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FINANCIAL MARKETS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Akio Yasuhara, B.A., M.A.

*****

The Ohio State University
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Theories of financial markets developed rapidly in the 1960's and the 1970's. New approaches are still emerging. Much of the innovatory thrust develops from the model of individual portfolio choice based on two parameters of the distribution of portfolio returns (mean and variance) first introduced by Markowitz (1952, 1959) and Tobin (1958). This approach culminated in a model of market equilibrium and capital-asset pricing (referred to as the CAPM) developed by Sharpe (1964), Mossin (1966), Lintner (1965) and others. The CAPM posits empirically testable market-equilibrium relations between the expected rates of return on individual securities and on the market portfolio.\(^1\) Shortcomings from not incorporating the third or higher moments of the asset-return distributions were pointed out by Borch (1969, 1974) and Feldstein (1969).\(^2\) Still the CAPM is widely used because of its intuitively appealing measure of risk and empirically testable implications. These attractive features of the CAPM contrast with the limited empirical applicability of the state-preference approach.

The state-preference approach was first introduced by Debreu (1959). He used the concept of state-contingent
claims on commodities to show the existence and Pareto optimality of a market equilibrium under uncertainty. In his model, a market for contingent claims is open in the current period to trade contracts to deliver all commodities available in the future. Each contract is to be effected only in the specified state of nature. But real-world economies usually do not have a complete set of contingent claims on commodities. In this respect, Arrow (1964) has shown that contingent claims on income supplemented by a sequence of spot markets of commodities would be sufficient for Pareto optimality. Arrow notes that stock-shares and insurance contracts provide examples of contingent claims on income.

The state-preference approach to financial markets assumes that individual utility varies with the combination of the income levels that the individual would receive in specific states of nature and his probability assessments of the occurrence of these states. The approach imposes only one specific restriction on the form of the utility function: the concavity property. However, this generality makes it difficult to define a specific measure of risk and reduces the power of generating specific testable hypotheses. The state-preference framework has been utilized mainly to analyze the existence and the Pareto optimality of market equilibrium.
I.1. Corporate Capital Structure

Both the CAPM and the state-preference approaches have been applied to many specific problems in finance. For individual firms, two central issues may be identified: the problems of choosing an optimal capital structure and the problems of selecting an optimal mix of investments.

Modern capital structure analysis begins with the path-breaking analysis of Modigliani and Miller (1958). They compare two firms in the same risk class, one is unlevered and the other is levered by default-risk-free bonds. Their analysis shows an irrelevance of financing decisions either to the value of the firm or the wealth of shareholders. As long as investors can rearrange the corporate capital structure by issuing personal debt to offset any change in the corporate financing decisions (home-made leverage), arbitrage eliminates any difference between the values of levered and unlevered firm.

Among the assumptions made in this analysis are: no corporate taxes; no default risk; exogenously given investment decisions; homogeneous expectations about the earnings of firm; perfect capital markets. Modigliani and Miller (1963) show that the tax deductibility of interest payments on debt capital induces firms to utilize as much debt capital as possible. Tax deductibility is equivalent to a government subsidy on the use of debt capital. However, unlimited debt issues bring on an increasing chance of bankruptcy.
When debt carries default risk, the home-made-leverage proposition for the irrelevance theorem must be modified so that investors can issue debt using firm's shares as collateral, letting them forfeit their shares in the case of default. Under the no-tax assumption, this modified opportunity for forming home-made leverage re-establishes the Modigliani-Miller irrelevance theorem. Stiglitz (1969) proved this in a state-preference model, while Rubinstein (1972) proved it in a CAPM framework.

Bankruptcy also introduces bankruptcy costs which work against the use of debt capital. Kraus and Litzenberger (1973) examined the interaction between preferential tax treatment on debt and bankruptcy costs in a state-preference model. They show that the value of a levered firm is the sum of (1) the value of the firm when unlevered, (2) the tax subsidy to debt capital and (3) the value of after-tax bankruptcy cost. The latter two effects work against each other and the optimal debt-equity ratio stops short of the corner solution which is implied by effect (2) alone. Empirical estimates of bankruptcy costs are provided by Baxter (1967) for individual bankruptcies and Warner (1977) for railroad bankruptcies. Though the Baxter study shows a high average bankruptcy cost, Warner's study of eleven railroad bankruptcies between 1933 and 1955 suggests that bankruptcy costs may not be a significant factor on the corporate financing decisions.⁶
The analyses mentioned so far assumes exogenously fixed investment decisions, which simultaneously fixes the earnings of firms. Methodologically, they are all comparative-static analysis. Maximizing firm value and maximizing shareholder wealth are equivalent goals. This helps to establish separation between corporate financing decisions and investment decisions. However, as Fama and Miller (1972) (1972), Stiglitz (1972) and Fama (1978) point out, revisions of investment strategies or additional debt issues would affect the wealth of existing securityholders. For instance, after having raised debt capital under the assumption of undertaking a specific investment project, the management may transfer bondholder wealth to shareholders by switching the investment plan to a riskier project. If the wealth transfer is large enough to compensate shareholders for the loss in the firm value, the investment switch would increase shareholder wealth. Rational bondholders would anticipate the investment switch and charge a premium for the risk of the debt. Given this, shareholders may have to undertake the riskier project for their wealth maximization. This breaks the equivalence between the firm-value maximization and the shareholder-wealth maximization. Another example is that a firm issuing new debt can expose existing bondholders to a higher default risk. This also increases shareholder wealth without changing investment decisions. These problems arise in a dynamic situation where revisions and
additions in investment or financing decisions can be made. It is worth mentioning that if all investors have equal shares in risky securities available in the market the wealth transfer does not constitute a problem. Each investor is a loser in one security and a winner in another. The separation theorem with the market portfolio of risky securities in the CAPM meets this condition. It applies to a perfect market where all assets are perfectly divisible. In the general framework of the state-preference approach, investors would hold different amounts of securities depending on their preferences and the market prices, and hence wealth transfers are always a potential problem.

Provisions for compensating wealth losses due to changes in financial or investment decisions attached to securities could prevent the problem. Fama and Miller (1972) call these provisions "me-first" rules. A seniority system in bond issues is a typical example of "me-first" rules. Also conceivable are restrictive covenants which prohibit certain courses of future actions of firms to secure securityholders' wealth. The agency-cost theory, which has recently been developed by Alchian and Demsetz (1972), Jensen and Meckling (1976) and others, provides insight into these activities. The agency-cost theory explicitly recognizes conflicts in objectives between the parties (agency and principal) in an economic contract. The conflict tends to reduce the maximum-attainable levels
of parties' objectives. Bonding and monitoring activities can be viewed as an effort to reduce these potential losses.

A significant contribution of the agency-cost theory is that it shows incentives for the contracting parties voluntarily to introduce bonding and monitoring activities based on their economic calculations. The agency-cost approach not only provides an alternative explanation for corporate debt-equity decisions, but also sheds light on restrictive covenants (Smith and Warner [1979]) and the existence of such complex securities as warrants or convertible bonds. Detailed analysis of financial contracts in the agency-cost perspective are expected.

1.2. Optimal Investment Decisions

Parallel to developments in corporate capital-structure theories has been concern for the investment decisions of firms. The question raised by Diamond (1967) is whether or not the stock-share market provides price signals sufficient to guide firms to Pareto-optimal investment decisions. Debrue (1959) proves the Pareto optimality of competitive production and allocation under uncertainty when contingent claims on commodities cover all possible states of the economy. If, instead of contingent claims on future commodities, contingent claims on income (called Arrow securities) cover all possible states, Pareto-optimal investments also result in a competitive environment (Arrow
Therefore, the first question is whether the stock-share market alone can substitute perfectly for the market of Arrow securities. Cass and Stiglitz (1970) provide the conditions under which this perfect substitution holds: namely, (a) the number of firms be at least as large as the number of possible states and (b) the vectors of possible returns from shares be linearly independent so that they span the space of income in all states. If these conditions are not met, the stock-share market is said to be incomplete. In incomplete markets even competition may fail to equalize the marginal rates of substitution of income across investors, which means a Pareto-nonoptimal allocation.

Diamond investigated the investment problem in an incomplete market when firms maximize the market values of their stocks. Diamond shows that when the production technology of a firm is expressed by a multiplicative form with two components, where one is stochastic and depends on the state of nature alone and the other is nonstochastic and depends on the inputs alone, the firm can obtain from the stock market sufficient price information to calculate unambiguously the value of a new investment. This type of production technology is named "decomposable", the value of a new investment may not be clearly defined. The market value of the existing stock reflects the total value of the returns in all states from investment but it does not
necessarily reveal the value of returns in each individual state. This implies that, if a new investment changes returns in all states by the same proportion, the firm can calculate the value of the new investment from the market value of the existing stock. This is the "decomposable" case. However, if, for instance, the new investment changes the return only in one particular state, it may not be possible to evaluate the new investment from stock-market information alone. Note that even the equilibrium investment pattern in the "decomposable" case may not be Pareto-optimal in the standard sense, but only constrained Pareto-optimal, constrained in the sense that the marginal rates of substitution and technical transformation among the incomes are equalized only in those states that can be spanned by the vectors of the returns from stocks. As the second-best theorem in welfare economics suggests, this may not be fully Pareto-optimal.

Two interrelated approaches to the investment problem have followed the Diamond's analysis. Stiglitz (1972), Fama (1972), Jensen and Long (1972), and Merton and Subrahmanyan (1974) investigate the issue in the CAPM framework. In this approach, the main focus is on whether or not firm-value maximization leads to Pareto-optimal decisions. Taking the market price of risk-bearing in terms of the expected rate of return as given, they find a divergence between firm-value-maximizing and Pareto-optimal
investment decisions. Stiglitz (1972), Jensen and Long (1972) and Merton and Subrahmanyam (1974) have interpreted the divergence as due to the violation of competitiveness and Fama (1972) interpreted it due to externality of a firm's investment decision on the value of the other firms.

The second approach to the investment problem focuses on the unanimity of shareholder opinion regarding investment changes. Radner (1974), Ekern and Wilson (1974), Leland (1972), LeRoy (1976), and Stiglitz and Grossman (1977) use this approach. Once ambiguity in defining the market value of a new investment in an incomplete market is recognized, we need to shift our attentions to the decision-making process of shareholders. Radner, Ekern and Wilson, and Leland calculate changes in the utility levels of shareholders due to an investment change of a firm and generalize Diamond's concept of decomposability as the condition for unanimity; namely that the return from the new investment can be spanned by the vectors of returns of existing investments of all firms. But this is precisely the case where the value of the new investment can be obtained from prices in a competitive stock market.

However, Stiglitz and Grossman (1977) add a further complication to this result. Suppose that investors anticipate that the new investment changes the other firms' values. This implies anticipated capital gains and losses, the magnitude of which depends on each investor's portfolio.
reshuffling decisions. Even if investors' anticipations about stock-price changes are homogeneous, the impact of these price changes on investors' optimal utility levels differ depending upon their intended changes in the holdings of stock shares over time. When the analysis is extended this far, the problem is not unique to incomplete markets. As LeRoy (1976) points out, we are no longer taking any market prices as given. Anticipated changes in the other firms' stock prices imply the changes in the underlying implicit prices of Arrow securities. Traditional-sense competitiveness is violated. Therefore, the problem is latent even in complete markets. Baron (1979) reviews the CAPM approach in this light, pointing out that taking the risk-free interest rate and the market price of risk-bearing as given in the CAPM implies changes in underlying implicit Arrow-security prices in response to a new investment, which are responsible for the divergence between the firm-value maximizing and Pareto-optimal investment decisions.

I.3. Purpose of the Dissertation

This dissertation has three main parts. Each part relates to a specific problem featured in this survey of the modern literature of corporate finance.

The first part discusses the role of innovative securities in incomplete security markets. Incomplete markets fail to guarantee a Pareto-optimal distribution of
uncertain returns from a given set of production decisions of firms. Introducing innovative securities is viewed as a response to incentives for investors to complete markets. The market for stock shares is carefully examined to show that it tends to be incomplete in itself. Then, Modigliani-Miller irrelevance theorems are re-examined in the context of our analysis. It is shown that irrelevance theorems can be viewed as a consequence of rearranging the ways of distributing investment returns without introducing a truly innovative security. This approach proves the irrelevance theorem even for markets that are incomplete and for cases where investors have different expectations about returns on investments in individual firms. The proof does not limit the focus to the firms in the same risk class. The result supports and further extends the Stiglitz's generalization (1974) of the irrelevance theorem and the Fama condition (1978) that perfect substitutes exist for all securities. 

The next part of the dissertation investigates the implications of special characteristics of the CAPM. The CAPM is characterized by three assumptions (a) investors have preference orderings over the expected value and the variance of their portfolio return, (b) risk-free loan opportunities exist and (c) investors share homogeneous expectations about the returns from assets and homogeneous assessments of the probabilities of the state of nature. We show that the incompleteness problem does not arise in the
CAPM. The unique specification of utility functions implies that the competitive distribution of the returns from investments of firms is always Pareto-optimal no matter how many securities exist relative to the number of the possible states of nature.

Then a special focus falls on the possibility of obtaining a closed-formed solution for market valuation of risky securities. The CAPM has gained popularity for its market-equilibrium relations between expected rates of returns on individual securities and on the market portfolio of risky securities. This is a closed-form solution in the sense that information about only potentially measurable factors, such as the risk-free interest rate and the market portfolio rate of return, is sufficient to explain the expected rates of return on individual securities (Rubinstein [1974] and Jensen [1972]). Information about investors' preference orderings is not needed. We examine the conditions under which closed-form solutions are derived in the general state-preference model. In the CAPM the separation property in investors' optimal portfolio choice serves as the basis of its closed-form solutions. We investigate the possibility that the closed-form solutions in the general state-preference model converge to CAPM closed-form solutions without the separation property. We show that when a linear relation holds between the market-portfolio rate of return and the probability-adjusted
Arrow-security price, CAPM closed-form solution obtains without separation property.

The final part of the dissertation develops a theory of convertible bonds in the context of the agency-cost theory. The theory is developed under the assumption of complete markets. While the completeness of financial markets assures the Pareto-optimality of the distribution of the uncertain returns from a given set of investments by firms, it does not resolve the wealth-redistribution problem that arises from investment-decision switches in a dynamic economy. This is the issue pointed by Fama and Miller (1972) and Stiglitz (1972) and reviewed in the context of the agency-cost theory by Jensen and Meckling (1976). An investment-decision switch after having raised capital by debt issue could result in a redistribution of debtholder wealth to shareholders. Unless debtholders have protection against detrimental wealth transfers, they expect the worst case. The market value of debt reflects this expectation, which in turn forces wealth-maximizing shareholders to undertake Pareto nonoptimal investments. Our analysis shows the existence of incentives for shareholders to avoid this by providing a convertibility provision to debtholders. We show that convertibility provisions can eliminate the agency cost of wealth redistributions without altering the firm's debt-equity ratio. We also determine a range for the conversion ratio within which rational debtholders would not
convert and, at the same time, they are protected from detrimental wealth transfers.

These three parts of the dissertation are tied together by a common thread. When the variety of existing securities is not yet fully sufficient to distribute returns from investments or to guide investment decisions, financial markets create new types of financial instruments to help reduce the losses associated with Pareto-nonoptimal results.

The concept of complete (or incomplete) markets has been developed to describe the Pareto-optimality (or non-optimality) of the distribution of returns from a given set of firms' investments. Under a given set of investments, uncertainty traces to the randomness of the state of nature, over which no one has control. The first two parts of the dissertation deal with this situation. The irrelevance theorem of corporate financing decisions and special characteristics of the CAPM are examined in the context of completing markets.

The third part deals with situations where the uncertainty of future investment decisions is anticipated by debtholders. The uncertainty here is nonrandom: it arises out of decisions to be made by firms. Without a convertibility provision debtholders anticipate the worst possible situation as a way to defend themselves. The convertibility provision introduces a mechanism for redistributing wealth between debtholders and shareholders.
contingent on the investment decisions of firms. Our analysis suggests that Pareto nonoptimalities that trace to uncertain decisions of economic agents could be resolved by new types of financial instruments which make returns contingent on the agents' decisions. This parallels the idea that the Pareto nonoptimality of distribution of given uncertain returns due to incompleteness could be resolved by innovative securities.
1Jensen (1972) gives an excellent survey of further theoretical and empirical developments of the CAPM. A later survey of the status of the CAPM is given by Ross (1978).

2It is found that if either (1) the distribution of asset returns is a stable distribution, or (2) the utility functions are quadratic, the two-parameter approach is justified. See Tobin (1958) and Fama (1971).

3The concavity means that the set of income combinations which are preferred to or indifferent to a given level of utility is convex. This concavity property is regarded as equivalent to risk aversion (Arrow [1964]).

4Examples are Baron (1979), Borch (1968), Friesen (1979), Green and Scheshinski (1975), Hart (1975), Radner (1972), Starr (1972) and Stigum (1972).

5Excellent surveys of the capital-structure theories are found in Chen (1978) and Chen and Kim (1979).

6Miller (1977) and Jensen and Meckling (1976) also express doubt about the significance of bankruptcy costs in explaining the corporate capital structure. Both base their opinions on the Warner study. Miller examines the effect of personal income taxes as an alternative explanation, and Jensen and Meckling provide the agency-cost approach.

7Kim and McConnell (1977) study the wealth transfer due to non-synergistic corporate mergers. Corporate mergers co-insure existing risky debts. The value of the co-insurance in a nonsynergistic merger is a wealth transfer from shareholders to debtholders. Rubinstein (1973) also points out this fact.

Myers (1977) shows the possibility that a financially-distressed firm may not undertake even socially desirable investment projects.

8Fama (1978) emphasizes this point and claims that "me-first" rules are unnecessary to re-establish the equivalence between firm-value maximization and shareholder-wealth maximization.
Arrow (1950) has examined the conditions under which competitive equilibrium is Pareto-optimal and a Pareto-optimal allocation can be supported by a competitive market system. The existence of competitive equilibrium and the extension of general-equilibrium analysis to the case of uncertain economies is by Debreu (1959). Here, Pareto optimality is conditional on the expectations (Radner [1968]). Starr (1973) calls this "Arrow optimal" and distinguishes it from "expost Pareto optimal" which applies to actually realized allocation of resources.

Maximizing firm value and maximizing shareholder wealth are consistent goals when financial markets are complete, and the resulting investments are Pareto optimal. Shareholders approve or disapprove investment decisions unanimously in complete markets. Friesen (1979) proves these propositions.

When Stiglitz (1972) emphasizes that the investment problem in the CAPM arises due to the incompleteness of financial markets, he seems to be unaware of this point.

Fama (1978), Theorem 3.
This chapter re-examines the Modigliani and Miller theorem concerning the irrelevance of corporate financial decisions on the value of a firm in the context of incomplete financial markets.

The distinction between complete and incomplete financial markets began to appear in the finance literature when Arrow (1964) developed the model of state-contingent financial claims on income (referred to as Arrow securities) in a state-preference framework. Arrow has shown the Pareto optimality of the distribution of uncertain future income and the investment decisions by firms when the current markets trade state-contingent claims on income that span all the possible states of nature. A complete set of Arrow securities enables investors to trade claims on the income in each possible future state of nature for their current income. This establishes an equalization across investors of their marginal rates of substitution between current income and future income receivable in each possible state of nature. Though in real-world economies we do not observe such contingent claims, corporate stock shares and insurance policies are recognized as substitutes for Arrow
securities. The condition for any given financial market to be a perfect substitute for markers in Arrow securities is that the vectors of the returns from the securities traded in the market span the income space of all possible states of nature. This requires that the number of securities be at least as large as the number of possible states of nature (Cass and Stiglitz [1970]). Markets which do not meet this condition are said to be "incomplete". Incomplete markets fail to assure a Pareto-optimal distribution of uncertain future income and fail to give the appropriate guiding signals to investment decisions (Diamond [1967], Stiglitz [1972a], Radner [1974], Ekern and Wilson [1974], Ekern and Wilson [1974], Baron [1979] and others).

The relation between corporate financing decisions and the value of firms has been investigated since the path-breaking analysis by Modigliani and Miller (1958). They compared the values of a levered firm and an unlevered firm in the same risk class, where the same risk class means that both firms generate the same level of income in each possible state of nature. Under the assumptions of no transactions cost, no taxes and no default risk, the values of the two firms must be the same. Any difference in values induces arbitrage. Every investor can create in his/her own portfolio home-made leverage by combining the stock shares of the unlevered firm with personal debt. He/she can buy shares of the unlevered firm and finance a fraction of
these shares with personal debt. This investment strategy can be designed to create a pattern of return identical to the simple purchase of shares in the levered firm. Since arbitrage must eliminate any difference between the current values of these two investment strategies, the Modigliani-Miller irrelevance theorem is established. Production-investment decisions have been kept fixed in these analyses. The focus has been on how the debt-equity ratio affects the value of the firm. The underlying premise is that maximizing firm value is consistent with maximizing shareholder's wealth under given production-investment decisions.

