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CHARACTERIZATION OF CONSTRAINTS AND FORCES ACTING BETWEEN LOOSELY COUPLED BODIES WITH APPLICATION TO HUMAN JOINT MECHANICS

The Ohio State University

Ph.D. 1981

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CHARACTERIZATION OF CONSTRAINTS AND FORCES
ACTING BETWEEN LOOSELY COUPLED BODIES WITH
APPLICATION TO HUMAN JOINT MECHANICS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

by

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* * * * *

The Ohio State University
1981

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CHAPTER 1
INTRODUCTION

1.1 Background

The predictive modeling of the dynamics of multiple coupled rigid bodies has only become feasible in the last few decades. This has been due primarily to advances in computer technology which allow direct integration of the equations of motion. While closed form solutions to rigid body dynamics problems are limited to very simple cases mainly consisting of single bodies or highly constrained motion of several bodies, methods of direct integration allow predictive modeling of a large number of coupled bodies with complex internal and external interactive forces.

A number of formulations for such models have been developed using both the Lagrangian and Newton-Euler methods [2, 17, 47, 51, 109, 132, 144]. Graph or systems synthesis techniques have been formulated which may be used to develop the equations of motion by merely specifying the link structure of the system [1, 77, 92, 99, 101, 131].

These models have applicability in such areas as mechanical linkage systems, articulating space structures and human body dynamics. In the latter area several articulated gross motion models describing the human body have been used to study body dynamics during automobile and
helicopter crashes, pedestrian accidents, ejections of pilots from high performance aircraft and during parachute opening shock [57,62,66,94,97,117].

In conducting simulations of human body dynamics that can provide meaningful information about motion in time, interactive forces and the probability of injury, it is not merely sufficient to have the chained element structure of the model, but one must also have a comprehensive data base describing the physical properties of the model segments and the nature of the coupling between the segments. Among the various types of data required for these specifications are the moments of inertia and masses of each segment, surface definitions for contact force application, joint properties and initial condition information. The development of these models has, in turn, spurred activity in the development of these data bases. The initial properties of human body segments, particularly the mass and the center of gravity, have been measured and calculated by a number of investigators. The studies most applicable to the investigation of multiple segment body dynamics are those by Hatze [54], which develops a method for calculating the properties of 17 body segments from a number of anthropometric measurements, and McConville [80], which gives regression equations for 17 body segments derived from the stereometric analysis of the body surface of 31 subjects. Surface definitions are obtained from anthropometric data or measured directly for specific simulation applications. Data on joints as well as
that for specifying initial conditions is very limited and its generation has had limited progress.

One of the chief hurdles to progress in these latter two areas is the difficulty in conceptualizing three-dimensional motion and developing analytical characterizations of the kinematics of loosely coupled systems. Loosely coupled here is defined to mean a joint which cannot be characterized by a pin, ball and socket, sliding, or simple combination of these articulations. All models of the human body currently in use assume joints to be pin or ball and socket, and in most cases do not allow separation of the joint centers. The pin joints are either implemented by reducing the degrees of freedom or introducing constraints which limit the motion in the joint.

The principal objective of this research is to address this shortcoming by developing a method, and appropriate intermediate analytic tools, for the quantitative description of joint articulations and internal forces from applied external force and resulting three-dimensional kinematic data. This method must be of a form which allows implementation in mathematically defined computer based body models as well as being applicable to the definition of structures of mechanisms for mechanical design for analogous joint structures.

As a secondary objective, which is technically fully complementary to the joint kinematics analysis, is the development of a consistent methodology for generating
initial condition data for multisegmented models. These data consist of the rotational positions and velocities of the body segments at initial time and in some cases translational and rotational accelerations, velocities and displacements of prescribed motion segments.

The initial step in addressing these objectives was to examine methods for prescribing three-dimensional motion of a body in space. According to a theorem by Euler, any three-dimensional motion of a rigid body from one position to another can be specified in terms of a pure translation and a pure rotation. A specific approach employing this theorem was developed by Chasles called screw axis motion in which the displacement of a body is viewed as a rotation about an axis fixed in space, called the screw axis, and a translation along this axis.

This approach is useful primarily for two reasons: (1) it provides a method of motion visualization in terms of an axis in space about which the body rotates and along which it translates. (2) If the motion is simple or highly constrained it can provide a direct means for defining joint mechanisms.

The method, however, has serious shortcomings in the analysis of relative motion of two bodies if the motion is complex and if there are small errors in the data specifying sequential body positions. These shortcomings are typically manifested by very erratic changes in the screw axis position during the course of body motion. The use of an axis
in the motion description is, however, appealing for it provides a certain direct physical insight into the motion process and for pure rotational motion is a most concise and understandable way of prescribing and visualizing the motion.

The most commonly used method of effecting body rotations is by orthonormal matrix transformations. These matrices are convenient to use, but, except for the most single cases, provide no direct insight into body rotation. The axis and angle of body rotation are opaquely embedded in the orthonormal matrix structure.

Since one of the goals of this research has been to emphasize methods of rotational analysis which maximize physical insight into body rotations, an approach has been chosen and systematically developed which stresses axis of rotation concepts and clearly differentiates between rotation operations and transformations between coordinate systems.

The methods of rotational analysis are applied in the development of techniques for characterizing three-dimensional body motion in terms of the rotation of the body about an arbitrary fixed point in the body and the three-dimensional trajectory of the rotation point in the fixed body or reference system.

This analysis is applied to three-dimensional human shoulder joint motion obtained by measuring three points on the arm using a three-dimensional spatial sonic digitizing
system. The points are obtained during the course of forced arm motion with the components of force and position of application being simultaneously measured.

1.2 Scope of Research

Rotational operators, and corresponding transformations to the rotated states, are developed and their operational equivalence demonstrated. Using vector analysis a rotation operator is developed which in matrix form is shown to be orthonormal. It is next shown that three rotations about coordinate system axes using $3 \times 3$ orthonormal matrices can produce the same net rotation operator. Finally, the use of quaternions as rotation operators is examined and their operational equivalence to the orthonormal matrix operation demonstrated. Considerable emphasis is placed on the quaternion formulation because it explicitly contains the axis of rotation and has only four parameters with one equation of constraint as opposed to nine parameters with six orthogonality conditions as constraint equations for the rotation matrix. In addition, the quaternion has the property of being readily factorable. Functionally, this means that any rotation can be broken down into a number of rotations. Since the quaternion explicitly contains the unit axis of rotation, it is possible to associate with each rotation an arc on a unit sphere described by the endpoint of a unit rotated vector. The factorization of the quaternion then results in respective arcs, which, when connected in sequence, connect the ends of
the original rotation arc. This method is discussed by Branets [9] and is an excellent method for visualizing sequential rotations.

A consistent and unambiguous rotation scheme is introduced which clearly defines the state or coordinate system in which a rotation is applied and how to transform rotational operators between states. While operationally the rotations and transformations are effected by relatively straightforward mathematical manipulations such as orthonormal matrix or unitary quaternion multiplications, the sequence of these operations is highly confusing if space fixed and body fixed rotations are mixed. This potential confusion is eliminated with the developed notation and mixed body and space specified rotations can be readily applied.

The time derivatives of the rotation matrix and quaternion are developed and their equivalence demonstrated. These kinematic relationships provide a means of obtaining body position from a knowledge of an initial state and the rate of angular body velocity.

For completeness, the equations of rigid body motion are developed which, when integrated, provide the rates of angular body velocity, and in turn the body position from the kinematic relationships. These derivations are carried out primarily to illustrate the use of quaternions and their applicability to rotational analysis.

Two methods of screw axis analysis are developed.
While the screw axis method for analyzing body motion is fairly standard, the methods developed here are partially original and were formulated to extract as much physical insight as possible from the motion description. The first method uses a rotation operator and a knowledge of the properties of the operator to find the angle of rotation and rotation angle. The second method is strictly based on vector displacement arguments.

The ideas introduced in the screw axis motion analysis are applied in formulating a body motion description in which body rotation is taken about a specified and fixed point in the body. This point is allowed to move in space as the body moves from one position to the next. By summing the rotation point displacements for sequential body displacements, a trajectory for the rotation point is described in space. The choice of the rotation point in the body determines the trajectory for a given body motion. A method for finding the optimal body rotation point for a given rotation point trajectory is developed. Finally a method is developed for analytically expanding the trajectory as a combination of a Fourier series and a polynomial.

A method for body position specification from a knowledge of position in space of three rigidly fixed non-collinear points on the body is formulated. Transformations are developed in terms of sequential elementary matrix rotations to be used for the reduction and presentation of the kinematic and internal joint movement data.
The experimental and data collection methods as described in Engin [28-37] are adapted to the present research. The adapted methods and the collected data on resistive joint properties and joint kinematics are discussed.

The computer programs used in this study are listed in the Appendix. They are written in Fortran and were all specifically written for this study. Two International Mathematical and Statistical Libraries, Inc. (IMSL) library programs were used for the Fourier and polynomial coefficient analysis.
CHAPTER 2

ROTATION OPERATORS AND TRANSFORMATIONS

2.1 The Rotation Operator

The specification of the three-dimensional position of one body with respect to some reference system or another body requires six independent parameters. According to a theorem by Euler the body position can be uniquely specified in terms of a body rotation and translation. The most common explicit form used for specifying a body rotation is by an orthonormal 3 x 3 matrix the elements of which are the cosines of the angles between axes of the rotated system and the reference system. As a consequence of the orthonormality the inverse is the transpose of the matrix. There exists a complementary cosine matrix which relates components of vectors in the body to those in the reference system. This cosine matrix, which we will henceforth refer to as a transformation matrix, which is also orthonormal and, as will be shown later, is the inverse of transpose of the rotation matrix. While this similarity of form and simple relationship between the rotation and transformation matrix operators is highly fortuitous for purposes of analytic and numerical manipulation, it has led to considerable confusion in the literature where quite often
rotation operators are used in place of transformation operators and vice versa.

In the present development it is assumed that the two operators while of identical form are intrinsically different. The rotation operator is viewed as acting on a vector, which is usually taken as embedded in a body, and rotating that vector and the body in which it is embedded to a new position. This operator defines the kinematics of the body which is a physical process. The transformation matrix, on the other hand, describes no physical process as it is strictly a mathematical entity which gives the components of a vector (or higher order tensor) in one system if the components in the other system are known.

The usual approach used in most texts to develop the transformation matrix and the rotation operator is to use geometric arguments to identify the form of the transformation matrix and then to make the observation that the inverse of the transformation operator can be viewed as a rotation operator. Since both the transformation and rotation operator have identical form the properties deduced for the transformation operator also hold for the rotation operator.

The emphasis in this study is on body kinematics and, therefore, an approach was chosen which accentuates the physical process of body motion. In this approach a rotational operator is developed, its properties deduced and the transformation between the rotated and unrotated systems is obtained from the rotation operator.
Consider the rotation of a vector $\mathbf{b}$, which extends from point 0 to point A, about the unit vector $\hat{\mathbf{u}}$ as shown in Fig. 2.1.1. The resultant vector is $\mathbf{b'}$, which extends from point 0 to point B, and has the same magnitude as $\mathbf{b}$. The rotated vector can be expressed as

$$\mathbf{b'} = \mathbf{OC} + \mathbf{CB} = (\mathbf{\hat{u}} \cdot \mathbf{b})\mathbf{\hat{u}} + \mathbf{CB}. \quad (2.1.1)$$

The vector from C to B can be written as

$$\mathbf{CB} = \mathbf{CD} + \mathbf{DB} = \frac{\cos\theta}{|\mathbf{CA}|} \mathbf{CB} |\mathbf{CA}| + \frac{\sin\theta}{|\mathbf{CA}|} \mathbf{CB} \mathbf{\hat{u}} \times \mathbf{CA} \quad (2.1.2)$$

where DB is in the CBA plane and is perpendicular to CA.

Figure 2.1.1 Rotation of Vector $\mathbf{b}$ to $\mathbf{b'}$ about $\mathbf{\hat{u}}$.
Since during the rotation the magnitude of $b$ does not change

$$|\vec{b}| = |\vec{b'}|$$

or

$$|\vec{OC} - \vec{CB}| = |\vec{OC} - \vec{CA}|$$

and

$$\vec{OC}^2 - 2\vec{OC} \cdot \vec{CB} + \vec{CB}^2 = \vec{OC}^2 - 2\vec{OC} \cdot \vec{CA} + \vec{CA}^2,$$

but $\vec{OC}$ is perpendicular to $\vec{CB}$ and $\vec{CA}$, therefore, $\vec{OC} \cdot \vec{CB} = 0$ and $|\vec{CB}| = |\vec{CA}|$ or $|\vec{CB}|/|\vec{CA}| = 1$. Substituting this into Eq. 2.1.2 and using

$$\vec{CA} = \vec{b} - (\vec{u} \cdot \vec{b})\vec{u}$$

we can write Eq. 2.1.1 as

$$\vec{b'} = \vec{u} \cdot \vec{b} + \cos \theta (\vec{b} - \vec{u} \cdot \vec{b}) + \sin \theta \vec{u} \times \vec{b}$$

$$= (\vec{u} \cdot \vec{b} + \cos \theta (\vec{I} - \vec{u} \cdot \vec{u}) + \sin \theta \vec{u} \times )\vec{b}$$

This equation can be put in matrix form by using matrix operators for the scalar and vector products given by

$$\vec{A} \cdot \vec{B} = \vec{A}^{T} \vec{B}$$
and

\[ \vec{A} \times \vec{B} = \begin{pmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{pmatrix} \]

where \( \vec{A} \) and \( \vec{B} \) are 3 x 1 matrices and \( \vec{A} \) is defined as the 3 x 3 matrix in the second term of Eq. 2.1.5.

Using these matrix operators Eq. 2.1.3 can be written as

\[ \vec{b}' = (\vec{u}^T \mu + \cos\theta (I - \vec{u}^T \mu) + \sin\theta \vec{u}) \vec{b} \]

\[ = \begin{bmatrix} \mu_1^2 (1 - \cos\theta) + \cos\theta & \mu_1 \mu_2 (1 - \cos\theta) - \mu_3 \sin\theta & \mu_1 \mu_3 (1 - \cos\theta) + \mu_2 \sin\theta \\ \\
\mu_2 \mu_1 (1 - \cos\theta) + \mu_3 \sin\theta & \mu_2^2 (1 - \cos\theta) + \cos\theta & \mu_2 \mu_3 (1 - \cos\theta) - \mu_1 \sin\theta \\ \\
\mu_3 \mu_1 (1 - \cos\theta) - \mu_2 \sin\theta & \mu_3 \mu_2 (1 - \cos\theta) + \mu_1 \sin\theta & \mu_3^2 (1 - \cos\theta) + \cos\theta \end{bmatrix} \vec{b} \]

\[ = R \vec{b}. \]

The \( R \) is a 3 x 3 matrix operator which acts on the \( \vec{b} \) (or equivalently the 3 x 1 matrix) to produce another vector, \( \vec{b}' \), which has the same magnitude as \( \vec{b} \), but is rotated through an angle \( \theta \) about \( \mu \) in a positive sense.

We now seek an operator which will rotate \( \vec{b}' \) into \( \vec{b} \). Consider Fig. 2.1.2 which differs from Fig. 2.1.1 in that a perpendicular is drawn from CB to point A instead of from
\[ \mathbf{b} = \mathbf{OC} + \mathbf{CA} \]
\[ = (\hat{\mathbf{u}} \cdot \mathbf{b'}) \hat{\mathbf{u}} + \mathbf{CA}. \]  \hspace{1cm} 2.1.7

The vector from \( \mathbf{C} \) to \( \mathbf{A} \) can be expressed as
\[ \mathbf{CA} = \mathbf{CE} + \mathbf{EA} \]
\[ = \cos\theta \frac{\mathbf{CA}}{|\mathbf{CB}|} + \sin\theta \frac{\mathbf{CA}}{|\mathbf{CB}|} (-\hat{\mathbf{u}} \times \mathbf{CB}). \]  \hspace{1cm} 2.1.8

As before \(|\mathbf{CA}|/|\mathbf{CB}| = 1 \) and \( \mathbf{CB} = \mathbf{b'} = (\hat{\mathbf{u}} \cdot \mathbf{b'}) \hat{\mathbf{u}} \), therefore
\[ \mathbf{b} = \hat{\mathbf{u}} \hat{\mathbf{u}} \cdot \mathbf{b'} + \cos\theta (\mathbf{b'} - \hat{\mathbf{u}} \hat{\mathbf{u}} \cdot \mathbf{b'}) - \sin\theta \hat{\mathbf{u}} \times \mathbf{b'} \]
\[ = (\hat{\mathbf{u}} \hat{\mathbf{u}} \cdot + \cos\theta (I-\hat{\mathbf{u}} \hat{\mathbf{u}} \cdot )-\sin\theta \hat{\mathbf{u}} \times ) \mathbf{b'}. \]  \hspace{1cm} 2.1.9
Using the matrix notation as given by Eqs. 2.1.4 and 2.1.5 this can be expressed as

\[ b = (I - \mu \mu^T + \cos^2(\mu \mu^T) - \sin \theta \mu \mu^T) b' \]

\[ \begin{bmatrix}
\mu_1^2(1-\cos \theta) + \cos \theta & \mu_1 \mu_2(1-\cos \theta) + \mu_3 \sin \theta & \mu_1 \mu_3(1-\cos \theta) - \mu_2 \sin \theta \\
\mu_2 \mu_1(1-\cos \theta) - \mu_3 \sin \theta & \mu_2^2(1-\cos \theta) + \cos \theta & \mu_2 \mu_3(1-\cos \theta) + \mu_1 \sin \theta \\
\mu_3 \mu_1(1-\cos \theta) + \mu_2 \sin \theta & \mu_3 \mu_2(1-\cos \theta) - \mu_1 \sin \theta & \mu_3^2(1-\cos \theta) + \cos \theta
\end{bmatrix} b' = R^T b' \]

where \( R^T \) is the inverse of \( R \). Examination of \( R \) and \( R^T \) as given in Eqs. 2.1.6 and 2.1.10 shows that

\[ R^T = R^T \]

which says that the inverse of the matrix rotation operator is its transpose and we can write

\[ R^T R = R R^T = I \]

This condition implies that \( R \) is orthonormal. Eq. 2.1.3 is known as Rodrigues formula, and in matrix form as given by Eq. 2.1.6, is an orthonormal operator which rotates an arbitrary vector, in this case \( b \), through angle \( \theta \) about a directed unit rotation axis, \( \mu \), in a positive sense, to give \( b' \).

2.2 Body Rotations and Coordinate System Transformations

The rotation of a body may be uniquely specified by the rotation of a coordinate basis rigidly embedded in the body. The rotation of the basis is equivalent to the
simultaneous rotation of three independent vectors and consequently the rotation and transformation properties that hold for the basis hold for individual vectors and vice versa. For the sake of conciseness and better physical insight the following development will be carried out with operations only on one vector, but keeping in mind that these same operations hold for the total basis as well.

Consider the rotation of a vector which we denote by \( r^1 \) which is embedded in a body, moves with the body and is specified in the reference system denoted by the superscript \( r \) in parenthesis. The resulting vector after rotation is given by

\[
r_2 = R(r)r_1.
\]

This rotation has produced a new body orientation in which the body basis or coordinate system is no longer aligned with the reference system.

We now seek a transformation for vector components between the reference system and the body system, the orientation of which was produced by \( R \). We assume a linear transformation of the form

\[
r(b) = A^br^r.
\]

where \( A^br \) is a 3 x 3 matrix, \( r^r \) is a vector in the reference system and \( r(b) \) is the same vector in the body system. We apply the condition that the components of the vector embedded in the body do not change during the rotation, that is, \( r_1 = r_2 \). Using Eqs. 2.2.1 and
2.2.2 and applying this condition we get

\[ r_2^\dagger (b) = b^R r_2^R = b^R R^R r_1^R (r) = b^R R^R r_2^R (b) \]

which implies that

\[ b^R R^R (r) = I \]

Multiplying from the right by \( R^{(r)T} \) we find that

\[ b^R R^{(r)T} \]

This shows that the coordinate transformation is also an orthonormal matrix operator. It also shows that the transpose of the rotation operator which rotates a body from orientation (1) to orientation (2) is the transformation matrix which acts on the components of a vector given in the (1) orientation to give the same vector components in the (2) orientation.

We next consider several sequential rotations and the properties of the transformations between various orientations of the coordinate systems. We denote a matrix rotation operator by \( R^{(a)}_{i \mu} (\phi) \), where the superscript denotes the basis or coordinate system in which the rotation is performed, the subscript \( i \) is the rotation order, \( \mu \) is the axis about which the rotation is made and \( \phi \) is the angle through which the rotation is taken.

Consider a rotation in the (0) basis of a vector \( r_0^{(0)} \) about the \( \mu \) axis through angle \( \phi \), to produce a vector \( r_1^{(0)} \) in the same basis. This rotation is denoted
by
\[ \mathbf{r}_1^0 = R_{1\mu}^0 (\phi) \mathbf{r}_0 \]  \hspace{1cm} 2.2.4

and according to Eqs. 2.2.2 and 2.2.3
\[ \mathbf{r}_1^1 = R_{1\mu}^0 T \mathbf{r}_1^0 \]  \hspace{1cm} 2.2.5

which relates the components of a vector \( \mathbf{r} \) between the \((0)\) and \((1)\) bases. We now consider a second rotation of the body embedded vector in the \((1)\) basis about \(\nu\) and through angle \(\phi_2\). The resulting vector in basis \((1)\) is
\[ \mathbf{r}_2^1 = R_{1\nu}^1 (\phi_2) \mathbf{r}_1^1 \]  \hspace{1cm} 2.2.6

and according to Eqs. 2.2.2 and 2.2.3
\[ \mathbf{r}_2^2 = R_{1\nu}^1 T \mathbf{r}_2^1 \]  \hspace{1cm} 2.2.7

This same rotation can also be performed in the \((0)\) basis by
\[ \mathbf{r}_2^0 = R_{1\gamma}^0 (\phi_2') \mathbf{r}_1^0 \]  \hspace{1cm} 2.2.8

where \(\gamma\) is the axis of rotation and \(\phi_2'\) the angle of rotation in the \((0)\) basis.

Both \(R_{1\nu}^1\) and \(R_{1\gamma}^0\) generate rotations which result in the same body orientation. To find how they are related we apply the condition that the components of the embedded body vector do not change during the rotation; i.e.
\[ \mathbf{r}_2^1 = \mathbf{r}_0 \]. Sequentially applying Eqs. 2.2.7, 2.2.5, 2.2.8 and 2.2.4 we get
but since the components of the embedded vector do not change this implies that

\[
R_{2y}^{(1)} = R_{1\mu}^{(0)} R_{2\gamma}^{(0)} R_{1\mu}^{(0)} R_{2\gamma}^{(0)} \tag{2.2.10}
\]

and we see that \( R_{2y}^{(1)} \) and \( R_{2\gamma}^{(0)} \) are related by a similarity transformation. By taking the trace of the rotation operator given by Eq. 2.1.6 we find that

\[
\text{trace } R = 1 + \cos \theta, \tag{2.2.11}
\]

but since the trace of a matrix is invariant under a similarity transformation we can equate

\[
\text{trace } R_{2y}^{(1)}(\phi_2) = \text{trace } R_{2\gamma}^{(0)}(\phi_2) \tag{2.2.11}
\]

and then by substituting from Eq. 2.2.11 we find

\[
1 + \cos \phi_2 = 1 + \cos \phi_2
\]

or

\[
\phi_2 = \phi_2.
\]

Which says that the angle of the rotation is independent of the coordinate system in which it is taken.

Taking a third rotation of the body embedded vector in the (2) basis about \( \alpha \) and through angle \( \phi_3 \) we have

\[
\tau_3^{(2)} = R_{3\alpha}^{(2)}(\phi_3) \tau_2^{(2)} \tag{2.2.12}
\]

and according to Eqs. 2.2.2 and 2.2.3
The same rotation performed in the \((0)\) basis is given by

\[
\mathbf{r}_3^{(0)} = \mathbf{r}_3^{(0)}(\phi_3) \mathbf{r}_2^{(0)}. \tag{2.2.14}
\]

Using the same method as for two successive rotations, we find that \(R_{3\alpha}^{(2)}\) and \(R_{3\delta}^{(0)}\) are related by

\[
R_{3\alpha}^{(2)} = R_{2\nu}^{(1)} R_{1\mu}^{(0)} R_{3\delta}^{(0)} R_{1\mu}^{(0)} R_{2\nu}^{(1)}
= (R_{1\mu}^{(0)} R_{2\nu}^{(1)}) R_{3\delta}^{(0)} (R_{1\mu}^{(0)} R_{2\nu}^{(1)}). \tag{2.2.15}
\]

Again the rotation operators are related by a similarity transformation with the transformation operator expressed in the body system. By using Eq. 2.2.10 the similarity transformation can be expressed in terms of rotations given in the initial unrotated system by

\[
R_{3\alpha}^{(2)} = (R_{2\gamma}^{(0)} R_{1\mu}^{(0)}) R_{3\delta}^{(0)} (R_{2\gamma}^{(0)} R_{1\mu}^{(0)}). \tag{2.2.16}
\]

We can express the components of the triply rotated vector in terms of three rotations given in the original \((0)\) coordinate system by

\[
\mathbf{r}_3^{(0)} = R_{3\delta}^{(0)}(\phi_3) R_{2\gamma}^{(0)}(\phi_2) R_{1\mu}^{(0)}(\phi_1) \mathbf{r}_0^{(0)}. \tag{2.2.17}
\]

By substituting from Eq. 2.2.10 for \(R_{2\gamma}^{(0)}\) and Eq. 2.2.15 for \(R_{3\delta}^{(0)}\) these vector components can be expressed in terms of body rotations as
The corresponding component transformations are according to Eq. 2.2.5 given by

\[ r_3 (0) = R_{1\mu} (\phi_1) R_{2\nu} (\phi_2) R_{3\alpha} (\phi_3) r_0 (0) . \quad 2.2.18 \]

\[ r^+(3) = R_{1\mu} (\phi_1) T_{2\nu} (0) (\phi_2) T_{3\delta} (0) (\phi_3) T_\gamma (0) \quad 2.2.19 \]

and

\[ r^+(3) = R_{3\alpha} (\phi_3) T_{2\nu} (1) (\phi_2) T_{1\mu} (0) (\phi_1) T_\gamma (0) . \quad 2.2.20 \]

### 2.3 Rotation Matrix from Elementary Rotations

The rotational orientation of a three-dimensional body can be uniquely specified by three parameters. In Sec. 2.1 we developed a rotation operator in which the three independent parameters were the angle of rotation and a three-dimensional unit vector specifying the axis of rotation and consisting of two independent parameters. It is also possible to specify an orientation as resulting from three elementary rotations, where an elementary rotation is defined as a rotation taken about one of the coordinate axes, where no two sequential rotations are taken about the same axis. These types of rotation specifications are often convenient because the elementary rotation operators have relatively simple explicit forms. Additionally, motions of bodies in space are often restricted by certain external constraints. These constraints may allow the rotational motion to be separated or factored into components about well defined axes.
Various combinations of rotation sequences have been developed in the literature. The most familiar ones are the Euler angles which have been traditionally used to study gyroscopic dynamics. The Krylov angles is another set which is equivalent to the yaw, roll and pitch angles used in aircraft and ship attitude specifications. We first want to show that three non-consecutive rotations by elementary rotation operators lead to a rotation operator specifying a body rotation about an arbitrary axis \( \hat{u} \) as given by Eq. 2.1.6.

Consider a rotation of a body about the \( \hat{u} \) axis through angle \( \vartheta \) as shown in Figure 2.3.1. To carry out this rotation we will first perform a rotation about \( Z^{(0)} \) through an angle \( \phi_1 \) which will place the original \( X^{(0)} \) axis into the plane common to \( Z^{(0)} \) and \( \hat{u} \). This elementary rotation is given by

\[
R_{1Z}^{(0)}(\phi_1) = \begin{pmatrix} \cos \phi_1 & -\sin \phi_1 & 0 \\ \sin \phi_1 & \cos \phi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

The next rotation is taken about the current body \( Y \) axis, denoted by \( Y^{(1)} \), through angle \( -\phi_2 \) which aligns the original \( X \) axis with \( \hat{u} \). This elementary rotation is given by

\[
R_{2Y}^{(1)}(-\phi_2) = \begin{pmatrix} \cos \phi_2 & 0 & -\sin \phi_2 \\ 0 & 1 & 0 \\ \sin \phi_2 & 0 & \cos \phi_2 \end{pmatrix}.
\]
The body X axis, denoted by $X^{(2)}$, is now aligned with $\hat{\mu}$ and the desired body rotation about $\hat{\mu}$ through angle $\theta$ is equivalent to a body rotation about its X axis through angle $\theta$ and is given by the elementary rotation operator

$$R_{3X}^{(2)}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}.$$ 

These elementary rotation operators have been given in the body system, but the total rotation operator as given by Eq. 2.1.6 must be expressed in the original fixed system.
This is accomplished by inverting Eq. 2.2.15 and substituting the rotation operators given by Eqs. 2.3.1, 2.3.2, and 2.3.3 to give

$$R_{3\mu}^{(0)}(\theta) = R_{1Z}^{(0)}(\phi_1) R_{2Y}^{(1)}(-\phi_2) R_{3X}^{(2)}(\theta) R_{2Y}^{(1)} T(-\phi_2) R_{1Z}^{(0)} T(\phi_1).$$

2.3.4

By expanding this and observing from Fig. 2.3.1 that

$$\mu = [\cos\phi_1 \cos\phi_2, \sin\phi_1 \cos\phi_2, \sin\phi_2]$$

we get

$$R_{3\mu}^{(0)}(\gamma) =$$

$$\begin{bmatrix}
  \mu_1^2(1-\cos\theta) + \cos\theta & \mu_1\mu_2(1-\cos\theta) - \mu_3 \sin\theta & \mu_1\mu_3(1-\cos\theta) + \mu_2 \sin\theta \\
  \mu_2\mu_1(1-\cos\theta) + \mu_3 \sin\theta & \mu_2^2(1-\cos\theta) + \cos\theta & \mu_2\mu_3(1-\cos\theta) - \mu_1 \sin\theta \\
  \mu_2\mu_1(1-\cos\theta) - \mu_3 \sin\theta & \mu_3\mu_2(1-\cos\theta) + \mu_1 \sin\theta & \mu_3^2(1-\cos\theta) + \cos\theta
\end{bmatrix}.$$

2.3.6

The significance of this development is the demonstration of the equivalence of a rotation specification in terms of three elementary rotation matrices to that given in terms of an axis of rotation and the angle of rotation about that axis. This latter method of rotation specification will be further developed and exploited in the following development.

It is worthwhile, however, to dwell on the properties and interpretation of Eq. 2.3.4. The equation consists of three rotational operators and three angles of rotation.
These three angles are the three independent parameters required to uniquely specify a body's rotational position in three-dimensional space. By the use of Eq. 2.3.5 we eliminated the angles \( \phi_1 \) and \( \phi_2 \) and substituted \( \hat{\mu} \) which resulted in the rotation operator being specified by the parameters \( \theta \) and \( \mu_1, \mu_2, \) and \( \mu_3 \) with the constraint \( \mu_1^2 + \mu_2^2 + \mu_3^2 = 1 \) which again left three independent parameters as required for the rotational position description.

A direct derivation of Eq. 2.3.4 may also be obtained by first noting that for an orthonormal rotation matrix

\[
\mathbf{R}(-\theta) = \mathbf{R}(\theta)^T
\]

2.3.7

which follows directly from Eq. 2.1.6. Using this result we can rewrite Eq. 2.3.4 as

\[
\mathbf{R}^{(0)}_{31}(\theta) = \mathbf{R}^{(0)}_{12}(\phi_1)\mathbf{R}^{(1)}_{21}(\phi_2)\mathbf{R}^{(2)}_{12}(\theta)\mathbf{R}^{(1)}_{21}(\phi_2)\mathbf{R}^{(0)}_{12}(\phi_1) .
\]

This equation could have been directly written if we had taken the following sequence of rotations in the \((0)\) system. We first perform a rotation of the body (and \( \hat{\mu} \)) about \( z^{(0)} \) through angle \( -\phi_1 \) which places \( \hat{\mu} \) into the \( X^{(0)}Z^{(0)} \) plane. We next rotate about the \( Y^{(0)} \) axis through angle \( \phi_2 \) which puts \( \hat{\mu} \) into coincidence with \( X^{(0)} \). The \( \theta \) angle rotation is now applied as a rotation about the \( X^{(0)} \) axis which coincides with \( \hat{\mu} \). The body is now rotated back to its original position by applying a rotation about \( Y^{(0)} \) through angle \( -\phi_2 \) and about \( Z^{(0)} \) through angle \( \phi_1 \).
This latter method of deriving Eq. 2.3.4 provides a geometric interpretation for the rotation operator specified in terms of rotations about given coordinate axes.

It is also pointed out that Eq. 2.3.4 is a similarity transformation with the two rotation operators corresponding to angles $\phi_1$ and $\phi_2$ forming the similarity operator.

