/n}{2}). \]

In general, for sufficiently large \( n \)

\[ (3.5) \quad |\cos \theta - \cos \theta(\ell-1),n| \geq \frac{2}{\pi} (n^{-\alpha} + (i-1)\frac{\pi}{n}) \sin(\theta - \frac{n^{-\alpha} + (i-1)\pi/n}{2}). \]

Since the number of \( \theta_{kn} \)'s between \( \theta \) and \( \theta - n^{-\alpha} \) is at most

\[ [(\theta - n^{-\alpha})n/\pi], \]

by (3.4) and (3.5) it follows that

\[ (3.6) \quad \left| \sum_{\theta_k \leq \theta - n^{-\alpha}} H_{\theta_k}(\cos \theta) \right| \]

\[ \leq \frac{\pi}{2n^2} \left[ (\theta - n^{-\alpha}) \frac{n}{\pi} \right] \sum_{i=1}^{\left\lfloor (\theta - n^{-\alpha}) \frac{n}{\pi} \right\rfloor} \frac{1}{\sin^2(\theta - \frac{n^{-\alpha} + (i-1)\pi/n}{2}) (n^{-\alpha} + (i-1)\pi/n)^2}. \]

Let \( M = \min(\sin \theta, \sin \frac{\theta}{2}) \). Then \( M > 0 \) and

\[ M \leq \sin(\theta - \frac{n^{-\alpha} + (i-1)\pi/n}{2}), \]
for $1 \leq i \leq \lfloor (\theta - n^{-\alpha})n/\pi \rfloor$. Hence, from (3.6), we have

$$
(3.7) \quad \left| \sum_{\theta kn \leq \theta - n^{-\alpha}} H_{kn} (\cos \theta) \right| \leq \frac{\pi^2}{2n^2M^2} \sum_{i=1}^{\infty} \frac{1}{(n^{-\alpha} + (i-1)\pi/n)^2}
$$

Finally, a simple calculation shows that

$$
\sum_{i=1}^{\infty} \frac{1}{(n^{1-\alpha} + (i-1)\pi)^2} \leq O(1)\frac{1}{n^{1-\alpha}}.
$$

Hence

$$
\left| \sum_{\theta kn \leq \theta - n^{-\alpha}} H_{kn} (\cos \theta) \right| \leq O(1)\frac{1}{n^{1-\alpha}}
$$

for all $n$ sufficiently large. Similarly

$$
\left| \sum_{\theta kn \geq \theta + n^{-\alpha}} H_{kn} (\cos \theta) \right| \leq O(1)\frac{1}{n^{1-\alpha}}
$$

for all $n$ sufficiently large. And the lemma is proved.

To prove the second asymptotic property of $H_n(\sigma, x)$, we need the following inequality (see, e.g. [27], vol. III, pp. 38).

**Lemma 2.** For $k = 1, 2, \ldots, n$ and $\theta \in (0, \pi)$, we have

$$
\left| \frac{\cos n\theta}{n(\cos \theta - \cos \theta_{kn})} \right| \leq \frac{2}{\sin \theta_{kn}}.
$$

Now

**Lemma 3.** For $\theta \in (0, \pi)$, $\frac{1}{2} < \alpha < 1$ and all $n$ sufficiently large, we have
\[
\left| \sum_{\theta - n^\alpha < \theta_{kn} < \theta} H_{kn} (\cos \theta) - \sum_{\theta - n^\alpha < \theta_{kn} < \theta} \frac{(-\cos n\theta)}{n(\theta - \theta_{kn})} \right|^2 \leq \frac{c}{\sin^2 \theta \cdot n^{2\alpha - 1}}
\]

and

\[
\left| \sum_{\theta < \theta_{kn} < \theta + n^\alpha} H_{kn} (\cos \theta) - \sum_{\theta < \theta_{kn} < \theta + n^\alpha} \frac{(-\cos n\theta)}{n(\theta - \theta_{kn})} \right|^2 \leq \frac{c}{\sin^2 \theta \cdot n^{2\alpha - 1}}
\]

where \( c \) is a positive constant depending on \( \alpha \) and \( \theta \).