Later, the implications of corporate income taxes (Modigliani and Miller [1963]), default risk and bankruptcy costs (Kraus and Litzenberger [1973]), and personal income taxes (Miller [1977]) have been investigated.\(^1\)

Stiglitz (1969, 1974) generalizes the Modigliani-Miller irrelevant theorem under fixed investment-production decisions. Stiglitz's analysis assumes no bankruptcy risk, no taxes and no transactions costs. However, his model is extended to cover more than two periods, dividend payments on stock shares, and different maturity bonds. Stiglitz carefully distinguishes a limited version and the general version of the theorem. The original theorem by Modigliani and Miller is a limited version in the sense that it proves the equalization of the values of the firms in the same risk class, but it does not necessarily imply that the equalized
value of the firms is invariant to changes in financial policies. There may be a determinate debt-equity ratio for the economy as a whole. The general version, on the other hand, states that any changes in the scale of bond issues of any individual firm are completely offset by the actions of investors, and therefore that the distribution of future uncertain income would not change. The value of firms remains unchanged. This also implies that the debt-equity ratio for the economy as a whole is indeterminate. Stiglitz (1969 and 1974) presents the general version of the Modigliani-Miller irrelevance theorem. The proof of this general version of the theorem does not require that firms be in the same risk class. Stiglitz proves that the theorem is valid in incomplete markets and with heterogeneous expectations about the firms' future incomes.

In his proof of the generalized Modigliani-Miller irrelevance theorem, Stiglitz shows that the budget constraints of investors would not shift when a firm changes its financial structure. This invariance in budget constraints implies that each investor's optimal configuration of current and future incomes is also invariant. The new demands for financial securities would exactly offset the changes in the financial structure of the firm. Investors can create home-made leverages.

Analysis presented in this chapter is an attempt to integrate the concept of incomplete financial markets and
the irrelevance theorem of corporate financial decisions. We extend the Stiglitz results to cover cases where: (1) the firms may issue contingent financial securities other than stock shares and (2) firms change not only the scale of their issues of financial instruments but also the state-contingency pattern of returns on the securities. The analysis is conducted under the assumption of given production-investment decisions, so that the impact of production-investment decisions on the values of outstanding securities is left out.

The new concept featured in the analysis is "innovative" securities. Competitive trade in a given set of securities determines an equilibrium distribution of the uncertain future incomes generated by the fixed investments of firms. Innovative securities are new financial instruments which help further to redistribute the uncertain incomes generated by fixed investments of firms. If financial markets are complete, no new security can be innovative. By definition, a sufficient number of innovative securities can make the distribution of income Pareto optimal in incomplete markets. We show that this implies that innovative securities must change the underlying Arrow-security prices, which, in turn, alters the valuation of the existing securities. From this perspective, the irrelevance theorem for corporate financing decisions can be viewed as a case of rearranging the distributive
mechanism of firm incomes without introducing a single innovative security into the market. It is not surprising that the irrelevance theorem should hold even in incomplete markets as well as in complete markets. Our approach helps us stretch the scope beyond the irrelevance of the debt-equity ratio to any combination of securities. It also shows that we can establish the general irrelevance theorem in the case where investors have heterogeneous expectations about the uncertain incomes of firms. The novelty of our analysis lies in our way of integrating the concept of innovative securities into the irrelevance theorem.

The analysis develops within a two-period, state-preference model. Taxes and transactions costs are assumed away. Section 1 briefly reviews the complete Arrow-security market model. Section 2 discusses the concept of incompleteness in the stock-share market. Section 3 introduces our key concept, "innovative" securities. This concept is used to re-examine and generalize the Modigliani-Miller theorem in sections 4, 5, and 6.

II.1. Arrow-Security Market Model

Suppose I individual investors and J firms exist. The number of all possible states of nature at t = 1 is S. The production-investment decisions of firms are given. Firm j (j = 1, 2, ..., J) generates income e_{js} in state s at time t = 1. In general, e_{js} \neq e_{jr} (s, r = 1, 2, ..., S)
which means the firm's income at $t = 1$ is uncertain. The financial market determines the distribution of these firms' income among investors by trading securities during the initial period ($t = 0$). The Arrow-security market trades state-contingent claims on income. Each Arrow security promises to deliver one unit of income in the specified state. This promise is void in the other states. Firms and investors can issue or buy Arrow securities, but they must be solvent. The solvency of firm $j$ means that the amount of Arrow security for state $s$ that the firm can issue, $y_{js}$, is limited by its income in that state:

$$y_{js} \leq e_{js}.$$ 

We assume each firm issues Arrow securities up to its state-contingent income levels, so that the equality holds in the above relation. This explains the firm's supply of Arrow securities. $S$ Arrow securities are issued.

Each investor has a preference ordering over the current-date consumption, $c_{i0}$, and his/her income in all possible states of nature which can be spent for consumption at $t = 1$, $c_{is}$ ($s = 1, 2, \ldots, S$). The preference also depends on his/her assessment of the probabilities of the possible states, $f^i = (f^i_1, f^i_2, \ldots, f^i_S)$:

$$(1) \quad u^i = u^i(c_{i0}, c_{is}; s = 1, \ldots, S; f^i).$$
We assume this preference is twice-differentiable and concave. Let $w_{io}$ be the i-th investor's exogenously given income at $t=0$ and $x_{is}$ be the amount of the Arrow security for state $s$ that investor $i$ purchases. A negative $x_{is}$ designates issuance of this security. For simplicity, we assume his/her consumption at $t=1$ is thoroughly financed by the income from Arrow securities, $c_{is} = x_{is}$. Arrow securities are traded in a competitive market at prices $p_s$ ($s=1, 2, \ldots, S$). Individual $i$ maximizes his/her utility subject to the budget constraint:

$$c_{io} + \sum_{s=1}^{S} p_s \cdot c_{is} = w_{io}.$$  

The control variables are $c_{io}$ and $c_{is}$ ($= x_{is}$: portfolios of Arrow securities). These equations and decision rule describe the investors' consumption-portfolio decisions.

First-order conditions for the i-th investor's optimal consumption-portfolio choices are:

$$u_{io}^i = \lambda^i,$$

$$u_{s}^i = p_s \lambda^i \quad (s=1, 2, \ldots, S)$$

where $u_{io}^i$ is the marginal utility of current consumption state $s$, at $t=1$, and $\lambda^i$ is the shadow price of additional resources that could relax the budget constraint.
These conditions imply the equalization of the marginal rates of substitution of current and future consumptions among all investors:

\[ \frac{u^i_s}{u^i_o} = P_s \quad (i = 1, 2, \ldots, I \text{ and } s = 1, 2, \ldots, S). \]

The price of the Arrow security for state \( s \), \( P_s \), equals to the marginal rates of substitution between \( c_{io} \) and \( c_{is} \). The above conditions, together with the market clearing conditions:

\[ \sum_{i=1}^{I} c_{is} = \sum_{j=1}^{J} y_{js} \quad (s = 1, 2, \ldots, S), \]

also imply that the competitive equilibrium distribution of firms' income in the Arrow-security market is Pareto-optimal (Arrow [1964]).

The value of a firm is the value of the financial securities issued by that firm. Since firm \( j \) issues \( y_{js} \) of the Arrow security for state \( s \) \( (s = 1, 2, \ldots, S) \), the value of the firm, \( V_j \), is defined as:

\[ V_j = \sum_{s=1}^{S} P_s \cdot y_{js} = \sum_{s=1}^{S} \frac{u^i_s}{u^i_o} e_{js}. \]

The substitutions that are expressed in the second equality come from (3) and the solvency conditions \( (y_{js} = e_{js}) \).
Equation (4) states that the value of the firm is the value of the state-contingent income generated by the firm evaluated by appropriate marginal rates of consumption substitution for investors. This suggests that, under a given pattern of firms' contingent incomes, the value of the firm could change only when the marginal rates of consumption substitution change.

Before we leave this section, it should be noted that the return receivable from each unit of an Arrow security is one unit of income at t=1 in a specified state. These contingent payoffs are known to all investors at t=0. Since solvency is assumed to be met by all firms, investors do not have to form subjective expectation about state-contingent future returns from their portfolios.

II.2. Incomplete Stock-Share Market

Real-world firms issue common-stock shares, preferred stock shares, several types of debts and other complex securities. Each security represents a claim against some portion of the income generated by a firm.

The simplest market structure is examined in this section. It permits only common-stock shares to distribute firms' incomes. If investor i buys $x_{ij}$ of the shares of firm j in the current-date market, he can expect to receive $100 \cdot x_{ij}$ percent of the firm's income no matter which state of nature occurs. Shares are traded in a competitive market
at \( t=0 \) with prices \( q_j(j=1,2,\ldots,J) \). Since each unit of shares represents only a portion of the claim on the total income of the firm, to figure out the amount of income receivable on the amount of shares that investor \( i \) has, he has to form expectations about the level of the firm's uncertain income. Let \( e^i_{js} \) be the income level of firm \( j \) in state \( s \) that is expected by investor \( i \). Then, from his/her \( x_{ij} \) of shares of firm \( j \), individual \( i \) expects to receive \( x_{ij} \cdot e^i_{js} \) of income in state \( s \) (\( s=1,2,\ldots,S \)). His total expected income from the stock-share portfolio in state \( s \) is, therefore:

\[
\sum_{j=1}^{J} x_{ij} e^i_{js}, \quad (s = 1, 2, \ldots, S).
\]

If this is spendable only for second-period consumption, \( c_{is} \), the optimal consumption portfolio decision of investor \( i \) can be characterized as follows:

\[
\text{maximize } u^i(c_{io}, \sum_{j=1}^{J} x_{ij} e^i_{js}; s = 1, 2, \ldots, S; f^i),
\]

subject to:

\[
c_{io} + \sum_{j=1}^{J} q_j x_{i} = w_{io}.
\]

Decision variables are \( c_{io} \) and financial security investments, \( x_{ij} \). The share prices, \( q_j(j=1,2,\ldots,J) \), are taken as given in the competitive market. We assume short sales are allowed (\( x_{ij} < 0 \)).
The market-clearing condition is:

\[ \sum_{i=1}^{I} x_{ij} = 1, \quad (j = 1, 2, \ldots, J). \]

First order conditions for the investor's optimal consumption-portfolio decision are:

\[ u^i_0 = \lambda^i, \text{ and} \]

\[ \sum_{s=1}^{S} u^i_s \cdot e^i_{js} = \lambda^i \cdot q_j \quad (j = 1, 2, \ldots, J). \]

where \( \lambda^i \) is the shadow price of additional income that could relax the budget constraint.

An infinitesimal increment in the amount of the shares of firm \( j \) individual \( i \) holds increases his/her expected income in state \( s \) by \( e^i_{js} \) and its marginal impact on utility is measured by \( u^i_s \cdot e^i_{js} \). Since the marginal increase in the amount of the shares has this impact in all possible states simultaneously, the sum of these marginal impacts on the utility level is balanced against the marginal cost, \( \lambda^i \cdot q_j \).

First-order conditions can be rewritten as:

\[ q_j = \sum_{s=1}^{S} \frac{u^i_s}{u^i_0} \cdot e^i_{js}, \quad (j=1, 2, \ldots, J). \]

The value of firm \( j \) is the value of its stock shares. Since the total amount of stock shares is:

\[ \sum_{i=1}^{I} x_{ij} = 1. \]
The price of the stock shares, $q_i^*$, in (5), is the value of the firm in this simple stock-share market.

Though the value of the firm (5) resembles the value of the firm derived in (4) for the Arrow-security market, two essential differences exist. First, the income of the firm in (5), $e_{js}^i$, is the income expected by investor $i$. Expectations may differ among investors. Even in the face of possible heterogeneity in expectations, the value of the firm is defined as (5). In other words, each investor adjusts his/her portfolio based on his/her subjective expectations of the firm's income in such a way that his/her valuation of the return from the share equals to its market price. Second, values of firms in (5) do not necessarily imply that each marginal rate of consumption substitution, $u_s^i/u_o^i$, is equalized across all investors. The Arrow-security market equilibrium establishes such an equality.

The number of equations in (5) is $J$. Therefore, for each investor $i$, to determine his $S$ marginal rates of consumption substitution uniquely, the number of firms, $J$, must be at least the number of possible states of nature, $S$. Further, for a unique solution the vectors of the expected income:

\[
\begin{bmatrix}
e_i^1 \\
e_i^2 \\
\vdots \\
e_i^J \end{bmatrix} \quad \ldots \quad \begin{bmatrix}
e_j^1 \\
e_j^2 \\
\vdots \\
e_j^S \end{bmatrix}
\]
must be linearly independent (Cass and Stiglitz [1970]). This means that the rank of the matrix of the firms' income level is $S$. In addition to these conditions, complete equalization of marginal rates of consumption substitution across all investors requires that expectations about the firms' incomes be homogeneous among all investors. When the rank condition and homogeneous expectations hold, the stock-share market is said to be "complete". As a complete marker, it uniquely determines marginal rates of consumption substitution for all investors:

\[
\begin{pmatrix}
\frac{u_i}{u_1} & \frac{u_i}{u_2} & \cdots & \frac{u_i}{u_S}
\end{pmatrix}
\begin{pmatrix}
e_{11} & \cdots & e_{1S} \\
\vdots & & \vdots \\
e_{JS} & \cdots & d_{JS}
\end{pmatrix}^{-1}
\begin{pmatrix}
q_1 \\
\vdots \\
q_J
\end{pmatrix}, \quad (i=1, 2, \ldots, I),
\]

where $J = S$.

When $J > S$, we can pick up $S$ firms whose vectors of incomes are linearly independent to determine marginal rates of consumption substitution. The uniquely determined marginal rates of consumption substitution in the complete stock-share market are equivalent to the Arrow-security prices, $P_s(s = 1, 2, \ldots, S)$, in the previous section.

However, it is unlikely that a stock-share market would be complete in itself. Even if expectations are homogeneous, the number of possible states of nature is
likely to exceed the number of firms. Let us imagine two sources of uncertainty that affect the levels of firms' incomes. The first consists of economy-wide uncertainty that influences all firms simultaneously: the second reflects uncertainty particular to each firm. Suppose $S_I$ states of nature are the result of the economy-wide uncertainty. $S_I$ may be less than the number of firms, $J$. By construction, firm-particular uncertainty is independent from economy-wide uncertainty. Let us imagine that each firm has two possible states of its firm-particular uncertainty. Then, this already creates $2^J$ possible states of nature for each of the $S_I$ possible states created by the economy-wide uncertainty. Hence, the number of all possible states of nature $S = S_I \times 2^J$. $S$ must, in this case, exceed the number of firms.

The Capital-Asset-Pricing model emphasizes the possibility that risk due to the firm-particular uncertainty can be diversified away (see Sharpe [1970] or other finance textbooks). But in our state-preference model, firm-particular uncertainty cannot safely be ignored.

When the stock-share market is incomplete, marginal rates of consumption substitution may differ among investors. This can be interpreted as indicating that each investor has his/her own set of implicit Arrow-security prices. Trading stock shares alone does not let investors
eliminate the differences among their sets of implicit Arrow-security prices.

II.3. "Innovative" Securities

Let us suppose the stock-share market is incomplete. We now examine the impact of introducing a new security for investors to trade among themselves.

Let $a_s$ be the income at $t=1$ from one-unit holding of the new security ($s = 1, 2, \ldots, S$). We ask each investor to buy one marginal unit of this new security and to indicate how much he/she is willing to pay for it. We start it from the stockshare-market equilibrium. The marginal unit of the new security promises to pay $a_s$ in state $s$ and it gives investor $i$ the marginal increase in utility of $u^i_s \cdot a_s$ (where $u^i_s$ is evaluated at the original stockshare market equilibrium). The investor's gain in the utility is:

$$
\sum_{s=1}^{S} u^i_s \cdot a_s.
$$

On the other hand, the payment for acquiring the new security costs him $u^i_o$ for each unit income he gives up. Hence, if the hypothetical price of the new security is $q_a$, the marginal cost in terms of utility is:

$$
-u^i_o \cdot q_a.
$$

These marginal benefits and costs define the net benefits as:
Since \( u^i > 0 \), investor \( i \) is willing to buy the new security when:

\[
S u^i < \sum_{s=1}^{S} u_s^i a_s = u_o^i \left( - q_a + \sum_{s=1}^{S} \frac{u_s^i}{u_o^i} a_s \right).
\]

If the inequality were reversed, he/she would rather be an issuer of the new security. The maximum price, \( q_a^i \), for investor \( i \) to be a buyer is simultaneously the minimum price to be an issuer. This price is:

\[
q_a < \sum_{s=1}^{S} \frac{u_s^i}{u_o^i} a_s.
\]

An important thing is that \( q_a^i \) may differ among investors in our incomplete stockshare market. This traces to the fact that an incomplete market does not equalize marginal rates of consumption substitution across investors.

Whether or not \( q_a^i \) \((i = 1 \ldots I)\) are the same for all investors also depends on the vector of returns, \((a_1, \ldots, a_s)\), on the new security. We examine this issue for the homogeneous-expectations case where \( e^i_{js} = e^j_{js} \) for all \( i \) and \( s \). Let us suppose the vector \((a_1, \ldots, a_s)\) happens to be linearly related to the vectors of the incomes of firms. This means that:

\[
(7) \quad du^i = -u_o^i q_a^i + \sum_{s=1}^{S} u_s^i a_s = u_o^i \left( - q_a^i + \sum_{s=1}^{S} \frac{u_s^i}{u_o^i} a_s \right).
\]
\[
\begin{pmatrix}
a_1 \\
\vdots \\
a_S
\end{pmatrix} = \sum_{j=1}^{J} b_j \begin{pmatrix}
e_{ji} \\
\vdots \\
e_{jS}
\end{pmatrix}
\]

where \( b_j (j = 1, 2, \ldots, J) \) are constants.

The critical price level, \( q^i_a \), is:

\[
q^i_a = \sum_{s=1}^{S} \frac{u^i_s}{u^i_0} \sum_{j=1}^{J} b_j \begin{pmatrix}
e_{j1} \\
\vdots \\
e_{jS}
\end{pmatrix}
= \sum_{j=1}^{J} b_j q_j.
\]

The last equality follows from the equilibrium condition for the stock-share market, (5). Equation (10) states that, with linear dependence and homogeneous expectations, the critical price level is the same for all investors. This implies that all investors simultaneously want to become buyers \( (q_a < q^i_a) \), or simultaneously want to become issuers \( (q_a > q^i_a) \). Therefore, the only possible market equilibrium price of the new security is \( \sum_{j=1}^{J} b_j q_j \), in which case no one would bother either to issue or to buy the new security. It would not come into existence precisely because it does not redistribute the firms' uncertain incomes beyond the original stock-share market equilibrium.
The same logic can be applied to the case where investors have heterogeneous expectations about firms' incomes. Let us again suppose the new security is a linear combination of existing stockshares. Under heterogeneous expectations, this means investor i perceives the return vector of the new security as:

\[
\begin{bmatrix}
  a_1^i \\
  \vdots \\
  a_S^i
\end{bmatrix}
= \sum_{j=1}^{J} b_j \cdot \begin{bmatrix}
  e_{j1}^i \\
  \vdots \\
  e_{jS}^i
\end{bmatrix}, \quad (i = 1, 2, \ldots, I).
\]

Under this condition, the critical price level for investor i is:

\[
q_a^i = \sum_{s=1}^{S} \frac{u_i^s}{u_0^i} \sum_{j=1}^{J} b_j \cdot \begin{bmatrix}
  e_{j1}^i \\
  \vdots \\
  e_{jS}^i
\end{bmatrix} = \sum_{j=1}^{J} b_j q_j \quad \text{by (5)}
\]

which, again, is equalized across all investors. With homogeneous and heterogeneous expectations, a new security whose return vector is a linear combination of the firms' incomes is of no significance for any investor. It does not help to redistribute the firms' incomes, and, therefore, does not change the marginal rates of consumption substitution of
investors. Prices of existing stockshares do not change. The value of all firms remains unchanged.

We call a new security "innovative" only if it is not linearly related to the existing stockshares and implies a different level $q_a^{i}$ for different investors. Differences in $q_a^{i} (i = 1, 2, \ldots, I)$ imply that the financial-market equilibrium with the new security requires changes in the marginal rates of consumption substitution. Some investors become issuers, while others become buyers of the new security. At the new equilibrium (denoted by ","):

$$q_j = \sum_{s=1}^{S} \frac{u_s^{i'}}{u_s^{i}} \cdot e_{js}^i \quad (j = 1, 2, \ldots, J)$$

for the stock-share prices and additional:

$$q_a = \sum_{s=1}^{S} \frac{u_s^{i'}}{u_o^{i'}} \cdot a_s$$

for the new security price.