2.4 Some Properties of the Rotation Matrix Operator

We will now briefly consider some additional properties of the rotation matrix operator. As shown in Sec. 2.2, the angle of rotation can be found from the rotation matrix from

$$\text{trace } R = r_{11} + r_{22} + r_{33} = 1 + 2 \cos \theta$$  \hspace{1cm} 2.4.1

where $r_{ij}$ are the elements of $R$.

By taking the differences of the off diagonal elements of $R$ as given by Eq. 2.1.6, we further find that

$$u_1 = \frac{r_{32} - r_{23}}{2 \sin \theta}$$

$$u_2 = \frac{r_{13} - r_{31}}{2 \sin \theta}$$

$$u_3 = \frac{r_{21} - r_{12}}{2 \sin \theta}.$$  \hspace{1cm} 2.4.2
Since the rotation operator rotates vectors about \( \hat{u} \), the \( \hat{u} \) vector itself remains unchanged if acted upon by the \( R \) operator. This can be directly demonstrated by the use of Eq. 2.1.6 which yields

\[
R \hat{u} = \hat{u} \quad 2.4.3
\]

This means that \( \hat{u} \) is an eigenvector of \( R \). A more general approach to the calculation of the rotation axis vector, if the explicit form of the rotation operator as given by Eq. 2.1.6 is not assumed, is to solve the eigenvalue and eigenvector equation

\[
\hat{R}v = \lambda v \quad 2.4.4
\]

It is shown by Goldstein[49] that if \( R \) is a 3 x 3 orthonormal matrix it possesses three eigenvalues of unit magnitude. He further demonstrates that only the +1 eigenvalue corresponds to a physically realizable body rotation. The solution for the eigenvector corresponding to the eigenvalue +1 results in an eigenvector \( \hat{u} \) which is proportional to \( \hat{u} \) as given by Eq. 2.4.3.

The rotation matrix \( R \) must be given in a specific basis or coordinate system. The components of \( \hat{u} \) in \( R \) must be expressed with respect to that coordinate system. If \( R \) is to be expressed in a different coordinate system it must be transformed by a similarity transformation as discussed in Sec. 2.2. This transformation will in effect transform the \( \hat{u} \) rotation vector which is implicitly embedded in the \( R \) matrix to the new coordinate system.
The similarity transformation, however, will not affect the angle of rotation which remains invariant. This fact was assumed in the development in Sec. 2.2, but is proven here for completeness.

We first show that the trace of the product of two matrices is invariant under commutation

\[
\text{trace} (AB) = \text{trace} \left( \sum_{j=1}^{3} a_{ij} b_{jk} \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} b_{ji} \]

\[
= \text{trace} \left( \sum_{i=1}^{3} b_{ji} a_{ik} \right) = \text{trace} (BA).
\]

2.4.5

We now take the trace of the similarity transformation of \( R' \) by \( C \) where \( C \) is an orthonormal transformation.

\[
\text{trace}(R) = \text{trace} \left( C^T R' C \right) = \text{trace} \left( C (C^T R') \right)
\]

\[
= \text{trace} \left( C C^T R' \right) = \text{trace} (R')
\]

2.4.6

where we have used the results of Eq. 2.4.5 and that \( C C^T = 1 \).

2.5 Quaternion Properties

A quaternion is a four dimensional number first introduced by Hamilton in 1843. It can be represented by the sum of a scalar and a three-dimensional vector as
where \( q_0, q_1, q_2, \) and \( q_3 \) are real numbers and the \( \hat{e}_1(0), \hat{e}_2(0) \) and \( \hat{e}_3(0) \) can be viewed as imaginary units which in three-dimensional space correspond to the unit basis vectors with \( 0 \) denoting the specific basis.

Two quaternions \( q = (q_0, q_1, q_2, q_3) \) and \( q' = (q'_0, q'_1, q'_2, q'_3) \) are equal, that is \( q = q' \), if and only if \( q_0 = q'_0, q_1 = q'_1, q_2 = q'_2 \) and \( q_3 = q'_3 \).

Addition is defined by 
\[
q + q' = (q_0 + q'_0, q_1 + q'_1, q_2 + q'_2, q_3 + q'_3)
\]

and multiplication by a scalar \( \alpha \) obeys 
\[
\alpha q = (\alpha q_0, \alpha q_1, \alpha q_2, \alpha q_3)
\]

The negative is given by \( -q = (-1)q \) from which follows the subtraction property 
\[
q - q' = (q_0 - q'_0, q_1 - q'_1, q_2 - q'_2, q_3 - q'_3)
\]

For quaternion multiplication the product of two quaternions \( p \) and \( q \) is denoted by \( pq \). The quaternion products of unit vectors are defined as shown in Table 2.5.1 where the column elements are multiplied from the left into the top row elements to give the product result.
We now consider an explicit expression for the product of \( p \) and \( q \). For the time being dropping the basis designation we can write

\[
pq = (p_0 + p_1 \hat{e}_1 + p_2 \hat{e}_2 + p_3 \hat{e}_3)(q_0 + q_1 \hat{e}_1 + q_2 \hat{e}_2 + q_3 \hat{e}_3)
\]

\[
= p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 + p_0(q_1 \hat{e}_1 + q_2 \hat{e}_2 + q_3 \hat{e}_3)
\]

\[
+ q_0(p_1 \hat{e}_1 + p_2 \hat{e}_2 + p_3 \hat{e}_3) + \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ e_1 & e_2 & e_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} 2.5.5
\]

which is a sum of a scalar and a vector and, consequently, is a quaternion.

By forming the quaternion product

\[
pq = (p_0 + p)(q_0 + q) = p_0q_0 + p_0q + q_0p + pq
\]

and comparing this to Eq. 2.5.5 we find that the quaternion
The product of two vectors is
\[ \mathbf{p} \mathbf{q} = \mathbf{p} \times \mathbf{q} - \mathbf{p} \cdot \mathbf{q} . \]  \hfill (2.5.6)

The conjugate of a quaternion \( q \) is denoted by \( q^* \) and is defined by
\[ q^* = q_0 - q = q_0 - q_1 \mathbf{e}_1 - q_2 \mathbf{e}_2 - q_3 \mathbf{e}_3 . \]  \hfill (2.5.7)

From this it follows that
\[ q q^* = (q_0 + \mathbf{q})(q - \mathbf{q}) = q_0^2 + q_0 q - q_0 q - q q^* \]
\[ = q_0^2 + q \cdot q \]
\[ = N(q) \]  \hfill (2.5.8)

where \( N(q) \) is the norm of \( q \). Note also that \( qq^* = q^*q \).

From Eq. 2.5.8 we can write
\[ q \frac{q^*}{N(q)} = 1 \]
and we see that the inverse of \( q \) is
\[ q^{-1} = \frac{q^*}{N(q)} . \]  \hfill (2.5.9)

If the quaternion has \( N(q) = 1 \) it is called a unit or normalized quaternion and \( q^{-1} = q^* \). The magnitude of \( q \) is the square root of the norm, i.e.
\[ |q| = \sqrt{N(q)} = \sqrt{qq^*} . \]  \hfill (2.5.10)
2.6 Quaternions as Rotation Operators

We will now consider vector and normalized quaternion products. Let \( q \) be a quaternion with \( N(q) = 1 \) and take

\[
q x = (q_0 + q)x = (q_0 + q)(x_P + x_T)
\]

2.6.1

where \( x_P \) and \( x_T \) are the components of \( x \) parallel and transverse to \( q \), respectively. We first consider the transverse component and use the quaternion vector multiplication rule given by Eq. 2.5.6

\[
(q_0 + q)x_T = q_0x_T + qx_T = q_0x_T + q \times x_T - q \cdot x_T
\]

2.6.2

The last term equals zero since \( \hat{q} \) and \( \hat{x}_T \) are perpendicular, and we see that the result is a vector perpendicular to \( \hat{q} \). We denote this vector by \( \hat{x}_T ' \) as shown in Fig. 2.6.1 and examine its magnitude by taking

\[
(q_0^2 + q^2)^{1/2} = (q_0x_T^2 + 2q_0x_T \cdot (q \times x_T) + (q \times x_T) \cdot (q \times x_T))^{1/2}
\]

\[
= (q_0^2x_T^2 + q^2x_T^2 - (q \cdot x_T)^2)^{1/2}
\]

\[
= (q_0^2 + q^2)^{1/2} |x_T|
\]

2.6.3

since \( N(q) = 1 \). Thus we see that the result of multiplying a vector from the left by a quaternion is to change its transverse component with respect to \( q \) to \( q_0x_T + q \times x_T \) which is a vector also transverse to \( q \), is rotated in a
positive sense about $\hat{q}$ through an angle $\alpha$ given by

$$\alpha = \tan^{-1} \left( \frac{\hat{q} \times \hat{x}_T}{q_0 |\hat{x}_T|} \right) = \tan^{-1} \left( \frac{\hat{q}}{q_0} \right), \quad 2.6.4$$

since $|\hat{q} \times \hat{x}_T| = |\hat{q}| |\hat{x}_T|$, and has the same magnitude as $\hat{x}_T$ according to Eq. 2.6.3.

We now consider the effect of multiplying a vector by the conjugate quaternion from the right

$$\hat{x}_T^\ast = (\hat{x}_p + \hat{x}_T)(q_0 - \hat{q}) \quad . \quad 2.6.5$$
Considering only the transverse component we get

$$\mathbf{x}_T(q_0 - q) = \mathbf{x}_Tq_0 - \mathbf{x}_Tq = q_0\mathbf{x}_T + q \times \mathbf{x}_T + q \cdot \mathbf{x}_T. \quad 2.6.6$$

The last term is zero since $q$ and $\mathbf{x}_T$ are perpendicular and we see that the resultant vector is identical to that obtained in Eq. 2.6.2. We thus conclude that the result of a double application of a quaternion from the left and its conjugate from the right on the transverse component of a vector is to produce a rotation of the transverse component through a total angle of $2\alpha$ which we denote by $\theta$.

From Eq. 2.6.4 we can write that

$$\theta = 2 \tan^{-1} \left| \frac{q}{q_0} \right|. \quad 2.6.7$$

We now examine how the vector component of $\mathbf{x}$ parallel to $\mathbf{q}$ is transformed under quaternion multiplication.

From Eq. 2.6.1 for the parallel $\mathbf{x}$ component we have

$$(q_0 + q)\mathbf{x}_p = q_0\mathbf{x}_p + q\mathbf{x}_p = q_0\mathbf{x}_p + q \times \mathbf{x}_p - q \cdot \mathbf{x}_p$$

$$= q_0\mathbf{x}_p - q \cdot \mathbf{x}_p. \quad 2.6.8$$

since $\mathbf{q}$ and $\mathbf{x}_p$ are parallel. The result is a quaternion with a vector component parallel to $\mathbf{q}$. We now multiply this quaternion from the right by $q^*$.
\[(q_0 \times q - q \cdot x_p)(q_0 - q) = q_0 x_p - q_0 x_p q - q - x_p q_0 + q \cdot x_p q\]

\[= q_0 x_p - q_0 (x_p \times q) + q_0 x_p \cdot q - q_0 q \cdot x_p + q q \cdot x_p\]

\[= q_0 x_p + q q \cdot x_p\]

\[= (q_0^2 + q^2) x_p = N(q) x_p = x_p\]  \hspace{1cm} 2.6.9

where we have used \[\bar{x} \cdot x_p = q^2 x_p\] which follows since \[x_p\] is parallel to \[q\], and we see that the component of \[x\] parallel to \[q\] remains unchanged.

In summary, the effect of the operation

\[x_1 = qx_0 q^*\]  \hspace{1cm} 2.6.10

by the unit quaternion \[q\] on a vector \[x_0\] is to produce another vector \[x_1\] which has the same magnitude as \[x_0\], but is rotated through an angle \(\theta\) in a positive sense about \[q\].

2.7 Equivalence of Quaternion and Matrix Operations

In the last section we demonstrated that if \[q\] is a normalized quaternion and \[x_0\] is any vector then

\[x_1 = qx_0 q^*\]  \hspace{1cm} 2.7.1

where \[x_1\] has the same magnitude as \[x_0\], but is rotated at a positive sense about \[q\] through angle \(\theta\) which is given by Eq. 2.6.7.

We want to show that this quaternion operator is equivalent to the matrix rotation operator given by Eq. 2.1.6. We proceed by expanding Eq. 2.7.1 and using the
quaternion algebraic properties given in Sec. 2.5

$$x_1 = (q_0 + q)x_0 (q_0 - q) = (q_0 + q)(q_0x_0 - x_0 \times q + x_0 \cdot q)$$

$$= (q_0^2x_0 - q_0(x \times q) + q_0(x \cdot q) + q_0q\dot{x}_0 = q(x_0 \times q) + q(x_0 \cdot q))$$

$$= q_0^2x_0 + 2q_0(q \times x_0) + q \cdot x_0) \times -q^2x_0 + q(q \cdot x_0)$$

$$= q_0^2x_0 + 2q_0(q \times x_0) + 2q(q \cdot x_0) - q^2x_0 \quad .$$ 2.7.2

Using the scalar and vector product matrix forms as given by Eqs. 2.1.4 and 2.1.5 this can be expressed as

$$x_1 = \begin{pmatrix} q_0^2 & -q_0 & 0 & 0 \\ 0 & q_0^2 & -q_0 & 0 \\ 0 & 0 & q_0^2 & -q_0 \end{pmatrix} x_0 \begin{pmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{pmatrix} x_0 \begin{pmatrix} q_1q_1 & q_1q_2 & q_1q_3 \\ q_2q_1 & q_2q_2 & q_2q_3 \\ q_3q_1 & q_3q_2 & q_3q_3 \end{pmatrix} x_0$$

or

$$\begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 2(q_1q_2 - q_0q_3) \\ 0 \end{bmatrix} \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 \\ 2(q_1q_1 - q_0q_3) \\ 2(q_1q_3 + q_0q_2) \\ 2(q_2q_3 - q_0q_1) \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} = R \times x_0 \quad .$$ 2.7.4

We have shown that the magnitude of $x$ is preserved during a quaternion rotation. Taking the scalar product of $x_1$ with itself in matrix notation and using Eq. 2.7.4
but since we must have \(|\mathbf{x}_1| = |\mathbf{x}_0|\), it follows that

\[
\mathbf{R}^T \mathbf{R} = \mathbf{I}
\]

which shows that \(\mathbf{R}\), as expected, is orthonormal and is equivalent to the rotation matrix given by Eq. 2.1.6 except that the rotation parameters are the quaternion elements.

Consider the trace of the quaternion rotation matrix

\begin{align*}
\text{trace}\mathbf{R} &= 3q_0^2 - q_1^2 - q_2^2 - q_3^2 = 4q_0^2 - 1 \quad . \\
\end{align*}

Equating this to the trace of the rotation matrix as given by Eq. 2.4.1 we get

\[
4q_0^2 - 1 = 1 + 2\cos \theta
\]

and solving for \(q_0\)

\[
q_0 = \pm \cos \frac{\theta}{2}
\]

By substituting the matrix elements of \(\mathbf{R}\) from Eq. 2.7.4 into the unit rotation axis vector given by Eq. 2.4.2 and using Eq. 2.7.9 we get
Substituting for the quaternion elements $q_0$, $q_1$, $q_2$ and $q_3$ from Eqs. 2.7.9 and 2.7.10 into Eq. 2.5.1 the quaternion can be expressed as

$$q = \pm \left( \cos \frac{\theta}{2} + \left( q_1 \hat{e}_1 + q_2 \hat{e}_2 + q_3 \hat{e}_3 \right) \right)$$

$$= \pm \left( \cos \frac{\theta}{2} + \hat{q} \sin \frac{\theta}{2} \right) \ . \ 2.7.11$$

This shows that the rotation axis or rotation eigenvector is parallel to the vector part of the rotation quaternion. We now examine the consequences of the sign of Eq. 2.7.11. First choosing the positive sign, we see that $\hat{q} = \hat{q} \sin \frac{\theta}{2}$ points in the direction of a positive rotation and the angle of the rotation is given by Eq. 2.6.7

$$\theta = 2\tan^{-1} \left( \frac{|q|}{q_0} \right) = 2\tan^{-1} \frac{|\hat{q} \sin \frac{\theta}{2}|}{\cos \frac{\theta}{2}}$$

which is satisfied only for positive $\theta$. 
If we choose the negative sign the direction of a positive rotation is given by

\[ q = -\hat{\omega} \sin \frac{\theta}{2} \]

and the angle of the rotation is given again by Eq. 2.6.7

\[ \theta = 2\tan^{-1}\left(\frac{|-\hat{\omega} \sin \frac{\theta}{2}|}{-\cos \frac{\theta}{2}}\right) \]

\[ = -2\tan^{-1}\left(\frac{|-\hat{\omega} \sin \frac{\theta}{2}|}{\cos \frac{\theta}{2}}\right) \]

which is satisfied only for negative \( \theta \).

The negative angle of rotation is equivalent to a positive angle of rotation for a rotation vector oppositely directed. Thus, the sign of the quaternion is arbitrary in defining a rotation and in further development we choose the positive sign.

We can apply the quaternion rotation operator to any vector including the basis vectors defining a coordinate system. As with rotations with matrix operators in Sec. 2.2 we choose to examine the transformation properties of quaternions using a single vector rather than the total triad of basis vectors.

According to Eq. 2.6.10, the rotation of a vector embedded in a body can be expressed by

\[ r_1(0) = q(0)^* r_0 q(0) \]

\[ 2.7.12 \]

where the superscript denotes the basis in which the quaternion and the basis vectors are prescribed. We seek a transformation using quaternions which would allow us to
transform components of vectors between different bases produced by body rotations. That such a transformation exists and its form are strongly suggested by the demonstrated equivalence of the quaternion rotation operation to the matrix rotation operation and the similarity of form of the matrix transformation to the matrix rotation operator. More specifically, according to Eqs. 2.2.4, 2.2.5 and 2.1.11, one is the inverse of the other. We assume a quaternion transformation for vector components of the form

\[ r^{(1)} = p_1 r^{(0)} p_2 \]  

2.7.13

where \( r^{(0)} \) are components in the (0) basis, \( r^{(1)} \) are the components in the (1) basis and the \( p^{(0)} \)'s are quaternions. Since the components of a vector embedded in the body do not change when the body is rotated it follows that \( r^{(1)}_0 = r^{(0)} \). Writing Eq. 2.7.13 for \( r^{(1)}_1 \) and substituting for \( r^{(0)} \) from Eq. 2.7.12 we get

\[ r^{(1)}_1 = p_1 r^{(0)}_1 p_2 = p_1 q^{(0)} r^{(0)}_1 q^{(0)*} p_2 \]  

2.7.14

Since \( r^{(1)}_1 = r^{(0)}_0 \), Eq. 2.7.14 is satisfied if

\[ p_1^{(0)} q^{(0)} = 1 \]

and

\[ p^{(0)*} p_2^{(0)} = 1 \]

Using Eq. 2.5.9 with \( N(q) = 1 \) we get that

\[ p_1^{(0)} = q^{(0)*} \]

and

\[ p_2^{(0)} = q^{(0)} \]

and the component transformation Eq. 2.7.13 becomes

\[ r^{(1)} = q^{(0)*} r^{(0)} q^{(0)} \]  

2.7.15
Here, as for the matrix rotation operator, the vector component transformation equation is the inverse of the rotation operation.

The rotation operation on a vector by quaternions is given in terms of quaternions in the same basis as the vector. This poses no problem if all the rotations are made in the original or space fixed system. However, it is often desirable to specify a rotation in one of the rotated systems. To examine how this can be done we consider the effect of a quaternion rotation operation on a quaternion. If $p$ and $q$ are quaternions then

\[
q^*(0)p(0)q(0)^* = q^*(0)(p_0 + p_1^*(0))q(0)^* = q^*(0)p_0q(0)^* + q^*(0)p_1^*(0)q(0)^*
\]

\[
= p_0 + p_1 = p_1^{(0)}
\]

where the resultant quaternion has the same scalar term, but the vector component, $p_1^{(0)}$, is the $p_1^{(0)}$ vector rotated about $q(0)$.

We now consider the operation

\[
q(0)^*p(0)q(0) = q(0)^*p_0q(0) + q(0)^*p_1q(0)
\]

\[
= p_0 + p_1^{(1)} = p_1^{(1)}q
\]

which is the original $p(0)$ quaternion in the $(1)$ basis generated by the $q$ quaternion.

We conclude that quaternions transform as vectors, i.e.

\[
p_1^{(1)} = q(0)^*p(0)q(0)
\]
2.8 Sequential Rotations and Transformations Using Quaternions

We denote a quaternion vector rotation operation from Eq. 2.7.1 as follows

\[ r_1 = q_{1u}^0 (\phi) r_0 q_{1u}^0 (\phi)^* \]

where the superscripts denote the basis in which the vectors and quaternions are given, the first quaternion subscript denotes the sequence of rotation application, 1 being the first rotation, 2 the second, etc., \( u \) the axis of positive rotation and \( \phi \) the angle through which the rotation is taken. Using the same notation Eq. 2.7.15 becomes

\[ r_1 = q_{1u}^0 (\phi) * r_0 q_{1u}^0 (\phi) \]

Assuming the first rotation to have been made through an angle \( \phi_1 \), we now take a second rotation through angle \( \phi_2 \) about \( z \) in the \( (0) \) basis to get

\[ r_2 = q_{2\delta}^0 (\phi_2) r_1 q_{2\delta}^0 (\phi_2)^* \]

\[ = q_{2\delta}^0 (\phi_2) q_{1u}^0 (\phi_1) r_0 q_{1u}^0 (\phi_1)^* q_{2\delta}^0 (\phi_2)^* \]

This same rotation can be taken in the \( (1) \) basis about the \( \gamma \) axis.

\[ r_2 = q_{2\gamma}^0 (\phi_2) r_1 q_{2\gamma}^0 (\phi_2)^* \]

and the transformation of vector components between bases
(1) and (2) is then given according to Eq. 2.7.15 by

\[ r^-(2) = q_{2\delta}^{(1)}(\phi_2) r^+(1) q_{2\delta}^{(1)}(\phi_2) \quad 2.8.5 \]

Substituting for \( r^+(1) \) from Eq. 2.8.2 we get

\[ r^-(2) = q_{2\gamma}^{(1)}(\phi_2) q_{1\mu}^{(0)}(\phi_1) r^+(0) q_{1\mu}^{(0)}(\phi_1) q_{2\gamma}^{(1)}(\phi_2) \]

\[ = (q_{1\mu}^{(0)}(\phi_1) q_{2\mu}^{(1)}(\phi_2)^*) r^+(0) q_{1\mu}^{(0)}(\phi_1) q_{2\gamma}^{(1)}(\phi_2) \quad 2.8.6 \]

which is a vector component transformation in terms of the body or eigenquaternions. The transformation in terms of quaternions given in the fixed system can be obtained by the inverse operation of that given by Eq. 2.8.3 and is

\[ r^-(2) = (q_{2\delta}^{(0)}(\phi_2) q_{1\mu}^{(0)}(\phi_1) r^+(0) q_{2\delta}^{(0)}(\phi_2) q_{1\mu}^{(0)}(\phi_1)) \quad 2.8.7 \]

We note that the sequence of the operations is reversed when body and fixed referenced quaternions are used. Also, Eq. 2.8.7 can also be derived from Eq. 2.8.6 by using the quaternion transformation Eq. 2.7.18.

We now make a third rotation in the \((0)\) basis through angle \(\phi_3\) about \(\beta\)

\[ r_3^- = q_{3\delta}^{(0)}(\phi_3) q_{2\delta}^{(0)}(\phi_2) q_{1\mu}^{(0)}(\phi_1) r^+(0) q_{1\mu}^{(0)}(\phi_1) q_{2\delta}^{(0)}(\phi_2)^* \]

\[ = q_{3\delta}^{(0)}(\phi_3)^* \quad 2.8.8 \]
The inverse gives the vector component transformation

\[ r^{-1}(3) = (q_{3\beta}^{(0)}(\phi_3)q_{2\beta}^{(0)}(\phi_2)q_{1\mu}^{(0)}(\phi_1))^* r(0)(q_{3\alpha}^{(0)}(\tau_3)q_{2\gamma}^{(0)}(\tau_2)q_{1\mu}^{(0)}(\phi_1)) \]

Using the previous observation that the sequence of operations is reversed when the rotations are given in the body basis we can write

\[ r^{-1}(3) = (q_{1\mu}^{(0)}(\phi_1)q_{2\gamma}^{(1)}(\phi_2)q_{3\alpha}^{(2)}(\phi_3))^* r(0)(q_{1\mu}^{(0)}(\phi_1)q_{2\gamma}^{(1)}(\phi_2)q_{3\alpha}^{(2)}(\phi_3)) \]

where \( \alpha \) is the axis of the third rotation in the (2) basis.

This expression can be directly applied to calculate the Euler, Bryant, Krylov or other rotation matrices which are specified in terms of sequenced rotations about body coordinate axes. The Euler angles are given by sequential rotations about the body Z, X and Z axes; the Bryant angles about the body X, Y and Z axes; and the Krylov angles about the body Z, Y and X axes. The latter angles correspond to rotations often denoted also as yaw, roll and pitch.

When we derived the rotation and transformation properties of vectors using 3 x 3 matrix operators, we found that the matrix operators had to be transformed by similarity transformations between different bases. In using quaternions this duality of transformations is not necessary since the quaternions transform identically to
vectors as given by Eq. 2.7.18. For this reason all the vector component transformation equations developed in this section can also be used as quaternion transformation equations.

2.9 Kinematic Quaternion Equations

We consider the effect on a vector of a quaternion operator as given by Eq. 2.7.11 for a small rotation angle $\Delta\theta$. This quaternion we express as

$$\Delta q = 1 + \frac{\Delta\theta}{2} \hat{u} .$$

Applying this quaternion to the vector $r$ we get

$$r_1 = \Delta qr \Delta q^* = (1 + \frac{\Delta\theta}{2} \hat{u}) r (1 - \frac{\Delta\theta}{2} \hat{u})$$

$$= r + \frac{\Delta\theta}{2} (\hat{u} \times r) - \frac{\Delta\theta}{2} \hat{u} \cdot r - \frac{\Delta\theta^2}{4} \hat{u} (r \times \hat{u})$$

$$- \frac{\Delta\theta}{2} (r \times \hat{u}) + \frac{\Delta\theta}{2} \hat{u} \cdot \hat{u} - \frac{\Delta\theta^2}{4} (\hat{u} r)$$

$$= r + \Delta\theta (\hat{u} \times r)$$

where the second order terms in $\Delta\theta$ have been discarded.

The change in $r$ produced by this rotation vector is

$$\Delta r = r_1 - r = \Delta\theta (\hat{u} \times r) .$$

Taking this change per unit time we get

$$\frac{\Delta r}{\Delta t} = \frac{\Delta\theta}{\Delta t} (\hat{u} \times r)$$
and in the limit
\[ \frac{dr}{dt} = \omega \times r \]  
2.9.5

where
\[ \omega = \frac{d\theta}{dt} \]  
2.9.6

We can now write Eq. 2.9.1, with the definition of angular velocity given by Eq. 2.9.6, in differential form as
\[ dq = 1 + \frac{1}{2} \omega dt \]  
2.9.7

We will now consider the rotation of a body as a function of time in terms of the rotation of the basis vectors attached to the body. Initially, these basis vectors are coincident with the fixed basis vectors denoted by \( (0) \). The rotated basis vectors at time \( t \) are then given in the \( (0) \) basis by
\[ e_1^{(0)} = q^{(0)}(t)e_0^{(0)}q^{(0)}(t)^* \]  
2.9.8

We next apply an infinitesimal rotation quaternion \( dq^{(0)} \), as shown in Fig. 2.9.1, in the \( (0) \) basis which produces a further rotation through time \( dt \) resulting in the basis vector
\[ e_2^{(0)} = dq^{(0)}(dt)q^{(0)}(t)e_0^{(0)}q^{(0)}(t)^*dq^{(0)}(dt)^* \]  
2.9.9

The single quaternion which produces the total rotation to \( t+dt \) as shown in Fig. 2.9.1 is then given by
Figure 2.9.1 Equivalence of Quaternion Rotation
\( q(t + dt) \) to \( q(t) \) and \( dq \)

\[
q^{(0)}(t + dt) = dq^{(0)}(dt)q^{(0)}(t) .
\]

According to Eq. 2.7.15 the infinitesimal quaternion \( dq^{(0)} \) is given in the \((1)\) or body basis by

\[
dq^{(1)} = q^{(0)}(t) * dq^{(0)}q^{(0)}(t) .
\]

By multiplying this equation from the left by \( q^{(0)}(t) \) we see that \( q^{(0)}(t + dt) \) can also be expressed as

\[
q^{(0)}(t + dt) = q^{(0)}(t) dq^{(1)} .
\]

Eqs. 2.9.10 and 2.9.12 can be written in explicit form using Eq. 2.9.7

\[
q^{(0)}(t + dt) = (1 + \frac{1}{2} \omega^{(0)}dt)q^{(0)}(t)
\]

\[
= q^{(0)}(t)(1 + \frac{1}{2} \omega^{(1)}dt)
\]

where \( \omega^{(0)} \) is the angular velocity in the fixed basis and \( \omega^{(1)} \) is the angular velocity in the body basis and are given by
\[ \omega(0) = \omega(0)^\wedge(0) + \omega(0)^\wedge(0) + \omega(0)^\wedge(0) \]  
\[ 2.9.15 \]

and

\[ \omega(1) = \omega(1)^\wedge(1) + \omega(1)^\wedge(1) + \omega(1)^\wedge(1) \]  
\[ 2.9.16 \]

respectively.

From the definition of the derivative we can write

\[ \frac{dq(0)(t)}{dt} = \lim_{\Delta t \to 0} \frac{q^0(t + \Delta t) - q^0(t)}{\Delta t} \]

\[ = \frac{1}{2} \omega(0) q^0(t) \]  
\[ 2.9.17 \]

or

\[ = \frac{1}{2} q^0(t) \omega(0) \]  
\[ 2.9.18 \]

where we have substituted from Eqs. 2.7.12 and 2.7.13, respectively.

Another method for deriving Eq. 2.9.17 is to take

\[ r_1 = qrq^* \]  
\[ 2.8.19 \]

where the vectors and the quaternion are taken in the fixed system and differentiating with respect to time

\[ \dot{r}_1 = \dot{q} r q^* + \dot{q} r q^* \]  
\[ 2.9.20 \]

where the change in \( r \) is due only to the time dependent rotation produced by \( q \). Substituting from Eq. 2.9.19 for \( \dot{r}q^* \) and \( qr \) we get
\[ \dot{\mathbf{r}}_1 = \dot{\mathbf{q}}^* \dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_1 \dot{\mathbf{q}}^* \]  
2.9.21

Differentiating \( \dot{\mathbf{q}}^* = 1 \) we get

\[ \dot{\mathbf{q}}^* + \dot{\mathbf{q}}^* = 0 \]  
2.9.22

Substituting from this into Eq. 2.9.21

\[ \dot{\mathbf{r}}_1 = \dot{\mathbf{q}}^* \dot{\mathbf{r}}_1 = \dot{\mathbf{r}}_1 \dot{\mathbf{q}}^* \]

\[ = p \mathbf{r}_1 = \dot{\mathbf{r}}_1 p \]

\[ = 2 (p \times \dot{\mathbf{r}}_1) \]  
2.9.23

where \( \dot{\mathbf{p}} = \dot{\mathbf{q}}^* \). Taking the conjugate of \( \dot{\mathbf{p}} \) and from Eq. 2.9.22 it follows that

\[ \dot{\mathbf{p}} = -\dot{\mathbf{p}}^* \]

or

\[ \dot{\mathbf{p}}_0 + \dot{\mathbf{p}} = -(\dot{\mathbf{p}}_0 - \dot{\mathbf{p}}) \]

which can only be true if \( \dot{\mathbf{p}}_0 = 0 \), and thus, it further follows that \( \dot{\mathbf{p}} = \dot{\mathbf{p}}_0 + \dot{\mathbf{p}} \) is a vector. By comparing Eq. 2.9.23 to Eq. 2.9.5 we see that

\[ \dot{\mathbf{p}} = \frac{1}{2} \dot{\omega} = \mathbf{q} \]

and upon multiplying by \( \mathbf{q} \) from the right we get

\[ \dot{\mathbf{q}} = \frac{1}{2} \dot{\omega} \mathbf{q} \]  
2.9.25

which is the same result as Eq. 2.9.17. To obtain the expression with respect to the body basis the \( \dot{\omega} \), which is here given with respect to the fixed basis, must be transformed according to Eq. 2.7.15.
To find an explicit form for Eq. 2.9.18, we expand it, first noting that from Eq. 2.7.18

\[ q(1) = q(0) * q(0) q(0) = q(0) \]  

2.9.26

to get

\[ q(0) = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \]

2.9.27

By equating the quaternion components, Eq. 2.9.27 can be written in matrix form as follows:

\[
q = \begin{bmatrix}
q_0 \\
q_1 \\
q_2 \\
q_3
\end{bmatrix} = \begin{bmatrix}
0 & -\omega_1^{(1)} & -\omega_2^{(1)} & -\omega_3^{(1)} \\
\omega_1^{(1)} & 0 & \omega_3^{(1)} & -\omega_2^{(1)} \\
\omega_2^{(1)} & -\omega_3^{(1)} & 0 & \omega_1^{(1)} \\
\omega_3^{(1)} & -\omega_1^{(1)} & \omega_2^{(1)} & 0
\end{bmatrix} \begin{bmatrix}
q_0 \\
q_1 \\
q_2 \\
q_3
\end{bmatrix} = T(1)q
\]  

2.9.28
According to Eq. 2.9.26 the quaternion is invariant between the (0) and (1) bases systems. It is an eigen-quaternion of the rotation moving the body from (0) to (1), and its vector component lies along the axis of this rotation and consequently is unchanged by the rotation. For this reason we have dropped the superscript basis identifier on the quaternion.

