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ERROR ESTIMATES FOR GAUSS-JACOBI QUADRATURE FORMULA AND PADE APPROXIMANTS OF STIELTJES SERIES

The Ohio State University

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ERROR ESTIMATES FOR GAUSS-JACOBI QUADRATURE FORMULA
AND PADE APPROXIMANTS OF STIELTJES SERIES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in Mathematics
at The Ohio State University

By
Radwan Abdul-Rahman Al-Jarrah, B.Sc., M.Sc.

* * * * *

The Ohio State University

1980

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To My Mother and Father
ACKNOWLEDGMENTS

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CHAPTER 0

INTRODUCTION

Throughout this thesis, we will be referring to systems of orthogonal polynomials and some of their properties. Thus, we start by describing briefly their definition and some properties (for more details see e.g., [9;CHI], [30;CHII]).

Let $\alpha$ be a non-negative measure having the whole real line or a part of the real line as its support. Let the support of $\alpha$ contain infinitely many points and let the improper Stieltjes integrals

$$\int_{-\infty}^{\infty} x^n \alpha(x) \quad n = 0,1,2,...$$

all exist.

Then, it is well known that there exists a uniquely determined sequence of orthogonal (normalized in this case) polynomials $\{p_n(\alpha;x)\}$ with respect to this measure; these polynomials are determined by the properties:

(a) $p_n(\alpha;x) = \gamma_n(\alpha)x^n + ...$ is a polynomial of exact degree $n$ and $\gamma_n(\alpha) = \gamma_n > 0$.

(b) $\sum_{-\infty}^{\infty} p_n(\alpha)p_m(\alpha) = \delta_{nm}$, where $\delta_{nm}$ is the Kronecker symbol:

$$\delta_{mn} = 0 \text{ if } m \neq n \text{ and } \delta_{nn} = 1 \text{ if } m = n.$$
It is also known that all zeros $x_{kn}(d\alpha) = x_{kn}$, $k = 1, 2, \ldots, n$, of $P_n(d\alpha;x)$ are real, simple and are contained in the smallest interval overlapping the support of $d\alpha$. We shall assume, as usual, that $x_1 > x_2 > \ldots > x_n$.

If $d\alpha$ is an absolutely continuous measure, then $d\alpha(x) = \alpha'(x) dx$ and $\alpha'(x)$ is called a weight function. In this case, $\alpha'(x)$ will be denoted by $w(x)$ and $P_n(d\alpha)$ by $P_n(w)$.

The work in this thesis consists of two parts.

In part I, we estimate the error in approximating the integral

$$\int_{-\infty}^{\infty} f(x) d\alpha(x)$$

by the Gauss-Jacobi quadrature formula, assuming that $f$ is an entire function and $d\alpha$ is an absolutely continuous measure satisfying certain growth conditions.

The second chapter of this part deals with the special weights $e^{-x^2}$, $e^{-x^4}$, $e^{-x^6}$ and the Pollaczeck weight $|\Gamma(\lambda + ix)|^2$ ($\lambda > 0$).

In part II, we estimate the error in approximating the Stieltjes transform

$$\int_a^b \frac{1}{l - \frac{1}{z} x} d\alpha(x)$$

by the $[n,n+j]$ ($j \geq -1$) Padé approximants, where $z$ is in the complex plane cut along the interval $[a,b]$ ($-\infty < a < b < \infty$). In one case of this approximation, we assume that $b < \infty$ and in the other we assume that $b = \infty$. 
PART I

ERROR ESTIMATES FOR GAUSS-JACOBI

QUADRATURE FORMULA WITH WEIGHTS HAVING

THE WHOLE REAL LINE AS THEIR SUPPORT
CHAPTER I.1

ERROR ESTIMATES FOR ENTIRE INTEGRANDS

§I.1.1. Mechanical quadrature and the Gauss-Jacobi quadrature formula.

Let \([a, b]\) be a finite or infinite interval, and let
\[ S_n: x_1 < x_2 < ... < x_n, \quad a < x_1, x_n < b \]
be the set of \(n\) distinct points in \([a, b]\). Furthermore, let \(\Lambda_n: \lambda_1, \lambda_2, ..., \lambda_n\) be a set of real numbers. If \(f(x)\) is an arbitrary function defined in \([a, b]\), we write
\[
Q_n(f) = \sum_{\nu=1}^{n} \lambda_\nu f(x_\nu).
\]

We call the numbers \(x_\nu\) the abscissas and the numbers \(\lambda_\nu\) the Cotes numbers of the "mechanical quadrature" \(Q_n(f)\).

An important special case is the following. Let the set \(\{S_n\}\) be an arbitrary set of distinct numbers in \([a, b]\), and let \(\alpha(x)\) be a given non-decreasing function. We define the Cotes numbers \(\lambda_\nu\) by requiring that \(Q_n(d\alpha;f) = \int_a^b f(x)d\alpha(x)\) shall hold if \(f(x)\) is an arbitrary \(\pi_{n-1}\). Obviously, under such a condition

* Throughout this thesis, \(\pi_n\) will denote an arbitrary polynomial of degree less than or equal to \(n\).
\[ \lambda_{vn} = \int_{a}^{b} \lambda_{vn}(x) d\alpha(x) \; ; \; v = 1, 2, \ldots, n \; ; \text{where} \; \lambda_{vn}(x) \text{ denote the fundamental polynomials of the Lagrange interpolation corresponding to the abscissas } \{s_n\}, i.e., \text{if } \pi_n \text{ is not identically zero, vanishing at} \]
\[ x = x_{vn} \; , \; v = 1, 2, \ldots, n \; , \text{then} \; \lambda_{vn}(x) = \frac{\pi_n(x)}{\pi'_n(x_{vn})(x-x_{vn})}, \; v = 1, 2, \ldots, n. \]

In this case, \( Q_n(d\alpha;f) \) is called a quadrature of the interpolatory type.

The Gauss-Jacobi quadrature formula (GJQF), is defined by the interpolatory quadrature formula

\[ (I.1.1.1) \quad Q_n(d\alpha;f) = \sum_{k=1}^{n} \lambda_n(d\alpha;x_{kn})f(x_{kn}) \]

and it has the property that for every \( \pi_{2n-1} \) we have \( Q_n(d\alpha;\pi_{2n-1}) = \sum_{k=1}^{n} \pi_{2n-1}(x) d\alpha(x) \), see e.g., [9; chapter I], [30; chapter III].

The GJQF is a special case of the interpolatory type when the abscissas \( \{s_n\} \) are the zeros of the orthogonal polynomials \( p_n(d\alpha;x) \) associated with the measure \( d\alpha \). The coefficients \( \lambda_n(d\alpha;x_{kn}) \) in the formula \( (I.1.1.1) \) are called the Christoffel numbers (see [18], [22], [5], [31], [25], [21; vol. 2, pp.1-31]). The Christoffel numbers \( \lambda_n(d\alpha;x_{kn}) \) are also the values of the function (see [28])

\[ \lambda^{-1}_n(x) = \sum_{\nu=0}^{n-1} \nu^2 p_{\nu}(d\alpha;x) \text{ at } x = x_{kn} \; , \; k = 1, 2, \ldots, n. \]
We also have \[ \sum_{\nu=1}^{n} \lambda_{\nu}(x) = \int_{a}^{b} d\alpha(x) = \alpha(b) - \alpha(a). \]

The nodes \( x_{\nu} \) are called the Gaussian abscissae with respect to \( d\alpha \).

The special case \( a = -1, b = +1, d\alpha(x) = dx \) is particularly important. Here, the abscissas \( x_{\nu} \) are the zeros of the \( n \)th Legendre polynomials, and the sum of the Christoffel numbers is 2. This is the case originally considered by Gauss and Jacobi. Another important special case, namely \( a = -1, b = +1, d\alpha(x) = (1 - x^{2})^{-1/2}dx \) is due to Mehler [25].

The GJQF ([1.1.1.1]) is a very important formula in applications and one of the most widely investigated in approximating a definite integral of the form \( \int_{a}^{b} f(x)w(x)dx \). This formula converges to the true value of the integral for almost any conceivable function which one meets in practice. For more on the convergence theorems of mechanical quadrature and the GJQF, see e.g., [9;chapter III], [30;chapter XV] and [26;part III].

In the remaining sections of this part of our thesis, we will be estimating error (remainder) terms in approximating an integral of the form \( \int_{-\infty}^{\infty} f(x)w(x)dx \) by the GJQF. We shall assume that \( f \) is an entire function and that \( w \) is a weight function having the whole real line as its support.

Freud, in his paper [11], obtained some error estimates for the GJQF with weights having a finite support.
His result is summarized by the following theorem.

**THEOREM:** Let the support of $d\alpha$ be $[-1,1]$ and 
\[
\log \alpha'(\cos \theta) \in \mathbb{R}[-\pi,\pi].
\]

Let $f(z)$ be an analytic function in $|z + \sqrt{z^2 - 1}| < r$. Then
\[
\left| \int_{-1}^{1} f \, d\alpha - Q_n(d\alpha; f) \right| \leq \frac{2}{r^{2n+1} + r^{2n-1}} \times [1 + o(1)] \times \\
\times \int_{e(r)} |d\alpha; z - \sqrt{z^2 - 1}|^2 |dz|;
\]

where $e(r)$ denotes the ellipse $|z + \sqrt{z^2 - 1}| = r$ and
\[
\Phi(d\alpha; \xi) = \exp\left( \frac{1}{4\pi} \int_{-\pi}^{\pi} \log(\alpha'(\cos \theta) |\sin \theta|) \right) \times \\
\times \frac{1 + \xi e^{-i\theta}}{1 - \xi e^{-i\theta}} \, d\theta \quad (|\xi| < 1).
\]

§1.1.2. **Main results**

Let $W$ be the class of all weight functions of the form
\[
w_Q(x) = \exp(-2Q(x)), \quad -\infty < x < +\infty,
\]
where

(i) $Q(x)$ is an even, differentiable function, except possibly at $x = 0$, increasing for $x > 0$. 
(ii) There exists $p < 1$ such that $x^p Q'(x)$ is increasing.