Innovative securities close differences in marginal rates of consumption substitution. They help to complete the market and to reach a Pareto-optimal distribution of the firms' incomes. It is clear that if the stock-share market is already complete, no additional security can be innovative.
We can think of a case where the number of firms is equal to (or more than) the number of possible states of nature and investors have heterogeneous expectations about the firm's incomes. Then each investor's set of marginal rates of consumption substitution would be uniquely determined. But the difference in expectations leaves the marginal rates of consumption substitution unequal among investors, which implies a Pareto-nonoptional distribution. In this situation, the role of innovative securities lies in forcing investors to modify their expectations and equalize their expectations. When an innovative security changes the marginal rates of consumption substitution, unless investors expectations are modified, equilibrium in the markets for existing stock shares will not be re-established. When an innovative security plays this role, it may be said to be "speculative".

II.4. Re-examination of the Original Modigliani-Miller Theorem

Innovative securities discussed in the previous section were assumed to be traded only among investors, not issued by firms. The importance of innovative securities lies in their ability to enable investors to redistribute the firms' incomes beyond the original market equilibrium with only stock shares. It is shown next that the same idea applies to instances where firms issue securities other than stock-shares and change the composition of securities. The
Modigliani-Miller theorem concerns a special case of this where firms issue debts and stock shares. We re-examine the Modigliani-Miller irrelevance theorem in the context of incomplete markets with the concept of innovative securities. We show that our way of viewing the problem lets us further generalize the theorem beyond the treatment of Stiglitz (1969 and 1974), which is the most general treatment in the existing literature.  

Let us first construct a model of the financial market in which firms have already issued a multiplicity of securities including stock shares. For simplicity, suppose each firm has two securities outstanding, the first of which is stock shares. The second security of a firm may or may not be a fixed-income obligation (debt). The two securities define the way that the firms' future income is to be distributed to the security holders. Let the vector of return which is distributed to the holders of the second security be publicly announced as:

\[
\begin{pmatrix}
  e_{j1}^{(2)} \\
  \vdots \\
  \vdots \\
  e_{jS}^{(2)}
\end{pmatrix}, \quad (j = 1, 2, \ldots, J),
\]

where "2" in the parentheses indicates that \(e_{js}\) is income to the second security. If firm \(j\) issues debt, the elements of \(e_{js}^{(2)} (S = 1, \ldots, S)\) are a constant, \(e_j:\)
But we do not have to make this constant-payoff assumption. The analysis which follows includes debt as a special case.

The income to the shareholders is defined as the difference between the firms' income and the return to the holders of the second security. If different investors expect the firms' incomes differently (heterogeneous expectations), investor i's expected return to the stock shares of firm j, denoted by $e_{js}^i (1)$ ($s = 1, 2, \ldots, S$), is defined as:

\[
\begin{pmatrix}
  e_{j1}^i (1) \\
  \vdots \\
  \vdots \\
  e_{jS}^i (1)
\end{pmatrix}
= 
\begin{pmatrix}
  e_{j1}^i \\
  \vdots \\
  \vdots \\
  e_{jS}^i
\end{pmatrix}
- 
\begin{pmatrix}
  e_{j1}^i (2) \\
  \vdots \\
  \vdots \\
  e_{jS}^i (2)
\end{pmatrix}
\]

(11)

This way of dividing the firms' income depends on the amount of securities investors hold. Let $N_j^i (2)$ be the amount of the second security of firm j that investor i holds. Then $N_j (2) = \sum_{i=1}^{I} N_j^i (2)$ is the total amount of the security issued. Each unit of the security pays the return of:
Then the portion of the security that individual $i$ holds, $x_{ij}(2)$, is $N_{j}^{i}(2)/N_{j}(2)$ and we have:

$$
\sum_{i=1}^{I} x_{ij}(2) = 1.
$$

Portfolio decisions of investors can be expressed in terms of $x_{ij}(2)$ as we did for stockshares in the previous section.

Changing the corporate financial structure means changing the vector of returns on the second security. Since income to stock shares is defined as the residual, (11), we have:

$$
\begin{bmatrix}
  de_{j1}(1) \\
  \vdots \\
  de_{jS}(1)
\end{bmatrix}
\begin{bmatrix}
  de_{j1}(2) \\
  \vdots \\
  de_{jS}(2)
\end{bmatrix} = \begin{bmatrix}
  e_{j1}(2) \\
  \vdots \\
  e_{jS}(2)
\end{bmatrix}.
$$

This implies that, regardless of differences in individuals' expectations about firms' incomes, as long as the return vector of the second security is publicly announced to investors, changes in income on stock shares expected by all investors are affected in the same ways.
With this much preparation, the optimal consumption portfolio choice of investor may be described as the solution to:

$$\max u^i(c_{io}, \sum_{k=1}^{2} \sum_{j=1}^{J} x_{ij}^{(k)} e_{js}^{(k)}; s=1, \ldots, S; f^i),$$

subject to:

$$w_{io} = c_{io} + \sum_{k=1}^{2} \sum_{j=1}^{J} q_j^{(k)} x_{ij}^{(k)}.$$  

Control variables are $c_{io}$ and $x_{ij}^{(k)}$, with given prices for both securities, $q_j^{(1)}$ and $q_j^{(2)}$.

First-order conditions for optimal choice are:

$$q_j^{(1)} = \frac{\sum_{s=1}^{S} u_{js}^{i}}{u_{io}^{i}} e_{js}^{(1)}, \text{ and}$$

$$q_j^{(2)} = \frac{\sum_{s=1}^{S} u_{js}^{i}}{u_{io}^{i}} e_{js}^{(2)}, \text{ (j = 1, 2, \ldots, J).}$$

Together with the market-clearing condition:

$$\sum_{i=1}^{I} x_{ij}^{(k)} = 1, \text{ (j = 1, 2, \ldots, J and k = 1,2),}$$

equation (13) defines for all investors the market-equilibrium distribution of the firms' incomes. The second set of equations in (13) is the additional condition of market equilibrium in the second security. By the same reasoning as in section 2, the completeness of the market requires: (1) that the number of securities, $2J$, be at least as large as the number of possible states, $S$,,
(b) that the vectors of the return of 2J securities be linearly independent, and (c) that all investors have homogeneous expectations.

The value of firm j is the total market value of the two securities:

\[ \bar{V}_j = q_j(1) + q_j(2) \]

\[ (14) \]

\[ \bar{V}_j = \sum_{s=1}^{S} \frac{u^i_s}{u^o} [e^i_{js}(1) + e^i_{js}(2)] \text{ by (13)} \]

\[ = \sum_{s=1}^{S} \frac{u^i_s}{u^o} e^i_{js} \text{ by (11)}. \]

This has the same form as the solution for the Arrow-security market and the stock-share market. The original limited version of the Modigliani-Miller irrelevance theorem is derived from (14). This limited version states that two firms, say j and l, in the same risk class have the same firm value. The same risk class means \( e^i_{js} = e^i_{ls} \) \((2 = 1, 2, \ldots, S)\). Since we have the possibility of heterogeneous expectations, the same risk-class assumption here is interpreted to mean that \( e^i_{js} = e^i_{ls} \) \((s = 1, 2, \ldots, S)\) holds for all investors; that is, all investors perceive the two firms in the same way. Then (14) implies:

\[ \bar{V}_j = \sum_{s=1}^{S} \frac{u^i_s}{u^o} \cdot e^i_{js} = \sum_{s=1}^{S} \frac{u^i_s}{u^o} \cdot e^i_{ls} = \bar{V}_l. \]
This holds regardless of how each firm divides its income into the holders of the two securities. Values of the firms in the same risk class are equalized. Note that the second security does not have to be a fixed-income security (debt) and that as long as these firms are perceived by investors to belong to the same risk class, the heterogeneity of investors' expectations does not invalidate the limited-version irrelevance theorem.

However, this limited-version irrelevance theorem does not indicate whether the firm value would stay unchanged when the firm changes its financial structure (the general version of the theorem first presented by Stiglitz [1969 and 1974]). The remaining part of this chapter re-examines the general-version irrelevance theorem. We allow heterogeneous expectations, incomplete markets, and non-fixed-income securities in the following analysis. We do not have to focus on firms in the same risk class in the following generalization of the original Modigliani-Miller theorem.

II.5. General-Version Irrelevance Theorem

The impact of a change in corporate financial structure on the value of a firm is analyzed as follows. We start from equilibrium with a given set of corporate financial structures and examine the impact of a marginal change in the financial structure of firm j on any investor.
Investor $i$ holds both securities of firm $j$ in amounts $x_{ij}(1)$ and $x_{ij}(2)$. Incomes receivable on both securities change simultaneously when the corporate financial structure changes, as modeled in equation (12). Given his/her shares in both securities of firm $j$, we ask investor $i$ how much gain or loss in utility he experiences when firm $j$ changes its financial structure infinitesimally. First, we examine the impact of the return change on security 2 and, then, the impact of the corresponding return change on stock shares. In each case we determine the size of security-price change which would exactly compensate him/her for the gain or loss. Later, we combine this information to re-examine the generalized Modigliani-Miller irrelevance theorem formulated by Stiglitz.

The financial-structure change increases or decreases the return on the second security infinitesimally by $d e_{js}(2)$. The infinitesimal change in the investor $i$'s utility from this is:

$$x_{ij}(2) \sum_{s=1}^{S} u_{s}^{i} d e_{js}(2),$$

where $u_{s}^{i}$ is evaluated at the initial equilibrium. Let $dq_{j}(2)$ be the hypothetical change in the security price. The impact of $dq_{j}(2)$ on utility is $x_{ij}(2) \cdot u_{o}^{i} dq_{j}(2)$. The net change in his/her utility due to the price and return changes in the second security is:

$$(15) \quad x_{ij}(2) \cdot u_{o}^{i} \{-dq_{j}(2) + \sum_{s=1}^{S} \frac{u_{s}^{i}}{u_{o}^{i}} d e_{js}(2)\}.$$
If the change in the price, \( dq_j(2) \), exactly offsets the gain (or loss) from the change in the return, investor \( i \) is willing to keep the same share in the security. In this case, his/her utility level remains unchanged. Otherwise he/she would like to change the amount of shares held.

For investor \( i \) to keep his/her utility level unchanged, the critical level of \( dq_j(2) \) is defined as:

\[
(16) \quad d_{ij}^* = \sum_{s=1}^{S} \frac{u_{s}^i}{u_{o}^i} \ de_{js}(2).
\]

This critical level of price change may differ, in general, among investors, depending on the set of marginal rates of consumption substitution for each investor and the specified change in the return vector.

The financial-structure change has another impact on investor's utility due to the corresponding change in the return of the stock shares. A change in the return vector of the second security necessarily implies a corresponding change in returns on stock shares, (12). The impact from this change on the utility level of investor \( i \) is:

\[
x_{ij}(1) \ u_{o}^i \ \{- dq_j(1) + \sum_{s=1}^{S} \frac{u_{s}^i}{u_{o}^i} \ de_{js}(1)\}
\]

\[
= x_{ij}(1) \ u_{o}^i \ \{- dq_j(1) - \sum_{s=1}^{S} \frac{u_{s}^i}{u_{o}^i} \ de_{js}(2)\} \quad \text{by (12)}.
\]

With (16) substituted into the second term in the bracket, this shows that if the price change of stock shares is just
the opposite of the critical size of the price change of the second security, the impact on the utility from the change in the stock-share income, as well as the change in the return to the second security, is nil. In that case, investor i prefers to hold his/her initial shares in both securities. However, \( dq_j(1) = -dq_j(2) \) does not make the two impacts cancel each other unless \( x_{ij}(1) = x_{ij}(2) \). Interestingly, \( x_{ij}(1) = x_{ij}(2) \) in the Capital-Asset-Pricing Model when both securities are risky securities. The Capital-Asset-Pricing Model is a market-equilibrium model developed by Sharpe (1964), Mossin (1966), Lintner (1965) and others, from individual's portfolio selection based on two parameters of the probability distribution of portfolio rate of return (mean and variance) introduced by Tobin (1958) and Markowitz (1952, 1959). Under assumptions of: (a) homogeneous beliefs about the distribution of security returns, (b) unlimited risk-free opportunities for lending and borrowing and (c) no transaction costs, the Capital-Asset-Pricing Model predicts that the optimal combination of risky securities is the same across all investors. This implies that the market-equilibrium shares of risky securities represent the individual's optimal combination of risky securities, which in turn implies that if investor i holds \( x \% \) of firm j's shares he/she holds \( x \% \) of firm j's other securities.
In general, if the critical level of the change in the second security's price, $dq^{i}_j(2)$, differs among investors, the change in the financial-market equilibrium is established by means of a new set of security prices. In this process, the marginal rates of consumption substitution must change. This, of course, changes the equilibrium values of individual firms.

When the critical price change $dq^{i}_j(2)$ ($i=1, 2, ..., I$) is the same for all investors, the new equilibrium is reached by adjusting the prices of stock shares and the second security by exactly $dq^{j}_j(1) = -dq^{i}_j(2)$ and $dq^{j}_j(2) = -dq^{i}_j(2)$ without altering marginal rates of substitution. In this equilibrating process, no investor gains or loses. Since the value of firm $j$ is defined as the sum of the two securities, the firm value is unchanged:

$$dV_j = dq^{j}_j(1) + dq^{j}_j(2)$$

$$= dq^{i}_j(2) - dq^{i}_j(2) = 0;$$

that is the financial decision becomes irrelevant to the value of firm $j$. The values of the other firms also remain unchanged. This is the general-version irrelevance theorem.

We now show that the condition under which general-version irrelevance of financial decisions occurs is the same as the condition for a new security not to be innovative. The irrelevance theorem has traditionally been
discussed in terms of the debt-equity ratio, which means that the second-security return vector is:

\[
\begin{pmatrix}
e_{j1}^{(2)} \\
\vdots \\
e_{js}^{(2)}
\end{pmatrix}
= 
\begin{pmatrix}
\bar{e}_j \\
\vdots \\
\bar{e}_j
\end{pmatrix}
\]

A change in debt-equity implies a scale change in \( \bar{e}_j \), say 100\( .z \) percent:

\[
\begin{pmatrix}
de_{j1}^{(2)} \\
\vdots \\
de_{js}^{(2)}
\end{pmatrix}
= z 
\begin{pmatrix}
\bar{e}_j \\
\vdots \\
\bar{e}_j
\end{pmatrix}
\]

Therefore, by (16):

\[
dq_j^{(2)} = \sum_{s=1}^{S} \frac{u_s^i}{u_o^i} z \cdot \bar{e}_j = zq_j^{(2)}
\]

where the second equality holds because the initial market price of the debt (2nd security) was, by (13):

\[
q_j^{(2)} = \sum_{s=1}^{S} \frac{u_s^i}{u_o^i} \bar{e}_j.
\]

\( dq_j^{(2)} \) is equalized across all investors. Therefore, generalized-version irrelevance holds. A change in the scale of debt issue does not alter the value of the firm.
This result is identical to the general version of the Modigliani-Miller theorem presented by Stiglitz (1969 and 1974).

II.6. Further Generalization

The logic of our analysis lets us further generalize the irrelevance theorem to cover more general types of securities. If the financial-structure change is a scale change of existing securities:

\[
\begin{pmatrix}
    de_{j1}(2) \\
    \vdots \\
    de_{js}(2)
\end{pmatrix}
=
\begin{pmatrix}
    e_{j1}(2) \\
    \vdots \\
    e_{js}(2)
\end{pmatrix}
\]

then even if the return to the second security is state-contingent, \( e_{js} \neq e_{jw} \) for \( s, w = 1, 2, \ldots, S \):

\[
dq_{ij}(2) = z \sum_{s=1}^{S} \frac{u_i^s}{u_o^i} e_{js}(2)
\]

\[
= zq_{j}(2) \quad \text{by (13)}.
\]

That is, the equalization of \( dq_{ij}(2) \) holds across investors, and therefore the irrelevance theorem holds. Changing the scale of any security issue does not affect the value of the firm.

A few words about bankruptcy are in order. When a firm intends to change the scale of its issue of an existing
security, say by 100.\% percent, the contractual payment on
the security in certain states may not be credible because
the promised payment would exceed income levels in these
states. This difficulty would be recognized by investors.
Let
\[ m^i_j = \text{minimum} \{ e^i_{j1}, \ldots, e^i_{jS} \}; \]
that is, \( m^i_j \) is the lowest possible level of the income of
firm \( j \) that is expected by investor \( i \). Then as the scale
of the issue of the second security by firm \( j \) increases, the
investor(s) who has the lowest \( m^i_j \) begins to expect a
default on the security. Then (18) no longer holds. There­
fore, we cannot establish the equalization of \( dq^i_j(2) \). This
invalidates the irrelevance theorem. For example, suppose
that investor \( i \) expects firm \( j \)'s income levels as \( (10, 5, 8, 15) \) in four possible states and that the current promised
contingent payoffs on security 2 is \( (8, 5, 7, 7) \). This mean
means that the current return to stock shares is \( (2, 0, 1, 8) \). Hence, even a 1\% increase in the scale of the second
security makes investor \( i \) expect the return on the second
security to be \( (8.08, 5, 7.07, 7.07) \) which is not 1\% higher
than \( (8, 5, 7, 7) \) due to the expected default in the second
state of nature. Investor \( i \) would view the expanded second
security as an innovative security.

The irrelevance theorem requires the equalization
of \( dq^i_j(2) \) across investors. This holds, in general, whenever
changes in the return vector on the second security is a linear combination of all existing security-return vectors:

\[
\Delta e_{jS}(2) = \sum_{k=1}^{2} \sum_{l=1}^{J} b_{l}(k) q_{l}(k), \quad (s=1, 2, \ldots, S)
\]

where \(b_{l}(k)\): a constant.

Under this linear-combination condition:

\[
\Delta q_{iJ}^{i}(2) = \sum_{k=1}^{2} \sum_{l=1}^{J} b_{l}(k) q_{l}(k)
\]

which is equal across all investors. This is the most general form of the irrelevance theorem. This general irrelevance theorem does not require that the financial-structure change be a scale change. The configuration of the state dependency of the return of a security would change without altering the value of firms as long as the linear dependency (19) holds. What (19) says essentially is that the existing securities can be combined to form a portfolio which is a perfect substitute for the new security created by the financial-structure change. No "innovative" security is created. Note that condition (19) always holds in complete markets.

II.7. Conclusion

Our analysis has provided a new way of viewing the Modigliani-Miller Theorem. Utilizing the concept of
"completing markets" let us generalize the theorem beyond Stiglitz (1974). The incompleteness of markets, heterogeneous expectations and unrestricted types of securities do not alter the validity of the theorem under the condition that financial rearrangements do not introduce innovative securities. It is easy to extend our analysis into multi-security cases. The idea underlying the irrelevance theorem is that no "innovative" security be created by the changes in the financial structure.

To explain why economies need financial markets, we show that financial markets improve the efficiency of resource allocation over time. This same explanation clarifies why a particular type of security may be needed. The irrelevance theorem does not illuminate such issues. Instead, it explains the redundancy of some security issues. Rather it suggests that we must search for other alternative meaningful explanations for corporate financial decisions. The agency-cost approach, pursued by Jensen and Meckling (1976), Smith and Warner (1979) and others, tries to explain corporate capital structure by monitoring and bonding costs in financial contracting. This seems to be a promising attempt. Also important may be an explicit formulation of shareholders' intention to maintain control over the firm. Especially, if some shareholders have specific ideas of future investment plans which are not shared by the other investors they would try to limit the
use of equity capital within their financial ability. The formulation of this aspect requires the analysis of asymmetric information in a dynamic context.
FOOTNOTES TO CHAPTER II

1 Chen and Kim (1979) provide an excellent survey of the development of theories of corporate financial decisions.

2 As Fama and Miller (1972) and Stiglitz (1972b) point out, additional financial or investment decisions may change the wealth of the holders of outstanding securities and, therefore, may break the consistency between maximizing the firm value and maximizing shareholder's wealth. This new aspect has been integrated into the theory of agency cost by Jensen and Meckling (1976) and Smith and Warner (1979). The agency-cost theory makes explicit the conflicts among the goals of the parties in a financial contract which may prevent the parties from achieving their mutually attainable maximum gains. The agency-cost theory gives an insight into costly bonding and monitoring activities as an effort to guarantee the maximum gains.

3 Hakkanson (1978), Ross (1976) and Friesen (1979) view call and put options written on the primary stock shares as an effort to complete the market. On the other hand, the option pricing models developed by Black and Scholes (1973) use arbitrage relations to derive closed-form solutions. The arbitrage approach assumes that a linear combination of existing securities is a perfect substitute for options. This implies that options are not innovative in this context.

4 See Fama's comment on Stiglitz's analysis (Fama [1978]).
Chapter III. THE CAPITAL-ASSET PRICING MODEL

The purpose of this chapter is to investigate the implications for the Pareto-optimality of the allocation of wealth and the market valuation of assets of the assumptions underlying the Capital-Asset-Pricing model (referred to as the CAPM). A special focus is on the pricing implications of the CAPM; i.e., the possibility of obtaining the closed-form solution for risk premiums for risky assets in terms of rate of return on the risk-free security and the market portfolio. This solution gives the CAPM empirical testability. The closed-form solution means that the values of assets are expressed, in market equilibrium, in terms of a set of potentially measurable factors such as prices, values, risk-free interest rate, etc. which does not include specific information about investors' preference orderings. We view the CAPM as a special case of a more general market model and examine the possibility of obtaining a general solution without imposing the specific form of investors' utility functions assumed in the CAPM. In the CAPM, the efficient-portfolio frontier for each investor is spanned by the risk-free asset and the market portfolio which is the optimal combination of risky assets.
The choice of this risky-asset portfolio is independent of investors' preference orderings. This separation property is the basis of the CAPM closed-form solution. We investigate conditions under which the general solution converges to the CAPM solution without the separation property for investors' optimal investment plans.

