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GRAPHS, REPRESENTATIONS, AND
SPINOR GENERA

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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* * * * *

The Ohio State University

1981

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Department of Mathematics
Dedicated to the Memory of my Father,

who provided early encouragement to my interest in Mathematics
ACKNOWLEDGMENTS

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Studies in Number Theory. Professors J.S. Hsia and H. Zassenhaus

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>DEDICATION</td>
<td>ii</td>
</tr>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>iii</td>
</tr>
<tr>
<td>VITA</td>
<td>iv</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>viii</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
</tbody>
</table>

**Chapter**

**0. PRELIMINARIES.**

| §1. Local Fields and Quadratic Defect.                                  | 5    |
| §2. Quadratic Spaces and Orthogonal Groups.                            | 6    |
| §3. Lattices on Quadratic Spaces.                                      | 8    |
| §4. Genus and Spinor Genus.                                            | 10   |
| §5. Splitting Lattices and Spinor Exceptions.                          | 12   |

**I. THE LOCAL GRAPH.**

| §1. The Graph X                                                        | 15   |
| §2. The Subgraph $X_y$                                                 | 26   |

**II. THE GLOBAL GRAPH**

| §1. Construction of the Global Graph.                                 | 43   |
| §2. Unit Groups and the Global Graph.                                 | 47   |
| §3. Representations and the Graph at Dyadic Primes.                   | 49   |

**III. SPINOR GENERA AND THE GLOBAL GRAPH**

<p>| §1. Number of Proper Spinor Genera Represented in $|R(L,p)|$             | 58   |</p>
<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Positive Definite Ternary Lattices with $\text{disc } L \geq -1000$, $h(L) = 3$, and $g(L) = 2$</td>
<td>80</td>
</tr>
<tr>
<td>2. Integers Represented by Only One Lattice in $S'_t$ ($2 \leq t \leq 5$)</td>
<td>82</td>
</tr>
<tr>
<td>3. Positive Definite Ternary Lattices with $\text{disc } L \geq -1000$, $h(L) = 4$, and $g(L) = 2$</td>
<td>85</td>
</tr>
<tr>
<td>4. Integers Represented by Only One Lattice in $S_t$ or $S'_t$ (Table 3)</td>
<td>86</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>1.</td>
<td>A Closed Path in X when ( \text{ind } V &gt; 1 )</td>
</tr>
<tr>
<td>2.</td>
<td>The Graph ( X_y ) when ( F ) is Non-dyadic</td>
</tr>
<tr>
<td>3.</td>
<td>The Graph ( X_y ) when ( F ) is Dyadic and ( c \cdot \text{disc } V = \Delta F^2 )</td>
</tr>
<tr>
<td>4.</td>
<td>The Graph ( X_y ) for a Dyadic Local Field ( F )</td>
</tr>
</tbody>
</table>
Introduction

One of the classical problems in the theory of quadratic forms is that of finding simple invariants that classify forms up to integral equivalence. Since it is unlikely that a complete solution to this problem will appear in the foreseeable future, much attention has been focused on the more manageable related problem of classifying integral quadratic forms up to spinor equivalence. A principal objective of this dissertation is to contribute to the solution of this problem by providing a reasonably effective method for determining whether two integral forms in the same genus are, in fact, in the same proper spinor genus.

Our approach to this problem is inspired by a construction of Schulze-Pillot: given a ternary lattice $L$ over $\mathbb{Z}$ and a prime $p$, he produces a graph $Z(L,p)$ whose vertices are lattices $M \in \text{gen } L$ such that $M_q = L_q$ for all primes $q \neq p$ ([SP], [SP₁]). In Chapters I and II, we show how a similar construction produces a graph $R(L,p)$ for any lattice $L$ of rank three or more over a ring of algebraic integers $R$ and for almost any prime ideal of $p$ of $R$. Like that of Schulze-Pillot, this graph is based on a fundamental construction of Kneser ([K]). In these two chapters, we also show how some general representation-theoretic results of Schulze-Pillot can be extended to the
case of ternary lattices over a ring of algebraic integers.

In Chapter III, we discuss the relationship between the graphs constructed in the preceding chapters and proper spinor genera. Our point of departure here is a result of Schulze-Pillot which implies that the set of vertices of $Z(L,p)$ contains no more than two proper spinor genera. Although a somewhat different proof is required, this result also holds for the graph $R(L,p)$. As a by-product, the proof we provide yields a simple criterion for determining whether a graph contains lattices from only one or from two proper spinor genera. We also show how, given lattices $L$ and $M$ in the same genus, one can find a prime $p$ such that the set of vertices of $R(L,p)$ contains lattices only from $\text{spn}^+_L$ and $\text{spn}^+_M$. Together, these results provide a quite effective procedure for determining whether $L$ and $M$ are properly spinor equivalent. A comparison of this procedure with a related method for determining spinor equivalence introduced by Cassels ([C], [C]) concludes the discussion.

Chapter III also contains a discussion of representations by the proper spinor genera in $R(L,p)$. Since any lattice $K$ with $\text{rk } L - \text{rk } K \geq 3$ which is represented by $\text{gen } L$ is represented by every proper spinor genus in $\text{gen } L$ ([H]), we confine our attention to the representation of lattices $K$ with $\text{rk } L - \text{rk } K = 2$. In this setting, if a graph contains lattices from two proper spinor genera, we provide a criterion for determining whether both, neither, or only one of these proper
spinor genera represents \( K \). This provides a link between the graphical approach and the "spinor character theory" developed in [BH].

In Chapter IV, the discussion turns from spinor genera to representations by the classes within a given spinor genus. In particular, we are concerned with finding examples of ternary lattices that are "spinor regular" in the sense that they represent every integer allowed by spinor-genus considerations. Using a method introduced by Watson ([W]), we are able to produce some non-trivial examples of this phenomenon. As a by-product, we obtain one new example of a regular lattice (i.e., a ternary lattice which represents every integer permitted by its genus).

Each of the genera considered in Chapter IV was split into spinor genera using graphs \( Z(L,p) \) given in the Appendix to that Chapter. These graphs, as well as those in Chapter III, were generated by the Amdahl 470 computer at The Ohio State University Instructional and Research Computer Center. The program used to generate them is given in Appendix A.

In general, this dissertation adds to the evidence that graphs can be useful tools in investigating the properties of quadratic forms. In addition to providing an approach to certain representation questions, they can also be used for purposes of classification, at least up to proper spinor equivalence. No doubt, the future will provide further examples of the application of graphical techniques to the
theory of quadratic forms. A few possible avenues for future investigations are mentioned in the Epilogue. Thus, one may hope that this work will stimulate continued investigation of the relationship between graphs and quadratic forms.
The purpose of this chapter is to establish notational conventions and to summarize some of the elementary concepts that shall be used throughout this dissertation. In order to achieve a more coherent exposition, other basic concepts will be introduced as they are needed. It is assumed that the reader is familiar with the elementary properties of groups, rings, fields, modules, and vector spaces. In addition, an acquaintance with algebraic number fields and their completions (as developed, for example, in the first three chapters of [OM]) is assumed. Unless otherwise indicated, we shall adopt the terminology and notations of [OM].

§1. Local Fields and Quadratic Defect

Let $F$ be a local field. Denote the ring of integers by $R$, the unit group by $u$ and the unique prime ideal by $p$. For any $\alpha \in F$, let $\text{ord}_p \alpha$ be the rational integer $t$ such that $\alpha R = p^t$. The residue class field $\overline{F}$ is defined by $\overline{F} = R/p$. Clearly, $\overline{F}$ is finite: let $N_p$ denote the number of elements in $\overline{F}$.

If $\alpha \in F$, one defines the quadratic defect of $\alpha$ by

$$\delta(\alpha) = 0 \lambda R,$$

where $\lambda$ ranges over $\{\lambda \in F | \alpha = \lambda + \eta^2 \text{ for some } \eta \}$.
Thus, \( \hat{a}(\alpha) = 0 \) if and only if \( \alpha \) is a square in \( F \).

At the other extreme, if \( \text{ord} (\alpha) \) is odd, then \( \hat{a}(\alpha) = \alpha R \).

If \( \alpha = \pi^{2r} \varepsilon \), where \( \pi \) is a prime element of \( F \) and \( \varepsilon \) is a unit, then \( \hat{a}(\alpha) = \pi^{2r} \hat{a}(\varepsilon) \). Thus, one need only determine the quadratic defects of units. In fact, one need only find the defect for a representative unit from each square class, since for any \( \varepsilon, \delta \in u \), \( \hat{a}(\varepsilon \delta^2) = \delta^2 \hat{a}(\varepsilon) = \hat{a}(\varepsilon) \). To this end, the following result, whose proof is given in [OM], is useful:

**Local Square Theorem:** Given \( \alpha \in R \), there exists \( \beta \in R \) such that \( 1 + 4\pi \alpha = (1 + 2\pi \beta)^2 \).

An immediate consequence is that \( 4R \subseteq \hat{a}(\varepsilon) \) for any non-square unit \( \varepsilon \). One can show that there is only one square class of units with defect equal to \( 4R \): let \( \Delta = 1 + 4\rho \) be a typical element of this square class. Clearly, \( \rho \) is a unit. If \( F \) is nondyadic, then \( 4R = R \), so that \( u^2 \) and \( \Delta u^2 \) are the only unit square classes in \( F \). On the other hand, a dyadic local field will also contain units of defect \( p^{2r+1} \) for all natural numbers \( r<\text{ord} 2 \) (see [OM] §63:2).

### §2. Quadratic Spaces and Orthogonal Groups

Now, let \( F \) be an arbitrary field. A **quadratic space** is a vector space \( V \) over \( F \) together with a map \( q:V \to F \) such that for any \( x, y \in V \) and any \( \alpha, \beta \in F \), we have \( q(\alpha x + \beta y) = \alpha^2 q(x) + \beta^2 q(y) + \alpha \beta b(x, y) \), where \( b:V \times V \to F \) is a symmetric bilinear form. Note that \( b(x, x) = 2 \cdot q(x) \) for all \( x \in V \).

A quadratic space \( V \) is said to be **regular** if \( \text{rad} V = \{x \in V \mid b(x, V) = 0\} = \{0\} \). Unless otherwise indicated, we
shall assume that any space mentioned in this paper is regular. If \( \{ e_1, e_2, \ldots, e_n \} \) is a basis for \( V \), then \( V \) may be associated with a quadratic form (i.e., a homogeneous polynomial of degree two) given by

\[
f(x_1, \ldots, x_n) = q(x_1^2 + \ldots + x_n^2).
\]

The discriminant of \( V \) is defined to be \((-1)^{\frac{n(n-1)}{2}} \det(b(e_i, e_j))\). If \( V \) is regular, its discriminant is not zero. A change of basis causes the discriminant to change by a square factor in \( \mathbb{F}/\mathbb{F}^2 \), thus, the discriminant of a regular space is well-defined as an element of \( \mathbb{F}/\mathbb{F}^2 \). When \( n \) is odd, the discriminant of \( V \) assumes a value \( 2 \cdot P(q(e_i), b(e_j, e_k)) \), where \( P \) is a polynomial in \( \frac{n(n+1)}{2} \) variables with rational integral coefficients.

Define the half-discriminant of \( V \) to be just \( P(q(e_i), b(e_j, e_k)) \). (For details, see [K1].) The symbol \( \text{disc} V \) shall denote the discriminant of \( V \) when \( n \) is even and the half-discriminant when \( n \) is odd.

An isotropic vector is a vector \( x \in V \) such that \( x \neq 0 \) but \( q(x) = 0 \). An isotropic space is just a space containing an isotropic vector; if \( q(V) = 0 \), one says that \( V \) is totally isotropic. A regular two-dimensional isotropic space is called a hyperbolic plane. If \( H \) is a hyperbolic plane, there is a basis \( \{ e, f \} \) for \( H \) with \( q(e) = q(f) = 0 \) and \( b(e, f) = 1 \). Any quadratic space \( V \) has an orthogonal decomposition

\[
V = H_1 \perp \ldots \perp H_t \perp V_0,
\]

where \( H_i \) is a hyperbolic plane for \( i \in \mathbb{I} \), and \( V_0 \) is anisotropic. The number \( t \) is called the Witt index of \( V \),
The orthogonal group $O(V)$ is the group of linear
transformations $\sigma$ of $V$ such that $q(\sigma x) = q(x)$ for all $x \in V$. Given any anisotropic vector $y \in V$, define the symmetry $S_y \in O(V)$ by

$$S_y(x) = x - \frac{b(x, y)}{q(y)} \cdot y$$

for all $x \in V$.

The significance of the symmetries is that they generate the orthogonal group: any element of $O(V)$ can be written as a product of symmetries. If an orthogonal transformation is the product of an even number of symmetries, it is called a rotation; the rotations form a subgroup of $O(V)$, which is denoted $O^+(V)$.

If $\sigma = S_{y_1} S_{y_2} \ldots S_{y_r} \in O(V)$, we define the spinor norm of $\sigma$ by

$$\theta(\sigma) = q(y_1) q(y_2) \ldots q(y_n) \cdot \mathbb{F}^2.$$

Using the Clifford algebra, one can show that $\theta : O(V) \to \mathbb{F}/\mathbb{F}^2$ is a well-defined map ([OM]). The kernel of the restriction of $\theta$ to $O^+(V)$ is denoted $O'(V)$.

§3. Lattices on Quadratic Spaces

Let $V$ be a quadratic space over a local or global field $F$ with ring of integers $\mathcal{O}$ and unit group $\mathcal{O}$. A lattice on $V$ is a finitely generated $\mathcal{O}$-module $L$ such that $\mathcal{O}\mathcal{L} = V$. Given any lattice $L$ on $V$, one can find a basis $\{e_1, \ldots, e_n\}$ for $V$ and fractional ideals $\mathfrak{r}_1, \ldots, \mathfrak{r}_n$ such that $L = \mathfrak{r}_1 e_1 + \ldots + \mathfrak{r}_n e_n$. 
If \( L = \mathbb{R}e_1 + \ldots + \mathbb{R}e_n \), one says that \( L \) is a free lattice; in this case one calls \( \{e_1, \ldots, e_n\} \) a basis for \( L \).

Clearly, any lattice over a principal ideal domain is free; in particular, \( L \) is free whenever \( F \) is a local field.

The discriminant of the free lattice \( L = \mathbb{R}e_1 + \ldots + \mathbb{R}e_n \) is just \((-1)^{\frac{n(n-1)}{2}} \cdot \det (b(e_i, e_j))\). Up to unit squares, the discriminant is independent of the choice of basis. If \( n \) is odd, the half-discriminant of \( L \) is defined in the same manner as the half-discriminant of \( V \). The symbol \( \text{disc} L \) shall mean the half-discriminant of \( L \) when \( n \) is odd and the discriminant of \( L \) when \( n \) is even. We shall say a free lattice \( L \) is "good" whenever \( q(L) \leq R \) and \( \text{disc} L \) is a unit. If \( L \) is a (not necessarily free) lattice over a global field, we say \( L \) is "good at \( p \)" for a prime ideal \( p \) whenever \( L_p \) is a good \( R_p \)-lattice.

Given a lattice \( L \), one defines its norm \( nL \) and its scale \( sL \) as the fractional ideals generated by \( q(L) \) and \( b(L,L) \), respectively. A lattice \( L \) is said to be \( r \)-maximal, where \( r \) is an ideal, if \( nL \nsubseteq r \) and \( nM \nsubseteq r \) for any lattice \( M \) properly containing \( L \). In particular, good lattices are always \( R \)-maximal. Now, let \( L \) and \( M \) be two \( R \)-maximal lattices on a quadratic space \( V \). Then, there is an orthogonal decomposition

\[
V = H_1 \perp \ldots \perp H_t \perp V_o
\]

where \( H_1, \ldots, H_t \) are hyperbolic planes and \( V_o \) is anisotropic.
such that

\[ L = (L_1 H) \cdots I (L_n H) I (L_n V), \text{ and} \]
\[ M = (M_1 H) \cdots I (M_n H) I (M_n V). \]

If \( F \) is a local field, then \( L_n V = M_n V = \{x \in V | q(x) \in R\} \)
(see [OM], §91:1). Also, \( L_1 H \approx M_1 H = A(0,0) \) for \( 1 \leq \text{Is} \leq 3 \)
([OM], §82:23). Here, as elsewhere in this paper, \( A(\alpha, \beta) \)
denotes a unimodular free lattice \( L = \mathbb{R} e_1 + \mathbb{R} e_2 \) with
\[ (b(e_i, e_j)) = \begin{pmatrix} \alpha & 1 \\ 1 & \beta \end{pmatrix}. \]

§4. **Genus and Spinor Genus**

Let \( L \) be a lattice on a quadratic space \( V \) over a field \( F \).

Then, the **class** of \( L \) is defined by
\[ \text{cls} L = \{cL | c \in \text{O}(V)\}, \]
and the **proper class** of \( L \) by
\[ \text{cls}^+ L = \{cL | c \in \text{O}^+(V)\}. \]

If \( F \) is a local field, \( \text{cls}^+ L = \text{cls} L \) since \( \text{O}(L) \) always contains
a symmetry. A complete set of invariants for classifying
lattices over local fields is given in [OM]. Unfortunately,
there is no local-global principle for lattices: if \( F \) is a
global field, one may have \( M_p \in \text{cls} L_p \) for all primes \( p \) without
having \( M \in \text{cls} L \). This leads one to define the **genus** of \( L \),
denoted \( \text{gen} L \), as the set of all lattices \( M \) with \( M_p \in \text{cls} L_p \)
for all \( p \).

This can be conveniently expressed using the group \( J_V \) of
**split rotations**. The elements of \( J_V \) are tuples \( \Sigma = (\Sigma_p) \in \text{O}^+(V_p), \)
with the restriction that for any lattice \( L \) on \( V \), \( \Sigma_p \in \text{O}^+(L_p) \)
for almost all \( p \). If \( L \) is a lattice on \( V \) and \( \Sigma \in J_V \), then \( \Sigma L \)
shall denote the lattice \( M \) on \( V \) such that \( M_p = \Sigma_p L_p \) for all \( p \).
(M can be constructed using [OM], §81:14.) Now, set \( \mathcal{G} = \text{gen } L = \{ \Sigma L \mid \Sigma \in J \}. \) Some useful subgroups of \( J \) are described below:

\[
\begin{align*}
J' & = \{ \Sigma \in J \mid \Sigma_p \in O'(V_p) \text{ for all } p \}, \\
J & = \{ \Sigma \in J \mid L \mathcal{L} \mathcal{L} \text{ for all } p \}.
\end{align*}
\]

Finally, \( P \) is the image of \( O'(V) \) under the diagonal embedding into \( J \).

The \textit{spinor genus} and \textit{proper spinor genus} of a lattice \( L \) may now be defined as follows:

\[
\text{spn } L = \{ \sigma \Sigma_L \mid \sigma \in O(V), \Sigma \in J' \}, \quad \text{and} \quad \text{spn}^+ L = \{ \sigma \Sigma_L \mid \sigma \in O^+(V), \Sigma \in J' \}.
\]

Clearly, \( \text{cls } L \subseteq \text{spn } L \subseteq \text{gen } L, \) and \( \text{cls}^+ L \subseteq \text{spn}^+ L \subseteq \text{gen } L. \)

Let \( h(L) \) and \( g(L) \) denote the number of classes and spinor genera in \( \text{gen } L, \) respectively; \( h^+(L) \) and \( g^+(L) \) shall denote, respectively, the numbers of proper classes and proper spinor genera in \( \text{gen } L. \) One can show that all these numbers are finite. In particular, since \( \Sigma L \in \text{spn}^+ L \) if and only if \( \Sigma \in P \mathcal{J}'_V J_L, \) one has \( g^+(L) = [J_V : P \mathcal{J}']_V J_L. \) When \( \text{rk } L \geq 3, \) there is an alternate formula for \( g^+(L) \) involving subgroups of the idele group \( J \). In particular, define

\[
\mathcal{G} = \{ (i) \in J \mid i \in \theta(O^+(L)) \text{ for all finite } p \}.
\]

Let \( P \) denote the image of \( D = \theta(O'(V)) \) under the diagonal embedding into \( J \). Then, when \( \text{rk } L \geq 3, \) the map \( \theta: J \to J \) given by \( \theta(\Sigma) = (\theta(\Sigma_p)) \) induces an isomorphism of \( J/P \mathcal{J} V \) onto \( J/P \mathcal{G} \). Hence, \( g^+(L) = [J : P \mathcal{J}]. \) (For details,
§5. Splitting Lattices and Spinor Exceptions

Let $V$ be a quadratic space with $n = \dim V \geq 3$ over an algebraic number field $F$, and let $L$ be a lattice on $V$. Suppose $K$ is a lattice of rank $n-2$ which is represented by $\delta = \text{gen } L$ (i.e., $K_p$ is represented by $L_p$ for all $p$). Set $\delta_K = \text{disc } U$, where $U = F K_1$, and let $E = F(\sqrt{\delta})$. Let $N = N_{E/K}^{\text{gen } L}$, and set $H_K = N_{K_p} \delta_K^{\text{gen } L}$; note that $H_K$ depends only on the isometry class of $FK$ (in fact, only on $\delta_K$). Let $J(V,K) = J_{U \cap V}^{\text{gen } L}$, and note that $J(V,K)$ also depends only on the isometry class of $FK$ (see, for example, [BH]). Now, the map $\theta: J_V \rightarrow J_F$ described above induces an isomorphism from $J_V/J(V,K)$ onto $J_F/H_K$, and $[J_F:H_K] = 2$ ([H]). When $[J_F:H_K] = 2$, $K$ is called a splitting lattice for $\delta$, since it splits $\delta$ into the two $K$-half genera, $H_L,K = \{E \in J(V,K) \}$ and $H_L,K' = \{E \in J_V \backslash J(V,K) \}$. Each half-genus contains exactly half the proper spinor genera in $\delta$. If $K \subset \langle c \rangle$, one calls $c$ a splitting integer; it splits $\delta$ into $c$-half genera. The notations $\delta, E, H, J(V,c), H_c, \text{ and } H_c'$ have the obvious meanings.

Now, if $K$ is a splitting lattice, either every proper spinor genus in $\delta$ represents $K$ or every proper spinor genus in one $K$-half-genus represents $K$. In the latter case, $K$ is called a spinor exceptional lattice: the half-genus that represents $K$ is called the good $K$-half-genus, and the other is
called the bad \( K \)-half-genus. When \( K \not\equiv \{\}, \), \( c \) is called a spinor exceptional integer, and splits \( \mathfrak{g} \) into good and bad \( c \)-half-genera.

A splitting lattice \( K \) is spinor exceptional if and only if the following conditions are satisfied:

1. \( \theta(J_L) \subseteq N_K \)

2. \( \theta(L_p : K_p) = N_K(p) \) for all finite spots \( p \).

Here, \( N_K(p) \) is the \( p \) component of \( N_K \) and \( \theta(L_p : K_p) \) is the subgroup of \( F_p \) generated by \( \{ \theta(\sigma) \mid \sigma \in O^+(V_p), \sigma K_p \subseteq L_p \} \).

For details, see [SP], [H].

A splitting lattice \( K \) is primitively spinor exceptional if it is primitively represented only by the proper spinor genera in one half-genus. All the definitions and assertions above concerning spinor exceptions also hold for primitive exceptions, except that condition (2) must be replaced by

\[(2^*) \theta^*(L_p : K_p) = N_K(p) \) for all finite \( p \).

Here, \( \theta^*(L_p : K_p) \) is the subgroup of \( F_p \) generated by \( \{ \theta(\sigma) \mid \sigma \in O(V_p), \sigma K_p \subseteq L_p \) with \( L_p / \sigma K_p \) torsion-free\}.

Note that every spinor exceptional lattice is contained in a primitive spinor exceptional lattice ([BH]).

A set \( S = \{K_1, \ldots, K_r\} \) of splitting lattices is said to be independent if \( [F(\sqrt{\delta_{K_1}}, \ldots, \sqrt{\delta_{K_r}}) : F] = 2^r \). In general, \( g^+(L) \geq 2^r \); if \( g^+(L) = 2^r \), we say that \( S \) is complete. If \( \{K_1, \ldots, K_r\} \) is a complete independent set of splitting lattices, then for any set \( I \subseteq \{1, \ldots, r\} \),
there is precisely one spinor genus $\mathcal{S}$ such that $\mathcal{S} \leq R_{L,K_i}$ for $i \in I$ and $\mathcal{S} \leq R_{L,K_j}$ for $j \notin I$. In particular, if $K_1, \ldots, K_r$ are (primitive) spinor exceptional lattices for $G$, there is precisely one spinor genus $\mathcal{S}$ which (primitively) represents $K_i$ for all $i \in I$ and does not (primitively) represent $K_j$ for $j \notin I$. For details, see [BH].
Chapter I

THE LOCAL GRAPH

In this chapter, we show how the construction by Schulze-Pillot of a graph $X$ whose vertices are good ternary lattices over $\mathbb{Z}_p$ ([SP], [SP_1]) can be extended to the case of good lattices on a quadratic space of dimension three or more over an arbitrary local field of odd characteristic. Whenever our results are merely straightforward extensions of those in [SP] and [SP_1], we shall provide somewhat abbreviated proofs.

Throughout this chapter, let $F$ be a local field with $\text{char } F \neq 2$; let $R$ be its ring of integers and $\mathfrak{u}$ its unit group. Let $\pi$ be a prime element of $F$; then $\mathfrak{p} = \pi R$ is the unique prime ideal of $F$. Let $V$ be a quadratic space over $F$ with $\dim V \geq 3$, and let $L$ be a good lattice on $V$.

§1. The Graph $X$

1.1 Proposition: $V$ is isotropic.

Proof: If $F$ is non-dyadic, this is clear. Thus, we may assume $F$ is dyadic. Then,

$L = B_1 \perp \ldots \perp B_t \perp X,$

where $B_1, \ldots, B_t$ are binary unimodular lattices and $\text{rk } X \leq 1.$

15
Now, by §93:11 of [OM], $B_i \sim A(0,0)$ or $B_i \sim A(2,2\rho)$ for $i \leq \text{isst}$. If $t \geq 2$, the result follows from the fact that $A(2,2\rho) \perp A(2,2\rho) \sim A(0,0) \perp A(0,0)$. If $t = 1$ and $L = A(2,2\rho) \perp \langle 2\varepsilon \rangle$ where $\varepsilon \in \mathfrak{a}$, then a Hasse symbol calculation shows $V \sim A(0,0) \perp \langle 2\Delta\varepsilon \rangle$, so that $V$ is isotropic.

Q.E.D.

1.2 Proposition: If $L$ and $M$ are two good lattices on $V$, then $[L:L\cap M] = [M:L\cap M] = (N\rho)^r$ for some $r \in \mathbb{N}$.