We now consider the explicit form of Eq. 2.9.17 which is the quaternion differential equation in terms of space fixed or (0) basis angular velocity components. Carrying out an expansion of Eq. 2.9.17 the same way as we did for Eq. 2.9.18, we get that

\[
\mathbf{q} = \begin{pmatrix}
\dot{q}_0 & 0 & -\omega^{(0)}_1 & -\omega^{(0)}_2 & -\omega^{(0)}_3 \\
\dot{q}_1 & \omega^{(0)}_1 & 0 & -\omega^{(0)}_3 & -\omega^{(0)}_2 \\
\dot{q}_2 & \omega^{(0)}_2 & \omega^{(0)}_2 & 0 & -\omega^{(0)}_1 \\
\dot{q}_3 & -\omega^{(0)}_2 & -\omega^{(0)}_2 & \omega^{(0)}_1 & 0
\end{pmatrix} \mathbf{q},
\]

2.9.29

The two matrices \( \mathbf{T}^{(1)} \) and \( \mathbf{T}^{(0)} \) are quite similar; the only difference being that the lower right 3 x 3 partitioned matrix is a transpose of the other. It also is worthwhile to note that this submatrix has the form of the matrix operator for the vector product as given by Eq. 2.1.5.
We now return to Eq. 2.9.28 and note that a solution to it is
\[ q = e^{T(1) t} q(0) \] 2.9.30

We check that this is a solution by differentiating \( q \)
\[ q(1) = T(1) e^{T(1) t} q(0) = T(1) q(0) \]

We can expand \( e^{T(1) t} \) to give
\[ e^{T(1) t} = I + T(1) t + \frac{(T(1) t)^2}{2!} + \frac{(T(1) t)^3}{3!} + \ldots \] 2.9.31

By multiplying \( T(1) \) by itself, we find that
\[ (T(1))^2 = \frac{1}{4} \omega^2 I \] 2.9.32

and substituting this result into Eq. 2.9.31 we can write
\[ e^{T(1) t} = \left(1 - \frac{(\omega t)^2}{2!} + \frac{(\omega t)^4}{4!} - \ldots\right) I + \frac{1}{3!} - \frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{5!} - \ldots \]
\[ = \cos \frac{\omega t}{2} I + \frac{2}{\omega} \sin \frac{\omega t}{2} T(1) \] 2.9.33

Substituting this result into Eq. 2.9.30 we get
\[ q(t) = \cos \frac{\omega t}{2} q(0) + \frac{2}{\omega} \sin \frac{\omega t}{2} T(1) q(0) \]

and further substituting from Eq. 2.9.28
\[ q(t) = \cos \frac{\omega t}{2} q(0) + \frac{2}{\omega} \sin \frac{\omega t}{2} q(0) \] 2.9.34
This is the solution to Eq. 2.9.28 if we assume that $\omega^{(1)}$ is constant. Let us now return to Eq. 2.7.11 and consider $\theta$ as a function of time.

$$q(\theta(t)) = \cos \frac{\theta(t)}{2} + \dot{\theta} \sin \frac{\theta(t)}{2}.$$  \hspace{1cm} 2.9.35

Differentiating with respect to time

$$\dot{q}(t) = -\frac{\dot{\theta}}{2} \sin \frac{\theta(t)}{2} + \frac{\dot{\theta}^2}{2} \cos \frac{\theta(t)}{2},$$  \hspace{1cm} 2.9.36

taking $q^{(0)}$ and $\dot{q}^{(0)}$ from Eqs. 2.9.35 and 2.9.36, respectively, and substituting into Eq. 2.9.34 we get

$$q(t) = \cos \frac{\omega t}{2} \left( \cos \frac{\theta(0)}{2} + \dot{\theta} \sin \frac{\theta(0)}{2} \right)$$

$$- \frac{2}{\omega} \sin \frac{\omega t}{2} \left( \frac{\omega}{2} \sin \frac{\theta(0)}{2} - \frac{\omega}{2} \cos \frac{\theta(0)}{2} \right)$$

$$= \begin{bmatrix}
\cos \frac{1}{2} \left( \omega t + \theta(0) \right) \\
\dot{\omega}^{(1)} \sin \frac{1}{2} \left( \omega t + \theta(0) \right) \\
\omega^{(1)} \sin \left( \frac{1}{2} \omega t + \theta(0) \right) \\
\dot{\omega}^{(1)} \sin \left( \frac{1}{2} \omega t + \theta(0) \right) \\
\dot{\omega}^{(1)} \sin \left( \frac{1}{2} \omega t + \theta(0) \right)
\end{bmatrix}$$  \hspace{1cm} 2.9.37

which gives the quaternion as a function of time for a constant $\omega^{(1)}$. The quaternion, in turn, can be used as a rotation operator as given by Eq. 2.6.10 to give the
rotation of a vector attached to the body or prescribe the rotation of the body basis vectors as given by Eq. 2.9.8. It also can be used to transform vector components between body positions at t=0 to those at time t using Eq. 2.7.15.

Any body rotation can be factored into a number of rotations. This follows directly from the observations that products of quaternions are quaternions or that products of rotation matrices always result in orthonormal matrices. We briefly examine here the time rate of change of the product of two rotation quaternions. Let

\[ q = q_1 q_2 \]  \hspace{1cm} 2.9.38

and differentiating with respect to time

\[ \dot{q} = \dot{q}_1 q_2 + q_1 \dot{q}_2 \]  \hspace{1cm} 2.9.39

Substituting for \( \dot{q}_1 \) and \( \dot{q}_2 \) from Eq. 2.9.17

\[ \dot{q} = \frac{1}{2} \omega_1 q_2 + \frac{1}{2} q_1 \omega_2 q_2 \]

\[ = \frac{1}{2} (\omega_1 + q_1 \bar{\omega}_2 q_1^*) q \]  \hspace{1cm} 2.9.40

This has the same form as Eq. 2.9.17 with the angular velocity being the sum of the angular velocity \( \bar{\omega}_1 \) associated with the first rotation produced by \( q_1 \) and \( q_1 \omega_2 q_1^* \), the angular velocity associated with \( q_2 \), but rotated with the body by \( q_1 \) as shown in Fig. 2.9.2. This again clearly illustrates the non-commutative property of rotation operators where each operation is dependent on the preceding operation(s).
2.10 Kinematic Matrix Rotation Operators

We now wish to derive a differential matrix rotation operator. We start by expressing the change in a vector $\mathbf{r}$ produced by a rotation operator for a small angular displacement $\Delta \theta$ as

$$\Delta \mathbf{r} = \mathbf{r}' - \mathbf{r} = (\mathbf{R}(\Delta \theta) - \mathbf{I})\mathbf{r}$$  \hspace{1cm} 2.10.1

where

$$\mathbf{r}' = \mathbf{R}(\Delta \theta)\mathbf{r}$$  \hspace{1cm} 2.10.2

Dividing Eq. 2.10.1 by $\Delta t$ and taking the limit we get

$$\dot{\mathbf{r}} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \left[ \frac{\mathbf{R}(\Delta \theta) - \mathbf{I}}{\Delta t} \right] \dot{\mathbf{r}} = \mathbf{D} \dot{\mathbf{r}}$$  \hspace{1cm} 2.10.3
where we have substituted from Eq. 2.1.6 for $R(\Delta \theta)$ and taken $\omega_i = \frac{d\theta}{dt}$ which results in

$$
D = \begin{pmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{pmatrix}
$$

2.10.4

and is called the rate of rotation or angular velocity matrix. In vector form it becomes

$$
\dot{r} = \omega \times r
$$

2.10.5

The $D$ matrix can also be derived by taking

$$
\dot{r} = R^{-1} \dot{r}_0
$$

2.10.6

and differentiating with respect to time

$$
\ddot{r} = R \ddot{r}_0
$$

2.10.7

where $\dot{r}_0$ is taken as constant as the total change in $\dot{r}$ is assumed to be due to the rotation. From Eq. 2.10.6

$$
\dot{r}_0 = R^{-1} \dot{r} = R^T \dot{r}
$$

2.10.8

and substituting for $r_0$ from this expression into Eq. 2.10.7 we get

$$
\ddot{r} = R \ddot{r}^T
$$

2.10.9

and by comparison to Eq. 2.10.3 we deduce that the angular velocity matrix is

$$
D = \ddot{r}^T R
$$

2.10.10
To derive the angular acceleration matrix we use Eq. 2.10.5 and differentiate with respect to time

\[ \ddot{\mathbf{r}} = \frac{d}{dt} \left( \mathbf{\hat{u}} \right) \times \dot{\mathbf{r}} + \mathbf{\hat{u}} \times \dot{\mathbf{r}} \]  
2.10.11

and since

\[ \frac{d}{dt}(\mathbf{\hat{u}}) = \mathbf{\hat{\omega}} + \mathbf{\dot{\omega}}, \quad \mathbf{\hat{u}} = \mathbf{\omega} \times \mathbf{\hat{r}} \]

we have

\[ \ddot{\mathbf{r}} = (\mathbf{\hat{\omega}} + \mathbf{\dot{\omega}}) \times \dot{\mathbf{r}} + \mathbf{\dot{\omega}} \times (\mathbf{\hat{\omega}} \times \dot{\mathbf{r}}) \]

\[ = [\mathbf{\omega} (\dot{\mathbf{u}} \times ) + \mathbf{\dot{\omega}} (\dot{\mathbf{u}} \times ) + \mathbf{\ddot{\omega}} \times (\dot{\mathbf{u}} \times )] \dot{\mathbf{r}} \]  
2.10.12

Using the matrix form for a vector product as given by Eq. 2.1.5 this can be written as

\[ \ddot{\mathbf{r}} = \begin{bmatrix} \left(u_1^2-1\right)\ddot{\mathbf{r}}^2 & \left(u_1u_2\ddot{\mathbf{r}}^2 \mathbf{\hat{u}} - u_3 \mathbf{\hat{r}}^2 \mathbf{\hat{u}} \right) & \left(u_1u_3\ddot{\mathbf{r}}^2 + u_2 \mathbf{\hat{u}}^2 + u_2 \mathbf{\hat{u}} \right) \\
\left(u_1u_2\ddot{\mathbf{r}}^2 + u_3 \mathbf{\hat{r}}^2 \mathbf{\hat{u}} \right) & \left(u_2^2 - 1\right)\ddot{\mathbf{r}}^2 & \left(u_2u_3\ddot{\mathbf{r}}^2 - u_1 \mathbf{\hat{r}}^2 \mathbf{\hat{u}} \right) \\
\left(u_1u_3\ddot{\mathbf{r}}^2 - u_2 \mathbf{\hat{r}}^2 \mathbf{\hat{u}} \right) & \left(u_2u_3\ddot{\mathbf{r}}^2 - u_1 \mathbf{\hat{r}}^2 \mathbf{\hat{u}} \right) & \left(u_3^2 - 1\right)\ddot{\mathbf{r}}^2 \\
\end{bmatrix} \dot{\mathbf{r}} \]

\[ = \ddot{\mathbf{D}} \dot{\mathbf{r}} \]  
2.10.13

where \( \mathbf{D} \) is the angular acceleration matrix.
Returning to Eq. 2.10.10 and rewriting it as
\[ \dot{R} = D R \]  
2.10.14
we get a differential equation for the nine direction cosines which are known as Poisson's equations.

If we assume that \( D \) is constant, i.e. the angular velocity is constant, a solution to the equation is given by
\[ R = e^{Dt}R(0) \]  
2.10.15
That this satisfies Eq. 2.10.14 can be shown by resubstitution
\[ \dot{R} = De^{Dt}R(0) = DR \] 
Thus, given \( R \) at \( t = 0 \) and constant \( \dot{\omega} \) we can calculate \( R \) for later times. If \( \dot{\omega} \) is a slowly changing function of time, then over a limited time interval Eq. 2.10.15 may provide a good estimate of body rotation and, hence, body position.

2.11 Rigid Body Equations of Motion

The kinetic energy is a scalar quantity and for an element of mass \( dm \) at \( r \) is given by
\[ T = \frac{1}{2} \dot{r}^2 dm \]  
2.11.1
where \( \dot{r} \) is taken with respect to an inertial system. The total kinetic energy of a rigid body is obtained by integration of the total body mass. It is convenient to integrate the body mass with respect to a point fixed in
the body denoted by P as shown in Fig. 2.11.1. The center of mass is at 0 and its position with respect to P is given by \( \mathbf{r}_0 \).

According to Fig. 2.11.1 the position of \( \text{dm} \) can be expressed as

\[
\mathbf{r} = \mathbf{r}_p + \mathbf{\rho}.
\]

2.11.2

\[\text{Figure 2.11.1 Body Mass Element } \text{dm} \text{ with Respect to Point } P \text{ in the Body}\]
The total rate of change or velocity of \( \dot{r} \) is the sum of the velocity of point \( P \) and the velocity due to rotation about \( P \) as given by Eq. 2.10.5

\[
\dot{r} = \dot{r}_p + \omega \times \rho .
\]

Substituting this into Eq. 2.11.1 and integrating over the entire body

\[
T = \frac{1}{2} \int \left( \dot{r}_p^2 + \omega \times \rho \right) \cdot \left( \dot{r}_p^2 + \omega \times \rho \right) \, dm
\]

\[
= \int \frac{1}{2} \dot{r}_p \, dm + \int \dot{r}_p \cdot (\omega \times \rho) \, dm + \frac{1}{2} \int (\omega \times \rho)^2 \, dm .
\]

The mass distribution is independent of the motion of the point \( P \) in the body resulting in the first term corresponding to the translational kinetic energy of the whole body. In the second term we let \( \rho = r_0 + \rho_0 \), and note that \( \int \rho_0 \, dm = 0 \), since 0 is the center of mass, and find that the kinetic energy can be expressed as

\[
T = \frac{1}{2} m v_p^2 + v_p \cdot (\omega \times r_0)_m + \frac{1}{2} \int (\omega \times \rho)^2 \, dm .
\]

If \( P \) is taken to coincide with the center of mass then \( r_0 = 0 \) and the second term vanishes.

The last term of Eq. 2.11.4 can be conveniently expanded using the scalar and vector product matrix operators given by Eqs. 2.1.4 and 2.1.5 and the short hand
convention for the vector product

\[ \vec{A} \times = \vec{A} = \begin{pmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & A_1 \\ -A_2 & A_1 & 0 \end{pmatrix} \]  \hspace{1cm} 2.11.5

Applying these to the last term

\[ \int (\rho \times \omega) \cdot (\rho \times \omega) \, dm = \int (\rho \omega) T (\rho \omega) \, dm \]

\[ = \int \omega T \rho^{-T} \rho \omega \, dm = \omega T \left( \int \rho^{-T} \rho \, dm \right) \omega \]

\[ = \omega \cdot I_p \cdot \omega \]

where

\[ I_p = \int \rho^{-T} \rho \, dm \]

\[ = \begin{bmatrix} \int \rho_3^2 \, dm & -\int \rho_1 \rho_2 \, dm & -\int \rho_1 \rho_3 \, dm \\ -\int \rho_1 \rho_2 \, dm & \int \rho_1^2 + \rho_2^2 \, dm & -\int \rho_2 \rho_3 \, dm \\ -\int \rho_1 \rho_3 \, dm & -\int \rho_2 \rho_3 \, dm & \int \rho_1^2 + \rho_2^2 \, dm \end{bmatrix} \]  \hspace{1cm} 2.11.7

The diagonal elements are denoted by \( I_{11}, I_{22} \) and \( I_{33} \) and are called the moments of inertia and the off-diagonal elements, the products of inertia.

The kinetic energy can now be expressed as

\[ T = \frac{1}{2} m v_p^2 + \vec{v}_p \cdot (\vec{r}_0 \times \vec{r}_0) m + \frac{1}{2} \omega \cdot I_p \cdot \omega \]  \hspace{1cm} 2.11.8
The angular momentum of a mass element dm at \( \vec{r} \) moving with velocity \( \vec{v} \) with respect to an inertial reference system is given by

\[
\vec{L} = \vec{r} \times \vec{v} \quad 2.11.9
\]

As shown in Fig. 2.11.1, \( \vec{r} \) can be written as a sum of the vectors from the inertial origin to point P and from P to dm. The velocity is given by Eq. 2.11.3.

Substituting these relations and integrating over the total body, we get

\[
\begin{align*}
L &= \int_m (\vec{r}_P + \vec{\rho}) \times (\vec{v}_P + \vec{\omega} \times \vec{\rho}) dm \\
&= \vec{r}_P \times (\vec{v}_P + \vec{\omega} \times \vec{r}_0) \int_m dm - (\vec{r}_P \times \vec{\omega}) \times \int_m \vec{\rho}_0 dm \\
&+ \vec{r}_0 \times \vec{v}_P \int_m dm - \int_m \vec{\rho} \times (\vec{\rho} \times \vec{\omega}) dm \quad 2.11.10
\end{align*}
\]

where we have taken \( \vec{\rho} = \vec{r}_0 + \vec{\rho}_0 \).

The variables in the first term are all independent of the mass distribution and can, therefore be taken outside the integral. In the second term, the integral equals zero since \( \vec{\rho}_0 \) is the center of mass. In the third term, the arguments are again independent of the mass distribution and can be taken outside the integral. The argument of the last term can be expressed in matrix form as
\[ \mathbf{\dot{p}} \times (\mathbf{\dot{p}} \times \mathbf{\omega}) = \mathbf{\dot{p}} \mathbf{\dot{p}} \mathbf{\omega} = - \mathbf{\omega}^T \mathbf{\dot{p}} \mathbf{\omega} \]  
\text{2.11.11}

where we have used the property \( \mathbf{\dot{p}} = - \mathbf{\omega}^T \) which follows from Eq. 2.11.5.

Substituting the results of Eq. 2.11.11 into Eq. 2.11.10 and using the definition for \( I_P \) from Eq. 2.11.7 the angular momentum can be expressed as

\[ \mathbf{L} = \mathbf{r}_P \times (\mathbf{v}_P + \mathbf{\omega} \times \mathbf{r}_0)m + (\mathbf{r}_0 \times \mathbf{v}_P)m + I_P \cdot \mathbf{\omega} \quad \text{2.11.12} \]

If the point \( P \) is taken to coincide with the center of mass at point \( 0 \), then \( \mathbf{r}_0 = 0 \) and

\[ \mathbf{L} = \mathbf{r}_P \times \mathbf{v}_P m + I_0 \cdot \mathbf{\omega} \quad \text{2.11.13} \]

where \( I_0 \) is the inertia tensor with respect to the center of mass. \( I_0 \) can be related to \( I_P \) by expanding the last term in Eq. 2.11.4 after letter \( \mathbf{\dot{\rho}} = \mathbf{\dot{\rho}}_0 + \mathbf{\dot{r}}_0 \) as follows.

\[
\frac{1}{2} \int_m (\mathbf{\omega} \times \mathbf{\dot{\rho}}) \cdot (\mathbf{\omega} \times \mathbf{\dot{\rho}}) \, dm = \frac{1}{2} \int_m (\mathbf{\omega} \times (\mathbf{\dot{\rho}}_0 + \mathbf{\dot{r}}_0) \cdot (\mathbf{\omega} \times (\mathbf{\dot{\rho}}_0 + \mathbf{\dot{r}}_0) \, dm
\]

\[
= \frac{1}{2} \int_m (\mathbf{\omega} \times \mathbf{\dot{\rho}}_0) \cdot (\mathbf{\omega} \times \mathbf{\dot{\rho}}_0) \, dm + \int_m (\mathbf{\omega} \times \mathbf{\rho}_0) \cdot (\mathbf{\omega} \times \mathbf{\rho}_0) \, dm
\]

\[
+ \frac{1}{2} \int_m (\mathbf{\omega} \times \mathbf{\dot{r}}_0) (\mathbf{\omega} \times \mathbf{\dot{r}}_0) \, dm .
\]

The first term results in an expression identical to Eq. 2.11.6 except that the integrations are taken with respect to the center of mass. The second term is zero.
since \( \int_m \dot{r}_0 \, dm = 0 \). In the last term, the argument is independent of the mass distribution and we can thus write

\[
\frac{1}{2} \omega \cdot I_p \cdot \dot{\omega} = \frac{1}{2} \omega \cdot I_0 \cdot \dot{\omega} + \frac{1}{2} (\dot{r}_0 \times \omega) \cdot (\dot{r}_0 \times \omega) m
\]

\[
= \frac{1}{2} \omega \cdot I_0 \cdot \dot{\omega} + \frac{1}{2} (\dot{r}_0 \omega)^T (\dot{r}_0 \omega) m
\]

\[
= \frac{1}{2} \omega \cdot I_0 \cdot \dot{\omega} + \frac{1}{2} \omega \cdot D \cdot \dot{\omega}
\]

and

\[
I_p = I_0 + D \quad 2.11.14
\]

where

\[
D = m (\dot{r}_0 T \dot{r}_0) = m \begin{pmatrix}
\dot{r}_{02}^2 + \dot{r}_{03}^2 & -\dot{r}_{01} \dot{r}_{02} & -\dot{r}_{01} \dot{r}_{03} \\
-r_{01} \dot{r}_{02} & r_{01}^2 + r_{03}^2 & -r_{02} \dot{r}_{03} \\
-r_{01} \dot{r}_{03} & -r_{01} \dot{r}_{03} & r_{01}^2 + r_{02}^2
\end{pmatrix} \quad 2.11.15
\]

The time derivative of the angular momentum with respect to an inertial system is equal to the torque acting on the mass. Taking the time derivative of \( L \) as given by Eq. 2.11.9

\[
\dot{L} = \int_m (\ddot{r} \times \dot{r} + \dot{r} \times \ddot{r}) \, dm = \int_m (\ddot{r} \times \ddot{r}) \, dm \quad 2.11.16
\]

From Fig. 2.11.1

\[
\ddot{r} = \dddot{r}_p + \dddot{\rho} \quad 2.11.17
\]
Differentiating this with respect to time
\[ \dot{r} = \ddot{r}_p + \dot{\rho} = \ddot{r}_p + \omega \times \dot{\rho} \] 2.11.18

and differentiating again
\[ \dddot{r} = \dddot{r}_p + \ddot{\omega} \times \rho + \omega \times \ddot{\rho} \] 2.11.19

Substituting Eqs. 2.11.17 and 2.11.19 into Eq. 2.11.16
\[ \dddot{r}_p + \dot{\omega} \times \rho + \dot{\omega} \times (\omega \times \rho) \] 2.11.19

Substituting Eqs. 2.11.17 and 2.11.19 into Eq. 2.11.16
\[ \int \dot{r}_p \times (r_p + \omega \times \rho + \omega \times (\omega \times \rho)) \, dm \]
\[ = \int \dot{r}_p \times (r_p + \omega \times \rho \times (\omega \times \rho)) \, dm + \int \begin{array}{c} \rho \times \dot{r}_p \end{array} \, dm \]
\[ + \int \begin{array}{c} \rho \times (\dot{\omega} \times \rho) \end{array} \, dm + \int \rho \times (\omega \times (\omega \times \rho)) \, dm \]. 2.11.20

The third term can be written as
\[ \int \rho \times (\omega \times \rho) \, dm = -\int \rho \times (\rho \times \omega) \, dm \]
\[ = -\int \rho \times \omega \, dm = \int \rho \times \rho \, dm \cdot \omega = \rho \cdot \omega \] 2.11.21

where we have used \( \rho = \rho \) and Eq. 2.11.7.

The last term can be rewritten using the vector identity
\[ \dot{a} \times (b \times c) = (a \cdot c)b - (a \cdot b)c, \]
forming
\[ \rho \times (\omega \times (\omega \times \rho)) = (\omega \times (\rho \times \omega)) - (\omega \times \omega) (\rho \times \rho) = (\omega \cdot \rho) (\rho \times \omega) \]

\[ \omega \times (\rho \times (\omega \times \rho)) = (\rho \times \omega) (\omega \times (\rho \times \rho)) - (\rho \times \rho) (\rho \times \omega) = (\omega \cdot \rho) (\rho \times \omega) \]

and subtracting the latter equation from the former we get

\[ \rho \times (\omega \times (\omega \times \rho)) = \omega \times (\rho \times (\omega \times \rho)) \]

\[ = -\omega \times (\rho \times (\rho \times \omega)) \]

\[ = \omega \times (\rho \cdot \rho \omega) \]

\[ = \omega \times (\rho T \rho \omega) . \]

By noting the definition of \( I_p \) from Eq. 2.11.7, using the results of Eq. 2.11.21 and letting \( \dot{\rho} = \dot{\rho}_0 + \dot{r}_0 \) in the first two terms Eq. 2.11.20 can be rewritten as

\[ L = m (\dot{r}_p \times (\dot{r}_p + \omega \times \dot{r}_0 + \omega \times (\omega \times \dot{r}_0)) + \dot{r}_0 \times \dot{r}_p) \]

\[ + I_p \cdot \omega + \omega \times I_p \cdot \omega . \]

According to Eq. 2.11.19, the vector product with \( r_p \) can be interpreted as the absolute acceleration of the center of mass, \( \ddot{r}_0 \), which is a vector from the origin to the center of the mass. Eq. 2.11.22 can now be written as

\[ L = m (\dot{r}_p \times \dot{r}_c + \dot{r}_0 \times \dot{r}_p) + I_p \cdot \ddot{\omega} + \omega \times I_p \cdot \ddot{\omega} . \]

From Newton's law, the acceleration of the center of mass is related to the applied external force by
and the total external torque acting on the body with respect to the origin is the sum of the torque about P and that due to F acting at P, or

\[ \mathbf{M} = \mathbf{M}_p + \mathbf{r}_p \times \mathbf{F} = \mathbf{M}_p + m(\mathbf{r}_p \times \mathbf{r}_c) \]  

2.11.25

Since the rate of change of angular momentum, as given by Eq. 2.11.23, is equal to the external moment acting on it, as given by Eq. 2.11.25, equating these we get

\[ \mathbf{M}_p = m(\mathbf{r}_0 \times \mathbf{r}_p) + \mathbf{I}_p \cdot \dot{\omega} + \dot{\omega} \times \mathbf{I}_p \cdot \omega \]  

2.11.26

If \( \mathbf{r}_p \) is taken to coincide with the body's center of mass, then the equation becomes

\[ \mathbf{I}_0 \cdot \dot{\omega} + \dot{\omega} \times \mathbf{I}_0 \cdot \omega = \mathbf{M}_0 \]  

2.11.27

and in matrix form

\[ \mathbf{I}_0 \dot{\omega} + \dot{\omega} \mathbf{I}_0 \omega = \mathbf{M}_0 \]  

2.11.28

If the transformation to principal axes from the coordinate system in which the original inertia tensor is calculated is given by \( \mathbf{A}^{0} \), then the vector components transform according to
and second order tensors according to
\[ \mathbf{B}^{(p)} = \mathbf{A}^{p0} \mathbf{B}(0) (\mathbf{A}^{p0})^T. \]  

Noting that \( \omega \) transforms as a second order tensor
and from the definition of the principal axes system

\[ \mathbf{I}^{(p)} = \mathbf{A}^{p0} \mathbf{I}_0(0) (\mathbf{A}^{p0})^T = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \]  

Eq. 2.11.28 can be written as

\[
\begin{align*}
I_1 \omega_1 - (I_2 - I_3) \omega_2 \omega_3 &= M_1 \\
I_2 \omega_2 - (I_3 - I_1) \omega_3 \omega_1 &= M_2 \\
I_3 \omega_3 - (I_1 - I_2) \omega_1 \omega_2 &= M_3
\end{align*}
\]  

which are Euler's equations of motion for a simple rigid body.
CHAPTER 3

METHODS OF THREE-DIMENSIONAL BODY MOTION ANALYSIS

3.1 Screw Axis Analysis

It has been demonstrated by Euler\textsuperscript{[129]} that a general rigid body displacement in three dimensional space can be totally characterized by a combination of a translational displacement and a rotation. A number of methods are available for quantitatively describing such rigid body displacements. One of the most convenient for analytic purposes is the use of a cosine matrix to define the rotation and a vector to define the translation. While this method is analytically sufficient, it does not provide much physical insight into the actual motion which would be especially useful if the motion is partially constrained. A method that can provide more physical insight, and especially when certain modes of motion are constrained, is the screw axis technique originally formulated by Chasles. In this technique a displacement vector in the body is separated into two components, one along the body rotational axis and the other in the plane normal to the rotational axis. The location of the rotational axis is specified so that the total component of body displacement in the plane can be attributed to a rotational displacement about this axis.
Consider the displacement of a vector \( \mathbf{r}_0 \) rigidly embedded in a moving body with its tail given by \( \mathbf{p}_0 \) and tip by \( \mathbf{q}_0 \) as in Fig. 3.1.1. The motion can be taken as a displacement of \( \mathbf{p}_0 \) to \( \mathbf{p}_1 \) followed by a rotation, \( R \). The original point \( \mathbf{q}_0 \) has now moved to \( \mathbf{q}_1 \) and is given by

\[
\mathbf{q}_1 = \mathbf{p}_1 + \mathbf{r}_1 = \mathbf{p}_1 + R\mathbf{r}_0 = \mathbf{p}_1 + R(\mathbf{q}_0 - \mathbf{p}_0) .
\] 3.1.1
This equation can be used to find the location of a point on a rigid body in space given the initial location of the point, \( \vec{q}_0 \), the original and displaced coordinates of a point on the body, \( \vec{p}_0 \) and \( \vec{p}_1 \), and the rotation matrix, \( \mathbf{R} \).

We can also use the equation to solve for \( \vec{p}_0 \) such that

\[
\vec{p}_1 - \vec{p}_0 = s \hat{u}
\]  \hspace{1cm} 3.1.2

where \( \hat{u} \) is a unit vector in the direction of positive body rotation and

\[
s = |\vec{p}_1 - \vec{p}_0| \quad .
\]  \hspace{1cm} 3.1.3

In this case, we assume that \( \vec{q}_0 \), \( \vec{q}_1 \) and \( \mathbf{R} \) are given.

From Eqs. 2.4.1 and 2.4.2 we have that

\[
\theta = \cos^{-1} \left[ \frac{r_{11} + r_{22} + r_{33}^{-1}}{2} \right]
\]  \hspace{1cm} 3.1.4

\[
\begin{aligned}
\hat{u}_1 &= \frac{r_{32} - r_{23}}{2 \sin \theta} \\
\hat{u}_3 &= \frac{r_{13} - r_{31}}{2 \sin \theta} \\
\hat{u}_3 &= \frac{r_{21} - r_{12}}{2 \sin \theta}
\end{aligned}
\]  \hspace{1cm} 3.1.5

By substituting \( \vec{p}_1 \) from Eq. 3.1.2 into Eq. 3.1.1 and rearranging we get

\[
(R \mathbf{I} - \mathbf{I}) \vec{p}_0 - s \hat{u} = R \vec{q}_0 - \vec{q}_1 \quad .
\]  \hspace{1cm} 3.1.6
We define a vector \( \mathbf{\hat{u}} \) which lies in the YZ plane and express \( \mathbf{p}_0 \) as

\[
\mathbf{p}_0 = \mathbf{\hat{u}} + m \mathbf{\hat{u}}
\]

3.1.7

where \( m \) is the distance of \( \mathbf{p}_0 \) from the YZ plane along \( \mathbf{\hat{u}} \). Substituting Eq. 3.1.7 into Eq. 3.1.5 and noting that \( u_1 = 0 \), we can write

\[
\begin{bmatrix}
    s \\
    u_2 \\
    u_3
\end{bmatrix} =
\begin{bmatrix}
    q_{01} \\
    q_{02} \\
    q_{03}
\end{bmatrix}

3.1.8

and

\[
\begin{bmatrix}
    s \\
    u_2 \\
    u_3
\end{bmatrix} =
\begin{bmatrix}
    q_{01} \\
    q_{02} \\
    q_{03}
\end{bmatrix}

3.1.9

where

\[
S =
\begin{bmatrix}
    -u_1 & r_{12} & r_{13} \\
    -u_2 & r_{22^{-1}} & r_{23} \\
    -u_3 & r_{32^{-1}} & r_{33^{-1}}
\end{bmatrix}
\]

3.1.10

The three-dimensional motion is now characterized by \( s \), the translation of the body along the \( \mathbf{\hat{u}} \) axis; \( u_2 \) and \( u_3 \) which are the coordinates at which the screw axis intersects the YZ plane and is called the YZ plane piercing point; \( \theta \), the angle through which the body rotates about \( \mathbf{\hat{u}} \).
and two of the rotational axis \( \hat{\mu} \) components (where the third can be determined from \( \mu_1^2 + \mu_2^2 + \mu_3^2 = 1 \)). These are six independent parameters, exactly the number needed to specify a rigid body position in three dimensions.