The CAPM has grown out of the portfolio-selection theories developed by Markowitz (1953, 1959) and Tobin (1958). They viewed the individual investor's portfolio decision as a choice over two parameters: the expected value and the variance (or the standard deviation) of the rate of return on the individual's portfolio of assets and liabilities. Variance is regarded as measuring the risk in the portfolio. Ignoring higher-order moments as the measure of the portfolio risk has been criticized by Borch (1969, 1974), Feldstein (1969) and others. But the simplicity of using only the variance has helped the two-parameter approach to gain popularity among financial economists. Sharpe (1964), Mossin (1966), Lintner (1965) and others have developed a market-equilibrium model of financial markets based on demand equations derived from the Markowitz-Tobin model. The market-equilibrium model is now called the CAPM.

In addition to two-parameter utility functions, the CAPM assumes: (a) the existence of risk-free loan opportunities, (b) homogeneous investor beliefs about the
stochastic distribution of the asset rates of return, (c) costless transactions, and (d) no restrictions on short sales. In principle, the CAPM covers all assets, both financial and physical. Taking the supplies of assets and the risk-free interest rate as given, the CAPM determines the optimal asset-portfolio proportions for individual investors and the market-equilibrium prices of assets. The CAPM assumptions produce specific predictions about investor behavior and asset prices.

The CAPM predicts that all investors would hold the same combination of risky assets in their optimal portfolios, which implies a separation of choice between the risk-free asset and the portfolio of risky assets. More precisely, the proportionate value of each risky asset in the total value of risky-asset portfolio is the same for all investors. This equi-proportionate diversification suggests the emergence of mutual funds. As a matter of fact, CAPM investors can form their optimal portfolios from only two mutual funds: the risk-free asset and the market-value-weighted portfolio of risky assets which is called the market portfolio.

Unique portfolio decisions of investors in the CAPM determine the market-equilibrium prices of assets. The price of any asset, say asset j, is determined so that the excess of the expected value of the rate of return, \( E_j \), over the risk-free rate of return, \( R_f \), is directly proportional
to the difference between the expected value of the market-portfolio rate of return over the risk-free rate, $E_m - R_f$:

$$E_j - R_f = \beta_j (E_m - R_f),$$

where the coefficient $\beta_j$ captures the risk of asset $j$ relative to the risk of the market portfolio. This relation is derived as follows. The market portfolio of risky assets represents the optimal risky-asset portfolio for all investors. Its risk premium, $E_m - R_f$, represents compensation for risk-bearing as measured by the variance of the rate of return on the market portfolio, $\sigma_m^2$. This determines the market price of a unit of risk in terms of the expected rate of return as:

$$\frac{(E_m - R_f)}{\sigma_m^2}.$$

On the other hand, the risk of asset $j$ is measured by its marginal contribution to the risk of the portfolio, which is the covariance of the asset $j$'s rate of return to the market-portfolio rate of return, $\sigma_{jm}^2$. Therefore, the risk premium on asset $j$ should be:

$$E_j - R_f = \text{(the price of risk-bearing)} \times \text{(the risk of asset } j)$$

$$= \{\frac{(E_m - R_f)}{\sigma_m^2}\} \sigma_{jm}^2.$$

By definition $\beta_j = \sigma_{jm}^2 / \sigma_m^2$, we obtain proportionality between the risk premium on asset $j$ and the risk premium on the market portfolio. The beta coefficient $\beta_j$, measures the
volatility of the rate of return of asset \( j \) relative to the risk of the market portfolio.

An important property of this market-equilibrium relation is that the coefficients \( \beta_j \), as well as \( E_j, R_f, \) and \( E_m \) are determined only from the information about the stochastic distribution of asset returns and the market value of assets. Hence, the CAPM provides a closed-form expression for expected rates of asset returns. These factors are all potentially measurable, which makes the model empirically testable (Jensen [1972]).

We are interested in the possibility of obtaining a solution similar to the CAPM solution in a model with an alternative set of assumptions. We re-examine the CAPM as a special case of the general state-preference model of asset markets and show that a generalized solution for expected rates of asset return is also found in the state-preference model. The state-preference model assumes that investor utilities depend on current and future state-contingent consumption plans. Unlike the CAPM, the state-preference model does not impose a specific form on the investors' utility functions other than properties required for the stability of investors' portfolio choices. Even with this generality, we show that a solution for risk premiums similar to the CAPM solution can be obtained.

However, the solution for the state-preference model requires that the market be complete, in the sense that the
variety of the asset returns be sufficient to span the income space of all possible states of nature and that investors share the same beliefs about the stochastic distribution of these returns. The latter assumption is called homogeneous expectations about the levels of return and homogeneous assessments of the probabilities of the uncertain state of nature. Homogeneous beliefs are also assumed in the CAPM. The generality of the state-preference model lies in the fact that it allows investors to have different combinations of risky assets in their optimal portfolios. The CAPM's closed-form solution is based on its unique implication for the optimal portfolio decisions of investors, namely the separation of the choice of risky-asset portfolio and risk-free asset. This separation means that the optimal combination of risky assets is identical for all investors and must be identical to the market portfolio of risky assets. A solution can also be obtained without this separation property.

We show that the fundamental property for deriving a closed-form solution for risk premiums is that the allocation of wealth in both the CAPM and the complete state-preference-market model is Pareto-optimal. Pareto-optimality assures that the marginal rates of substitution between current-date income and income in every possible future state of nature are equalized across all investors. Marginal rates of income substitution are the bases for
valuing income in uncertain state in terms of the current
income. The Pareto-optimal allocation implies that valua-
tion bases for income in all possible states of nature are
equalized across all investors.

This property enables us to calculate the risk pre-
mium on any asset in terms of potentially measurable
factors. Pareto-optimality obtains in the CAPM even though
asset markets may not be complete. The special assumptions
of the CAPM generate this result. On the other hand, the
Pareto-optimality of wealth allocation in the state-
preference model requires the completeness of the asset
markets.

We show that when the market is complete, the pre-
mium on a risky asset, in the state-preference model, can
be expressed in terms of the covariance between the asset
rate of return and an artificial random variable constructed
from the implicit Arrow-security prices which are equal to
the marginal rates of income substitution. Examining this
relation lets us establish conditions under which the pre-
mium can be expressed by a beta coefficient in the same way
the CAPM allows. An important finding is that these cir-
cumstances could arise without the separation property
holding for investors' portfolio decisions. The specific
utility-function form assumed in the CAPM is a sufficient
condition for this to be always assured.
The first section of this chapter briefly reviews the state-preference model in chapter II applied to markets of all assets, financial and physical. Pareto-optimality of the allocation of wealth in complete markets is also reviewed. The second section reviews the theoretical structure of the CAPM, emphasizing the separation property and the derivation of the closed-form solution. We show that the CAPM equilibrium allocation of wealth is always Pareto-optimal even in incomplete markets. Then, in the third section, we derive a general solution for the expected values of rates of return on assets in the state-preference model. This lets us identify the circumstance under which a CAPM-type of solution may be obtained in the general state-preference model without the separation property.

III.1. The State-Preference Model of Financial Markets

Let us recall the simple state-preference model of Chapter II to apply it to competitive asset markets and review the market-equilibrium allocation of wealth. We do not impose any specific form on investors' utility functions. Later in this chapter, we compare the CAPM equilibrium with this model. Each investor's utility level generally depends on his current consumption, the income in each possible state of nature in the future, which is spent for consumption, and his assessments of the probabilities of the possible states. We treat supplies of assets as exogenous
to focus on the equilibrium allocation of wealth. We review the concept of "complete" and "incomplete" markets.

Let us suppose \( I \) investors trade \( J+1 \) assets, \( J \) of which are risky assets whose returns are state-dependent, while the other asset is risk-free. A risk-free asset is one whose return is the same in all states. For convenience, we let the 0-th asset be the risk-free asset. The market prices of assets are given by \( q_j \) (\( j=0,1,2,\ldots,J \)). Investors take these prices as given when making their optimal portfolio plans. Investors trade assets at \( t=0 \) and receive returns on assets at \( t=1 \). Let \( x_j^0 \) be the exogenously given supply of asset \( j \) (\( j=0,1,2,\ldots,J \)) and \( x_{ij} \) be the amount of asset \( j \) that investor \( i \) demands. Investor \( i \) purchases \( x_{ij} \) of asset \( j \) at price \( q_j \) and expects to receive \( e_j^i \) of return on each unit of \( x_{ij} \) at \( t=1 \). The return \( e_j^i \) is uncertain. The level of return achieved depends on the state of nature. Suppose \( S \) possible states of nature exist. Then the value of \( e_j^i \) is:

\[
e_j^i = e_{js}^i \quad \text{in the event of state } s \ (s=1,2,\ldots,S).
\]

The subscript \( i \) on an asset return indicates that it is the individual \( i \)'s subjective expectation. Homogeneous expectations across investors is a condition where:

\[
e_{js}^i = e_{js} \quad \text{for all } i=1,2,\ldots,I.
\]
Let $\tilde{w}_{i0}$ be exogenously given initial wealth of investor $i$ at $t=0$. It is spent either on consumption at $t=0$, $c_{i0}$, or put into the asset portfolio, $x_{ij}$. The investor's budget constraint at $t=0$ may be expressed as:

$$
(20) \quad \tilde{w}_{i0} = \sum_{j=0}^{J} q_j x_{ij} + c_{i0}.
$$

Let us examine the case where the return on the portfolio of assets at $t=1$ is spent entirely for consumption. Let $c_{is}$ be the consumption at $t=1$ when state $s$ occurs ($s=1,2,\ldots,S$). Then, $c_{is}$ can be expressed as:

$$
(21) \quad c_{is} = \sum_{j=0}^{J} x_{ij} e^i_s \quad (s=1,2,\ldots,S).
$$

Investor $i$'s preference ordering over the current and contingent future consumption plan, $c_{i0}$ and $c_{is}$ respectively, also depends on his assessment of the probabilities of the state, $f^i=(f^i_1,\ldots,f^i_S)$ with $f^i_s > 0$ and $\sum_{s=1}^{S} f^i_s = 1$:

$$
(22) \quad u^i = u^i(c_{i0}, c_{is}, s=1,2,\ldots,S; f^i).
$$

We assume $u^i$ is concave (to assure the stability of the individual's equilibrium) and differentiable (for simplicity). The investor maximizes the level of utility (22) subject to the budget constraint (20) and the feasibility relation (21). Decision variables are $c_{i0}$ and $x_{ij}$. The subjective probability assessment, $f^i$, is given. The market prices of
assets, $q_j$, are given. We assume no transactions costs and
taxes, perfect divisibility of consumption and assets, and
that investors can take short positions (i.e., $x_{ij} < 0$) in
assets.

Substituting (21) into the utility function (22),
the investor's optimal consumption-portfolio decision is
characterized as:

\[
(23) \quad \text{maximize } u^i(c_{i0}, \sum_{j=0}^{J} x_{ij} e_{js}^i; s=1,2,...,S; f^i)
\]

subject to $\bar{w}_{i0} = \sum_{j=0}^{J} q_j x_{ij} + c_{i0}$.

Let $e^i$ be the $(J+1)$-by-$S$ matrix of the returns on assets in
all possible states (the columns index for states and the
rows for assets). Then, using equation (5) in chapter II,
the first-order optimal condition may be stated as:

\[
(24) \quad \begin{pmatrix}
    \frac{u_i^i}{u_0^i} \\
    \vdots \\
    \frac{u_S^i}{u_0^i}
\end{pmatrix}
= \begin{pmatrix}
    q_0 \\
    \vdots \\
    q_J
\end{pmatrix} \quad (i=1,2,...,I).
\]

Since the 0-th asset is risk-free which means $e_{0s}^i = e_0$ for
all states $s=1,2,...,S$, condition (24) for the risk-free
asset is:

$$
\sum_{s=1}^{S} \left( \frac{u_s^i}{u_0^i} \right) e_0 = q_0.
$$
By defining the risk-free rate of interest, $R_f$, as:

$$R_f = e_0 / q_0 - 1,$$

we have:

$$\sum_{s=1}^{S} \frac{u_i^s / u_i^0}{(1 + R_f)^{-1}}.$$

The conditions in (24) and the budget constraint (20) determine the demands for assets, $x_{ij}$ $(j=0, l, \ldots, J)$. The market-clearing condition is:

$$\sum_{i=1}^{I} x_{ij} = \bar{x}_j \quad (j=0, l, \ldots, J).$$

Equations (24) and (26) determine the allocation of wealth in equilibrium markets.

Pareto-Optimality

The competitive market allocation is said to be optimal in the sense of Pareto when the following condition holds: no investor can gain by redistributing assets without causing a sacrifice by some other investors. Pareto-optimality requires that the marginal rates of income substitution (in our case, equivalent to consumption substitution), $u_i^s / u_i^0$, between current and the future state-contingent income in state $s$ be equalized across all investors for the following reason. Suppose that between two investors, say investor 1 and 2, marginal rates of income substitution are not equalized:
for some state \( s \). For instance, suppose for state \( s=1 \), we have

\[
\frac{u_s^1}{u_0^1} > \frac{u_s^2}{u_0^2}.
\]

This means that investor 1 values the marginal unit of income in state 1 higher than investor 2 does. Investor 1 is willing to pay up to \( \frac{u_1^1}{u_0^1} \) at \( t=0 \) to receive an additional unit of income in state 1, while investor 2 is willing to pay only up to \( \frac{u_1^2}{u_0^2} \). This implies that investor 2 would like to give up one additional unit of income in state 1 if he could receive more than \( \frac{u_1^2}{u_0^2} \). This implies that investor 2 would like to give up one additional unit of income in state 1 if he could receive more than \( \frac{u_1^2}{u_0^2} \) of current income. Therefore, an exchange of current income and future income in state 1 would take place between the two individuals. This redistribution improves the utility level of at least one of the two investors without hurting anyone else. This possibility for unambiguous welfare improvement means that the initial distribution could not be Pareto-optimal. Pareto-optimality requires equalization of marginal rates of income substitution across investors.

Completeness of the Market

Let us now examine the Pareto-optimality of the market-equilibrium allocation of wealth described by
equations (24) and (26). Equalization across investors of marginal rates of income substitution in the asset-market model requires: (a) that the set of the marginal rates of income substitution for each investor \( i \) be uniquely determined, and (b) that the uniquely determined marginal rates of income substitution be equalized across investors.

Requirement (a) means that the optimal consumption-portfolio condition (24) for investor \( i \) must be uniquely solved for \( (u_{1i} / u_{0i}^i, \ldots, u_{Si} / u_{0i}^i) \) which requires:

\[
\text{rank} \left( e_i^i \right) = S.
\]

This condition demands that the number of assets, \( J+1 \), must be at least as large as the number of the possible states of nature, \( S \). If this "rank condition" is satisfied, we can choose \( S \) assets among \( J+1 \) assets such that the rank of the return matrix drawn from these \( S \) assets is \( S \).

Suppose the rank condition is satisfied. For simplicity, let us assume that the number of assets is exactly the same as the number of all possible states, \( J+1=S \). Then, from (24) we can develop the solution for the marginal rates of income substitution as

\[
\begin{pmatrix}
\frac{u_{Si}^i}{u_{0i}^i} \\
\vdots \\
\frac{u_{Si}^i}{u_{0i}^i}
\end{pmatrix}
= \left( e_i^i \right)^{-1}
\begin{pmatrix}
q_0 \\
\vdots \\
q_J
\end{pmatrix}.
\]

(27)
Equation (27) tells us that, for the set of marginal rates of income substitution to be equalized across all investors (requirement (b)), investors must have homogeneous expectations about the returns on assets:

\[
\begin{bmatrix}
e^i
\end{bmatrix} = \begin{bmatrix}e\end{bmatrix} \quad \text{for all } i=1,2,\ldots,I.
\]

The concept of "complete" market applies to markets which meet these rank and homogeneous-expectation conditions.8

The competitive allocation of wealth in a market of Arrow securities makes the marginal rates of income substitution equal the prices of Arrow securities, \(p_s\):

\[
\frac{u^i_s}{u^i_0} = p_s \quad (s=1,2,\ldots,S).
\]

Therefore, equalization across investors of marginal rates of income substitution holds in an Arrow-security market in which a complete set of \(S\) state-contingent claims on income is traded. Compared with an Arrow-security market, we can say that when observable asset markets are complete, we have a correspondence between (27) and the prices of Arrow securities9:

\[
\begin{bmatrix}
p_1 \\
\vdots \\
p_S
\end{bmatrix} = \begin{bmatrix}
\frac{u^i_1}{u^i_0} \\
\vdots \\
\frac{u^i_S}{u^i_0}
\end{bmatrix} = \begin{bmatrix}e\end{bmatrix}^{-1} \begin{bmatrix}q_1 \\
\vdots \\
q_J
\end{bmatrix}.
\]

In incomplete markets, each investor has his own set of implicit Arrow-security prices. This set is
identical to his marginal rates of income substitution and may differ among investors.

We must note, however, that the completeness requirement for Pareto-optimality presumes that we allow the utility functions of investors to be general as stated by (22). It may be possible to have a Pareto-optimal allocation in an incomplete market if we impose a specific form on investor utility functions. The CAPM does assume a specific form of utility functions. The next section examines CAPM assumptions for implications about the Pareto-optimality of the market allocation of wealth.

III.2. The Structure of the CAPM

The Capital-Asset-Pricing model assumes:

(A-1) the utility functions of investors depend on current consumption and the expected value and the variance of t=1 income (which is, in our model, also t=1 consumption),

(A-2) investors share the same beliefs about the stochastic distribution of asset returns,

(A-3) unlimited opportunities for risk-free borrowing and lending exist, and

(A-4) transactions costs and restrictions on short sales do not exist.

Assumption (A-4) has also been assumed in the general asset market model discussed in section 1 of this chapter.
Assumption (A-3) is equivalent to assumption (A-4) applied to the 0-th asset in the state-preference model in the previous section. Assumption (A-2) is satisfied in the state-preference model by assuming:

\[ f^i = (f^i_1, \ldots, f^i_S) = (f_1, f_2, \ldots, f_S) \quad \text{and} \quad [e^i] = [e] \quad \text{for all investors } i=1,2,\ldots,I. \]

Homogeneous expectations about the levels of asset returns is one of the two necessary conditions for the completeness of the market. The other condition is the rank condition on the matrix of asset returns which requires \( J+1=S \). This is not assumed in the CAPM.

Now, let us examine the consequences of Assumption (A-1). The expected value of \( t=1 \) consumption is denoted by \( E(c) \). This value may be expressed as:

\[
E(c) = \sum_{S=1}^{S} f^i_S c_{iS}.
\]

The variance, which is denoted by \( \sigma^2(c) \) may be expressed as:

\[
\sigma^2(c) = \sum_{S=1}^{S} f^i_S (c_{iS} - E(c))^2.
\]

Written in symbols, assumption (A-1) states that equation (22) in section 1 has the special form:

\[
u^i = u^i(c_{i0}; c_{iS}, S=1,2,\ldots,S; f_i) = u^i(c_{i0}; E(c_i), \sigma^2(c_i)).
\]
The variance measures the uncertainty of returns from the assets which are spent for consumption at \( t=1 \). Equation (30) states that the investor's preference ordering over alternative asset portfolios can be summarized by his preferences with respect to the expected value and the risk of the asset-portfolio return.