Proof: Since $L$ and $M$ are $\mathbb{R}$-maximal, there is an orthogonal decomposition $V = H_1 \perp \ldots \perp H_t \perp V_0$, with $H_1, \ldots, H_t$ hyperbolic planes and $V_0$ anisotropic, such that $L = (L\cap H_1) \perp \ldots \perp (L\cap H_t) \perp (L\cap V_0)$, and $M = (M\cap H_1) \perp \ldots \perp (M\cap H_t) \perp (M\cap V_0)$.

Choose a basis $\{e_1, f_1, \ldots, e_t, f_t, z_{2t+1}, \ldots, z_n\}$ for $L$ such that $\{e_i, f_i\}$ is a basis for $L\cap H_i$ for all $i \leq \text{isst}$ with $q(e_i) = q(f_i) = 0$ and $b(e_i, f_i) = 1$, and $\{z_{2t+1}, \ldots, z_n\}$ is a basis for $L\cap V_0$. Then,

$$M = R(\pi^{\alpha_1} e_1) + R(\pi^{-\alpha_1} f_1) + \ldots + R(\pi^{\alpha_t} e_t) + R(\pi^{-\alpha_t} f_t) + Rz_{2t+1} + \ldots + Rz_n$$

where, without loss of generality, $\alpha_1, \ldots, \alpha_t \in \mathbb{N} \cup \{0\}$. Then, $[L:L\cap M] = \prod_{i=1}^t [R : p_i^\alpha_i] = (N\rho)^{\alpha_1} + \ldots + \alpha_t$.

Similarly, $[M:L\cap M] = \prod_{i=1}^t [R : p_i^{-\alpha_i}] = (N\rho)^{\alpha_1} + \ldots + \alpha_t$.

Q.E.D.
1.3 Notation: If \([L:L\cap M] = [M:L\cap M] = (N_p)^r\), we write \(d(L,M) = r\). Clearly, \(d(\sigma L, \sigma M) = d(L,M)\) for any \(\sigma \in O(V)\); just apply \(\sigma\) to the bases in the preceding proof. Let \(X\) denote a graph whose vertices are the good lattices on \(V\), with two vertices \(L\) and \(M\) connected by an edge if and only if \(d(L,M) = 1\). In this case, we say that \(L\) and \(M\) are neighbors in \(X\). The symbol \(|X|\) shall denote the set of vertices of \(X\). The following propositions give a method for constructing neighbors of \(L\) in \(X\).

1.4 Proposition: Let \(x \in p^{-1}L\setminus L\) with \(q(x) \in R\). Set \(L_x = \{y \in L \mid b(x,y) \in R\}\), and let \(L_1 = L_x + Rx\). Then, \(L_1\) is a neighbor of \(L\).

Proof: We show first that \(L_x = L \cap L_1\). Clearly, \(L_x \subseteq L \cap L_1\). On the other hand, suppose \(z = y + cx \in L \cup L_1\), where \(y \in L_x\) and \(c \in R\). Then, \(b(z,x) = b(y,x) + 2cq(x) \in R\), so that \(z \in L_x\).

Now, define a map \(f:L \to R/p\) by \(f(z) = \pi \cdot b(x,z)\). Then, \(f\) is surjective, since otherwise we would have \(L = L_x\) properly contained in \(L_1\), contradicting the \(R\)-maximality of \(L\). Since \(\ker(f) = L_x\), we have
\[
L/(L \cap L_1) = L/L_x \cong R/p.
\]
In addition,
\[
L_1/(L \cap L_1) = (L_x + Rx)/L_x \cong Rx/(L_x \cap Rx) = Rx/px \cong R/p.
\]
Thus, \([L:L \cap L_1] = [L_1:L \cap L_1] = N_p\). To complete the proof, note that \(\text{disc } L_1 \cdot R = \text{disc } L_x \cdot N_p^{-2} = \text{disc } L \cdot R = R\), so that \(L_1\) is good. Q.E.D.
1.5 Proposition: Let \( L, M \in |X| \) with \( d(L, M) = r \geq 1 \). Then, there exists \( x \in (p^{-1}L \cap M) \setminus L \). If \( L = L_x + Rx \) for any such \( x \), then \( d(L_1, M) = r-1 \). In particular, if \( d(L, M) = 1 \), then \( L_1 = M \).

Proof: As in the proof of 1.2, we choose a basis \( \{e_1, f_1, \ldots, e_t, f_t, z_{2t+1}, \ldots, z_n\} \) for \( L \) such that

\[ \pi^{-\alpha_s} e_1, \pi^{-\alpha_s} f_1, \ldots, \pi^{-\alpha_s} e_t, \pi^{-\alpha_s} f_t, z_{2t+1}, \ldots, z_n \]

is a basis for \( M \). Since \( L \neq M \), \( \alpha_s > 0 \) for some \( 1 \leq s \leq t \); thus, we may take \( x = \pi^{-1} f_s \).

Now, let \( x \) be an arbitrary vector in \( (p^{-1}L \cap M) \setminus L \). If \( z \in M \cap L \), then \( b(x, z) \in R \) since \( x \in M \), so that \( z \in L_x \). Thus, \( M \cap L = M \cap L_x \). On the other hand,

\[ M \cap L_1 = M \cap (L_x + Rx) = (M \cap L_x) + Rx = (M \cap L) + Rx. \]

Hence, we have

\[ (M \cap L_1)/(M \cap L) = (M \cap L) + Rx)/(M \cap L) \cong Rx/(M \cap L \cap Rx) = Rx/pRx \cong R/p. \]

Thus, \([M \cap L_1 : M \cap L] = (NP) \). Hence,

\[ [M : M \cap L_1] = [M : M \cap L] \cdot [M \cap L_1 : M \cap L] = [M : M \cap L_1] \cdot (NP)^{-1}. \]

Since \([M : M \cap L] = (NP)^r \), we have \([M : M \cap L_1] = (NP)^{r-1}. \)

Q.E.D.

1.6 Corollary: \( X \) is connected.

Proof: Let \( L, M \in |X| \) with \( d(L, M) = r \). If \( r = 1 \), \( L \) and \( M \) are connected by an edge. If \( r > 1 \), use Proposition 1.5 and induction to construct a sequence of lattices connecting \( L \).
and M.

Q.E.D.

Our next goal is to determine the number of neighbors of L in X. We shall need two lemmas:

1.7 Lemma: Let \( x, x' \in p^{-1}L\backslash L \) with \( q(x), q(x') \in R \). Set \( L_1 = L_x + Rx \) and \( L_1' = L_x' + Rx' \). Then, \( L_1 = L_1' \) if and only if \( cx - x' \in L \) for some unit c.

Proof: (Necessity) If \( L_1 = L_1' \), then \( x' = y + cx \) where \( y \in L \) and \( c \in R \). If \( c \in p \), then \( x' \in L \), contrary to hypothesis. Hence, \( c \in u \).

(Sufficiency) Let \( x' = cx + y \) with \( c \in u \) and \( y \in L \). Then, \( q(x') = c^2 \cdot q(x) + c \cdot b(x, y) + q(y) \). Since \( q(x'), q(x), q(y) \in R \), we have \( b(x, y) \in c^{-1}R = R \), so that \( y \in L_x \). Hence, \( x' \in Rx + L_x = L_1 \). If \( z' \in L_x \), then

\[
b(x, z') = b(c^{-1}(x'-y), z') = c^{-1}(b(x', z') - b(y, z')) \in R.
\]

Hence, \( z' \in L_x \), so that \( L_1' = L_x' + Rx' \leq L_1 \). Since \( L_1' \) is R-maximal, \( L_1' = L_1 \).

Q.E.D.

1.8 Lemma: Let \( x \) be a maximal vector in \( L \) with \( q(x) \in p \). Then, there exists a (maximal) vector \( x' \in L \) such that \( q(x') \in p^2 \) and \( x' - x \in pL \).
Proof: If \( q(x) \in p^2 \), set \( x' = x \). Hence, we may assume that \( q(x) = \pi \varepsilon \) where \( \varepsilon \in \mathbb{u} \). Now, if \( n = \dim L \) is even, \( L \) is unimodular, so that \( b(x,L) = R \) by §82:17 of [OM]. If \( n \) is odd, let \( \{e_1, f_1, \ldots, e_t, f_t, z\} \) be a basis for \( L \) as in the proof of Proposition 1.2, and suppose \( x = a_1 e_1 + b_1 f_1 + \ldots + a_t e_t + b_t f_t + cz \). Then, \( q(x) = a_1 b_1 + \ldots + a_t b_t + c^2 q(z) \). If \( c \not\in \mathbb{u} \), then \( a_i \in \mathbb{u} \) or \( b_i \in \mathbb{u} \) for some \( i \in \{1, \ldots, t\} \) since \( x \) is maximal. If \( c \in \mathbb{u} \), then \( a_i b_i \in \mathbb{u} \) for some \( i \in \{1, \ldots, t\} \) since otherwise we would have \( q(x) \not\in \mathbb{u} \). Thus, once again \( b(x,L) = R \). In particular, there exists \( y \in L \) such that \( b(x,y) = \varepsilon \). Set \( x' = x - \pi y \). Then, \( x' - x \in pL \) and \( q(x') = \pi \varepsilon - \pi \varepsilon + \pi^2 q(y) \in p^2 \).

Q.E.D.

1.9 Proposition: Let \( \overline{V} = L/pL \), considered as a quadratic space over \( \overline{F} = R/p \). Then, there is a one-to-one correspondence between the isotropic lines in \( \overline{V} \) and the neighbors of \( L \) in \( \mathcal{X} \).

Proof: Let \( M \) be a neighbor of \( L \) in \( \mathcal{X} \). By Proposition 1.5 there exists \( x \in (p^{-1}L \cap M) \backslash L \) such that \( M = L_x + Rx \). Since \( x \in p^{-1}L \), \( \overline{x} \) is a non-zero vector in \( \overline{V} \), and \( \overline{q}(\overline{x}) = \pi^2 q(x) \) = 0. Map \( M \) to the isotropic line \( \overline{F}(\overline{x}) \). Now, if \( x' \in p^{-1}L \backslash L \) with \( q(x') = 0 \), then by Lemma 1.7, \( M = L_{x'} + Rx' \) if and only if \( x - cx' \in L \) for some \( c \in \mathbb{u} \). Hence, \( \pi x - c\pi x' \in pL \), so that \( \overline{F}(\overline{x}) = \overline{F}(\overline{x'}) \) if and only if \( M = L_{x'} + Rx' \). Thus, the map described above is well-defined and injective.
To see that it is surjective, choose an isotropic vector \( \overline{x} \in \overline{V} \). Let \( x \) be a representative in \( L \) of the coset \( \overline{x} \). Clearly, \( x \) must be maximal in \( L \), and \( q(x) \in p \) since \( \overline{x} \) is isotropic. By Lemma 1.8, there exists \( x' \in L \) with \( x' - x \in pL \) and \( q(x') \in p^2 \). Then, \( \overline{x'} = \overline{x} \) and \( y = \pi^{-1}x' \in p^{-1}L\setminus L \) with \( q(y) \in R \). Thus, \( M = L_y + Ry \) is a neighbor of \( L \) whose image under the map above is \( \overline{Fx'} = \overline{Fx} \).

Q.E.D.

1.10 Corollary: Let \( N(L) \) denote the number of neighbors of \( L \) in \( X \).

(i) If \( n = \dim V \) is odd, \( N(L) = \frac{(np)^{n-1} - 1}{np - 1} \).

(ii) If \( \dim V = 2m \) and \( \text{ind} \ V = m \), then

\[
N(L) = \frac{[(np)^m - 1][(np)^{m-1} + 1]}{np - 1}
\]

(iii) If \( \dim V = 2m \) and \( \text{ind} \ V = m-1 \), then

\[
N(L) = \frac{[(np)^m + 1][(np)^{m-1} - 1]}{np - 1}
\]

Proof: This is an immediate consequence of Proposition 1.9 and the formulae for the number of isotropic vectors in \( \overline{V} \) from [K1], §12.

Q.E.D.
1.1 Proposition: Suppose \( \text{ind } V = 1 \), and let \( L \) and \( M \) be good lattices on \( V \) with \( d(L, M) = r \). Let \( x \in (p^{-1}L \cap M) \setminus L \) and set \( L_1 = L_x + Rx \).

(i) Suppose \( x' \in (L + M) \cap p^{-1}L \) with \( x' \not\in L \), and set \( L_1' = L_x + Rx' \). Then, \( L_1' = L_1 \).

(ii) Suppose \( x' \in p^{-1}L \setminus (L + M) \), and set \( L_1' = L_x + Rx' \). Then, \( d(L_1', M) = r + 1 \).

Proof: (i) By Lemma 1.8, we may assume \( x \in M \). Let 
\( \{e, f, z_3, \ldots, z_n\} \) be a basis for \( L \) as in Proposition 1.2 so that \( \{\pi^r e, \pi^r f, z_3, \ldots, z_n\} \) is a basis for \( M \). Then, we have

\[
[M \cap p^{-1}L : MnL] = [(\pi^r e + \pi^{-1} f + Rz_3 + \ldots + Rz_n) : \\
(\pi^r e + Rf + Rz_3 + \ldots + Rz_n)] = [p^{-1} : R] = N_p.
\]

On the other hand, \( MnL \subseteq M \cap p^{-1}L \) and \( MnL_1' \subseteq M \cap p^{-1}L \).

Also, from the proof of Proposition 1.5, we have

\[
[MnL : MnL] = [MnL_1' : MnL] = N_p.
\]

Hence, \( MnL_1' = MnL_1 = M \cap p^{-1}L \). Since \( x' \in L_1' \cap M \), we have \( x' \in L_1' \cap M \subseteq L_1 \), so that \( x' = y + cx \) for some \( y \in L_x \), \( c \in R \).

Since \( x' \in L \), \( c \in u \). Hence, \( L_1' = L_1 \) by Lemma 1.7.

(ii) Note first that \( MnL_1' = MnL_1 x' \). To see this, suppose \( z \in MnL_1' \); then \( z = y + cx' \) where \( y \in L_x \), and \( c \in R \). If \( c \in u \), then \( x' = c^{-1} (z-y) \in L + M \), contrary to hypothesis. Since clearly \( MnL_1 x' \subseteq MnL_1' \), the result follows.
On the other hand, we claim that $M_nL$ properly contains $M_nL_x$. If not, then $b(x',M_nL) \not\subseteq R$, so that $x' \in (M_nL)^# = M^# + L^#$. Now, if $\dim V = 4$ or if $F$ is a non-dyadic local field, then good lattices are unimodular, so that $L^# = L$ and $M^# = M$. Thus, $x' \in M + L$, contrary to hypothesis. If $F$ is dyadic and $\dim V = 3$, let $\{e, f, z\}$ be a basis for $L$ such that $\{e, -f, z\}$ is a basis for $M$, as in the proof of Proposition 1.2. Then,

$$L^# = Re + Rf + \frac{1}{2}Rz,$$

and

$$M^# = R(\pi^re) + R(\pi^{-r}f) + \frac{1}{2}Rz.$$ 

Hence we have

$$L^# + M^# = Re + R(\pi^{-r}f) + \frac{1}{2}Rz.$$ 

Thus, we have $x' = ae + \pi^{-r}bf + \frac{1}{2}cz$, for some $a, b, c \in R$. Since $x' \not\in L + M = Re + R(\pi^{-r}f) + Rz$, $c \not\in 2R$. On the other hand, since $x' \in p^{-1}L$, we have $b \in p^{r-1}$ and $c \in 2p^{-1}$. Let $b = \pi^{r-1}b_1$ and $c = 2\pi^{-1}c_1$, where $b_1 \in R$ and $c_1 \in \mathbb{Z}$. Then,

$$q(x') = \pi^{-r}a^1b_1 + \frac{1}{4}c_1^2q(z) = \pi^{-1}a^1b_1 + \pi^{-2}c_1^2q(z) \not\in R,$$

contrary to hypothesis.

Summarizing, we have $M_nL' = M_nL_x \not\subseteq M_nL$.

Hence, we have

$$[M:M_nL'] = [M: M_nL][M_nL: M_nL_1] > [M:M_nL].$$ 

Since $[M:M_nL] = (Np)^r$, we have $[M:M_nL'] \geq (Np)^{r+1}$, i.e. $d(L_1', M) \geq r + 1$.

Suppose now that $d(L_1, M) = S > r + 1$. Since $L$ is a neighbor of $L_1$, there exists $x_1 \in p^{-1}L_1 \setminus L_1$ such that $L = (L_1')_{x_1} + Rx_1'$. 


If \( x \in L + M \), then \( d(L, M) + s-1 > r \). On the other hand, if \( x \notin L + M \), then \( d(L, M) \geq s+1 > r \). In either case, this is contrary to hypothesis, so that \( d(L, M) = r+1 \).

Q.E.D.

1.12 Corollary: Suppose \( \text{ind } V = 1 \), and let \( L_0, L_1, \ldots, L_r \) be pairwise distinct good lattices on \( V \) with \( d(L_i, L_{i+1}) = 1 \) for \( 0 \leq i \leq r-1 \). Then, \( d(L_0, L_r) = r \).

Proof: Proceed by induction on \( r \).

(i) If \( r = 1 \), there is nothing to show.

(ii) Suppose the result holds for all \( s \leq r-1 \).

In particular, \( d(L_0, L_{r-2}) = r - 2 \) and \( d(L_0, L_{r-1}) = r - 1 \). By Proposition 1.5, there exists \( x \in L_r \cap p^{-1}L_{r-1} \) such that \( L_r = (L_{r-1})_x + Rx \). If \( x \in L_0 + L_{r-1} \), then \( L_r = L_{r-2} \) by Proposition 1.11, Part (i), contrary to hypothesis. Hence, \( x \notin L_0 + L_{r-1} \), so that \( d(L_0, L_r) = r \) by Proposition 1.11, Part (ii).

Q.E.D.

1.13 Corollary: If \( \text{ind } V = 1 \), then \( X \) is a tree; i.e. \( X \) is a connected graph containing no closed paths.

Proof: We have already shown that \( X \) is connected. If \( X \) contains a closed path, there is a sequence of pairwise distinct lattices \( L_0, L_1, \ldots, L_r \) with \( d(L_i, L_{i+1}) = 1 \) for \( 0 \leq i \leq r-1 \) and \( d(L_r, L_0) = 1 \), contradicting the preceding result.

Q.E.D.
1.14 Remark: If \( \text{ind } V > 1 \), then \( X \) is not a tree. To see this, let \( L \) be a good lattice on \( V \) with a basis \( \{ e_1, f_1, \ldots, e_t, f_t, z_{2t+1}, \ldots, z_n \} \) as in the proof of Proposition 1.2. Write \[
L = (R_e + R_f) \perp (R_{e_2} + R_{f_2}) \perp K.
\]
Define lattices \( L_1, L_2 \), and \( M \) by
\[
L_1 = (p e_1 + p^{-1} f_1) \perp (R_e + R_f) \perp K,
\]
\[
L_2 = (R_{e_2} + R_{f_2}) \perp (p e_2 + p^{-1} f_2) \perp K, \quad \text{and}
\]
\[
M = (p e_1 + p^{-1} f_1) \perp (p e_2 + p^{-1} f_2) \perp K.
\]
Then, \( L, L_1, M, \) and \( L_2 \) form a closed path as shown in the diagram below:

![Diagram](figure1.png)

**FIGURE 1**
A closed path in \( X \) when \( \text{ind } V > 1 \)

We conclude this section with a generalization of a Theorem of Schulze-Pillot ([SP], [SP\textsuperscript{1}]) which is basic to the discussion of spinor genera and the global graph in Chapter III. The proof given by Schulze-Pillot relies on his identification of the graph \( X \) for ternary \( \mathbb{Z}\)-lattices with the Bruhat-Tits building for a binary vector space over \( \mathbb{Q}_p \). We shall give a somewhat different proof which eliminates the use of this identification, and readily generalizes to the case \( \text{dim } V \geq 3 \).
1.15 Theorem: Let $L$ be a good lattice on $V$ and choose $\sigma \in \mathcal{O}^+(V)$. Then, $\theta(\sigma) \in uF^2$ if and only if $d(L, \sigma L) \equiv 0 \, (\text{mod} \, 2)$.

Proof: Choose a basis $\{e_1, f_1, \ldots, e_t, f_t, z_{2t+1}, \ldots, z_n\}$ for $L$ as in the proof of Proposition 1.2 such that $M = \sigma L$ has the basis $\{\pi^a e_1, \pi^{-a} f_1, \ldots, \pi^t e_t, \pi^{-t} f_t, z_{2t+1}, \ldots, z_n\}$. Let $r = d(L, M) = \sum_{i=1}^{t} a_i$. Set $e_j' = \sigma e_j$, $f_j' = \sigma f_j$ and $z_k' = \sigma z_k$ for $1 \leq j \leq t$ and $2t+1 \leq k \leq n$. Then, there exists a transformation $\lambda \in \mathcal{O}^+(M)$ such that $\lambda(\pi^a e_j) = e_j'$, $\lambda(\pi^{-a} f_j) = f_j'$ and $\lambda(z_k) = z_k'$ for $1 \leq j \leq t$ and $2t+1 \leq k \leq n$. Note that $\theta(\lambda) \leq \theta(\mathcal{O}^+(M)) = uF^2$. Now, let $\rho_j = S e_j - f_j S e_j - \pi f_j$ so that $\rho_j(e_j) = \pi e_j$ and $\rho_j(f_j) = \pi^{-1} f_j$ for $1 \leq j \leq t$. Clearly, $\theta(\rho_j) = \pi$. Set $\rho = \prod_{j=1}^{t} \rho_j^a$, and note that $\theta(\rho) = \pi^r$. Since $\sigma = \lambda \rho$, we have $\theta(\sigma) \in \pi^r uF^2$. Hence, $\theta(\sigma) \in uF^2$ if and only if $r \equiv 0 \, (\text{mod} \, 2)$.

§2. The Subgraph $X_y$

For the remainder of this chapter, let $\dim V = 3$. Given $y \in V$, let $X_y$ denote the subgraph of $X$ whose vertices are good lattices on $V$ which contain $y$; two lattices are joined by an edge in $X_y$ if and only if they are joined by an edge in $X$. The set of vertices of $X_y$ shall be denoted $|X_y|$. The remainder of this chapter is devoted to determining the shape of $X_y$. 
1.16 Proposition: $X_y$ is a tree.

Proof: Since $X$ contains no closed paths, neither does $X_y$. To see that $X_y$ is connected, let $L, M \in |X_y|$ with $d(L, M) = r$. Choose $x \in (p^{-1}L \cap M) \setminus L$, and construct $L_1 = L_x + Rx$. Then, by Propositions 1.4 and 1.5, $d(L_1, L) = 1$ and $d(L_1, M) = r - 1$. Since $x, y \in M$, $b(x, y) \in R$ so that $y \in L_x \subseteq L_1$. Continuing in this way we obtain of sequence $L = L_0, L_1, \ldots, L_r = M$ of lattices in $X_y$ linking $L$ to $M$.

Q.E.D.

1.17 Proposition: Let $L \in |X_y|$ with $b(y, L) \subseteq p$. Then, $y$ is contained in every neighbor of $L$ in $X$.

Proof: If $M$ is a neighbor of $L$, then $M = L_x + R_{x'}$ for some $x \in (p^{-1}L \cap M) \setminus L$. Now

$$b(y, x) \subseteq b(y, p^{-1}L) = p^{-1} b(y, L) \subseteq R.$$ 

Hence, $y \in L_x \subseteq M$.

Q.E.D.

In order to obtain further information about the shape of $X_y$, one must consider, among other factors, the nature of the underlying local field $F$. When $F$ is nondyadic the results and proofs of Schulze-Pillot can be extended without difficulty (see [SP], §2.25; [SP1]). These results, with abbreviated proofs, are given below. Note that, when $F$ is nondyadic, $b(y, L) = R$ if and only if $y$ is maximal in $L$. 

1.18 Proposition: Let $F$ be nondyadic, and suppose $y$ is a maximal vector in $L$ with $q(y) \in \mathfrak{m}$.

(i) If the space $(Fy)^\perp$ is anisotropic, then $y$ is contained in no other good lattices on $V$, and $X_y$ is a single point.

(ii) If $(Fy)^\perp$ is isotropic, then $y$ is contained in precisely two neighbors of $L$. In this case $X_y$ is a line through $L$.

Proof: Since $Ry$ is unimodular, it splits $L$ ([OM], §82:15). Hence, $L = Ry \perp M$, where $M$ is a maximal lattice in $(Fy)^\perp$.

(i) If $(Fy)^\perp$ is anisotropic, then $M$ is the unique $R$-maximal lattice on $(Fy)^\perp$. The result follows.

(ii) If $(Fy)^\perp$ is isotropic, it is a hyperbolic plane. Then, $M$ has a basis $\{e,f\}$ such that $q(e) = q(f) = 0$ and $b(e,f) = 1$. For $r \in \mathbb{Z}$, let $L_r = (p^r e + p^{-r} f) \perp Ry$. Clearly, the set $\{L_r \mid r \in \mathbb{Z}\}$ forms a line through $L = L_0$. Since $Fe$ and $Ff$ are the only isotropic lines in $(Fy)^\perp$, this set contains all the lattices in $|X_y|$. Q.E.D.

1.19 Proposition: Suppose $F$ is nondyadic, and let $y$ be a maximal vector in $L$ with $q(y) \in \mathfrak{p}$. Then, $y$ is contained in one and only one neighbor of $L$ in $|X|$. If $q(y) \notin \mathfrak{p}^2$, these two are the only lattices in $|X_y|$. 
Proof: Choose $x \in \mathbb{P}^{-1} L \setminus L$ with $q(x) \in R$ and set $L_1 = L + Rx$. Then, $y \in L_1 \cap L = L$ if and only if $b(x,y) \in R$ so that $\overline{b(x,y)} = 0$. In addition, $\overline{q(y)} = \overline{q(\pi x)} = 0$. Thus, if $\overline{y}$ and $\overline{\pi x}$ were linearly independent, $\overline{V}$ would contain a totally isotropic two-dimensional subspace. Then, $\overline{V}$ is not regular, which is impossible since $L$ is unimodular. Hence, $F(\pi x) = \overline{F y}$; i.e., $L_1$ is the neighbor of $L$ corresponding to the isotropic line $\overline{F y}$. Conversely, if $F(\pi x) = \overline{F y}$, then $\overline{\pi x} = y + \pi y'$ for some $y' \in L$. Hence, $b(x,y) \in R$, so that $y \in L_1 x \subseteq L_1$.

Finally, if $q(y) \notin \mathbb{P}^2$, then $y$ must be maximal in any lattice in $|X_y|$. Hence, $|X_y|$ contains only two lattices.

Q.E.D.