**Proof of Lemma 3.** It is clearly sufficient to prove only the first of these two inequalities. To simplify the notations, let \( E_n(\theta) = \{ k : \theta - n^\alpha < \theta_{kn} < \theta \} \) and

\[
(1) \quad \Sigma_n (\theta) = \sum_{k \in E_n(\theta)} H_{kn} (\cos \theta).
\]

Then

\[
\begin{align*}
\Sigma_n (\theta) & = \sum_{k \in E_n(\theta)} \left( \frac{\cos n\theta}{n(\theta - \theta_{kn})} \right)^2 \\
& = \sum_{k \in E_n(\theta)} (1 - \cos^2 \theta) \left( \frac{\cos n\theta}{n(\cos \theta - \cos \theta_{kn})} \right)^2 + \cos \theta \sum_{k \in E_n(\theta)} \left( \frac{\cos n\theta}{n(\cos \theta - \cos \theta_{kn})} \right)^2 - \sum_{k \in E_n(\theta)} \left( \frac{\cos n\theta}{n(\theta - \theta_{kn})} \right)^2.
\end{align*}
\]

Since \( \cos n\theta = (-1)^k \sin n(\theta - \theta_{kn}) \), it follows that

\[
\begin{align*}
\Sigma_n (\theta) & = \sum_{k \in E_n(\theta)} \left( \frac{\cos n\theta}{n(\theta - \theta_{kn})} \right)^2 \\
& = \frac{1}{n^2} \sum_{k \in E_n(\theta)} \sin^2 n(\theta - \theta_{kn}) \left( \frac{\sin^2 \theta}{(\cos \theta - \cos \theta_{kn})^2} \right)^2 - \frac{1}{(\theta - \theta_{kn})^2} + \frac{\cos \theta}{n} \sum_{k \in E_n(\theta)} \frac{\cos^2 n\theta}{n(\theta - \theta_{kn})}.
\end{align*}
\]
Hence

\[ | \sum_{n \in \mathbb{N}} (1) \theta - \sum_{k \in \mathbb{N}} \cos \frac{n \theta}{n(n-\theta kn)} | \]

\[ \leq \frac{1}{n^2} \sum_{k \in \mathbb{N}} \sin^2(n(n-\theta kn)) \left| \frac{\sin^2 \theta}{\cos \theta - \cos \theta kn} - \frac{1}{(\theta-\theta kn)^2} \right| \]

\[ + \frac{1}{n} \sum_{k \in \mathbb{N}} \left| \frac{\cos n \theta}{n \left( \cos \theta - \cos \theta kn \right)} \right| \]

\[ = A_n(\theta) + B_n(\theta). \]

The estimation of \( B_n(\theta) \) is relatively easy. By Lemma 2 we have

\[ B_n(\theta) \leq \frac{2}{n^2} \sum_{k \in \mathbb{N}} \frac{1}{\sin \theta kn}. \]

Since the number of \( k \)'s in \( \mathbb{N} \) is \( < n^{1-\alpha}/n \), it follows that

\[ B_n(\theta) \leq \frac{2}{n^M_n} n^{-\alpha} \]

where \( M_n = \min_{|t-\theta|<n^{-\alpha}} \sin t + \sin \theta (n+\infty). \)