The choice of a piercing point in the YZ plane was arbitrary as the \( \hat{\mu} \) vector could have been taken in the XY or ZX planes as well with \( m \), then being the distance along \( \hat{p}_0 \) to the XY or XZ planes, respectively. It should also be noted that the location of the intersection point of \( \hat{p}_0 \) and \( \hat{\mu} \) along the \( \hat{\mu} \) axis is arbitrary and is determined by the choice of \( m \) in Eq. 3.1.7.

Another method for developing the screw axis analysis is illustrated in Fig. 3.1.2. We can form the vector relation

\[
\hat{p}_0 = \hat{R} + \frac{\Delta R}{2} + \hat{q}_0 \tag{3.1.11}
\]

where \( \hat{p}_0 \) is a vector from the origin to the point on the \( \hat{\mu} \) axis lying closest to \( \hat{q}_0 \) and \( \Delta R \) is the projection of \( \Delta q \) on a plane normal to \( \hat{\mu} \). From examination of the vector geometry we can write

\[
2 \frac{\Delta R}{|\Delta R|} = \tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} \tag{3.1.12}
\]

\[
\hat{R} = \frac{\hat{\mu} \times \Delta R}{|\Delta R|} \tag{3.1.13}
\]

and

\[
\Delta R = \Delta \hat{q} - \hat{\mu} s \tag{3.1.14}
\]

Substituting these into Eq. 3.1.11 we get
Figure 3.1.2 Vector Construction of Screw Axis Motion Analysis

Recalling that $\Delta q = \vec{q}_1 - \vec{q}_0$ and that $s = \vec{u} \cdot (\vec{q}_1 - \vec{q}_0)$, $\vec{p}_0$ can be expressed in terms of the initial and final coordinates of a body point, the rotation vector and the angle of rotation. Thus

$$\vec{p}_0 = \frac{1}{2} \left[ \frac{1 + \cos \theta}{\sin \theta} \hat{\omega} \times (\vec{q}_1 - \vec{q}_0) - (\vec{q}_1 - \vec{q}_0) \cdot \hat{\omega} + \vec{q}_1 + \vec{q}_0 \right].$$ 3.1.16
To locate the \( \hat{\mu} \) axis piercing points we can form the following relations according to Fig. 3.1.3

\[
\begin{align*}
\dot{\pmb{p}}_0 &= \dot{u}_1 + m_1 \hat{\mu} = u_{12} \hat{e}_2 + u_{13} \hat{e}_3 + m_1 \hat{\mu} \\
&= \dot{u}_2 - m_2 \hat{\mu} = u_{21} \hat{e}_1 + u_{23} \hat{e}_3 - m_2 \hat{\mu} \\
&= \dot{u}_3 + m_3 \hat{\mu} = u_{31} \hat{e}_1 + u_{32} \hat{e}_2 + m_3 \hat{\mu}
\end{align*}
\]

These can be directly solved for the \( u_{ij} \)'s and \( m_i \)'s. The \( m_1, m_2 \) and \( m_3 \) are the distances along \( \hat{\mu} \) from \( \dot{\pmb{p}}_0 \) to the \( XY, XZ \) and \( YZ \) planes, respectively, and are given explicitly by

\[
\begin{align*}
m_1 &= \frac{\pmb{p}_{01}}{\mu_1} \\
m_2 &= \frac{\pmb{p}_{02}}{\mu_2} \\
m_3 &= \frac{\pmb{p}_{03}}{\mu_3}
\end{align*}
\]

A particularly concise expression for the \( u_{ij} \)'s can be written if we take \( [u_{ij}] = U \) and let

\[
\dot{v} = \dot{\pmb{p}}_0 \times \hat{\mu}
\]

then Eq. 3.1.17 with conditions expressed in Eq. 3.1.18 can be solved for the \( u_{ij} \)'s using
Figure 3.1.3 Piercing Point Locations for the Screw Axis

\[
\mathbf{u} = \begin{bmatrix}
0 & -\frac{v_3}{u_1} & \frac{v_2}{u_1} \\
\frac{v_3}{u_2} & 0 & -\frac{v_1}{u_2} \\
-\frac{v_2}{u_3} & \frac{v_1}{u_3} & 0
\end{bmatrix}
\]

3.1.20
3.2 Displacement of Arbitrarily Picked Point of Rotation

If a body's motion in space is subject to constraints, certain symmetries may result in the trajectories of points on the body or on extensions of the body in space. Screw axis analysis is one method of extracting such symmetry information, especially if the body rotation vector tends to stay closely aligned with the translational displacement trajectory. However, if large translational displacements occur in the plane normal to the body rotation vector, the instantaneous center of rotation undergoes large and random appearing fluctuations. The reason for these random appearing fluctuations is that components of body motion in the plane normal to the rotation vector are fully attributed to rotation about some point through the angle of total body rotation. If the motion components in the plane are predominately due to translational displacements, the rotational radius will be very large and, thus, the center of rotation will be far removed from the observed displacement vector. If small errors are made in the measurement of body point displacement vectors, and since these errors are multiplied by a ratio of rotational radius vector to displacement vector, large relative errors can result between positions or adjacent instantaneous rotation axes.

In some cases, limited information is available about the constraints acting on the body and it is desirable to consider another method for characterizing the three dimensional motion of a body in space. We will seek to
find a point on the body about which the rotation of the body is assumed to take place but which will undergo minimal translational displacement.

Consider a body displaced from position 0 to position 1. We follow a point on that body and denote its initial position as \( \dot{q}_0 \) and final position as \( \dot{q}_1 \). We specify a point on the body \( T \) about which we will take the rotation as shown in Figure 3.2.1.

The displacement of the body point can be separated into components along the body rotation axis and in the plane normal to \( \mu \). These components are

\[
\mu \cdot (\dot{q}_1 - \dot{q}_0)\mu
\]

and

\[
\Delta R = (q_1 - q_0) - \mu \cdot (q_1 - q_0)\mu,
\]

respectively.

\( \dot{p}_0 \) defines the point of intersection of the normal plane to \( \mu \) which passes through \( \dot{q}_0 \) and is the center of instantaneous rotation of the body.

We locate another point on this plane denoted by \( \dot{p}_0' \) which is the projection of \( \dot{T} \) on the plane. Since \( \dot{p}_0' \) lies on an extension of \( \dot{T} \) along \( \mu \), body rotation about \( \mu \) at \( \dot{p}_0' \) and \( \dot{T} \) are equivalent. From Fig. 3.2.1 we can write

\[
\dot{p}_0' = \dot{T} + \mu \cdot (\dot{p}_0 - \dot{T})\mu
\]

and
where $\vec{\rho}$ is the vector from the projection of our chosen rotation point on the plane normal to $\hat{u}$ to $\hat{q}_0$. Since the rotation vector is normal to the above plane, the application of the rotation operator, $R$, on $\vec{\rho}$ results in a new vector $\vec{\rho}'$ of the same magnitude as $\vec{\rho}$, lying in the same plane and rotated an angle $\theta$ positively about $\hat{u}$. The vector

\[ \vec{\rho} = \hat{q}_0 - \hat{p}_0 \]

\[ \hat{\mu} \cdot (\hat{q}_1 - \hat{q}_0) \hat{\mu} \]
can be viewed as the component of \( \Delta R \) strictly due to rotational motion about \( \hat{\mu} \) passing through \( \hat{T} \). The other planar component is then given by

\[
\Delta R_T = \Delta R - \Delta R_R
\]

and is the translational component in the plane normal to \( \hat{\mu} \).

The total translational motion is the sum of the displacement along the rotational axis and that given by Eq. 3.2.6

\[
\Delta R_D = \hat{\mu} \cdot (q_1 - q_0) \hat{\mu} + \Delta R_T
\]

\[
= (q_1 - q_0) + (R - I)(T - q_0)
\]

where use has been made of Eqs. 3.2.3, 3.2.4, 4.2.5 and 3.2.6 and the relation

\[
\hat{R} \hat{\mu} = \hat{\mu}
\]

since \( \hat{\mu} \) is an eigenvector of \( \hat{R} \).

3.3 Transition Transformation

Rotational positions of a body are measured with respect to a fixed reference system denoted by \((r)\). However, for much of our analysis, we are concerned with rotations from one body position to another and would like to characterize this transition transformation strictly in terms of an initial and final state.
Consider the two rotations in Fig. 3.3.1 of vector $\mathbf{r}(r)$ in the $(r)$ system to positions $\mathbf{r}_0$ and $\mathbf{r}_1$ carried out by rotation operators $\mathbf{R}_0(r)$ and $\mathbf{R}_1(r)$, respectively. These rotations can be denoted in matrix form by

$$
\mathbf{r}_0 = \mathbf{R}_0(r) \mathbf{r}_0
$$

$$
\mathbf{r}_1 = \mathbf{R}_1(r) \mathbf{r}_1.
$$

According to Eq. 2.2.5 the component transformation equations are

$$
\mathbf{r}^{(0)} = (\mathbf{R}_0(r))^{T} \mathbf{r}(r)
$$

$$
\mathbf{r}^{(1)} = (\mathbf{R}_1(r))^{T} \mathbf{r}(r).
$$

By eliminating $\mathbf{r}(r)$ in Eqs. 3.3.2 we can write

$$(1)$$

$$(0)$$

$$(r)$$

Figure 3.3.1 Relative Body Rotation Between Two Rotation Orientations
and again by Eq. 2.2.5 we also have

\[ r^{(0)} = (A^{10}) r^{(0)} \]

3.3.3

and again by Eq. 2.2.5 we also have

\[ r^{(0)} = (A^{10}) r^{(0)} = R^{(0)} r^{(0)} \]  

3.3.4

where \( R^{(0)} \) is a rotation operator in the \( (0) \) system which rotates \( r^{(0)} \) into \( r^{(1)} \). In terms of the two operators in the \( (r) \) system it can be expressed as

\[ R^{(0)} = R^{(r)} R^{(r)} \]  

3.3.5

The \( R^{(0)} \) operator is given in the \( (r) \) system according to Eq. 2.2.10 by

\[ R^{(r)} = R^{(r)} R^{(r)} R^{(r)} = R^{(r)} R^{(r)} \]  

3.3.6

3.4 Calculation of Optimal Rotation Point

The trajectory formed by the \( \Delta R_D \)'s depends on the choice of \( T \), the point about which all body rotation is assumed to occur. \( T \) may be adjusted according to the physical reasonableness of the \( \Delta R_D \) trajectory using either qualitative or quantitative analytic constraint criteria. It may be sufficient that the trajectory lie within the boundaries of a physically jointed structure or that it not exceed the dimensions of an identified constraining element.
An analytic constraint may also be defined which may be used to find a $\mathbf{T}$ that minimizes the difference between the $\Delta R_D$ trajectory. If the constraint trajectory is given by $R(X(0),Y(0),Z(0))$ then we can form

$$L = \sum_{n=1}^{N} \left[ \sum_{i=1}^{n} \Delta R_D(i) - R_n \right]^2 \tag{3.4.1}$$

where the $R_n$ corresponds to the nearest point on the constraint trajectory to the head of the $\sum_{i=1}^{n} \Delta R_D(i)$ vector. By differentiating $L$ with respect to the components of $\mathbf{T}$ and setting these equal to zero, three equations are obtained for the three unknown optimal components of $\mathbf{T}$.

To derive an expression for $\Delta R_D(i)$, we examine the first few terms. Rewriting Eq. 3.2.7 in terms of rotation operators in the $(r)$ system for $i = 1$ we get

$$\Delta R_D(1) = (q_1 - q_0) + (R_1^{(r)} R_0^{(r)} T - I)(\mathbf{T} - q_0)$$

where Eq. 3.3.6 has been used to express the transition rotation in the $(r)$ system.

The second displacement can be similarly expressed. The $\mathbf{T}$ vector, however, must be updated since it has been displaced by $\Delta R_D(1)$ due to the first body displacement.

$$\Delta R_D(2) = (q_2 - q_1) + (R_2^{(r)} R_1^{(r)} T - I)(\mathbf{T} + \Delta R_D(1) - q_1)$$

$$= (q_2 - q_1) + (R_3^{(r)} R_2^{(r)} - R_2^{(r)} T)(\mathbf{T} - q_2).$$
Updating $\tilde{T}$ by adding $\Delta R_D(1)$ and $\Delta R_D(2)$ and multiplying through by the rotation operators, the third displacement is found

$$\Delta R_D(3) = q_4 - q_3 + \left( R_4^{(r)} R_1^{(r)} T - R_3^{(r)} R_1^{(r)} T \right) (T - q_0).$$

By observing the progression of the last three displacement expressions a general relationship can be stated as follows

$$\Delta R_D(i) = q_{i+1} - q_i + \left( R_{i+1} R_1^{(r)} T - R_i^{(r)} R_1^{(r)} T \right) (T - q_0).$$

We now consider the summation of these displacement vectors by examining partial sums of them. Define

$$\Delta R_{Dn} = \sum_{i=1}^{n} \Delta R_D(i).$$

Then for $n=1$ we see from Eq. 3.4.2 that $\Delta R_{D1} = \Delta R_D(1)$.

For $n=2$

$$\Delta R_{D2} = (q_2 - q_0) + R_2^{(r)} R_2^{(r)} (T - I) (T - q_0) + (q_2 - q_1) + R_2^{(r)} R_0^{(r)} (T - R_1^{(r)} R_1^{(r)} T) (T - q_0)$$

$$= (q_2 - q_0) + (R_2^{(r)} R_0^{(r)} T - I) (T - q_0),$$

and for $n=3$
Observing the progression of these partial sums, a general expression for the partial sum to \( n \) can be written as

\[
\delta R_{dn} = \sum_{i=1}^{n} \delta R_d(i) = (q_n - q_0) + (R_n R_0 T - I)(T - q_0). \tag{3.4.5}
\]

Letting

\[
R = R_n R_0 T, \tag{3.4.6}
\]

and substituting for \( \delta R_{dn} \) from Eq. 3.4.5 into Eq. 3.4.1 we get

\[
L = \sum_{n=1}^{N} \left[ (q_n - q_0 + (R - I)(T - q_0) - R_n)^2 \right] \tag{3.4.7}
\]

The expression within the parentheses can be rewritten as

\[
\begin{align*}
\left( q_n - R_n R q_0 \right) + (R - I)T &= \left( q_n - R_n R q_0 \right) + (R - I)T \\
&= (q_n - R_n R q_0)^2 + (R - I)T \cdot (q_n - R_n R q_0) \\
&= (q_n - R_n R q_0) \cdot (R - I)T - (R - I)T \cdot (R - I)T
\end{align*}
\]
Consider the second term which can be written as
\[
R T \cdot (q' - R_{n} - R_{q_{0}}) - T \cdot (q' - R_{n} - R_{q_{0}})
\]
\[
= R T (q'_{n} - R_{n} - R_{q_{0}}) \cdot T - (q'_{n} - R_{n} - R_{q_{0}}) \cdot T
\]
\[
= \left[ (R T - I)(q'_{n} - R_{n}) + (R - I)q_{0} \right] \cdot \mathbf{T}
\]

Where use has been made of the fact that if \( R \) is an orthonormal operator, then
\[
b \cdot R a = R a \cdot b = R^T b \cdot a \quad 3.4.8
\]

where \( a \) and \( b \) are arbitrary vectors.

The third term can also be rearranged
\[
(R T - I)(q'_{n} - R_{n}) + (R - I)q_{0} \cdot \mathbf{T}
\]
\[
= \left[ (R T - I)(q'_{n} - R_{n}) + (R - I)q_{0} \right] \cdot \mathbf{T}
\]

Where again use has been made of Eq. 3.4.8. Again applying Eq. 3.4.8, this time to the last term, Eq. 3.4.7 can be expressed as
\[
L = \sum_{n=1}^{N} \left[ (q'_{n} - R_{n} - R_{q_{0}})^{2} + 2 \left[ (R T - I)(q'_{n} - R_{n}) + (R - I)q_{0} \right] \cdot \mathbf{T}
\]
\[
+ (2I - R - R^T) T \cdot \mathbf{T} \right] \quad 3.4.10
\]

Expanding and differentiating this equation with respect to \( t \); the following equation is obtained.
\[ \frac{\partial L}{\partial r_1} = \sum_{n=1}^{N} \left\{ 2 \left( (r_{11}-1) (q_n - \bar{q}_n)_1 + r_{21} (q_n - \bar{q}_n)_2 + r_{31} (q_n - \bar{q}_n)_3 \right) \right. \\
\left. + (r_{11}-1) q_{01} + r_{12} q_{02} + r_{13} q_{03} \right\} \] 3.4.10

\[ + 4(1 - r_{11}) t_1 - 2(r_{12} + r_{21}) t_2 - 2(r_{13} + r_{31}) t_3 \]

where the \( r_{ij} \) are elements of \( R \).

Differentiating \( L \) with respect to \( t_2 \) and \( t_3 \) and setting all three equations equal to zero, three linear equations with three unknown values of \( t \) result. These equations can be written in the concise form

\[ -DT_0 = b \] 3.4.11

where

\[ D = \begin{pmatrix}
\sum_{n=1}^{N} 2(r_{11}-1) & \sum_{n=1}^{N} (r_{12} + r_{21}) & \sum_{n=1}^{N} (r_{13} + r_{31}) \\
\sum_{n=1}^{N} (r_{12} + r_{21}) & \sum_{n=1}^{N} 2(r_{22}-1) & \sum_{n=1}^{N} (r_{23} + r_{32}) \\
\sum_{n=1}^{N} (r_{13} + r_{31}) & \sum_{n=1}^{N} (r_{23} + r_{32}) & \sum_{n=1}^{N} 2(r_{23}-1)
\end{pmatrix} \] 3.4.12
and

\[
\begin{align*}
\mathbf{b}_1 &= \sum_{n=1}^{N} \left[ (r_{11}-1)(q_n-R_n)_1' + 21(q_n-R_n)_2' + r_{31}(q_n-R_n)_3' \\
&\hspace{1cm} + (r_{11}-1)q_0 + r_{12}q_0 + r_{13}q_0 \right] \\
\mathbf{b}_2 &= \sum_{n=1}^{N} \left[ r_{12}(q_n-R_n)_1' + (r_{22}-1)(q_n-R_n)_2' + r_{32}(q_n-R_n)_3' \\
&\hspace{1cm} + r_{21}q_0 + (r_{22}-1)q_0 + r_{23}q_0 \right] \\
\mathbf{b}_3 &= \sum_{n=1}^{N} \left[ r_{13}(q_n-R_n)_1' + r_{23}(q_n-R_n)_2' + (r_{33}-1)(q_n-R_n)_3' \\
&\hspace{1cm} + r_{31}q_0 + r_{32}q_0 + (r_{33}-1)q_0 \right]
\end{align*}
\]

The optimal value of $\mathbf{T}$ is then obtained by multiplying Eq. 3.4.11 by the inverse of $\mathbf{D}$,

\[
\mathbf{T}_0 = \mathbf{D}^{-1}\mathbf{b}
\]

The point $\mathbf{R}_n$ lies on the trajectory or space curve given by $\mathbf{R}$ and corresponds to the point closest to $\mathbf{R}_{Dn}$ as shown in Fig. 3.4.1. The vector from $\mathbf{R}_{Dn}$ to the nearest point on the space curve is normal to the curve, and if $\mathbf{t}$ is a tangent vector along the curve, then there is an $\mathbf{R}_n$ for which the following relation holds

\[
\mathbf{t} \cdot (\mathbf{R}_n - \mathbf{R}_{Dn}) = 0
\]
The tangent to the space curve is given by

\[
\mathbf{t} = \frac{d\mathbf{R}}{ds}
\]

where \( s \) is a trajectory parameter. Substituting Eq. 3.4.16 into Eq. 3.4.15 and writing the resulting equation out explicitly we get

\[
\frac{dx}{ds}(x - \Delta x_{D_n}) + \frac{dy}{ds}(y - \Delta y_{D_n}) + \frac{dz}{ds}(z - \Delta z_{D_n}) = 0
\]

where \( \mathbf{R} = [x, y, z]^T \) and the \( x, y, \) and \( z \) are functions of \( s \). Eq. 3.4.17 must be solved for \( s \), which then, in turn, is substituted into \( \mathbf{R} \) to give \( \mathbf{R}_n \).
As an example, consider a trajectory described by a rotation of a constant magnitude vector, \( \mathbf{r}_0 \), about the axis \( \mathbf{u} = \left[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right] \) through an angle \( \theta \) as shown in Fig. 3.4.2.

This rotation can be expressed as a rotation about \( \mathbf{u} \) through an angle of 45° followed by a rotation about \( \mathbf{u} \) through an angle \( \phi \). Acting on vector \( \mathbf{r}_0 \), the rotation results in the vector \( \mathbf{r}_2 \) according to

\[
\mathbf{r}_2 = R^{(0)}_{2u}(\phi)R^{(0)}_{1x}(45°)\mathbf{r}_0.
\]

The first rotation, \( R^{(0)}_{1x} \), aligns the rotated z axis with \( \mathbf{u} \) and, therefore, the second rotation can be expressed

Figure 3.4.2 Example Space Curve Define by Constant Radius Rotation about \( \mathbf{u} \) Axis
in the rotated system by

\[
R_{2z}^{(1)}(\phi) = \begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\] 3.4.19

According to Eq. 2.2.10, this can be transformed to the original systems \((0)\) by

\[
R_{2z}^{(0)}(\phi) = R_{1x}^{(0)}(45^\circ) R_{2z}^{(1)}(\phi) R_{1x}^{(0)T}(45^\circ).
\] 3.4.20

Substituting Eq. 3.4.20 into Eq. 3.4.18 we get

\[
\begin{pmatrix} r_2 \end{pmatrix} = R_{1x}^{(0)}(45^\circ) R_{2z}^{(1)}(\phi) \begin{pmatrix} r_0 \end{pmatrix}.
\] 3.4.21

The \(R_{1x}^{(0)}(45^\circ)\) is given by

\[
R_{1x}^{(0)}(45^\circ) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos 45^\circ & -\sin 45^\circ \\
0 & \sin 45^\circ & \cos 45^\circ
\end{pmatrix}
\] 3.4.22

Substituting transformations given by Eq. 3.4.19 and Eq. 3.4.22 into Eq. 3.4.21 we get

\[
\begin{pmatrix} r_2 \end{pmatrix} = \begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\frac{1}{\sqrt{2}} \sin \phi & \frac{1}{\sqrt{2}} \cos \phi & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \sin \phi & \frac{1}{\sqrt{2}} \cos \phi & \frac{1}{\sqrt{2}}
\end{pmatrix} \begin{pmatrix} r_0 \end{pmatrix}.
\] 3.4.23

Choosing \(\begin{pmatrix} r_0 \end{pmatrix} = [r_0, 0, 0]^T\) we get
\[ R = r_2 = \begin{bmatrix} r_0 \cos \phi \\ \frac{r_0}{\sqrt{2}} \sin \phi \\ \frac{r_0}{\sqrt{2}} \sin \phi \end{bmatrix} \] 3.4.24

Taking the derivative of \( \dot{R} \) with respect to \( \phi \)

\[ \frac{d\dot{R}}{d\phi} = \begin{bmatrix} -r_0 \sin \phi \\ \frac{r_0}{\sqrt{2}} \cos \phi \\ \frac{r_0}{\sqrt{2}} \cos \phi \end{bmatrix} \] 3.4.25

Substituting Eqs. 3.4.24 and 3.4.25 into Eq. 3.4.17 we get

\[
(-r_0 \sin \phi)(r_0 \cos \phi - \Delta x_{Dn}) + \left( \frac{r_0}{\sqrt{2}} \cos \phi \right) \left( \frac{r_0}{\sqrt{2}} \sin \phi - \Delta y_{Dn} \right) \\
+ \left( \frac{r_0}{\sqrt{2}} \cos \phi \right) \left( \frac{r_0}{\sqrt{2}} \sin \phi - \Delta z_{Dn} \right) = 0
\]

The solution of this for \( \phi_n \) is

\[ \phi_n = \tan^{-1} \left( \frac{\Delta z_{Dn} + \Delta y_{Dn}}{\sqrt{2} \Delta x_{Dn}} \right) \] 3.4.26

Thus, for a given \( n \), the components of \( \Delta R_{Dn} \) are substituted into Eq. 3.4.26 which gives a value of \( \phi_n \).
This \( \phi_n \) is, in turn, substituted into Eq. 3.4.24 to give \( \bar{R}_n \).

3.5 Calculation of Rotation Point Trajectory

The trajectory of a point on the body in space about which rotation is calculated may, in general, follow an arbitrary path and could be very difficult to characterize analytically with well behaved functions. The trajectory composed of vector elements connecting the points at which the rotations are taken are piecewise continuous and do not have continuous derivatives. What we seek is an approximating trajectory which has continuous derivatives as well as a tractable analytic form that can be used for the calculation of the optimal center of rotation and provide a concise quantitative description of the center of rotation motion.

A particularly convenient form is possible if we can separate the motion into independent components with one component along the average rotational axis and the other in the plane normal to this axis. The motions considered in this study are amenable to this type of separation since one of the experimental requirements is that the forced motion be such that the point of force application be constrained to the plane normal to the gravitational vector. While this constrains one point of the upper arm to this plane, some complex motion of the upper arm is still possible and is actually induced by the anatomical joint structure as the arm is rotated. Nonetheless, the
motion is relatively uniform and the individual transition rotation axes are reasonably closely distributed about their average.

A quantitative indicator of how much the motion of the upper arm deviates from the plane is the standard deviation of the rotation vector about the mean rotation direction. The mean or average rotation vector is given by

\[ \hat{\mathbf{u}}_a = \frac{1}{N} \sum_{i=1}^{N} \hat{\mathbf{u}}_i \]  

where \( \hat{\mathbf{u}}_i \) are the individual normal rotation vectors.

The variances of the distributions of the components of \( \hat{\mathbf{u}}_i \) are given by

\[ \sigma_j^2 = \frac{1}{N} \sum_{i=1}^{N} \left( \mathbf{u}_{aj} - \hat{\mathbf{u}}_{ij} \right)^2 \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \left( \mathbf{u}_{aj}^2 - 2\mathbf{u}_{aj}\hat{\mathbf{u}}_{ij} + \hat{\mathbf{u}}_{ij}^2 \right) \]

\[ = \frac{1}{N} \left( N\mathbf{u}_{aj}^2 - 2\mathbf{u}_{aj} \sum_{i=1}^{N} \hat{\mathbf{u}}_{ij} + \sum_{i=1}^{N} \hat{\mathbf{u}}_{ij}^2 \right) . \]

Substituting for the jth component from Eq. 3.5.1 for the sum in the second term we get

\[ \sigma_j^2 = \frac{1}{N} \left( \sum_{i=1}^{N} \hat{\mathbf{u}}_{ij}^2 - N\mathbf{u}_{aj}^2 \right) . \]  

3.5.2

The standard deviation is defined as the square root of the variance and is thus given by
We would like to find a rotation which rotates the $y(r)$ axis into alignment with $\hat{u}_a(r)$. The axis of rotation to produce this alignment is given by

$$\hat{u}_a(r) = y(r) \times \hat{u}_a(r)$$  \hspace{1cm} (3.5.4)

and the angle of rotation by

$$\hat{z} = \cos^{-1} (y(r) \cdot \hat{u}_a(r))$$  \hspace{1cm} (3.5.5)

Substituting the angle of rotation from Eq. 3.5.5 and the normalized rotation axis vector from Eq. 3.5.4 into Eq. 2.1.6 yields the rotation operator $R(r)$ which rotates $y(r)$ into alignment with $\hat{u}_a(r)$

$$\hat{u}_a(r) = R(r) y(r)$$  \hspace{1cm} (3.5.6)

We can also apply this rotation operator to rotate the $x(r)$ and $z(r)$ axes vectors to produce a new coordinate system which we will designate by $(y)$. According to Eq. 2.2.5 the components of a vector in the $(r)$ system are transformed to the $(y)$ system by $R(r)^T$. Using this transformation individual rotation vectors can be transformed to the $(y)$ system by

$$\hat{u}_i(y) = R(r)^T \hat{u}_i(r) = A^y \hat{u}_i(r)$$  \hspace{1cm} (3.5.7)
We want to examine the distribution of the rotation vectors in the (y) system. To do this, we first apply $A_{yr}$ to both sides of Eq. 3.5.1

$$A_{yr} \hat{\mu}_a = \frac{1}{N} \sum_{i=1}^{N} A_{yr} \hat{\mu}_i = \frac{1}{N} \sum_{i=1}^{N} \hat{\mu}_i = \hat{\mu}_a$$  \hspace{1cm} 3.5.8

let

$$\hat{\mu}_a = \alpha \hat{\mu}_a$$  \hspace{1cm} 3.5.9

where

$$\alpha = \sqrt[\nu_a(\gamma)]{\hat{\mu}_a(\gamma)}$$  \hspace{1cm} 3.5.10

then substituting for $\hat{\mu}_a$ from Eq. 3.5.9 into Eq. 3.5.8 we get

$$\alpha A_{yr} \hat{\mu}(\gamma) = \hat{\mu}_a(\gamma)$$  \hspace{1cm} 3.5.11

By multiplying Eq. 3.5.6 by $R(r)T$ and substituting in Eq. 3.5.11 and noting that $A_{yr} = R(r)T$ we get

$$\hat{\mu}_a(\gamma) = \begin{bmatrix} \mu_{a1} \\ \mu_{a2} \\ \mu_{a3} \end{bmatrix} = \alpha y(\gamma) = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix}$$  \hspace{1cm} 3.5.12

The variances in the (y) system are then given by
\[
\sigma_1^2 = \frac{1}{N} \sum_{n=1}^{N} \left( \mu_{i1} \right)^2 \\
\sigma_3^2 = \frac{1}{N} \sum_{n=1}^{N} \left( \mu_{i3} \right)^2
\]

and the standard deviations by \( \sigma_i \).

The \( \sigma_1^2 \) and \( \sigma_3^2 \) can be viewed as the semimajor axes of an ellipse in the \( X(Y)Z(Y) \) plane as shown in Fig. 3.5.1.

**Figure 3.5.1 Standard Deviative Ellipse for the Distribution of Rotation Axes**
This ellipse indicates the distribution of the rotation axes about the average rotation axis during body motion. If $\sigma_{1}^{(y)}$ and $\sigma_{3}^{(y)}$ are small, a plane trajectory for the rotation point may be assumed.

Consider a trajectory described by the rotation points as shown in Fig. 3.5.2 which are assumed to lie in the $X^{(y)} Z^{(y)}$ plane. If we choose the $Z$ axis in the $(y)$
system as it is rotated to define the position of each rotation point, then the trajectory of these points in the reference system \( r \) can be expressed as

\[
\hat{r}_i = \hat{T}(r) + \hat{L}(r) + R(r) Y(\phi_i) \begin{bmatrix} 0 \\ 0 \\ r_3(\phi_i) \end{bmatrix}
\]

where \( \hat{T}(r) \) is the assumed center of rotation, \( \hat{L}(r) \) is an average local center of rotation, \( R(r) \) is the rotation operator as given in Eq. 3.5.6 and which here acts to transform vector components from the \( (y) \) system to the \( (r) \) system, \( R_y(\phi_i) \) is a rotation operator in the \( (y) \) system which rotates a vector in the \( X^Y Z^Y \) plane into alignment with a given rotation point \( i \), and the last vector has the amplitude of the sum of the \( X \) and \( Z \) components of the \( i \) rotation point in the \( (y) \) system and lies along the \( Z^Y \) axis. We can perform another transformation on the rotation point data, as we will do later, which aligns the \( Z \) axis with the first rotation point. For this point \( \phi_1 = 0 \), and the rotation point vector in this system will have zero \( X \) and \( Y \) components.

The \( \hat{T}(r) \) and \( \hat{L}(r) \) could be combined into one vector but are kept separate since \( |\hat{T}| \gg |\hat{L}| \) and \( \hat{L} \) defines an approximate center of local rotation.

We will now calculate a trajectory for the assumed rotational point \( \hat{T} \), with respect to the \( (y) \) system and develop a method for calculating \( \hat{L} \).
Transforming the total translational displacement vector given by Eq. 3.5.7 from the reference system $r$ to the $y$ system according to Eq. 3.5.7 we get

$$\Delta R_D(y) = A^y r \Delta R_D(r)$$

$$= A^y r \left( \bar{\sigma}_1(r) - \bar{\sigma}_0(r) + (R + I) \bar{\sigma}_1(r) - \bar{\sigma}_0(r) \right).$$

3.5.15

We denote the component along the average screw axis, which in the $y$ system lies along the $Y(y)$ axis, by $\Delta R_Y(y)$ and the vector component lying in the $X(y) Z(y)$ plane by $\Delta R_p(y)$.