1.20 Remark: As Propositions 1.18 and 1.19 show, when $F$ is nondyadic, the shape of the subgraph $X_y$ is precisely the same as in [SP] and $[SP_{1}]$. In particular, if $c = q(y) \in u \cup \pi u$, $X_y$ has the shape indicated below:

(a) If $c \in u$ and $c \text{ disc } V \neq F^2$

(b) If $c \in u$ and $c \text{ disc } V = F^2$

(c) If $c \in \pi u$

FIGURE 2

The graph $X_y$ when $F$ is non-dyadic

If $q(y) \in \mathbb{P}^2$, see Proposition 1.37 and Remark 1.38 for a method of constructing $X_y$. 
Now, suppose $F$ is a dyadic local field, and set $e = \text{ord}_p(2)$. When $e = 1$ (i.e., when $2$ is a prime), Schulze-Pillot's results completely describe $X_y$. If $e > 1$, these results no longer suffice (see [SP], §2.25, [SP]). The results needed to describe $X_y$ in this more general case are given below.

1.21 Notation: For the remainder of this chapter, if $L$ is a given good lattice on $V$, let $\{e, f, z\}$ be a basis for $L$ with $q(e) = q(f) = 0$, $b(e, f) = 1$, $q(z) \in \mathfrak{a}$, and $b(e, z) = b(f, z) = 0$.

Also, let $DV = \text{disc } L$, where $L$ is any good lattice on $V$. Then, $DV$ depends only on $V$, up to multiplication by unit squares. Since the quadratic defect is not affected by multiplication by unit squares, the symbol $\alpha(DV)$ is unambiguous for any $\alpha \in F$.

1.22 Proposition: Let $y \in L$ with $q(y) \in \mathfrak{a}$ and $b(y, L) = R$. Then, $y$ is contained in one and only one neighbor of $L$.

Proof: Let $y = c_1 e + c_2 f + c_3 z$, where $c_1, c_2, c_3 \in \mathbb{R}$. Since $b(y, z) = 2c_3 q(z) \in \mathfrak{p}$, we may assume that $c_2 = b(y, e) \in \mathfrak{a}$. Hence, $b(y, e) \neq 0$ so that $H = Fy + Fe$ is a hyperbolic plane in $V$. Let $\{\vec{v}_1, \vec{v}_2\}$ be a basis for $H$ with $q(\vec{v}_1) = q(\vec{v}_2) = 0$ and $b(\vec{v}_1, \vec{v}_2) = 1$. If $\vec{y} = a_1 \vec{v}_1 + a_2 \vec{v}_2$, where $a_1, a_2 \in F$, then $q(\vec{y}) = a_1 a_2$. Now, since $H$ is regular, it splits $V$; thus, $\{\vec{v}_1, \vec{v}_2\}$ extends to a basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $V$ with $q(\vec{v}_3) \neq 0$ and $b(\vec{v}_1, \vec{v}_3) = b(\vec{v}_2, \vec{v}_3) = 0$. 
Let \( \vec{w} = b_1 \vec{v} + b_2 \vec{v} + b_3 \vec{v} \in \vec{V} \). Then, \( \overline{q}(\vec{w}) = 0 \) and \( \overline{b}(\vec{y}, \vec{w}) = 0 \) are equivalent, respectively, to the equations

(i) \( b_1 b_{\overline{1}} + b_2 b_{\overline{2}} + b_3 b_{\overline{3}} = 0 \), and

(ii) \( a_1 b_{\overline{1}} + a_2 b_{\overline{2}} = 0 \).

Solving for \( b_2 \) and \( b_3 \), we obtain

\[
\begin{align*}
  b_2 & = -b_1 \left( \frac{a_2}{a_1} \right), \\
  b_3 & = -b_1 b_{\overline{2}} \frac{1}{\overline{q}(\vec{v})} = b_1 \frac{a_2}{\overline{q}(\vec{v})}.
\end{align*}
\]

Thus, since \( \text{char } \tilde{F} = 2 \), both \( b_3 \) and \( b_2 \) are uniquely determined by the choice of \( b_1 \). Furthermore, if \( (b_1, b_2, b_3) \) is a solution of (i) and (ii), so is \( (\lambda b_1, \lambda b_2, \lambda b_3) \) for any \( \lambda \in \tilde{F} \). Hence, the vectors \( \vec{w} \in \vec{V} \) with \( \overline{q}(\vec{w}) = 0 \) and \( \overline{b}(\vec{y}, \vec{w}) = 0 \) lie on a single isotropic line in \( \vec{V} \). Let \( w \) be a representative of \( \vec{w} \) in \( L \), and set \( x = \pi^{-1}w b(y, x) - \pi^{-1}b(y, w) \in R \), so that \( y \in L_x \subseteq L_x + Rx = L_1 \).

Now, suppose \( y \in L_1' = L_1 + Rx \) for some \( x' \in \pi^{-1}L \backslash L \) with \( q(x') \in R \). Then, \( y \in L_1' \cap L = L_x' \), so that \( b(y, \pi x') \in p \), i.e. \( \overline{b}(\vec{y}, \pi x') = 0 \). Hence \( \overline{F}(\pi x') = \overline{F}w \) so that \( L_1' = L_1 \).

Q.E.D.

1.23 Lemma: \( \vec{V} \) does not contain a two-dimensional totally isotropic subspace.
Proof: If $V$ contained such a subspace, one would have the half-discriminant of $V$ equal to 0, so that $\text{disc } L \in p$, contrary to hypothesis.

Q.E.D.

1.24 Proposition: Let $y$ be a maximal vector in $L$ with $q(y) \in p$. Then, $b(y,L) = R$ and $y$ is contained in one and only one neighbor of $L$.

Proof: That $b(y,L) = R$ was shown in the proof of Lemma 1.8. Now, suppose $y \in L_1 = L_x + Rx$ where $x \in p^{-1}L \setminus L$ with $q(x) \in R$. Then, $y \in L_1 \cap L = L_x$, so that $b(x,y) \in R$. Hence, $F(y,\pi x) = 0$ so that $\overline{Fy} + \overline{F(\pi x)}$ is an isotropic subspace of $V$. By Lemma 1.22, $\overline{y}$ and $\pi x$ must lie on the same isotropic line in $\overline{V}$. Hence, by Proposition 1.9, $y$ is contained in at most one neighbor of $L$.

On the other hand, by Lemma 1.8, there is a vector $y' \in L$ with $y' - y \in pL$ and $q(y') \in p^2$. Letting $x = \pi^{-1}y'$, we have $b(y,x) = b(y,\pi^{-1}y + \pi^{-1}(y'-y)) \in 2\pi^{-1}p + R = R$.

Thus, $y \in L_x \subseteq L_x + Rx = L_1$.

Q.E.D.

1.25 Corollary: If $q(y) \in \mu$ with $q(y) = \tau^2q(y) \not\in R$, $y$ is maximal in $L$. Hence, by Proposition 1.24, $y$ is contained in precisely one neighbor of $L$: call it $M$. A similar argument shows that $y$ is
contained in only one neighbor of M, namely L.

Q.E.D.

1.26 Proposition: Suppose \( y \in V \) with \( c := q(y) \in \mathfrak{u} \). Then, there exists a lattice \( M \in \mathcal{X}_Y \) with \( b(y,M) = 2R \) if and only if \( c \cdot \text{disc} V = \mathbf{F}^2 \) or \( c \cdot \text{disc} V = \Delta \mathbf{F}^2 \).

Proof: (Necessity) Suppose \( y \in M \), with \( b(y,M) = 2R \). Since \( b(y,y) = 2q(y) \in 2\mathfrak{u} \), \( R_y \) is \( 2R \)-modular; hence, \( M = R_y \perp N \) ([OM], §82:15). Now, since \( N \) is \( R \)-maximal in \((Fy)^{-1}\) and \( N \) is unimodular, \( N \cong A(0,0) \) or \( N \cong A(2,2\rho) \) ([OM], §93:11). Thus, \( \text{disc} V = c \cdot \mathbf{F}^2 \) or \( \text{disc} V = c \Delta \cdot \mathbf{F}^2 \); i.e., \( c \cdot \text{disc} V = \mathbf{F}^2 \) or \( c \cdot \text{disc} V = \Delta \mathbf{F}^2 \).

(Sufficiency) If \( L \in \mathcal{X}_| \), \( L = N \perp Rz \) where \( N \cong A(0,0) \) and \( b := q(z) \in \text{disc} V \cdot \mathbf{F}^2 \). Let \( \{u,v\} \) be a basis for \( N \) adapted to \( A(0,0) \). Then, either \( R(u+v) + Rz \) or \( R(u-v) + Rz \) is anisotropic. In either case, \( L \) has anisotropic sublattice \( N' \cong A(2,2\rho) \). Since \( N' \) is unimodular, \( L = N' \perp Rz' \) for some vector \( z' \) with \( b' := q(z') \in \Delta \cdot \text{disc} V \). If \( q(y) \in \text{disc} V \), then there exists a transformation \( \sigma \in O(V) \) with \( \sigma z = y \). If \( q(y) \in \Delta \cdot \text{disc} V \), then \( \sigma(y) = z' \) for some \( \sigma \in O(V) \). In either case \( M = \sigma L \) is the desired lattice.

Q.E.D.

1.27 Proposition: Suppose \( c := q(y) \in DV \cdot \mathfrak{u}^2 \). Then, \( \{M \in \mathcal{X}_Y : b(y,M) = 2R\} \) lie on a line in \( \mathcal{X}_Y \).
Proof: By Proposition 1.26, there exists \( L \in \mathcal{X}_y \) with \( b(y, L) = 2R \). As in the preceding proof, \( L = Ry \perp N \), where \( N \npreceq A(0,0) \). Let \( \{u, v\} \) be a basis for \( N \) with \( q(u) = q(v) = 0 \) and \( b(u, v) = 1 \). For \( r \in \mathbb{Z} \), set \( L_r = (pu + rv) \perp Ry \).

Clearly, the set of lattices \( \{L_r \mid r \in \mathbb{Z}\} \) form a line in \( \mathcal{X}_y \) through \( L_0 = L \). Also, \( b(y, L_r) = 2q(y)R = 2R \) for all \( r \in \mathbb{Z} \).

Conversely, suppose \( M \in \mathcal{X}_y \) with \( b(y, M) = 2R \). Then, once again \( M = N' \perp Ry \) where \( N' \) is an \( R \)-maximal unimodular lattice on the hyperbolic plane \( (Fy)^\perp \). Thus, \( N' \simeq A(0,0) \), and since \( Fu \) and \( Fv \) are the only isotropic lines in \( (Fy)^\perp \), \( N' = pu + rv \) for some \( r \in \mathbb{Z} \). Hence, \( M = L_r \) for some \( r \in \mathbb{Z} \).

Q.E.D.

1.28 Proposition: If \( c = q(y) \in \Delta \cdot D \cdot u^2 \), then there is one and only one lattice \( M \in \mathcal{X}_y \) such that \( b(y, M) = 2R \).

Proof: The existence of \( M \) was established in Proposition 1.25. To show uniqueness, we note that \( M = Ry \perp N \), when \( N \) is an \( R \)-maximal lattice on \( (Fy)^\perp \). Since \( (Fy)^\perp \) is anisotropic, \( N \) is the unique \( R \)-maximal lattice on \( (Fy)^\perp \). The result follows.

Q.E.D.

1.29 Lemma: If \( c = q(y) \in u \) with \( b(c \cdot D \cdot u) = p^{2r+1} \), where \( 0 \leq r < e \), then \( p^r \leq b(y, L) \) for all \( L \in \mathcal{X}_y \).
Proof: Choose $L \in \mathcal{X}_y$, and let $\{e, f, z\}$ be the basis mentioned above. Let $y = c_1 e + c_2 f + c_3 z$, where $c_1, c_2, c_3 \in R$, and suppose $b(y, L) = p^s$. Then, $c_1 = b(y, f) \in p^s$ and $c_2 = b(y, e) \in p^s$. Since $q(y) = c_1 c_2 + c_2^2 q(z) \in u$, either $s = 0$ or $c_3 \in u$. If $s = 0$, $b(y, L) = R$, and we are done. If $c_3 \in u$, we have

$$2r + 1 = \hat{a}(c \cdot DV) = \hat{a}((c_1 c_2 + c_3^2 q(z)) \cdot q(z))$$

$$= \hat{a}(c_1 c_2 q(z) + (c_3 q(z))^2) \in c_1 c_2 R \in p^{2s}.$$ 

Hence, $2r + 1 \geq 2s$ so that $r \geq s$. Thus, $p^{r+1} \leq p^s + 2R \leq b(y, L)$.

Q.E.D.

1.30 Lemma: If $L \in \mathcal{X}_y$ with $b(y, L) = p^r$, where $0 \leq r < e$, then $y \in M$ for some neighbor $M$ of $L$ with $b(y, M) \leq p^r$.

Proof: If $b(y, e) \in p^{r+1}$, let $M = (p^{-1}e + pf) \perp Rz$, and if $b(y, f) \in p^{r+1}$, let $M = (pe + p^{-1}f) \perp Rz$. In both cases, $b(y, M) \leq p^r$. Hence, we may assume that $b(y, e) = \pi^r e_1$ and $b(y, f) = \pi^r e_2$ where $e_1, e_2 \in u$. Set $e = e_1 e_2^{-1}$. Since char $\overline{F} = 2$, one can find $\alpha \in u$ such that $\alpha^2 e \equiv -q(z) \pmod{p}$. Choose $\beta \in R$ so that $\alpha \beta e \equiv -\alpha^2 e + q(z) \pmod{p}$, and set $v = \alpha e + (\alpha + \pi \beta) ef + z$. Then, we have

$q(v) = \alpha (\alpha + \pi \beta) e + q(z) = \alpha^2 e + q(z) + \alpha \beta e \pi$

$$= \pi \left( \frac{\alpha^2 e + q(z)}{\pi} + \alpha \beta e \right) \in \pi p = p^2.$$ 

Also, letting $y = \pi^r e_1 e + \pi^r e_2 f + ce_3$, one obtains

$$b(v, y) = \pi^r e_1 \alpha + \pi^r e_2 (\alpha + \pi \beta) e + 2cq(z)$$

$$= \pi^r e_1 \alpha + \pi^r e_2 (\alpha + \pi \beta) + 2cq(z)$$

$$= 2\pi^r e_1 \alpha + 2cq(z) + \pi^{r+1} e_1 \beta \in p^{r+1}.$$ 


Set $x = \pi^{-1}v$, and note that $q(x) = \pi^{-2}q(v) \in R$. Hence, $M = L_x + Rx$ is a neighbor of $L$. Now, $b(y,x) = \pi^{-1}b(y,v) \in \mathcal{P}$ $\subseteq R$ so that $y \in L_x \subseteq M$. Since $b(y,L_x) \subseteq b(y,L) = \mathcal{P}$, we have $b(y,M) = b(y,L_x) + b(y,Rx) \subseteq \mathcal{P}$.

Q.E.D.

1.31 Proposition: Suppose $c = q(y) \in u$ with $\hat{q}(c \cdot DV) = p^{2r+1}$, where $0 < r < e$. Then, $|X_y|$ contains two lattices $L$ and $M$ with $d(L,M) = 1$ and $b(y,L) = b(y,M) = p^r$.

Proof: Choose a lattice $L \in |X_y|$; by Lemma 1.29, $b(y,L) = p^s$ for some $0 \leq s \leq r$. If $s = r$, then $L$ has a neighbor $M \in |X_y|$ with $b(y,M) \subseteq \mathcal{P}$ by Lemma 1.30. Since $\mathcal{P} \subseteq b(y,M)$ by Lemma 1.30, $b(y,M) = p^r$. Hence, it suffices to prove the following: if $L \in |X_y|$ with $b(y,L) = p^s$ where $s < r$, then $|X_y|$ contains a lattice $L'$ with $b(y,L') \subseteq p^{s+1}$.

By Lemma 1.30, $L$ has a neighbor $L'$ in $|X_y|$ with $b(y,L') \subseteq p^s$. As in the proof of Proposition 1.2, let $\{e, f, z\}$ be a basis for $L$ such that $L = (pe + p^{-1}f) \perp Rz$. Set $y = ce + cf + c^2z$, where $c, c, c \in R$. Then,

$$c_1 = b(y,f) = \pi b(y, p^{-1}f) \in \pi \mathcal{P} = p^{s+1},$$

$$c_2 = b(y,e) \in \mathcal{P}.$$

Since $q(y) = c_1 c_2 + c_3^2 q(z)$ is a unit, $c_3$ must be a unit.

Now, observe that

$$q(y) \cdot DV = (c_1 c_2 + c_3^2 q(z)) \cdot q(z)$$

$$= \left( \begin{array}{cc} c_1 c_2 & c_3 \\ c_3 & 2q(z) \\ c & 3^2q(z) \\ 3 & 1 \end{array} \right) + 1 \cdot (c_3 q(z))^2.$$
Hence, if \( c_1 \in \pi^{s+1} u \) and \( c_2 \in \pi^s u \), then \( \Delta(q(y) \cdot DV) = \pi^{2s+1} \) ([OM], §63:5), contrary to hypothesis. Hence, either \( c_1 \in \pi^{s+2} \) or \( c_2 \in \pi^{s+1} \). In the second case, we have

\[
b(y,e) = c_2 \in \pi^{s+1}, \quad b(y,f) = c_1 \in \pi^{s+1}, \quad \text{and} \quad b(y,z) \in 2R \subseteq \pi^{s+1}.
\]

Thus, \( b(y,L) \subseteq \pi^{s+1} \), contrary to hypothesis. Hence, \( c_1 \in \pi^{s+2} \), and an argument similar to that above shows that \( b(y,L_1) \subseteq \pi^{s+1} \).

Q.E.D.

1.32 Definition: Given \( L, M, N \in |X| \), we shall say that \( M \) is between \( L \) and \( N \) in \( X \) if and only if \( M \neq L, M \neq N \), and \( M \) is a vertex on the path in \( X \) from \( L \) to \( N \). (This path is unique since \( X \) is a tree.)

1.33 Proposition: Let \( L, M \in |X| \) with \( b(y,L), b(y,M) \subseteq \pi^s \), where \( 0 \leq s \leq e \). If \( N \) is between \( L \) and \( M \) in \( X \), then \( b(y,N) \subseteq \pi^{s+1} \).

Proof: As in the proof of Proposition 1.2, let \( \{e, f, z\} \) be a basis for \( L \) such that \( M = (\pi^r e + \pi^{-r} f) \downarrow Rz \), and let

\[
y = c_1 e + c_2 f + c_3 z.
\]

Then, \( c_2 = b(y,e) \in b(y,L) = \pi^s \) and

\[
c_1 = b(y,f) = \pi^r b(y,\pi^{-r} f) \in \pi^r b(y,M) = \pi^{r+s}.
\]

Set \( c = \pi^{r+s} c_1 \) and \( c_2 = \pi^s c_2 \), where \( c_1, c_2 \in R \). Now, \( N = (\pi^{r-j} e + \pi^{-r+j} f) \downarrow Rz \) for some \( j \in \{1, \ldots, r-1\} \). Thus, we have

\[
b(y,\pi^{r-j} e) = \pi^{r-j} c_2 = \pi^{s+(r-j)} c_1 \in \pi^{s+1},
\]

\[
b(y,\pi^{-r+j} f) = \pi^{-r+j} c_1 = \pi^{-r+j+s} c_1 \in \pi^{s+j} c_1 \in \pi^{s+1}, \quad \text{and}
\]

\[
b(y,z) \in 2R \subseteq \pi^{s+1}.
\]
Hence, \( b(y,N) \leq p^{s+1} \).

Q.E.D.

1.34 Proposition: If \( c := q(y) \in u \) with \( \hat{a}(c \cdot D^y) = p^{2r+1} \), where \( 0 \leq r < e \), then \( |X_y| \) contains precisely two lattices \( L \) and \( M \) with \( d(L,M) = 1 \) and \( b(y,L) = b(y,M) = p^r \).

Proof: By Proposition 1.31, it contains at least two lattices \( L \) and \( M \) with \( b(y,L) = b(y,M) = p^r \). Suppose \( J \in |X_y| \) with \( J \neq L, J \neq M \) and \( b(y,J) = p^r \). Without loss of generality, we may assume \( d(J,L) > 1 \). Choose \( N \) between \( L \) and \( J \) in \( X \). Then, \( J \in |X_y| \) and \( b(y,J) \leq p^{r+1} \) by Proposition 1.33. This is impossible by Lemma 1.29.

Q.E.D.

1.35 Proposition: Let \( L \in |X_y| \) with \( b(y,L) = p^s \) where \( 1 \leq s \leq e \). Then, \( y \in L \) for every neighbor \( L_1 \) of \( L \) and \( b(y,L_1) \leq p^{s-1} \).

Proof: By Proposition 1.17, \( y \in L \). As in the proof of Proposition 1.2, let \( \{e, f, z\} \) be a basis for \( L \) such that \( L_1 = (pe + p^{-1}f) \perp Rz \). Then we have

\[
\begin{align*}
  b(y,pe) &= \pi b(y,e) \in \pi p^s = p^{s+1}, \\
  b(y,p^{-1}f) &= \pi^{-1} b(y,f) \in \pi^{-1} p^s = p^{s-1}, \text{ and} \\
  b(y,z) &= b(y,L) = p^s.
\end{align*}
\]

Hence, \( b(y,L_1) \leq p^{s-1} \).

Q.E.D.
1.36 Remark: The preceding results enable us to determine the shape of $X_Y$ when $F$ is dyadic. For purposes of illustration, we shall assume that $e = 3$ and $N_p = 2$. Then, by Corollary 1.10, every lattice in $|X|$ has $N_p + 1 = 3$ neighbors.

(a) We shall now show in detail how to construct $X_Y$ when $c_0 = q(y) \in u$ with $c \cdot \text{disc } V = \Delta F^2$. By Proposition 1.28, there is precisely one lattice $L \in |X_Y|$ with $b(y, L) = 2R$. By Proposition 1.35, every neighbor $L_1$ of $L$ contains $y$ with $b(y, L_1) \leq 2p^{-1} = p^{e-1} = p^2$. Since $L$ is the only lattice in $|X_Y|$ with $b(y, L) = 2R$, we must have $b(y, L_1) = p^2$ for all neighbors $L_1$ of $L$. Now, if $L_2 \neq L$ is a neighbor of $L_1$, then $y \in L_2$ with $b(y, L_2) \leq p^{2-e-1} = p$. If $b(y, L_2) \leq p^2$, then $b(y, L_1) \leq p^3 = 2R$ by Proposition 1.33, a contradiction. Hence, $b(y, L_2) = p$. A similar argument shows that, if $L_3 \neq L_1$ is a neighbor of $L_2$, then $y \in L_3$ and $b(y, L_3) = R$. By Proposition 1.22, $y$ is contained in only one neighbor of $L_3$, which must be $L_2$. Hence, $X_Y$ has the shape sketched below.

Here, $b(y, L) = 2R$.

The graph $X_Y$ when $F$ is dyadic and $c \cdot \text{disc } V = \Delta F^2$.

The sketches given in Figure 4 can be obtained in a similar manner.
(b) $c: = q(y) \in u$, $c \cdot \text{disc } V = \mathbb{F}^2$

where, $b(y, L_0) = 2R$ for all $r \in \mathbb{Z}$.

(c) $c: = q(y) \in u$ with $\hat{a}(c \cdot DV) = p^5 = p^{2 \cdot 2 + 1}$

where, $b(y, L) = b(y, M) = p^2$.

(d) $c: = q(y) \in u$ with $\hat{a}(c \cdot DV) = p^3$

where, $b(y, L) = b(y, M) = p$.

(e) $c: = q(y) \in u$ with $\hat{a}(c \cdot DV) = p$

where, $b(y, L) = b(y, M) = R$.

(f) $c: = q(y) \in \pi u$

---

**FIGURE 4**

The graph $x_y$ for a dyadic local field $F$. 

---
When $q(y) \in \mathbb{P}^2$, the following result from [SP], [SP$_1$] is needed to determine the shape of $X_y$.

1.37 Proposition: Let $F$ be an arbitrary local field, and $V$ a ternary quadratic space over $F$ containing a good lattice $L$. Let $y \in V$ with $q(y) \in \pi^{2r}u \cup \pi^{2r+1}u$. Then, there is a lattice $M \in \mathcal{X}$ containing $\pi^{-r}y$.

Proof: Since $L$ contains an $R$-maximal hyperbolic lattice, there exists $x \in L$ with $q(x) = \pi^{-2r}q(y) = q(\pi^{-r}y)$. By Witt's Theorem, one can find $\sigma \in O(V)$ with $\sigma(x) = \pi^{-r}y$. Then, $M = \sigma L$ is the desired lattice.

1.38 Remark: If $q(y) \in \mathbb{P}^2$, one can construct $X_y$ as follows. Choose $r \in \mathbb{N}$ such that $q(\pi^{-r}y) \in u \cup \pi u$. By Proposition 1.36, there exists a lattice $M \in \mathcal{X}$ with $y' = \pi^{-r}y \in M$. Since $q(y') \in u \cup \pi u$, one can construct $X_{y'}$, as in Remark 1.35. Now, if $L_1$ is a neighbor of a lattice $M' \in \mathcal{X}$ with $y' \not\in L_1$, then $L_1$ contains $\pi y' = \pi^{-r+1}y$ as a maximal vector. By Proposition 1.23, $M'$ is the only neighbor of $L_1$ containing $\pi^{-r+1}$. Any other neighbor of $L_1$ contains $\pi^{-2+r}y$ as a maximal vector. Continue in this way for $\pi^{-r+3}y$, $\ldots$, $\pi^{-1}y$, $y$. 


Chapter II

THE GLOBAL GRAPH

In this chapter, $F$ shall denote a global field and $R$ its ring of integers. Let $V$ be a quadratic space over $F$ with $\dim V \geq 3$. Let $L$ be a lattice on $V$ with $\eta L \subseteq R$, and choose a prime ideal $\mathfrak{p}$ such that $L$ is good at $\mathfrak{p}$ (i.e., $L_{\mathfrak{p}}$ is good). Then, define $R(L,\mathfrak{p})$ to be the graph whose vertices are lattices $M \in \text{gen } L$ with $M_{\mathfrak{q}} = L_{\mathfrak{q}}$ for all primes $\mathfrak{q} \neq \mathfrak{p}$; two vertices $M$ and $M'$ are joined by an edge in $R(L,\mathfrak{p})$ if and only if $d(M, M', \mathfrak{p}) := d(M_{\mathfrak{p}}, M'_{\mathfrak{p}}) = 1$. In this case, one says that $M$ and $M'$ are neighbors in $R(L,\mathfrak{p})$. The set of vertices of $R(L,\mathfrak{p})$ is denoted $|R(L,\mathfrak{p})|$. A moment's reflection shows that $R(L,\mathfrak{p})$ is isomorphic to the local graph $X$ formed by the good lattices on $V_{\mathfrak{p}}$. In particular, $R(L,\mathfrak{p})$ is connected, and if $\text{ind } V_{\mathfrak{p}} = 1$, it is a tree.