(iii) The sequence $\{q_n\}$ defined as the (unique) positive solution of the equation

\begin{equation}
(I.1.2.1) \quad q_n Q'(q_n) = n,
\end{equation}

satisfies the condition

\begin{equation}
(I.1.2.2) \quad \frac{q_{n+1}}{q_n} > c_1 > 1, \quad n = 1, 2, \ldots
\end{equation}

for some constant $c_1$ independent of $n$.

Remarks:
(a) Observe that whenever $Q(x) = Q_\alpha(x) = \frac{1}{2} |x|^\alpha$ ($\alpha \geq 1$),

then $w_\alpha \in W$.

(b) We also remark, see [16], that (I.1.2.2) will be satisfied

if we assume that $Q''$ exists and $\frac{xQ''(x)}{Q'(x)} \leq c_2$, $-\infty < x < \infty$,

for some constant $c_2$.

We now turn to our main results.

**THEOREM I.1.2.1**: Let $w_\alpha \in W$. Then, there exists a constant $A \in (0,1)$, depending on $Q$ only, such that whenever $f(z)$ is an entire function satisfying the condition

\begin{equation}
(I.1.2.3) \quad \limsup_{R \to \infty} \frac{\max_{|z|=R} (\log|f(z)|)}{2Q(R)}
\end{equation}
we have

\[
(I.1.2.4) \quad \lim_{n \to \infty} \sup_{x} \left\{ \int_{-\infty}^{\infty} f(x)e^{-2Q(x)} \, dx - Q_n(w_q;f) \right\}^{1/n} < 1.
\]

**THEOREM I.1.2.2:** Let \( w_q \in W \). Let \( f(z) \) be an entire function satisfying the condition

\[
(I.1.2.5) \quad \lim_{R \to \infty} \frac{\max_{|z|=R} (\log |f(z)|)}{2Q(R)} = 0.
\]

Then

\[
(I.1.2.6) \quad \lim_{n \to \infty} \left\{ \int_{-\infty}^{\infty} f(x)e^{-2Q(x)} \, dx - Q_n(w_q;f) \right\}^{1/n} = 0.
\]

§I.1.3. Auxiliary results

In order to prove our main results stated in the previous section, we are going to make use of the following three theorems due to G. Freud.

**THEOREM I.1.3.1:** Let \( f(z) \) be an analytic function in a domain \( \Phi \) containing the Gaussian abscissae \( x_{kn} \) \((k = 1, 2, \ldots, n)\) and \( x_{j,n+1} \) \((j = 1, 2, \ldots, n+1)\). If \( p_n(d\alpha;x) = \gamma_n x^n + \ldots \) is the orthogonal polynomial of degree \( n \) associated with the measure \( d\alpha \), we have

\[
(I.1.3.1) \quad Q_{n+1}(d\alpha;f) - Q_n(d\alpha;f) = \frac{\gamma_{n+1}}{\gamma_n} \frac{1}{2\pi i} \oint_{C_n} \frac{f(z)}{p_n(d\alpha;z)p_{n+1}(d\alpha;z)} \, dz.
\]
where $c_n \in D$ is a simple closed curve containing the zeros of $p_n(d\alpha)$ and $p_{n+1}(d\alpha)$ in its interior and the error term of the quadrature formula is

\[
(I.1.3.2) \quad \int f d\alpha - Q_n(d\alpha; f) = \sum_{\nu=n}^{\infty} \frac{\gamma_{\nu+1}}{\gamma_{\nu}} x \frac{1}{2\pi i} \oint_{c_{\nu}} \frac{f(z)}{p_{\nu}(d\alpha;z)p_{\nu+1}(d\alpha;z)} dz.
\]

**Proof:** See [11].

**THEOREM I.1.3.2:** Let $w_Q(x) = \exp(-2Q(x))$, $-\infty < x < \infty$, be a weight function, where $Q(x)$ is an even differentiable function, except possibly at $x = 0$, increasing for $x > 0$, for which $x^p Q'(x)$ is increasing for $p < 1$. We have then

\[
(I.1.3.3) \quad c_3 q_n \leq x_{1n} \leq c_4 q_n
\]

where $c_3$, $c_4$ are constants not depending on $n$, and $q_n$ $(n > 0)$ is as in (I.1.2.1).

**Proof:** See [13].

**THEOREM I.1.3.3:** For every even weight function $w(x)$, we have

\[
(I.1.3.4) \quad \max_{1 \leq k \leq n-1} \frac{\gamma_{k-1}}{\gamma_k} \leq x_{1n} \leq 2 \max_{1 \leq k \leq n-1} \frac{\gamma_{k-1}}{\gamma_k}.
\]

**Proof:** See [12].
The following more precise version of Theorem I.1.3.2, also due to Freud, will be needed in the next chapter.

**THEOREM I.1.3.4:** Let \( Q(x) , \ -\infty < x < \infty , \) be a convex, even and differentiable function. Let \( \{ q_n \} \) be as defined by (I.1.2.1), and \( w_Q(x) = \exp(-2Q(x)) \). Then

\[
(I.1.3.5) \quad \frac{1}{4} q_{n-1} \leq x_{ln} \leq \frac{1}{4} q_{n-1}
\]

and

\[
(I.1.3.6) \quad \frac{1}{4} q_{k} \leq \frac{\gamma_{k-1}}{\gamma_{k}} \leq 2 q_{k} .
\]

Since the proof of this result, published in Freud's lecture notes, is not widely known, we shall give here an outline of the proof. In order to prove this Theorem, we prove first the following Lemmas.

**LEMMA I.1.3.5:** Let \( w(x) , \ -\infty < x < \infty , \) be an even weight function having a continuous first derivative. Then

\[
(I.1.3.7) \quad \frac{k \gamma_{k}}{\gamma_{k-1}} = - \sum_{-\infty}^{\infty} p_{k}(w;x)p_{k-1}(w;x)w'(x)dx .
\]

**Proof:** Let \( I = \sum_{-\infty}^{\infty} p_{k-1}(w;x)p_{k}'(w;x)w(x)dx \). Since \( p_{k}'(w;x) =

\[
= k \frac{\gamma_{k}}{\gamma_{k-1}} p_{k-1}(w;x) + \pi_{k-2}(x) ,
\]

we have
Before evaluating $I$ using integration by parts, observe that

$$\int_{-\infty}^{\infty} |x|^r w(x) \, dx < \infty$$
implies $\liminf_{x \to \infty} |x|^r w(x) = 0$ and

$$\liminf_{x \to \infty} |x|^r w(x) \, dx = 0.$$ Thus, there are sequences $\{a_n\}$ and $\{b_n\}$ tending to $-\infty$ and $\infty$ respectively such that $(a_n)^r w(a_n) \to 0$ and $(b_n)^r w(b_n) \to 0$ as $n \to \infty$ for every $j = 0, 1, \ldots, 2k-1$. Integrating by parts, we obtain

$$\int_{a_n}^{b_n} p_{k-1}(w;x)p_k'(w;x)w(x) \, dx = p_{k-1}(w;x)p_k(w;x)w(x) \bigg|_{a_n}^{b_n} -$$

$$- \int_{a_n}^{b_n} p_{k-1}'(w;x)p_k(w;x)w(x) \, dx - \int_{a_n}^{b_n} p_{k-1}(w;x)p_k(w;x)w'(x) \, dx.$$ Letting $n \to \infty$, we find that

$$\int_{-\infty}^{\infty} p_{k-1}(w;x)p_k'(w;x)w(x) \, dx = -\int_{-\infty}^{\infty} p_k(w;x)p_{k-1}'(w;x)w(x) \, dx -$$

$$- \int_{-\infty}^{\infty} p_{k-1}(w;x)p_k(w;x)w'(x) \, dx.$$ The first integral on the right hand side of this relation is zero by orthogonality. Hence,

$$(I.1.3.8) \quad I = \frac{k\gamma_k}{\gamma_{k-1}}.$$ and $(I.1.3.7)$ follows from $(I.1.3.8)$ and $(I.1.3.9)$. 

$$(I.1.3.9) \quad I = -\int_{-\infty}^{\infty} p_{k-1}(w;x)p_k(w;x)w'(x) \, dx$$
**LEMMA I.1.3.6:** Let \( w(x) \), \(-\infty < x < \infty \), be a weight function having continuous first derivative and \( \int_{-\infty}^{\infty} x^n w'(x) \, dx \) exists for \( n = 0,1,2,\ldots \). Then

\[
(1.1.3.10) \quad 2k + 1 + \int_{-\infty}^{\infty} \frac{xw'(x)}{w(x)} p_k^2(w;x) w(x) \, dx = 0 .
\]

**Proof:** Let \( \pi \) be a polynomial of degree \( k \) with leading coefficient \( \gamma_k \), so that

\[
\pi(x) = p_k(w;x) + \pi_{k-1}(x) .
\]

By orthogonality relations and [9;Lemma I.1.1],

\[
\int_{-\infty}^{\infty} \pi^2(x) w(x) \, dx = 1 + \int_{-\infty}^{\infty} \pi_{k-1}^2(x) w(x) \, dx \geq 1 .
\]