We now turn to the calculation of the vector $L$ which is to serve as an approximate center of rotation for the $\Delta R_p$ components. We do not require a precise center but rather a local offset point from which to reference the positions of the rotation points.

For each $\Delta R_{p_i}$ we will calculate an $\tilde{L}_i$ and then calculate an average $\tilde{L}$ position. The $\tilde{L}_i$ is the vector lying in the $X(y) Z(y)$ plane, normal to $\Delta R_{p_i}$ and extending from the mid-point of $\Delta R_{p_i}$ to the center of rotation, or the $\bar{\sigma}_1(y)$ axis, as shown in Fig. 3.5.3. From the figure it follows that

$$|\tilde{L}_i| = \frac{|\Delta R_{p_i}|}{(2 \tan(\bar{\sigma}_1/2))}$$
Figure 3.5.3 Instantaneous Rotation Points for Rotation Point Displacements in the Y System

where the $\theta_{i}$ are the rotation angles given by Eq. 3.1.4 and the vectors are taken in the $(y)$ system. The $\vec{L}$ vector is the average rotational center for the chords $\Delta R_{p_{i}}$ and is given by
\[
\mathbf{L}^\prime(y) = \frac{1}{N} \left[ \left( \frac{\Delta R_{p1}}{2} + \mathbf{L}_1 \right) + \left( \frac{\Delta R_{p2}}{2} + \mathbf{L}_2 \right) + \left( \frac{\Delta R_{p3}}{2} + \mathbf{L}_3 \right) + \ldots + \left( \frac{\Delta R_{p(N-1)}}{2} + \mathbf{L}_{N-1} \right) \right] + \frac{\Delta R_{pN}}{2} + \mathbf{L}_N \\
= \frac{1}{N} \left[ \sum_{i=1}^{N} \left( \frac{\Delta R_{p1}}{2} + \mathbf{L}_1 \right) + \Delta R_{p(N-1)} + 2\Delta R_{p(N-2)} + \ldots + (N-1)\Delta R_{p1} \right] 
\]

The \( A^Y_e \) transformation aligned the \( (r) \) system \( y \) axis with the average screw axis vector \( \mathbf{\mu}_a \), but imposed no constraint on the alignment of the \( X^Y \) or \( Z^Y \) axes. We now choose an alignment of these axes such that the \( Z^Y \) axis coincides with the \( L^Y \) direction, but is oppositely directed. This is effected by the rotation such that

\[
R^Y_y(\psi) \begin{bmatrix} L_1 \\ 0 \\ L_3 \end{bmatrix}^{(Y)} = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}^{(Y)} 3.5.18
\]

where

\[
R^Y_y(\psi) = \begin{pmatrix} \cos \psi & 0 & \sin \psi \\ 0 & 1 & 0 \\ -\sin \psi & 0 & \cos \psi \end{pmatrix} 3.5.19
\]
The transformation of vector components between the system produced by the \( R_y(y) \) rotation denoted by \( (y') \) and the \( (y) \) system is according to Eq. 2.2.2 given by

\[
\mathbf{r}'(y') = R_y(y) \mathbf{T} \mathbf{r}(y)
\]

\[
= A_y' r'(y) \quad .
\] 3.5.21

By eliminating \( \mathbf{r}(y) \) between Eqs. 3.5.7 and 3.5.21 we get

\[
\mathbf{r}(y) = A_y' y A_y \mathbf{r}'(r)
\]

\[
= A_y' r'(r) \quad .
\] 3.5.22

Using Eq. 3.5.22 and from Fig. 3.5.2 we can express the location of the first displaced rotation point by

\[
\mathbf{R}_1 = \mathbf{T}(r) + A_y' L(y') + A_y' R_y(y') r(z_1)
\]

\[
= \begin{bmatrix}
0 \\
0 \\
\mathbf{r}(z_1)
\end{bmatrix}
\] 3.5.23
where

\[
\begin{bmatrix}
0 \\ \\
0 \\ r(\phi_1)
\end{bmatrix}
\cdot
\begin{bmatrix}
\Delta R_P^{(y')} \\
\Delta \cdot \\
r(\phi_1)
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
r(0)
\end{bmatrix}
\]

and for any \( \phi_n \) we can write

\[
\begin{bmatrix}
0 \\ \\
0 \\ r(\phi_n)
\end{bmatrix}
\cdot
\begin{bmatrix}
\Delta R_P^{(y')} \\
\Delta \cdot \\
r(\phi_n)
\end{bmatrix}
= \sum_{n=1}^{N} \begin{bmatrix}
\Delta R_P^{(y')} \\
\Delta \cdot \\
r(0)
\end{bmatrix}
\]

which is shown in Fig. 3.5.4. The position for any rotation point of the trajectory in the \((r)\) system can then be expressed as

\[
\begin{bmatrix}
0 \\
0 \\
r(\phi_n)
\end{bmatrix}
\cdot
\begin{bmatrix}
\Delta R_P^{(y')} \\
\Delta \cdot \\
r(\phi_n)
\end{bmatrix}
= \sum_{n=1}^{N} \begin{bmatrix}
\Delta R_P^{(y')} \\
\Delta \cdot \\
r(0)
\end{bmatrix}
\]

where \( \Delta R_n^{(y')} \) is strictly a function of \( \phi_n \) which is given by

\[
\phi_n = \tan^{-1}\left( \frac{\sum_{j=1}^{n} \Delta R_{Pj}^{(y')}}{r(0) + \sum_{j=1}^{n} \Delta R_{Pj}^{(y')}} \right)
\]

for \( n \geq 1 \) and \( \phi_0 = 0 \).
Figure 3.5.4 Rotation Point for the n-th Displacement

and $R_{\gamma}(Y')(\phi_n)$ has the same form as in Eq. 3.5.19 except $\phi_n$ is substituted for $\gamma_0$.

Eq. 3.5.25 differs from Eq. 3.5.14 in that the initial alignment of $r(Y')(\phi_n)$ coincides with the $-L(Y')$ direction and $\gamma_0=0$. The $r(\phi_n)$ is a discrete valued function of $\phi_n$ and consequently $\bar{R}_n$ is also a discrete valued function. We would like to have $\bar{R}$ as a continuous function of $\gamma$. 
Consider the rotational points plotted in rectangular coordinates as shown in Fig. 3.5.5. If we have chosen $L$ to be a reasonable center of rotation the changes in $r(\phi_n)$ as $\phi$ varies from 0 to its maximum value, $\phi_m$, will be small. Also, local fluctuations between adjacent $r_n$ values will be small. Under these conditions, we can use linear interpolation between adjacent $r_n$ points to generate a set of equally spaced points in $\phi$ for the $r$'s.

For later computational convenience we choose to break up the 0 to $\phi_m$ interval into $2^m$ subintervals. The number of rotational points calculated during the course of an upper arm sweep ranges from about 40 to 100. Thus, to provide several interpolated points between each of the rotation points, a value of 8 was chosen for $m$ leading to 256 points for the total trajectory.

![Figure 3.5.5](image)

**Figure 3.5.5** The Rotation Point Trajectory in the $xz$ Plane of the $Y$ System Expressed in Polar Coordinates $r(\phi_n)$ and $\phi_n$ and Plotted in Rectangular Coordinates
Using the explicit form for $R_y(y')$ from Eq. 3.5.19 in Eq. 3.5.24 and defining

$$\Delta R_p(n) = \sum_{i=1}^{N} \Delta R_p(y')$$

we can solve for $r(\phi_n)$

$$r(\phi_n) = \left( \left[ \Delta R_p(n) \right]^2 + (r(0) + \left[ \Delta R_p(n) \right]^2) \right)^{1/2} \quad 3.5.28$$

The equation of a straight line between two adjacent points $n$ and $n+1$ is given by

$$r_{n,n+1} = \frac{(\phi - \phi_n)r_{n+1} - (\phi - \phi_{n+1})r_n}{\phi_{n+1} - \phi_n} \quad 3.5.29$$

where $\phi_n$ is given by Eq. 3.5.26 and $r_n$ by Eq. 3.5.28. By applying this between adjacent points from $n=0$ to $n=m$, we generate an array of equally spaced in $\phi$ values for $r$ which we denote by $r_\ell$ where $\ell$ varies from 0 to 256, and the incremental value in $\phi$ is denoted by $\Delta \phi$ and is given by

$$\Delta \phi = \frac{\phi_m}{256} \quad 3.5.30$$

This discrete, equally spaced array, is highly amendable to Fourier series expansion in the form
An International Mathematical and Statistical Libraries (IMSL) program which calculates the Fast Fourier Transform for a real valued equally spaced array was used to obtain the $A_n$ and $B_n$ coefficients. It was found that to maintain calculation accuracy up to 20 terms in Fourier, the expansion given by Eq. 3.5.31 had to be retained.

To reduce the number of necessary terms, several transformations of the data were made as shown in Fig. 3.5.6. The original data has the typical form given by $r(\phi)$. The large discontinuities at $\phi_0$ and $\phi_m$ introduce relatively large "high frequency" contributions. These were considerably reduced by subtracting $r_0$ from $r(\phi)$ producing the form shown for $r(\phi) - r_0$. To remove the discontinuity at $\phi = \phi_m$, the points were rotated about an axis normal to

$$r(\phi) = \sum_{n=0}^{128} \left[ A_n \cos \frac{2\pi n \phi}{\phi_m} + B_n \sin \frac{2\pi n \phi}{\phi_m} \right]$$

3.5.31

where

$$A_n = \frac{2}{\phi_m} \int_0^{\phi_m} r(\phi) \cos \frac{2\pi n \phi}{\phi_m} \, d\phi$$

3.5.32

and

$$B_n = \frac{2}{\phi_m} \int_0^{\phi_m} r(\phi) \sin \frac{2\pi n \phi}{\phi_m} \, d\phi$$

3.5.33

An International Mathematical and Statistical Libraries (IMSL) program which calculates the Fast Fourier Transform for a real valued equally spaced array was used to obtain the $A_n$ and $B_n$ coefficients. It was found that to maintain calculation accuracy up to 20 terms in Fourier, the expansion given by Eq. 3.5.31 had to be retained.

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Figure 3.5.6 Transformations Performed on $r(\phi_0)$ to Improve Fourier Expansion Accuracy

$r(\phi)$ and $\phi$ according to

$$r'(\phi) = -(r - r_0) \sin \alpha + (r - r_0) \cos \alpha$$  \hspace{1cm} 3.5.34

where

$$\alpha = \tan^{-1} \left( \frac{r(\phi_m) - r_0}{\phi_m} \right).$$  \hspace{1cm} 3.5.35

The Fourier expansion of the data in the form given by Eq. 3.5.34 required only three to five terms to attain the same accuracy as was obtained originally using 20 terms with the data in the form given by Eq. 3.5.29. After performing the Fourier expansion, inverse transformations were made to obtain $r(\phi)$. 
The displacement of the rotation point along the average rotation axis is given by the summation of the \( \Delta R_y \) individual displacements. An analytic form for this displacement was obtained by a third order least squares polynomial fit to \( \phi \) of the form

\[
S(\phi) = B_0 + B_1\phi + B_2\phi^2 + B_3\phi^3 \quad . \tag{3.5.36}
\]

The total analytic rotation point displacement is obtained by recombining the displacement in the plane normal to the average rotation plane given by Eq. 3.5.25 with \( r(\phi) \) given by Eq. 3.5.31 with that along the average rotation axis given by Eq. 3.5.36.
4.1 General Considerations for Determining Three-Dimensional Body Position

The determination of the stresses acting across a joint coupling two segments as a function of the relative three-dimensional positions of the two segments requires knowledge of the forces and moments acting on one segment and the relative segment positions if the other segment can be taken as fixed. In the present study, it is assumed that for the calculation of internal joint forces, the segments are coupled by a ball and socket joint. Also, forces and moments contributing to motion about the long axis of the moving segment are neglected. This results in it being possible to characterize the joint resistive properties by two orthogonal moment components in the moveable segment coordinate system.

The positions of segments were determined using two methods. For rotations, where the axes and angles of rotation were known a priori, a three-dimensional coordinate measuring system was used. This system consisted of an assemblage of linear microphones which could locate a point in three-dimensional space by detecting times of arrival of acoustic signals generated by a spark emitter.
placed at the point of interest. A minimal of three spark emitters were placed at fixed positions on a body and their coordinates recorded. From these coordinates, the body position was established.

A method for establishing body positions given three points on the body is described in Section 4.2. Section 4.3 shows that any rotational position for a ball and socket joint with arbitrary rotational position about the long body axis can be characterized by two variables. Rotations of the upper arm with respect to the fixed torso are described in Section 4.4 and total seat and upper arm rotational transformations are given in Section 4.5. Section 4.6 describes the transformations of the measured force data and shows that transformation to the upper arm system reduces the net force vector to two independent components.

4.2 Specification of Relative Segment Positions

The position of a segment in three-dimensional space is uniquely determined by three noncollinear points. These three points are specified by nine independent parameters, however, since the points are fixed in the rigid body, three equations of constraint exist fixing the relative distances between any two points. Thus, the nine parameters consisting of the three point coordinates of the three points are reduced to six independent parameters by the three equations of constraint. This is equal to the number
of independent parameters required to specify the position of a rigid body in three-dimensional space.

The rotational orientation of a segment can be specified with respect to some reference system by a cosine matrix in which each row consists of the cosines that each orthogonal coordinate axis of the segment forms with the triad of reference axes. Translational position is specified by the vector from the origin of the reference coordinate system to the origin of the segment coordinate system.

A method is developed for calculating the cosine matrix, $A$, and the translational displacement vector of a segment with respect to the reference system given three noncollinear points $A_1$, $A_2$ and $A_3$ on the segment. As shown in Section 2.2, $A$ equals $R^T$ where $R$ is the matrix rotation operator which can be viewed as producing the rotation of the segment to its measured position from original coordinate system alignment with the reference system.

Consider the point, or spark element, placement, as shown in Fig. 4.1 1 with the locations designated by $\vec{A}_1$, $\vec{A}_2$ and $\vec{A}_3$ in the laboratory reference system ($r$).

We first form the vector

$$\vec{R}_1 = \vec{A}_2 - \vec{A}_1$$

(normalize it,

$$\vec{R}_1 = \frac{\vec{R}_1}{|\vec{R}_1|}$$

4.2.1)
then form a second vector

\[ \mathbf{R}_{2I} = \mathbf{A}_3 - \mathbf{A}_2 \]  

and take

\[ \mathbf{R}_2 = \mathbf{R}_{2I} - (\mathbf{R}_1 \cdot \mathbf{R}_{2I}) \mathbf{R}_1 \]  

which is a vector normal to \( \mathbf{R}_1 \).

After normalizing \( \mathbf{R}_2 \) we can calculate \( \mathbf{R}_3 \) from

\[ \mathbf{R}_3 = \mathbf{R}_1 \times \mathbf{R}_2 \]  

The cosine matrix relating the rotational orientation of the segment embedded orthogonal vector triad to the
laboratory reference axes is then given by

\[
A_{br} = \begin{pmatrix}
\hat{R}_1 \\
\hat{R}_2 \\
\hat{R}_3
\end{pmatrix}
\]  \hspace{1cm} 4.2.5

and the components of a vector in the two systems with common origin are related by

\[
\mathbf{r}^{(b)} = A_{br} \mathbf{r}^{(r)}
\]  \hspace{1cm} 4.2.6

The center of the body coordinate system is defined by

\[
\mathbf{R}_A = \mathbf{A}_2 + (\mathbf{R}_1 \cdot \mathbf{R}_2) \mathbf{R}_1
\]  \hspace{1cm} 4.2.7

It is desirable to define another axes system embedded in the segment associated with certain surface features or landmarks of the segment. Again, three vectors on the segment \( \mathbf{B}_1, \mathbf{B}_2 \) and \( \mathbf{B}_3 \) can be measured and the matrix \( \mathbf{B} \) and the vector \( \mathbf{R}_B \) calculated. The components of a vector in this system are related to those in the reference, assuming common origin, by

\[
\mathbf{r}^{(\ell)} = \mathbf{B}^{\ell r} \mathbf{r}^{(r)}
\]  \hspace{1cm} 4.2.8

The coordinates of a vector in the body and landmark systems are related by
for a common origin, where \( C^b_\ell \) remains constant if the segment remains rigid.

4.3 Generalized Rotation Described by Two Parameters

The calculation of the moments acting on a segment as a function of the rotational orientation of that segment must be expressed in terms of variables which characterize that rotational orientation. In the general case, three independent variables are required to define the relative rotational orientation of one body with respect to another. If a cosine matrix is used, the three independent parameters are obtained from the nine matrix elements with the six orthogonality conditions between the rows or columns of the matrix. If sequential elementary rotations are used, then the three parameters are the angles of rotation taken about nonconsecutive axes. A third method is by use of Rodriguez formula or quaternion operation. For any one of these methods, an axis of rotation, \( \hat{u} \), and angle of rotation, \( \phi \), must be specified. Since only three independent parameters are required, the constraint condition \( \sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2} = 1 \) is used.

In the present study, only two orientational parameters were used, as the internal joint resistive moment properties...
were assumed to be independent of rotations about the segment long axis, henceforth, taken as the segment Z axis. This Z axis can be rotated to any position by a rotation of the segment about an axis, \( \hat{u} \), lying in the fixed \( X(0) \) \( Y(0) \) plane as shown in Fig. 4.3.1. If \( \hat{u} \) makes a positive angle \( \phi' \) with the \( X(0) \) axis, then \( u_1 = \cos \phi' \), \( u_2 = \sin \phi' \) and \( u_3 = 0 \); and the angle of rotation about \( \hat{u} \) is \( \phi \), then according to Eq. 2.1.6

![Figure 4.3.1. Two Parameter Rotation of the Body z Axis to an Arbitrary Position about Fixed ç Axis.](image-url)
\[
Z_1^{(0)} = R_{1\nu}^{(0)}(\theta) Z_0^{(0)}
\]

\[
\begin{bmatrix}
\cos^2\phi'(1-\cos\theta) + \cos\theta & \cos\phi' \sin\phi'(1-\cos\theta) & \sin\phi' \sin\theta \\
\cos\phi' \sin\phi'(1-\cos\theta) & \cos^2\phi'(1-\cos\theta) + \cos\theta & -\cos\phi' \sin\theta \\
-\sin\phi' \sin\theta & \cos\phi' \sin\theta & \cos\theta
\end{bmatrix} \begin{bmatrix} Z_0^{(0)} \\
\end{bmatrix}
\]

4.3.1

\(R_{1\nu}^{(0)}(\theta)\) can also be derived by applying a sequence of elementary rotations to the segment in the \(X^{(0)}Y^{(0)}Z^{(0)}\) reference system as shown in Fig. 4.3.2. The segment is first rotated through \(-\phi'\) about the \(Z^{(0)}\) axis by the operator \(R_{1Z}^{(0)}(-\phi')\) to align \(\hat{y}\) with \(X^{(0)}\). The segment is then rotated about \(X^{(0)}\) through angle \(\theta\) using \(R_{2X}^{(0)}(\theta)\), and then a final rotation is made about \(Z^{(0)}\) to return \(\hat{z}\) to its original position with \(R_{3Z}^{(0)}(\phi')\).

The total rotation operator is

\[
R_{1\nu}^{(0)}(\theta) = R_{3Z}^{(0)}(\phi') R_{2X}^{(0)}(\theta) R_{1Z}^{(0)}(-\phi')
\]

\[
= R_{1Z}^{(0)}(\phi') R_{2X}^{(0)}(\theta) R_{1Z}^{(0)}T(\phi')
\]

4.3.2

since \(R_{1Z}^{(0)}(\phi')\) and \(R_{3Z}^{(0)}(\phi')\) are identical, \(R_{1Z}^{(0)}(-\phi') = R_{1Z}^{(0)}T(\phi')\), and \(R_{1\nu}^{(0)}(\theta)\) and \(R_{2X}^{(0)}(\theta)\) are related by a similarity transformation analogous to Eq. 4.3.1.

From Eq. 4.2.5 and Eq. 4.3.1, the components of a vector in the segment coordinate system \(X^{(1)}Y^{(1)}Z^{(1)}\) are
Figure 4.3.2. Alternate Two Parameter Rotation Sequence, related to the components in the fixed system \( X^{(0)}Y^{(0)}Z^{(0)} \) by

\[
\mathbf{r}^{(1)} = R_{(0)}^{(0)}T(\phi) \mathbf{r}^{(0)} = A^{10} \mathbf{r}^{(0)} \quad . \quad 4.3.3
\]

The components of the segment \( Z \) axis in the fixed system are then given by

\[
Z_1^{(0)} = (A^{10})^T Z_1^{(1)} = \begin{bmatrix}
\sin \phi \sin \theta \\
-\cos \phi \sin \theta \\
\cos \theta
\end{bmatrix} \quad \quad \quad 4.3.4
\]
Then for a rotational orientation given by $A^{10}$ we can find the equivalent rotation as described above by equating the $Z_1^{(0)}$ components to the third row elements of $A^{10}$

$$\sin\phi'\sin\theta = A_{31}$$
$$-\cos\phi'\sin\theta = A_{32}$$
$$\cos\theta = A_{33}$$

and solving for $\phi'$ and $\theta$

$$\phi' = \tan^{-1}\left(\frac{-A_{31}}{A_{32}}\right)$$
$$\theta = \cos^{-1} A_{33}$$

4.4 Torso Positioning

The data gathering procedure requires forcing a moveable segment, the upper arm, through an arc in a plane normal to the gravitational vector while monitoring the positions of the moving segment; the location, direction and magnitude of force applications; and the position of the fixed segment.

Since the arcs of motion can be confined to a horizontal plane, the fixed segment must be repositioned for each sweep. This is accomplished by a combination of rotations of the seat to which the subject and the proximal segment, in this case, the torso, are assumed to be rigidly fixed.
Consider the seat as shown in Fig. 4.4.1 with the \( X^0, Y^0, Z^0 \) coordinate system denoting its initial position.

The relative orientation of the seat within the laboratory reference frame \( X^r, Y^r, Z^r \) is given by \( A^{0r} \) where

\[
\vec{r}(0) = A^{0r} \vec{r}(r)
\]

and \( \vec{r}(0) \) and \( \vec{r}(r) \) denote the same vector in the chair system and laboratory reference system, respectively.

The subject is assumed to be rigidly attached to the seat, therefore, rotations of the seat result in identical rotations of the body. Due to the seat design, only rotations used to place the torso in position for the testing of the right shoulder for sweeps below the shoulder level are \( R_{1Z}^{(0)}(90^\circ) \) and \( R_{2Y}^{(0)}(\alpha - 90^\circ) \) where \( \alpha \) is the angle of the seat \( z \) axis above a horizontal plane. The seat coordinate axes and the relative positions of the seat rotational axes are shown in Fig. 4.4.2.

![Diagram of seat and initial torso coordinate system alignment](image)
Figure 4.4.2. Relative Place of Seat Rotation Axes.

Explicitly, these are

\[
R_{1Z}^{(0)}(+90^\circ) = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
R_{2Y}^{(0)}(\alpha - 90^\circ) = \begin{pmatrix}
\cos(\alpha - 90^\circ) & 0 & \sin(\alpha - 90^\circ) \\
0 & 1 & 0 \\
-\sin(\alpha - 90^\circ) & 0 & \cos(\alpha - 90^\circ)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
sin\alpha & 0 & -\cos\alpha \\
0 & 1 & 0 \\
cos\alpha & 0 & sin\alpha
\end{pmatrix}
\]

4.4.2

4.4.3
The sequential application of these operators rotates a vector \( \vec{r}_0 \) into \( \vec{r}_2 \) according to

\[
\vec{r}_2 = R_{2Y}^{(0)} (\alpha - 90^\circ) R_{1Z}^{(0)} (90^\circ) \vec{r}_0
\]

and according to Eq. 4.2.5

\[
\vec{r}^{(2)} = R_{1Z}^{(0) T} (90^\circ) R_{2Y}^{(0) T} (\alpha - 90^\circ) \vec{r}^{(0)} = A^{20} \vec{r}^{(0)}
\]

which relates the components of the same vector in the \( X^2Y^2Z^2 \) and \( X^0Y^0Z^0 \) bases.

The seat has design constraints which do not allow \( \alpha \) to exceed 90° resulting in the use of two seat rotational sequences. The first sequence, described by Eq. 4.4.4, allows arm sweeps in which the upper arm elevation does not exceed shoulder height. For arm sweeps with higher lateral elevations, the rotations \( R_{1Z}^{(0)} (-90^\circ) \) and \( R_{2Y}^{(0)} (\alpha - 90^\circ) \) are sequentially applied in the same manner as in Eq. 4.4.4.

The respective rotational operators for below and above shoulder positions are designated as \( R_{b}^{(0)} \) and \( R_{a}^{(0)} \) and given explicitly by

\[
R_{b}^{(0)} = \begin{pmatrix}
0 & -\sin \alpha & -\cos \alpha \\
+1 & 0 & 0 \\
0 & -\cos \alpha & \sin \alpha \\
\end{pmatrix}
\]

\[
R_{a}^{(0)} = \begin{pmatrix}
0 & \sin \alpha & -\cos \alpha \\
1 & 0 & 0 \\
0 & \cos \alpha & \sin \alpha \\
\end{pmatrix}
\]
From Eq. 2.2.3 it follows that

\[
A_{20}^D = (R_D^{(0)})^T \\
A_{20}^a = (R_a^{(0)})^T
\]

and

\[
A_{20} = (R_{20}^{(0)})^T
\] 4.4.8

The instantaneous position of the torso with respect to the original seat orientation is given by \( A_{20}^D \).

For consistency of notation, we denote this transformation henceforth as \( \hat{A}_{t0} \) and let

\[
\hat{r}(t) = \hat{A}_{t0} \hat{r}(0)
\] 4.4.9

which relates vector components in the torso system to the original seat position orientation.

By eliminating \( \hat{r}(0) \) between Eqs. 4.4.1 and 4.4.9 we can write

\[
\hat{r}(t) = \hat{A}_{t0} A_0 \hat{r} \hat{r}(r) \\
= \hat{A} \hat{r} \hat{r}(r)
\] 4.4.10

4.5 Rotation of Right Upper Arm with Respect to Torso

Initially, the coordinate systems of the right upper arm and the torso are assumed to be coincident. The Z axis is taken to lie along the long axis of the arm and the arm is first rotated with respect to the torso negatively through angle \( \alpha \) about the \( X^{(0)} \) axis as shown in Fig. 4.5.1. The explicit operator is
This results in an elevation of the arm through angle $\alpha$ and placement of the upper arm axis, $z^{(1)}$, at an angle $\alpha$ from the fixed $z^{(0)}$ and $y^{(1)}$ pointing vertically downward.

The rotations of the torso described in the previous section and the above described rotation of the arm with respect to the torso result in the $z^{(1)}$ axis lying in a horizontal plane. In this orientation, rotations of the upper arm about the $y^{(1)}$ axis result in arm motion in the
plane normal to the gravitational vector or, in other words, through an isogravitational arc.

During the experiment, a force is applied to the upper arm which constrains motion of the upper arm to a horizontal plane and forces it through maximally tolerable arcs. The resulting motion is described by the rotation operator

\[
\begin{align*}
R^{(1)}_{-2Y}(\phi) &= \begin{pmatrix}
\cos\phi & 0 & -\sin\phi \\
0 & 1 & 0 \\
\sin\phi & 0 & \cos\phi
\end{pmatrix} \\
\end{align*}
\]

where a positive \( \phi \) corresponds to a ventral motion.

To calculate the vector component transformation matrix between the upper arm system and that of the torso, we first calculate the position of a vector in the upper arm after undergoing the two described rotations.

\[
\begin{align*}
\mathbf{r}'^{(0)} &= R^{(0)}_{2Y}(\phi) R^{(0)}_{1X}(-\alpha) \mathbf{r}^{(0)} \\
&= R^{(0)}_{1X}(-\alpha) R^{(1)}_{-2Y}(\phi) R^{(0)}_{1X}(-\alpha) \mathbf{r}^{(0)} \\
&= R^{(0)}_{1X}(-\alpha) R^{(1)}_{-2Y}(\phi) \mathbf{r}^{(0)} \\
\end{align*}
\]

Then according to Eq. 4.2.5 the components are transformed according to
The components of the upper arm Z axis in the torso system are given by

\[ r(2) = R_{-2Y}^{(1)}T(\phi) \quad R_{1X}^{(0)}T(-\alpha) \quad r(0) = A^{20} \quad r(0) \]

\[
\begin{pmatrix}
\cos \phi & \sin \phi \sin \alpha & \sin \phi \cos \alpha \\
0 & \cos \alpha & -\sin \alpha \\
-\sin \phi & \cos \phi \sin \alpha & \cos \phi \cos \alpha
\end{pmatrix}
\begin{pmatrix}
r(0)
\end{pmatrix}
\]

\[ 4.5.4 \]

The position of the upper arm is calculated by monitoring the motion of three points rigidly attached to the arm during the forced sweep in a horizontal plane by the method described in Section 4.2. Transformation matrix \( A^a_r \) is obtained which relates vector components in the upper arm system to those in the reference system

\[
\begin{pmatrix}
r(0)
\end{pmatrix}
= (A^{20}r(2)) =
\begin{pmatrix}
-sin\phi
\cos \phi \sin \alpha
\cos \phi \cos \alpha
\end{pmatrix}
\]

\[ 4.5.5 \]

In Section 4.4, we derived the transformation matrix relating vector components in the torso system to those in the reference system

\[
r(a) = A^a_r \quad r(r)
\]

\[ 4.5.6 \]
By eliminating \( r(t) \) between Eqs. 4.5.6 and 4.5.7 we can relate vector components in the upper arm system to those in the torso by

\[
\mathbf{r}(a) = \mathbf{A}^{ar} \mathbf{A}^{-tr} \mathbf{r}(t) = \mathbf{A}^{at} \mathbf{r}(t)
\]  

4.5.8

If the elements of this matrix are taken as \( \mathbf{A}_{ij} \) then we can solve for \( a \) and \( \phi \) by equating elements of \( \mathbf{A}^{at} \) and \( \mathbf{A}^{20} \) as given by Eq. 4.4.4,

\[
\begin{align*}
-sin\phi &= A_{31} \\
\cos\phi\sin\alpha &= A_{32} \\
\cos\phi\cos\alpha &= A_{33}
\end{align*}
\]

and solving for \( \alpha \) and \( \phi \)

\[
\begin{align*}
\phi &= \sin^{-1} A_{31} \\
\alpha &= \tan^{-1} \left( \frac{A_{32}}{A_{33}} \right)
\end{align*}
\]  

4.5.9

We can also relate the angles corresponding to a given upper arm Z axis orientation obtained by the rotation scheme described in Section 4.2 to that developed here by comparing upper arm Z axis components in the torso system. This is accomplished by equating vectors \( \mathbf{Z}_1^{(0)} \) and \( \mathbf{Z}_2^{(0)} \) given by Eqs. 4.3.4 and 4.4.5, respectively, leading to
\[ \phi' = \tan^{-1} \left( \frac{\tan \phi}{\sin \theta} \right) \]

and

\[ \theta = \cos^{-1} (\cos \phi \cos \alpha) \]

We now consider the combined rotations of the torso, which is rigidly attached to the moveable seat, and the upper arm. In developing the form of the transformation we will use the rotations for below shoulder elevation testing. The rotational operations for above shoulder elevation testing are, however, the same and only the rotational angles are different. The below the shoulder transformation will be developed, and the transformation for above the shoulder merely given.

The application of a rotation operator in any basis to a vector in that basis produces another vector in the same basis. If a sequence of rotation operators is to be applied to a vector, they must all be in the same basis or coordinate system since a rotation operator does not transform a vector out of its original basis.

We wish to apply sequential rotations to the upper arm which is originally aligned with the initial seat testing position. The first two rotations are given by Eq. 4.4.4 and are with respect to the initial seat position coordinate system or basis designated by \( x^{(0)} y^{(0)} z^{(0)} \). The first arm rotation is given by Eq. 4.5.1, however, this is in the coordinate system aligned with the torso after the first two rotations. Finally, a rotation of the arm is
performed according to Eq. 4.5.2 which is given in the upper arm coordinate system that is three times rotated from the initial seat position.