One question that arises naturally at this point is the following: which classes in $\text{gen } L$ are represented by some lattice in $|R(L,\mathfrak{p})|$? The following result, a straightforward generalization of the fundamental insight underlying Kneser's introduction of graphical techniques in [K], provides a partial answer:
2.1 Proposition: \(| R(L, p) |\) contains a representative of every class in \(\text{spn} L\).

**Proof:** This is an immediate consequence of the following.

2.2 Lemma: If \(\dim V \geq 3\) and \(\text{ind } V_{p} > 0\), and if \(M \in \text{spn} L\), then there exists \(M' \in \text{cls } M\) such that \(M'_{q} = L_{q}\) for all primes \(q \neq p\).

**Proof:** See [K, §24.3, or [OM], §104.5.

2.3 Remark: A similar result holds for proper classes in \(\text{spn}^+ L\), with essentially the same proof.

§1. Construction of the Global Graph

In general, one can construct the neighbors of \(L\) in \(R(L, p)\) by first using the techniques from Chapter I to obtain the neighbors of \(L_p\) in the local graph. Then, for each neighbor \((L_p)_1\) of \(L_p\), one may use the construction in §81:14 of [OM] to obtain a lattice \(L_1\) on \(V\) with \((L_1)_p = (L_p)_1\) and \((L_1)_q = L_q\) for all \(q \neq p\). Clearly, \(L_1\) is a neighbor of \(L\) in \(R(L, p)\). Since \(R(L, p) \cong X\), this procedure produces all the neighbors of \(L\) in \(R(L, p)\). If \(p\) is principal, one can construct the neighbors of \(L\) directly, using the following generalization of a result from [SP], [SP_].

2.4 Proposition: Suppose \(p = \pi R\) and let \(M\) be a neighbor of \(L\) in \(R(L, p)\). Then, there exists \(x \in \pi^{-1}L \setminus \mathbb{L}\) with \(q(x) \in R\)
such that $M = L_x + Rx$. Here $L_x = \{ y \in L \mid b(x,y) \in R \}$.

**Proof:** By Proposition 1.5, there exists $y \in \pi^{-1}L_p \setminus L_p$ with $q(y) \in R_p$ such that $M_p = (L_p)_x + R_p y$. Since $L$ is dense in $L_p$, one can find a vector $x \in \pi^{-1}L$ with $y - x \in pL_p$.

Then, $q(x) \in R_p$, and since $x \in \pi^{-1}L$, $q(x) \in R$. Also, by Lemma 1.7, $x \in M_p$. Hence, by Proposition 1.5, $M_p = (L_p)_x + R_p x = (L_x + Rx)_p$. Finally, since $\pi^{-1}L_q = L_q$, $(L_x + Rx)_q = L_q = M_q$. Hence, $M = L_x + Rx$.

Q.E.D.

2.5 **Remark:** The condition that $p$ be principal is necessary in Proposition 2.4; otherwise, it is not possible to find a vector $x \in \pi^{-1}L \setminus L$ such that $x \in L_q$ for all $q \neq p$. To see this, suppose $x$ is such a vector, and choose $\alpha \in R$ such that $\alpha x$ is maximal in $L$. Since $R$ is a dedekind domain, $\alpha R = q_1 \cdots q_t$ so that $x \in q_1^{-1} \cdots q_t^{-1} L$. Since $x \in L_q$ for all $q \neq p$, we must have $t = 1$ and $q_1 = p$. Thus, $p = \alpha R$ is principal.

Since $L/pL \cong L_p/pL_p$, the neighbors of $L$ in $R(L,p)$ are in one-to-one correspondence with the isotropic lines in $L/pL$. In general, one can obtain this correspondence explicitly by considering the local graph. If $p$ is principal, one can obtain a more straightforward description, using the following extension of a result of Schulze-Pillot ([SP], [SP1]).
2.6 Proposition: Let \( x \) be a maximal vector in \( L \) with 
\[ q(x) \in p. \]
Then, there exists \( x' \in L \) with \( x - x' \in pL \) and 
\[ q(x') \in p^2. \]

Proof: By Lemma 1.8, there exists \( x'' \in L_p \) with 
\[ x - x'' \in pL_p \] and \( q(x'') \in p^2. \)
Since \( L \) is dense in \( L_p \), one can find a vector \( x' \in L \) with \( x'' - x' \in p^2L_p \). Then, 
\[ x - x' \in pL_p \] and \( q(x') = q(x'' + (x' - x'')) \in p^2. \)
Since \( x, x' \in L \), the result follows.

Q.E.D.

2.7 Remark: Suppose \( p = \pi R \). Given an isotropic line \( l \) in 
\( L/pL \), one can find, by Proposition 2.6, a vector \( x' \in L \) with 
\[ x' \in L \] and \( q(x') \in p^2. \)
Letting \( x = \pi^{-1}x' \), form the 
neighbor \( L_1 = L_x + Rx \) which corresponds to \( l \). Repeating this 
procedure for each isotropic line in \( L/pL \), one obtains all 
the neighbors of \( L \) in \( R(L, p) \).

2.8 Proposition: If \( M \in \mathcal{R}(L, p) \), then \( R(M, p) = R(L, p) \).
If \( M \in \text{cls} L \), the neighbors of \( M \) fall into the same classes 
as the neighbors of \( L \).

Proof: The first statement is clear. For the second, 
choose \( \sigma \in \text{O}(V) \) such that \( \sigma L = M \). Then, for any neighbor 
\( L_1 \) of \( L \), one has 
\[ d(M, \sigma L_1, p) = d(\sigma L, \sigma L_1, p) = d(L, L_1, p) = 1. \]
The result follows. Q.E.D.
2.9 **Remark:** If $M \in \text{cls}^{+}L$, a similar result governs which proper classes are neighbors of $M$.

2.10 **Remark:** By Proposition 2.8, one can determine all the classes represented in $|R(L,p)|$ by constructing all the neighbors of $L$, then all the neighbors of those neighbors, and so on until at some step one obtains only representatives of classes from preceding steps. When $V$ is definite, one can determine whether two lattices $M$ and $M'$ on $V$ lie in the same class as follows: Let $\{e_1, \ldots, e_s\}$ be a generating set for $M'$ as an $R$-module. Now, there are only finitely many subsets $S_k = \{x_{1k}, \ldots, x_{sk}\}$ of $M$ with $q(x_{ik}) = q(e_i)$ for $1 \leq i \leq s$. If for some $k$, $b(x_{ik}, x_{jk}) = b(e_i, e_j)$ for $1 \leq i, j \leq s$ and $\sum \alpha_i x_{ik} = 0$ precisely when $\sum \alpha_i e_i = 0$, then $M' \cong M$. Otherwise, $M' \not\cong M$. If $V$ is indefinite, then $\text{cls} M = \text{spn} M$ for all lattices $M$ on $V$. Thus, one need only determine how many spinor genera are represented in $R(L,p)$. This question will be addressed in Chapter III.

2.11 **Remark:** If $\text{ind} V_p = 1$, then $R(L,p)$ is a tree, so that the construction in the preceding remark allows one to give a complete description of $R(L,p)$. When $\text{ind} V_p > 1$, a somewhat more delicate analysis is needed.
2.12 Remark: If $F = Q$, $\dim V = 3$, and $V$ is positive definite, $R(L,p\mathbb{Z})$ is identical to the graph $Z(L,p)$ from [SP], [SP1]. In this case, rather than using the somewhat tedious procedure outlined in Remark 2.10, one can determine whether two lattices are equivalent by comparing the Eisenstein reduced forms for each (see, for example, [Jo], p. 188). This is the method of determining class used in the computer program for constructing $Z(L,p)$ given in Appendix A.

§2. Unit Groups and the Global Graph

In addition to the restrictions at the beginning of this chapter, let $V$ be positive definite. Then, for any lattice $M$ on $V$, the unit group $O(M)$ is finite. (To see this, just let $M' = M$ in Remark 2.10.) In this section, we shall establish a relation between the orders of the unit groups $O(K)$ for $K \in |R(L,p)|$ and the division of neighbors in $R(L,p)$ into classes. We must first introduce some notation.

2.13 Notation: If $K \in |R(L,p)|$, then $N(L,K,p)$ denotes the number of neighbors of $L$ in $\text{cls} K$ in the graph $R(L,p)$. We shall adopt the notation $|O(K)|$ for the order of $O(K)$. By the proof of Proposition 2.8, $N(L,K,p)$ depends only on $\text{cls} L$, $\text{cls} K$ and $p$. 
2.14 Proposition: If \( K \in |R(L,p)| \), then \(|O(K)| \cdot N(L,K,p) = |O(L)| \cdot N(K,L,p)| \).

Proof: For any \( \sigma \in O(V) \), one has
\[
S_{x} = \{ \sigma' \in O(V) \mid \sigma'K = \sigma K \} = \sigma O(K).
\]
Hence, \( |S_{x}| = |O(K)|, \) so that
\[
\sum_{\sigma \in O(V)} \frac{1}{d(L,\sigma K,p)} = |O(K)| \cdot N(L,K,p).
\]

A similar argument shows that
\[
\sum_{\rho \in O(V)} \frac{1}{d(K,\rho L,p)} = |O(L)| \cdot N(L,K,p).
\]

On the other hand,
\[
\sum_{\rho \in O(V)} \frac{1}{d(K,\rho L,p)} = \sum_{\sigma \in O(V)} \frac{1}{d(\sigma K,L,p)} = 1.
\]

The result follows immediately. Q.E.D.

2.15 Remark: This result can be used to reduce somewhat the work involved in constructing \( R(L,p) \). For this purpose, it is often convenient to express the result in the form of a proportion: if \( N(L,K,p) \neq 0 \), then
\[
\frac{N(L,K,p)}{N(K,L,p)} = \frac{|O(L)|}{|O(K)|}.
\]
If \( \dim V = 3 \) and \( p \) is a (dyadic) prime with \( N_p = 2 \), then the following result of Schulze-Pillot holds, with essentially the same proof as in [SP] and [SP₁].

2.16 Proposition: Let \( M \in |R(L,p)| \) with \( d(L,M,p) = 1 \). Then, if \( |O(L)| \neq |O(M)| \), we have
\[
N(L,M,p) = [O(L):O(L) \cap O(M)].
\]

2.17 Remark: The proof depends very strongly on the fact that \( L \) has \( N_p + 1 = 3 \) neighbors in \( R(L,p) \). Schulze-Pillot provides counter-examples which show that the conditions that \( |O(L)| \neq |O(K)| \) and that \( p \) be dyadic cannot be eliminated.

2.18 Remark: Proposition 2.16 often provides a fairly efficient method for constructing \( R(L,p) \) when \( N_p = 2 \).

§3. Representations and the Graph at Dyadic Primes

In addition to the assumptions at the beginning of this chapter, we assume in this section that \( V \) is positive definite with \( \dim V = 3 \). We also assume that \( p \) is dyadic. In this setting, the graph \( R(L,p) \) can be used to generalize some representation-theoretic results of Schulze-Pillot ([SP], [SP₁]). We first introduce some notation.

2.19 Notation: Let \( h(L,p) \) denote the number of classes represented in \( |R(L,p)| \). Clearly, \( h(L,p) \leq h(L) \).
Let \( q(R(L,p)) = \bigcup_{M \in |R(L,p)|} q(M) \). Using Proposition 2.1 and the theory of spinor exceptional integers, one can determine \( q(R(L,p)) \) by purely local considerations if one knows which spinor genera are represented by lattices in \(|R(L,p)|\) (see [SP^2]). This can be determined using the techniques from Chapter III.

If \( y \in L \), then \( R_y(L,p) \) denotes the subgraph of \( R(L,p) \) whose vertices are lattices \( M \in |R(L,p)| \) with \( y \in M \). Two vertices are connected by an edge in \( R_y(L,p) \) if and only if they are connected by an edge in \( R(L,p) \). As usual, \( |R_y(L,p)| \) denotes the set of vertices of \( R_y(L,p) \). Clearly, \( R_y(L,p) \sim X_y \), the graph formed by good lattices on \( V_y \) which contain \( y \). In particular, the shape of \( R_y(L,p) \) is as described in Remarks 1.20 and 1.36.

Finally, for \( \alpha \in F \), let \( \delta_p(\alpha) \) denote the quadratic defect of \( \alpha \) considered as an element of the local field \( F_p \).

2.20 Lemma: Let \( M \in |R(L,p)| \). Then, there exists a lattice \( M' \in \text{cls} M \) such that \( M' \in |R(L,p)| \) and \( d(L,M',p) < h(L,p) \).

Proof: Choose \( M' \in \text{cls} M \cap |R(L,p)| \) such that \( d = d(L,M',p) \) is minimal. If \( d \geq h(L,p) \), consider the chain \( L = L_0, L_1, \ldots, L_d = M' \) of vertices in \( R(L,p) \) with \( d(L_i, L_{i+1}) = 1 \) for \( 0 \leq i \leq d-1 \). Since \( d+1 > h(L,p) \), we must
have $L_i \cong L_j$ for some $i, j$ with $0 \leq i < j \leq d$. Choose $\alpha \in O(V)$ with $\sigma L_j = L_i$. Then, one has

$$d(L_i, \sigma M', p) = d(\sigma L_j, \sigma M', p) = d(L_j, M', p) = d - j.$$

Hence, since $R(L, p)$ is a tree,

$$d(L, \sigma M', p) = d(L_j, L_i', p) + d(L_i', \sigma M', p) = i + (d-j) < d.$$

Since this contradicts the choice of $M'$, we must have $d(L, M') < h(L, p)$.

Q.E.D.

2.21 **Proposition:** Suppose $h(L, p) \leq e + 1$. Let $c \in q(R(L, p))$ such that $c \in R_{n \uparrow p}$ and $\hat{d}(c \cdot D_{V_p}) \leq 4R_p$. Then, $c \in q(L)$.

**Proof:** Choose $M \in |R(L, p)|$ with $c \in q(M)$, and let $y \in M$ with $c = q(y)$. By Proposition 1.26, there exists $K \in |R_y(M, p)|$ such that $b(y, K_p) = 2R_p$. By Lemma 2.20, one can find $L' \in \text{cls } L$ such that $L' \in |R(K, p)| = |R(M, p)|$ and $d(K, L', p) < h(L, p) \leq e + 1$. Hence, by Remark 1.36 (i.e., by repeated applications of Proposition 1.35), $y \in L'$. Hence, $c \in q(L') = q(L)$.

Q.E.D.

2.22 **Corollary:** Suppose that $p = \pi R$ and that $|R(L, p)|$ contains a representative of every class in gen $L$. Suppose $h(L) \leq e + 1$, and let $c \in R \cap \alpha \text{ disc } V$ for some $\alpha \in u_p$ with $\hat{d}_p(\alpha) \leq 4R$. If $c$ is represented by all completions $L_q$ of $L$, then $c$ is represented by $L$. 

Proof: Since \( c \in \text{disc} V \subset \mathbb{F}^2 \) and \( \mathbb{P} \), we may write \( c = \pi^2 r c' \) where \( r \in \mathbb{N} \cup \{0\} \) and \( c' \in \mathbb{R} \cap \mathbb{P} \). Since \( \mathbb{P} \) is isotropic and \( \mathbb{P} \)-maximal, \( c \in q(\mathbb{L}_p) \), and since \( \pi \in \mathbb{U} \) for all primes \( q \neq p \), \( c' \in q(\mathbb{L}_q) \). Hence, \( c' \) is represented by \( \mathbb{L} \). Since every class in \( \text{gen} \mathbb{L} \) is represented in \( |\mathbb{R}(\mathbb{L}, \mathbb{P})| \), \( c' \in q(\mathbb{R}(\mathbb{L}, \mathbb{P})) \). Hence, by Proposition 2.21, \( c' \in q(\mathbb{L}) \). Thus, \( c = \pi^2 r c' \in q(\mathbb{L}) \).

Q.E.D.

2.23 Remark: By Proposition 2.1, a sufficient condition for every class in \( \text{gen} \mathbb{L} \) to be represented in \( |\mathbb{R}(\mathbb{L}, \mathbb{P})| \) is that \( \text{gen} \mathbb{L} = \text{spn} \mathbb{L} \). The condition is satisfied by a large number of lattices. (See, for example, [K] and [EH].)

2.24 Notation: If \( \mathbb{L} = \mathbb{R}_1 + \mathbb{R}_2 + \mathbb{R}_3 \) is a ternary free lattice, we shall write
\[
\mathbb{L} = \langle q(e_1), q(e_2), q(e_3), b(e_2, e_1, e_3), b(e_1, e_2, e_3) \rangle.
\]
If \( \mathbb{L} = \mathbb{R}_1 \perp \mathbb{R}_2 \perp \mathbb{R}_3 \), we write simply
\[
\mathbb{L} = \langle q(e_1), q(e_2), q(e_3) \rangle.
\]

2.25 Example: Let \( \mathbb{F} = \mathbb{Q} \) and \( \mathbb{L} = \langle 1, 1, 4, 0, 1, 1 \rangle \). Then, using [BI], we see that \( h(\mathbb{L}) = 2 = \text{ord}_2(2) + 1 \), where \( M = \langle 1, 1, 3, 0, 1, 0 \rangle \) represents the other class in \( \text{gen} \mathbb{L} \). Since \( \text{disc} \mathbb{L} = -11 \), \( \text{spn} \mathbb{L} = \text{gen} \mathbb{L} \) by [K] and [EH]. By Corollary 2.22, \( \mathbb{L} \) represents every natural number in \(-11 \cdot \mathbb{Q}_2^2 \cup 5(-11) \cdot \mathbb{Q}_2^2 \) that is represented by all
completions \( L_p \).

Now, consider the natural number \( 11 = (-1)(-11) \). Since \( \hat{a}_2(-1) = 2Z_2 \not\in 4Z_2 \), Corollary 2.22 does not apply. In fact, \( 11 = 1 \cdot 1^2 + 3(-2)^2 + 1(-2) \in q(M) \), so that \( 11 \in q(M_p) = q(L_p) \) for all \( p \), but \( 11 \not\in q(L) \). Similarly, \( \hat{a}_2(-5) = 2Z_2 \), and \( 55 = (-5)(-11) \in q(L_p) \) for all \( p \). But, \( 55 \not\in q(L) \). Thus, the condition \( \hat{a}_p(\alpha) \subseteq 4R \) cannot be eliminated in Corollary 2.22. Similarly, the condition \( \hat{a}(c \cdot \mathcal{D}_p) \subseteq 4R_p \) is essential in Proposition 2.21.

2.26 Proposition: Suppose \( h: = h(L, p) \leq e \), and let \( c \in R \cap \mathcal{U}_p \) with \( \hat{a}(c \cdot \mathcal{D}_p) \subseteq p^{2h-1} \). If \( c \in q(R(L, p)) \), \( c \in q(L) \).

Proof: Let \( M \in |R(L, p)| \) with \( c \in q(M) \). Choose \( y \in M \) such that \( q(y) = c \). By Proposition 1.31, there is a lattice \( K \in |R_y(M, p)| \) such that \( b(y, K_p) \subseteq p^{h-1} \). By Lemma 2.20, there exists \( L' \in \text{cls} L \) such that \( L' \in |R(K, p)| \) and \( |R(M, p)| \) with \( d(K, L, p) \leq h-1 \). Then, by Remark 1.36, \( y \in L' \). Thus, \( c \in q(L') = q(L) \).

Q.E.D.

2.27 Corollary: Suppose that \( p = nR \) and that every class in \( \text{gen} L \) is represented in \( |R(L, p)| \). Suppose \( h: = h(L) \leq e \), and let \( c \in R \cap \alpha \text{ disc } V \) where \( \alpha \in \mathcal{U}_p \) with \( \hat{a}_p(\alpha) \subseteq p^{2h-1} \). Then, if \( c \) is represented by all completions \( L_q \) of \( L \), it is represented by \( L \).
Proof: As in the proof of Corollary 2.22, let
c = \pi^2 r c', where c' ∈ R \cap \mathbb{u}_p. Then, c' ∈ q(L_q)
for all primes q, so that c' ∈ q(R(L,p)). Hence, c' ∈ q(L)
by Proposition 2.26. Finally, we have c = \pi^2 r c' ∈ q(L).
Q.E.D.

2.28 Proposition: Suppose that p = \pi R and that gen L
= cls L ∪ cls M, where M ∈ |R(L,p)|. If N(L,M,p) = Np + 1,
then M represents every number represented by every
completion M_q of M. Furthermore, if N(M,L,p) = 1, then M
primitively represents every number primitively represented
by all its completions.

Proof: If c ∈ R is (primitively) represented by M_q
for all primes q', then c is (primitively) represented by
either M or L. In the former case, we are done. If c is
(primitively) represented by L, choose a (maximal) vector
y ∈ L with q(y) = c. By Remark 1.36, y ∈ M' for some
neighbor M' of L. But every neighbor of L is in cls M;
in particular M' ∈ cls M. Thus, M represents c and the
first part of the result is proved.

To prove the second part, suppose y is maximal in L.
If y is maximal in M', we are done. Otherwise, y = \pi y'
for some y' ∈ M' . Since y is maximal in L, y' ∉ L. Thus,
by Remark 1.36, y' is contained in precisely one neighbor
of M'. On the other hand, since b(y,M') ⊆ p, y is
contained in every neighbor of $M'$. Thus, $y$ is maximal in some neighbor $M'' \neq L$ of $M'$. Since $N(M', L, p) = 1$, we must have $M'' \cong M$. Hence, $M$ primitively represents $c$.

Q.E.D.

2.29 Remark: In general, such representation-theoretic results as those above can only be obtained using the graph at a dyadic prime $p$. When $p$ is nondyadic, the subgraph $R_y(L, p)$ may consist of only one point, making it impossible to duplicate the proofs given above. However, it is still possible in some cases to obtain useful information from such graphs, as the following example illustrates.

2.30 Example: ("Ramanujan's form"): Consider the $\mathbb{Z}$-lattice $L = \langle 1, 1, 10 \rangle$. Ramanujan provided a list of integers represented by gen $L$ but not by $L$, and remarked that these integers seemed to follow no definite pattern ([R]). Using a method developed by Schulze-Pillot ([SP], §7:7, [SP]), we shall show that if $c$ is such an integer, then $c \equiv 2 \pmod{3}$.

For suppose $c \equiv 2 \pmod{3}$ is represented by gen $L$ but not by $L$. Then, since gen $L = \text{cls } L \cup \text{cls } K$ where $K = \langle 2, 2, 3, 0, 2, 0 \rangle$ ([BI]), $c$ is represented by $K$.

Choose $y \in K$ such that $q(y) = c$; by Proposition 1.18, $Z_y(K, 3)$ is a line through $K(0) = K$, so that we may write $|Z_y(K, 3)| = \{K(n) \mid n \in \mathbb{Z}\}$, where $K_3(n)$ is as in the proof of Proposition 1.18. Since $c \notin q(L)$, $K(n) \cong K$. 
for all \( n \in \mathbb{Z} \). Using the construction from Remark 2.7, one finds that \( K \) has precisely two neighbors from \( \text{cls} \ K \) in \( \mathbb{Z}(L,3) \): both must lie on the line \( \mathbb{Z}(K,3) \). In particular, if \( \{e_1, e_2, e_3\} \) is a basis for \( K \) adapted to \( <2, 2, 3, 0, 2, 0> \), then

\[
K(l) = \mathbb{Z}(e_1 - \frac{2}{3}e_3) + 2e_2 + \mathbb{Z}(e_1 + \frac{1}{3}e_3), \text{ and}
\]

\[
K(-l) = \mathbb{Z}(\frac{1}{3}e_1 + \frac{2}{3}e_2) + 2e_3 + \mathbb{Z}(-e_1 + e_3).
\]

By similar arguments, letting \( \{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\} \) be a basis for \( K^{(n)} \), we have

\[
e_1^{(n)} = e_1^{(n-1)} - \frac{2}{3}e_3^{(n-1)} = \frac{1}{3}e_1^{(n+1)} + \frac{2}{3}e_3^{(n+1)},
\]

\[
e_2^{(n)} = e_2^{(n-1)} = e_2^{(n+1)}, \text{ and}
\]

\[
e_3^{(n)} = e_3^{(n-1)} + \frac{1}{3}e_3^{(n-1)} = -e_1^{(n+1)} + e_3^{(n+1)}.
\]

Letting \( y = c_1^{(n)}e_1^{(n)} + c_2^{(n)}e_2^{(n)} + c_3^{(n)}e_3^{(n)} \), where \( c_1^{(n)}, c_2^{(n)}, c_3^{(n)} \in \mathbb{Z} \) for all \( n \in \mathbb{Z} \), we have

\[
c_1^{(n+1)} = \frac{1}{3}c_1^{(n)} - c_3^{(n)},
\]

\[
c_1^{(n-1)} = c_1^{(n)} + c_3^{(n)},
\]

\[
c_3^{(n+1)} = \frac{2}{3}c_1^{(n)} + c_3^{(n)}, \text{ and}
\]

\[
c_3^{(n-1)} = -\frac{2}{3}c_1^{(n)} + \frac{1}{3}c_3^{(n)} \text{ for all } n \in \mathbb{Z}.
\]

Adding the first two equations, we obtain

\[
3(c_1^{(n+1)} + c_1^{(n-1)}) = 4c_1^{(n)} \text{ for all } n \in \mathbb{Z}.
\]
Thus, if \( c_1^{(n)} \equiv 0 \pmod{3^r} \) for all \( n \in \mathbb{Z} \) and for some \( r \in N \cup \{0\} \), then \( c_1^{(n)} \equiv 0 \pmod{3^{r+1}} \) for all \( n \in \mathbb{Z} \). Since \( c_1^{(n)} \equiv 0 \pmod{3^0} \) clearly holds for all \( n \in \mathbb{Z} \), by induction \( c_1^{(n)} \equiv 0 \pmod{3^r} \) for all \( r \in N \). Hence, \( c_1^{(n)} = 0 \) for all \( n \in \mathbb{Z} \). Similarly, \( c_3^{(n)} = 0 \) for all \( n \in \mathbb{Z} \).