Let, for \( t > 0 \),

\[
F(t) = \int_{-\infty}^{\infty} \left( t^k p_k(w;x) \right)^2 w(x) \, dx .
\]

By change of variable, we find that

\[
(1.1.3.11) \quad F(t) = t^{2k+1} \int_{-\infty}^{\infty} p_k^2(w;x) w(tx) \, dx .
\]

Since \( F(t) \geq 1 \) and \( F(1) = 1 \), the function \( F \) has a minimum at \( t = 1 \). Using (I.1.3.11), we find that

\[
F'(t) = (2k + 1)t^{2k} \int_{-\infty}^{\infty} p_k^2(w;x) w(tx) \, dx + t^{2k+1} \int_{-\infty}^{\infty} xp_k^2(w;x) w'(tx) \, dx
\]

and (I.1.3.10) follows by substituting here the number 1 for \( t \).
LEMMA I.1.3.7: If $Q(x)$, $-\infty < x < \infty$, is a convex, even and differentiable function and if $\{q_k\}$ is defined by (I.1.2.1), we have for all real $x$ and $k = 1,2,\ldots$,

(I.1.3.12) \[ |x| \leq q_k + \frac{q_k}{k} x Q'(x) \]

and

(I.1.3.13) \[ |Q'(x)| \leq \frac{k}{q_k} + \frac{xQ'(x)}{q_k} \]

Proof: If $|x| < q_k$, (I.1.3.12) is clearly true.

Suppose, next, that $|x| > q_k$. Since $|x|Q'(|x|) = xQ'(x)$ and $Q'(|x|) \geq Q'(q_k)$, we have

\[ |x| = \frac{|x|Q'(|x|)}{Q'(|x|)} \leq \frac{xQ'(x)}{Q'(q_k)} \leq q_k + \frac{xQ'(x)}{Q'(q_k)} = q_k + \frac{q_k}{k} x Q'(x) \]

and (I.1.3.12) is again true.

Likewise, if $|x| \leq q_k$, we have $|Q'(x)| = Q'(|x|) \leq Q'(q_k) = \frac{k}{q_k}$ and (I.1.3.13) is satisfied. If $|x| > q_k$, we have

\[ |Q'(x)| = Q'(|x|) = \frac{xQ'(x)}{|x|} \leq \frac{x}{q_k} Q'(x) \leq \frac{k}{q_k} + \frac{xQ'(x)}{q_k} \]

so that (I.1.3.13) holds again.

Now, we are ready to prove Theorem I.1.3.4.

Proof of Theorem I.1.3.4: It is well known that (see e.g., [9;§I.2]):

(I.1.3.14) \[ \frac{\gamma_{k-1}}{\gamma_k} = \int_{-\infty}^{\infty} x p_k(w;x)p_{k-1}(w;x)w(x)dx \]
By (I.1.3.14) and (I.1.3.12), we have

(I.1.3.15) \[
\frac{\gamma_{k-1}}{\gamma_k} \leq \int_{-\infty}^{\infty} |x| |p_{k-1}(w_Q;x)| |p_k(w_Q;x)| w_Q(x) \, dx \\
\leq \int_{-\infty}^{\infty} \left( q_k + \frac{q_k}{k} x Q'(x) \right) |p_{k-1}(w_Q;x)| |p_k(w_Q;x)| w_Q(x) \, dx.
\]

Using the Cauchy-Schwartz inequality, we obtain

(I.1.3.16) \[
\int_{-\infty}^{\infty} |p_{k-1}(w_Q;x)| |p_k(w_Q;x)| w_Q(x) \, dx \leq \\
\leq [\int_{-\infty}^{\infty} p_{k-1}^2(w_Q;x) w_Q(x) \, dx]^{1/2} [\int_{-\infty}^{\infty} p_k^2(w_Q;x) w_Q(x) \, dx]^{1/2}
\]

and by (I.1.3.10)

(I.1.3.17) \[
\int_{-\infty}^{\infty} x Q'(x) |p_{k-1}(w_Q;x)| |p_k(w_Q;x)| w_Q(x) \, dx \\
\leq [\int_{-\infty}^{\infty} x Q'(x) p_{k-1}^2(w_Q;x) w_Q(x) \, dx]^{1/2} [\int_{-\infty}^{\infty} x Q'(x) p_k^2(w_Q;x) w_Q(x) \, dx]^{1/2}
\]

Combining these inequalities with (I.1.3.14), we find that

(I.1.3.18) \[
\frac{\gamma_{k-1}}{\gamma_k} \leq 2q_k.
\]

Next, by (I.1.3.7) and (I.1.3.12),
\[
\frac{k}{2} \gamma_k \leq \int_{-\infty}^{\infty} \left| Q'(x) \right| \left| p_{k-1}(w';x) \right| \left| p_k(w';x) \right| w(x) \, dx
\]
\[
\leq \int_{-\infty}^{\infty} \left( \frac{k}{q_k} + \frac{x Q'(x)}{q_k} \right) \left| p_{k-1}(w';x) \right| \left| p_k(w';x) \right| w(x) \, dx
\]
and using again inequalities (I.1.3.16) and (I.1.3.17),
\[
\frac{k}{2} \gamma_k \leq \frac{k}{q_k} + \frac{k}{q_k} = \frac{2k}{q_k}.
\]
It follows that \( \frac{\gamma_{k-1}}{\gamma_k} \geq \frac{1}{4} q_k \) and combining with (I.1.3.18), we obtain
\[
(I.1.3.19) \quad \frac{1}{4} q_k \leq \frac{\gamma_{k-1}}{\gamma_k} \leq 2q_k.
\]
Let \( \Gamma_n = \max_{1 \leq k \leq n} \frac{\gamma_{k-1}}{\gamma_k} \).

Since \( \{q_k\} \) is an increasing sequence, we have by (I.1.3.19),
\( \Gamma_n \leq 2q_n \) and, on the other hand, \( \Gamma_n \geq \frac{\gamma_{n-1}}{\gamma_n} \geq \frac{1}{4} q_n \). Thus, we have
\[
(I.1.3.20) \quad \frac{1}{4} q_n \leq \Gamma_n \leq 2q_n \text{ for } n = 1, 2, \ldots.
\]
By Theorem I.1.3.3, we have
\[
\Gamma_{n-1} \leq x_{ln} \leq 2\Gamma_{n-1}
\]
and the rest of the proof follows from (I.1.3.20).

We will also make use of the following two lemmas whose proofs are due to the author. The second one is known in a more general form, see
[19], and will be used in the next chapter.

**Lemma 1.1.3.8:** Let \( Q(x) \), \(-\infty < x < \infty\), be an even differentiable and increasing for \( x > 0 \), for which \( x^\rho Q'(x) \) is increasing for some \( \rho < 1 \). Then there exists a positive constant \( c_5 \) such that

\[
\frac{Q(x)}{xQ'(x)} \leq c_5 \quad \text{for } x \geq x_o > 1 \ (x_o \text{ fixed}).
\]

**Proof:** For \( x \geq x_o > 1 \), we have

\[
\begin{align*}
Q(x) &= Q(x_o) + \int_{x_o}^{x} Q'(t) dt = Q(x_o) + \int_{x_o}^{x} \frac{t^\rho Q'(t)}{t^\rho} dt \\
&\leq Q(x_o) + x^\rho Q'(x) \int_{x_o}^{x} t^{-\rho} dt \\
&\leq Q(x_o) + x^\rho Q'(x) \frac{1}{1 - \rho} \left( \frac{1}{x^\rho - 1} - \frac{1}{x_o^\rho - 1} \right). 
\end{align*}
\]

Therefore,

\[
\frac{Q(x)}{xQ'(x)} \leq \frac{Q(x_o)}{xQ'(x)} + \frac{1}{1 - \rho} - \frac{1}{(1 - \rho)x_o^\rho - 1} \frac{1}{x^\rho - 1}.
\]

Since \( x^\rho Q'(x) \) is increasing, it follows that \( xQ'(x) \) is also increasing which completes the proof.

**Lemma 1.1.3.9:** Let \( w(x) \), \(-\infty < x < \infty\), be an even weight function. Then we have
(I.1.3.22) \[ \sum_{k=1}^{[n/2]} x_{kn}^2 = \sum_{k=1}^{n-1} \left( -\left( \frac{k-1}{k} \right) \right)^2. \]

**Proof:** From the fact that \( w \) is an even weight function, it follows that (see e.g., [30; §2.3(2)]) \( p_n(w;x) \) is an even or an odd polynomial according as \( n \) is even or odd. Hence, we can write

(I.1.3.23) \[ p_n(w,x) = \gamma_n x^n - \beta_n x^{n-2} + \ldots. \]

Recalling the recursion formula for orthogonal polynomials generated by an even weight function (see e.g., [30; §3.2(1)] or [9; §I.2]), we have

(I.1.3.24) \[ x p_n(w;x) = \frac{\gamma_n}{\gamma_{n+1}} p_{n+1}(w;x) + \frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(w;x). \]

Combining (I.1.3.23) and (I.1.3.24), we get by comparing the coefficients of \( x^{n-1} \) in both sides of (I.1.3.24)

\[ -\beta_n = -\frac{\beta_{n+1}}{\gamma_{n+1}} \gamma_n + \frac{\gamma_{n-1}}{\gamma_n} \]

i.e.,

\[ \frac{\beta_{n+1}}{\gamma_{n+1}} = \frac{\beta_n}{\gamma_n} + \left( \frac{\gamma_{n-1}}{\gamma_n} \right)^2 \]

which implies that

\[ \frac{\beta_n}{\gamma_n} = \sum_{k=1}^{n-1} \left( -\left( \frac{k-1}{k} \right) \right)^2. \]

Since it is easy to see that \( \frac{\beta_n}{\gamma_n} = \left[ \frac{n}{2} \right] \sum_{k=1}^{[n/2]} x_{kn}^2 \), the proof of the Lemma is completed.