To estimate \( A_n(\theta) \), observe that \( \cos \theta - \cos \theta kn = -(\theta-\theta kn) \cdot \sin \theta kn \) for some \( \theta kn \) between \( \theta \) and \( \theta kn \). Since \( \sin \theta kn \geq M_n \) and \( |\sin^2 \theta - \sin^2 \theta kn| \leq 2 |\theta - \theta kn| \), it follows that

\[ A_n(\theta) = \frac{1}{n^2} \sum_{k \in \mathbb{N}} \frac{\sin n(\theta-\theta kn)}{\theta - \theta kn}^2 \left| \frac{\sin^2 \theta - \sin^2 \theta kn}{\sin^2 \theta kn} \right| \]

\[ \leq \frac{2}{n^M_n^2} \sum_{k \in \mathbb{N}} \frac{\sin n(\theta-\theta kn)}{\theta - \theta kn}^2 |\theta - \theta kn|. \]
\[
\leq \frac{2n^{-\alpha}}{M_n^2} \sum_{k \in \mathbb{E}_n(\theta)} 1
\]

\[
\leq \frac{2}{\pi M_n^2} n^{1-2\alpha}
\]

and the lemma is proved.

Now the proof of Theorem 2'. We shall write, as usual, \(x = \cos \theta, x_{kn} = \cos \theta_{kn}\), where \(\theta_{kn} = \frac{2k-1}{2n} \pi, k = 1, 2, \ldots, n\), and

\[
R_n(\theta) = H_n(\sigma_x, x).
\]

Then, for \(\theta \epsilon (0, \pi)\) and all \(n\) sufficiently large,

\[
R_n(\theta) = \sum_{\theta_{kn} < \theta} H_{kn}(\cos \theta) - \sum_{\theta_{kn} > \theta} H_{kn}(\cos \theta)
\]

\[
= \sum_{\theta_{kn} < \theta} -\alpha x_{kn} H_{kn}(\cos \theta) - \sum_{\theta_{kn} > \theta} -\alpha H_{kn}(\cos \theta)
\]

\[
+ \left( \sum_{\theta_{kn} \leq \theta - n} -\alpha \right) H_{kn}(\cos \theta)
\]

\[
\leq \sum_{n} (\theta) - \sum_{n} (\theta) + \sum_{n} (\theta).
\]

We have clearly, by Lemma 1 and Lemma 3, for all \(n\) sufficiently large

\[
\left| \sum_{n} (\theta) \right| \leq \left| \theta_{kn} - \theta \right| \leq \frac{c_1}{n^{1-\alpha}},
\]

\[
\left| \sum_{n} (\theta) \right| - \left| \theta_{kn} - \theta \right| \leq \frac{c}{n^{2\alpha-1}}.
\]
\[ |\Sigma_{n}^{(2)}(\theta) - \sum_{\theta < \theta_{kn} < \theta + n^{-\alpha}} \left( \frac{\cos n\theta}{n(\theta - \theta_{kn})} \right)^2| \leq \frac{c}{n^{2\alpha-1}}, \]

where \( c_1 \) and \( c \) are positive constants depending on \( \alpha \) and \( \theta \).

Therefore, with \( \alpha = 2/3 \), we have

\[ R_n(\theta) = -\frac{2}{3} \sum_{\theta - n^{-2/3} < \theta < \theta + n^{-2/3}} \left( \frac{\cos n\theta}{n(\theta - \theta_{kn})} \right)^2 + O(n^{-1/3}) \]

as \( n \to \infty \).

Let \( i, j \) and \( \ell \) be such that

\[ \theta - n^{-2/3} < \theta < \theta + n^{-2/3}, \quad \theta < \theta _{kn} < \theta + n^{-2/3} \]

A simple calculation shows that

\[ i = \left[ \frac{n(\theta - n^{-2/3})}{\pi} + \frac{1}{2} \right], \quad j = \left[ \frac{n\theta}{\pi} + \frac{1}{2} \right], \quad \ell = \left[ \frac{n(\theta + n^{-2/3})}{\pi} + \frac{1}{2} \right]. \]