Thus, letting \( c_2 = c_2^{(0)} \), we have \( y = c e_2 \). Hence, \( c = q(y) = 2c_2^2 e q(L) \), contrary to hypothesis. Therefore, if \( c \equiv 2 \pmod{3} \) and \( c \) is represented by \( \text{gen } L \), then \( c \in q(L) \). The desired result follows.
Chapter III

SPINOR GENERA AND THE GLOBAL GRAPH

Let $V$ be a quadratic space over an algebraic number field $F$ with $\dim V \geq 3$. Let $L$ be a lattice on $V$, and suppose $p$ is a prime such that $L$ is good at $p$. From the preceding chapter, we know that if $M \in |R(L,p)|$, then $|R(L,p)|$ contains a representative of every class in $\text{spn} M$ and of every proper class in $\text{spn}^+ M$. Thus, in order to find which (proper) classes are represented in $|R(L,p)|$, it suffices to determine which (proper) spinor genera are represented in $|R(L,p)|$. This chapter addresses the problem of making such a determination.

§1. Number of Proper Spinor Genera Represented in $|R(L,p)|$

3.1 Notation: Let $g(L,p)$ and $g^+(L,p)$ denote, respectively, the number of spinor genera and the number of proper spinor genera represented by $|R(L,p)|$.

3.2 Proposition: $g^+(L,p) \leq 2$.

Proof: Choose $M \in |R(L,p)|$ and let $M = \Sigma L$, where $\Sigma \in J_V$. Then, $\Sigma_q = 1$ for $q \neq p$. If $d(L,M,p)$ is even, then $\theta(\Sigma_p) \in \mathbb{U}_p \mathbb{F}_p^2$ by Theorem 1.15. Choose $\rho \in O^+(L_p)$ with 

58
\( \theta(p) = \theta(\Sigma_p) \) (see [OM], §91:8), and define \( A \in J_V \) by \( A_q = \Sigma_q \) = 1 for \( q \neq p \) and \( A_p = \Sigma_p \). Then, \( A \in J_V \), and \( M = A L \), so that \( M \in \text{spn}^+ L \). If \( d(L,M,p) \) is odd, then \( d(L_1,M,p) \) is even for any neighbor \( L_1 \) of \( L \). Thus, \( M \in \text{spn}^+ L \), by an argument similar to that above. Hence, \( |R(L,p)| \) contains only lattices from \( \text{spn}^+ L \) and \( \text{spn}^+ L_1 \).

Q.E.D.

3.3 Corollary: \( g(L,p) \leq 2 \).

Proof: One may give a proof similar to that above, or simply observe that \( g(L,p) \leq g^+(L,p) \).

Q.E.D.

3.4 Remark: If \( g^+(L,p) = 2 \) and \( J, K \in |R(L,p)| \), then, by the proof of Proposition 3.2, \( J \in \text{spn}^+ K \) if and only if \( d(K,J,p) \) is even. Similarly, if \( g(L,p) = 2 \), \( J \in \text{spn} K \) if and only if \( d(K,J,p) \) is even. In particular, when \( g(L,p) = 2 \) (\( g^+(L,p) = 2 \)), neighbors are in different (proper) spinor genera.

3.5 Remark: If \( V \) is indefinite, then every lattice in \( |R(L,p)| \) is (properly) equivalent to either \( L \) or a neighbor of \( L \), since \( \text{cls} L = \text{spn} L \) and \( \text{cls}^+ L = \text{spn}^+ L \). Thus, one can find lattices from every (proper) class represented in \( |R(L,p)| \) simply by constructing one neighbor of \( L \).
3.6 Notation: Let \( \pi \) be a prime element of \( F_p \), and let \( \{ e_1, f_1, \ldots, e_t, f_t, z_{2t+1}, \ldots, z_n \} \) be a basis for \( L_p \) as described in the proof of Proposition 1.2. Define \( \Sigma(p) \in J_V \) by \( \Sigma(p)_p = S e_1 f_1 e_1 f_1 \) and \( \Sigma(p)_q = 1 \) for all primes \( q \neq p \). Clearly, \( M = \Sigma(p)L \) is a neighbor of \( L \) in \( R(L, p) \). Note that the action of \( \Sigma(p) \) does not depend on the choice of \( \pi \).

Define \( j(p) \in J_F \) by \( j(p) = \theta(\Sigma(p)) \). In particular, \( j(p)_p = \pi \) and \( j(p)_q = 1 \) for all \( q \neq p \). Since \( \theta(\Omega^+(L, p)) \geq \mu_p^F \) ([OM], §91:8), the coset \( j(p)J_F^G \) is also independent of the choice of \( \pi \).

3.7 Proposition: \( g^+(L, p) = 1 \) if and only if \( j(p) \in P_D J_F^G \). Otherwise, \( g^+(L, p) = 2 \).

Proof: It suffices to show \( \Sigma(p)L \in \text{spn}^+ L \) if and only if \( j(p) \in P_D J_F^G \). From the proof of §102.7 in [OM], we know that \( \Sigma(p)L \in \text{spn}^+ L \) if and only if \( \Sigma(p) \in P_V J_V^V J_L^L \). But since \( \theta \) induces an isomorphism from \( J_V^V/P_V J_V^V J_L^L \) onto \( J_F^F/P_D J_F^G \), \( \Sigma(p) \in P_V J_V^V J_L^L \) if and only if \( j(p) = \theta(\Sigma(p)) \in P_D J_F^G \). The result follows.

Q.E.D.

3.8 Remark: Since \( g(L, p) \leq g^+(L, p) \), we have \( g(L, p) = 1 \) whenever \( j(p) \in P_D J_F^G \). Unfortunately, this condition is in general merely sufficient and not necessary. If \( \text{spn} L \)
= \text{spn}^+ L, then \( g(L,p) = g^+(L,p) \) so that the above condition becomes both necessary and sufficient. For example, if there is a lattice \( M \in \text{gen} L \) with an odd-dimensional orthogonal component \( M \), then \(-1 \in O^-(M)\), so that \( \text{spn}^+ M = \text{spn} M \). Hence, \( \text{spn}^+ L = \text{spn} L \) (see [OM], §102:2). In particular, \( g(L,p) = g^+(L,p) \) whenever \( \dim V \) is odd or \( \text{gen} L \) contains a diagonalizable lattice.

3.9 Example: Consider the \( \mathbb{Z} \)-lattice \( L = \langle 1, 1, 9, 0, 0, 1 \rangle \). Now, \( g(L) = h(L) = 2 \), and \( M = \langle 1, 3, 3, 3, 0, 0 \rangle \) represents the other class in \( \text{gen} L \) (see [C], p. 228). We wish to determine \( g(L,2) \). Observe that \( \theta(O^+(L)) = \mathbb{Q}_{p}^2 \) for all primes \( p \neq 3 \) and that \( \theta(O^+(L)) = \mathbb{Q}_3^2 \cup 3\mathbb{Q}_3^2 \) (see [EH], [K], and §92:5 of [OM]). Now, suppose \( \sqrt{d \cdot j(2)} \in \mathbb{Q} \) for some \( d \in D = \mathbb{Q}^+ \). Since \( J_F \) contains all squares, we may assume \( d \) is a square-free integer. Since \( d \cdot 1 \in \mathbb{Q}_{p}^2 \) for \( p \notin \{2,3\} \), we have \( d = 2^a 3^\beta \) where \( a, \beta \in \{0,1\} \). Since \( d \cdot 2 \in \mathbb{Q}_2^2 \), \( a = 1 \). Thus, \( 2 \cdot 3^\beta \cdot 1 \in \mathbb{Q}_3^2 \cup 3\mathbb{Q}_3^2 \), which is impossible for any value of \( \beta \). Hence, \( j(2) \notin \mathbb{Q}_{D}^{G} \), so that \( g(L,2) = 2 \) by Proposition 3.7 and the accompanying Remark.

Now, by Remark 3.4, every neighbor of \( L \) in \( Z(L,2) \) is in \( \text{spn} M = \text{cls} M \) and every neighbor of \( M \) is in \( \text{spn} L = \text{cls} L \). Since \( R (L,p) \) contains at least two lattices for any \( y \in L \) (see Remark 1.36), \( q(L) \subseteq q(M) \). Similarly, \( q(M) \subseteq q(L) \), so that \( q(L) = q(M) \). Thus, we have an example of two classes.
of positive definite ternary quadratic forms in the same
genus which represent the same integers. Hsia provides a
different development of this example, using the theory of
spinor exceptional representations ([H_2]).

Note that M primitively represents 4 and L does not.
Thus, L and M are classified by the sets of integers they
primitively represent. Hsia has conjectured that positive
definite ternary quadratic forms in the same genus are
always classified by the sets of integers they primitively
represent ([BH], [H_2]).

§2. Splitting Lattices and the Global Graph

Let K be a splitting lattice for \( \mathfrak{g} = \text{gen } L \). In this
section we shall investigate under what conditions \(|R(L,p)|\)
contains lattices from only one K-half-genus. A general
criterion is provided by the following.

3.10 Proposition: \(|R(L,p)|\) contains lattices from only one
K-half-genus if and only if \( j(p) \in H_K \).

Proof: Since \( \theta: J^V \rightarrow J_F \) induces an isomorphism from
\( J^V/J(V,K) \) onto \( J_F/H_K \), we have \( \Sigma(p) \in J(V,K) \) if and only
if \( j(p) \in H_K \). Hence, \( \text{spn}^+ L \) and \( \text{spn}^+ \Sigma(p)L \) are in the
same K-half-genus if and only if \( j(p) \in H_K \). The result
follows from Remark 3.4.

Q.E.D.
Although it is not difficult to apply this result directly in any given case, one can use it to obtain a still simpler criterion. We must first introduce some notation and a lemma.

3.11 Notation: Define a map \( \phi_K : J_F \rightarrow \{ \pm 1 \} \) by \( \phi_K((a_q)) = \prod q \left( \frac{a_q}{q} \right)^{\delta_K} \). Since \( a_q, \delta_K \in \mathfrak{u}_q \) for almost all \( q \), this is actually a finite product. In particular, we have \( \phi_K(j(p)) = \left( \frac{\pi}{p} \right)^{\delta_K} \). This image does not depend on the choice of \( \pi \).

To see this, observe that \( \mathfrak{u}_p^2 \subset \Theta (O^+(L,F)) \subset N_K(p) \) for any unit \( \epsilon \), so that either \( \delta_K \in \mathfrak{p}_p \) (in which case \( \left( \frac{\pi}{p} \right)^{\delta_K} = 1 \)) or \( \delta_K \notin \mathfrak{p}_p \) for any prime \( \pi \) (see [OM], §63:13).

3.12 Lemma: \( \phi_K \) induces an isomorphism from \( J_F / P_F N_K \) onto \( \{ \pm 1 \} \).

Proof: We show first that \( \phi_K \) is surjective. Since \( K \) is a splitting lattice, \( \delta_K \notin \mathfrak{p}_F \) ([H]), so that we may choose a prime \( q \) such that \( \delta_K \in \mathfrak{u}_q \setminus \mathfrak{u}_q^2 \); then \( \phi_K(j(q)) = -1 \).

We must now show that \( \ker \phi_K = P_F N_K \). Since \( [J_F : P_F N_K] = 2 \) ([OM], §65:21), it suffices to show that \( P_F N_K \subset \ker \phi_K \). Suppose \( \alpha = a \cdot b \in P_F N_K \) where \( a \in \mathfrak{f} \) and \( b \in N \). Then \( \phi_K(\alpha) = \phi_K(a \cdot b) = \phi_K(a) \phi_K(b) = 1 \cdot 1 = 1 \).
\[ b = (b_q) \in N_K. \] Then,
\[ \phi(a) = \phi((a)) \cdot \phi(b) = \prod_{q} \left( \frac{\alpha, \delta_K}{q} \right) \cdot \prod_{q} \left( \frac{b_q, \delta_K}{q} \right). \]

The first product is 1 by Hilbert's Reciprocity Law; since \[ b_q \in N_K(q) = \{ \gamma \in F_q \mid (\gamma, \delta_K) = 1 \} \] for all \( q \), the second product is clearly 1, also.

\[ \Omega \text{ E.D.} \]

3.13 Notation: If \( \alpha \in u_q \), define \( \overline{\alpha} \) by
\[
\left( \overline{\alpha} \right)_q = \begin{cases} 
1 & \text{if } \alpha \in u_q^2 \\
-1 & \text{if } \alpha \notin u_q^2
\end{cases}
\]

Now, since either \( \delta_K \in F_p^2 \) or \( \left( \frac{\delta_K}{p}, \pi \right) = -1 \) for all primes \( \pi \in F_p \), we may assume \( \delta_K \in u_p \). Furthermore, \( \left( \frac{\delta_K}{p} \right) = \left( \frac{\delta_K, \pi}{p} \right) = \phi_K(j(p)). \)

When \( p \) is non-dyadic, \( \left( \overline{\alpha} \right)_p = 1 \) if and only if \( \overline{\alpha} \in F_p^2 \). In particular, if \( F = Q, p \neq 2 \), and \( \alpha \in u_p \cap Z \), then \( \left( \frac{\alpha}{p^2} \right) \) reduces to the Legendre symbol \( \left( \frac{\alpha}{p} \right) \).

3.14 Proposition: \( |R(L, p)| \) contains lattices from only one \( K \)-half-genus if and only if \( \left( \frac{\delta_K}{p} \right) = 1 \).

Proof: We know that \( \left( \frac{\delta_K}{p} \right) = \phi_K(j(p)) \). By Lemma 3.12 and Proposition 3.10, it suffices to show that \( j(p) \in H_K \) if and only if \( j(p) \in P_F N_K \).

Suppose first that \( j(p) \in H_K \). Then, \( j(p) = d \cdot a \cdot b \) where \( d \in D \), \( a = (a_q) \in N_K \), and \( b = (b_q) \in J_F^6 \). Let \( T \) be the set of all real spots \( q \) such that \( V_q \) is anisotropic.
Then, $\delta_K < 0$ for all $q \in T$. Since $(a_q', \delta_q) = 1$ for $q \in T$, we must have $a_q > 0$ for all $q \in T$. Since $l = j(p)_q q 0$ and $d > q 0$ for all $q \in T$ ([OM], §101:8), we may assume $b_q = 1$ for all $q \in T$. If $q \not\in T$, then $\theta(O^+(V_q)) \subseteq N_K(q)$ (see the proof of Theorem 2 in [H]), so that we may assume $b_q = 1$ for $q \not\in T$. Thus, we have $b = (1)$, so that $j(p) = d \cdot a \in P_D N_K \subseteq P_F N_K$.

On the other hand, suppose $j(p) \in P_F N_K$. Choose

$l = h_1, \ldots, h_t \in F$ so that $F = \cup_{i=1}^t h_i D$ and $h_i D \cap h_j D = \emptyset$ for $i \neq j$. Then, we have

$$P_F N_K = P_D N_K \cup h_2 P_D N_K \cup \cdots \cup h_t P_D N_K.$$ 

Hence, if $j(p) \not\in P_D N_K$, $j(p) = h_j \cdot d \cdot b$, where $j \geq 2$, $d \in D$, and $b = (b_q) \in N_K$. Once again, if $q \in T$, $(b_q, \delta_q) = 1$ so that $b_q > 0$. Since $l = j(p)_q q 0$ and $d > q 0$ for all $q \in T$, we have $h_j > q 0$ for all $q \in T$. Hence, $h_j \in D$ so that $h_j D \cap h_i D \neq \emptyset$, a contradiction. Thus, $j(p) \in P_D J_K \subseteq P_F J_K$.

Q.E.D.

3.15 Remark: If $(-K_p) = -1$, then $|R(L, p)|$ contains lattices from two proper spinor genera, one in each $K$-half-genus, so that $g^+(L, p) = 2$. If $(-K_p) = 1$, then $|R(L, p)|$ either represents only one proper spinor genus or represents two proper spinor genera from the same $K$-half-genus.

Now, suppose $\{K_1, \ldots, K_r\}$ is a complete independent set of splitting lattices. Let $I_p$ be the subset of $\{1, \ldots, r\}$
such that \( \left( \frac{\delta_{K_i}}{\mathfrak{p}} \right) = 1 \) if \( i \in I_p \) and \( \left( \frac{\delta_{K_i}}{\mathfrak{p}} \right) = -1 \) if \( i \notin I_p \).

Then, \( |R(L,p)| \) represents the unique proper spinor genus \( \mathfrak{s} \) such that \( \mathfrak{s} \in \mathbb{H}_L,K_i \) for \( i \in I_p \) and \( \mathfrak{s} \in \mathbb{H}_L,K_i \) for \( i \notin I_p \).

If \( I_p \neq \{1, \ldots, r\} \), then \( \mathfrak{s} \neq \text{spn}^+L \), so that \( g(L,p) = 2 \).
If \( I_p = \{1, \ldots, r\} \), then \( \mathfrak{s} = \text{spn}^+L \), so that \( g(L,p) = 1 \).

3.16 Example: Let \( L^{(1)} \) be the ternary \( z \)-lattice \(<4, 5, 400> \), and let \( \mathcal{G} = \text{gen } L^{(1)} \). Then, \( g(L) = g^+(L) = 4 \). Using the criteria from \([SP_1]\), one finds that 1 and 5 are spinor exceptional integers for \( \mathcal{G} \). Now, since \( \delta_1 = 1 \cdot \text{disc } (qL^{(1)}) = -5 \) and \( \delta_5 = -1 \), \( \{1, 5\} \) is a complete independent set of spinor exceptional integers. Since \( \left( \frac{\delta_1}{3} \right) = \left( \frac{5}{3} \right) = 1 \) and
\[
\left( \frac{\delta_5}{3} \right) = \left( \frac{-1}{3} \right) = -1,
\]

\(|Z(L^{(1)},p)| \) represents two spinor genera which are in the same 1-half-genus but in opposite 5-half-genera. Thus, by computing \( Z(L^{(1)},3) \), we obtain representatives of all the classes in \( \mathfrak{s}_1 = \text{spn } L^{(1)} \) and another spinor genus \( \mathfrak{s} \). Using Remark 3.4, one finds that \( \mathfrak{s}_1 = \text{cls } L^{(1)} \) \( \cup \text{cls } L^{(2)} \cup \text{cls } L^{(3)} \), where \( L^{(2)} = <1, 80, 100> \) and \( L^{(3)} = <16, 20, 29, 0, -16, 0> \). Also, \( \mathfrak{s}_2 = \text{cls } K^{(1)} \) \( \cup \text{cls } K^{(2)} \cup \text{cls } K^{(3)} \), where \( K^{(1)} = <1, 20, 400> \), \( K^{(2)} = <9, 9, 100, 0, 0, -2> \), and \( K^{(3)} = <4, 45, 45, -5, 0, 0> \).

By inspection, one can see that \( \mathfrak{s}_1 \) is in both good half-genera. Thus, \( \mathfrak{s}_2 \) is in the good 1-half-genus and the bad 5-half-genus (this is also clear by inspection).
Since \((-\frac{5}{7}) = (-\frac{5}{3}) = 1\) and \((-\frac{1}{7}) = (-\frac{1}{3}) = -1\), no new spinor genera would be obtained by computing \(Z(L^{(1)}, 7)\). On the other hand, \((-\frac{1}{7}) = (-\frac{5}{11}) = -1\), so that \(|Z(L^{(1)}, 11)|\) represents a spinor genus \(g_3\) which is in both bad half-genera. Computing \(Z(L, 11)\), we find that \(g_3 = \text{cls } J^{(1)} \cup \text{cls } J^{(2)} \cup \text{cls } J^{(3)}\), where \(J^{(1)} = \langle 16, 20, 25 \rangle\), \(J^{(2)} = \langle 4, 20, 105, -20, 0, 0 \rangle\), and \(J^{(3)} = \langle 4, 21, 100, 0, 0, -4 \rangle\). One can confirm by inspection that indeed \(g_3\) represents neither 1 nor 5.

Finally, one can find the classes in \(g_4\) (which represents 5 but not 1) by computing either \(Z(K^{(1)}, 11)\) or \(Z(J^{(1)}, 3)\): \(g_4 = \text{cls } M^{(1)} \cup \text{cls } M^{(2)} \cup \text{cls } M^{(3)}\), where \(M^{(1)} = \langle 4, 25, 80 \rangle\), \(M^{(2)} = \langle 4, 20, 101, 0, -4, 0 \rangle\) and \(M^{(3)} = \langle 5, 16, 100 \rangle\).

For further information on the graphs mentioned above, see Appendix B. A sketch of \(Z(L, 3)\) along with some details of its construction are provided in [BH].

§3. **Existence of a Prime Linking Two Given Spinor Genera**

For any \(M \in \text{gen } L\), we shall say \(\text{cls } L\) and \(\text{cls } M\) are linked at \(p\) if \(|R(L, p)|\) contains lattices from \(\text{cls } M\).

Thus, \(L\) and \(M\) are linked at \(p\) if and only if there exists \(M' \in \text{cls } M\) such that \(L_q = M'_q\) at all spots \(q \neq p\). Linkage of proper classes, spinor genera, and proper spinor genera at \(p\) are defined analogously. By Proposition 2.1, \(\text{cls } L\) and \(\text{cls } M\) are linked at \(p\) if and only if \(\text{spn } L\) and \(\text{spn } M\)
are linked at \( p \). A similar result relates the linkage of proper classes to the linkage of proper spinor genera. Finally, note that if \( \text{spn}^+ L \) and \( \text{spn}^+ M \) are linked at \( p \), then so are \( \text{spn} L \) and \( \text{spn} M \). Thus, it is natural to ask the following question: Given \( M \in \text{gen} L \), does there exist a prime \( p \) which links \( \text{spn}^+ L \) and \( \text{spn}^+ M \)? The answer to this question is "yes." In order to show this, we shall need a few concepts from class field theory, which we introduce below. Our notation and terminology follow closely that in [Ja], Chapter IV.

3.17 Definition (Multiplicative congruences in \( \mathbb{F} \)): Let \( \alpha, \beta \in \mathbb{F} \). If \( p \) is a real spot on \( F \), we shall say \( \alpha \equiv \beta (\text{mod } x^p ) \) if and only if \( (\alpha/\beta) \geq p^0 \) (i.e., if and only if \( \alpha \) and \( \beta \) have the same sign as elements of \( F^p \) ). If \( p \) is a finite prime and \( n \in \mathbb{N} \), we shall say \( \alpha \equiv \beta (\text{mod } x^p^n ) \) if and only if \( \alpha \in \mathbb{R}^p \) and \( \beta^{-1} \in \mathbb{P}^n \). Note that this is somewhat different from the usual notion of congruence.

Now, define a modulus \( m \) to be a formal product \( \prod_p n(p) \) taken over all (finite and infinite) primes of \( F \) with \( n(p) \in \mathbb{N} \cup \{0\} \) for finite primes, \( n(p) \in \{0, 1\} \) for real primes, \( n(p) = 0 \) for complex primes. In addition, we require \( n(p) = 0 \) for almost all primes \( p \). Let \( m_0 = \prod_{p \text{ finite}} n(p) \) and \( m = \prod_{p \text{ real}} n(p) \); clearly, we may write \( m = m_0 \cdot m \). Finally, we shall write \( \alpha \equiv \beta (\text{mod } x^m ) \) whenever
\[ \alpha \equiv \beta \pmod{x \, p^n(p)} \] for all primes \( p \) with \( n(p) > 0 \).

3.18 **Definition** (Ray classes): Given a modulus \( m \) of \( F \), let \( I_F \) denote the subgroup of the group \( I_F \) of fractional ideals of \( F \) generated by ideals that are relatively prime to \( m \).

Also, define

\[ F_m = \{ a \in F \mid a, b \in R \text{ and } (aR, m_0) = (bR, m_0) = R \}, \quad \text{and} \quad F_{m,1} = \{ a \in F \mid a \equiv 1 \pmod{x \, m} \}. \]

The multiplicative group \( F_m \) is called the **ray modulo** \( m \).

Define \( i: F \to I_F \) by \( i(\alpha) = (\alpha) \) and let \( S = i(F) \).

Clearly, \( S_m \) is a subgroup of \( I_F \): the group \( I_F^{m,1} / S_m \) is called the **ray class group** mod \( m \); its elements are called **ray classes**.

We are now ready to state and prove the main result of this section.

3.19 **Theorem**: Given \( M \in \text{gen } L \), there exists a prime \( p \) that links \( sp^n + L \) and \( sp^n + M \). In particular, there exists \( M' \in sp^n + M \) such that \( M' \in |R(L,p)| \).

**Proof**: Choose \( \Sigma \in J \) such that \( M = \Sigma L \). Let \( X \) be the set of all real spots of \( F \), and let \( T \) be a finite set of (finite) primes of \( F \) such that

1. \( T \) contains all dyadic primes,
2. \( L_q \) is unimodular for all finite \( q \not\in T \), and
3. \( L_q = M_q \) for all \( q \not\in T \).
For each \( q \in T \), choose \( x_q \in \Theta (\Sigma n) \cdot \hat{F}_q^2 \cap R_q \) and set \( \alpha_q = \text{ord}_q 4 + 1 \). By the Strong Approximation Theorem ([OM], §21:2), there exists \( c \in R \) such that \( c - x_q \in q \alpha_q \) for all \( q \in T \). By adding an element of \( N \cap (\cap q \alpha_q) \) to \( c \) if necessary, we may assume that \( c > 0 \) for all \( q \in X \). Thus, \( c \in D \) ([OM], §101:8).

Now, we may write \( (c) = (\cap q \gamma_q) \cdot a \), where \( a \) is relatively prime to \( q \) for all \( q \in T \). Define a modulus \( m = (\cap q \alpha_q) \cdot \cap q \). By a density result from class field theory ([Ja], Chapter V, Theorem 10.3), each ray class in \( \hat{F}_m / S_m \) contains infinitely many primes. In particular, we may choose a prime \( p \in a \cdot S_m \); suppose \( p = a \cdot (b) \) where \( b \in \hat{F}_m \). Then, we have

\[
(cb) = (\cap q \gamma_q) \cdot a \cdot (b) = (\cap q \gamma_q) \cdot p.
\]

Clearly, \( cb \in n_q \) for all finite \( q \notin T \cup \{p\} \) and \( \text{ord}_p (cb) = 1 \).

Since the right hand side of the above equation is integral, \( cb \in R \). Also, since \( b \in \hat{F}_m \), \( b \equiv 1 \pmod{\cap q} \) for all \( q \in X \); hence, \( b > q \) so that \( cb > 0 \) for all \( q \in X \). Thus, \( cb \in D \). Finally, for all \( q \in T \), we have

\[
\text{ord}_q (cb - x_q) = \text{ord}_q (cb - c + c - x_q) \\
= \min (\text{ord}_q (c(b-1)), \text{ord}_q (c-x_q)).
\]

Since \( b \equiv 1 \pmod{\cap q} \), \( b-1 \in q \alpha_q \) for all \( q \in T \), so that \( \text{ord}_q (c(b-1)) \geq \text{ord}_q (b-1) \geq \alpha_q \). Since \( \text{ord}_q (c-x_q) \geq \alpha_q \) for all \( q \in T \) by the choice of \( c \), we must have \( \text{ord}_q (cb-x_q) \geq \alpha_q \).
for all \( q \in T \). Thus, \( cb \equiv x (\mod x^q q^a q) \), which means that
\[
\frac{cb}{x^q q^a q} \equiv 1 + q^a q = 1 + 4q \text{ for all } q \in T.
\]
Hence, by the Local Square Theorem, \( cb \equiv x F^2 \text{ for all } q \in T \).