* Here, \([ ]\) is the greatest integer function.
Before concluding this section, we state here the analogue of (I.1.3.6), which is interesting on its own and will be referred to in part II of this thesis.

THEOREM I.1.3.5: Let \( Q(x) \), \( 0 \leq x < \infty \), be a convex and differentiable function on \((0,\infty)\). Let \( \{q_n\} \) be as defined by (I.1.2.1) and \( w_q(x) = \exp(-2Q(x)) \). Then

\[
(I.1.3.6)' \quad \frac{kq_k}{4k + q_k|p_{k-1}(0)p_k(0)w_Q(0)|} \leq \frac{v_{k-1}}{v_k} \leq 2q_k.
\]

Proof: The proof of this Theorem is a straightforward imitation of the proof of Theorem I.1.3.4 with a slight modification due to the fact that the function \( Q(x) \) is no longer an even function.

§I.1.4. Proofs of main results

We are now ready to prove our main results of this chapter.

Proof of Theorem I.1.2.1: The proof of this theorem will be mainly based on estimating the error term in (I.1.3.2).

Suppose (I.1.2.3) holds with \( A = 2^{-(M+1)} \ast \), where \( M \) is equal to the greatest integer not exceeding \( 1 + \frac{1 + 2c_5 + \log c_4}{\log c_1} \), with

\[\ast\] Other values of \( A \), smaller than this one, may be chosen and the theorem will remain true.
\(c_1, c_2,\) and \(c_3\) are as in (I.1.2.2), (I.1.3.3), and (I.1.3.21) respectively. Let

\[
I_n = \frac{\gamma_{n+1}}{\gamma_n} \frac{1}{2\pi i} \oint_{c_n} \frac{f(z)}{p_n(w_{Q};z)p_{n+1}(w_{Q};z)} dz
\]

and

\[
\Delta_n = \oint_{-\infty}^{\infty} f(x)e^{-2Q(x)} dx - Q_n(w_{Q};f) = \sum_{\nu = n}^{\infty} I_{\nu}.
\]

Hence, to complete the proof of the theorem, it remains to estimate the complex integrals, \(I_{\nu}\).

Let us choose the path of integration \(c_n\) in (I.1.3.1) to be the circumference of a circle of radius \(R_n\), with center at the origin, where \(R_n\) is chosen such that

(I.1.4.1) \(R_nQ'(R_n) = 2^M(n+1)\) for \(n = 1, 2, \ldots\).

From (I.1.2.1) and the fact that its solution is unique, it follows that

\[
R_n = q^M(2^{n+1} - 1).
\]

Combining this with (I.1.2.2), we conclude

(I.1.4.2) \(R_n > c_1^M q_{n+1}\) for \(n = 1, 2, \ldots\).

We shall also need an inequality for \(\frac{1}{\left| p_n(w_{Q};z) \right|} \). Since \(w_{Q}\) is an even weight function, it follows that (see e.g., [30; §2.3(2)])
\[ p_n(w_Q; z) = \gamma_n z^{-n-2} \prod_{k=1}^{[n/2]} (z^2 - x_k^{2n}) \]

\[ = \gamma_n z^n \exp\left( \sum_{k=1}^{[n/2]} \log(1 - \frac{x_k^{2n}}{|z|^2}) \right) \]

where \( z \) is any complex number different from zero and \( x_{1n} > x_{2n} > \ldots \)

\( \ldots > x_{[n/2],n} \) are the positive zeros of \( p_n(w_Q) \). Thus, \( p_{n/2} \) are the positive zeros of \( p_n(w_Q) \). Thus,

\[ \frac{1}{|p_n(w_Q; z)|} \leq \frac{1}{\gamma_n |z|^n} \exp\left( \sum_{k=1}^{[n/2]} \log(1 - \frac{x_k^{2n}}{|z|^2}) \right) \]

\[ \leq \frac{1}{\gamma_n |z|^n} \exp\left( \sum_{k=1}^{[n/2]} \log\left( 1 - \frac{x_k^{2n}}{|z|^2} \right) \right) \]

\[ \leq \frac{1}{\gamma_n |z|^n} \exp\left( \sum_{k=1}^{[n/2]} \left( 1 - \frac{x_k^{2n}}{|z|^2} \right) \right) \]

where \( \frac{x_k^{2n}}{|z|^2} < 1 \) for \( k = 1, 2, \ldots, [n/2] \). Using (I.1.4.3) with \( c_n \)

replaced by the circle \( |z| = R_n \), we find that

\[ |I_n| = \left| \frac{\gamma_{n+1}}{\gamma_n} \frac{1}{2\pi i} \oint_{R_n} \frac{f(z)}{p_n(w_Q; z) p_{n+1}(w_Q; z)} \frac{dz}{p_n(w_Q; z) p_{n+1}(w_Q; z)} \right| \]

\[ \leq \frac{\gamma_{n+1}}{\gamma_n} \frac{1}{2\pi} \oint_{R_n} \left| f(z) \right| \left| \frac{dz}{p_n(w_Q; z) p_{n+1}(w_Q; z)} \right| \]

\[ \leq \frac{\gamma_{n+1}}{\gamma_n} \frac{1}{2\pi} \oint_{R_n} \left| f(z) \right| \left| \frac{dz}{|p_n(w_Q; z)| |p_{n+1}(w_Q; z)|} \right| \]

\[ \leq \frac{\gamma_{n+1}}{\gamma_n} \frac{1}{2\pi} \oint_{R_n} \left| f(z) \right| \left| \frac{dz}{|p_n(w_Q; z)| |p_{n+1}(w_Q; z)|} \right| \]

\[ \leq \frac{\gamma_{n+1}}{\gamma_n} \frac{1}{2\pi} \oint_{R_n} \left| f(z) \right| \left| \frac{dz}{|p_n(w_Q; z)| |p_{n+1}(w_Q; z)|} \right| \]

\[ \leq \frac{\gamma_{n+1}}{\gamma_n} \frac{1}{2\pi} \oint_{R_n} \left| f(z) \right| \left| \frac{dz}{|p_n(w_Q; z)| |p_{n+1}(w_Q; z)|} \right| \]

A straightforward calculation under our assumptions will show that

\[ \frac{x_k, n+1}{|z|^2} < 1 \ (k = 1, 2, \ldots, n+1) \] holds for \( |z| = R_n \) and hence all the

zeros of \( p_n(w_Q) \) and \( p_{n+1}(w_Q) \) are inside the circle of radius \( R_n \).
\[
\left| I_n \right| < \frac{1}{\gamma_n} \frac{1}{2n} \frac{2\pi R}{\gamma_n} \max_{|z|=R_n} \frac{|f(z)|}{\gamma_n R_n n^{n+1}} \times \exp\left( \sum_{k=1}^{[n/2]} \frac{x_{k,n}}{R_n^2} + \frac{(n+1)/2}{R_n^2} \right) \times \left( 1 - \frac{x_{k,n+1}}{R_n^2} \right)
\]

Since \( x_{k,n+1} < x_{1,n+1} \) for \( n = 1, 2, \ldots, n+1 \), we get

\[
(I.1.4.4) \quad \left| I_n \right| < \frac{1}{\gamma_n} \frac{1}{2n} \max_{|z|=R_n} \frac{|f(z)|}{\gamma_n R_n^2 n} \times \exp((n+1) \frac{x_{1,n+1}}{R_n^2}) \times \left( 1 - \frac{x_{1,n+1}}{R_n^2} \right)
\]

From our hypothesis \((I.1.2.3)\) and the definition of \( A \), it follows that there exists an integer \( N \) such that \( \max_{|z|=R_n} |f(z)| \leq e^{4AQ(R_n)} \leq \exp(2 \cdot 2^{-M} Q(R_n)) \), for all \( n \geq N \). Combining this with \((I.1.4.4)\), we obtain

\[
(I.1.4.5) \quad \left| I_n \right| < \frac{1}{\gamma_n} \frac{1}{2n} \exp(2 \cdot 2^{-M} Q(R_n)) + \frac{x_{1,n+1}}{R_n^2} + (n+1) \frac{x_{1,n+1}}{R_n^2} \) \quad \text{for all} \quad n \geq N.
\]

From \((I.1.3.3)\), \((I.1.4.2)\) and the choice of \( M \) we can easily verify
that

\[ \frac{x_{1,n+1}^2}{R_n^2} \leq 1 - \left( \frac{x_{1,n+1}}{R_n} \right)^2 \]

(I.1.4.6)

From (I.1.4.1) and (I.1.3.21), we get

\[ 2^{-M} Q(R_n) \leq c_5(n+1) \]

(I.1.4.7)

which implies that

\[ \frac{1}{\gamma_n} \leq \frac{1}{\gamma_o} c_4 \prod_{k=2}^{n+1} q_k \]

(I.1.4.8)

By inserting (I.1.4.6), (I.1.4.7), (I.1.4.8) and (I.1.4.2) in (I.4.5), we obtain

\[ |I_n| \leq \frac{1}{\gamma_o} \frac{c_4^{2n} \prod_{k=2}^{n+1} q_k^2}{2^{2M} c_1^{2} q_{n+1}^2} \exp\{2c_5(n+1) + (n+1)\} \]

\[ < \frac{1}{\gamma_o} \left( \frac{c_4^2 \exp(4c_5 + 2)}{2^{2M} c_1} \right)^n \quad \text{for all } n \geq N. \]

Denoting the positive quantity \( c_4^2 \exp(4c_5 + 2) \) by \( B \), we can easily check that the inequality \( B < 1 \) holds. Hence,

\[ |I_n| < \frac{1}{\gamma_o} B^n \quad \text{for all } n \geq N, \]

(I.1.4.10)
and from the definition of $\Delta_n$, it follows that

$$|\Delta_n| \leq \sum_{v=n}^{\infty} |I_v| \leq \frac{1}{\gamma_o} \sum_{v=n}^{\infty} B^v \leq K B^n, \text{ for all } n \geq N$$

where $K$ is a positive constant independent of $n$. Therefore,

$$\limsup_{n \to \infty} |\Delta_n|^{1/n} < \limsup_{n \to \infty} (B^{1/n}) = B < 1.$$ 

This completes the proof of Theorem I.1.2.1.