From these specifications for the utility function (30), we can easily find the marginal utilities of current consumption, \( c_{i0} \), and future consumption, \( c_{is'} \):

\[
\begin{align*}
    u^i_o = U^i_o, \quad & \text{where } U^i_o = \partial U^i / \partial c_{i0}, \text{ and } \\
    (31) \quad u^i_s = U^i_{E_s} + U^i_{\sigma^2_s} \cdot 2f_s (c_{is'} - E(c_i)), \\
    \text{where } U^i_E = \partial U^i / \partial E(c_i) \text{ and } U^i_{\sigma^2} = \partial U^i / \partial \sigma^2(c_i).
\end{align*}
\]

These marginal utilities may also be expressed in terms of rates of asset return. Since consumption at \( t=1 \) in state \( s, c_{is'} \), is entirely financed by the portfolio return, we have:

\[
    c_{is'} = \sum_{j=0}^{J} x_{ij} e_{js'}.
\]

The rate of return on asset \( j \) in state \( s \) is defined as:

\[
    r_{js} = e_{js} / q_{j} - 1.
\]

From this definition, we can define the expected value and covariances of the rate of return on each asset as:
\[ E_j = \sum_{s=1}^{S} f_s r_{js}, \text{ and} \]

\[ \sigma_{jk} = \sum_{s=1}^{S} f_s (r_{js} - E_j)(r_{ks} - E_k) \quad (j, k = 0, 1, 2, \ldots, J). \]

For the risk-free asset, \( j = 0 \), \( E_0 = R_f \) and \( \sigma_{0k} = 0 \) for \( k = 1, 2, \ldots, J \). With this notation, marginal utilities in equation (31) become:

\[ u^i_s = \sum_{s=1}^{S} f_s (r_{js} - E_j)(r_{js} - E_j). \]

Or, in a matrix-and-vector form:

(32)

\[
\begin{bmatrix}
    u^i_1 \\
    \vdots \\
    u^i_S
\end{bmatrix} = \begin{bmatrix}
    f_1, \ldots, 0 \\
    \vdots \\
    0, \ldots, f_S
\end{bmatrix} \begin{bmatrix}
    1 \\
    \vdots \\
    1
\end{bmatrix} + 2 \begin{bmatrix}
    f_1, \ldots, 0 \\
    \vdots \\
    0, \ldots, f_S
\end{bmatrix} \begin{bmatrix}
    x_{i1} q_1 \\
    \vdots \\
    x_{iS} q_J
\end{bmatrix}.
\]

Next, we use this specific form of marginal utility to determine the characteristics of the CAPM equilibrium allocation of wealth.

The market-equilibrium allocation is defined by the investors' optimal consumption-portfolio plans and the market-clearing conditions, equations (24) and (26) in the previous section. Substituting equations for marginal utilities, (31) or (32), into the optimal portfolio condition (24) implies the separation property for investors' optimal portfolio plans.
With the definition of the rate of asset return, 
\[ r_{js} = \frac{e_{js}}{q_j} - 1 \]
the condition for the investors' optimal consumption-portfolio decisions, (24), is rewritten as:

\[
\begin{pmatrix}
    r_{11} & \cdots & r_{1S} \\
    \vdots & \ddots & \vdots \\
    r_{J1} & \cdots & r_{JS}
\end{pmatrix}
\begin{pmatrix}
    u_{i1}/u_{01} \\
    \vdots \\
    u_{iS}/u_{0S}
\end{pmatrix}
= (1 - \sum_{s=1}^{S} u_{is}/u_{0s}) \begin{pmatrix}
    1 \\
    \vdots \\
    1
\end{pmatrix}
= \frac{R_f}{1+R_f} \begin{pmatrix}
    1 \\
    \vdots \\
    1
\end{pmatrix}
\]

The last equality in (33) follows from (25). It merely applies condition (24) to the risk-free asset. By substituting the specific forms for marginal utility in the CAPM (the first equation in (31) with respect to \( c_{i0} \) and (32) with respect to \( c_{is} \)) into (33), we obtain the optimal consumption-portfolio condition for investor \( i \) as:

\[
\frac{R_f}{1+R_f} \begin{pmatrix}
    1 \\
    \vdots \\
    1
\end{pmatrix}
= \frac{U_{E}}{u_{0}} \begin{pmatrix}
    E_1 \\
    \vdots \\
    E_J
\end{pmatrix}
+ \frac{2U_{Q2}}{u_{0}} \begin{pmatrix}
    r_{11} & \cdots & r_{1S} \\
    \vdots & \ddots & \vdots \\
    r_{J1} & \cdots & r_{JS}
\end{pmatrix}
\begin{pmatrix}
    f_{1} & \cdots & 0 \\
    \vdots \\
    0 & \cdots & f_{S}
\end{pmatrix}
\]

\[
\begin{pmatrix}
    r_{11} - E_1 & \cdots & r_{J1} - E_J \\
    \cdots \\
    r_{IS} - E_1 & \cdots & r_{JS} - E_J
\end{pmatrix}
\begin{pmatrix}
    x_{i1} q_{1} \\
    \vdots \\
    x_{iJ} q_{J}
\end{pmatrix}
\]

\[
= \frac{U_{E}}{u_{0}} \begin{pmatrix}
    E_1 \\
    \vdots \\
    E_J
\end{pmatrix}
+ \frac{2U_{Q2}}{u_{0}} \begin{pmatrix}
    x_{i1} q_{1} \\
    \vdots \\
    x_{iJ} q_{J}
\end{pmatrix}
\]

\[
\begin{pmatrix}
    x_{i1} q_{1} \\
    \vdots \\
    x_{iJ} q_{J}
\end{pmatrix}
\]
where \([\sigma]\) is the \(J\)-by-\(J\) variance-covariance matrix of the risky-asset rates of return. Equation (34) may be simplified by noting the optimal portfolio condition for the risk-free asset. The condition for the risk-free asset, (25) can be applied to the CAPM to yield:

\[
(1+R_f)^{-1} = \frac{U_i^F}{U_0^F}.
\]

Substituting this into (34), we obtain:

\[
\begin{pmatrix}
E_1 - R_f \\
\vdots \\
E_J - R_f
\end{pmatrix}
= -(1+R_f) \frac{2U_i}{U_0} \frac{\sigma^2}{\sigma^2} \begin{pmatrix}
\sigma_1 \\
\vdots \\
\sigma_J
\end{pmatrix}
\begin{pmatrix}
x_{i1}\sigma_1 \\
\vdots \\
x_{iJ}\sigma_J
\end{pmatrix}
\]

Condition (35) governing the optimal consumption-portfolio decisions in the CAPM establishes separation.

Assuming non-singularity of the variance-covariance matrix of asset rates of return,\(^{10}\) we can solve (35) for the values of asset demands:

\[
\begin{pmatrix}
x_{i1}\sigma_1 \\
\vdots \\
x_{iJ}\sigma_J
\end{pmatrix}
= -(1+R_f)^{-1} \frac{2U_i}{U_0} \frac{\sigma^2}{\sigma^2} \begin{pmatrix}
\sigma_1 \\
\vdots \\
\sigma_J
\end{pmatrix}^{-1}
\begin{pmatrix}
E_1 - R_f \\
\vdots \\
E_J - R_f
\end{pmatrix}
\]

In terms of the proportion of each asset in the total value of the risky-asset portfolio, equation (36) gives us:
Since the right-hand side of (37) is independent of the investor index, $i$, shares of each risky assets in the total risky-asset portfolio are equalized across investors. This means that the optimal choice of risky-asset combination is the same for all investors and it is separated from the choice between the risk-free asset and the risky-asset portfolio. This is, of course, the separation property associated with the CAPM. Letting $m_j$ be the market share of risky asset $j$ in the aggregate market value of risky assets, we obtain from (37) the following relation which holds for all investors:

$$
\left(\begin{array}{c}
x_{11}q_1 \\
\vdots \\
x_{ij}q_j \\
\end{array}\right) \frac{1}{\sum_{j=1}^{J} x_{ij}q_j} = \left[\sigma^{-1}\right]^{-1} \left(\begin{array}{c}
E_1-R_f \\
\vdots \\
E_J-R_f \\
\end{array}\right) / (1, \ldots, 1) \left[\sigma^{-1}\right]^{-1} \left(\begin{array}{c}
E_1-R_f \\
\vdots \\
E_J-R_f \\
\end{array}\right).
$$

(37)

Investors can form this optimal portfolio from only two mutual funds: (1) the risk-free asset and (2) the aggregate portfolio of given risky assets defined by (38).
Pareto-Optimality

We next examine the implications of separation, (37) or (38), for the Pareto-optimality of the resulting equilibrium allocation of wealth. Pareto-optimality in asset allocations requires equal marginal rates of income substitution for all investors. This condition is established in complete markets even when we do not impose a specific form on utility functions.

In the CAPM equilibrium, the marginal rates of income substitution can be calculated from the first equation in (31), \( u_i^0 = u_0^i \), and from (32) using the separation property, (37). Combining these equations, we obtain:

\[
\begin{bmatrix}
    u_1^i / u_0^i \\
    \vdots \\
    u_S^i / u_0^i
\end{bmatrix}
= \frac{1}{1+R_f} \left\{ \begin{bmatrix} f_1 \\ \vdots \\ f_S \end{bmatrix} - \begin{bmatrix} f_1 & \cdots & 0 \\ \vdots \\ 0 & \cdots & f_S \end{bmatrix} \begin{bmatrix} E_1 - r_1 \\ \cdots \\ E_S - r_S \end{bmatrix} \right\}^{-1} \begin{bmatrix} E_1 - R_f \\ \vdots \\ E_S - R_f \end{bmatrix}
\]

It is clear from this equation that the right-hand side is independent of the investor index, \( i \), which means that marginal rates of income substitution are equal for all investors. The market allocation of wealth across assets meets the condition for Pareto-optimality. Note that this optimality is established without specifically assuming that asset markets are complete. We assume homogeneity in investor beliefs about the stochastic distribution of asset returns, but we do not assume the rank condition needed for
the completeness. Even when the number of assets is less than the number of all possible states of nature, we can count on the specific form of utility functions, (A-1), of the CAPM to assure a Pareto-optimal allocation of wealth. The separation property tells us that investors used only two mutual funds. CAPM investors combine these two mutual funds in a way that establishes a Pareto-optimal allocation.

**Closed-Form Valuation**

Besides assuring Pareto-optimality, the separation property of the CAPM provides a closed-form expression for the risk premium on each risky asset. It is expressed in terms of the market portfolio of risky assets.

The market portfolio is the portfolio of risky assets in which the shares of assets are the same as their market share, \( m_j \) (\( J=1,2,\ldots,J \)) in (38). Note that this concept of the market portfolio applies both in the state-preference model and in the CAPM. However, the CAPM has the special property that the market share, \( m_j \), is also the share that asset \( j \) takes in each individual investor's optimal portfolio of risky assets.

The rate of return to the market portfolio, \( \tilde{r}_m \), is random with \( \tilde{r}_m = r_{ms} \) when state \( s \) occurs:

\[
\begin{bmatrix}
  r_{m1} \\
  \vdots \\
  r_{mS}
\end{bmatrix}
= 
\begin{bmatrix}
  r_{11} & \cdots & r_{1J} \\
  \vdots & \ddots & \vdots \\
  r_{1S} & \cdots & r_{JS}
\end{bmatrix}
\begin{bmatrix}
  m_1 \\
  \vdots \\
  m_J
\end{bmatrix}
\]  

(40)
Correspondingly, the expected value and the variance of the market-portfolio rate of return may be written as:

\[
E_m = (m_1, \ldots, m_J) \begin{bmatrix} E_1 \\ \vdots \\ E_J \end{bmatrix}, \quad \text{and} \\
\sigma_m^2 = (m_1, \ldots, m_J) \sigma \begin{bmatrix} m_1 \\ \vdots \\ m_J \end{bmatrix}
\]

respectively.

Let us now use the market portfolio to derive the closed-form solution for the risk premium on each risky asset in the CAPM. First, from condition (35) for an optimal consumption-portfolio plan and the separation property (38), we have:

\[
\begin{bmatrix} E_1 - R_f \\ \vdots \\ E_J - R_f \end{bmatrix} = - (1 + R_f) \frac{2U^i}{U_0} \left( \sum_{j=1}^{J} x_{ij} q_j \right) \sigma_j \begin{bmatrix} m_1 \\ \vdots \\ m_J \end{bmatrix}
\]

This shows that the risk premium on each asset may be expressed in terms of the covariance of its rate of return to the market-portfolio rate of return, \( \sigma_{jm} \):

\[
E_j - R_f = - (1 + R_f) \frac{2U^i}{U_0} \sum_{j=1}^{J} x_{ij} q_j \sigma_{jm} \quad (j = 1, 2, \ldots, J).
\]

where \( \sigma_{jm} = \sum_{k=1}^{J} \sigma_{jk} m_k \).
From both of these equations, we get:

$$
\begin{pmatrix}
E_1 - R_f \\
\vdots \\
E_J - R_f
\end{pmatrix}
= (E_m - R_f)
\begin{pmatrix}
\sigma_{1m}/\sigma_m^2 \\
\vdots \\
\sigma_{jm}/\sigma_m^2
\end{pmatrix}.
$$

That is, the risk premium on each asset is $\sigma_{jm}/\sigma_m^2$ times the risk premium on the market portfolio. The coefficient $\sigma_{jm}/\sigma_m^2$ measures the risk of asset $j$ relative to the risk of the market portfolio and called the "beta" of asset $j$, $\beta_j$. Equation (42) is the closed-form solution for the risk premium on each asset in the sense that every element in (42) is potentially measurable. This measurement does not require any specific information of the preference orderings of investors (Rubinstein [1974]). Information on preference orders is embodied in the prices of assets. This imparts empirical testability to the CAPM (Jensen [1972]).

III.3. Risk Premiums in the State-Preference Model

Separation, Pareto-optimality and the closed-form solution in the CAPM follow from assumptions (A-1) through (A-4). Three of these four assumptions, (A-2) through (A-4), also hold in the state-preference model with complete markets. Clearly, properties particular to the CAPM stem from assumption (A-1), which imposes a specific form on the utility functions.

In this section, we investigate the possibility of obtaining formulas similar to the CAPM solution (42) for
the risk premium for individual assets in the state-preference model developed in section 1 of this chapter. We focus particularly on conditions under which the CAPM solution holds in complete markets even though we do not assume a special form of utility function. This lets us determine a condition which, although it does not imply the separation property, still defines the risk-premium vector in terms of parameters of the market portfolio.

Empirical tests of the CAPM focus on the validity of the closed-form solution (42) (Jensen [1972]). However, the separation property, which underlies the solution (42), requires that all investors have the same proportions of risky assets in their portfolios. This phenomenon can not be observed in real-world markets. Therefore, it is worth investigating the validity of the closed-form solution (42) under utility functions which allow investors to hold different equilibrium combinations of risky assets.

**General Solution for Risk Premium**

First, we derive a general equation for the risk premium for each asset in the state-preference model with complete markets. Completeness requires that investors have identical expectations about future levels of asset returns. However, completeness does not require identical assessments of the probabilities of the state of nature. Market equilibrium and Pareto-optimality in the state-
preference model are conditioned on a given set of probability assessments. This means that the expected values and the variance-covariance matrix for rates of asset returns may differ across investors even in complete markets. To focus on the role of the CAPM assumption (A-1) in deriving the closed-form solution for risk premiums, we, nevertheless, assume that all investors share the same beliefs about the stochastic distribution of uncertain returns on assets. This means homogeneous expectations, \( \{ e^i \} = \{ e \} \), and homogeneous probability assessments, \( f^i = f \).

Under these conditions, the optimal consumption-portfolio plan implies:

\[
\begin{bmatrix}
    p_1 \\
    \vdots \\
    p_S
\end{bmatrix} = \begin{bmatrix}
    q_1 \\
    \vdots \\
    q_J
\end{bmatrix},
\]

where \( p_s = u_s^i / u_0^i \): the implicit Arrow-security price. Note that when condition (43) is applied to the risk-free asset, we obtain:

\[
S \sum_{s=1}^{S} p_s = (1 + R_f)^{-1}.
\]

Combining equations (43), (44) and the definition of asset rate of return, we can write the risk-premium vector for assets as:
where \([r]\) = the \(J+1\)-by-\(S\) matrix of asset rates of return.

Note that:

\[
\frac{p_s}{\sum_{t=1}^{S} p_t} > 0, \quad \text{and} \quad \sum_{s=1}^{S} \left(\frac{p_s}{\sum_{t=1}^{S} p_t}\right) = 1.
\]

Since these are the axioms of modern probability theory, each \(\frac{p_s}{\sum_{t=1}^{S} p_t}\) can be interpreted as if it were a probability measure of state \(s\). Since \(p_s = \frac{u_i^s}{u_i^0}\) for each investor, the probability measure may be regarded as the relative marginal value of income in each state. Equation (45) shows that the risk premium on each asset is the weighted sum of the levels of the uncertain rate of return. The weights combine differences in the probabilities of states and the relative marginal values of income in each state.

The risk premium in equation (45) can be usefully restated. Let us define a new random variable \(a\) whose value, \(a_s\), depends on the state of nature as:

\[
a_s = \frac{p_s}{f_s} \quad (s=1,2,\ldots,S).
\]

This new variable may be called the probability-adjusted Arrow security price. Since the Arrow-security price is the
present value of one unit of income in every state, \( a_s \) may also be called the probability-adjusted present value of state-contingent income.

The expected value of \( a \) is:

\[
E_a = \sum_{s=1}^{S} f_s a_s = \sum_{s=1}^{S} P_s
\]

and its covariance with rates of asset return are:

\[
\sigma_{ja} = \sum_{s=1}^{S} f_s (r_{js} - E_j)(a_s - E_a) \quad (j=1,2,\ldots,J).
\]

Note that \( \sigma_{0a} = 0 \). Using this new random variable and the relation between the risk-free rate of return and Arrow-security prices (44), equation (45) is rewritten as:

\[
\begin{bmatrix}
E_0 - R_f \\
\vdots \\
E_j - R_f \\
\vdots \\
E_N - R_f
\end{bmatrix}
= \frac{1}{S} \begin{bmatrix}
\sum_{s=1}^{S} f_s (p_s / f_s - \sum_{s=1}^{S} p_s)
\vdots \\
\vdots \\
\sum_{s=1}^{S} f_s (p_s / f_s - \sum_{s=1}^{S} p_s)
\end{bmatrix}
= -(1+R_f) \begin{bmatrix}
\sigma_{0a} \\
\vdots \\
\sigma_{Ja}
\end{bmatrix}.
\]

Equation (47) states that an asset's risk premium is proportional to the covariance of its rate or return with the probability-adjusted implicit Arrow-security price.
The premium on the market portfolio can be found by premultiplying the vector of market shares, \((m_1, \ldots, m_J)\), to equation (47):

\[
(48) \quad E_{m-R_f} = -(1+R_f) \begin{bmatrix} m_1 \\ \vdots \\ m_J \end{bmatrix} \begin{bmatrix} \sigma_{1a} \\ \vdots \\ \sigma_{Ja} \end{bmatrix} \\
= -(1+R_f) \sigma_{ma}.
\]

Combining (47) and (48) lets us see that:

\[
(49) \quad \begin{bmatrix} E_1 - R_f \\ \vdots \\ E_J - R_f \end{bmatrix} = (E_{m-R_f}) \begin{bmatrix} \sigma_{1a}/\sigma_{ma} \\ \vdots \\ \sigma_{Ja}/\sigma_{ma} \end{bmatrix}.
\]

This solution for the risk premiums on risky assets in the state-preference model corresponds to the CAPM solution (42).

To derive (49) does not require separation for investors' optimal portfolio plans. What we require is the rank condition and homogeneous beliefs for investors, which assures Pareto-optimality. The difference between (49) and (42) is that the probability-adjusted implicit Arrow-security price serves in place of the rate of return on the market portfolio. In complete markets, implicit Arrow-security prices are calculated in terms of returns of assets and individual asset prices. Also using (43), the \(a_s\) are
calculated from asset prices and returns on assets. Therefore, the general solution (49) uses elements that are potentially measurable.

III.4. Alternative Conditions for the CAPM

Closed-Form Solution

To round out the analysis in this chapter, we need to establish conditions under which the solution (49) converges to the CAPM solution (42) without assuming the specific form of utility functions. It should be clear that if the new random variable, $a$, can be constrained to have a particular relation to the rate of return on the market portfolio, we should be able to reduce (49) to (42).

From (49) the requisite condition may be seen to require:

$$
\begin{pmatrix}
\sigma_{1a}/\sigma_{ma} \\
\vdots \\
\sigma_{ja}/\sigma_{ma}
\end{pmatrix} =
\begin{pmatrix}
\sigma_{1m}/\sigma_m^2 \\
\vdots \\
\sigma_{jm}/\sigma_m^2
\end{pmatrix}.
$$

From the definitions of variance and covariances, this condition is rewritten as:

$$
\frac{1}{\sigma_{ma}^2} [r] \begin{pmatrix}
f_1, \ldots, 0 \\
\vdots \\
0, \ldots, f_S
\end{pmatrix} \begin{pmatrix}
a_1 - E_a \\
\vdots \\
a_S - E_a
\end{pmatrix} = \frac{1}{\sigma_m^2} [r] \begin{pmatrix}
f_1, \ldots, 0 \\
\vdots \\
0, \ldots, f_S
\end{pmatrix} \begin{pmatrix}
r_{m1} - E_m \\
\vdots \\
r_{mS} - E_m
\end{pmatrix}.
$$
Completeness of the market assures invertibility of \([r]\).
The diagonal matrix of the probabilities are also invertible.