Hence, letting \( d = cb \), we have \( d \cdot \theta(\Sigma_q) \in d \cdot x F_q^2 = F^2 \text{ for all } q \in T \). Also, since \( \theta(\Sigma_q) \in \theta(\Sigma(L_q)) = u_q F_q^2 \) for all finite \( q \notin T \), we have \( d \cdot \theta(\Sigma_q) \in u_q F_q^2 \) for all finite \( q \notin T \cup \{p\} \), and \( d \cdot \theta(\Sigma_p) \in \pi u_p F_p^2 \), where \( \pi \) is a prime element of \( F_p \). Thus we have
\[
\theta(\Sigma) \in (d^{-1}) j(p) J_F^G \leq j(p) \cdot P D F^G.
\]
Since \( j(p) = \theta(\Sigma(p)) \) and \( \theta : J_V^+ J_L \rightarrow J_F^G \) induces an isomorphism from \( J_V^+ J_L \) onto \( J_F^G \), we have \( \Sigma \in \Sigma(p) \). Thus, \( M' = \Sigma(p) L \) is the desired lattice in \( \text{spn}^+ M \cap | R(L,p) | \).

Q.E.D.

3.20 Corollary: Given \( M \in \text{gen} L \), there exists a prime linking \( \text{cls}^+ L \) with \( \text{cls}^+ M \), \( \text{spn} L \) with \( \text{spn} M \), and \( \text{cls} L \) with \( \text{cls} M \).

Proof: This is an immediate consequence of the theorem and the observations at the beginning of this section.

3.21 Remark: Let \( g = g(L) = [J_F^+ P J_F^G] \). By the theorem, there exist primes \( p_1, \ldots, p_g \) such that the cosets
\[
j(p_1) P D F^G, \ldots, j(p_g) P D F^G
\]
are distinct.
Then, one can determine every (proper) class in \( \text{gen} L \).
by constructing the graphs $R(L, p_1), \ldots, R(L, p_g)$.

3.22 Remark: Note that the lattice $M' \in \text{spn}^+ M \cap |R(L, p)|$ constructed in the proof of Theorem 3.19 is actually a neighbor of $L$ in $R(L, p)$. Hence, by Proposition 3.7 and Remark 3.4, $\text{spn}^+ M = \text{spn}^+ L$ if and only if $j(p) \in P_D^6$.

Thus, one can determine whether two given lattices $L$ and $M$ in the same genus are properly spinor equivalent by using the procedure outlined in the proof to find $p$ and then determining whether $j(p) \in P_D^6$.

3.23 Example: Let $L = \mathbb{Z} e_1 \perp \mathbb{Z} e_2 \perp \mathbb{Z} e_3 \perp \mathbb{Z} e_4$ where $q(e_1) = q(e_2) = 1$ and $q(e_3) = q(e_4) = 16$. Then, $g(L) = g^+(L) = 2$ ([EH], Remark 4.7). Let

$$M = \begin{pmatrix} 2 & 2 & 0 & 2 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 5 & 0 \\ 0 & 0 & 0 & 2 & 16 \end{pmatrix}.$$ 

One can show that $M \in \text{gen} L$ (using, for example, §93:29 of [OM]); we wish to determine whether $M \in \text{spn} L$. Since $M' = \mathbb{Z}(e_1 + e_2) + \mathbb{Z}(e_1 - e_2) + \mathbb{Z}(\frac{1}{2}e_3 + e_1) + \mathbb{Z}e_4 \in \text{cls} M$, we shall assume $M = M'$. Then, $[L:LN] = [M:LN] = 2$, so that $L_p = M_p$ for all $p \neq 2$. Thus, letting $T$ denote the finite set of primes from the proof of Theorem 3.19, we may assume $T = \{2\}$. 
Now, let \( \sigma \in O^+(V) \) be the transformation whose matrix with respect to the basis \( \{e_1, e_2, e_3, e_4\} \) is

\[
\begin{pmatrix}
2/3 & -1/3 & 8/3 & 0 \\
1/3 & -2/3 & -8/3 & 0 \\
1/6 & 1/6 & -1/3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Define \( \Sigma \in J_V \) by \( \Sigma_p = 1 \) for \( p \neq 2 \) and \( \Sigma_2 = \sigma \); then \( M = \Sigma L \).

Using a formula due to Zassenhaus (see [Z] or [OM], p. 137n), we have

\[
\theta(\Sigma_2) = \det \left( \frac{1 + \Sigma_2}{2} \right)^{2} = 6 \cdot \varrho_2^2.
\]

Thus, \( c = 6 \) so that \( a = 32 \) (again, using notation from the proof of Theorem 3.19). Thus, since \( \alpha_2 = \operatorname{ord}_2(4) + 1 = 3 \), we need to find a prime in the arithmetic sequence \( \{3 + 8t \mid t \in \mathbb{N}\} \); in particular, we may choose \( p = 3 \).

Hence, \( M \in \operatorname{spn} L \) if and only if \( j(3) \in J_D \). Now,

\[
\theta(O^+(L_p)) = \varrho_p \varrho_2^2 \quad \text{for} \quad p \neq 2 \quad \text{([OM], §92:5)} \quad \text{and} \quad \theta(O^+(L_2)) = \varrho_2^2 + 5\varrho_2^2 + 2\varrho_2^2 + 10\varrho_2^2 \quad \text{([EH])}.
\]

Suppose \( d \cdot j(3) \in J_D \) for some \( d \in D = \varrho^+ \). Then, as in Example 3.10, we may assume \( d = 2^\alpha 3^\beta \) where \( \alpha, \beta \in \{0, 1\} \). Since \( d \cdot 3 \in \varrho_3 \varrho_2^2 \), we must have \( \beta = 1 \). Since \( 2^\alpha \cdot 3 \cdot 1 \notin \theta(O(L_2)) \) for any value of \( \alpha \), \( d \cdot j(3) \notin J_Q^6 \), a contradiction. Hence, \( j(3) \notin P_D J_Q^6 \), so that \( M \notin \operatorname{spn} L \).
3.24 Remark: The procedure outlined above for determining whether lattices are properly spinor equivalent works for indefinite as well as definite lattices. In this case, since $\text{spn}^+ L = \text{cls}^+ L$, it enables us to determine whether two lattices in the same genus are actually in the same proper class. For an example, see [BH].

§ 4. Relationship to a Method of Cassels

The procedure described in the preceding section is closely related to a method for determining whether two $\mathbb{Z}$-lattices are properly spinor equivalent introduced by Cassels ([C], Chapter 11 §4; [C]). In this section, we shall first briefly describe Cassels's method and then discuss its relationship with that outlined above.

3.25 Notation: Let $L$ and $M$ be $\mathbb{Z}$-lattices in the same genus. Let $P$ be a set of primes in $\mathbb{Z}$ such that $2 \notin P$ and $\theta (O^+(L_p)) = \prod_{p \notin P} Q_p^2$ for all $p \notin P$. Define $\Omega = \prod_{p \in P} Q_p$ and $S = \prod_{p \in P} \theta (O^+(L_p))$. Let $T = \{ t \in \mathbb{D} \mid t \in \mathbb{U}_P \text{ for all } p \notin P \}$; we shall identify $T$ with its image under the diagonal embedding into $\Omega$. Thus, $S$ and $T$ are subgroups of $\Omega$. Let $P^*$ denote a set of primes with $2 \in P^*$ and $L_p$ unimodular for all $p \notin P^*$. In addition, we require $P^* \supset P$. Finally, define $I = I(L,M) = [L:L \cap M] = [M:L \cap M]$. 
3.25 Proposition: \( \Omega/ST \cong J_Q / P_D J^6_Q \).

Proof: Define \( \Psi: \Omega \to J_Q / P_D J^6_Q \) by \( \Psi((b_p)) = \overline{j} = \overline{(j_p)} \), where \( j_p = b_p \) for all \( p \in P \) and \( j_p = 1 \) for all \( p \notin P \). We shall first show that \( \Psi \) is surjective. To see this, suppose \( j = (j_p) \in J_Q \). Choose \( d \in D \) such that \( j_p \cdot d \in \mathcal{D} \) for all \( p \notin P \). Since \( \theta(0^+(L_p)) = u_p^{-1} \) for all \( p \notin P \), we may choose \( i = (i_p) \in J^6_Q \) such that \( j_p \cdot d \cdot i_p = 1 \) for all \( p \notin P \). Thus, clearly \( j = \overline{d\cdot i} \in \Psi(\Omega) \).

We must now show that \( \ker \Psi = ST \). Clearly, \( \Psi(T) \subseteq P_D \) and \( \Psi(S) \subseteq J^6_Q \). On the other hand, suppose \( d \in D \) and \( i = (i_p) \in J^6_Q \). Choose \( d' \in D \) such \( d' \cdot d \in \mathcal{D} \) for all \( p \notin P \). Hence, \( d \cdot d' \in T \). Since \( \theta(0^+(L_p)) = u_p^{-1} \) for all \( p \notin P \), we may choose \( i' = (i'_p) \in J^6_Q \) such that \( d' \cdot d \cdot i' \cdot i_p = 1 \) for all \( p \notin P \). Choose \( s = (s_p) \in S \) so that \( s_p = i'_p \cdot i_p \) for all \( p \in P \). Then, clearly \( \Psi(d \cdot d' \cdot s) = \overline{d \cdot d' \cdot i} \cdot \overline{i} = \overline{d} = \overline{1} \). The desired result follows.

Q.E.D.

3.26 Remark: Choose a prime \( p \notin P \). We claim that

\[ \Psi((p, \ldots, p)) = \overline{i(p)} \].

To see this, let \( i(p) \) be the idele with \( i(p)_q = p \) for \( q \in P \) and \( i(p)_q = 1 \) for \( q \notin P \). Then, \( \Psi((p, \ldots, p)) = \overline{i(p)} \) by definition, and

\[
p \cdot i(p)_q \in \begin{cases} q^2, & \text{if } q \in P \\ u_q q^2, & \text{if } q \notin P, q \neq p \\ p \cdot u_q q^2, & \text{if } q = p. \end{cases}
\]
Hence, there exists $i' \in J_F$ such that $p \cdot i' \cdot i(p) = j(p)$. This establishes the claim.

Now, suppose that $(p, I(L,M)) = 1$ for all $p \in P^*$. If $I(L,M) = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then one can easily extend the argument above to show that $\psi((I, \ldots, I)) = j(p_1)^{\alpha_1} \cdots j(p_r)^{\alpha_r}$. In addition, by judiciously choosing local bases, one has $M = \sum (p_1^{\alpha_1} \cdots \sum (p_r^{\alpha_r}) L$. From such considerations, together with Proposition 3.7, we obtain the following.

3.27 Theorem (Cassels): Let $M \in \text{gen } L$ with $(p, I(L,M)) = 1$ for all $p \in P^*$. Then, the proper spinor genus of $M$ depends only on the image of $I(L,M)$ under the diagonal embedding into $\Omega/ST$ and $\text{spn}^+ L$. In particular, $M \in \text{spn}^+ L$ if and only if the image of $I(L,M)$ is in $ST$.

3.28 Remark: Allowing for a difference in terminology, the proof sketched above is in the same spirit as that given by Cassels in [C], Chapter 11, §4. Thus, Proposition 3.7 above may be regarded as an extension of a special case of Cassels's theorem to lattices over the ring of integers of an algebraic number field. It is not immediately clear how to extend Cassels's full result to this case. Hence, at least in its present form, Cassels's method can be applied only to $\mathbb{Z}$-lattices. The procedure developed in the preceding section does not require this restriction.
When $L$ and $M$ are $z$-lattices, our method may be described as follows: Use the construction from the proof of Theorem 3.19 to obtain a $z$-lattice $M' \in \text{span}^+ M$ such that $I(L,M') = p$ for some $p \not\in P^*$. Then, apply Cassels's method to the lattices $L$ and $M'$. Unlike Cassels's method, this procedure does not require a determination of the exact value of $I(L,M)$, which sometimes involves a non-trivial amount of calculation. In addition, Cassels's method fails whenever $p \mid I(L,M)$ for any $p \in P^*$, whereas the procedure from the preceding section is not subject to this limitation.
Chapter IV

SPINOR-REGULAR LATTICES

Let $F$ be an algebraic number field with ring of integers $\mathcal{O}_F$. Given a ternary quadratic lattice $L$ over $\mathcal{O}_F$ with $nL \subseteq \mathcal{O}_F$, define

$$q(\text{gen } L) = \{ c \in \mathcal{O}_F | c \in q(L) \text{ for all spots } P \text{ of } F \}.$$ 

Equivalently, we have $q(\text{gen } L) = \bigcup_{M \in \text{gen } L} q(M)$ (see [OM], §102:5). When $q(L) = q(\text{gen } L)$, one says that $L$ is regular. Examples of regular positive definite ternary lattices over $\mathbb{Z}$ are given in [JP], [W], [SP], and [H]. In Example 3.9 above, we give the only known instance in which a positive definite ternary genus contains two distinct classes of regular lattices ([H]). It is possible to construct an indefinite ternary genus $L$ with arbitrarily large class number in which every lattice is regular (and, in fact, primitively regular) (see [BH]).

Define $q(\text{spn } L) = \bigcup_{M \in \text{spn } L} q(M)$. We shall say that $L$ is spinor-regular if and only if $q(L) = q(\text{spn } L)$. If $\text{spn } L = \text{cls } L$, we say that $L$ is trivially spinor-regular. Thus, every indefinite ternary lattice is trivially spinor-regular, so that in searching for non-trivial examples of
spinor-regularity, we may confine our attention to positive-definite ternary lattices.

When \( q(\text{spn } L) = q(\text{gen } L) \), we shall say that \( \text{spn } L \) is a regular spinor genus. Note that this definition is somewhat different from that in [BH]: that a spinor genus \( \mathcal{S} \) represents a complete set of spinor exceptional integers does not in general imply that \( \mathcal{S} \) represents every integer permitted by genus considerations. Clearly, a lattice is regular if and only if it is spinor-regular in a regular spinor genus.

The remainder of this chapter is devoted to a search for non-trivial spinor-regular lattices over \( \mathbb{Z} \). We shall first consider all lattices \( L \) with \( \text{disc } L \geq -1000 \), \( g(L) = 2 \) and \( h(L) = 3 \) or \( h(L) = 4 \). Our source for these is the table provided by Brandt and Intrau [BI]. The technique we use to show that a lattice is spinor-regular is an adaptation to the language of lattices of a method outlined by Watson in [W].

4.1 Notation: Suppose \( L = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 \) and \( c_1, c_2, c_3 \in \mathbb{Z} \). Define \( q_L(c_1, c_2, c_3) = q(c_1e_1 + c_2e_2 + c_3e_3) \).

Clearly, the value of \( q_L(c_1, c_2, c_3) \) depends strongly on the choice of basis; thus we only use this notation when a basis has been unambiguously specified.
4.2 Proposition: There are no non-trivial spinor-regular positive definite $\mathbb{Z}$-lattices $L$ with $\text{disc } L \geq -1000$, $h(L) = 3$ and $g(L) = 2$.

Proof: Using the list of discriminants which admit a genus with two spinor genera from $[H_2]$, one finds that the lattices $L$ with $\text{disc } L \geq -1000$, $g(L) = 2$ and $h(L) = 3$ fall into only five genera. These are listed in the table below. The table also shows how each genus $\mathcal{G}_t$ splits into spinor genera. This information was obtained from the graphs given in Appendix C. (One can also split these genera into spinor genera using the theory of spinor exceptional representations. See [BH], [H_2].)

**TABLE 1**
POSITIVE DEFINITE TERNARY LATTICES
WITH $\text{disc } L \geq -1000$, $h(L) = 3$, and $g(L) = 2$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$L_t$</th>
<th>$S_t$</th>
<th>$S'_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$L_1 = \langle 1,3,7,0,-1,0 \rangle$</td>
<td>$K_1 = \langle 1,1,27,0,0,-1 \rangle$</td>
<td>$J_1 = \langle 3,3,4,3,3,3 \rangle$</td>
</tr>
<tr>
<td></td>
<td>$L_2 = \langle 1,4,9,-4,0,0 \rangle$</td>
<td>$K_2 = \langle 1,1,32 \rangle$</td>
<td>$J_2 = \langle 2,2,9,-2,-2,0 \rangle$</td>
</tr>
<tr>
<td></td>
<td>$L_3 = \langle 1,7,9,0,0,-1 \rangle$</td>
<td>$K_3 = \langle 4,4,4,-1,-1,-1 \rangle$</td>
<td>$J_3 = \langle 1,9,9,-9,0,0 \rangle$</td>
</tr>
<tr>
<td></td>
<td>$L_4 = \langle 1,7,12,0,0,-1 \rangle$</td>
<td>$K_4 = \langle 1,1,108,0,0,-1 \rangle$</td>
<td>$J_4 = \langle 3,3,13,3,3,3 \rangle$</td>
</tr>
<tr>
<td></td>
<td>$L_5 = \langle 1,7,36,0,0,-1 \rangle$</td>
<td>$K_5 = \langle 4,9,9,-9,0,0 \rangle$</td>
<td>$J_5 = \langle 7,7,7,5,5,5 \rangle$</td>
</tr>
</tbody>
</table>
Since \( \text{cls } L_t = \text{spn } L_t \) for \( 1 \leq t \leq 5 \), the lattices \( L_1, L_2, \ldots, L_5 \) are trivially spinor-regular. For each \( t \in \{1, \ldots, 5\} \), we shall show that \( K_t \) and \( J_t \) are not spinor-regular by producing integers \( c_t \in q(J_t) \setminus q(K_t) \) and \( d_t \in q(K_t) \setminus q(J_t) \).

(1) Let \( c_1 = 10 = q_{J_1}(1, 0, 1) \). Since \( 10 < 27 \), \( 10 \in q(K_1) \) only if there exist \( x, y \in \mathbb{Z} \) with \( x^2 - xy + y^2 = 10 \). Fix the value of \( y \) for a moment and observe that

\[
x^2 - xy + y^2 = \left(\frac{y}{2}\right)^2 - \left(\frac{y}{2}\right)y + y^2 = \left(\frac{3}{4}\right)y^2.
\]

Thus, if \( |y| \geq 4 \), we have \( x^2 - xy + y^2 > 10 \). Now, one can easily check that, for any integer \( y \) with \( |y| \leq 4 \), there are no integers \( x \) with \( x^2 - xy + y^2 = 10 \). Hence, \( c_1 = 10 \notin q(K_1) \). Let \( d_1 = 1 \); since \( K_1 \) and \( J_1 \) are given in Eisenstein-reduced form, clearly \( 1 \in q(K_1) \setminus q(J_1) \).

The arguments in the other cases are similar. Values for \( c_t \) and \( d_t \) for these genera are given in Table 2.

Q.E.D.
<table>
<thead>
<tr>
<th>$t$</th>
<th>$c_t$</th>
<th>$d_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$21 = q_{J_2}$ (0, 2, -1)</td>
<td>$5 = q_{K_1}$ (1, 2, 0)</td>
</tr>
<tr>
<td>3</td>
<td>$1 = q_{J_3}$ (1, 0, 0)</td>
<td>$7 = q_{K_3}$ (1, 1, 0)</td>
</tr>
<tr>
<td>4</td>
<td>$61 = q_{J_4}$ (0, 1, 2)</td>
<td>$7 = q_{K_4}$ (2, -1, 0)</td>
</tr>
<tr>
<td>5</td>
<td>$7 = q_{J_5}$ (1, 0, 0)</td>
<td>$4 = q_{K_5}$ (1, 0, 0)</td>
</tr>
</tbody>
</table>
4.3 Remark: There are no spinor exceptional integers for \( \mathfrak{g}_1 \) and \( \mathfrak{g}_3 \): hence, \( \mathfrak{g}_1 \) and \( \mathfrak{g}_3 \) are regular spinor genera (as are \( \mathfrak{g}'_1 \) and \( \mathfrak{g}'_3 \)). Since \( \text{cls} L = \mathfrak{g}_1 \) and \( \text{cls} L = \mathfrak{g}_3 \), \( L \) and \( L_3 \) are regular.

Clearly, \( 1 \) is a spinor exception for \( \mathfrak{g}_5 \). Then, since \( g(L) = 2 \), \( \{1\} \) is a complete independent set of spinor exceptions, so that every spinor exception is a perfect square. Since \( m^2 = q_{L_5} (m, 0, 0) \in q(\mathfrak{g}_5) \) for any \( m \in \mathbb{Z} \), \( \mathfrak{g}_5 \) is a regular spinor genus. Thus, since \( \text{cls} L = \mathfrak{g}_5 \), \( L_5 \) is a regular lattice.

Finally, \( 2 \) and \( 3 \) are spinor exceptions for \( \mathfrak{g}_2 \) and \( \mathfrak{g}_4 \), respectively. Since \( 2 = q_{K_2} (1, 1, 0) \in q(\mathfrak{g}'_2) \) and \( 3 = q_{J_4} (1, 0, 0) \in q(\mathfrak{g}'_4) \), \( \mathfrak{g}_2 \) and \( \mathfrak{g}_4 \) are not regular spinor genera. Hence, \( L_2 \) and \( L_4 \) are not regular lattices. Since \( K_2, J_2, K_4, \) and \( J_4 \) are not spinor regular by Proposition 4.2, they are not regular lattices, either. Thus, \( \mathfrak{g}_2 \) and \( \mathfrak{g}_4 \) contain no regular lattices.

4.4 Proposition: The only classes of non-trivial positive definite spinor-regular lattices \( L \) with \( \text{disc} L \geq -1000 \), \( g(L) = 2 \) and \( h(L) = 4 \) are \( \text{cls} <1, 3, 37, -3, -1, 0> \) and \( \text{cls} <3, 7, 7, 5, 3, 3> \). These two lie in opposite spinor genera of the same genus.
Proof: Using the list of discriminants which admit a genus of two spinor genera from \([H_2^2]\), one finds that the positive definite lattices \(L\) with disc \(L \geq -1000\), \(g(L) = 2\) and \(h(L) = 4\) fall into ten genera, which are listed in the table on the following page. The table also shows the division of each genus into spinor genera, as obtained from the graphs in Appendix C.

Note that \(M_2\) is trivially spinor-regular. Since

\[
1 = q_{L_2}^{-1} (1, 0, 0) \notin q(J_2), \quad 2 = q_{J_2}^{-1} (1, 0, 0) \notin q(K_2), \quad \text{and}
\]

\[
21 = q_{K_2}^{-1} (0, 1, -1) \notin q(L_2), \quad \text{none of the lattices in } S_2 \text{ is spinor regular.}
\]

For each \(t \in \{1, 3, 4, \ldots, 10\}\) Table 4 gives integers \(c_t \in q(L_t) \setminus q(K_t)\), \(d_t \in q(K_t) \setminus q(L_t)\), \(c'_t \in q(J_t) \setminus q(M_t)\), \(d'_t \in q(M_t) \setminus q(J_t)\), wherever such integers exist. The proofs that these numbers have the desired properties are similar to that given in the proof of Proposition 4.2.

Table 4 shows that the only possible non-trivial spinor-regular lattices in Table 3 are \(K_5\) and \(J_3\). We shall show that both of these are indeed spinor-regular.
## TABLE 3

**POSITIVE DEFINITE TERNARY LATTICES**

With $\text{disc} \ L \geq -1000$, $h(L) = 4$, and $g(L) = 2$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$L_t$</th>
<th>$J_t$</th>
<th>$K_t$</th>
<th>$M_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\langle 1,1,63,0,0,-1 \rangle$</td>
<td>$\langle 1,3,16,0,-1,0 \rangle$</td>
<td>$\langle 3,4,4,-1,0,0 \rangle$</td>
<td>$\langle 3,3,7,0,0,-3 \rangle$</td>
</tr>
<tr>
<td>2</td>
<td>$\langle 1,1,64 \rangle$</td>
<td>$\langle 2,5,8,-4,0,-2 \rangle$</td>
<td>$\langle 2,2,17,-2,-2,0 \rangle$</td>
<td>$\langle 2,2,21,-2,-2,0 \rangle$</td>
</tr>
<tr>
<td>3</td>
<td>$\langle 1,1,80 \rangle$</td>
<td>$\langle 1,9,9,-2,0,0 \rangle$</td>
<td>$\langle 3,4,9,-3,0,-3 \rangle$</td>
<td>$\langle 3,3,16,0,0,-3 \rangle$</td>
</tr>
<tr>
<td>4</td>
<td>$\langle 1,3,28,-3,-1,0 \rangle$</td>
<td>$\langle 1,10,10,8,1,1 \rangle$</td>
<td>$\langle 4,4,7,1,2,4 \rangle$</td>
<td>$\langle 4,4,7,1,2,4 \rangle$</td>
</tr>
<tr>
<td>5</td>
<td>$\langle 1,1,144,0,0,-1 \rangle$</td>
<td>$\langle 3,7,7,5,3,3 \rangle$</td>
<td>$\langle 3,3,37,-3,-1,0 \rangle$</td>
<td>$\langle 3,3,37,-3,-1,0 \rangle$</td>
</tr>
<tr>
<td>6</td>
<td>$\langle 1,2,64 \rangle$</td>
<td>$\langle 2,4,17,-4,0,0 \rangle$</td>
<td>$\langle 3,3,17,-2,-2,-2 \rangle$</td>
<td>$\langle 3,3,17,-2,-2,-2 \rangle$</td>
</tr>
<tr>
<td>7</td>
<td>$\langle 1,9,17,-6,0,0 \rangle$</td>
<td>$\langle 2,9,9,0,-2,-2 \rangle$</td>
<td>$\langle 4,5,9,-4,-4,0 \rangle$</td>
<td>$\langle 4,5,9,-4,-4,0 \rangle$</td>
</tr>
<tr>
<td>8</td>
<td>$\langle 1,1,144 \rangle$</td>
<td>$\langle 4,5,10,2,4,4 \rangle$</td>
<td>$\langle 2,2,37,-2,-2,0 \rangle$</td>
<td>$\langle 2,2,37,-2,-2,0 \rangle$</td>
</tr>
<tr>
<td>9</td>
<td>$\langle 1,1,252,0,0,1 \rangle$</td>
<td>$\langle 1,12,19,-12,-1,0 \rangle$</td>
<td>$\langle 3,3,28,0,0,-3 \rangle$</td>
<td>$\langle 3,3,28,0,0,-3 \rangle$</td>
</tr>
<tr>
<td>10</td>
<td>$\langle 1,16,16,5,1,1 \rangle$</td>
<td>$\langle 4,9,10,-9,-4,0 \rangle$</td>
<td>$\langle 4,4,16,-2,-2,-1 \rangle$</td>
<td>$\langle 4,4,16,-2,-2,-1 \rangle$</td>
</tr>
</tbody>
</table>
To see that $K^s$ is spinor-regular, let $\{e_1, e_2, e_3\}$ be a basis for $L^s$ adapted to $<1,1,144,0,0,-1>$. Then,

$$\{-e_1, e_1 + 2e_2, -e_2 + \frac{1}{2}e_3\}$$

is a basis for $K^{(1)} \in \text{cls } K^s$ adapted to $<1,3,37,-3,-1,0>$. Since $L^s \cap K^{(1)} = Z e_1 + Z(2e_2) + Z e_3$, $q_{L^s}(c_1, c_2, c_3) = q(c_1 e_1 + c_2 e_2 + c_3 e_3) \in q(K^{(1)}) = q(K)$ whenever $2 | c_2$. Similarly, $K^{(2)} = Z(-e_2)$

$$+ Z(e_2 + 2e_1) + Z(\frac{1}{2} e_1 - e_2) \in \text{cls } K^s$$

with $L^s \cap K^{(2)} = Z(2e_1) + Z e_2 + Z e_3$. Hence, $q_{L^s}(c_1, c_2, c_3) \in q(K)$ whenever $2 | c_1$. Finally, $K^{(3)} = Z(-e_1 - e_2) + Z(e_1 - e_2)$

$$+ Z(e_2 + \frac{1}{2} e_3) \in \text{cls } K^s$$

with

<table>
<thead>
<tr>
<th>t</th>
<th>c_t</th>
<th>d_t</th>
<th>c'_t</th>
<th>d'_t</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>22</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>24</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>10</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>*</td>
<td>45</td>
<td>7</td>
<td>*</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>35</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>56</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>40</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>13</td>
<td>10</td>
<td>7</td>
</tr>
</tbody>
</table>

**TABLE 4**

**INTEGERS REPRESENTED BY ONLY ONE LATTICE**

**IN $s_t$ OR $s'_t$ (TABLE 3)**
$L_5 \cap K^{(3)} = z(e_1 + e_2) + z(e_1 - e_2) + ze$. Hence, if $c \equiv c \equiv 1 \pmod{2}$, then

$$q_{L_5}(c_1, c_2, c_3) = q(c_1 e_1 + c_2 e_2 + c_3 e_3)$$

$$= q\left(\frac{c_1 + c_2}{2} (e_1 + e_2) + \frac{c_1 - c_2}{2} (e_1 - e_2) + c_3 e_3\right)$$

$$\in q(K^{(3)}) = q(K)$$

Thus, $q(L_5) \subseteq q(K_5)$ so that $K$ is spinor-regular.