We will apply all four rotations to the upper arm in the \(X(0)Y(0)Z(0)\) system, designating the first upper arm rotation by \(R_{3\mu}(\alpha)\) and the second by \(R_{4\nu}(\gamma)\) as follows

\[
R_4(0) = R_{4Y}(\phi) R_{3\mu}(-\alpha) R_{2Y}(\alpha - 90^\circ) R_{1Z}(90^\circ) R_0(0)
\]

where \(\mu\) designates the \(X(2)\) axis in the \(X(0)Y(0)Z(0)\) basis and \(\gamma\) the \(Y(3)\) axis in the \(X(0)Y(0)Z(0)\) basis. We will now apply similarity transformations to \(R_{3\mu}(\gamma)\) and \(R_{4\nu}(\gamma)\) to express them in terms of rotation operators given by Eq. 4.5.1 and Eq. 4.5.2. Using Eq. 2.2.16 and letting

\[
R_{1\mu} = R_{1Z} \quad \text{and} \quad R_{2\nu} = R_{2Y}
\]

we can write

\[
R_{3\mu}(0) = \left(\begin{array}{c}
R_{2Y} \\
R_{1Z}
\end{array}\right) R_{3X} \left(\begin{array}{c}
R_{2Y} \\
R_{1Z}
\end{array}\right)
\]

\[
= R_{2Y} R_{1Z} R_{3X} \quad 4.5.12
\]

where \(R_{2Y} R_{1Z}\). Taking \(R_{3X} = R_{2Y}(1)\) in Eq. 2.2.10 and using the transformation properties as given by Eq. 2.2.15 we can write

\[
R_{4\nu}(0) = \left(\begin{array}{c}
R_{2Y} \\
R_{3X}
\end{array}\right) R_{4Y} \left(\begin{array}{c}
R_{2Y} \\
R_{3X}
\end{array}\right)
\]

\[
= R_{2Y} R_{3X} \quad 4.5.13
\]
By substituting for $R(0)$ and $R(0)$ from Eqs. 4.5.12 and 4.5.13, respectively, we get

$$r_4 = R(0) R X R Y R Z R(0) R X R Y r_0$$

and according to Eq. 2.2.3

$$A^{as} = A^{40} = R(3) T(2) T(0) T(0) T$$

The result of the total sequence of rotational operations is shown in Fig. 4.5.2 where $X_{S} Y_{S} Z_{S}$ is the initial seat position and the $X_{A} Y_{A} Z_{A}$ is the coordinate system moving with the upper arm. Multiplying the rotation matrices and taking the transpose we get

$$A_{b}^{as} = \begin{pmatrix}
-sin\phi & cos\phi & 0 \\
0 & 0 & -1 \\
-cos\phi & -sin\phi & 0
\end{pmatrix}$$

By examination of Fig. 4.5.2, it can be seen that the sequential rotations $R_{1X}(90^\circ)$ and $R_{2Z}(90^\circ + \phi)$ produce the same alignment and transformation matrix between the initial seat and upper arm positions.
The reason this transformation matrix depends only on \( \phi \) is because the arm rotation axis has been aligned with the vertical which is the seat original position \(-Z\) vector. Since in the arm system this is identified as the \( Y\) axis, the last two rows are interchanged and multiplied by \(-1\).

The transformation described by Eq. 4.5.16 was used for testing sweeps below the shoulder level. For above, the shoulder sweeps \( A_{a}^{20} \) from Eq. 4.4.8 was substituted for \( R_{12}^{(0)T} R_{2Y}^{(0)T} \) in Eq. 4.5.15 to give

\[
A_{a}^{as} = \begin{pmatrix}
\sin\phi & -\cos\phi & 0 \\
0 & 0 & -1 \\
\cos\phi & \sin\phi & 0
\end{pmatrix}
\]  
4.5.17
4.6 Force and Vector Component Transformations

The location and orientation of the general force applicator (GFA) is determined from monitoring the three-dimensional positions of the three spark elements attached to the GFA. These spark element positions are measured in the laboratory reference system denoted by \( \{r\} \). The force transducer origin and orientation is fixed with respect to the spark element calculated coordinate system \( \{g\} \) and is denoted by \( \{f\} \) as shown in Fig. 4.6.1.

The transformation of vector components from the \( \{f\} \) system to the \( \{g\} \) is accomplished by

\[
\vec{r}(g) = A^g_{\{f\}} \vec{r}(f).
\]

The \( A^g_{\{f\}} \) transformation matrix is obtained from measurements of the GFA dimensions and relative positions of the spark elements and from the scheme for forming a coordinate system from three ordered vector positions as described in Section 4.2. Vector components in this system are related to those in the laboratory reference system, \( \{r\} \), by

\[
\vec{r}(g) = A^g_{\{r\}} \vec{r}(r)
\]

where \( A^g_{\{r\}} \) is obtained according to Eq. 4.2.5. The origin of the transducer coordinate system is specified in the \( \{g\} \) system by the fixed vector \( \vec{r}(g) \). The origin of the GFA coordinate system is calculated as described in Section 4.2 and is given by \( R_{gc} \) in the laboratory reference system.
The origin of the transducer system, or the force application point, is then given in the laboratory reference system by

$$
\mathbf{r}_{\text{fc}} = \mathbf{r}_{\text{gc}} + (A^g r)^T \mathbf{r}^g
$$

4.6.3

where use has been made of Eq. 4.6.2.

By eliminating $r^g$ between Eqs. 4.6.1 and 4.6.2 we get a transformation relating vector components in the $(f)$ and $(r)$ systems given by

$$
\mathbf{t}^r = (A^g r)^T \mathbf{t}^g \quad \mathbf{r}^g = A^r r^f \quad \mathbf{r}^f
$$

4.6.4
which can be applied to measured force components in the transducer system to give those components in the laboratory reference system (r)

\[ F^+(r) = A_{rf}^+ F(r) \]  

4.6.5

Those force components and the point of force application can also be expressed with respect to the torso system denoted by (t) and shown in Fig. 4.6.2. The transformation matrix between systems (r) and (t) is obtained by using spark points whose relative positions with

![Figure 4.6.2. Coordinate System for the Torso.](image-url)
respect to the initial seat position are known to calculate $A_{sr}^I$ according to the scheme described in Section 4.2 and $A_{ts}$ obtained from explicit rotation operators for seat motion to the testing position as described in Section 4.4. The rotational position of the seat is the same as that of the torso, since during the experiments, the torso is rigidly attached to the seat and the seat is rotated about known axes to specific positions at which it is then fixed for the duration of an arm sweep.

If the transformation matrix between the torso and reference system is given by $A^tr$, then the components of a vector in the two systems are related by

$$\vec{r}(t) = A^tr \vec{r}(r) \quad . \quad 4.6.6$$

We want to express the force components and the point of their application with respect to the torso $(t)$ system. In the $(r)$ system, the vector from the origin of the torso system $(t)$ to the origin of the force transducer system $(f)$ is

$$\vec{R}_{ft} = \vec{R}_{fc} - \vec{R}_{tc} \quad 4.6.7$$

where $\vec{R}_{fc}$ is shown in Fig. 4.6.1 and $\vec{R}_{tc}$ in Fig. 4.6.2. Substituting for $\vec{R}_{fc}^T$ from Eq. 4.6.3 and applying Eq. 4.6.6 to transform the vectors to the torso system we get

$$\vec{R}_{ft}^T = A^tr (\vec{R}_{fc} + (A^tr)^T \vec{r}(g) - \vec{R}_{tc}) \quad . \quad 4.6.3$$
Using Eq. 4.6.6 to transform the force components from
the reference system \((r)\) as given by Eq. 4.6.5, we get
\[
F(t) = A_{tr} A_{rf} F^f(t)
\]
\[
= A_{tr} (A_{gr})^T A_{gf} F^f(t).
\]  
4.6.9

The position of the upper arm is determined using
spark elements attached to the arm. The transformation \(A^{ra}\)
and center of arm coordinate system \(R_{ac}\) are calculated as
described in Section 4.2 and given by Eqs. 4.2.5 and 4.2.7,
respectively. Using Eq. 4.2.6 and Eq. 4.6.6 we can relate
vector components in the upper arm system with respect to
the torso by
\[
\mathbf{r}^a(t) = A_{ar} (A_{tr})^T \mathbf{r}^r(t)
\]
\[
= A_{at} \mathbf{r}^r(t).
\]  
4.6.10

The point of force application with respect to the
upper arm origin, but expressed in \((r)\) system components,
is given by
\[
\mathbf{r}_a = \mathbf{r}_c - \mathbf{r}_a
\]  
4.6.11

where \(\mathbf{r}_a\) is the vector from \((r)\) origin to the upper arm
coordinate system \((a)\) origin given by Eq. 4.2.7. Substituting
from Eq. 4.6.3 for \(\mathbf{r}_c\) and using Eq. 4.2.6 to transform to
\((a)\) system components we have
The force to the upper arm is applied to the point on the $z^{(a)}$ axis at $Z_0$. The moment vector acting on the upper arm in the upper arm system is

$$\mathbf{M}(a) = R_{fa}(a) \times F(a) = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} F_1(a) \\ F_2(a) \\ F_3(a) \end{bmatrix} = -Z_0 \begin{bmatrix} -F_2 \\ F_1 \\ 0 \end{bmatrix}. \quad 4.6.13$$

We have specified the relative rotational orientation between the torso and the upper arm in terms of two rotations through angles $\alpha$ and $\phi$ as given in Section 4.5 which yielded the $A_{ac}(\alpha, \phi)$ transformation given by Eq. 4.5.3.

We now consider the relations among $\alpha$ and $\phi$ resulting from the transformation of moment components between $(t)$ and $(a)$.

Applying the transformation given by Eq. 4.3.1 to $\overrightarrow{M}(t)$ we get the moment components in the arm system

$$\begin{bmatrix} M_1(t) \\ M_2(t) \\ M_3(t) \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \sin \alpha & \sin \phi \cos \alpha \\ 0 & \cos \alpha & -\sin \alpha \\ \sin \phi & \cos \phi \sin \alpha & \cos \phi \cos \alpha \end{bmatrix} \begin{bmatrix} M_1(a) \\ M_2(a) \\ M_3(a) \end{bmatrix}. \quad 4.6.14$$
From the last component we get the equation of constraint

\[ \tan \alpha M_1(t) + \sin \alpha M_2(t) + \cos \alpha M_3(t) = 0 \]

which is equivalent to

\[ \widehat{Z}_a(t) \cdot \widehat{M}_a(t) = 0 \]

and implies that \( \widehat{M}_a(t) \) lies in a plane normal to the arm long axis, \( \widehat{Z}_a(t) \), and, thus, cannot produce torsional moments in the upper arm.
CHAPTER 5
EXPERIMENTAL DATA COLLECTION

5.1 Body and Upper Arm Positioning

To maintain the gravitationally induced moments acting at the shoulder joint as constant as possible, an experimental system and procedures were used in which forced motion of the arm would be constrained to a horizontal plane [28, 29]. Since a number of sweeps of arm motion at different elevation angles to the torso were needed, a seat was used which had powered rotational capabilities for yaw and pitch motion. For the present set of tests, the seat was used at two yaw angles.

For right upper arm sweeps below the shoulder level, the seat was first rotated 90° positively with respect to the vertical and then at incremental levels about the original position pitch axis. These values of incremental rotation are given in Table 5.1.1. The positive sense of the pitch axis was taken to the subject's right in his original position.

Right upper arm sweeps above the shoulder level were made after an initial -90° yaw rotation and the incremental positive angle pitch rotations are given in Table 5.1.1.
TABLE 5.1.1: Seat Rotation Angles for the Positions Tested

<table>
<thead>
<tr>
<th>YAW ANGLE</th>
<th>PITCH ANGLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>90°</td>
<td>0°</td>
</tr>
<tr>
<td>90°</td>
<td>15°</td>
</tr>
<tr>
<td>90°</td>
<td>30°</td>
</tr>
<tr>
<td>90°</td>
<td>45°</td>
</tr>
<tr>
<td>90°</td>
<td>60°</td>
</tr>
<tr>
<td>-90°</td>
<td>15°</td>
</tr>
<tr>
<td>-90°</td>
<td>30°</td>
</tr>
<tr>
<td>-90°</td>
<td>45°</td>
</tr>
<tr>
<td>-90°</td>
<td>60°</td>
</tr>
</tbody>
</table>

A ventral sweep (with the arm moving into the front of the body) and a dorsal sweep (with the arm moving behind the body) was performed at each of the positions except at 0° pitch angle position where two sweeps in each direction were made. Sweeps at ±90° yaw and 75° pitch were attempted, but these positions resulted in excessive obstruction of the spark elements with respect to the microphones and insufficient data could be recorded for position analysis.

The force of the arm was applied manually by an instrumented force applicator constrained to move in a horizontal plane. This constraint was applied by mounting the force applicator to a vertical column which in turn was rigidly attached to a carriage above the experimental
apparatus. This carriage was free to roll on a dual set of horizontal perpendicular tracks. The force applicator was free to rotate freely about the axis of the vertical column.

5.2 Sonic Digitizing System

A commercial sonic three-dimensional spatial digitizing system manufactured by Science Accessories Corporation was used to determine the coordinates of spark elements mounted on a cuff attached to the upper right arm and to the force applicator. Coordinates could be measured in a volume of 75 cm high, 150 cm wide and 180 cm deep. The nominal accuracy of the system was ±.05 cm and depended on such factors as temperature gradients and the proximity of the spark elements to the boundaries of the volume.

In preliminary testing of the system's accuracy, a spark element was rigidly mounted and multiple readings taken. Variations in the position were of the order of ±.05 cm. Sparkers were also mounted on a rigid body which was moved through various trajectories and orientations in front of the board. Calculation of the distances between these rigidly mounted spark elements varied less than ±.1 cm.

It was found that on some occasions, especially when the spark elements were near the volume boundaries, spurious position values were recorded. Also, when at least three microphones of the digitizer did not receive inputs from
a spark element, zero coordinates were given to that element. Since the spark elements on the force applicator and arm cuff, respectively, maintained constant relative positions, logic based on the distance between spark elements was used to discard such bad data sets.

5.3 Coordinate Systems for the Upper Arm

The position of the upper arm is obtained by means of a transformation from the position of the cuff assumed to be rigidly attached to the upper arm. The position of the cuff is calculated from the measured positions of four spark elements attached to the cuff. Since only three points in space are required to uniquely specify the position of a body in three dimensions, the information from four sparkers provides additional information. A least squares method can be used which would give the optimum estimation of body position using the four points. This method was not used here for two reasons. First, it was felt that one additional point to the minimally required three points would provide only a marginal improvement in position determination. Secondly, during the actual data taking procedure a spark element position is quite often not recorded due to the poor orientation or location of the cuff with respect to the sonic digitizer board, spark misfire or acoustic obstruction. A method that was chosen is totally deterministic and uses the position of three of the four spark elements if any three of the four spatial positions are recorded.
The approach used is to calculate four transformation matrices according to the method described in Section 4.2 using three permuted combinations of the four spark elements. The four transformations are defined according to the coordinate point sequence used in their calculation as given in Table 5.3.1. The relative alignment of these coordinate systems with respect to the four defining points is shown in Fig. 5.3.1 where only the x and y axes are drawn, and the z axis is generated by the cross product of the x and y axes.

These transformations change as the cuff is moved, but since the spark elements are rigidly connected, we can form transformations which remain constant between any two of the coordinate systems. Let us pick $A^{1r}$ as the standard transformation for defining cuff orientation. We now define the three transformations

<table>
<thead>
<tr>
<th>Transformations</th>
<th>Coordinate Point Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^{1r}$</td>
<td>1, 2, 4</td>
</tr>
<tr>
<td>$A^{2r}$</td>
<td>2, 3, 4</td>
</tr>
<tr>
<td>$A^{3r}$</td>
<td>1, 3, 4</td>
</tr>
<tr>
<td>$A^{4r}$</td>
<td>1, 2, 3</td>
</tr>
</tbody>
</table>

**TABLE 5.3.1:** Reference to Cuff Coordinate System Transformations Using Permuted Sets of Three Points from a Total Set of Four
Figure 5.3.1. Relative Alignment of Four Coordinate Systems Used for Standard Coordinate System Reconstruction.

\[ A_{12}^{\text{lr}} = A_{1r}^{\text{lr}} (A_{2r}^{\text{lr}})^T \]
\[ A_{13}^{\text{lr}} = A_{1r}^{\text{lr}} (A_{3r}^{\text{lr}})^T \]
\[ A_{14}^{\text{lr}} = A_{1r}^{\text{lr}} (A_{4r}^{\text{lr}})^T \]

which are constant and transform vector components in the (2), (3) and (4) systems to the (1) system, respectively. These transformations are calculated the first time that the positions of all four sparkers are recorded and are then retained. At subsequent body positions new transformations matrices \( A_{1r}^{\text{lr}}, A_{2r}^{\text{lr}}, A_{3r}^{\text{lr}} \) and \( A_{4r}^{\text{lr}} \) may be calculated. By multiplying Eqs. 5.3.1 through 5.3.3 by the transpose of the latter three transformations,
respectively, we get three equations for $\mathbf{A}^{1r}$,

$$\mathbf{A}^{1r} = \mathbf{A}^{12}(\mathbf{A}^{2r})^T$$  \hspace{1cm} 5.3.4

$$\mathbf{A}^{1r} = \mathbf{A}^{13}(\mathbf{A}^{3r})^T$$  \hspace{1cm} 5.3.5

$$\mathbf{A}^{1r} = \mathbf{A}^{14}(\mathbf{A}^{4r})^T$$  \hspace{1cm} 5.3.6

In the process of data collection, if all four points are recorded, $\mathbf{A}^{1r}$ can be calculated directly or any of the equations can be used. Aside from errors introduced in the measurement and calculation processes, all methods of calculation should yield the same result. If, however, one of the points is missing, then there is only one unique solution. For example, if point 3 is not available, $\mathbf{A}^{1r}$ can be calculated directly, but $\mathbf{A}^{2r}$, $\mathbf{A}^{3r}$, and $\mathbf{A}^{4r}$ cannot be calculated. If we have chosen $\mathbf{A}^{1r}$ as the standard transformation, then this presents no problem. If point 1 is not available, we can only compute $\mathbf{A}^{2r}$ and must use Eq. 5.3.4 to reconstruct $\mathbf{A}^{1r}$. Likewise, if points 2 or 4 are missing, we must use Eqs. 5.3.5 and 5.3.6, respectively, to reconstruct $\mathbf{A}^{1r}$.

The transformations in Table 5.3.1 were calculated using a set of data in which the positions of all four points were recorded. If no relative motion occurs among the spark elements, then the matrices $\mathbf{A}^{12}$, $\mathbf{A}^{13}$ and $\mathbf{A}^{14}$ should remain constant in time and during cuff motion. A check was performed to see if, indeed, these matrices remained
constant by calculating $A_{12}^1$, $A_{13}^1$ and $A_{14}^1$ at the initial time that all four spark element positions were recorded, and then multiplying these transformations by $(A_{12}^1)^T$, $(A_{13}^1)^T$ and $(A_{14}^1)^T$, respectively, at each subsequent time that all four positions were obtained. If the recorded positions did reflect a totally rigid structure, then the resulting products should have yielded identity matrices. It was found that during the course of a sweep that up to a 3 degree drift in coordinate system alignment occurred. The causes of this drift could not be totally isolated, but did appear to be related to spark element orientation to the sonic digitizer board and position within the measurement volume of the system.

The origins for the four coordinate systems were calculated as described in Section 4.2, and the vectors from the (2), (3) and (4) system origins to the (1) system origin were calculated and transformed to the (1) system basis in which they remained constant during the cuff motion. They are shown in Fig. 5.3.1 and were used to find the origin of the (1) system when point 1, 2 or 4 was missing using

$$R_{ic}^{-1}(r) = (A_{1r}^{-1})^T R_{11}^{-1}(1) + R_{ic}^{-1}(r), \quad 5.3.7$$

where $R_{ic}^{-1}(r)$ is a vector to the (i)th system origin, $A_{1r}^{-1}$ is the transformation from the reference system to the (1) system and $R_{11}^{-1}(1)$ is the vector from the (i) system origin to (1) system origin in the (1) system.
The position of the cuff with its four spark elements is shown on the right upper arm in Fig. 5.3.2. A vector from the origin of the cuff system to the acrmiale is denoted by $\mathbf{R}_{ca}$ and a coordinate system is defined at this point based on anatomical, surface palpable landmarks. The specific landmarks and the landmark coordinate system is shown in Fig. 5.3.3 with the lateral humeral epicondyle.

Figure 5.3.2. Cuff Position and Coordinate System of Right Upper Arm.
taken as point 1, the acromiale is point 2 and the medial humeral epicondyle as point 3. This coordinate system definition with origin at the acromiale is the same as that used by McConville[80].
5.4 Coordinate Systems for the Force Applicator

The general transformation scheme for relating forces measured by the general force applicator (GFA) and its position to the forces and point of force application as seen from the arm system were presented in Section 4.6. The specifics of the transformations between the force transducer coordinate system and the GFA are given here. Fig. 5.4.1 shows the relative orientation of the force transducer system, \( (f) \), and the GFA system, \( (g) \). The transformation of vector components from the \( (g) \) system to the \( (f) \) system is given by

\[
\mathbf{R}_{fg} = \begin{bmatrix}
0 & 0.70711 & 0.70711 \\
0 & -0.70711 & 0.70711 \\
1.0000 & 0 & 0
\end{bmatrix}.
\]

The vector \( \mathbf{\vec{z}}(g) \), from the origin of the \( (f) \) system to the origin of the \( (g) \) system and given in the \( (g) \) system is

\[
\mathbf{\vec{z}}(g) = \mathbf{R}_c \ + \mathbf{L}
\]

\[
= (0, 6.60, -6.60) + (29.62, 0, 0)
\]

\[
= (29.62, 6.60, -6.60) \text{ cm}.
\]
5.5 Coordinate Systems for the Torso and Laboratory

The coordinates of the spark elements were always measured with respect to the reference system, \((r)\). This, of course, was the sonic digitizer coordinate system. In the course of data collection, this system had to be repositioned to minimize sparker element to sonic digitizer microphone obstruction. This repositioning of the reference
system necessitated the establishment of yet another coordinate system which would be stationary within the laboratory. This system was called the inertial system and was designated by (i). It was implemented by hanging two plumb lines, one with two knots and the other with one, in such a location that the know positions could be measured (using a spark element held at the position of the knots) for all potential board positions which may be used. They were also placed so that the calculated coordinate system axes would coincide with the torso vertical and lateral axes with the seat in its initial position with zero yaw and pitch angles.

The (r) system (the sonic digitizer board) is shown with respect to the (i) system in Fig. 5.5.1. The (1) coordinate system is calculated according to the method described in Section 4.2. The (i) system is related to this system by the transformation

\[
\mathbf{A}^{il} = \begin{bmatrix}
0 & 0 & -1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{bmatrix}
\]

which aligns \( z^{(i)} \) with the positive vertical. The vector from the (r) system origin to the (i) system origin is given by \( \mathbf{R}_{ic} \).
5.6 Experimental Procedure

The experimental procedure used in this study was very similar to that used by Engin [28,29], but with a few exceptions. Among these were: (1) the upper arm trajectories were such that in the transformation matrix relating the arm position to the torso position, and in which the rotation about the long bone axis is ignored, one
of the two angles remains constant. (2) Motion to the horizontal plane is constrained by the attachment of the general force applicator to a carriage which can only move in the horizontal plane. (3) The motion of the arm and the resultant resistive moment calculations were taken with respect to a three-dimensionally defined coordinate system for the arm based on anatomical features as described in Section 5.3.

The experimental system with the volunteer subject is shown in Fig. 5.6.1. The sonic digitizing board with its microphones mounted along the edges is in the background. The GFA, attached to the vertical column which rides on a carriage above, is in the foreground. A rigid cuff is attached to the subject's right upper arm and on it, as well as on the GFA, are visible spark elements. In Fig. 5.6.2, the seat is rotated -90° in yaw and 60° in

Figure 5.6.1. Laboratory Apparatus and Subject.
in pitch, resulting in an upper arm sweep 60° above full lateral extension.

The GFA was pushed manually against the arm at or above the elbow and an attempt was made to maintain about a 90° angle between the arm and the GFA. The range of rotation was determined by the volunteer who would vocally indicate when he wanted the force application to cease.
CHAPTER 6

EXPERIMENTAL RESULTS

6.1 Resistive Moments in the Shoulder Joint

The various seat positions used during testing are listed in Table 5.1.1. At each of these testing positions the arm was elevated in the body frontal plane to a horizontal position. It was then forced either in the dorsal direction ($\mathbf{P}$ negative) or ventral direction ($\mathbf{S}$ positive), with the motion constrained by the force applicator to maintain the arm in the horizontal plane. The transformations for relating the arm position to the torso were developed in Chapter 4 and were of a form such that arm position during a forced sweep was totally defined by one variable, $\mathbf{\psi}$, with the other one, $\mathbf{\alpha}$, remaining constant. The $\mathbf{\alpha}$ was the angle between the seat vertical and the horizontal plane.

Since the motion resistive properties of the shoulder joint are independent of the seat angle and are assumed to be a function only of the relative arm to torso orientation, the measured moment data are presented in terms of the specification of arm elevation in the frontal plane of the body as shown in Fig. 6.1.1. Initial position or zero elevation angle is taken to be that with the right arm fully extended laterally. The arm sweeps are identified as either
being above or below this position by an angle $\phi$ while in the frontal plane.

The data analyzed in this study was from one subject with single sweeps made at each arm elevation position dorsally and ventrally except for the zero elevation level where two sweeps were made. Prior to sweep initiation, the force transducers were offset to give zero readings with the arm in the cuff of the force applicator. This was done to remove the force contribution due to gravity.
The z vector of the arm was used to specify its position with respect to the torso. This z vector was based on anatomical landmarks and was coincident with the vector from the lateral humeral epicondyle to the acromiale. The measure of the angular distance of this vector from the horizontal plane and change in this distance indicated whether the arm was indeed horizontal and if there was any drift from the horizontal plane. The angular measure of the arm z axis above the horizontal plane is given by $\angle 9$.

The resistive moment components, $M_x$ and $M_y$, in the arm, taken with respect to the acromiale as origin, and the deviation from the horizontal plane for the arm z axis for two repeated dorsal sweeps are shown in Fig. 6.1.2. During these, and all subsequent sweeps, the subject attempted to maintain an arm long axis rotational position in which the arm y axis (vector direction from lateral to medial humeral epicondyle) would be approximately aligned with the vertical.

It is seen that $M_x$ remains relatively constant with no significant trends during most of the sweep. The change beyond $-60^\circ$ may be caused by a rotation of the arm about its long axis resulting in some of the moment being projected along the x axis. $M_y$ increases in magnitude steadily with decreasing $\phi$, reaching a maximum tolerable level at $-70^\circ$ to $-80^\circ$ of about $-21$ N-m. The average level of rotation was about $9^\circ$ below the horizontal plane with $\phi$ staying relatively constant, but with a slight tendency towards the horizontal as $\phi$ approached zero.
Figure 6.1.2 Dorsal Sweep Data for Arm at Zero Inclination to Full Lateral Extension.
Fig. 6.1.3 shows two ventral sweeps at zero arm elevation. There is no clear trend for $M_x$, but $M_y$ clearly becomes significant beyond $140^\circ$ reaching a maximum of about 12 N-m. This was not a voluntary limit as data collection of this sweep was prematurely terminated due to body obstruction of the spark elements. The rotation is on the average about $5^\circ$ above the horizontal plane and approaches the plane as $\phi$ decreases.

The moment components and deviation from the horizontal for a ventral sweep at $15^\circ$ below full lateral extension are shown in Fig. 6.1.4. $M_x$ stays relatively small, but shows a tendency to rotate the arm upward as $\phi$ becomes more negative and reaches about $-5$ N-m at $-70^\circ$. $M_y$ decreases as the arm is rotated and reaches $-14$ N-m at $-70^\circ$. The arm motion is on the average $9^\circ$ below the horizontal plane and appears constant throughout the sweep.

Fig. 6.1.5 shows the moment components and the arm deviation from the horizontal for the same elevation for a ventral sweep. $M_x$ remains relatively constant and $M_y$ increases steadily to 11 N-m at $153^\circ$. Arm position with respect to the plane appears unsteady, but does not vary more than $6^\circ$ from the plane.

The data for dorsal motion $30^\circ$ below full lateral extension is shown in Fig. 6.1.6. $M_x$ decreases for $\phi$ less than $-40^\circ$ and reaches $-8$ N-m at $-67^\circ$. $M_y$ likewise decreases reaching a maximum of $-11$ N-m at $-67^\circ$. The motion takes
Figure 6.1.3 Ventral Sweep Data for Arm at Zero Inclination to Full Lateral Extension.
Figure 6.1.4 Dorsal Sweep Data for Arm 15° Below Full Lateral Extension.
Figure 6.1.5 Ventral Sweep Data for Arm 15° Below Full Lateral Extension.
Figure 6.1.6  Dorsal Sweep Data for Arm 30° Below Full Lateral Extension.
place on the average at 8° below the horizontal, but appears to approach the horizontal for increasing φ.

The data for a ventral sweep at the same elevation are shown in Fig. 6.1.7. The relative body-board position was such that for this sweep only data through 129° could be collected. The trend of the data is the same as for the ventral motion at 15° shown in Fig. 6.1.5.

The moment components and the arm deviation from the horizontal at the arm 45° below the full lateral extended position for a dorsal sweep are shown in Fig. 6.1.3. Mx shows no significant change with φ and the maximum tolerable angle is only -60°. My decreases with decreasing φ reaching -7.5 N·m at -60°. Rotation is on the average 7° below the horizontal, but appears to be approaching the horizontal as φ increases.

Data for a ventral sweep at this same elevation is shown in Fig. 6.1.9. This sweep tended to force the arm toward the subject's face resulting both in artificially limited motion and possibly some active muscle contribution to the joint resistive response. Mx showed no significant trend, and while My increased with the rotation, the resistive moment attained was relatively low. The arm z axis deviation from the horizontal plane was quite large and erratic, reaching as high as 20° at 118° and 129°.

Fig. 6.1.10 shows the moment components and the arm deviation from the horizontal for the arm 60° below lateral extension for a dorsal sweep. Mx is quite small and shows
Figure 6.1.7 Ventral Sweep Data for Arm 30° Below Full Lateral Extension.
Figure 6.1.8 Dorsal Sweep Data for Arm 45° Below Full Lateral Extension.
Figure 6.1.9 Ventral Sweep Data for Arm 45° Below Full Lateral Extension.
Figure 6.1.10 Dorsal Sweep Date for Arm 60° Below Full Lateral Extension.
no clear trend with changing $\phi$. $M_y$ decreases regularly with decreasing $\phi$ and begins to drop abruptly at $-60^\circ$ reaching $-11.5$ N-m at $-66^\circ$. The arm z axis is on the average $3^\circ$ above the horizontal plane and shows no consistent trend with $\phi$.

The moment components and the arm deviation from the horizontal for a dorsal sweep for the arm at $15^\circ$ above full lateral extension are shown in Fig. 6.1.11. $M_x$ is relatively constant to about $-50^\circ$ when it starts to decrease with decreasing $\phi$. $M_y$ begins to decrease at $-40^\circ$ and reaches a peak negative value of $-12$ N-m at $-63^\circ$. The motion of the arm is on the average $10^\circ$ below the horizontal and shows no trend with $\phi$. For ventral motion at the same arm elevation, the data are shown in Fig. 6.1.12. This sweep as well as all the other ventral sweeps with initial arm elevations above the full lateral extension level resulted in the arm being forced into the chest. In these cases the measured resistive moment resulted from a combination of forces acting within the joint structure and externally between the upper arm and the chest. This combination of internal and external forces produced much greater measured net moments with much greater $x$ components than for the dorsal sweeps and ventral sweeps with initial arm elevations below the full lateral extension level. The $M_x$ component as shown in Fig. 6.1.12 has an initially significant negative value, and decreases steadily to $-15$ N-m at $154^\circ$. $M_y$ increases steadily reaching $20$ N-m at $154^\circ$. 
Figure 6.1.11 Dorsal Sweep Data for Arm 15° Above Full Lateral Extension.
Figure 6.1.12 Ventral Sweep Data for Arm 15° Above Full Lateral Extension.
The data for the dorsal arm sweep at 30° above the full lateral extension are shown in Fig. 6.1.13. $M_x$ steadily drops to -11 N-m at -77°. $M_y$ is relatively large already near zero sweep angle and reaches -20 N-m at -77°. The arm rotation is an average 5° below the horizontal plane and shows no clear trend with $\phi$. The ventral sweep data for the same arm elevation are shown in Fig. 6.1.14. A large negative and relatively constant $M_x$ value was measured between 90° and 140° reflecting a tendency to push the arm upward as it moved across the front of the body. At 140° it began to decrease reaching -18 N-m at 158°. $M_y$ increases rapidly after 120° reaching 32 N-m at 158°. The arm deviation from the horizontal averages 4° below and shows no trend with $\phi$.

Fig. 6.1.15 shows data for the arm 45° above the full lateral extension for a dorsal sweep. $M_x$ decreases slowly, but steadily, with decreasing $\phi$ to -14 N-m at -85°. $M_y$ shows a similar decrease with $\phi$ and reaches -20.5 N-m at -85°. The arm z axis is an average of 6° below the horizontal during the sweep and approaches the horizontal as $\phi$ increases. Fig. 6.1.16 shows the data for the same arm elevation but for a ventral sweep. Both $M_x$ and $M_y$ increase in magnitude considerably above 130°. $M_x$ reaches a peak value of -39 N-m and $M_y$ of 30 N-m at 160°. This particular sweep brings the arm across the lower chest and a substantial amount of the resistive moment magnitudes may be attributable to chest wall interaction. The level of the arm sweep was
Figure 6.1.13  Dorsal Sweep Data for Arm 30° Above Full Lateral Extension.
Figure 6.1.14 Ventral Sweep Data for Arm 30° Above Full Lateral Extension.
Figure 6.1.15 Dorsal Sweep Data for Arm 45° Above Full Lateral Extension.
Figure 6.1.16 Ventral Sweep Data for Arm 45° Above Full Lateral Extension.
an average $12^\circ$ below the horizontal and showed on trends with $\phi$.