**Proof of Theorem I.1.2.2:** The proof of this theorem is similar to that of Theorem I.1.2.1 with a few extra modifications due to the condition (I.1.2.5) that is more restrictive than its counterpart (I.1.2.3).

Let us recall from the previous proof of Theorem I.1.2.1 that we have $B = \frac{c_1^2 \exp(4c_5 + 2)}{c_2^{2M}}$ and $A = 2^{-(M+1)}$. These two equations along with the assumption (I.1.2.5) enable us, by choosing $M$ sufficiently large, to state:

For any $\epsilon > 0$, there exists a positive number $N$ such that

$$|\Delta_n|^{1/n} \leq K \epsilon \text{ for all } n \geq N \text{ and } K \text{ as in Theorem I.1.2.1}.$$ 

Since the choice of $\epsilon$ is arbitrary, we can immediately conclude from this last inequality that (I.1.2.6) holds.

This completes the proof of Theorem I.1.2.2.
CHAPTER I.2

APPLICATIONS

§I.2.1. Introduction

In this chapter, we are going to treat in more detail four special cases of Theorem I.1.2.1, namely, when the weight function \( w_Q \) is given explicitly and the ratio \( \frac{\gamma_{n-1}}{\gamma_n} \) is given either explicitly or asymptotically in each case.

The weight functions that will be considered are:

(i) The Hermite weight function \( e^{-x^2} \), \( (Q(x) = \frac{1}{2} x^2) \),

(ii) The weight function \( e^{-x^4} \), \( (Q(x) = \frac{1}{2} x^4) \),

(iii) The weight function \( e^{-x^6} \), \( (Q(x) = \frac{1}{2} x^6) \),

and

(iv) The Pollaczek weight function

\[
|\Gamma(\lambda + ix)|^2 \approx e^{-\pi |x| + (2\lambda - 1)\ln |x| + \ln(2\pi)} \quad , \quad \lambda > 0
\]

\[
(Q(x) = Q_\lambda(x) \approx \frac{\pi |x|}{2} - \frac{2\lambda - 1}{2} \ln |x| - \ln \sqrt{2\pi})
\]

Using the same technique that we have used in the proof of Theorem I.1.2.1, we are going to find the largest possible value that the
constant $A$ of Theorem I.1.2.1 can take in each of the above stated cases, in order that the theorem remains valid.

§1.2.2. The case of the Hermite weight function

$$w_q(x) = e^{-x^2}, \quad Q(x) = \frac{1}{2}x^2, \quad -\infty < x < \infty.$$  

Our main result is given by the following theorem

**Theorem I.2.2.1:** Let $f(z)$ be an entire function satisfying the condition

$$\beta = \lim \sup_{R \to \infty} \frac{\max (\log |f(z)|)}{|z|^2} < \rho$$

where $\rho (\approx 0.70541786)$ is the solution of the equation

$$\frac{1}{6}(1 + 2x - \sqrt{1 - 2x + 4x^2}) \exp\left\{\frac{1+2x - \sqrt{1-2x+4x^2}}{1 - 4x + \sqrt{1-2x+4x^2}}\right\} = 1.$$  

Then we have

$$\lim \sup \int_{-\infty}^{\infty} f(x)e^{-x^2}dx - Q_n(e^{-x^2};f)|^{1/n} < 1.$$  

**Proof:** Since $w_q(x) = e^{-x^2}$, it is well known that the $n$th orthogonal polynomial generated by this weight function is the $n$th Hermite polynomial $h_n(x)$ ($h_n(x)$ are normalized in this case) and it is also well known, see e.g., [30;§5.5], that
(I.2.2.4) \[ \gamma_n^2 = \frac{z^n}{\sqrt{n \cdot n!}} \]

which easily implies

(I.2.2.5) \[ \frac{\gamma_{n-1}}{\gamma_n} = \sqrt{\frac{n}{2}} . \]

By combining (I.2.2.5) and (I.1.3.4), we get

(I.2.2.6) \[ x_{1,n+1} \leq \sqrt{2n} . \]

By combining (I.2.2.5) and (I.1.3.22), we find that

(I.2.2.7) \[ \sum_{k=1}^{[n/2]} x_{kn}^2 = \frac{n(n-1)}{4} , \quad n = 2,3,4,\ldots . \]

Proceeding as in the proof of Theorem I.1.2.1, we can see that

\[ |h_n(z)| = |\gamma_n| |z|^{n-2 \cdot \lfloor n/2 \rfloor} \sum_{k=1}^{\lfloor n/2 \rfloor} |z^2 - x_{kn}^2| \]

\[ = |\gamma_n| |z|^n \exp\left( \sum_{k=1}^{\lfloor n/2 \rfloor} \log|1 - \frac{x_{kn}^2}{z^2}| \right) \]

\[ \geq |\gamma_n| |z|^n \exp\left( \sum_{k=1}^{\lfloor n/2 \rfloor} \log(1 - \frac{x_{kn}^2}{|z|^2}) \right) \]

\[ \geq |\gamma_n| |z|^n \exp\left( - \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{x_{kn}^2}{|z|^2} \right) \cdot (|z|^2 - x_{kn}^2) \]
for every complex number $z$ such that $x_{ln} < |z|$. We have next

$$\frac{1}{|z|^2 - x_{kn}^2} \leq \frac{1}{|z|^2 - x_{ln}^2}$$

and so

$$|h_n(z)| \geq |\gamma_n| |z|^n \exp\left(-\frac{1}{|z|^2 - x_{ln}^2} \sum_{k=1}^{[n/2]} x_{kn}^2\right).$$

Using (I.2.2.7), we find that

$$|h_n(z)| \geq |\gamma_n| |z|^n \exp\left(-\frac{n(n-1)}{4(|z|^2 - x_{ln}^2)}\right).$$

Therefore, we find

$$\frac{1}{|h_n(z)|} \leq \frac{1}{|\gamma_n| |z|^2 \exp\left(-\frac{n(n-1)}{4(|z|^2 - x_{ln}^2)}\right)}.$$

It follows that

$$\frac{1}{|h_n(z)h_{n+1}(z)|} \leq \frac{1}{\gamma_n \gamma_{n+1}} \frac{1}{|z|^{2n+1}} \exp\left(-\frac{n(n-1)}{4(|z|^2 - x_{ln}^2)} + \frac{(n+1)n}{4(|z|^2 - x_{ln,n+1}^2)}\right)$$

$$\leq \frac{1}{\gamma_n \gamma_{n+1}} \frac{1}{|z|^{2n+1}} \exp\left(-\frac{n^2}{2(|z|^2 - x_{ln,n+1}^2)}\right).$$

Since $\beta = \limsup_{R \to \infty} \frac{|z|=R}{R^2}$, for any $\delta > 0$, we can find $N_\delta$ such that

$$(I.2.2.8) \quad |f(z)| \leq \exp((\beta + \delta)|z|^2), \quad \text{for all } |z| \geq N_\delta.$$
Denoting by $I_n$, as we did in the proof of Theorem I.1.2.1, the expression
\[ \frac{\gamma_{n+1}}{\gamma_n} \frac{1}{2\pi i} \oint_c \frac{f(z)}{h_n(z)h_{n+1}(z)} \] taking the path of integration to be the circle $|z| = R$, where

(I.2.2.9) \[ R \geq \frac{x_1^{\frac{n+1}{1-\epsilon}}}{1-\epsilon}, \quad (0 < \epsilon < 1) \]

we find that on the circle $|z| = R$

(I.2.2.10) \[ \left| \frac{1}{h_n(z)h_{n+1}(z)} \right| \leq \frac{1}{\gamma_n \gamma_{n+1}} \frac{1}{R^{2n+1}} \exp\left(\frac{n^2}{2(R^2 - \xi_{1,n+1}^2)}\right) \]

Using (I.2.2.10), (I.2.2.4) and (I.2.2.8), we conclude that $R \geq N_6$,

\[ |I_n| \leq \frac{\sqrt{\pi} n!}{2^n} \max_{|z|=R} |f(z)| \exp\left(\frac{n^2}{2\epsilon R^2}\right) \]

\[ \leq \frac{\sqrt{\pi} n!}{2^n} \frac{1}{R^{2n}} \exp\left((\beta + \delta)R^2 + \frac{n^2}{2\epsilon R^2}\right) . \]

$R$ will be chosen next so as to minimize the right hand side of this inequality, and at the same time, to satisfy (I.2.2.9).
Consider the function

\[ h(R) = \frac{1}{R^{2n}} \exp\left((\beta + \delta)R^2 + \frac{n^2}{2\epsilon R^2}\right). \]

By differentiating \( h(R) \) and setting \( h'(R) = 0 \), we get

\[ (I.2.2.11) \quad 2(\beta + \delta)\epsilon R^4 - 2n\epsilon R^2 - n^2 = 0. \]

If we denote by \( R_n \) the solution to this equation, we find that

\[ R_n^2 = \frac{1 + (1 + \frac{2(\beta + \delta)}{\epsilon})^{1/2}}{2(\beta + \delta)} n. \]

(We can easily check that \( f(R) \) attains its minimum value at \( R = R_n \).)