To see that $J$ is spinor-regular, choose a basis

$\{f_1, f_2, f_3\}$ for $M$ adapted to $<3, 3, 16, 0, 0, -3>$. Let

$J^{(1)} = z(f_1) + z(-f_1 - \frac{1}{2} f_3) + z(f_1 + f_2 - \frac{1}{2} f_3)$,

$J^{(2)} = z(f_2) + z(-f_2 - \frac{1}{2} f_3) + z(f_1 + f_2 - \frac{1}{2} f_3)$, and

$J^{(3)} = z(f_3) + z(f_3) + z(f_1 - \frac{1}{2} f_3) + z(f_1 - \frac{1}{2} f_3)$.

Then, $J^{(1)} = J^{(2)} = J^{(3)} = J$ with $J^{(1)} \cap M_5 = z(f_1)$

$+ z(2f_2) + zf_3$, $J^{(2)} \cap M_5 = z(2f_1) + zf_2 + zf_3$ and $J^{(3)} \cap M_5$

$= z(f_1 + f_2) + z(f_1 - f_2) + zf_3$. Hence an argument similar to that above shows that $q(M_5) \subseteq q(J_5)$. Thus, $J_5$ is spinor-regular.

Q.E.D.

4.5 Corollary: The lattice $K_5 = <1, 3, 37, -3, -1, 0>$

is regular. It is the only regular lattice in Table 3.

Proof: The spinor exceptions for the genus $S_5 := S_5 \cup S'$

are all of the form $m^2$ for $m \in \mathbb{Z}$. Since $m^2 = q_{K_5}(m, 0, 0)$

$\in q(S_5)$ for any $m \in \mathbb{Z}$, $S_5$ is a regular spinor genus.

Hence, since $K_5$ is spinor-regular in $S_5$, $K_5$ is regular.
Since $l \notin q(S')$, $S'$ is not a regular spinor genus; hence, $J_5$ is not regular. Finally, $l \in q(L_2) \setminus q(M_2)$ so that $M_2$ is not regular. Since $K_5$, $J_5$ and $M_2$ are the only spinor-regular lattices in the table, none of the other lattices there is regular.

Q.E.D.

4.6 Proposition: The lattices $L = \langle 1, 48, 144 \rangle$ and $K = \langle 9, 16, 48 \rangle$ are non-trivially spinor-regular.

Proof: Jones and Pall have shown that $L$ is regular and that $h(L) = 4$ with $K \in \text{gen } L$. From the graph $Z(L, 5)$ we learn that $\text{spn } L = \text{cls } L \cup \text{cls } M$, where $M = \langle 4, 48, 49, -48, -4, 0 \rangle$, and $\text{spn } K = \text{cls } K \cup \text{cls } J$, where $J = \langle 16, 25, 25, 14, 16, 16 \rangle$. Since $L$ is regular and $\text{spn } L = \text{cls } L$, $L$ is clearly non-trivially spinor-regular. To show that $K$ is spinor-regular, choose a basis $\{e_1, e_2, e_3\}$ for $J$ adapted to $\langle 16, 25, 25, 14, 16, 16 \rangle$. Let $K^{(1)} = Z(\frac{1}{2}(e_2 - e_3)) + Z(e_1) + Z(e_1 - e_2 - e_3)$. Then, $K^{(1)} \in \text{cls } K$ and $K^{(1)} \cap J = Z(e_1) + Z(e_1 - e_3) + Z(e_2 + e_3)$. Hence, $q_J(c_1, c_2, c_3) = q(c_1 e_1 + \frac{c_1 - c_2}{2} (e_2 - e_3) + \frac{c_1 + c_2}{2} (e_2 + e_3)) \in q(K^{(1)}) = q(K)$. Then, $q_J(c_1, c_2, c_3) \in q(K)$ whenever $c_1 \equiv c_2 \pmod{2}$. 


Let \( K^{(2)} = \mathbb{Z}(\frac{1}{2}(e - e)) + \mathbb{Z}(\frac{1}{2}(e + e)) + \mathbb{Z}(\frac{1}{2}(e + e) - 2e). \)

Then, \( K^{(2)} \not\subseteq \text{cls} K \) with \( K^{(2)} \cap J = \mathbb{Z}(e_1) + \mathbb{Z}e_2 + \mathbb{Z}(2e_3). \)

Thus, \( q_J(c_1, c_2, c_3) \in q(K^{(2)}) = q(K) \) whenever \( c \) is even.

Finally, let \( K^{(3)} = \mathbb{Z}(\frac{1}{2}(e - e)) + \mathbb{Z}(\frac{1}{2}(e + e) + e) + \mathbb{Z}(\frac{1}{2}(e + e) - e). \) Again, \( K^{(3)} \not\subseteq \text{cls} K; K^{(3)} \cap J = \mathbb{Z}(e_1 + e_2) + \mathbb{Z}(e_2 + e_3) + \mathbb{Z}(e_2 - e_3). \) Hence, \( q_J(c_1, c_2, c_3) = q(c_1(e_1 + e_2) + \frac{c_2 + c_3 - c_1}{2} (e_1 + e_2) + \frac{c_2 - c_3 - c_1}{2} (e_2 - e_3)) \in q(K^{(3)}) \) when \( c \) is odd and \( c \not\equiv c \mod 2). \) Thus,

\[ q(J) \subseteq q(K), \] so that \( K \) is spinor regular.

Q.E.D.
Epilogue

SUGGESTIONS FOR FURTHER RESEARCH

Perhaps the most pertinent conclusion that one can draw from the foregoing is that graphical techniques can be successfully exploited to obtain significant results on the classification of and representations by integral quadratic forms. A few possible avenues for further investigation are suggested below.

1. If $K$ is a sublattice of $L$, define a subgraph $R_K(L,p)$ of $R(L,p)$ analogous to the subgraph $R'_y(L,p)$. This subgraph is connected. To see this, suppose $M \in R(L,p)$ with $K \subseteq L \cap M$, and choose $x \in (L_p^{-1} \cap L \cap M) \setminus L_p$ (such an $x$ exists by Proposition 1.5). Then, $b(x,K_p) \subseteq b(x,M_p) \subseteq R_p$, so that $K_p \subseteq (L_p)x \subseteq (L_p)x + R_p = (L_1)_p$. Hence, $K \subseteq L_1$, where $L_1$ is a neighbor of $L$ between $L$ and $M$. Continuing this process, we obtain a chain of lattices in $|R_K(L,p)|$ linking $L$ and $M$.

One would like to obtain further information on the shape of this subgraph, which could be used to obtain results on representations of lattices by lattices analogous to the results on representations of integers by...
lattices in Chapter II.

(2) Proposition 3.7 provides a simple method for determining $g^+(L,p)$. Is there a similar result that would allow one to determine $g(L,p)$ whenever $L$ is good at $p$? By Remark 3.8, we need only consider lattices $L$ with $g^+(L,p) = 2$.

(3) Using Theorem 3.19, one can associate an infinite set of primes to each proper spinor genus $\mathcal{G}$ in $G = \text{gen } L$ in the following manner: If $\mathcal{G} = \mathcal{G}_0 = \text{spn}^+ L$, define

$$P_\mathcal{G} = \{p: \mathcal{G} \text{ and } \mathcal{G}_0 \text{ are linked at } p\}.$$ 

On the other hand, set

$$P_{\mathcal{G}_0} = \{p: |R(L,p)| \text{ contains only lattices from } \mathcal{G}_0\}.$$ 

Can one, by judiciously choosing subsets of these sets, obtain a partition of $\mathcal{G}$ into finer objects than proper spinor genera? In particular, is it possible in some cases to somehow associate an infinite set of primes to each (proper) class in $\text{gen } L$?

(4) Observe that in both examples of genera $\mathcal{G}$ containing non-trivial spinor-regular lattices developed above, every spinor genus in $\mathcal{G}$ contains a spinor-regular lattice. Is this true in general? If not, how might one construct a counter-example?
(5) The method used in Chapter IV to show that a lattice $K$ is spinor-regular relies on the construction of lattices $K^{(1)}$, $K^{(2)}$, $K^{(3)} \in \text{cls } K$ with $[J:K^{(i)} \cap J] = [K^{(i)}:K^{(i)} \cap J] = 2$ (1≤i≤3) for any lattice $J \in \text{spn } K \setminus \text{cls } K$, i.e., on constructing "neighbors" of $J$ in $\text{cls } K$ at the prime 2. We use quotation marks because in the cases considered in Chapter IV, $K$ is not good at 2.

Thus, one would like to replace the condition $\text{disc } L_{p} \subseteq \mathfrak{n}_{p}$ with a somewhat weaker condition that still permits the construction of a graph analogous to $R_{y}(L,p)$ whose shape can be determined. Such a construction may lead to a coherent theory of spinor-regular lattices, rather than the isolated examples that are available at present.
APPENDIX A

A COMPUTER PROGRAM FOR CONSTRUCTING GRAPHS

This Appendix contains a computer program that constructs the graph $Z(L,p)$ for a positive definite ternary $\mathbb{Z}$-lattice $L$ and a prime $p$ such that $\text{disc } L \leq \nu_p$. The algorithm used is that outlined in Chapter II (see especially Remarks 2.7, 2.10, and 2.12). The program was used to construct the graphs in Appendixes B and C.

The program is written in the programming language PL/C as described in the text of Conway and Gries [CG]. The purpose of each subroutine is given in the comments preceding that subroutine.
**GRAPHS**

**PROCEDURE OPTIONS(MAIN)**

```
***GRAPHS PROCEDURE OPTIONS(MAIN)***
//**PURPOSE: TO CONSTRUCT THE GRAPHS Z(L,P) FOR SEVERAL Z-LATTICES L AND PRIMES P.**
//**METHOD: SEE COMMENTS AT THE BEGINNING OF THE SUBROUTINE *GRAPH*.**
//**INPUT: SEVERAL CARDS, EACH AS DESCRIBED IN COMMENTS PRECEDING GRAPHS, BUT WITH "1" IN COLUMN 10. A FINAL CARD WITH "0"**
//**OUTPUT: SEE COMMENTS IN *GRAPH*.**
```

DCL I FIXED DEC; /* '1' IF CARD CONTAINS DATA, ELSE '0' */
GET EDIT (I) (F(10));
 LP1: DO WHILE (I > 0);
   CALL GRAPH*
   CALL EDIT (I) (F(10));
 END [LP1];
END GRAPHS;

**GRAPH**

**PROCEDURE**

```
***GRAPH PROCEDURE***
//**PURPOSE: GIVEN A TERNARY QUADRATIC Z-LATTICE L AND A PRIME P, FIND REPRESENTATIVES L(II,...,L?H) OF EACH CLASS IN THE GRAPH ZIL.PI AND ALSO FIND THE NUMBERS OF NEIGHBORS N(ILII,L(J),P) FOR ALL 1<i<j<h.**
//**METHOD: (1) FIND ALL NEIGHBORS OF L. (2) FIND NEIGHBORS OF EACH NEW CLASS THAT ARISES FROM (1). (3) CONTINUE THIS PROCESS UNTIL NO NEW CLASSES ARISE.**
//**INPUT: ONE CARD WITH COEFFICIENTS OF L IN COLUMNS 12-20, 62-70. THE PRIME P IN COLUMNS 72-80.**
//**OUTPUT: THE EISENSTEIN REDUCED FORM IN EACH CLASS IN ZIL.PI AND THE H BY H MATRIX N(LIL),L(J),P).**
```

DCL (L(6,50)), /* COEFFICIENTS OF LATTICES IN Z(L,P) */
   (P(50,50)), /* A GIVEN PRIME */
   (H), /* NUMBER OF CLASSES IN ZIL.PI */
   (I,J,K,M) FIXED DEC(9), /* SUBSCRIPTS, LOOP COUNTERS */
/* INITIALIZE L(*) AND N(*) TO 0. */
L = 0; N = 0;
/* READ COEFFICIENTS OF L, READ P. */
GET EDIT 10 (F(10));
CALL EDIT 10 (F(10));
BLK: BEGIN;
DCL NL(6,P+1) FIXED DEC(9); /* NEIGHBORS */
L = 1; H = 1;
LP1: DO WHILE (L < H);
   CALL NBHDL((*,I),P,NL); /* FIND NEIGHBORS OF L(*,I). */
   CALL NBHDL((*,J),P,NL); /* DETERMINE CLASS OF EACH NEIGHBOR NL(*,J). */
   SSL: DO J = 1 TO P+1;
   K = 1;
   SSL: DO WHILE (K < H) & (L(1,K) = NL(1,J));
IF \( K = H \) THEN
\[
\text{SSL112:DO} \quad M = H+1; \quad \text{L}(0:H) = \text{NL}(0:H); \quad N(0:H) = 1; \quad \text{END SSL112};
\]
ELSE \( N(I,K) = N(I,K)+1 \);
\[
\text{END SSL111}; \quad I = I+1; \quad \text{END LP1}; \quad \text{END BLK};
\]
/* PRINT RESULTS. */
\[
\text{PUT PAGE EDIT} \quad \{\text{THE GRAPH} \} (X[30], A[14], F[4], A[1]); \quad \text{PUT SKIP(2) EDIT} \quad \{\text{NEIGHBORS} \} (X[30], A[15], F[4], A[1]);
\]
\[
\text{PUT SKIP(2)}; \quad \text{LP2:DO I = 1 TO } H; \quad \text{PUT SKIP(2) EDIT} \quad \{\text{NEIGHBORS} \} (X[30], A[15], F[4], A[1]); \quad \text{END LP2}; \quad \text{PUT SKIP(3)};
\]
/* PRINT MATRIX \( (N(I,J)) \). */
\[
\text{I = 0}; \quad \text{LP3:DO WHILE} \quad (9*I < H); \quad \text{IF } H <= 9 \quad \text{THEN PUT SKIP(3) EDIT} \quad \{\text{NUMBER OF* IN*} \} (A[13]); \quad \text{ELSE PUT PAGE EDIT} \quad \{\text{NUMBER OF* IN*} \} (A[13]); \quad \text{PUT SKIP EDIT} \quad \{\text{NEIGHBORS} \} (A[16]); \quad \text{M = MIN(H-9*I, 9)}; \quad \text{SL31:DO J = 9*I+1 TO 9*I+M; \quad PUT SKIP EDIT} \quad \{\text{NEIGHBORS} \} (X[30], A[15], F[4], A[1]); \quad \text{END SL31}; \quad \text{PUT SKIP EDIT} \quad \{\text{NEIGHBORS} \} (X[30], A[15], F[4], A[1]); \quad \text{PUT SKIP EDIT} \quad \{\text{NEIGHBORS} \} (X[30], A[15], F[4], A[1]); \quad \text{SL32:DO K = 1 TO } H; \quad \text{PUT SKIP(2) EDIT} \quad \{\text{L('+K',*)} \} (X[2], A[2], F[2], A[1]); \quad \text{SSL321:DO J = 9*I TO 9*I+M; \quad PUT EDIT} \quad \{\text{NEIGHBORS} \} (X[30], A[15], F[4], A[1]); \quad \text{END SSL321}; \quad \text{END SL32}; \quad I = I+1; \quad \text{END LP3}; \quad \text{END GRAPH};
NBHD:PROCEDURE(A,P,NL):

    /* PURPOSE: GIVBN THE TERNARY QUADRATIC Z-LATTICE L WITH COEFFICIENTS */
    /* FROM A AND THE PRIME P, FIND THE NEIGHBORS OF L IN Z(L,P). */
    /* METHOD: 1) FIND A VECTOR REPRESENTING EACH ISOTROPIC LINE IN L/PL. */
    /* 2) FOR EACH VECTOR X FROM (1), FIND Y SUCH THAT Y IS IN */
    /*     K + PL AND P*P|Q(Y). REPLACE X BY Y. */
    /* 3) FOR EACH X FROM (2), FIND THE NEIGHBOR L(X/P) + Z(X/P). */
    /* HERE, L(X/P) IS THE SET OF Y IN L WITH B(Y,X/P) IN Z. */

DCL (A(*), P, NL(*,*)) FIXED DEC(9); /* COEFFICIENTS FOR Z-LATTICE L */
    /* A PRIME */
    /* COEFFICIENTS FOR NEIGHBORS OF L */

    DCL (X(3,P+1), I) FIXED DEC(9); /* VECTORS FROM ISOTROPIC LINES */
        /* I SUBSCRIPT, LOOP COUNTER */

    CALL ISOTLN(A,P,X); /* FIND VECTORS FROM ISOTROPIC LINES */

    CALL DIVPSG(A,X(*,I),P); /* CALL DIVPSG(A,X(*,I),P) */
    CALL NHBR(A,X(*,I),P,NL(*,I));

END NBHD;
ISOTLN: PROCEDURE(A, P, X1);

/*******************************************/
/* PURPOSE: GIVEN A 2-LATTICE L (WHOSE COEFFICIENTS ARE GIVEN BY A), */
/* FIND A VECTOR FROM EACH ISOTROPIC LINE IN L/PL. */
/* METHOD: TEST A VECTOR Y (IN L) REPRESENTING EACH LINE IN L/PL TO */
/* SEE WHETHER Q(Y) IS DIVISIBLE BY P. IF SO STORE IN X1(N) */
/* FOR SOME 1< N < P*/
/*******************************************/

DCL (A(*)), /* COEFFICIENTS OF QUADRATIC FORM */ X(*,*) FIXED DEC(9), /* VECTORS REPRESENATING ISOTROPIC LINES */ P /* A PRIME */
DCL (Y(*)3), /*VECTOR TO BE TESTED FOR ISOTROPY */ N FIXED DEC(9), /* NUMBER OF ISOTROPIC LINES FOUND */
DCL QUAD ENTRY(uppercase) FIXED DEC(9), /* FIXED DEC(9) RETURNS FIXED DEC(9) */

/* INITIALIZE N AND Y */
N = 1; Y(1) = 0; Y(2) = 0; Y(3) = 1;

/* CHECK VECTOR Y = (0,0,1) */
IF MOD(QUAD(A,Y),P) = 0 THEN LPI1DO:
   X(*,N) = Y;
   N = N+1;
END LPI1;

/* CHECK VECTORS Y = (0,1,K) WHERE K = 0,1,...,P-1 */
LPI2DO Y(3) = 0 TO P-1
   IF MOD (QUAD(A,Y),P) = 0 THEN SLPI2DO:
      X(*,N) = Y;
      N = N+1;
   END SLPI2;
END LPI2;

/* CHECK VECTORS Y = (1,J,K) WHERE J,K = 0,1,...,P-1 */
Y(1) = 1;
LPI3DO Y(2) = 0 TO P-1
   IF MOD (QUAD(A,Y),P) = 0 THEN SLPI3DO:
      X(*,N) = Y;
      N = N+1;
   END SLPI3;
END LPI3;
END ISOTLN;
DIVPSQ:PROCEDURE(A,X,P);

/***************************************************************************/
/** PURPOSE: GIVEN A VECTOR X IN A QUADRATIC Z-LATTICE L WITH Q(X) */
/** DIVISIBLE BY THE PRIME P, FIND A VECTOR X* SUCH THAT X-X* */
/** IS IN P*L AND Q(X*) IS DIVISIBLE BY P*P. REPLACE X BY X*. */
/** METHOD: IF Q(X) IS ALREADY DIVISIBLE BY P*P, RETURN X. */
/** IF NOT, FIND I=1<3 SUCH THAT B(X,E(I)) IS NOT DIVISIBLE */
/** BY P, AND X(J)=1 FOR SOME J=1. THEN, FIND N SUCH THAT */
/** Q(X + N*E(I)) IS DIVISIBLE BY P*P (SUCH AN N EXISTS). */
/** REPLACE X BY X + N*E(I). */
/***************************************************************************/

DCL A(*), /* COEFFICIENTS OF QUADRATIC FORM */
    X(*) FIXED DEC, /* VECTOR WITH P|Q(X) */
    P FIXED DEC, /* A PRIME */

DCL E(3,3), /* BASIS VECTORS FOR LATTICE L */
    J, I FIXED DEC, /* SUBSCRIPTS, LOOP COUNTERS */

DCL QUAD ENTRY(1) FIXED DEC, /* QUADRATIC FORM */
    B1L ENTRY(1,1) FIXED DEC, /* B1L FORM */

/* CHECK WHETHER P|Q(X). */
IF MOD(QUAD(A*X),P*P) = 0
    THEN RETURN;
/* SET E(1,1) = (1,0,0), E(2,2) = (0,1,0), AND E(3,3) = (0,0,1). */
LP1:
    SET E(1,1) = (1,0,0), E(2,2) = (0,1,0), AND E(3,3) = (0,0,1);
    IF J = 1 TO 3;
        THEN E(I,J) = 1;
        ELSE E(I,J) = 0;
    END SL11;
    END LP1;
/* FIND X* IN X + P*L SUCH THAT P*P|Q(X*). */
IF (MOD(BIL(A,X,E(1,1)),P) = 0) AND (X(2) = 1) AND (X(3) = 1)
    THEN LP2:
        IF (MOD(QUAD(A,X,P*P)) = 0)
            THEN X(1) = X(1)*P;
        END LP2;
    ELSE IF (MOD(BIL(A,X,E(2,2)),P) = 0) AND (X(1) = 1) AND (X(3) = 1)
        THEN LP3:
            IF (MOD(QUAD(A,X,P*P)) = 0)
                THEN X(2) = X(2)*P;
            END LP3;
        ELSE IF (MOD(BIL(A,X,E(3,3)),P) = 0) AND (X(1) = 1) AND (X(2) = 1)
            THEN LP4:
                IF (MOD(QUAD(A,X,P*P)) = 0)
                    THEN X(3) = X(3)*P;
                END LP4;
        ELSE LP5:
            PUT SKIP LIST('IMPOSSIBLE TO FIND X IN LINE WITH P*P|Q(X)*');
            STOP;
        END LP5;
    END LP3;
END LP2;
END DIVPSQ;
PROCEDURE NHBRPROCEDURE(A,X,P,NL); /* PURPOSE: GIVEN A VECTOR X IN THE TERNARY QUADRATIC 2-LATTICE L WITH */ /* COEFFICIENTS GIVEN BY A, SUCH THAT P*P DIVIDES D(X), FIND */ /* THE COEFFICIENTS FOR THE NEIGHBOR NL = L(X/P) + Z[X/P]. */ /* HERE, L(X/P) IS THE SET OF Y IN L WITH B(Y,X/P) IN Z. */ /* METHOD: CHOOSE A PERMUTATION (I,J,K) OF (1,2,3) SUCH THAT X(I)=1. */ /* {1} IF P*B[(X,E[J])], NL HAS Z BASIS {X/P,E[J],P*X[K]}. */ /* {2} IF P*B[(X,E[K])], NL HAS Z BASIS {X/P,E[I],P*X[J]}. */ /* {3} OTHERWISE, CHOOSE N SO THAT N*B[(X,E(I))]-1 MOD P. THEN, NL HAS Z-BASIS */ /* {X/P,P*X[J],E(K) N*B[(X,E[K])]*E(I)}. */

DCL A(*); /* COEFFICIENTS FOR LATTICE L */ DCL P; /* A PRIME */ DCL NL(*) FIXED DEC(9); /* COEFFICIENTS FOR NEIGHBOR OF L */

DCL (E(3,3), /* BASIS VECTORS FOR L */ N, /* INVERSE OF B[(X,E(1)=J)] IN Z/P */ M, /* N*B[(X,E(1)=K)] REDUCED MOD P */ I,J,K) FIXED DEC(9); /* SUBSCRIPTS, LOOP COUNTERS */

DCL QUAD ENTRY(*,*) FIXED DEC(9),(*,*) FIXED DEC(9) RETURNS(FIXED DEC(9))
DCL BIL ENTRY(*,*) FIXED DEC(9),(*,*) FIXED DEC(9) RETURNS(FIXED DEC(9))

/ * FIX E(I,J),S. */ LP1: DD I = 1 TO 3;
 LP2: DD J = 1 TO 3;
 IF I = J
   THEN E(I:J) = 1;
 ELSE E(I:J) = 0;
 END LP2;
 END LP1;