Since \( n(\theta - \theta_{kn}) = \pi(n\theta/\pi + 1/2 - k) \), we have

\[ R_n(\theta) = \frac{1}{\pi^2} \sum_{k=i+1}^{j} \left( \frac{\cos n\theta_{kn}}{n^n_{\pi+\frac{1}{2}-k}} \right)^2 - \frac{1}{\pi^2} \sum_{k=j+1}^{\ell} \left( \frac{\cos n\theta_{kn}}{n^n_{\pi+\frac{1}{2}-k}} \right)^2 + O(n^{-1/3}) \]

or

\[ R_n(\theta) = \frac{1}{\pi^2} \sum_{k=0}^{j-1} \left( \frac{\cos n\theta_{kn}}{n^n_{\pi+\frac{1}{2}-k}} \right)^2 - \frac{1}{\pi^2} \sum_{k=0}^{\ell-j-1} \left( \frac{\cos n\theta_{kn}}{n^n_{\pi+\frac{1}{2}-k}} \right)^2 + O(n^{-1/3}). \]

Now it is easy to see that these finite sums can be replaced by infinite sums, with an error term of the order \( O(n^{-1/3}) \). This follows from the inequalities

\[ \sum_{k=N}^{\infty} \frac{1}{k^2} < \frac{2}{N+1} \quad \text{for} \quad N \geq 3, \]

\[ N+1 = j-1+1 = \left[ \frac{n\theta}{\pi} + \frac{1}{2} \right] + 1 - \left[ \frac{n(\theta - n^{-2/3})}{\pi} + \frac{1}{2} \right] \geq \frac{n^{1/3}}{\pi}, \]
and
\[ N+1 = j - j + 1 = \left[ \frac{n(n+n-2/3)}{\pi} + \frac{1}{2} \right] + 1 - \left[ \frac{n}{\pi} + \frac{1}{2} \right] \geq \frac{n^{1/3}}{\pi}. \]

Therefore,
\[ R_n(\theta) = \frac{1}{\pi^2} \sum_{k=0}^{\infty} \left( \frac{\cos n\theta}{\rho(n\theta) + k} \right)^2 - \frac{1}{\pi^2} \sum_{k=0}^{\infty} \left( \frac{\cos n\theta}{1 - \rho(n\theta) + k} \right)^2 + O(n^{-1/3}) \]

where \( \rho(x) = x/\pi + \frac{1}{2} - \left[ x/\pi + \frac{1}{2} \right] \). This expression for \( R_n(\theta) \) can be simplified further if we observe that
\[ \cos^2 n\theta = \sin^2 (n\theta/\pi + \frac{1}{2} - \left[ n\theta/\pi + \frac{1}{2} \right]) = \sin^2(\rho(n\theta)\pi) \]

and
\[ \cos^2 n\theta = \sin^2 (1 - (n\theta/\pi + \frac{1}{2}) + [n\theta/\pi + \frac{1}{2}]) = \sin^2((1 - \rho(n\theta))\pi). \]

With these expressions for \( \cos^2 n\theta \), we have
\[ R_n(\theta) = \frac{1}{\pi^2} \sum_{k=0}^{\infty} \left( \frac{\sin \rho(n\theta)\pi}{\rho(n\theta) + k} \right)^2 - \frac{1}{\pi^2} \sum_{k=0}^{\infty} \left( \frac{\sin(1 - \rho(n\theta))\pi}{1 - \rho(n\theta) + k} \right)^2 + O(n^{-1/3}), \]

or
(3.8) \[ R_n(\theta) = G(\rho(n\theta)) + O(n^{-1/3}), \]

where
\[ G(t) = F(t) - F(1 - t), \]

and
\[ F(t) = \frac{1}{\pi^2} \sum_{k=0}^{\infty} \left( \frac{\sin \pi t}{t + k} \right)^2. \]

The function \( F(t) \) is clearly a continuous function on \((0,1)\) and
\[ \lim_{t \to 0} F(t) = 1, \quad \lim_{t \to 1} F(t) = 0. \]
Therefore, $F(t)$ is a continuous function on $[0,1]$ if we define\n$F(0) = 1$ and $F(1) = 0$.