For \( n \geq N \), we will have \( R_n \geq N_0 \). Aslo, from (I.2.2.6), it follows that

\[ R_n^2 \geq \frac{1 + (1 + \frac{2(\beta + \delta)}{\epsilon})^{1/2}}{4(\beta + \delta)} x^2_{1,n+1} \]

and consequently, condition (I.2.2.9) will be satisfied if

\[ (I.2.2.12) \quad \frac{4(\beta + \delta)}{1 + (1 + \frac{2(\beta + \delta)}{\epsilon})^{1/2}} = 1 - \epsilon. \]

Since \( R_n \) satisfies equation (I.2.2.11), we find that

\[ (\beta + \delta)R_n^2 = n + \frac{n^2}{2\epsilon R_n^2} \]

and it follows that
Using (1.2.2.12), we find that

\[ |I_n| \leq \sqrt{\pi} \frac{n!}{2^n} \frac{1}{R_n^{2n}} \exp\left(n + \frac{n^2}{e R_n^2}\right) \]

\[ \leq \sqrt{\pi} \frac{n!}{2^n} \left[ \frac{2(\beta + \delta)}{1 + (1 + \frac{2(\beta + \delta)}{\epsilon})^{1/2}} \right]^n \frac{1}{n^n} \cdot \exp\left(n + \frac{2(\beta + \delta)n}{\epsilon(1 + (1 + \frac{2(\beta + \delta)}{\epsilon})^{1/2})}\right). \]

Using (1.2.2.12), we find that

\[ |I_n| \leq \sqrt{\pi} \frac{n!}{2^n n^n} \frac{(1 - \epsilon)^n}{\epsilon} \exp\left(n + \frac{(1 - \epsilon)n}{2\epsilon}\right). \]

Using the Stirling formula \( n! \approx \sqrt{2\pi n} n^n e^{-n} \), we find that

(1.2.2.13) \[ |I_n| \leq K\sqrt{n} \left(\frac{1 - \epsilon}{\epsilon} \exp\left(\frac{1 - \epsilon}{2\epsilon}\right)\right)^n. \]

It is easy to see that \( g(\epsilon) = \frac{1 - \epsilon}{\epsilon} \exp\left(\frac{1 - \epsilon}{2\epsilon}\right) \) is a decreasing function on \((0,1)\). Consequently, if \( \epsilon_0 \) is the unique solution of \( g(\epsilon) = 1 \) then, for \( \epsilon_0 < \epsilon < 1 \), we have

\[ 0 < g(\epsilon) < 1. \]

Thus, \( \Sigma |I_k| \) is a convergent series. If \( \Delta_n = \Sigma_{k=n}^{\infty} I_k \), we have

\[ |\Delta_n| \leq \Sigma_{k=n}^{\infty} |I_k| \leq K \Sigma_{k=n}^{\infty} \sqrt{k} \left(\frac{1 - \epsilon}{\epsilon} \exp\left(\frac{1 - \epsilon}{2\epsilon}\right)\right)^k. \]

Since \( \Sigma_{k=n}^{\infty} k x^k \leq \frac{(n + 2)x^n}{(1 - x)^2} \) for \( 0 < x < 1 \), it follows that...
\[ |\Delta_n| \leq \frac{K(n+3)}{[1 - \frac{1 - \varepsilon}{4} \exp(\frac{1 - \varepsilon}{2\varepsilon})]^2} \left( \frac{1 - \varepsilon}{4} \exp(\frac{1 - \varepsilon}{2\varepsilon}) \right)^n \]

and so

\[
\limsup_{n \to \infty} |\Delta_n|^{1/n} \leq \frac{1 - \varepsilon}{4} \exp(\frac{1 - \varepsilon}{2\varepsilon}) < 1
\]

for every \( \varepsilon_0 < \varepsilon < 1 \).

Finally, it remains to justify the assumption (I.2.2.1), and the proof of the theorem will be complete. From (I.2.2.12) and the choice of \( \varepsilon_0 < \varepsilon < 1 \), we see that

\[
\frac{4(\beta + \delta)}{1 + (1 + \frac{2(\beta + \delta)}{\varepsilon_0})^{1/2}} < \frac{4(\beta + \delta)}{1 + (1 + \frac{2(\beta + \delta)}{\varepsilon})^{1/2}} = 1 - \varepsilon < 1 - \varepsilon_0
\]

and therefore,

\[
\frac{4(\beta + \delta)}{1 + (1 + \frac{2(\beta + \delta)}{\varepsilon_0})^{1/2}} < 1 - \varepsilon_0.
\]

Solving this inequality, we find that

\[
0 < \beta + \delta < \frac{(3\varepsilon_0 + 1)(1 - \varepsilon_0)}{8\varepsilon_0} = \rho \quad (\approx 0.70541786).
\]

Hence, we must have

\[
0 < \beta < \rho.
\]

On the other hand, by combining

\[
\frac{(3\varepsilon_0 + 1)(1 - \varepsilon_0)}{8\varepsilon_0} = \rho \quad \text{and} \quad \frac{1 - \varepsilon_0}{4} \exp\left(\frac{1 - \varepsilon_0}{2\varepsilon_0}\right) = 1
\]
we can see that \( \rho \) is the unique solution of (I.2.2.2).

Remark: If \( \beta \) where chosen such that \( \beta \geq \rho \), we can see that it is not possible to prove the convergence of \( \Sigma |I_n| \) by using the same technique and estimates used in the proof of Theorem I.2.2.1, and consequently we cannot have (I.2.2.3). It would be interesting to find whether the constant \( \rho \) is the best possible for the validity of Theorem I.2.2.1.

§ I.2.3. The case of the weight functions

(a) \( w_Q(x) = e^{-x^4} \quad Q(x) = \frac{1}{2}x^4 \quad -\infty < x < \infty \)

(b) \( w_Q(x) = e^{-x^6} \quad Q(x) = \frac{1}{2}x^6 \quad -\infty < x < \infty \).

Before stating the main results of this section, we first state a conjecture by G. Freud (see [15]), that we are going to use in two special cases that Freud has proved in the same paper [15].

CONJECTURE: We have for every positive even \( m \) and every \( p < -1 \)

\[
(I.2.3.1) \quad \lim_{n \to \infty} \frac{-1}{m} \frac{\gamma_{n-1}(w_{pm})}{\gamma_n(w_{pm})} = \left[ m \left( \frac{m-1}{2} \right) \right]^{-\frac{1}{m}} = \left[ \frac{\Gamma(m+1)}{\Gamma(\frac{m}{2})\Gamma(\frac{m}{2}+1)} \right]^{-\frac{1}{m}}
\]

where \( w_{pm}(x) = |x|^p \exp(-|x|^m) \).
Freud has proved that the relation (I.2.3.1) holds for \( m = 2, 4 \) and 6. Moreover, it is proven that this relation is valid whenever the limit on the left hand side exists.

We now turn to the main results.

**THEOREM I.2.3.1**: Let \( f(z) \) be an entire function satisfying the condition

\[
(\text{I.2.3.2}) \quad \beta = \lim_{R \to \infty} \sup_{|z|=R} \frac{\max (\log |f(z)|)}{R^4} < \rho
\]

where \( \rho (\approx .243136729) \) is such that

\[
(\text{I.2.3.3}) \quad \rho = \frac{(1 - \epsilon_o)}{8\epsilon_o}
\]

and \( \epsilon_o \) is the solution of \( \frac{1 - x}{4} \exp(\frac{1 - x}{2x}) = 1 \). Then we have

\[
(\text{I.2.3.4}) \quad \lim_{n \to \infty} \sup_{x} \left( \int_{-\infty}^{\infty} f(x)e^{-x^4}dx - Q_n(e^{-x^4};f) \right)^{1/n} < 1.
\]

**Proof**: Since \( w_Q(x) = e^{-x^4} \), we conclude from (I.2.3.1), with \( p = 0 \) and \( m = 4 \), that

\[
\lim_{n \to \infty} n^{-\frac{1}{4}} \gamma_{n-1} = \frac{1}{4\sqrt{12}}.
\]

So, if we choose an arbitrary positive number \( \eta \), then there exists a positive number \( N_\eta \) such that

\[
(\text{I.2.3.5}) \quad \frac{\gamma_{n-1}}{\gamma_n} \leq \left( \frac{1}{4\sqrt{12}} + \eta \right)^{\frac{1}{\sqrt{n}}} \quad \text{for all} \quad n \geq N_\eta.
\]
We also conclude from (I.1.3.6) that

\[ (I.2.3.6) \quad \frac{\gamma_{n-1}}{\gamma_n} \leq \frac{\eta}{\sqrt{8n}} , \quad \text{for all } n = 1, 2, 3, \ldots \]

and from (I.1.3.4) combined with (I.2.3.5) that

\[ (I.2.3.7) \quad x_{1,n+1} \leq 2\left(\frac{1}{\sqrt[4]{12}} + \eta\right)\sqrt{n} , \quad \text{for all } n \geq N_\eta . \]