Therefore, we can further transform the condition into:

\[
\frac{1}{\sigma_{ma}} \begin{pmatrix}
    a_1 - E_a \\
    \vdots \\
    a_S - E_a
\end{pmatrix} = \frac{1}{\sigma_m^2} \begin{pmatrix}
    r_{ml} - E_m \\
    \vdots \\
    r_{mS} - E_m
\end{pmatrix}.
\]

Substituting (48) into this equation, we obtain:

\[
\begin{pmatrix}
    a_1 - E_a \\
    \vdots \\
    a_S - E_a
\end{pmatrix} = - \frac{E_m - R_f}{\sigma_m^2} \cdot \frac{1}{1 + R_f} \begin{pmatrix}
    r_{ml} - E_m \\
    \vdots \\
    r_{mS} - E_m
\end{pmatrix}.
\]

Condition (50) for the CAPM solution states that the probability distribution of probability-adjusted Arrow-security prices must be such that the deviation of its value in state \(s\) is proportional to the deviation of the market-portfolio rate of return from its mean. The coefficient is the market price of risk-bearing discounted by the risk-free rate of return. When this condition (50) is met, the CAPM closed-form solution holds even though separation does not.

Recalling the definition of \(a_s\), equation (46), and its expected value, condition (50) may be restated in terms of Arrow-security prices as:
Equation (51) states that estimating the stochastic distribution of the market-portfolio rate of return provides information on the implicit Arrow-security prices. When the CAPM closed-form solution holds, the deviation of the relative marginal value of one unit of income in state $s$ from the probability of the state is proportional to the probability-weighted deviation of the market-portfolio rate of return from its mean. The coefficient of proportionality is the market price of risk-bearing. When the rate of return on the market portfolio is high in a particular state, the marginal valuation of income in that state would be small. Condition (51) captures this idea.

It is unlikely that condition (51) would always hold. We need to make sure that asset markets are complete, which is empirically a difficult task to accomplish. Even when markets can be shown to be complete, condition (51) may not hold. In general, we have the solution (49) instead of the CAPM solution. Condition (50) and (51) help us to see that if the CAPM solution is empirically validated even though separation is rejected, estimates of the parameters of stochastic distribution of the market-portfolio rate of return may let us estimate implicit Arrow-security prices. This interpretation frees us from
the need to use two-parameter form of utility function. Nor do we have to assume a specific class of stochastic distribution for asset returns. The assumptions of the CAPM always assure (51) because of the properties of separation and Pareto-optimality.

III.5. Conclusion

We have seen how the specific assumptions underlying the CAPM establish a Pareto-optimal allocation of wealth and a closed-form solution for risk premiums. The separation property of the CAPM stems from requiring a narrow class of two-parameter utility functions. Its neglect of the influence of the higher-order moments of portfolio rates of return may introduce errors into the analysis of the investors' optimal investment decisions.

Using a state-preference model, we have identified another theoretical basis for the pricing equation of the CAPM. The fundamental explanation for the CAPM's closed-form solution lies in two elements: (1) Pareto-optimality of the market-equilibrium allocation of wealth and (2) separation. Pareto-optimality can obtain in an unrestricted state-preference model as long as rank and homogeneous expectations assumptions assure the completeness of markets. Even without separation, it is possible for the CAPM closed-form solution to hold. What matters is whether the random rate of return on the market portfolio
depends linearly on the implicit Arrow-security prices. Our analysis provides an alternative justification for the CAPM solution.
FOOTNOTES TO CHAPTER III

They claim, for instance, that even if two portfolios have the same expected value and variance, if one has "higher skewness (third-order moment)" than the other, investors should prefer the high-skewness portfolio. Tobin (1958), Samuelson (1970), Fama (1971) and others investigate conditions under which the two-parameter approach is justified when investors are maximizing the expected value of their concurrent utility levels. Tobin shows that if either (1) the concurrent utility function is quadratic, or (2) the distribution of asset returns is normal, the two-parameter approach is justified. Fama generalizes the second case of Tobin into the class of symmetric stable distributions. Samuelson introduces the concept of "compact" distribution to justify the approach.

Cass and Stiglitz (1970) investigate conditions under which investors' optimal portfolios can be constructed by a number of mutual funds.

An excellent survey of the theoretical and empirical developments of the CAPM is found in Jensen (1972). Jensen states:

The main result . . . is that one can derive the individual's demand function for assets, aggregate these demand to obtain the equilibrium prices (or expected returns) of all assets, and then eliminate all the utility information to obtain market equilibrium prices solely as a function of potentially measurable market parameters. Thus the model becomes testable. (p. 363)

The state-preference model was first introduced by Debreu (1959). Debreu has examined markets which trade state-contingent claims on commodities and show that the existence and the Pareto-optimality of market equilibrium can be analyzed in the same manner as for the economy without uncertainty. Arrow (1964) constructed the financial market model of state-contingent claims on income, which is now often called Arrow (or Arrow-Debreu) security market model. The word "state-preference", in general, is used to mean that the model assumes that individual's preference

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ordering is defined over the state-contingent plans of their consumptions. The CAPM is a special case of the state-preference model. However, in our analysis, we use "state-preference" to indicate that we do not impose the CAPM-specific assumptions on the utility functions.

The distinction between complete and incomplete financial markets traces to Arrow (1964). Cass and Stiglitz (1970) examined these conditions for completeness of markets.

The possibility of forming the optimal portfolio with only two mutual funds already suggests this result.

If \( J+1 > S \) we choose the appropriate \( S \) assets to determine the marginal rates of income substitution. The condition (24) for the rest of assets give us arbitrage pricing relations in terms of the \( S \) assets.

Frisen (1979), Baron (1979) and others examine the optimality of firms' investment decisions in complete markets. Diamond (1964), Stiglitz (1972), Ekern and Wilson (1974), Radner (1974) and others investigate the problem of investment decisions in the incomplete markets.

Note that in complete markets:

\[
[e^i] = [e];
\]

that is, homogeneous expectations hold.

We assume that the inverse exists. Singularity of the variance-covariance matrix implies that some assets have returns which are linearly dependent on the others. In that case, we can eliminate these linearly dependent assets without a loss of generality of the analysis.
Chapter IV. A THEORY OF CONVERTIBLE BONDS

In the previous two chapters, we dealt with situations where firms' investment decisions and returns on assets are exogeneously given. There, uncertainty in the economy traces to the uncertainty in nature or randomness. Under this condition, we examined the role of financial instruments in distributing given returns on investments. Pareto-optimality requires that marginal rates of substitution between current and future income in every state of nature be equalized across investors. Complete markets assure that this condition holds. Also, the CAPM specific assumptions assure this condition to hold.

In this chapter, we examine the impact of investment decisions on debtholder and shareholder wealth in a dynamic model. The main argument is that, even if markets are complete for distributing given returns, endogenizing investment decisions introduces uncertainty about firms' future decisions, which would have significant impacts on the wealth of existing security holders. Markets would require new financial instruments which could resolve the wealth-transfer problem particular to uncertain investment decisions. We present an analysis of convertible bonds in this context.
In their path-breaking analysis, Modigliani and Miller (1958) showed that corporate financial decisions are irrelevant to the value of the firm, and therefore that investment-production decisions can be separated from financing decisions. Since then, the theory of the optimal corporate capital structure has been developed to incorporate preferential tax treatment of different types of claims, default risk, bankruptcy costs, and other considerations.\(^1\) The analysis has also been generalized to cover multi-period, multi-claim, heterogeneous-expectations cases, and imperfect markets, and the original irrelevance theorem has been confirmed in this broader context.\(^2\) However, some problems still remain to be analyzed. Preferential tax treatment of debt claims suggests that the optimal capital structure is to use as much debt as possible to take advantage of the implicit government subsidy on interest charges, a result we do not observe empirically. Bankruptcy costs may not be sufficiently large to explain meaningfully the optimal capital structure.\(^3\) Above all, the impact of bonds issued previously on the investment-production decision, which was pointed out by Fama and Miller (1972) and Stiglitz (1972), started a new line of analysis along with the recent development of agency-cost theories.

Fama and Miller (1972) and Stiglitz (1972) show that if debts were issued prior to investment-production decisions, shareholders can redistribute wealth to themselves.
from bondholders by switching the investment plan to a riskier project and exposing the firm to possible bankruptcy. This raises a serious obstacle to society in achieving firm-value maximizing investment-production decisions, which is an essential ingredient of Pareto-efficient resource allocation under uncertainty. Such problems were not recognized in the early literature where investment-production decisions of firms were exogenous. Neither do they arise if all investment-production and distribution decisions are simultaneously made, as in tâtonnement processes, and the final decisions at equilibrium are enforced without an alteration. Reversibility of past decisions on the firms' side and irreversibility of past decisions on the bondholders' side, are key elements in the problem. Due to this, even if the expectations which bond investors had at the time of their bond purchase turn out right, socially suboptimal investment-production decisions may result. The possibility of wealth redistribution exists even in perfect markets with rational expectations.\textsuperscript{4,5}

Fama and Miller also suggested that if "me-first" rules of the distribution of the value of a firm are adopted, the redistribution problem is resolved. Perfectly enforced "me-first" rules are a mechanism to compensate fully losses from wealth redistribution. Introducing a priority structure into debt claims or corporate capital-structure rearrangements are examples of "me-first" rules.\textsuperscript{6}
This chapter investigates the role of convertible bonds in resolving the wealth-redistribution problem. The problem is an example of "agency cost" as suggested by Jensen and Meckling (1976) and Smith and Warner (1979). Agency costs arise in any contract stipulating a delegation of control when principals and agents have different objectives. In our context, after bondholders have contributed to the capital of a firm under the assumption that the firm will choose a specific project, shareholders can switch the investment decision to maximize their own wealth. This works against bondholders. Rational bondholders would anticipate this switch. Then, shareholders are forced to take a socially suboptimal investment project. Maximizing the wealth of shareholders conflicts with the maximization of the firm's value. The difference between the value of the firm when it undertakes the socially best project and the value of the firm when it undertakes the suboptimal project is the agency cost. The agency cost may be reduced by introducing monitoring or bonding activities such as restrictive covenants.7

The following passage in Jensen and Meckling (1976) motivated the analysis:

... convertible securities can be thought of as securities with non-detachable warrants. It seems that the incentive effects of warrants would tend to offset, to some extent, the incentive effects of the existence of risky debts because the owner-manager would be sharing part of the proceeds associated with a shift in the distribution of returns with the warrant holders (p. 354).
Also, Smith and Warner (1979) state:

- The conversion privilege is like a call option written by the stockholders and attached to the debt contract. It reduces the variability of the firm's cash flows, because with a higher variance rate, the attached call option becomes more valuable. Therefore, the stockholders' gain from increasing the variance rate is smaller with the convertible debt outstanding than with non-convertible debt (p. 141).

These insightful suggestions have not yet materialized in a specific analysis of convertible bonds.

We show that the conversion ratio and the rationality of shareholders and bondholders are important factors in resolving the wealth-redistribution problem. With an arbitrarily high conversion ratio and naive expectations, a possibility of a reverse-direction redistribution of wealth from shareholders to bondholders could result. Even if all investors are rational, a high conversion ratio induces bondholders to exercise the convertibility option, which may disturb the debt-equity ratio of the firm from its desired level.

If resolving the wealth-distribution problem is the only purpose of convertible bond-issues, it is sufficient to make the convertibility option exercisable just after the investment-production decision is executed but before bondholders would know the state of the economy. Though the exercisability is usually not limited to that period in real-world markets, it is worth examining such a restricted-period convertible bond. The restriction lets us determine
the range of the conversion ratio within which bonds will never be converted by rational bondholders, but at the same time the convertibility option is effective in preventing the firm from choosing a suboptimal investment project. This non-exercise condition implies a possibility of a convertibility option that does not have a market value.\(^8\)

Our comparative-static analysis of restricted-period convertible bonds predicts that the optimal conversion ratio responds positively to both the debt/equity ratio and the value of alternative projects, and negatively to the difference between the values of the optimal and alternative projects. The effect of the investment outlay on the optimal conversion ratio is ambiguous. We also conduct a comparative-static analysis of the size of the optimal range for conversion ratio.

Our result that restricted-period convertible bonds do not have a value higher than those of regular bonds should not be taken to contradict what we observe in real-world markets. Usually, convertible bonds are exercisable even after bondholders learn which state of the economy has occurred. This extension of the exercise period tends to increase the value of convertible bonds. We can still determine a lower limit on the conversion ratio of extended-period convertible bonds that would prevent the wealth redistribution from an investment-project switch. However, the upper limit, which serves to prevent actual conversion,
may no longer be defined. That is, shareholders may not be able to expect to eliminate the agency cost of the wealth redistribution without altering the debt-equity ratio. In some future state of the economy, conversion may occur. This fact increases the value of the convertible bonds. The analysis of extended period convertible bonds is similar to the call-option pricing model.

We confine our analysis to a simple two-period, state-preference framework, with two possible states for the economy. The market is assumed to be complete in the sense that the variety of the securities traded in the market is enough to span the space of incomes in all possible states of the economy. In complete markets we can determine the prices of the Arrow-Debreu securities, claims on one unit of purchasing power contingent on the state of the economy. 9 The assumption of complete markets lets us draw conclusions in a context where convertible bonds have no role in completing the market, which is a different issue. 10 A two-period, two-state model is enough to illustrate the essential ideas.

It should be mentioned that the analysis does not address the question of optimal capital structure, which would require explicit monitoring and bonding costs in the model. We assume that the firm wants (or needs) to issue a given amount of debt to finance a new investment project. Given this objective, we investigate the role of the
convertibility option in achieving optimal investment-production decisions. We show that incentives exist for shareholders to issue convertible bonds and to set an appropriate conversion ratio.

IV. 1. The Model and Optimal Investment

The firm chooses between two mutually exclusive investment opportunities, project A and project B. The firm invests in the first period (t=1). Both projects cost the same expenditure in the first period, denoted by c. The value of the firm in the second period (t=2) is the sum of the second period cash inflow and the second-period market value of all future cash inflows from the project. Therefore, it depends on the choice of project and the state of the economy at t=2. The second-period value of the firm with project A is \(a_1\) in state 1 and \(a_2\) in state 2 and with project B is \(b_1\) in state 1 and \(b_2\) in state 2. Table 1 illustrates the combination:

<table>
<thead>
<tr>
<th></th>
<th>cost</th>
<th>second-period value of the firm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>state 1</td>
</tr>
<tr>
<td>project A</td>
<td>c</td>
<td>(a_1)</td>
</tr>
<tr>
<td>project B</td>
<td>c</td>
<td>(b_1)</td>
</tr>
</tbody>
</table>
When the investment decision is made, which one of these states will take place is not known. Shareholders and potential bondholders act on their expectations. We assume that expectations about the possible levels of the second-period value of the firm are homogeneous among shareholders and bondholders.\textsuperscript{11}

Four decisions are made sequentially in the first period: (1) the original shareholders decide what kind of bonds, regular or convertible, are to be issued, and if convertible, the level of the conversion ratio and the conversion period; (2) bond investors determine the value of bonds in the market; (3) the firm decides on the project; and (4) if the bonds are convertible, bondholders decide whether or not to exercise the convertibility option. If the conversion period includes the 2nd period, bondholders can exercise the convertibility option after they observe the realized state of the economy. The four decisions in the first period are to be made sequentially in the specified order. Keeping this sequence in mind, we can perform our analysis without introducing three or more periods into our model. Every time shareholders or bondholders make decisions, they anticipate the reactions of the other party in the following sense. When the firm decides the bond issue, stage (1), the firm may suggest which of the two projects will be undertaken, but it is not binding as to which project is actually adopted at stage (3). At the time
of their bond purchase, bondholders have to guess whether the firm will actually take the suggested project.

Because the market is complete, we can determine the prices of the Arrow-Debreu securities. An Arrow-Debreu security is the claim on one unit of purchasing power contingent on a particular state of the economy. Let $p_1$ be the price of a claim on one dollar receivable in state 1 and $p_2$ be the price of a claim on one dollar in state 2. $p_1$ is also equal to the marginal rate of substitution of any investor between a dollar income in state $i$ in period 2 and a dollar of income in the first period. Since the present value of $1$ receivable in both states in period 2 is $(p_1 + p_2) \times 1$, the risk-free interest rate, $R$, is defined as:

$$R = (p_1 + p_2)^{-1} - 1.$$  

These prices provide sufficient information to determine which project is socially more desirable. Since project A uses up current resources, that are worth $c$ to generate the second-period firm values, $a_i$ ($i = 1, 2$), the social value of the project, denoted by $V_A$, is:

$$V_A = p_1a_1 + p_2a_2 - c.$$  

This is also the first-period value of the firm with project A assuming no market distortions such as external effects or
market power. We assume that project A is socially desirable, \( V_A > 0 \). Similarly the value of the firm with project B is:

\[
V_B = p_1 b_1 + p_2 b_2 - c.
\]

Let us suppose that the value of project A is greater than the value of project B:

\[\text{(52)} \quad V_A > V_B.\]

This implies that if the firm bases the investment decision on the value of the firm, project A would be chosen and would be socially optimal.

If projects are financed entirely by equity, the value of the firm is distributed entirely to shareholders. Therefore, management would obviously choose project A. In this case, maximization of the wealth of shareholders and maximization of the value of the firm are consistent. However, if the investment cost is partly financed by a bond issue and one of the projects has a bankruptcy risk, this consistency may break down. Section 2 discusses this point.

IV.2. Agency Cost of Regular Bonds

Let us specify the details of the two projects. The risk of a project can be characterized by the deviation of the levels of its second-period value in the two possible
states. We assume that project B is riskier than project A in the sense that the period-2 value in state 2 for project B is the lowest:

\[ a_1, a_2, b_1 > b_2. \]

We introduce an additional condition on the levels of the second-period value of the firm later in our analysis.

The possibility of bankruptcy depends on the size of contractual bond repayments in period 2, relative to the value of the firm. The firm issues bonds to finance a part of the project cost, \( c \), and the size of the bond repayment, denoted by \( d \), is such that it brings the firm into bankruptcy only when project B is chosen and state 2 occurs:

\[ a_1, a_2, b_1 > d > b_2. \]

The size of debt, \( d \), is assumed to be fixed. No additional debt is issued. In this section, we examine the case where the bonds are regular single-payment bonds without a convertibility provision.

The possibility that shareholders could steal a part of the wealth of bondholders was pointed out by Fama and Miller (1972) and Stiglitz (1972). It arises in the following way. The value of the bonds in period-1 competitive market is the present value of the bond repayment. Shareholders can announce that the bond issue is to finance project A. Let us suppose bond investors believe that
project A would indeed be taken. Then, since bond investors would anticipate no bankruptcy risk, the value of the bond \( V_d \), would be

\[
(55) \quad V_d(A) = (p_1 + p_2)d.
\]

The argument in the brackets, \((A)\), denotes that bondholders assumed that project A is chosen. This valuation of bonds implies that bondholders contribute \( V_d(A) \) to finance the project. The shareholders' wealth in period 2 after repaying debt is \((a_1 - d)\) in state 1, and \((a_2 - d)\) in state 2. Pay-off Table 2 illustrates this:

**TABLE 2**

<table>
<thead>
<tr>
<th>state 1</th>
<th>state 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>bondholders</td>
<td>shareholders</td>
</tr>
<tr>
<td>project A</td>
<td>(d)</td>
</tr>
<tr>
<td>project B</td>
<td>(d)</td>
</tr>
</tbody>
</table>

Then, the shareholders current wealth after financing the rest of the project cost, denoted by \( V_s \), is:

\[
(56) \quad V_s(A,A) = p_1(a_1 - d) + p_2(a_2 - d) - (c - V_d(A))
\]

\[
= p_1a_1 + p_2a_2 - c
\]

\[
= V^*_A
\]
if project A is indeed chosen. The first argument in the bracket attached to $V_S$ in equation (56) means that bondholders assumed project A, and the second argument means that the firm indeed chooses project A. However, once bonds have been issued to raise $V_d(A)$, the best choice for shareholders may not be project A. If the firm switches from project A to project B after raising $V_d(A)$, the firm can confront the bondholders with a bankruptcy risk. Since the second-period value of the firm with project B in state 2, $b_2$, is smaller than the promised debt repayment, $d$, the firm would go bankrupt if this state occurs. Assuming no bankruptcy costs, bondholders receive $b_2$ and shareholders receive nothing in bankruptcy. (See Pay-off Table 1.)

Therefore, the current wealth of shareholders with project B is:

\[(57) \quad V_s(A,B) = p_1 (b_1 - d) - \{c - V_d(A)\} \]

\[= V_B + p_2 (d - b_2) > V_B.\]

This shows that the wealth of shareholders is larger than the value of the project (firm) by $p_2 (d - b_2)$. Bondholders will receive $d$ in state 1, but only $b_2$ ($< d$) in the bankruptcy. The difference between the promised repayment, $d$, and the actual repayment, $b_2$, evaluated in period 1, represents the wealth redistribution from bondholders to shareholders. A comparison between (56) and (57) shows that
if the wealth distribution more than offsets the difference in the values of the two projects $V_A$ and $V_B$:

$$V_A - V_B < p_2 (d-b_2),$$

maximization of the shareholders' wealth leads the firm to choose a project B, which is socially suboptimal. We assume that (58) holds. Condition (58) implies:

$$p_1 (a_1 - b_1) < -p_2 (a_2 - d) < 0;$$

that is, the value of project B in state 1 is higher than project A, $b_1 > a_1$. Project B is riskier than project A in this sense, as well as in the sense of our previous condition (53). However, we do not exclude the possibility that $a_2 > b_1 > a_1 > b_2$, in which case we cannot determine which project is riskier than the other in terms of the variance. Notice that project B may be socially undesirable, $V_B < 0$.