Fig. 6.1.17 shows data for the arm $60^\circ$ above the full lateral extension for a dorsal sweep. The sweeps at this elevation were by far longer than any others extending to $-107^\circ$, more than $20^\circ$ further than for the rotation plane at $45^\circ$ as shown in Fig. 6.1.15. This was also the only arm elevation position in which $M_x$ exceeded $M_y$ at maximum extension with $M_x$ reaching $-29$ N-m and $M_y$ $-24$ N-m. The arm rotation was almost constant at $9^\circ$ above the horizontal plane, but began to approach the plane as $\phi$ neared zero. Fig. 6.1.18 shows data for a ventral sweep for the same elevation position. In this position $M_y$ attained the highest magnitude of any position reaching $-55$ N-m at $150^\circ$. At the same angle $M_y$ reached a peak value of $22$ N-m. The arm rotation averaged $7^\circ$ below the horizontal and tended towards the plane as $\phi$ decreased.

Composite plots of the moment components were made to allow direct comparison of maximum sweep angle and the relative moment magnitudes at different arm elevation angles. Fig. 6.1.19 shows the $M_x$ components for dorsal motion for sweeps at and below the full lateral arm extension level. Maximum of sweep angle of $-74^\circ$ (average of two sweeps) was attained for $\beta=0^\circ$. The maximum sweep angle dropped with increasing $\beta$ to $-60^\circ$ for $\beta=45^\circ$ and then increased to $-66^\circ$ for $\beta=60^\circ$. Only the sweeps at $\beta$ equal to $15^\circ$ and $30^\circ$ had
Figure 6.1.17 Dorsal Sweep Data for Arm 60° Above Full Lateral Extension.
Figure 6.1.18 Ventral Sweep Data for Arm 60° Above Full Lateral Extension.
Figure 6.1.19 $M_X$ Moments for Dorsal Motion for Sweeps at and Below Full Lateral Arm Extension Level.

Figure 6.1.20 $M_Y$ Moments for Dorsal Motion for Sweeps at and Below Full Lateral Arm Extension Level.
any $\phi$ dependence with $M_x$ reaching about -11 N-m for $\theta=30^\circ$ and about half that for $\theta=15^\circ$.

The $M_y$ moments for the same sweeps are shown in Fig. 6.1.20. All of these components appear to have a very similar $\phi$ dependence except that the $\theta=0^\circ$ curves are displaced about 5 N-m below the other curves.

The $M_y$ components for ventral motion for sweeps at and below the full lateral arm extension level are shown in Fig. 6.1.21. The greatest resistive moment appeared to be at $\theta=0^\circ$, but the full sweep data was not obtained due to obstruction of the spark elements during the sweep. The maximum sweep angle was at $\theta=15^\circ$ and the minimum at $\theta=45^\circ$. The $M_x$ components were not plotted in composite form for

![Figure 6.1.21 M_y Moments for Ventral Motion for Sweeps at and Below Full Lateral Arm Extension Level.](image)
these sweeps as no significant $\phi$ dependence was observed.

The $M_x$ components for dorsal motion for sweeps above the full lateral arm extension level are shown in Fig. 6.1.22. The range of these sweeps increased regularly with $\theta$ from $-64^\circ$ for $\theta=15^\circ$ to $-107^\circ$ for $\theta=60^\circ$. The magnitudes likewise increased with increasing $\theta$. The $M_y$ components for the same sweeps are shown in Fig. 6.1.23. The maximum resistive $M_y$ moment attained increase with increasing $\theta$, but some crossover of the moment values occurs for intermediate $\phi$ values.

The $M_x$ components for ventral motion for sweeps above the full lateral arm extension level are shown in Fig. 6.1.24. The maximum range of the sweep increases with increasing $\theta$ until $45^\circ$, but then decreases at $60^\circ$. The maximum $M_x$ moments attained increase with $\theta$ reaching $-53$ N-m for $\theta=60^\circ$ at $149^\circ$. The $M_y$ components are shown in Fig. 6.1.25.

6.2 Shoulder Kinematics During Forced Motion

A methodology was developed in Chapter 3 to analyti-
cally characterize the motion of an arbitrarily picked rotation point on a segment moving in space. It was argued that if the moving segment motion is sufficiently constrained the rotation point displacement can be expressed in terms of orthogonal motions along an axis and a trajectory in the plane normal to the axis. This expansion technique was applied to the trajectories of the arm rotation point taken to be at the initial point of the acromiale landmark.
Figure 6.1.22 $M_x$ Moments for Dorsal Motion for Sweeps Above the Full Lateral Arm Extension Level.

Figure 6.1.23 $M_y$ Moments for Dorsal Motion for Sweeps Above the Full Lateral Arm Extension Level.
Figure 6.1.24 $M_x$ Moments for Ventral Motion for Sweeps Above the Full Lateral Arm Extension Level.

Figure 6.1.25 $M_y$ Moments for Ventral Motion for Sweeps Above the Full Lateral Arm Extension Level.
It was found that the methodology worked very well in some cases, but failed in others. The determining factor in whether the approach worked or failed was the stability or regularity of the rotation point displacement data. Several rotation point displacement trajectories are plotted to demonstrate the method.

In Fig. 6.2.1 is shown the trajectory in the torso xy plane of the chosen arm rotation point, defined to be

![Graph showing arm rotation point trajectory with angles φ=-15°, φ=-70°, and φ=55°](image)

**Figure 6.2.1** Arm Rotation Point Trajectory in the Torso xy Plane for Dorsal Motion - First Sweep.
initially coincident with the acromiale, for dorsal arm motion in a plane normal to the torso's z axis. The start of the trajectory was at (.0,.0) cm and the arm location in the horizontal plane was at $\varphi=55^\circ$. The initial motion is along the Y axis for about 3 cm and then in the X direction for about 4 cm. This initial rotation point displacement from (0,0) cm to about (4.5,4.5) cm corresponds to a $\varphi$ variation of 70° (from 55° to -15°). The trajectory beyond $\varphi=-15^\circ$ becomes very localized moving about the point (4.5,4.5) cm for $\varphi$ varying from -15° to -70°.

A trajectory of the rotation point for a repeated abduction sweep at the same arm elevation is shown in Fig. 6.2.2. The characteristics of the trajectory are similar to that shown in Fig. 6.2.1 for the same arm motion. In this case the rotation point is at (5.2,4.6) cm for arm motion between $\varphi=-10^\circ$ and $\varphi=-77^\circ$.

For both of these rotation point trajectories there is no place in the xy plane from which the data can be expressed in polar function form without having a multi-valued function. While this is not totally critical for the developed expansion method, it does lead to the discarding of data points and may thus define a misleading rotation trajectory.

The automatic selection of an average rotation point, $\bar{L}$, resulted in $\bar{L} = (-2.83,-1.68)$ cm for the data in Fig. 6.2.1 and $\bar{L} = (-3.28,0.22)$ cm for the data in Fig. 6.2.2. Transforming the data to polar coordinates with respect to
this point and keeping only the points which fall in ascending order of increasing polar angle the data array in Fig. 6.2.1 was reduced from 83 points to 18 points and that in Fig. 6.2.2 from 92 points to 20 points. The standard deviations of the rotation points about the analytic trajectory were .160 cm and .089 cm, respectively.
The rotation point displacement for ventral arm motion for two separate sweeps at the same arm elevation are shown in Fig. 6.2.3 and Fig. 6.2.4. The displacements correspond to arm rotations from 115° to 149° and 112° to 148°, respectively. Compared to the rotation point trajectories in Figs. 6.2.1 and Fig. 6.2.2 these are surprisingly regular and similar. The application of the trajectory calculating

Figure 6.2.3 Arm Rotation Point Trajectory in the Torso xy Plane for Ventral Motion - First Sweep.
methodology worked ideally with these. The average rotation points in the respective Figs. 6.2.3 and 6.2.4 are at $\mathbf{L} = (-3.24, -7.88)$ cm and $\mathbf{L} = (-2.87, 8.22)$ cm. Also 36 of 36 points, in the former, and 27 of 27 points, in the latter, were used in the trajectory calculation. The standard deviations of the rotation points about the analytic trajectory were .042 cm and .030 cm, respectively.
Fig. 6.2.5 shows the same rotation point trajectory as in Fig. 6.2.4 with respect to the y' coordinate system. It is clear from this figure that the transformation of data to polar coordinates with respect to point 0 presents no difficulties.

The expansion of the displacement along the screw axis for the total sweep motion was attempted for all the sweeps.

Figure 6.2.5  Arm Rotation Point Trajectory in the Torso and Y' Systems for Second Ventral Sweep Data
in the form of first, second and third order polynomials. The coefficients for these polynomials and the standard deviation of the fit for the four sweeps discussed above are listed in Table 6.2.1. For ventral motion all three orders gave a good fit. The polynomials for dorsal motion did not produce a good fit for any order.
### Table 6.2.1 Coefficients and Standard Deviation of Fit for First, Second and Third Order Polynomials for Screw Axis Displacement for Arm Sweeps at 0°.

**Screw Axis Displacement Expansions**

\[ \text{SAD} = B_0 + B_1 \phi + B_2 \phi^2 + B_3 \phi^3 \]

<table>
<thead>
<tr>
<th>Sweep Motion</th>
<th>1st Order</th>
<th>2nd Order</th>
<th>3rd Order</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$B_1$</td>
<td>Std. Dev.</td>
<td>$B_1$</td>
</tr>
<tr>
<td>Ventral</td>
<td>-0.153</td>
<td>0.11</td>
<td>-0.060</td>
</tr>
<tr>
<td></td>
<td>4.296</td>
<td></td>
<td>3.107</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.923</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ventral</td>
<td>-0.146</td>
<td>0.12</td>
<td>-0.090</td>
</tr>
<tr>
<td></td>
<td>4.199</td>
<td></td>
<td>3.410</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.340</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dorsal</td>
<td>-2.354</td>
<td>1.39</td>
<td>-2.288</td>
</tr>
<tr>
<td></td>
<td>1.647</td>
<td></td>
<td>1.609</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>-0.098</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dorsal</td>
<td>-2.655</td>
<td>1.38</td>
<td>-1.508</td>
</tr>
<tr>
<td></td>
<td>1.655</td>
<td></td>
<td>1.572</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>-1.003</td>
</tr>
<tr>
<td></td>
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</tbody>
</table>
CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

7.1 Applicability of the Resistive Moment Data

One of the main objectives of this research was to develop and demonstrate a methodology for obtaining data on the resistive properties of joints, with particular emphasis on human joints, in a form amenable to analytic expansion and application to motion simulation models. The objective was limited to consideration of only two degrees of freedom for the moment expansions. Nonetheless, this was one degree more than has been done previously. Many investigators have studied ranges of human limb motion and numerous anthropometric design handbooks abound with such data. They provide no information, however, about the forces acting in the joint structures at various limb positions. Also, a number of investigators have measured the resistive properties of pin or single degree of freedom joints such as the elbow and the knee. Again, these provide no useful information for the present study since the methods of single degree of freedom analysis cannot be extrapolated to two degrees or more.

Engin [28-37] has been the first to look at human joint resistive properties for large displacements in three dimensions. His methods of data collection, however, did not provide joint resistive moment data in a readily expandable form.
In the present research, a method has been developed and applied in which the moments are calculated as a function of a single variable, $\phi$, while the other variable, $\theta$, remains constant. Each moment curve can be expanded as a function in $\phi$ for a fixed $\theta$, and then these curves combined to give a two dimensional expansion in $\phi$ and $\theta$ for the moments. While the expansions were not attempted in this research, the relatively well-behaved form of the moment curves and the small variations in $\Delta \theta$ during the sweeps indicates that the method can produce data amenable to two dimensional expansion.

7.2 Applicability of the Kinematic Analysis

The rotation point trajectory synthesis and the method for finding optimal body point of rotation have direct applicability to the simulation of body kinematics and dynamics, design of human exoskeletal devices, design and evaluation of mechanical human analog joint mechanisms and the development of joint prostheses. The application of the kinematic analysis to the shoulder demonstrated that the method can be highly effective, but that it also has some pitfalls if blindly applied.

In its present form, it cannot deal directly with closed loop rotation point trajectories, nor does it necessarily pick the optimal point for polar expansion. It can, however, be applied in a piecewise fashion to provide partial trajectories which can then be reassembled. A closed loop portion of a trajectory can be readily calculated from an
interior point and combined with trajectory segments for open curves. Also, if the rotation point trajectory does move about a localized area, a point in this area may be chosen as a rotation point for the body.

The three-dimensional kinematics of joints have been studied by a number of investigators [13,15,21,39,68,143] who have characterized the observed motion graphically, in terms of instantaneous centers of rotation, by a full screw axis analysis or a combination of these. Methods for optimization of observation motion path have been applied by several investigators [10,45,112]. These optimizations have generally been made with respect to the instantaneous center of rotation. The present study has provided a means for optimizing the description of relative body motion with respect to any point on either of the bodies. This provides a direct method for synthesizing joint kinematics in terms of rotation about a fixed point in the body and motion of this point on a space curve.

7.3 Suggestions for Extended Research

This research work has resulted in development of methods for three dimensional kinematic analysis and joint mechanism synthesis, complex joint resistive property data collection and analysis in an expandable form, and in the generation and analysis of original data on shoulder joint kinematics. In all of these areas refinements and additional applications of the methodology are not only possible but desirable.
Among these are the incorporation of the developed quaternion operators into the joint kinematics rotational analysis. Aside from the demonstrated rotation conceptualization advantages of quaternions their use would also improve the computational accuracy and efficiency of the analysis program by avoiding trignometric angle calculations. The trignometric angle calculations for some body positions are highly sensitive and can lead to computational instabilities. Also the evaluation of trignometric functions is computationally more time consuming than the evaluation of algebraic relations which result with the quaternion method of analysis.

It was found that a considerable amount of information obtained from the kinematic and resistive moment analysis would have been highly useful during experimental data collection. Of particular interest during testing are the level of the arm long axis with respect to the horizontal plane, the orientations of the arm y axis during a sweep and whether total and consistent data sets have been obtained for a sweep. The analysis programs could be implemented to provide this and other data during testing and thus allow direct insight into joint kinematics mechanisms and how they are affected by different experimental procedures.

A further review of the collected data for optimization of joint rotation trajectories would be worthwhile, as would also the expansion of the collected moment data in two dimensions. It would also be desirable to repeat the
experiments that were conducted in this study several more
times on one individual so as to check for repeatability,
and then on a number of individuals so as to check for
variability.
APPENDIX

Computer Programs for Kinematic and Force Analysis

The following program was written in Fortran for execution on a CDC Cyber 74 computer system. The program reads data, either from a systems file or cards, consisting of individual records containing 34 data elements each corresponding to seven three-dimensional coordinate points and six voltages proportional to axial forces and bending moments measured by the force transducer. Separately calculated transformations relating seat, torso, inertial and board reference systems as well as transducer output voltages to stress components are provided as input data. Also provided as input data are average rotation axes which are obtained by an initial execution of the program with these inputs arbitrarily specified.

The program calculates the applied moments with respect to a rotation point on the moving body for each data set. It calculates the translational displacement for a fixed point on a body, the body rotation matrix, screw axis motion parameters and rotation point displacement for sequential body displacements. These individual displacements are then combined and a body rotation point trajectory calculated in terms of a Fourier series and a polynomial expansion.
The program is structured so that the main program, POSANL3, performs the data manipulations by calling various subroutines. All subroutines used, except for standard system functions, a Fast Fourier Transform and a Multiple Regression Analysis program, were specifically written for this analysis and are listed in the Appendix. These latter two programs were used for the plane trajectory and screw axis displacement calculation, respectively, and were library routines available at the Wright-Patterson Air Force Base Computer Center, and obtained from International Mathematical and Statistical Libraries (IMSL), Inc.
PROGRAM PR3GPAA PU5ANL 3 (INPUT, OUTPUT, TAPE=INPUT, TAPE=OUTPUT, TAPE=)

POSITION ANALYSIS PROGRAM 3
AUTHOR INTS KALEPS

REAL 10J(3,3),A1I(3,3),Q1(3),Q3(3),R0(3,3),MUR(3),HUU(3),ALT(3,3)
REAL 100(3,3),QFS(3),PQRS(3),FF(3,3),U(3,3),M(3),PUE(3),QVI(3)
REAL 0P(3),PFAI(3),PQAV(3),PQAKV(3),A1(3),DK(3),KOT(3,3),Q2(3)
REAL TR(3),QIV(2),TV(3),AVMUP(3),AVNHU(3),Y1(3),r.YT(3)
REAL CN(256),TOP(3),TOT(3),AVNHUY(3),QG(3)
REAL 1VUU(3),FY(3,3),AYR(3,3),NUM(3),AVEMU(3),SCCV(3)
REAL TYL(3),LS(3),RYP(3,3),APY(3,3),RT(3,3),OVL(3),QVS(3)
REAL ATI(3,3),CEI(3,3),AT(206,3),AYR(3,3)
REAL DRY(3),OPTS(3),CIT(3),ON(3),ATY(3,3),TO(3,3),QVD(3,3),QV3(3)
REAL POT(3),POT(3),QV(3),QP(3),SPR(3),SOPA(3),SOPR(3),SOP(3)
REAL PDAVT(3),P4AVF(3),PDAV(3),QPV(3),SPAA(3),SDPA(3),SCF(3)
REAL ZOT(3),OLTR(3),QIT(3),QIT(3),UT(3,3),MT(3,3),QPS(3)
REAL ATI(3,3),APL(3,3),FP(3,3),AFF(3),AFAC(3),RFL(3),ML(3)
REAL FF(3),FHM(3),AR(3),AFF(3),AFAC(3),RFL(3),ML(3)
REAL PTHAT(150),PTHRT(150),FT(3,3),AVNHU(3),AVNMR(3),ART(3,3)
REAL DBRA(150),CWY(150),CWT(150),CWTY(150),CWTP(150),ALC(3,3)
REAL DRY(3),CPY(3),LPY(3),CYP(150),CYP(150),ANC(256),BN(256)
REAL ALT(3,3),ATL(3,3),NT(3,3),MTH(3),MLH(3),FFL(3)
REAL PLOC(3),POLAC(3),POL(3),POLA(3)
COMMON/BLK1/FT
COMMON/BLK2/ANG,RIKFS
DIMENSION LAB(2)
C
DATA SET CODE
READ(1,100) LAB
C
 thrill 6x6 transformation matrix from voltages to forces and moments
READ(3,109) FT
C
AIR: REFERENCE TO INERTIAL TRANSFORMATION
READ(5,101) AIR
C
ATII: INERTIAL TO TORSO TRANSFORMATION
READ(3,101) ATI
C
ALC: CUFF TO LANDMARK TRANSFORMATION
READ(5,101) ALC
C
TOP: OPTIMUM BODY ROTATION POINT
READ(3,102) TOP
C
TAI: ROTATION POINT IN CUFF SYSTEM
READ(5,102) TAI
C
AVMUH: AVERAGE ROTATION AXIS IN REFERENCE
READ(4,102) AVMUH
C
NJO: ORDER OF POLYNOMIAL EXPANSION FOR DISPLACEMENT ALONG SCREW AXIS
C
NF: NUMBER OF TERMS IN FOURIER EXPANSION OF ROTATION POINT DISPLACEMENT
IN PLANE NORMAL TO AVERAGE SCREW AXIS
READ(5,104) N0,NF
C POTAVF: AVERAGE INSTANTANEOUS ROTATION POINT WITH RESPECT TO CUFF
READ(5,102) POTA V F
C POTAVF: AVERAGE INSTANTANEOUS ROTATION POINT WITH RESPECT TO REF
READ(5,102) POTAVF
C POTAVF: AVERAGE INSTANTANEOUS ROTATION POINT WITH RESPECT TO TO: SO
READ(5,102) POTAVF
C T H S : ANGLE OF ARM ABOVE OR BELOW FULL LATERAL EXTENSION POSITION
READ(5,105) T H S
WRITE(6,121) LAB
WRITE(6,111)((POTAVF(I,J),J=1,3),(POTAVF(I,J),J=1,3),(POTAVF(I,J),J=1,3)),
*I=1,3)
WRITE(6,112) TOP,TA,AVIMUR
WRITE(6,113) POTAVF,POTAVF,POTAVF
WRITE(6,114) N0,NF
MF=3
N00=0
N05=0
N19=0
N30=0
N45=0
N60=0

THEtas=0.
C CALCULATES AT(I) TRANSFORMATION FROM REFERENCE TO TORSO
CALL HATPROU(ATI,AIR,AR)
C CALCULATES AO: TRANSFORMATION FROM REFERENCE TO CUFF
C AO: CUFF ORIGIN WITH RESPECT TO REFERENCE
C FF: FORCE ACTING ON ARM IN REFERENCE
C FM: MOMENT ACTING ON ARM IN REFERENCE
C RF: POINT OF FORCE APPLICATION IN REFERENCE
C MFI: IF MF=1 DATA SET DEPLETED
CALL GETNEW(10,10,FF,FM,RF,MF)
C FF: FORCE ACTING ON ARM IN CUFF WRT REFERENCE ORIGIN
CALL HTVCPRO(A0,FF,FFC)
C FM: MOMENT COMPONENTS IN CUFF
CALL HTVCPRO(A0,FM,FHMC)
CALL HTVCPRO(A0,FFC,FHFC)
CALL HVTCPROM(TA,FAM)
C TA: ARM ROTATION POINT WRT CUFF ORIGIN BUT IN REFERENCE
CALL HVTCPRO(ATI,TA,TAR)
00 31 I=1,3
C TRI: ORIGIN OF TORSO IN REFERENCE
TRI(I)=QF(I)+TAF(I)
C RFAT: VECTOR FROM LANDMARK ORIGIN TO POINT OF FORCE APPLICATION
31 RFAT(I)=FF(I)-TAF(I)-QF(I)
C RFAC: FF IN CUFF
CALL HTVCPRO(A0,RFAT,RFAC)
C RFAT: RFAT IN LANDMARK
CALL HTVCPRO(ALC,RFAT,RFAC)
C ML: MOMENT IN LANDMARK SYSTEM
CALL HVTCPRO(A0,ML,ML)
C ALR: TRANSFORMATION FROM REFERENCE TO LANDMARK
CALL HATPROU(ALR,ART)
CALL TRAPS(RA,ART)
C ALT: TRANSFORMATION FROM TORSO TO LANDMARK
CALL HATPROU(ALT,ALT)
CF=(130./14.141592654)
C QH: KAR: OMEGA IN SEQUENTIAL ARM ROTATION MATRIX
QH=ASIN(ALT(2,2))
C PHF: PHI IN SEQUENTIAL ARM ROTATION MATRIX
PHF=ACOS(ALT(1,1))
C PHI: ANGLE WITH FULL LATERAL EXTENSION VECTOR FOR BELOW SHOULD
PHI=ACOS(ALT(3,2)*COS(THS)*ALT(3,3)*SIN(THS))
PH2: ANGLE WITH FULL LATERAL EXTENSION VECTOR FOR ABOVE SHOULDER
PH2 = ACOS(ALT(3,2) * COS(THS) - ALT(3,3) * SIN(THS))

TH1: ANGLE WITH NEGATIVE VERTICAL VECTOR FOR BELOW SHOULDER
TH1 = ACOS(ALT(3,2) * SIN(THS) - ALT(3,3) * COS(THS))

TH2: ANGLE WITH NEGATIVE VERTICAL VECTOR FOR ABOVE SHOULDER
TH2 = ACOS(-ALT(3,2) * SIN(THS) - ALT(3,3) * COS(THS))

OMEGA SPHERICAL ANGLE AS DEFINED BY ENGINEER
O = ACOS(-ALT(3,3))

PHI: PHI SPHERICAL ANGLE AS DEFINED BY ENGINEER
PH = ATAN(-ALT(1,3) / ALT(2,3))

TH1D: ANGLE ABOVE HORIZONTAL PLANE FOR BELOW SHOULDER IN DEG
TH1D = CF * TH1 - 90

TH2D: ANGLE ABOVE HORIZONTAL PLANE FOR ABOVE SHOULDER IN DEG
TH2D = CF * TH2 - 90

ABL: TRANSFORMATION FROM LANDMARK TO TCRSO
CALL TRANS(ALT, ATL)

MTT: SHOULDER MOMENT VECTOR IN TCRSO SYSTEM
CALL TVCPRO(ALT, ML, MTT)

DO 81 I = 1, 3
  FFLM(I) = -48 * FFL(I)
  MLM(I) = -45 * ML(I)

81

OMEGA = 160. * OMGKAR / 3.141592654
PHIK = 130. * PHIKAR / 3.141592654
WRITE(6, 110) PFL, FFLM, MLM
WRITE(6, 111) MTT

WRITE(6, 112) PHIA0, OMEGK AO
WRITE(6, 113) THI0, THI10, THI20

WRITE(6, 115) (ALT(I, J), J = 1, 3), I = 1, 3
DO 1 I = 1, 3
  DO 1 J = 1, 3

1 CONTINUE

IF(0) = 0
CALL GETNEWO(A1, 31, FF, FM, RF, MF)

IF(MF = 1) END OF DATA ENCOUNTERED

12

CALL TRANS(T1, ALT)
CALL TVCPRO(A1, FF, FFC)
CALL TVCPRO(A1, FM, FMC)
CALL TVCPRO(ALC, FFC, FFL)
CALL TVCPRO(ALT, TA, TAR)
DO 32 I = 1, 3

32
QMHKAR = ASIN(ALT(2, 2))
PHIKAR = ACOS(ALT(1, 1))
THI = ACOS(ALT(3, 2) * SIN(THS) - ALT(3, 3) * COS(THS))
TH2 = ACOS(-ALT(3, 2) * SIN(THS) - ALT(3, 3) * COS(THS))
PH1 = ACOS(ALT(3, 2) * COS(THS) * ALT(3, 3) * SIN(THS))
PH2 = ACOS(ALT(3, 2) * COS(THS) - ALT(3, 3) * SIN(THS))
PHI = ATAN(-ALT(1, 3) / ALT(2, 3))
QME = ACOS(-ALT(3, 3))

PHIQ = QME * PHI
QHEM = QHE * QME
QHO = QHE * QHO
QHE2 = QHE * QHE
TH10 = SQR(THI^2 - SQR(TH20))
TH20 = SQRT(TH10^2 - TH2^2)

CALL MVCP(ALT, HL, MT)
FIM(I) = FIM(I) + FFLM(I)
MTTM(I) = 0.4445 * MTM(I)

PMXAO = 1.0 * PMXAO / PMXAO
OMGAO = 1.0 * QMGAO / QMGAO
WRITE(6, 110) RFL, FFLM, MLM
WRITE(6, 117) MTM
WRITE(6, 120) PH10, TH10, PH20, TH20
WRITE(6, 115) RHEM, QME
WRITE(6, 116) (1, I) (* ALT(I, J), J = 1, 3, I = 1, 3)

C ROTATION TORSION MATRIX IN 3 SYSTEM
CALL HVPRDG(ALT, AL, R)

C ROTATION TORSION MATRIX IN REFERENCE
CALL HVPRD(ALT, A0, R)

C THETA ANGLE OF ROTATION AROUND SCREEN AXIS
THETA = ACOS((PR(1, 1) + RP(2, 2) + RP(3, 3) - 1) / 2)

C MUI ROTATION AXIS IN REFERENCE
MUI(1) = (PR(3, 2) - RP(2, 3)) / (2 * SIN(THETA))
MUI(2) = (RP(1, 3) - RP(3, 1)) / (2 * SIN(THETA))
MUI(3) = (RP(2, 1) - RP(1, 2)) / (2 * SIN(THETA))

THETAS = THETA + THETAS
THOD = THETA + THETAS

C MUI ROTATION AXIS IN TORSO
CALL HVCPR3D(ALT, MUI, MJ)

C AVNNMUI NORMALIZED AVERAGE INITIAL ROTATION AXIS IN REFERENCE
CALL VNPRAL(AVMUC, AVVM)
CALL JOT(AVNNMUI, AVNMU)

C BETA ANGLE INDIVIDUAL ROTATION AXES MAKE WITH AVERAGE AXIS
BETA = ACOS(Z)

N11 = N10 + 1
GO TO 67
IF((BETA > 60.0)) GO TO 67
GO TO 49
IF((BETA > 15.0)) GO TO 90
GO TO 39
IF((BETA > 15.0)) GO TO 90
GO TO 90

C N01: NUMBER OF AXES BETWEEN 0 AND 5 DEGREES
N01 = N00 + 1
GO TO 50

C N02: NUMBER OF AXES BETWEEN 5 AND 15 DEGREES
N02 = N01 + 1
GO TO 50

C N03: NUMBER OF AXES BETWEEN 15 AND 30 DEGREES
N03 = N02 + 1
GO TO 50

C N04: NUMBER OF AXES BETWEEN 30 AND 45 DEGREES
N04 = N03 + 1
GO TO 80
C 45 N4S1: NUMBER OF AXES BETWEEN 45 AND 50 DEGREES
G O TO 80
C 50 N4S0: NUMBER OF AXES AT GREATER THAN 50 DEGREES
G O TO 80
C 60 WITHE(6,130) BETA
C MUD: ROTATION AXIS IN CUFF
CALL MTVCPR(UT, MUR, MUQ)
C CALCULATES SCREW AXIS PARAMETERS IN REFERENCE
U: MATRIX CONTAINING PIERCING POINT VECTORS
M: DISTANCE FROM PIERCING POINTS TO INSTANTANEous CENTER OF ROT
P: INSTANTANEous CENTER OF ROTATION IN REFERENCE
S: DISPLACEMENT ALONG SCREW AXIS
CALL ALSMAXP(UF, Q0, Q1, THETA, U, M, P3, SA)
DO 15 I=1,3
C QIT: Q0 HRT TORSO ORIGIN IN REFERENCE
QIT(I)=Q0(I)-RTR(I)
C QIT: Q1 HRT TORSO ORIGIN IN REFERENCE
QIT(I)=Q1(I)-R7(I)
DO 2 I=1,3
C QV1: INST. CENTER OF ROT HRT TORSO ORIGIN IN REFERENCE
QV1(I)=QV(I)-Q0(I)
C QV1: VECTOR FROM Q0 TO INST. CENTER OF ROT IN REFERENCE
QV1(I)=QV(I)-QO(I)
C QV1: VECTOR FROM CUFF ORIGIN TO INST. CENTER OF ROT IN REFERENCE
QV1(I)=QV(I)-Q1(I)
C DO: AMOUNT OF CUFF ORIGIN DISPLACEMENT
DOA=SQRT((Q0(1)**2+Q0(2)**2+Q0(3)**2)
C POA: INST. CENTER OF ROT IN CUFF
CALL MTVCPR(AT, Q0, Q1)
C POT: INST. CENTER OF ROT IN TORSO
CALL MTVCPR(UT, V1, PO)
C NP0: ROTATION NUMBER
NP0=NP0+1
ANG=(136.*THETA)/3.141592654
THETAS(TPO)=THETAS
THETAT(NPO)=THSDEG
WRITE(15,123) NP0, THETA, THETAS, THSDEG, SAO
WRITE(6,145) SOUTH(I), I=1,3)
WRITE(6,146) (Q0(I), I=1,3), QOA
DO 3 I=1,3
POAS(I)=POAS(I)+Q0(I)
POAS(I)=POAS(I)+Q1(I)
C 3 POAS(I)=POAS(I)+POAS(I)
ADD(I)=ADD(I)+ADD(I)
C POA(I)=POA(I)+POA(I)+POA(I)
C 5 POA(I)=POA(I)+POA(I)+POA(I)
C 6 POA(I)=POA(I)+POA(I)+POA(I)
C
CALL MTVCPRD (ATF, OVS, TTI)

CALL MTVCPRO (AYR, OVS, OVS)

DO 13 I=1,3

OVS: SUM OF ROTATION POINT DISPLACEMENTS IN TORSO

OVS(I) = OVS(I) + OVS(I)

13 CONTINUE

CALL MTVCPRD (AYR, OVS, OVS)

CALL NTVCPRO (AYR, OVS, OVS)

IF (NPS.EQ.1) GO TO 52

GO TO 52

T1 DO 11 I=1,3

11 TOT(I) = TTI(I)

CONTINUE

Y1(1) = 0.

Y1(2) = 1.

Y1(3) = 0.