Combining (I.2.3.22), (I.2.3.6) and (I.2.3.5), we get for \( n \geq N_\eta \)

\[
\frac{[n/2]}{\sum_{k=1}^{n-1} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2} = \frac{N_{\eta-1}}{\left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2} + \frac{n-1}{\sum_{k=N_\eta}^{N_{\eta-1}} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2}
\]

\[
\leq \sqrt{8} \sum_{k=1}^{N_{\eta-1}} k^{1/2} + \left(\frac{1}{\sqrt[4]{12}} + \eta\right)\sqrt{2} \sum_{k=N_\eta}^{n-1} k^{1/2}
\]

\[
\leq 2\sqrt{2} \left[\frac{2}{3} (N_{\eta-1})^{3/2} - 1\right] + \frac{2}{3} \left(\frac{1}{\sqrt[4]{12}} + \eta\right)^2 (n^{3/2} - N_{\eta-1}^{3/2})
\]

\[
\leq \left[\frac{4\sqrt{2}}{3} - \frac{2}{3}\left(\frac{1}{\sqrt[4]{12}} + \eta\right)^2\right] N_{\eta-1}^{3/2} + \frac{2}{3}\left(\frac{1}{\sqrt[4]{12}} + \eta\right)^2 n^{3/2}
\]

i.e.,

\[ (I.2.3.8) \quad \frac{[n/2]}{\sum_{k=1}^{n-1} x_{k,n}^2} \leq K_\eta + \frac{2}{3} \left(\frac{1}{\sqrt[4]{12}} + \eta\right)^2 n^{3/2} \]

where \( n > N_\eta \) and \( K_\eta = \left[\frac{4\sqrt{2}}{3} + \eta\right] N_{\eta}^{3/2} \).

From (I.2.3.5) and (I.2.3.6), we obtain

\[
\frac{1}{\gamma_n} \leq \left(\frac{1}{\sqrt[4]{12}} + \eta\right)^{n-N_\eta-1} \cdot \sqrt{n(n-1)} \ldots (N_\eta+1)\eta_1 \cdot \frac{1}{\gamma_{N_\eta-1}} , \quad n > N_\eta ,
\]
and

\[ \frac{1}{\gamma_{N \eta - 1}} \leq (\frac{\sqrt{3}}{\sqrt{2}})^{N \eta} \frac{1}{\sqrt{(N \eta - 1)!}} \frac{1}{\gamma_0}, \]

respectively.

Hence, for \( n > N \eta \), we have

\[ (I.2.3.9) \quad \frac{1}{2} \leq \frac{1}{2} (4 \sqrt{6})^{N \eta} \left( \frac{1}{\sqrt{12}} + \eta \right)^{2n} \sqrt{n!}. \]

Denoting the \( n \)th orthonormal polynomial generated by the weight function \( e^{-x^4} \) by \( p_n(x) \) and proceeding as in the proof of Theorem I.2.2.1, we see that

\[ \frac{1}{|p_n(z)|} \leq \frac{1}{\gamma_n |z|^n} \exp\left( \frac{1}{|z|^2} - \frac{x_{1n}^2}{2n} \right) \sum_{k=1}^{[n/2]} x_{kn}^2. \]

Using (I.2.3.8), we find that

\[ \frac{1}{|p_n(z)|} \leq \frac{1}{\gamma_n |z|^n} \exp\left( \frac{1}{|z|^2} - \frac{x_{1n}^2}{2n} \right) \left( K_\eta + \frac{2}{3} \left( \frac{1}{\sqrt{12}} + \eta \right)^2 n^{3/2} \right), \]

which implies that

\[ \frac{1}{|p_n(z)p_{n+1}(z)|} \leq \frac{1}{\gamma_n \gamma_{n+1}} \frac{1}{|z|^{2n+1}} \exp\left( - \frac{2K_\eta + \frac{4}{3} \left( \frac{1}{\sqrt{12}} + \eta \right)^2 (n + 1)^{3/2}}{|z|^2 - x_{1n, n+1}^2} \right). \]

Since \( \beta = \lim \sup_{R \to \infty} \frac{\max |f(z)|}{|z|^R} \), for every \( \delta > 0 \), we can find \( N_\delta \) such that

\[ (I.2.3.10) \quad |f(z)| \leq \exp((\beta + \delta)|z|^4), \quad \text{for all} \quad |z| \geq N_\delta. \]
Denoting by $I_n$, as we did in the proof of Theorem I.1.2.1, the expression

\[ \frac{\gamma_{n+1}}{\gamma_n} \frac{1}{2\pi i} \oint_C \frac{f(z)}{p_n(z)p_{n+1}(z)} dz, \]

taking the path of integration to be the circle $|z| = R$ where

\[(I.2.3.11) \quad R^2 \geq \frac{x_{1,n+1}}{1-\varepsilon}, \quad (0 < \varepsilon < 1),\]

we find that on the circle $|z| = R$

\[(I.2.3.12) \quad \left| \frac{1}{p_n(z)p_{n+1}(z)} \right| \leq \frac{1}{\gamma_n \gamma_{n+1}} \frac{1}{R^{2n+1}} \exp\left(\frac{2K_{\eta} + \frac{4}{3}(\frac{1}{\sqrt{12}} + \eta)^2(n+1)^{3/2}}{\varepsilon R^2}\right).\]

Using (I.2.3.12), (I.2.3.9) and (I.2.3.10), we conclude that, $R \geq N_0$, 

\[|I_n| \leq \frac{1}{\gamma_0} (4\sqrt{6})^N \eta \left( \frac{1}{\sqrt{12}} + \eta \right)^{2n} \sqrt{n!} \frac{1}{R^{2n}} \cdot \exp\left(\beta + \delta \right) + \frac{2K_{\eta} + \frac{4}{3}(\frac{1}{\sqrt{12}} + \eta)^2(n+1)^{3/2}}{\varepsilon R^2}\]

\[\cdot \exp\left(\beta + \delta \right) R^4 + \frac{2K_{\eta} + \frac{4}{3}(\frac{1}{\sqrt{12}} + \eta)^2(n+1)^{3/2}}{\varepsilon R^2}\]

$R$ will be chosen next so as to minimize the right hand side of this inequality, and at the same time, to satisfy (I.2.3.11).

Consider the function

\[h(R) = \frac{1}{R^{2n}} \exp\left(\beta + \delta \right) + \frac{2K_{\eta} + \frac{4}{3}(\frac{1}{\sqrt{12}} + \eta)^2(n+1)^{3/2}}{\varepsilon R^2}.\]

By differentiating $h(R)$ and setting $h'(R) = 0$, we get
\[(I.2.3.13) \quad 2(\beta + \delta)R^6 - \frac{2}{\varepsilon}[K\eta + \frac{2}{3}(\frac{1}{\sqrt{12}} + \eta)^2(n + 1)^{3/2}] - nR^2 = 0.\]

For \(n\) sufficiently large \((n > \max(N_1, N_2))\), we will have \(R_n > N_2\).

Also, from \((I.2.3.7)\), it follows that \(R^2_n \geq \frac{x_{1,n+1}^2}{2\varepsilon^{1/3}(\beta + \delta)^{1/3}}\) and consequently, condition \((I.2.3.11)\) will be satisfied if

\[(I.2.3.14) \quad 2\varepsilon^{1/3}(\beta + \delta)^{1/3} = 1 - \varepsilon.\]

Since \(R_n\) satisfies equation \((I.2.3.13)\), we find that

\[\frac{1}{2}(\beta + \delta)R^4_n = n + \frac{K\eta + \frac{2}{3}(\frac{1}{\sqrt{12}} + \eta)^2(n + 1)^{3/2}}{\varepsilon R_n^2},\]

and it follows that

\[|I_n| \leq \frac{1}{\gamma_0^2(4\sqrt{6})^{N_1}}(\frac{1}{\sqrt{12}} + \eta)^{2n \sqrt{n!}} \frac{\varepsilon^{n/3}(\beta + \delta)^{n/3}}{[K\eta + \frac{2}{3}(\frac{1}{\sqrt{12}} + \eta)^2(n + 1)^{3/2}]^{n/3}} \cdot \exp\left(\frac{n}{2} + \frac{3K\eta + 2\left(\frac{1}{\sqrt{12}} + \eta\right)^2(n + 1)^{3/2}}{\varepsilon R_n^2}\right)\]

\[\leq \frac{1}{\gamma_0^2(4\sqrt{6})^{N_1}}(\frac{1}{\sqrt{12}} + \eta)^{2n \sqrt{n!}} \frac{\varepsilon^{n/3}(\beta + \delta)^{n/3}}{[K\eta + \frac{2}{3}(\frac{1}{\sqrt{12}} + \eta)^2(n + 1)^{3/2}]^{n/3}} \cdot \exp\left(\frac{n}{2} + \frac{\varepsilon^{1/3}(\beta + \delta)^{1/3}[3K\eta + 2\left(\frac{1}{\sqrt{12}} + \eta\right)^2(n + 1)^{3/2}]}{\varepsilon K\eta + \frac{2}{3}(\frac{1}{\sqrt{12}} + \eta)^2(n + 1)^{3/2}]^{1/3}}\right)\]
Using (I.2.3.14) and the Stirling formula, we find that

\[ |I_n| \leq K_1(\eta) \left( \frac{1}{\sqrt{12}} + \eta \right)^{\frac{3n}{2}} \exp\left(\frac{\frac{3}{2} \frac{1}{\sqrt{12}}}{\frac{1}{\sqrt{12}} + \eta} \right) \]

where \( K_1(\eta) \) is a constant depending on \( \eta \), \( \varepsilon_1(\eta) \to 0 \) \((\eta \to 0)\), \( \varepsilon_2(\eta) \to 0 \) \((\eta \to 0)\), and \( \varepsilon_3(n) \to 0 \) \((n \to \infty)\).