The following numerical example may help. Suppose $p_1 = .6$, and $p_2 = .3$, which means that the risk-free interest rate is $(p_1 + p_2)^{-1} -1 = .11$ (11 percent). Suppose the second period values of the two projects are:

<table>
<thead>
<tr>
<th></th>
<th>state 1</th>
<th>state 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>project A</td>
<td>$a_1 = 110$</td>
<td>$a_2 = 120$</td>
</tr>
<tr>
<td>project B</td>
<td>$b_1 = 134$</td>
<td>$b_2 = 33$</td>
</tr>
</tbody>
</table>
and the project cost is 90. Then the first-period value of the firm is:

\[ V_A = 0.6 \times 110 + 0.3 \times 120 - 90 = 12 \]

\[ V_B = 0.6 \times 134 + 0.3 \times 30 - 90 = 0.3 \]

which means A is socially more desirable than B. Whether condition (58) holds or not depends on the size of debt. Since \( V_A - V_B = 11.7 \), and \( p_2 = 0.3 \), if the debt size is such that:

\[ d - 33 > 11.7/0.3 = 39, \]

condition (58) holds. The wealth redistribution from the project switch from A to B will occur whenever the debt is more than 72.

If bondholders are rational in the sense that they are aware of the incentive to switch policy after the bonds are issued, they can avoid the detrimental wealth transfer. With rational expectations, bond investors anticipate the switch and the resulting bankruptcy in state 2. Rational bondholders now determine the value of bonds as:

\[ (59) \quad V_d(B) = p_1 d + p_2 b_2 \]

\[ = V_d(A) - p_2 (d - b_2) < V_d(A). \]
Under these conditions which project would be better for shareholders? If the firm takes project A, the shareholders' wealth is:

\[ V_s(B,A_r) = p_1(a_1 - d) + p_2(a_2 - d) - (c - V_d(B)) \]

\[ = V_A - p_2(d-b_2), \]

which tells that shareholders now lose \( p_2 (d - b_2) \). On the other hand, if project B is chosen, the shareholders' wealth is:

\[ V_s(B,B) = p_1(b_1 - d) - (c - V_d(B)) \]

\[ = V_B. \]

Condition (58) implies that:

\[ V_s(B,B) > V_s(B,A_r). \]

Therefore, shareholders are forced to take a socially sub-optimal project when bondholders are rational in their expectations. The loss to society, the agency cost, is measured by the difference between the values of the project, \( V_A - V_B \).

Notice that this result depends on an implicit assumption that shareholders and bondholders are separate individuals. If all investors have equal proportions of the shares and the bond of the firm in their portfolios, the redistribution of wealth would not matter.13 The
redistribution problem arises when the original shareholders do not have enough funds to finance the new projects but, at the same time, do not want to issue additional shares of stock.

This requires an explanation of the optimal debt/equity ratio which is not investigated in our analysis. We need an objective function, \( u \), for the original shareholders which depends on at least the value of shareholders' wealth, \( V_s \), and the debt/equity ratio:

\[
u = u(V_s, \text{debt/equity} , . . .).
\]

The objective function must have a positive partial derivative with respect to the shareholders' wealth, \( \partial u / \partial V_s > 0 \), and the partial derivative with respect to the debt/equity ratio with our given \( d \), must be zero, \( \partial u / \partial (\text{debt/equity}) = 0 \), to justify our assumption of optimal debt issue.

Further, to conduct our analysis of convertible bonds independently of the decision of debt size, no cross effect between shareholders' wealth and debt/equity ratio and no cross effect between the conversion ratio and debt/equity ratio is assumed. One possible explanation is that shareholders attach some value to control of the firm and they try to avoid dilution of their control.\(^{14}\) Or one might attempt to explain, in a more general framework, interactions between preferential tax treatment on debt, bankruptcy costs, or other considerations and agency
costs of debt. We simply take the conditions on u as given for the rest of our analysis.

IV.3. Convertible Bonds with a Restricted Convertibility Period

Jensen and Meckling (1976) and Smith and Warner (1979) suggest a possible remedy to our problem is issuing convertible bonds. A convertibility provision gives bondholders an option to switch debt claims to a certain portion of the shares in the stock. This will provide bondholders protection against detrimental wealth transfers in the case of bankruptcy. Before proceeding with our analysis of convertible bonds, it should be noted that restrictive covenants, bonding, or monitoring at some costs, could also prevent the firm from taking on the socially suboptimal project. Therefore, the final analysis should rest on the relative efficiency of all possible remedies in reducing agency-cost.

Convertible bonds are issued with two specifications: (1) conversion ratio and (2) the exercise period. Let z be the portion of equity that would accrue to bondholders as a class upon 100 percent conversion, $1 > z > 0$. We call z the conversion factor because, strictly speaking, z is not the conversion ratio. But, since we assume a fixed size of debt, a one-to-one correspondence exists between z and the conversion ratio.
If the sole purpose of convertible-bond issues is to resolve the wealth-distribution problem at issue, restricting the exercisable period to an interval just after the firm's project determination would be sufficient. We call this the restricted-period convertible-bond case. On the other hand, real-world convertible bond issues are similar to American options in that they are exercisable anytime up to maturity. We call this the extended-period convertible-bond case. The difference between the two cases in our context is that extended-period convertible bonds can be exercised even after the second-period state of the economy is known, as well as in the first-period. In this section and the next, we examine the restricted-period convertible-bond case. Section 5 examines extended-period convertible bonds.

Suppose that convertible bonds are only exercisable immediately after the firm takes one of the projects in the first period. We concentrate on the simplest case where all bonds are convertible. This avoids additional complexity that would arise with three groups of investors. Since the purpose of the convertible-bond issue is to let bondholders prevent any loss due to a post-financing investment-policy switch from project A to B, the conversion factor would be set at the level with which bondholders can retain their wealth under project B equal to the bond value with project A. This is the level of z which the analysis in
the previous section suggests. Since bondholders exercise
the convertibility option if the firm chooses project B,
they receive 100 z percent of the second-period value of
the firm, \( z(p_1 b_1 + p_2 b_2) \). Together with the value of bonds
under project A in equation (55) in section 2, this suggests
that \( z \) would be set at:

\[
(62) \quad (p_1 + p_2) d = z(p_1 b_1 + p_2 b_2), \text{ or equivalently}

V_d(A) = z(V_B + c).
\]

What is the value of the bonds with the conversion
factor defined by (62)? One possibility is that bondholders
tell the firm that since they are protected by the converti-
bility option in case of project B, they will pay up to the
value \( V_d(A) \), and shareholders naively accept this bid. Let
us examine this case now to show that the level of \( z \) defined
by (62) causes a problem for shareholders. Pay-off Table 3
illustrates shareholders' expected wealth in the second
period:

<table>
<thead>
<tr>
<th>TABLE 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>state 1</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Project A</td>
</tr>
<tr>
<td>Project B</td>
</tr>
</tbody>
</table>
The shareholders would expect their first-period wealth to be:

\[(63) \quad V_s(A) = p_1 (a_1 - d) + p_2 (a_2 - d) - \{c-V_d(A)\} = V_A\]

if the firm chooses project A, or:

\[(64) \quad V_s(B) = (1-z)(p_1b_1 + p_2b_2) - \{c-V_d(A)\} = V_B\]

if project B is chosen. The comparison between (63) and (64) leads shareholders to prefer project A. It not only yields a larger wealth to them, but also project B would invite new shareholders into the firm so that the debt/equity ratio would depart from its optimal level.

However, the above analysis is naive because it ignores the possibility that bonds will be converted even if project A is taken. Equation (63) and (64) were derived based on the assumption that shareholders do not anticipate conversion when project A is chosen. In fact, bondholders can steal some wealth from shareholders when the firm decides on project A by exercising the convertibility option. The amount of the wealth transfer, denoted by \(T\), in this case is:
\begin{align*}
T &= z(p_1a_1 + p_2a_2) - V_d(A) \\
&= \frac{V_d(A)}{(V_B+c)} (V_A + c) = V_d(A) \quad \text{by (62)} \\
&= V_d(A)\left(\frac{V_A+c}{V_B+c} - 1\right) > 0.
\end{align*}

Correspondingly, the shareholders' "actual" wealth, denoted by \( V_s^{\text{act}}(A) \), is less than \( T_A \) by the amount in (65):

\begin{align*}
V_s^{\text{act}}(A) &= V_A - V_d(A)\left\{\frac{V_A+c}{V_B+c} - 1\right\}.
\end{align*}

This is a counterpart of the wealth transfer from bondholders to shareholders when bondholders are naive (the result in the previous section). This implies that buying convertible bonds at the value defined by (62) and converting them into shares would be cheaper than a direct forward purchase. Does this mean that B was a better choice for the original shareholders, retrospectively? The comparison between (64) and (66) shows:

\begin{align*}
V_s^{\text{act}}(A) - V_s(B) &= V_A - V_d(A)\left\{\frac{V_A+c}{V_B+c} - 1\right\} - V_B \\
&= (V_A - V_B) (1-z) > 0;
\end{align*}

that is, project A is still better.

What could prevent the wealth transfer from shareholders to bondholders calculated in (65) above? The answer lies not only in the rationality of shareholder
expectations, but also in the competitiveness of the bond market. Competition among bond investors would incorporate the possible excess gain, \( T \), into the valuation of bonds, and the value of bonds is set at:

\[
\bar{V}_d = V_d(A) + T \\
= z(V_A + c).
\]

With this valuation, bondholders will convert their bonds into \( 100z \) percent of the shares. If they do not, they lose \( T \) to shareholders. Then, the corresponding wealth of the original shareholders is:

\[
\bar{V}_s(A) = (1-z) (p_{1A} + p_{2A}) - (c - \bar{V}_d) \\
= V_A.
\]

In this case, the original shareholders do not prefer project B since their wealth from project B is:

\[
\bar{V}_s(B) = (1-z) (p_{1B} + p_{2B}) - (c - \bar{V}_d) \\
= V_B - T < V_A.
\]

We conclude that if the conversion factor is set at \( z = \frac{V_d(A)}{V_B + c} \) and the bond market is competitive, then the socially optimal project A is chosen as the result of the maximization of the shareholders' wealth. This also results in the conversion of bonds.
The value of shareholders' wealth is higher than in the case where only nonconvertible bonds are issued, $V_A > V_B$. But the change in the debt/equity ratio resulted from the conversion would reduce the level of the objective function, $u$.

IV.4. Optimal Conversion Ratio

We have established three points: (i) nonconvertible bond issues to rational investors may bring in a conflict between the value-of-the-firm maximization and the shareholders' wealth maximization, (ii) a restricted-period convertibility provision with the conversion rate set at the level defined by (62) and a competitive bond market can resolve the conflict, but (iii) this solution makes the debt-equity ratio depart from its optimal value. However, note that the wealth of the original shareholders is larger when convertible bonds are issued than when only nonconvertible bonds are issued. This creates an incentive for the firm to issue convertible bonds. The firm faces a trade-off; management can take project B at the opportunity cost of the value of project A, or they take project A at the expense of suboptimal debt/equity ratio.

At least two ways exist to get rid of this agency cost. One is to attach another provision to the convertible bonds specifying that the convertibility option is ineffective when the firm takes project A. This reduces
convertibility to the status of a "covenant." This is not the usual call-ability on convertible bonds which forces conversions at the firm's option. In this case, bondholders can prevent the detrimental wealth transfer by exercising the convertibility option when project B is chosen, while the shareholders can enjoy the value of project A to its full amount without altering the debt-equity ratio. This solution is equivalent to issuing regular bonds with a restrictive covenant saying that the firm will use the receipts from the bond issue only to finance project A. With this arrangement, rational bondholders cannot expect to have additional wealth, $T$, and the competitive valuation of bonds returns to $V^d(A)$.

The second way out of the dilemma is to find a level of the conversion factor, $z$, which is:

(a) high enough to protect bondholders against the detrimental wealth transfer due to project B, and

(b) low enough to discourage bondholders from exercising the convertibility option if project A is undertaken.

As a result, maximizing shareholders' wealth leads to project A, which is consistent with the value-of-the-firm maximization, and ownership is not diluted. If such a level of $z$ exists, shareholders are willing to adopt it since it lets them enjoy the best investment opportunity without changing the debt/equity ratio.
We now prove that the upper and lower limits on the conversion factor which satisfy these conditions constitute a non-empty set. The search for these limits on $z$ can be translated into the following problem: Let $z^*$ be a level which satisfies conditions (a) and (b). Then rational bond investors should not expect any gain from exercising the convertibility option. Therefore, the value of bonds in the competitive market should be:

\[(67) \quad V_d = V_d(A).\]

This valuation of bonds is based on the fact that bondholders rationally expect that the firm would choose project A as the result of the shareholders' wealth maximization. Next, the non-exercise condition (b) is:

\[(68) \quad z^*(p_1a_1 + p_2a_2) > V_d = V_d(A).\]

Finally, the bondholder-protection condition (a), together with the value-of-the-firm maximization must imply that the shareholders' wealth is maximized with project A. The real wealth of shareholders under project A with the valuation of bonds (67) is $V_A$, and that with project B depends on whether bondholders will exercise the convertibility option or not. From the bondholders' point of view, if the option is not exercised, their wealth is:
$p_1^d + p_2^b_2$, 

while if exercised, their wealth is:

$$z^*(p_1^b_1 + p_2^b_2).$$

The protection by conversion must imply that the difference between these two, denoted by $T^*$, be positive:

$$T^* = z^*(p_1^b_1 + p_2^b_2) - (p_1^d + p_2^b_2) > 0,$$

$$= z^*(V_B+c) - V_d(A) + p_2(d-b_2) > 0,$$

or equivalently

$$z^* > \frac{V_d(A) - p_2(d-b_2)}{V_B + c}.$$

Given that bondholders would choose to convert the wealth of the shareholders under project B is:

$$(1-z^*)(p_1^b_1 + p_2^b_2) - \{c-V_d(A)\}$$

$$= V_B - \{z^*(V_B+c)-V_d(A)\}.$$

Based on the comparison between $V_A$ and this expression, the firm should take project A as the best choice for the shareholders; that is, or $z$ must be set such that:

$$(70) \quad V_A - V_B + \{z^*(V_B+c) - V_d(A)\} > 0,$$

or equivalently

$$z^* > \frac{V_d(A) - (V_A - V_B)}{V_B + c}.$$

We now look for a non-empty interval of the conversion factor, $z^*$, from (68), (69), and (70).
From (68) and our original assumption, $V_A > V_B$, it is clear:

$$z^*(V_B+c) < V_d(A).$$

This implies, with (70), that the conversion factor $z^*$ does not completely eliminate the potential wealth transfer from bondholders to shareholders if project B is chosen. However, unless the amount of the wealth transfer is large enough to reverse the inequality of (70), bondholders may safely assume that project B is not taken. If the only way shareholders can hurt bondholders is to hurt themselves, no agency cost exists. Our original conditions (58) for the existence of the problem at issue, $V_A - V_B < P_2(d-b_2)$, is now utilized to compare the lower limits on $z^*$ by (69) and (70) to conclude that (70) implies (69). Summing up, we have established:

$$V_d(A) \quad (71) \quad \text{the upper limit } (\bar{z}) = \frac{V_d(A)}{V_A+c}, \text{ and}$$

$$\text{the lower limit } (z) = \frac{V_d(A)-(V_A-V_B)}{V_B+c}.$$

The range defined by these limits is not empty since:

$$\bar{z}-z = \frac{V_d(A)}{V_A+c} - \frac{V_d(A)}{V_B+c} + \frac{V_A-V_B}{V_B+c}$$

$$= \frac{V_A-V_B}{V_B+c} \left(1 - \frac{V_d(A)}{V_A+c}\right) > 0.$$
By setting the level of the conversion factor within the range defined by (71), shareholders will choose the socially optimal project A as a consequence of the maximization of their wealth without losing the control of the firm to rational bondholders.

The analysis can be easily extended to the case where multiple mutually exclusive projects exist. The upper limit is still the same as defined by (71). The lower limit is modified to:

$$z = \max_i \left\{ \frac{v_d(A) - (V_A - V_i)}{V_i + c} \right\}$$

where i is the index of suboptimal projects, i = B, C, C, ... This implies that shareholders have to identify the project which gives the highest lower limit. If the search for z is costly, they can set the level of the conversion factor at a level close to the upper limit.

Let us use the numerical example in section 2 again. It has: p_1 = .6, p_2 = .3, c = 90, a_1 = 110, a_2 = 120, b_1 = 134, b_2 = 33, and V_A = 12 and V_B = .3. Suppose d = 80, which is large enough to cause the wealth-redistribution problem, condition (58). The upper and lower limits are:

$$\bar{z} = \frac{(.6 + .3) \times 80}{12 + 90} = .705, \text{ and}$$

$$z = \frac{(.6 + .3) \times 80 - (12 - .3)}{.3 + 90} = .668.$$
The firm can set the conversion factor at $z = .7$, which is within the range $(.705, .668)$. If bondholders convert under project A with the conversion factor, their wealth is 

\[ .7 \times (.6 \times 110 + .3 \times 120) = 71.4 , \] while bonds pay 80 if not converted and the value is \[ 80 \times (.6 + .3) = 72. \] Therefore, bondholders would not exercise the convertibility option under project A. If project B is taken, bondholders would receive 80 in state 1 and 33 ($b_2$) in state 2 if bonds are not converted. The first-period value of this combination is \[ .6 \times 80 + .3 \times 33 = 57.9. \] If they convert, the value of bonds is \[ .7 \times (.6 \times 134 + .3 \times 33) = 63.21 , \] which is larger than 57.9. Therefore, conversion occurs and the value of the shareholders' wealth is \[ (1-.7) \times (.6 \times 134 + .3 \times 33) = 27.09. \] This is compared with the value of the shareholders' wealth under project A, which is \[ .6 \times 110 + .3 \times 120 = 102. \] This is larger than 27.09 under project B. Therefore, the calculation establishes that incentives exist to induce the firm to take project A. Note that shareholders correctly anticipate what bondholders would prefer assuming bondholders correctly anticipate which project shareholders would prefer, and vice versa. Both shareholders and bondholders are rational in this sense. Bondholders' rational expectations let them value the bonds as if they were single-payment bonds without a convertibility privilege. Their value lies in eliminating the agency cost.
We can conduct a comparative-static analysis of the upper and lower limits on the conversion factor. The limits depend on the debt size (d), the values of the two projects (VA and VB), and the project cost (c). Given VA, VB and c, an increase in the debt size, d, increases both upper and lower limits:

\[
\frac{\partial z^u}{\partial d} = \frac{p_1 + p_2}{V_A+c} > 0, \text{ and } \frac{\partial z^l}{\partial d} = \frac{p_1+p_2}{V_B+c} > 0.
\]

at the same time, increasing d clearly reduces the size of the range:

\[
\frac{\partial (z^u - z^l)}{\partial d} = \left\{ \frac{1}{V_A+c} - \frac{1}{V_B+c} \right\} (p_1 + p_2) < 0.
\]

An increase in the debt size increases the size of potential wealth redistribution, p_2 (d - b_2), and protection of bondholders requires a higher conversion factor, which pushes up the lower limits. On the other hand, it increases the value of bonds, V_d(A), which allows the firm to increase the upper limit without risking bonds being converted.

An increase in the project cost c, given VA, VB and d, lowers both upper and lower limits:

\[
\frac{\partial z^u}{\partial c} = -\frac{V_d(A)}{(V_A+c)^2} < 0, \text{ and } \frac{\partial z^l}{\partial c} = -\frac{V_d(A)-(V_A-V_B)}{(V_B+c)^2} < 0.
\]

but its impact on the size of the range is ambiguous. Moreover, an increase in the project cost must increase
either equity or debt to finance the project. An additional debt issue will cancel out the downward shift of the range. If the additional project cost is entirely financed by debt, the impact on the upper and lower limits is:

\[
\left(\frac{\partial Z}{\partial d} + \frac{\partial Z}{\partial c}\right)dc = \left\{\frac{p_1+p_2}{V_{A}+c} - \frac{V_{d}(A)}{(V_{A}+c)^2}\right\} dc
\]

\[
= \frac{p_1+p_2}{V_{A}+c} (V_{A}+c-d) \quad dc > 0, \text{ and}
\]

\[
\left(\frac{\partial Z}{\partial d} + \frac{\partial Z}{\partial c}\right)dc = \left\{\frac{p_1+p_2}{V_{B}+c} - \frac{V_{d}(A)-(V_{A}-V_{B})}{(V_{B}+c)^2}\right\} dc
\]

\[
= \frac{p_1+p_2}{V_{B}+c} \left\{V_{B} + c-d + \frac{V_{A}-V_{B}}{(p_1+p_2)}\right\} dc > 0;
\]

that is, the range definitely shifts up.