We shall show next that $F'(t) < 0$ for $t \in (0,1)$, so that\n$F(t)$ is a strictly decreasing function on $[0,1]$. We have\n\[
\pi^2 F'(t) = 2 \sin \pi \left( \pi \cos \pi \sum_{k=0}^{\infty} \frac{1}{(t+k)^2} - \sin \pi \sum_{k=0}^{\infty} \frac{1}{(t+k)^3} \right).
\]
Hence\n\[
F'(t) < 0 \quad \text{if} \quad \frac{1}{2} < t < 1.
\]

On the other hand, since\n\[
x \cos x - \sin x < -\frac{x^3}{6} \quad \text{for} \quad 0 < x < \frac{\pi}{2},
\]
it follows that, for $0 < t < \frac{1}{2},$\n\[
\pi^2 F'(t) = 2 \sin \pi \left( \frac{1}{t^3} (\pi \cos \pi - \sin \pi) \right) + \pi \cos \pi \sum_{k=1}^{\infty} \frac{1}{(t+k)^2} - \sin \pi \sum_{k=1}^{\infty} \frac{1}{(t+k)^3} < 2 \sin \pi \left( - \frac{x^3}{6} + \pi \sum_{k=1}^{\infty} \frac{1}{k} \right) = 0.
\]

Consequently, we have $F'(t) < 0$ for all $t \in (0,1)$, and $G'(t) = F'(t) + F'(1-t) < 0$ for all $t \in (0,1)$.

So $G(t)$ is a continuous, strictly decreasing function on $[0,1]$\nwith $G(0) = 1$ and $G(1) = -1$.

From these properties of the function $G(t)$ and (3.8)\nfollows that\n\[
(3.9) \quad \limsup_{n \to \infty} H_n(\sigma_x, x) = \limsup_{n \to \infty} R_n(\theta) = G(\liminf_{n \to \infty} \rho(n\theta))
\]
and

\[ (3.10) \quad \lim_{n \to \infty} \inf H_n(\sigma, x) = \lim_{n \to \infty} \inf R_n(\theta) = G(\limsup_{n \to \infty} \rho(n\theta)). \]

The behavior of the sequence \((\rho(n\theta))\) where

\[ \rho(x) = \frac{x}{\pi} + \frac{1}{2} - \left[ \frac{x}{\pi} + \frac{1}{2} \right] \]

is well known.

If \(\theta/\pi\) is an irrational number, the sequence \((\rho(n\theta))\) is uniformly distributed in \((0, 1)\). We can therefore find increasing sequences of integers \((n_k)\) and \((m_k)\) such that

\[ \rho(n_k \cdot \theta) \to 0 (k \to \infty) \]

and

\[ \rho(m_k \cdot \theta) \to 1 (k \to \infty). \]

Therefore, if \(\theta/\pi = (\arccos x)/\pi\) is an irrational number, we have, from (3.9) and (3.10),

\[ \lim_{n \to \infty} \sup H_n(\sigma, x) = G(0) = 1 \]

and

\[ \lim_{n \to \infty} \inf H_n(\sigma, x) = G(1) = -1. \]

If \(\theta/\pi\) is a rational number, \(\theta = \frac{p}{q}\) where \(p/q \in (0, 1)\).

Let \(n = mq + i, 0 < i < q\). We have then

\[ \rho(n\theta) = \frac{n\theta}{\pi} + \frac{1}{2} - \left[ \frac{n\theta}{\pi} + \frac{1}{2} \right] = mp + \frac{2ip + q}{2q} - \left[ mp + \frac{2ip + q}{2q} \right]. \]

Thus the sequence \((\rho(n\theta))\) takes only \(q\) distinct values \(\frac{2i - 1}{2q}\).
j = 1, 2, ..., q. Consequently, we have in this case

\[
\limsup_{n \to \infty} H_n(\sigma_x, x) = G(\frac{1}{2q}) = F(\frac{1}{2q}) - F(1 - \frac{1}{2q}),
\]

\[
\liminf_{n \to \infty} H_n(\sigma_x, x) = G(1 - \frac{1}{2q}) = F(1 - \frac{1}{2q}) - F(\frac{1}{2q})
\]

and Theorem 2' is proved.
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