/ * DETERMINE I,J,AND K. */ IF X(I)
   THEN LP2: DD I = 1;
END
ELSE IF \( x(2) = 1 \)
THEN LP3: DO
   \( J = 2 \);
   \( K = 3 \);
   END LP3;
ELSE IF \( x(3) = 1 \)
THEN LP4: DO
   \( J = 2 \);
   \( K = 3 \);
   END LP4;
ELSE LP5: DO;
   PUT SKIP EDIT('**x, y** REJECTED**');
   {A[1], B[10], A[11]};
   PUT SKIP LIST ('**ALL COORDINATES == 1**');
   STOP;
END LP5;

/* CONSTRUCT NEIGHBOR. */
IF MOD(BIL(A, X, E(*, J))), P) = 0
THEN LP6: DO;
   {NL(1): = QUAD(A, X)/(P*P);}
   {NL(2): = A(J);}
   {NL(3): = P*P*AX;X;E(*, J));}
   {NL(4): = P*BIL(A, X, E(*, J)));}
   {NL(5): = BIL(A, X, E(*, J)));}
   {NL(6): = BIL(A, X, E(*, J)));}
   END LP6;
ELSE IF MOD(BIL(A, X, E(*, J))), P) = 0
THEN LP7: DO;
   {NL(1): = QUAD(A, X)/(P*P);}
   {NL(2): = A(K);}
   {NL(3): = P*P*AX;X;E(*, K));}
   {NL(4): = P*BIL(A, X, E(*, K)));}
   {NL(5): = BIL(A, X, E(*, K)));}
   {NL(6): = BIL(A, X, E(*, K)));}
   END LP7;
ELSE LP8: DO;
   N = 1;
   SL8: DO WHILE (MOD(N*BIL(A, X, E(*, J))), P) == 1):
      N = N + 1;
   END SL8;
   M = MOD(N*BIL(A, X, E(*, K))), P);
   {NL(1): = QUAD(A, X)/(P*P);}
   {NL(2): = P*P*AX;X;E(*, J));}
   {NL(3): = A(K) + M*BIL(A, X, E(*, J));}
   {NL(4): = P*BIL(A, X, E(*, K)));}
   {NL(5): = BIL(A, X, E(*, K)));}
   {NL(6): = BIL(A, X, E(*, J)));}
   CALL EISRED(NL);
   END LP8;
CALL EISRED(NL);
END NHBR;
EISRED: PROCEDURE (A);

******************************************************************************

# PURPOSE: GIVEN A TERNARY QUADRATIC FORM
# F(X,Y,Z) = A(1)*X^2 + A(2)*Y^2 + A(3)*Z^2
# + A(4)*X*Y + A(5)*X*Z + A(6)*Y*Z
# TO FIND THE EQUIVALENT EISENSTEIN REDUCED FORM.
#
# METHOD: THE FORM IS CHECKED FOR CONFORMITY TO EACH CONDITION FROM
# JONES'S "ARITHMETIC THEORY OF QUADRATIC FORMS", P. 180.
# IF IT FAILS TO SATISFY A CONDITION, IT IS REPLACED BY AN
# EQUIVALENT FORM THAT DOES SATISFY THAT CONDITION. THEN,
# THE NEW FORM IS CHECKED TO SEE IF IT IS EISENSTEIN REDUCED.
# THIS PROCESS IS CONTINUED UNTIL A REDUCED FORM IS OBTAINED.
#
******************************************************************************

DCL A(*) FIXED DEC(9); /* COEFFICIENTS OF QUADRATIC FORM */
DCL X FIXED DEC(9); /* TEMPORARY VBL FOR EXCHANGES, ETC. */
DCL STATE CHAR(10) VAR; /* 'REDUCED' OF 'UNREDUCED' */

/* CHECK THAT FORM IS POSITIVE DEFINITE. IF NOT, PRINT ERROR MESSAGE AND RETURN. */
IF (A(1)<0) OR (6*A(1)*A(2)-A(4)*A(6)<0)
AND (4*A(1)*A(2)+A(4)+A(5)+A(6)-A(4)*A(4)+A(1)+A(5)*A(5)*A(2)-A(6)*A(6)+A(3)<0)
THEN LPO: DO:
   PUT SKIP(2) EDIT ('* (A,1) NOT POSITIVE DEFINITE')
   (A(1), 6 F(10), A(2))
   RETURN;
END LPO;

/* INITIALIZE STATE */
STATE = 'UNREDUCED';
MAINLP: DO WHILE (STATE = 'UNREDUCED');
   STATE = 'REDUCED';
   /* CHECK CONDITION (1). */
IF \((A(4) \geq OCA(5) > OCA(6) < OCA(14) = 0\mid A(5) = 0)\) THEN LP1:
STATE = "UNREDUCED";
\(A(4) = -A(4);\)
\(A(5) = -A(5);\)
END LP1;

IF \((A(4) > OCA(6) < OCA(14) = 0\mid A(5) = 0)\) THEN LP2:
STATE = "UNREDUCED";
\(A(6) = -A(6);\)
END LP2;

IF \((A(5) > OCA(6) < OCA(14) = 0\mid A(6) = 0)\) THEN LP3:
STATE = "UNREDUCED";
\(A(5) = -A(5);\)
END LP3;

/* CHECK CONDITION (2). */

IF \((A(1) > A(2))\) THEN LP4:
STATE = "UNREDUCED";
\(X = A(1);\)
\(A(1) = A(2);\)
\(A(2) = X;\)
\(X = A(4);\)
\(A(4) = A(5);\)
\(A(5) = X;\)
END LP4;

IF \((A(2) > A(3))\) THEN LP5:
STATE = "UNREDUCED";
\(X = A(2);\)
\(A(2) = A(3);\)
\(A(3) = X;\)
\(X = A(5);\)
\(A(5) = A(6);\)
\(A(6) = X;\)
END LP5;

LP6: DO WHILE \((A(1) + A(2) + A(4) + A(5) + A(6) < 0)\)
STATE = "UNREDUCED";
\(A(3) = A(3) + A(1) + A(2) + A(4) + A(5) + A(6);\)
\(A(4) = A(4) + A(6) + 2*A(2);\)
\(A(5) = A(5) + A(6) + 2*A(1);\)
END LP6;
/* CHECK CONDITION 3. */

IF A(1) < ABS(A(5))
THEN LP1DO;
    STATE = 'UNREDUCED';
    X = FLOOR(A(5)/(2*A(1)*0.5));
    A(3) = A(2) + A(1)*X*X - A(5)*X;
    A(4) = A(3) + A(2)*X;
    A(5) = A(5) - 2*A(1)*X;
END LP1;

IF A(1) < ABS(A(6))
THEN LP8DO;
    STATE = 'UNREDUCED';
    X = FLOOR(A(6)/(2*A(1)*0.5));
    A(3) = A(2) + A(1)*X*X - A(6)*X;
    A(4) = A(4) - A(5)*X;
    A(5) = A(6) - 2*A(1)*X;
END LP8;

IF A(2) < ABS(A(4))
THEN LP9DO;
    STATE = 'UNREDUCED';
    X = FLOOR(A(4)/(2*A(2)*0.5));
    A(3) = A(3) + A(2)*X*X - A(4)*X;
    A(4) = A(4) + A(6)*X;
    A(5) = A(5) - 2*A(2)*X;
END LP9;

/* CHECK CONDITION 4. */

IF A(1) + A(2) + ABS(A(4)) > ABS(A(5))
THEN LP1DO;
    STATE = 'UNREDUCED';
    A(1) = A(4);
    A(5) = X;
END LP1;

IF A(2) + A(3) + ABS(A(5)) > ABS(A(6))
THEN LP1DO;
    STATE = 'UNREDUCED';
    A(2) = A(5);
    A(6) = X;
END LP1;

IF (A(1) + A(2)) + (A(4) + A(5) + A(6)) = 0
THEN LP1DO;
    STATE = 'UNREDUCED';
    A(4) = A(6) + A(2);
    A(5) = A(5) + A(6) + 2*A(1);
END LP1;

/* CHECK CONDITION 5. */
IF \( \{ A(4) \leq OCA(5) \leq OCA(6) \leq OCA(1) \leq A(6) \leq A(5) \leq 0 \} \)
THEN LP13:DDI:
STATE = 'UNREDUCED':
\[ A(4) = -A(4) - A(5); \]
\[ A(5) = -A(5); \]
\[ A(6) = -A(6); \]
END LP13;

IF \( \{ A(4) \leq OCA(5) \leq OCA(6) \leq OCA(1) \leq A(5) \leq A(6) \leq 0 \} \)
THEN LP14:DDI:
STATE = 'UNREDUCED':
\[ A(4) = -A(4) - A(6); \]
\[ A(5) = -A(5); \]
\[ A(6) = -A(6); \]
END LP14;

IF \( \{ A(4) \leq OCA(5) \leq OCA(6) \leq OCA(2) \leq A(4) \leq A(6) \leq 0 \} \)
THEN LP15:DDI:
STATE = 'UNREDUCED':
\[ A(4) = -A(4) - A(5); \]
\[ A(5) = -A(5); \]
\[ A(6) = -A(6); \]
END LP15;

/* CHECK CONDITION (6). */

IF \( \{ A(4) \leq OCA(6) \leq OCA(5) \leq OCA(1) \leq A(6) \leq (A(5) \geq 2A(4)) \} \)
THEN LP16:DDI:
STATE = 'UNREDUCED':
\[ A(4) = -A(4) - A(5); \]
\[ A(5) = -A(5); \]
\[ A(6) = -A(6); \]
END LP16;

IF \( \{ A(4) \leq OCA(5) \leq OCA(6) \leq OCA(1) \leq A(5) \leq A(6) \leq 2A(4) \} \)
THEN LP17:DDI:
STATE = 'UNREDUCED':
\[ A(4) = A(5) - A(4); \]
\[ A(5) = A(6) - A(4); \]
\[ A(6) = A(6) - A(4); \]
END LP17;

IF \( \{ A(4) \leq OCA(5) \leq OCA(6) \leq OCA(2) \leq A(4) \leq A(6) \leq 2A(5) \} \)
THEN LP18:DDI:
STATE = 'UNREDUCED':
\[ A(5) = A(6) - A(5); \]
\[ A(6) = A(6) - A(5); \]
END LP18;

END MAINLP;
END EISRED:
QUAD: PROCEDURE(A,X) RETURNS(FIXED DEC(9));

DCL (A(*)), X(*) FIXED DEC(9); /* COEFFICIENTS OF QUADRATIC FORM */
DCL QX FIXED DEC(9); /* Q(X) */
QX = A(1)*X(1)*X(1) + A(2)*X(2)*X(2) + A(3)*X(3)*X(3) + A(4)*X(1)*X(2);
RETURN (QX);
END QUAD;

BIL: PROCEDURE(A,X,Y) RETURNS(FIXED DEC(9));

DCL (A(*), X(*), Y(*)) FIXED DEC(9); /* VECTORS */
DCL BXY FIXED DEC(9); /* B(X,Y) */
BXY = 2*A(1)*X(1)*Y(1) + 2*A(2)*X(1)*Y(2) + 2*A(3)*X(1)*Y(3) + A(2)*X(2)*Y(2) + A(3)*X(2)*Y(3) + A(4)*X(1)*Y(2);
RETURN (BXY);
END BIL;
APPENDIX B

SOME COMPUTER-GENERATED GRAPHS
FROM CHAPTER III

The following table contains some computer-generated information on the graphs mentioned in Example 3.16, which should clarify the division into spinor genera of the genus described there. This information was obtained using the computer program given in Appendix A on the Amdahl 470 at The Ohio State University Instructional and Research Computer Center.
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NUMBER OF $v$ IN NEIGHBORS OF $l$ OF $v$ OF $l$ of $l$.

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</thead>
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<table>
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</tr>
</thead>
</table>

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<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
</table>
### The Graph Z(L, 11)

<table>
<thead>
<tr>
<th>L</th>
<th>4</th>
<th>5</th>
<th>400</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
</table>

### Classes in Z(L, 11)

<table>
<thead>
<tr>
<th>L(i)</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>L(1)</td>
<td>4</td>
<td>5</td>
<td>400</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L(2)</td>
<td>16</td>
<td>20</td>
<td>25</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L(3)</td>
<td>4</td>
<td>20</td>
<td>105</td>
<td>-20</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L(4)</td>
<td>4</td>
<td>21</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>L(5)</td>
<td>1</td>
<td>80</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L(6)</td>
<td>16</td>
<td>20</td>
<td>29</td>
<td>0</td>
<td>-16</td>
<td>0</td>
</tr>
</tbody>
</table>

### Number of* in Neighbors* per Class

<table>
<thead>
<tr>
<th>V</th>
<th>L(1)</th>
<th>L(2)</th>
<th>L(3)</th>
<th>L(4)</th>
<th>L(5)</th>
<th>L(6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L(1)</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L(2)</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>L(3)</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>L(4)</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>L(5)</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L(6)</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
### The Graph $Z(L, 3)$

$L = (16, 20, 25, 0, 0, 0) \times 3$

#### Classes in $Z(L, 3)$

<table>
<thead>
<tr>
<th></th>
<th>$L(1)$</th>
<th>$L(2)$</th>
<th>$L(3)$</th>
<th>$L(4)$</th>
<th>$L(5)$</th>
<th>$L(6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>16</td>
<td>25</td>
<td>21</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>100</td>
<td>80</td>
<td>100</td>
<td>105</td>
<td>101</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-20</td>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

#### Number of Neighbors *Class*

<table>
<thead>
<tr>
<th></th>
<th>$L(1)$</th>
<th>$L(2)$</th>
<th>$L(3)$</th>
<th>$L(4)$</th>
<th>$L(5)$</th>
<th>$L(6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
### THE GRAPH ZIL, 11

\[
L = (1 \quad 20 \quad 400 \quad 0 \quad 0 \quad 0)
\]

### CLASSES IN ZIL, 11

<table>
<thead>
<tr>
<th>L(1)</th>
<th>L(2)</th>
<th>L(3)</th>
<th>L(4)</th>
<th>L(5)</th>
<th>L(6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>25</td>
<td>16</td>
<td>9</td>
<td>45</td>
</tr>
<tr>
<td>400</td>
<td>101</td>
<td>80</td>
<td>100</td>
<td>100</td>
<td>45</td>
</tr>
</tbody>
</table>

### NUMBER OF NEIGHBORS IN CLASS OF

<table>
<thead>
<tr>
<th>V</th>
<th>L(1)</th>
<th>L(2)</th>
<th>L(3)</th>
<th>L(4)</th>
<th>L(5)</th>
<th>L(6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L(1)</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L(2)</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>L(3)</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>L(4)</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>L(5)</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L(6)</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
APPENDIX C

GRAPHS USED TO SPLIT THE GENERA IN CHAPTER IV INTO SPINOR GENERA

The following pages contain computer-generated information on the graphs used in Proposition 4.2, 4.4, and 4.6 to split the genera into spinor genera. These were produced by the Amdahl 470 at The Ohio State University Instructional and Research Computer Center using the program in Appendix A.
### The Graph $Z(L, 2)$

$L = (1, 1, 27, 0, 0, -1)$

**Classes in $Z(L, 2)$**

<table>
<thead>
<tr>
<th>L(1) =</th>
<th>L(2) =</th>
<th>L(3) =</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1, 27, 0, 0, -1</td>
<td>1, 3, 7, 0, -1, 0</td>
<td>3, 3, 4, 3, 3, 3</td>
</tr>
</tbody>
</table>

#### Number of Neighbors in Classes

$\begin{array}{ccc}
\text{v} & \text{L(1)} & \text{L(2)} & \text{L(3)} \\
\text{L(1)} & 0 & 3 & 0 \\
\text{L(2)} & 1 & 0 & 2 \\
\text{L(3)} & 0 & 3 & 0 \\
\end{array}$

### The Graph $Z(L, 3)$

$L = (1, 1, 32, 0, 0, 0)$

**Classes in $Z(L, 3)$**

<table>
<thead>
<tr>
<th>L(1) =</th>
<th>L(2) =</th>
<th>L(3) =</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1, 32, 0, 0, 0</td>
<td>1, 4, 9, -4, 0, 0</td>
<td>2, 2, 9, -2, -2, 0</td>
</tr>
</tbody>
</table>

#### Number of Neighbors in Classes

$\begin{array}{ccc}
\text{v} & \text{L(1)} & \text{L(2)} & \text{L(3)} \\
\text{L(1)} & 0 & 4 & 0 \\
\text{L(2)} & 2 & 0 & 2 \\
\text{L(3)} & 0 & 4 & 0 \\
\end{array}$
## The Graph $Z(L, 2)$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>7</th>
<th>9</th>
<th>0</th>
<th>0</th>
<th>-1</th>
</tr>
</thead>
</table>

### Classes in $Z(L, 2)$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>7</th>
<th>9</th>
<th>0</th>
<th>0</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>L(1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L(2)</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>L(3)</td>
<td>1</td>
<td>9</td>
<td>9</td>
<td>-9</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### Number of Neighbors in Class

```
V
L(1)  0  2  1
L(2)  3  0  0
L(3)  3  0  0
```

## The Graph $Z(L, 5)$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>108</th>
<th>0</th>
<th>0</th>
<th>-1</th>
</tr>
</thead>
</table>

### Classes in $Z(L, 5)$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>108</th>
<th>0</th>
<th>0</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>L(1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L(2)</td>
<td>1</td>
<td>7</td>
<td>12</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>L(3)</td>
<td>3</td>
<td>3</td>
<td>13</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

### Number of Neighbors in Class

```
V
L(1)  0  6  0
L(2)  2  0  4
L(3)  0  6  0
```
THE GRAPH Z(L, 5)

\[ L = (1, 1, 7, 36, 0, 0, -1) \]

CLASSES IN Z(L, 5)

\[
\begin{align*}
L(1) &= (1, 7, 36, 0, 0, -1) \\
L(2) &= (4, 9, 9, -9, 0, 0) \\
L(3) &= (7, 7, 7, 5, 5, 5)
\end{align*}
\]

NUMBER OF* IN NEIGHBORS *CLASS OF \*OF \*OF \*OF \*OF \*OF = L(1) L(2) L(3)

\[
\begin{align*}
V L(1) &= 0, 2, 4 \\
L(2) &= 6, 0, 0 \\
L(3) &= 6, 0, 0
\end{align*}
\]

THE GRAPH Z(L, 2)

\[ L = (1, 1, 63, 0, 0, -1) \]

CLASSES IN Z(L, 2)

\[
\begin{align*}
L(1) &= (1, 1, 63, 0, 0, -1) \\
L(2) &= (1, 3, 16, 0, -1, 0) \\
L(3) &= (3, 4, 4, -1, 0, 0) \\
L(4) &= (3, 3, 7, 0, 0, -3)
\end{align*}
\]

NUMBER OF* IN NEIGHBORS *CLASS OF \*OF \*OF \*OF \*OF \*OF = L(1) L(2) L(3) L(4)

\[
\begin{align*}
V L(1) &= 0, 3, 0, 0 \\
L(2) &= 1, 0, 2, 0 \\
L(3) &= 0, 2, 0, 1 \\
L(4) &= 0, 0, 3, 0
\end{align*}
\]
### The Graph \( Z(L, 3) \)

**\( L = \) ( 1 1 64 0 0 0 )**

**Classes in \( Z(L, 3) \)**

<table>
<thead>
<tr>
<th>( L(1) )</th>
<th>1</th>
<th>1</th>
<th>64</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L(2) )</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>-4</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>( L(3) )</td>
<td>1</td>
<td>4</td>
<td>17</td>
<td>-4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( L(4) )</td>
<td>2</td>
<td>2</td>
<td>17</td>
<td>-2</td>
<td>-2</td>
<td>0</td>
</tr>
</tbody>
</table>

### Number of Neighbors of Class \( V \)

**\( \) ( 1 ) ( 2 ) ( 3 ) ( 4 )**

| \( L(1) \) | 0 | 4 | 0 | 0 |
| \( L(2) \) | 1 | 0 | 2 | 1 |
| \( L(3) \) | 0 | 4 | 0 | 0 |
| \( L(4) \) | 0 | 4 | 0 | 0 |

### The Graph \( Z(L, 3) \)

**\( L = \) ( 1 1 80 0 0 0 )**

**Classes in \( Z(L, 3) \)**

| \( L(1) \) | 1 | 1 | 80 | 0 | 0 | 0 |
| \( L(2) \) | 1 | 9 | 9  | -2| 0 | 0 |
| \( L(3) \) | 4 | 5 | 5  | 0 | 0 | -4|
| \( L(4) \) | 2 | 2 | 21 | -2| -2| 0 |

### Number of Neighbors of Class \( V \)

**\( \) ( 1 ) ( 2 ) ( 3 ) ( 4 )**

| \( L(1) \) | 0 | 4 | 0 | 0 |
| \( L(2) \) | 2 | 0 | 2 | 0 |
| \( L(3) \) | 0 | 2 | 0 | 2 |
| \( L(4) \) | 0 | 0 | 4 | 0 |
### THE GRAPH $Z(L, 5)$

$L = (1, 1, 3, 28, -3, -1, 0)$

#### CLASSES IN $Z(L, 5)$

| L(1) | 1 | 3 | 28 | -3 | -1 | 0 |
| L(2) | 4 | 4 | 7  | 1  | 2  | 4 |
| L(3) | 1 | 10| 10 | 8  | 1  | 1 |
| L(4) | 3 | 4 | 9  | -3 | 0  | -3 |

### NUMBER OF NEIGHBORS IN EACH CLASS

| L(1) | 0 | 4 | 2 | 0 |
| L(2) | 2 | 0 | 0 | 4 |
| L(3) | 2 | 0 | 0 | 4 |
| L(4) | 0 | 4 | 2 | 0 |

### THE GRAPH $Z(L, 5)$

$L = (1, 1, 1, 144, 0, 0, -1)$

#### CLASSES IN $Z(L, 5)$

| L(1) | 1 | 1 | 144 | 0 | 0 | -1 |
| L(2) | 3 | 7 | 7  | 5 | 3 | 3 |
| L(3) | 1 | 3 | 37 | -3 | -1 | 0 |
| L(4) | 3 | 3 | 16 | 0  | 0  | -3 |

### NUMBER OF NEIGHBORS IN EACH CLASS

| L(1) | 0 | 6 | 0  | 0 |
| L(2) | 2 | 0 | 4  | 0 |
| L(3) | 0 | 4 | 0  | 2 |
| L(4) | 0 | 0 | 6  | 0 |
THE GRAPH $Z(L, 3)$

$L = (1 \quad 1 \quad 2 \quad 64 \quad 0 \quad 0 \quad 0)$

CLASSES IN $Z(L, 3)$

$L(1) = (1 \quad 1 \quad 2 \quad 64 \quad 0 \quad 0 \quad 0)$
$L(2) = (1 \quad 1 \quad 8 \quad 18 \quad -8 \quad 0 \quad 0)$

NUMBER OF IN
NEIGHBORS $\leftrightarrow$ CLASSES

$V \rightarrow L(1) \rightarrow L(2)$

$L(1) \quad 2 \quad 2$
$L(2) \quad 2 \quad 2$

THE GRAPH $Z(L, 5)$

$L = (1 \quad 1 \quad 9 \quad 17 \quad -6 \quad 0 \quad 0)$

CLASSES IN $Z(L, 5)$

$L(1) = (1 \quad 9 \quad 17 \quad -6 \quad 0 \quad 0)$
$L(2) = (2 \quad 5 \quad 16 \quad 0 \quad 0 \quad -2)$

NUMBER OF IN
NEIGHBORS $\leftrightarrow$ CLASSES

$V \rightarrow L(1) \rightarrow L(2)$

$L(1) \quad 4 \quad 2$
$L(2) \quad 4 \quad 2$
THE GRAPH \( z(L, 5) \)

\[
L = \begin{pmatrix}
1 & 1 & 144 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

CLASSES IN \( z(L, 5) \)

\[
L(1) = \begin{pmatrix}
1 & 1 & 144 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
L(2) = \begin{pmatrix}
1 & 9 & 16 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

NUMBER OF * IN NEIGHBORS OF CLASS

OF \( \rightarrow L(1) \) \( L(2) \)

\[
\begin{array}{c|c|c}
L(1) & 2 & 4 \\
L(2) & 2 & 4 \\
\end{array}
\]

THE GRAPH \( z(L, 5) \)

\[
L = \begin{pmatrix}
1 & 1 & 252 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

CLASSES IN \( z(L, 5) \)

\[
L(1) = \begin{pmatrix}
1 & 1 & 252 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
L(2) = \begin{pmatrix}
1 & 12 & 19 & -12 & -1 & 0 \\
7 & 7 & 7 & 5 & 7 & 7
\end{pmatrix}
\]

\[
L(3) = \begin{pmatrix}
3 & 3 & 28 & 0 & 0 & -3 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

NUMBER OF * IN NEIGHBORS OF CLASS

OF \( \rightarrow L(1) \) \( L(2) \) \( L(3) \) \( L(4) \)

\[
\begin{array}{c|c|c|c|c}
L(1) & 0 & 6 & 0 & 0 \\
L(2) & 2 & 0 & 4 & 0 \\
L(3) & 0 & 4 & 0 & 2 \\
L(4) & 0 & 0 & 6 & 0 \\
\end{array}
\]
### The Graph $Z(\mathcal{L}_5)$

\[ L = (1, 4, 9, 10, -9, -4, 0) \]

#### Classes in $Z(\mathcal{L}_5)$

<table>
<thead>
<tr>
<th>$L(1)$</th>
<th>$L(2)$</th>
<th>$L(3)$</th>
<th>$L(4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

#### Number of Neighbors of Class

\[ V \]

<table>
<thead>
<tr>
<th>$L(1)$</th>
<th>$L(2)$</th>
<th>$L(3)$</th>
<th>$L(4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

### The Graph $Z(\mathcal{L}_5)$

\[ L = (1, 48, 144, 0, 0, 0) \]

#### Classes in $Z(\mathcal{L}_5)$

<table>
<thead>
<tr>
<th>$L(1)$</th>
<th>$L(2)$</th>
<th>$L(3)$</th>
<th>$L(4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>48</td>
<td>144</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>25</td>
<td>25</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>48</td>
<td>49</td>
<td>-48</td>
</tr>
<tr>
<td>16</td>
<td>16</td>
<td>16</td>
<td>0</td>
</tr>
</tbody>
</table>

#### Number of Neighbors of Class

\[ V \]

<table>
<thead>
<tr>
<th>$L(1)$</th>
<th>$L(2)$</th>
<th>$L(3)$</th>
<th>$L(4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
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<tr>
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<td>0</td>
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</tbody>
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BIBLIOGRAPHY