CALL CROSS (Y1, AVNNMV, OVS)

RY1: JVI NORMALIZED

CALL NORMAL (OVS, RY1)

THETD: ANGLE BETWEEN Y AND AVERAGE ROTATION AXIS

THETD = ACOS (3)

RY1: ROTATION OPERATOR THAT ROTATES Y INTO AVERAGE ROTATION AXIS

CALL ROTAT (THETD, RYT(1), RYT(2), RYT(3))

AYR: TRANSFORMATION FROM REFERENCE TO Y (AVERAGE ROTATION SYSTEM)

CALL TRANSF (RY, AYR)

MUY1: ROTATION AXIS IN Y

CALL MTVCPRD (AYR, MUY, MUY)

DO 13 I=1,3

MUY: SUM OF ROTATION POINT DISPLACEMENTS IN REFERENCE

MUY(I) = MUY(I) + MUY(I)

13 CONTINUE

CALL MTVCPRD (AYR, MUY, MUY)

CALL NTVCPRO (AYR, MUY, MUY)

AVEY: AVEVY: AVERAGE AND STD DEV OF MUY IN Y

CALL AVEVY (MUY, AVEY, SDEVY)
AVENU AND SDEV: I ACCUMULATIVE AVERAGE AND STD DEV OF HUR IN REF
CALL AVE:V (AVR,H,PI,AVNUR,SOEVR)
CALL NORM:AL (AVNUR,AVNUR0)
CALL AVE:V (AVMUR,AVMUR0)
C AYI TRANSFORMATION FROM Y TO TORSO
CALL ROTPROD (ITY,XY,ITY)
C AVMUR NORMALIZED ACCUMULATIVE AVERAGE ROTATION AXIS IN TORSO
CALL ROTPROD (ITY,AVMUR,AVMUR0)
C DRY: ROTATION POINT DISPLACEMENT IN Y
CALL ROTPROD (ARY,ORY,DRY)
C CALCULATES AVERAGE ROTATION CENTER IN Y
CALL ROTCENT (DRY,THTR,PI,L)
CALL I (UPD) =DRY(1)
CALL I (UPD) =DRY(2)
CALL I (UPD) =DRY(3)
DO 14 I = 1,3
C OTT: TOTAL ROTATION POINT DISPLACEMENT STARTING AT TOT
14 OTT(I) =TOT(I) + ORS(I)
DO 7 I = 1,3
Q0(I) =Q1(I)
TR(I) =TR(I) * QF(I)
DO 7 J = 1,3
AO(I,J) =AI(I,J)
WRITE(6,142) PIU,POAV
WRITE(6,509)
WRITE(6,511) ((UT(I,J),J=1,3),NT(I),I=1,3)
WRITE(6,127)
WRITE(6,136) ((UF(I,J),J=1,3),(U(I,J),J=1,3),M(I),I=1,3)
WRITE(6,140)
WRITE(6,131)
WRITE(6,132) (PF(I),PDAV(I),SOPA(I),POA(I),POAV(I),SOPA(I),
* POT(I),PDAV(I),SOPOT(I),I=1,3)
WRITE(6,139)
WRITE(6,133)
WRITE(6,134) (MRU(I),AYNUR(I),MUT(I),AVNUR(I),MVY(I),AVNUR(I),
* SCEVY(I),I=1,3)
WRITE(6,122)
WRITE(6,141) ((TO(I),OTT(I),ORTS(I),ORT(I),DRY(I),DRS(I),
* DRI(I),I=1,3)
CALL GETHE:O (AI,01,FF,FR,RF,FL)
GO TO 50
C PHI0: ROTATION ANGLE TO PLACE FIRST ROTATION POINT ON Z AXIS
12 PHI0 =ATPHIL (1,1,L,31)
IF (PHI0) 21,21,22
21 IF (L(1)) 25,25,24
22 IF (L(3)) 25,25,23
25 PHI0 =PHI0 +3.141592654
GO TO 22
26 PHI0 =2.* PHI0 +3.141592654
GO TO 23
21 PHI0 =3.141592654
GO TO 50
23 CONTINUE
C PHI0 =PHI0 +3.141592654
C CALCULATES ROTATION MATRIX FROM Y TO Y PRIME
CALL ROTMAT (PHI0,C,R,1,0,RYP)
C AYPY: TRANSFORMATION FROM Y TO Y PRIME
CALL TRANSF (AY,AYPY)
C LYP: L VECTOR IN Y PRIME
CALL ATVPP (AYPY,L,LYP)
WRITE(6,511) (LYP(I),I=1,3)
C CALCULATES EXPANSION COEFFICIENTS FOR ROTATION POINT TRAJECTORY
C AN AND CN ARE FOURIER SERIES COEFFICIENTS FOR OI'S
C CH ARE COEFFICIENTS OF UP TO A THIRD ORDER POLYNOMIAL FOR OAD
CALL 2: WRIT (CRYP,3,DRYP,1,DRYP2,A,NP0,AN,SN,CN,NQ,0P)
WRITE(6,50) T,D
CALL 1: ATPRO0 (A,Y,LYP,ARYP)
CALL 1: ATPRO0 (ATY,ARYP,ATYP)
DO 9 I=1,256
R=AN(I)/2.
PH=R*I.
DO 10 J=1,NF

10 R=PH(J)*COS(J*I*QDF)+SN(J+1)*SIN(J*I*QD)
RC1=|SIN(ANG)**COS(ANG)
RC2=COS(ANG)**SIN(ANG)
U3(I)=SIN(PH)*RC2
V3(I)=COS(PH)*RC2+LYP(J)
WRITE(5,506) (V3(I),I=1,3),R,PH
CALL MTCPRO (ATYP,V3,ORT)
DO 9 K=1,3

C RT IS THE TRAJECTORY OF THE DEFINED ROTATION POINT IN THE TORSO
9 RT(I,J)=OVT(I,J)
WRITE(6,505) (CTC(I),I=1,3)
WRITE(6,503)
WRITE(6,504) (RT(I,J),J=1,3),I=1,256
WRITE(6,122)
WRITE(6,123) (ATR(I,J),J=1,3),I=1,3
WRITE(6,124) (TA(I),I=1,3)
WRITE(6,125) (FR(I),I=1,3)
WRITE(6,126) (AVHUR(I),I=1,3)
WRITE(6,127) (NQ0,NQ1,NQ2,NQ3,NQ4,NQ5)
WRITE(6,128) (NQ6)
WRITE(6,129) (AYR(I,J),J=1,3), (ARYY(I,J),J=1,3), I=1,3
WRITE(6,130) (AN(I),I=1,129)
WRITE(6,131) (OM(I),I=1,129)
100 FORMAT (2A13)
101 FORMAT (9(F5.0))
102 FORMAT (3(F10.0))
103 FORMAT (3,I3)
104 FORMAT (I3)
105 FORMAT (F10.0)
106 FORMAT (F12,4)
108 FORMAT (1X,"AIR",2EX,"ATI",2EX,"ALC"/1X,3(F10.3),4X,3(F10.3),4X,3(F10.3))
111 FORMAT (1X,"ORDER OF POLYNOMIAL FIT TO OAD=",I3,2X,"NUMBER OF TERMS IN FOURIER EXPANSION",I3)
112 FORMAT (1X,"ARM TO TORSO RELATIVE ANGLES: PHI=",F7.2,4X,"OMEGA=",F7.2)
SUBROUTINES

SUBROUTINE ALSKAXP(MUE,Q0,Q1,THETA,U,M,P0,3)

ALTERNATE METHOD FOR CALCULATING SCREW AXIS PARAMETERS.

REQUIRES NORMALIZED SCREW AXIS, MUE, COORDINATES OF
POINT ON BODY BEFORE, Q0, AND AFTER, Q1, DISPLACEMENT,
AND THE ANGLE OF BODY ROTATION, THETA, IN RADIANS.

DO2 IS THE BODY DISPLACEMENT ALONG THE SCREW AXIS.

P0 IS THE VECTOR FROM THE (3) SYSTEM ORIGIN TO THE
INTERSECTION OF THE SCREW AXIS WITH THE NORMAL FROM Q0.

THE DISTANCE ALONG THE SCREW AXIS, IN THE POSITIVE
SENSE, FROM P0 TO THE XY PLANE, XZ PLANE AND YZ PLANE
ARE GIVEN BY M(1), M(2) AND M(3), RESPECTIVELY. THE
COLUMNS 1, 2 AND 3, OF U GIVE THE XY, XZ AND YZ PLANE
PIECING POINT COORDINATES, RESPECTIVELY.

REAL MUE(3),Q0(3),Q1(3),M(3),U(3,3),P0(3),Q0(3),CR(3),V(3)

DO 10 I=1,3
DO(I)=Q1(I)-Q0(I)
CALL CROSS(MUE,Q0,QR)
CALL DOT(Q1,MUE,U)
DO 20 I=1,3
P(I)=((1+COS(THETA))*CR(I)/SIN(THETA)-S*MUE(I)*Q1(I)+Q0(I))/2.0
M(1)=P0(1)/MUE(1)
M(2)=P0(2)/MUE(2)
M(3)=P0(3)/MUE(3)
CALL CROSS(+Q0,MUE,V)
U(1,1)=0.
U(1,2)=V(3)/MUE(2)
U(1,3)=V(2)/MUE(3)
U(2,1)=V(3)/MUE(2)
U(2,2)=0.
U(2,3)=V(1)/MUE(3)
U(3,1)=V(2)/MUE(1)
U(3,2)=-V(1)/MUE(2)
U(3,3)=0.
RETURN
END
SUBROUTINE CRCONST (P1,P2,P3,P4,A,RC,M)

CALCULATES COORDINATE SYSTEM WITH RESPECT TO POINTS P1,P2, AND P4 AND CONSTRUCTS THE COORDINATE SYSTEM USING P3 IF ONE OF THE ORIGINAL POINTS IS MISSING. OUTPUTS OX3 TRANSFORM A AND ORIGIN RC.

REAL P1(3),P2(3),P3(3),P4(4),A(3,3),AC(3),PAV(3),O1(3),O2(3)
REAL O3(3),O4(3),A1L(3),A3L(3),AN1(3,3),AN2(3,3),AN3(3,3)
REAL AN1T(3,3),AL1(3,3),AL2(3,3),AL3(3,3),O12(3),O23(3),O14(3)
REAL P3L(3),P4L(3)
REAL O3L(3),O4L(3)
REAL O12(3),O23(3),O34(3)
REAL N1,N2,N3,N4
DATA FLAG/0./

IF (P01.EQ.0.) L=1
IF (P02.EQ.0.) L=1
IF (P03.EQ.0.) L=1
IF (P04.EQ.0.) L=1
IF (L.GE.2) GO TO 30

0010 L=1

IF (P01.EQ.0.) L=1
IF (P02.EQ.0.) L=1
IF (P03.EQ.0.) L=1
IF (P04.EQ.0.) L=1

10 L=1

IF (P01.EQ.0.) L=1
IF (P02.EQ.0.) L=1
IF (P03.EQ.0.) L=1
IF (P04.EQ.0.) L=1

0012=SQRT (O12(1)**2+O012(2)**2+O012(3)**2)
O023=SQRT (O023(1)**2+O023(2)**2+O023(3)**2)
O034=SQRT (O034(1)**2+O034(2)**2+O034(3)**2)
0013=SQRT (O013(1)**2+O013(2)**2+O013(3)**2)
IF (O012M.GT.10.) N12=1
IF (O013M.GT.10.) N13=1
IF (O013M.GT.10.) N13=1
IF (O012M.GT.10.) N12=1

NM1=N12+N14
NM2=N12+N23
NM3=N23+N34
NM4=N34+N41
IF (NN1.EQ.2) N1=1
IF (NN2.EQ.2) N2=1
IF (NN3.EQ.2) N3=1
IF (NN4.EQ.2) N4=1
WRITE (6,54) N1,N2,N3,N4
NM=N1+N2+N3+N4
IF (NN1.EQ.2) GO TO 30
IF (NN1.EQ.2) GO TO 30

10 L=1

IF (FLAG.EQ.2) GO TO 60
IF (NN1.EQ.1) GO TO 52
IF (NN1.EQ.1) GO TO 52
IF (NN1.EQ.1) GO TO 52
IF (NN1.EQ.1) GO TO 52

51 DO 13 = 1, 3
13 A1(I) = P1(I)
14 DO 15 = 2, 3
15 A3(I) = P4(I)

11 CALL SYSFOTH(A1,A2,A3,AN1,AC1)
IF (FLAG.EQ.1) GO TO 41
A2(I) = P2(I)
12 A3(I) = P3(I)
   CALL SYSFOR(A1, A2, A3, AN2, AN2I, A12)
   IF (FLAG .EQ. 1) GO TO 62
53 DO 13 I = 1, 3
   A1(I) = P1(I)
   A2(I) = P2(I)
   A3(I) = P3(I)
   CALL SYSFOR(A1, A2, A3, AN3, AN3I, AN3I, A13)
   IF (FLAG .EQ. 1) GO TO 63
54 DO 1 I = 1, 3
   A1(I) = P1(I)
   A2(I) = P2(I)
   A3(I) = P3(I)
   CALL SYSFOR(A1, A2, A3, AN4, AN4I, AN4I, A14)
   IF (FLAG .EQ. 1) GO TO 64
   CALL TRANSP(AN2, AN2I)
   CALL TRANSP(AN3, AN3I)
   CALL MATPROD(AN1, AN2I, A12)
   CALL MATPROD(AN1, AN3I, A13)
   CALL MATPROD(AN1, AN4I, A14)
   DO 21 I = 1, 3
   RC(I) = RC1(I) - RC2(I)
   UL3(I) = RC1(I) - RC3(I)
21 DO 24 J = 1, 3
   RC(I) = RC2(I) + R21(I)
   GO TO 23
22 A(I, J) = AN1(I, J)
   WRITE(6, 503)
   GO TO 63
62 CALL TRANSP(AN2, AN2I)
   CALL MATPROD(AN2, A12, R21)
   CALL MATPROD(AN1, AN2I, A14)
   DO 23 I = 1, 3
   RC(I) = RC2(I) - R21(I)
   GO TO 24
23 DO 25 I = 1, 3
   RC(I) = RC3(I) + P31(I)
   GO TO 24
24 CALL TRANSP(AN3, AN3I)
   CALL MATPROD(AN3, AN3I, A13)
   CALL MATPROD(AN1, AN3I, A14)
   DO 25 I = 1, 3
   RC(I) = RC3(I) - R31(I)
   GO TO 26
25 CALL TRANSP(AN4, AN4I)
   CALL MATPROD(AN4, AN4I, A14)
   CALL MATPROD(AN1, AN4I, A13)
   DO 25 I = 1, 3
   RC(I) = RC4(I) + R41(I)
   GO TO 26
26 M = 1
   GO TO 60
30 M = 2
60 CONTINUE
502 FORMAT (1X, "S32")
503 FORMAT (1X, "S3")
504 FORMAT (/ , 1X, "S13")
RETURN
SUBROUTINE CROSS(A,B,C)

** ------------------------------- **
** CALCulates Cross Or Vector Product Of Vectors A(3) **
** And B(3) **
** C = AxB **
** ------------------------------- **

REAL A(3), B(3), C(3)
C(1) = A(2)*B(3) - A(3)*B(2)
C(2) = A(3)*B(1) - A(1)*B(3)
C(3) = A(1)*B(2) - A(2)*B(1)
RETURN
END

SUBROUTINE CURVFIT(DRYP3, DRYP1, DRYP2, AL, NP0, AN, BN, CN, NO, DP)

** ------------------------------- **
** CALCulates Continuous Functions R(PHI) And SAD(PHI) To Define **
** Trajectory Of Assumed Rotation Point Composed Of Components **
** Normal And Along The Average Screw Axis **
** ------------------------------- **

REAL X(39,4), TEMP(4), XYBAR(4), A(730), ANOVA(1-), VAR3(1,3), 3(4,7)
REAL RHO1(256), MON2(256), RHOCL(256), RHOCL2(255)
INTEGER I,K(8)
INTEGER NBR(6)
COMPLEX X(129)
REAL S01(160), S02(160), S03(150), PHI(160), RHO(160), RHOCL(257)
REAL DRYP3(160), DRYP1(160), DRYP2(160), AN(129), BN(129), CN(1)
REAL PHOR(160), PHIR(160), RHOCL(256), RHOCL2(256),
REAL SADG(160), DSAD(160)
NN=4
S01=0.
S02=0.
S03=0.
DO 1 I=1,NP0
S01(I+1)=S01(I)*DRYP1(I)
S02(I+1)=S02(I)*DRYP2(I)
S03(I+1)=S03(I)*DRYP3(I)
1=MNP(I)+1
102 I=1,M
IF(S03(I).EQ.0.) GO TO 10
GO TO 21
10 WRITE(2,21) M0
21 PHI(I)=ATAN(S01(I)/S03(I))
2 RHO(I)=SQR(S01(I)**2+S03(I)**2)
WRITE(5,91) (PHI(I), RHO(I), I=1,M)
PHIR(I)=PHI(I)
PHOR(I)=RHO(I)
NP=1

If (phi(j), eq, phi(i)) go to 50
If (j, eq, np0+1) go to 60
j = j+1
Go to 50
np = np+1
Ah(j, np) = ho(j)
phi(np) = phi(j)
i = j
If (i, eq, np0+1) go to 50
Go to 70
continue
write(5,505) np
np = mix(np)/256.
l = 1
Nh(1) = nh(1)
continue
Go to 12
l = l+1
If (l, eq, pt, phi(n+1)) go to 33
n = n+1
If (n, eq, np) go to 40
Go to 29
ahl(1) = (l*dp-phil(n))*nh(n+1)+(l*dp-phil(n))**phil(n)
* (phil(n+1)-phil(n))
Go to 13
40
continue
Go to 12
l = l+1,
12
Rhoni(i) = Rhon(i)
ang = atan(Rhon(i), 256)/256.
o = 1
i = 1, 256
13
Rhoni(i) = i*sin(ang)*Rhoni(i)*cos(ang)
write(io, 400)
call fft_rui(rhon2, 256, x, wk, wk)
write(5, 501)
write(5, 502)(ahoni(i), i = 1, 256)
Go to 3
i = i+1,256
ahoni(i) = real(x(i) + conjg(x(i)))/256.
3
bn(i) = (aimag(conjg(x(i)) - x(i)))/256.
ax(i) = (29316306/256).
do = i = 1, 256
fhoc(i) = ahoni(i)/2.
do = i = 1, nn
4
nhoc(i) = rho(i) + ang(i+1)*cos(i*i+4) + nin(i+1)*sin(i*i+4)
do = i = 1, 256
rho(i) = sin(ang) + rho(i) + cos(ang)
rhoc2(i) = rho(i) + rho(i)
orho(i) = rho(i) * rho(i)**2
var = orhos */255
sdev = sqrt(var)
write(6, 537)
write(6, 506)(rhoc2(i), i = 1, 256)
write(6, 539)
write(6, 510)(okhoi(i), i = 1, 256)
write(6, 513)nn
write(6, 511) var, sdev
nhr(1) = nh*1
nhr(2) =
nhr(1) =
nhr(2) =
ix =
alpha = 0.1
n = 1
If (n = 2) 51,52,53
Go to 5
ix(i, 2) = sd2(i)
5
ix(i, 1) = phi(i)
CALL BECOVM(XY,IX,NBR,TEMP,XYBAR,A,IER)
K=1
CALL RLHUL(A,XYBAR,N,K,ALFA,ANOVA,B,IB,VARB,IER)
WRITE(6,504)
WRITE(5,505)((S(I,J),J=1,7),I=1,2)
CN(1)=0(2,1)
CN(2)=0(1,1)
CN(3)=0
CN(4)=0.
GO TO 52
50 6 I=1,M
XY(I,3)=SQ2(I)
GO 6 J=1,2
6 XY(I,J)=PHI(I)**J
WRITE(6,513)((XY(I,J),J=1,3),I=1,2)
CALL BECOVM(XY,IX,NBR,TEMP,XYBAR,A,IER)
K=2
IB=3
CALL RLHUL(A,XYBAR,N,K,ALFA,ANOVA,B,IB,VARB,IER)
WRITE(6,504)
WRITE(5,505)((S(I,J),J=1,7),I=1,3)
CN(1)=0(3,1)
CN(2)=0(1,1)
CN(3)=0(2,1)
CN(4)=0.
GO TO 52
53 GO 7 I=1,M
XY(I,4)=SQ2(I)
GO 7 J=1,3
7 XY(I,J)=PHI(I)**J
WRITE(6,514)((XY(I,J),J=1,4),I=1,3)
CALL BECOVM(XY,IX,NBR,TEMP,XYBAR,A,IER)
K=3
IB=3
CALL RLHUL(A,XYBAR,N,K,ALFA,ANOVA,B,IB,VARB,IER)
WRITE(6,504)
WRITE(5,505)((S(I,J),J=1,7),I=1,4)
CN(1)=0(4,1)
CN(2)=0(1,1)
CN(3)=0(2,1)
CN(4)=0(3,1)
54 GO 8 I=1,M
SAOC(I)*CN(1)+CN(2)*PHI(I)+CN(3)*PHI(I)**2+CN(4)*PHI(I)**3
OS10(I)=SAOC(I)-SQ2(I)
8 SSQ=SSQ+AD(I)**2
VAR2=SSQ/NPG
DEV2=SQR(T(VAR2)
WRITE(9,F19.9)
WRITE(6,516)(OSAD(I),I=1,7)
WRITE(6,517)VAR2,DEV2
500 FORMAT(1X,"ARRIVE AT FFTRC")
501 FORMAT(1X,"LEAVE FFTRC")
502 FORMAT(1X,16(F7.3))
503 FORMAT(1X,7(F3.3,F3.3,2X))
504 FORMAT(1X,7(F3.4))
505 FORMAT(1X,7(F3.4))
506 FORMAT(1X,"NUMBER IN REDUCED ARRAY",I3)
507 FORMAT(1X,"CALCULATED RHO VALUES USING FIRST 7 FOURIER SERIES TE
* RMS")
508 FORMAT(1X,13(F7.3))
509 FORMAT(1X,30X,"DIFFERENCE BETWEEN ORIGINAL RHO VALUES AND FOURIER SE
* TIES CALCULATED VALUES")
510 FORMAT(1X,16(F8.2))
511 FORMAT(1X,"VARIANCE",F13.5,2X,"STD CEVIATIONS",F13.5)
512 FORMAT(1X,2(F13.5,-X))
513 FORMAT(*,16(F7.3,F3.3))
514 FORMAT(*,16(F7.3,-X))
SUBROUTINE OATREAD(DATG1, DATG2, DATG3, DATA1, DATA2, DATA3, DATA4, * OATT, M)\n\nREAD THE ORIGINAL DATA FILE AND SEPARATES THE DATA INTO GPA COORDINATE AND ARP COORDINATE POINTS, AS WELL AS GPA TRANSUCER VOLTAGES
\nREAL DATA(3), DATG1(3), DATG2(3), DATG3(3), DATA1(3), DATA2(3)
REAL DATA(3), DATA4(3), DATA5(6)
READ(1, 110) DATA
IF(EOF(1), NE.0) RETURN
WRITE(6, 115) M
WRITE(6, 120) DATA
DO 10 I = 1, 3
   DATG1(I) = DATA(I + 1)
   DATG2(I) = DATA(I + 9)
   DATG3(I) = DATA(I + 3)
   DATA1(I) = DATA(I + 11)
   DATA2(I) = DATA(I + 17)
   DATA3(I) = DATA(I + 21)
10 DATA*(I) = DATA(I + 25)
DO 30 I = 1, 5
30 OATT(I) = DATA(I + 29)
110 FORMAT(1F10.6)
115 FORMAT(1X, "DATA SET ", I2)
120 FORMAT(1X, 2F10.4)
RETURN
END

SUBROUTINE DOT(A, B, C)

CALCULATES DOT OR SCALAR PRODUCT OF VECTORS A(3) AND B(3)

REAL A(3), B(3), C(1)
SUM = 0
DO 11 J = 1, 3
   SUM = SUM + A(J) * B(J)
   C(1) = SUM
RETURN
END
SUBROUTINE FORCECL(A,B,C,FT,V,FFR,FMR,RF,M)

CALCULATES FORCE(FFR) AND MOMENT(FMR) IN THE REFERENCE SYSTEM

REAL A(3), B(3), C(3), FT(6,6), V(6), F6(6), PF(3), AF(3,3), RF(3)
REAL AA(3,3), AB(3,3), RCF(3), AFT(3,3), FF(3), FM(3), FMR(3)
REAL ZT(J,3), XF(J), XG(J), XFG(J), XF1(J), XAT(J,3), XR(J)
DECLARE A(I), A2(I), A3(I)
DO 10 I = 1,5
SUM = 0.
DO 5 J = 1, 6
SUM = SUM + FT(I, J) * V(J)
F6(I) = SUM
5 CALL SYSFORM(A, B, C, AA, RF)
A1(I) = A1(2) = A1(3) = 0.
A2(I) = 2.
A2(2) = A2(3) = 1.
A3(I) = 0.
A3(2) = 1.
A3(3) = 1.
CALL SYSFORM(A1, A2, A3, AB, RCF)
CALL MATPFOG(A8, AA, AF)
CALL TRANSF(AF, FT)
DO 20 I = 1, 3
FF(I) = F6(I)
20 CALL MATPFOG(AFT, FF, FFR)
CALL MATPFOG(AFT, FM, FMR)
CALL TRANSF(AF, ABT)
XF(1) = XF(2) = 1.
XF(3) = 29.62
CALL MTVPRED(ABT, XF, XFG)
XF1(I) = 0.
XG1(2) = 9.332/SQRT(2.)
XG1(3) = -9.332/SQRT(2.)
DO 30 I = 1, 3
30 XF(I) = XF1(I) + XG1(I)
CALL TRANSF(AA, AAT)
CALL MTVPRED(AAT, XG, XR)
DO 40 I = 1, 3
40 RETURN
END
SUBROUTINE GETNW0(AN, RC, FF, FM, RF, MF)

CALCULATES BODY ORIENTATION COSINE MATRIX, AN, WITH RESPECT TO REFERENCE SYSTEM FROM THREE POINTS ON THE BODY. ALSO CALCULATES THE VECTOR, RC, FROM REFERENCE ORIGIN TO BODY SYSTEM ORIGIN.

ALSO CALCULATES THE FORCE AND MOMENTS AND THE POINT OF THEIR APPLICATION IN THE REFERENCE SYSTEM. THEY ARE RESPECTIVELY GIVEN BY FF, FM AND RF.

REAL A(3,3), B(3,3)
COMMON/SLK1/FT
DATA N/O/

CALL OATRAO(A,8,C,Q1»O2,O3,0*,,V*N), RETURNS(60)
CALL CCONSTR(01,02,03,04,AN,RC,MG)
IF(MG.GE.1) GO TO 93
WRITE(5,200)
WRITE(5,210) 01,32,03,04
WRITE(5,220) 0,1,2,3
WRITE(5,230) (FF(I),FP(I),RF(I),RC(I), AN(I,J), J=1,3), I=1,3)
WRITE(5,200) GO TO 70
60 NG=1
70 CONTINUE
200 FORMAT(1X,"UP AT (IX,"Pt"," ICF7.2 ),2X,"P2"," J(F7.2 ),2X,"P3"," 3 < F7.2 ) , 2X, "P L s * * , 3 t  F ' . 2 )
210 FORMAT(2X,"A " ,3(I7.2 ),2X,"P2=" ,3(I7.2 ),2X,"P3=" ,3(I7.2 ),2X,"P4="
*,3(I7.2 ) )
230 FORMAT(3X,"C =",3(I7.2 ))
240 FORMAT(7X,"FF","8X,"FM","8X,"RF","8X,"RC","15X,"AN ARRAY"/(1X,=F10.2,
*,3X,I(2X,F9.5))))
RETURN
END

SUBROUTINE INVERSE(A,E,C)

CALCULATES THE INVERSE, 3, AND DETERMINANT, C, OF 3X3 MATRIX A.

REAL A(3,3), B(3,3)
C=A(1,1)*A(2,2)*A(3,3)
1*A(1,2)*A(2,3)*A(3,1)
1*A(2,1)*A(1,3)*A(3,2)
1-A(1,3)*A(2,2)*A(3,1)
1-A(1,2)*A(2,3)*A(3,1)
1-A(2,1)*A(1,3)*A(3,2)
1-A(2,1)*A(1,2)*A(3,3)
SUBROUTINE MTVCPRO(A,B,C)

----------------------------------------
CALCULATES PRODUCT OF 3X3 MATRIX A, WITH 3X1 MATRIX B.
C = AB
----------------------------------------

REAL A(3,3),B(3),C(3)
CO 20 I=1,3
SUM=0
DO 10 J=1,3
10 SUM=SUM+A(I,J)*B(J)
20 C(I)=SUM
RETURN
END

SUBROUTINE NORMAL(A,B)

----------------------------------------
NORMALIZES VECTOR A(3)
----------------------------------------

REAL A(3),B(3)
C=A(1)**2+A(2)**2*A(3)**2
IF(C.EQ.0.) GO TO 30
SUM=1
DO 10 J=1,3
10 SUM=SUM+A(J)**2
DO 20 I=1,3
20 B(I)=A(I)/SQRT(SUM)
GO TO 30
30 GO TO 10
40 B(I)=A(I)
50 CONTINUE
RETURN
END

SUBROUTINE ROTAXOP(Q0,Q1,T,R,OR)

----------------------------------------
CALCULATES TOTAL TRANSLATIONAL DISPLACEMENT ,OR, OF
CHosen BODY ROTATION POINT T, IN (R) SYSTEM
----------------------------------------

REAL Q0(3),Q1(3),T(3,3),R(3,3),OR(3),L(3),D(J),RD(3,3)
CO 5 I=1,3
CO 5 J=1,3
5 DD(I,J)=A(I,J)
DO 10 I=1,3
10 DD(I,J)=DD(I,J)-1.
DO 21 I=1,3
21 L(I)=T(I)-Q0(I)
CALL MTVCPRO(RD,L,D)
DO 30 I=1,3
30 CF(I)=Q1(I)-OR(I)*D(I)
RETURN
END
SUBROUTINE RQTCE (UPY, THETA, NPL, L)

CALCULATES APPROXIMATE TRAJECTORY CENTER

REAL DHY(3), L(3)
IF (NPL) 20, 20, 30
20 SD1=0.
SD3=0.
SL1=0.
SL3=0.
SSDR=0.
SSD1=0.
N=0.
30 N=N+1
A=(DHY(3)/(2.*TAN(THETA/2.)))
B=(-DRY(1)/(2.*TAN(THETA/2.)))
SDR1=SDR1+DHY(1)
SDR3=SDR3+DRY(3)
SL1=SL1+A
SL3=SL3+B
L(1)=(SDR1/2.+SL1+SSCR1)/N
L(3)=(SDR3/2.+SL3+SSCR3)/N
SSDR=SDR1
SSD1=SDR3
L(2)=0.
RETURN
END

SUBROUTINE ROTMAT(A,L,M,N,D)

CALCULATES ROTATION MATRIX, D(3,3), FROM NORMALIZED
ROTATION AXIS VECTOR, (L,M,N), AND ANGLE OF ROTATION,
A, GIVEN IN RADIANS.

REAL A,L,M,N,D(3,3)
C=COS(A)
S=SIN(A)
D(1,1)=L**2+C*R*C
D(1,2)=L*N*C*R*S
D(1,3)=L*N*C*R*K+S
D(2,1)=L*M*C*R*K+S
D(2,2)=M**2+C*R*C
D(2,3)=M*N*C*R-L*S
D(3,1)=L*M*C*R-K+S
D(3,2)=M*N*C*R-L*S
D(3,3)=N**2+C*R+C
RETURN
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SUBROUTINE SYSFC(A1,12,A3,A1,R)

THE VECTOR P FROM REFERENCE SYSTEM ORIGIN TO BODY
CALCULATES BODY ORIENTATION FROM THE MATRIX A, WITH
RESPECT TO REFERENCE SYSTEM FROM THREE POINTS SPECIFIED
ON THE BODY IN THE REFERENCE SYSTEM. ALSO CALCULATES
THE VECTOR R FROM REFERENCE SYSTEM ORIGIN TO BODY
SYSTEM ORIGIN.

BODY SYSTEM AXES ARE DEFINED SUCH THAT THE X AXIS IS
ALIGNED WITH THE VECTOR FROM POINT 1 TO POINT 2
THE y AXIS IS NORMAL TO THE X AXIS AND EXTENDS FROM
THE X AXIS THROUGH POINT 3
THE y AXIS IS FORMED BY THE VETER PRODUCT OF X AND Y

THE ORIGIN IS DEFINED AT THE POINT OF INTERSECTION OF
THE X AND Y AXES.

REAL A1(3),A2(3),A3(3),R(3),R1(3),R1N(3),R2I(3),R2(3)
REAL R2N(3),R3N(3)
DO 16 I=1,3
16 R1(I)=A2(I)-A1(I)
    CALL NORMAL(R1,R1N)
DO 17 I=1,3
17 R2I(I)=A3(I)-A2(I)
    CALL DOT(R1N,R2I,0)
DO 18 I=1,3
18 R2I(I)=R2I(I)-0*R1N(I)
    CALL NORMAL(R2,R2N)
    CALL CROSS(R1N,R2N,R3N)
    CALL TRANSFORM(R1N,R2N,R3N,A)
DO 19 I=1,3
19 R(I)=A2(I)+0*R1N(I)
RETURN
END

SUBROUTINE TRANSP(A,3)

CALCULATES TRANSPOSE OF 3X3 MATRIX A
B = TRANSPOSE(A)

REAL A(3,3),B(3,3)
DO 10 I=1,3
DO 10 J=1,3
10 B(J,I)=A(I,J)
RETURN
END
BIBLIOGRAPHY


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