An increase in the value of project A, given d, c, and \(V_B\), lessens the possibility of wealth redistribution, but it increases attractiveness of the convertibility option under project A. Therefore, both upper and lower limits decline:

\[
\frac{\partial Z}{\partial V_{A}} = \frac{V_{d}(A)}{(V_{A}+c)^2} < 0, \quad \text{and} \quad \frac{\partial Z}{\partial V_{A}} = - \frac{1}{V_{B}+c} < 0.
\]

On the other hand, an increase in the value of project B, given d, c, and \(V_{A}\), increases only the lower limit:

\[
\frac{\partial Z}{\partial V_{B}} = 0, \quad \text{and} \quad \frac{\partial Z}{\partial V_{B}} = \frac{V_{A}+c-V_{d}(A)}{(V_{B}+c)^2} > 0.
\]
An increase in $V_B$ could result from an increase in $b_1$, or an increase in $b_2$, or both. If an increase in $b_2$ wipes out the wealth redistribution problem, convertible bonds are no longer required. But if the redistribution problem still remains to be resolved by convertible bonds, an increase in $b_2$ has two kinds of impact. On one hand, it tends to reduce the size of potential wealth transfer, $p_2(d - b_2)$. On the other hand, it makes project B more attractive to shareholders. The former impact implies a reduction of the lower limit, while the latter impact implies an increase in the conversion factor to keep project B still unattractive to shareholders. This comparative-static analysis shows that the former impact is outweighed by the latter. An increase in $V_B$ from an increase in $b_1$ has only the first impact which pushes up the lower limit on z and reduces the attractiveness of project B.

IV.5. Extended-Period Convertible Bonds

The analysis in the previous section clearly indicates that restricted-period convertible bonds are sufficient to eliminate the agency cost that would cause suboptimal investment decisions. By setting the level of the conversion factor within the range defined by (71), the firm can provide shareholders with the highest-value project without disturbing the debt/equity ratio from its optimal level. Convertible bonds serve only to eliminate
the agency cost. Bondholders need the convertibility option to be exercisable only immediately after the firm selects a project. Convertible bonds do not receive a premium value unless the conversion factor is set outside of the range defined by (71). However, ordinarily convertible bonds are exercisable, not only in the first period, but at any time until their maturity dates. We now examine some implications of this additional flexibility in conversion timing.

Suppose bonds are convertible in either period 1 or period 2. This implies that bondholders can exercise the convertibility option, not only just after they know which project has been undertaken, but also after they know which state has occurred in period 2. This requires rational bondholders to determine the timing of conversion. They calculate the benefits from exercising the convertibility option in all states of the economy in both periods under both project. If project A is taken and bonds are not converted until period 2, their expected pay-offs are:

<table>
<thead>
<tr>
<th>TABLE 4</th>
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<tbody>
<tr>
<td></td>
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<tr>
<td>period 2</td>
</tr>
<tr>
<td>state 2</td>
</tr>
<tr>
<td>exercised</td>
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<tr>
<td>not exercised</td>
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</tbody>
</table>
Bondholders convert bonds in period 2 in state $i$ if $z_{a_1} > d$, which means the first-period value of bonds under project $A$ is:

\[
(72) \quad p_{1}^{\text{max}} (z_{a_1},d) + p_{2}^{\text{max}} (z_{a_2},d) \\
= (p_1+p_2)d + p_{1}^{\text{max}} (z_{a_1}-d,0) + p_{2}^{\text{max}} (z_{a_2}-d,0) \\
= V_d(A) + p_{1}^{\text{max}} (z_{a_1}-d,0) + p_{2}^{\text{max}} (z_{a_2}-d,0).
\]

The second and third terms constitute the premium. The premium is the value of the convertibility option, which is exactly the same as the value of call options with the exercise price at $d$. The benefits under project $B$ need careful attention. Bankruptcy in state 2 implies that $d > b_2$. Since in state 2 the firm would be bankrupt, bondholders receive only $z_{b_2}$ (<d) if they convert. Thus, bondholders would not choose to convert and would receive $b_2$. Therefore, the value of the bonds, if not converted in period 1, under project $B$ is:

\[
(73) \quad p_{1}^{\text{max}} (z_{b_1},d) + p_{2}b_2 \\
= V_d(A) + p_{1}^{\text{max}} (z_{b_1}-d,0) - p_{2}(d-b_2).
\]

The value of bonds differs from its risk-free value, $V_d(A)$, by the difference between the option value in state 1, $p_{1}^{\text{max}} (z_{b_1}-d,0)$, and the value of the loss in bankruptcy, $p_{2}(d-b_2)$.

Next, we determine whether bondholders would exercise the convertibility option in period 1, expecting the
bond value (72) or (73) if not exercised in period 1. Under project A, if bondholders convert in period 1, they forego the chance of conversion in period 2 contingent on the state of the economy. Hence, the value of goods in period 1 is:

\[(74) \quad z(p_1a_1 + p_2a_2)\]

\[= (p_1 + p_2)d + p_1(za_1 - d) + p_2(za_2 - d)\]

\[= V_d(A) + p_1(za_1 - d) + p_2(za_2 - d).\]

The difference between (72) and (74) is:

\[p_1 \{\max (za_1 - d, 0)\} - (za_1 - d) + p_2 \{\max (za_2 - d, 0)\} - (za_2 - d) \geq 0\]

which implies bondholders should not exercise the convertibility option in period 1. By postponing the conversion decision until period 2, they can be at least as well off as they would be when bonds are converted in period 1. Since convertible bonds are equivalent to American call options with the exercise price at d and period 2 as the expiration date, this result is not a surprise. This result is valid regardless of the level of z. The same result holds under project B. If bonds are to be converted in period 1, the value of bonds is:

\[z(p_1b_1 + p_2b_2)\]

\[= V_d(A) + p_1(zb_1 - d) + p_2(zb_2 - d).\]

A comparison between this and (73) gives the difference:
which means conversion does not occur in period 1 under Project B. Conversion in period 1 incurs the loss of the option value, \( \max(z_b^1 - d, 0) - (a_1 - d) \geq 0 \), and the additional loss in bankruptcy that they have to suffer as shareholders, \( p_2(1-z)b_2 > 0 \). We conclude that the extended-period convertibility option is not exercised in period 1, and therefore that (72) or (73) gives the first-period value of bonds under project A and B, respectively. Rational shareholders should expect this.

We can now determine the lower limit on the conversion factor to eliminate incentives for wealth redistribution. The contribution by bondholders in financing the project (the value of bonds) is rationally determined in a competitive market. If project A is their rational expectation of the consequence of the shareholders' wealth maximization, the bond value must be determined by (72) and the shareholders' wealth is \( V_A \). Further, the project switch from A to B should not benefit shareholders. The second-period pay-off to shareholders under project B is \( b_1 - \max(z_b^1, d) \) in state 1 and zero in state 2. In state 1, however, convertible bonds should protect bondholders. This means \( \max(z_b^1, d) = z_b^1 \), or equivalently, \( z > d/b_1 \). Otherwise, bondholders never exercise the convertibility option under project B and the wealth redistribution occurs.
Therefore, the project switch from A to B after raising the amount of debt capital defined by (72) makes shareholders' wealth:

\[ p_1(1-z)b_1 - \{c - V_d(A) - p_1 \max (za_1 - d, 0) - p_2 \max (za_2 - d, 0) \} \]

\[ = V_B + V_d(A) + z(V_B + c) - p_2(1-z)b_2 + p_1 \max (za_1 - d, 0) + p_2 \max (za_2 - d, 0). \]

Shareholders must compare this value with their wealth with project A, which is \( V_A \), to conclude that project A maximizes their wealth. This requires:

The right hand side of (75) defines the lower limit on the conversion factor when the convertibility option is exercisable in both period 1 and period 2.

The first term in the lower limit (75) is the lower limit for restricted-period convertible bonds, equation (71) in section 4. The lower limit for extended-period convertible bonds (75) differs from this for restricted-period convertible bonds allow bondholders to exercise the convertibility option after they know the second-period value of the firm. This enables bondholders to avoid the extra loss in state 2 (when the firm is bankrupt) that they would have to take if they had to exercise conversion in
the first-period and become shareholders. This benefit serves to protect bondholders from the detrimental wealth redistribution and substitutes for the protection provided by the restricted-period convertibility option. The second term in (24), \(-p_2(1-z)b_2/(V_B+c) < 0\), represents this substitution effect. On the other hand, the competitive market value of extended-period convertible bonds is higher than that of restricted-period convertible bonds by the amount of the option value equation (22). If the firm takes project B bondholders' losses would be larger than \(V_d(A)\) by this option value. This fact raises the level of the conversion factor, which is represented by the third term in (75),

\[
(V_B+d)^{-1} \{p_1 \max(z_{a_1}-d,0) + p_2 \max(z_{a_2}-d,0)\} > 0.
\]

Let us now examine the upper limit. Whether or not the convertibility option is exercised in period 2 under project A depends on whether \(z_{a_i} > d\) \((i = 1,2)\) is satisfied. Avoiding conversion in any state requires:

\[
(76) \quad z < \min\{\frac{d}{a_1}, \frac{d}{a_2}\}.
\]

Unlike the restricted-period convertible-bond case, we cannot always establish a non-empty range which satisfies both the lower limit defined by (75) and the upper limit defined by (76). The following two examples demonstrate this point.
The first example is the same as the one in sections 2 and 4. We have \( a_1 = 110, a_2 = 120, \) and \( b = 80 \), which gives the following upper limit:

\[
\min \left\{ \frac{d}{a_1}, \frac{d}{a_2} \right\} = .666.
\]

If the conversion factor is set below .666, bondholders would not exercise the convertibility option, and the value of bonds is only \( V_d(A) = (.6 + .3) \times 80 = 72 \). The premium for the option is zero. The remaining question is whether we can find a level of \( z \) which is less than .666 and at the same time more than the lower limit (75). Since \( \frac{\{V_d(A) - (V_A-V_B)\}/(V_B+c)}{(b/2)} = .668 \) as we calculated in section 4, and the lower limit defined by (75) could be less than .668 by the amount \( p_2(1-z)b_2/(V_B+c) = (1-z) \times .1096 \), we can find a non-empty range in this case. For instance, \( z = .665 \) is low enough to prevent conversion since the lower limit is \( .668 - (1-.665) \times .1096 = .6313 \) which is smaller than .665.

However, the following second example, which has a combination \((a_1, a_2)\) slightly different from the first example but has \( V_A \) and the other data the same as before, illustrates a case in which it is not possible to find a level of \( z \) which simultaneously satisfies the lower and the upper limit. Suppose \( a_1 = 80 \) (instead of 110) and \( a_2 = 180 \) (instead of 120). \( a_1 = 80 \) is just enough to avoid
bankruptcy with \( d = 80 \). This combination, \( (a_1, a_2 = 80, 180) \), implies the same value of the firm as the first example, 
\[
VA = .6 \times 80 + .3 \times 180 - 90 = 12.
\]
The upper limit, (76) is now:
\[
\min \left\{ \frac{-d}{a_1}, \frac{d}{a_2} \right\} = .444.
\]
Let us suppose that the firm sets \( z \) at this maximum level. Since \( z = .444 \) eliminates the option value, the third term in the right hand side of (75) vanishes, and the lower limit is calculated as:
\[
\frac{V_d(A) - (V_A - V_B)}{V_B + d} - \frac{p_2 (1-z) b_2}{V_B + c}
\]
\[
= .668 - \frac{.3 \times (1-.444) \times 33}{90.3} = .607 > .444.
\]
This shows that even the highest allowable value of \( z \) in avoiding conversion is not high enough to prevent the firm from choosing the suboptimal project B. The extended-period option has to offer insurance against more than agency costs. Hence, this example suggests some guidelines for choosing between covenants and convertible bonds to reduce agency costs.

IV.6. Conclusion

Jensen and Meckling (1976) and Smith and Warner (1979) suggested, in the context of the agency-cost theory, the use of convertible bonds to resolve the wealth-transfer
problem pointed out by Fama and Miller (1972) and Stiglitz (1972). This chapter provides a concrete analysis of the role of convertible bonds in reducing the agency cost associated with the regular bond issues.

Convertible bonds can indeed eliminate the agency cost. Both restricted-period and extended-period convertible bonds were examined. If the sole purpose of convertible-bond issues is to make the share-holder-wealth-maximizing firm choose the socially optimal project, the convertibility option only exercisable immediately after the firm's project choice is sufficient. An attractive aspect of restricted-period convertible bonds is that the firm can set the level of the conversion factor low enough to maintain its desired debt/equity structure unaltered. We have shown the existence of a non-empty range for the conversion factor within which convertible bonds can eliminate the agency cost of suboptimal investment decisions without altering the debt-equity ratio.

If the convertibility option is also exercisable after bondholders observe the state of the economy, it is not always possible to find a level of the conversion factor which simultaneously serves the two purposes. Extended-period convertible bonds can eliminate the agency cost of suboptimal investment decisions but, perhaps, only at the cost of possible conversion. If
possible exercise of the convertibility option raises other agency costs, this limits the use of convertible bonds.
FOOTNOTES TO CHAPTER IV

1 As surveys of the literature, see Fama and Miller (1972), especially chapter 4, Chen (1978), and Chen and Kim (1979).

2 Stiglitz (1969, 1974) has proven the irrelevance theorem in a multi-period, state-preference framework with multiple claims. He showed that the theorem requires neither homogeneous expectations among security holders nor perfect markets. Fama (1978) re-examined the Stiglitz analysis and pointed out that the theorem is valid when no firms issue securities for which there are not perfect substitutes.

3 Jensen and Meckling (1976) express doubt about the significance of bankruptcy costs on the basis of Warner's study (1975) which indicates that bankruptcy costs average about 2.5% of the value of the firm three years prior to bankruptcy. On the other hand, other studies (for example, Baxter [1969]) show much higher percentage of bankruptcy costs. Chen and Kim (1979) point out that the part of bankruptcy cost due to the loss of tax credits on operating losses has not been included in any empirical estimates.

4 See the numerical example in Fama and Miller (1972).

5 Myers (1977) also shows the possibility that firms already in financial distress may not make even socially desirable investment projects.

6 Further analysis of the "me-first" rules was done by Kim, McConnell and Greenwood (1977).

7 Smith and Warner (1979) investigate different types of restrictive covenants.

8 The result is similar to the non-exercise of American call options before their expiration dates. A call option is the right to purchase a particular base security at a predetermined price, and the right may be waived if exercising it would be disadvantageous to its holders. American call options are exercisable any time before their expiration dates, while European call options are
exercisable only at their expiration dates. It has been shown that rational investors do not exercise American call options before their expiration dates and that American call options are priced the same as European call options of the same specifications (see Merton [1973]). The exercisability of American call options prior to their expiration dates does not have any market value. Both this and our result show that additional flexibility in exercising options does not have a market value if rational investors do not expect to exercise them.

This definition is due to Arrow (1964). Debreu (1959) introduces the idea of contingent claims on commodities. Case and Stiglitz (1970: in appendix) have shown that if (1) the number of securities traded in the market is not less than the number of all possible states, and (2) the vectors of the returns from these securities are linearly independent, then the competitive equilibrium prices of securities provide the information sufficient to calculate the prices of the Arrow-Debreu securities implicitly defined.

In an incomplete market, the Pareto efficiency of the allocation of resources under uncertainty is not guaranteed. See Diamond (1967), Leland (1974), Ekern and Wilson (1974), Baron (1979), Hart (1975) and others for the allocational inefficiency associated with incomplete markets.

Asymmetric information between bondholders and shareholders provides an explanation for empirically observed debt-equity ratio. See Stiglitz (1972) and the financial signaling model by Ross (1977).

Additional risky debt issues may dilute the value of debt previously issued and result in a wealth redistribution from old bondholders to shareholders. For instance, the firm may distribute the receipts from new debt issues to shareholders as cash dividends. This action reduces the ability to repay old debt. Since this process per se does not change the value of the firm with a given investment project, a wealth redistribution occurs from old bondholders to shareholders. Setting a priority rule in debt repayment can prevent this problem. This issue was examined by Kim, McConnell, and Greenwood (1977). However, our analysis focuses on wealth redistribution from an investment-project switch.

In the Capital-Asset-Pricing Model world with the separation theorem, all investors have the same proportions of risky assets as the market portfolio. Therefore, our wealth-redistribution problem would not arise.
14 See Alchian and Demsetz (1972).

15 See Black and Schole (1973).

16 See Merton (1973) for the proof that rational investors do not exercise American call options until their expiration dates.
Chapter V. CONCLUSION

Financial markets facilitate the intertemporal allocation of resources and risk-bearing by helping investors and firms to diversify and share risks. These functions of financial markets are essential for achieving an efficient allocation of resources under uncertainty.

Since the 1960s, we have seen rapid developments of theoretical and empirical studies of financial markets. Especially, the Capital-Asset-Pricing Model based on the Tobin-Markowitz portfolio-selection theory has significantly contributed to our understanding of financial markets. The Capital-Asset-Pricing Model provides us with empirically testable propositions about the pricing of all assets. Also influential are state-preference models of economies under uncertainty. They permit us to examine the existence and the Pareto-optimality of the competitive allocation of resources and wealth.

Having financial markets is not the only way of allocating resources and risk-bearing. Debreu presented a market model where state-contingent claims on commodities are traded. As an alternative to this, Arrow introduced markets of state-contingent claims on income. In the Debreu
and Arrow market models, the complete set of contingent claims allows a Pareto-optimal allocation. However, real-world markets are not exactly the same as the Arrow and Debreu markets. We have stock shares, bonds, insurance policies, futures contracts and other complex securities. The concept of "complete" markets is used to describe the situation where these market instruments can substitute for the complete set of Arrow or Debreu contingent claims.

When markets are incomplete, a Pareto-nonoptimal allocation results. This leaves marginal rates of substitution between current income and future state-contingent income unequalized across investors. The differences in marginal rates of income substitution among investors provide mixed signals to firms' investment decisions. In incomplete markets, the value of a new investment is not unambiguously determined, nor does the unanimity of shareholders hold.

Introducing new types of securities into incomplete markets would eliminate the differences among investors of marginal rates of income substitution. We call these new securities "innovative" securities. Innovative securities have unique state-contingent patterns of their returns, and combining existing securities can not imitate them. Only innovative securities can change the market-equilibrium allocation of wealth and potentially help to establish a Pareto-optimal allocation. If securities are not
innovative, they are redundant. Investors and firms have no incentive to issue them. If they are introduced, their prices are determined by pure arbitrage relations in terms of the prices of the existing securities.

The fundamental purpose of the studies of financial markets is to explain why certain types of securities must exist. We need to examine the welfare implications of financial securities. This dissertation re-examined the theories of financial decisions and the theoretical structure of the Capital-Asset-Pricing Model from this point of view. In Chapter II, we found that rearrangements of corporate capital structure would be irrelevant to the value of firms as long as the rearrangements do not introduce any innovative securities. Our approach lets us generalize the Modigliani-Miller irrelevance theorem.

In Chapter III, we found that the special assumptions of the Capital-Asset-Pricing Model always assure a Pareto-optimal allocation of wealth even where markets are incomplete. Once a Pareto-optimal allocation is established, no new security is innovative. The Capital-Asset-Pricing Model shows that only two mutual funds would be sufficient to establish the Pareto-optimality. This serves as the basis for the CAPM closed-form valuation of all assets, which is basically arbitrage pricing relations. This is an attractive feature of the Capital-Asset-Pricing Model. However, it also means that the Capital-Asset-Pricing Model
fails to give the fundamental rationale for the emergence of any new type of securities.

By replacing the CAPM assumption on utility functions by the completeness assumption, both of which assure Pareto-optimality, we could derive an alternative set of conditions for the CAPM closed-form solution for risks premiums on assets.

After examining the allocation of wealth under an exogeneously given set of investment decisions, we investigated, in Chapter IV, the problem of wealth transfer among the existing security holders due to investment-decision switches. Even when markets are complete in the static sense for distributing exogenously given investment returns, possible future revisions of investment or financing decisions may create a situation where the issues of regular bonds and stock shares alone would result in Pareto-nonoptimal investment decisions. Possibility of transferring wealth from debtholders to shareholders by an investment switch, when it is rationally anticipated by debtholders, forces shareholders to take on projects which do not maximize firm value. We have shown that convertible bonds can resolve this problem by providing debtholders the option to become shareholders conditional on investment decisions. This analysis suggests that completing markets in a dynamic situation, where nonrandom risk (uncertainty in the future course of firms' actions) exist, requires
innovative securities whose returns are contingent on the decisions of issuers.

Finally, we recall that our analysis assumes away transactions and information costs. Such costs as collecting and analyzing information, identifying which state has actually occurred, making contracts, and monitoring and bonding are all important factors. These costs constrain the competitive-market allocation of wealth and investments from reaching as high a Pareto-optimal outcome as would have obtained in their absence. The recent agency-cost approach attempts to explain costly financial contracting. The costs of identifying the state of nature may be one of the main reasons for the emergence of stock-share markets instead of the Arrow or Debreu contingent-claim markets. We leave the analysis of these subjects for the future research.
LIST OF REFERENCES


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