It is easy to see that \( g(\varepsilon) = \frac{1 - \varepsilon}{4} \exp\left(\frac{1 - \varepsilon}{2\varepsilon}\right) \) is a continuous decreasing function on \((0,1)\). Consequently, if \( \varepsilon_0 \) is the solution of \( g(\varepsilon) = 1 \), then we have \( 0 < g(\varepsilon) < 1 \) for \( \varepsilon_0 < \varepsilon < 1 \).

Thus, we can find, for small enough \( \eta \) and a large enough \( n \), a number \( \lambda < 1 \) such that
\[
\left(\frac{1}{4} + \epsilon_1(\eta)\right)e^{\frac{1}{4} + \epsilon_1(\eta)} (1 + \epsilon_2(\eta)) < \lambda < 1
\]

which would imply that \(\Sigma |I_k|\) is a convergent series. If

\[
\Delta_n = \int_{-\infty}^{\infty} f(x) e^{-x} dx - Q_n(e^{-x}; f) = \sum_{k=n}^{\infty} I_k
\]

then we have, for all \(n\) sufficiently large,

\[
|\Delta_n| \leq \sum_{k=n}^{\infty} |I_k| \leq K(\epsilon, \eta) \cdot n \cdot \left(\frac{1}{4} + \epsilon_1(\eta)\right) e^{\frac{1}{4} + \epsilon_1(\eta)} (1 + \epsilon_2(\eta) + \epsilon_3(n))
\]

where \(K(\epsilon, \eta)\) is a constant depending on \(\eta\) and \(\epsilon\). Hence

\[
\limsup_{n \to \infty} |\Delta_n|^{1/n} \leq \left(\frac{1}{4} + \epsilon_1(\eta)\right)e^{\frac{1}{4} + \epsilon_1(\eta)} (1 + \epsilon_2(\eta))
\]

Since \(\eta\) can be chosen arbitrarily small and \(\epsilon_1(\eta), \epsilon_2(\eta) \to 0\) \((\eta \to 0)\), it follows that

\[
\limsup_{n \to \infty} |\Delta_n|^{1/n} \leq \frac{1}{4} \exp(\frac{1}{2\epsilon}) < 1
\]

It remains to justify the choice of \(\rho\) in (I.2.3.3) and the proof of the theorem will be finished. From (I.2.3.14) and the choice \(\epsilon_o < \epsilon < 1\), we see that \(0 < 2\epsilon_o^{1/3} \beta^{1/3} < 2\epsilon_o^{1/3}(\beta + \delta)^{1/3} < 2\epsilon^{1/3}(\beta + \delta)^{1/3} = 1 - \epsilon < 1 - \epsilon_o\) and therefore

\[
0 < \beta < \frac{(1 - \epsilon_o)^3}{\delta \epsilon_o} = \rho
\]
Hence, we must have $0 < \beta < \rho$ where $\rho = \frac{(1 - x)^3}{8x}$ and $x$ is the solution of $\frac{1 - x}{4} \exp\left(\frac{1 - x}{2x}\right) = 1$.

THEOREM I.2.3.1: Let $f(z)$ be an entire function satisfying the condition

$$\beta = \limsup_{R \to \infty} \frac{\max (\log|f(z)|)}{R^p} < \rho$$

where $\rho$ ($\approx 0.17495268$) is such that $\rho = \frac{15(1 - \epsilon_0)^6}{128\epsilon_0}$ and $\epsilon_0$ is the solution of $\frac{1 - x}{4} \exp\left(\frac{1 - x}{2x}\right) = 1$. Then, we have

$$\limsup_{n \to \infty} \left( \int_{-\infty}^{\infty} f(x)e^{-x^6} \, dx - Q_n(e^{-x^6};f)\right)^{1/n} < 1$$

Proof: The proof of this theorem is, step by step, similar to that of Theorem I.2.3.1, with slight modifications of some of the inequalities and equations due to the change in the weight function to $e^{-x^6}$ instead of $e^{-x^6}$ of the previous theorem.

We outline here the proof without giving all the details.

From (I.2.3.1), with $p = 0$ and $m = 6$, we can see that if we choose an $\eta > 0$, then there exists an $N_\eta$ such that

$$\frac{\gamma_{n-1}}{\gamma_n} \leq \left(\frac{1}{\sqrt[6]{60}} + \eta\right) \frac{\gamma_n}{\sqrt{n}}$$

for all $n \geq N_\eta$.

From (I.1.3.6) and (I.1.3.4), we get
\[
\frac{\gamma_{n-1}}{\gamma_n} \leq 2 \sqrt[3]{\frac{n}{3}} \quad \text{for all } n = 1, 2, 3, \ldots
\]

and

\[
x_{1,n+1} \leq 2 \left(\frac{1}{\sqrt[6]{60}} + \eta\right) \sqrt[n]{3} \quad \text{for all } n \geq \eta,
\]

respectively.

Combining (I.1.3.22), (I.2.3.22) and (I.2.3.21), we get

\[
\sum_{k=1}^{[n/2]} x_{k,n}^2 \leq K_2(\eta) + \frac{3}{4} K_3(\eta) n^{4/3},
\]

where

\[
K_2(\eta) = \left[\sqrt{\eta} - \frac{3}{4} \left(\frac{1}{\sqrt[6]{60}} + \eta\right)^2\right] N_{\eta}^{1/3}
\]

and

\[
K_3(\eta) = \frac{1}{\sqrt[6]{60}} + \eta.
\]

From (I.2.3.18) and (I.2.3.14), we obtain for \( n \geq \eta \)

\[
\frac{1}{\gamma_n} \leq \frac{1}{\gamma_o} \frac{4}{3^{180}} \frac{N_{\eta}^{-1}}{K_3(\eta) \sqrt[n]{3}} \cdot
\]

Denoting the nth orthonormal polynomial generated by the weight function \( e^{-x^6} \) by \( p_n(x) \) and proceeding as in the proof of Theorem I.2.3.1, we can see that

\[
|I_n| \leq \frac{1}{\gamma_o} \frac{4}{3^{180}} \frac{N_{\eta}^{-1}}{K_3(\eta) \sqrt[n]{3}} \frac{1}{R^{2n}} \cdot
\]

\[
\cdot \exp\left(\beta + \delta R^6 + \frac{2K_2(\eta) + \frac{3}{2} K_3(\eta)(n + 1)^{4/3}}{\epsilon R^2}\right)
\]
and
\[ 3(\beta + 6)R^3 - \frac{1}{\varepsilon}[2K_2(\eta) + \frac{3K_3^2(\eta)(n + l)^{1/3}}{3\varepsilon(\beta + \delta)}] - nR^2 = 0. \]

By choosing \( R = R_n = \left[\frac{2K_2(\eta) + \frac{3K_3^2(\eta)(n + l)^{1/3}}{3\varepsilon(\beta + \delta)}}{4(\frac{\varepsilon(\beta + \delta)}{30})^{1/4}}\right]^{1/8} \) and
\[ 4(\frac{\varepsilon(\beta + \delta)}{30})^{1/4} = 1 - \varepsilon, \]
we obtain
\[ |I_n| \leq k_n^4(\eta) \sqrt{n} \left[ (\frac{1 - \varepsilon}{4} + \varepsilon_4(\eta)) \exp(\frac{1 - \varepsilon}{2\varepsilon} (1 + \varepsilon_5(\eta) + \varepsilon_6(n))) \right], \]

where \( k_n^4(\eta) \) is a constant depending on \( \eta, \varepsilon_4(\eta) \to 0 \ (\eta \to 0), \varepsilon_5(\eta) \to 0 \ (\eta \to 0), \) and \( \varepsilon_6(n) \to 0 \ (n \to \infty). \)

And the rest of the proof is clear at this point.

\section{1.2.4. The case of the Pollaczek weight function}

\[ w^{(\lambda)}(x) = |\Gamma(\lambda + ix)|^2; \ \lambda > 0, \ -\infty < x < \infty. \]

The Pollaczek polynomials are defined as follows:

\[ p_{-1}^{(\lambda)} = 0, \quad p_0^{(\lambda)} = 1 \]

and

\[ n_{p_n}^{(\lambda)}(x) = 2xp_n^{(\lambda)}(x) - (n - 2 + 2\lambda)p_{n-1}^{(\lambda)}(x) \]

with parameter \( \lambda > 0 \).

It had been shown that the weight function is
\[ w^{(\lambda)}(x) = |
\Gamma(\lambda + ix)|^2 \text{ which is defined on } (-\infty, \infty). \]
In this case, we have

\[ A_n = \int_{-\infty}^{\infty} [p_n^{(\lambda)}(x)]^2 w^{(\lambda)}(x) dx = \pi 2^{1-2\lambda} \frac{(2\lambda)(2\lambda+1) \ldots (2\lambda+n-1)}{n!} \]

\[ A_0 = \int_{-\infty}^{\infty} w^{(\lambda)}(x) dx = \frac{\pi \Gamma(2\lambda)}{2^{2\lambda-1}} \]

which implies that

\[ (\text{I.2.4.1}) \quad \frac{\gamma_{n-1}}{\gamma_n} = \frac{1}{2\sqrt{n^2 + 2\lambda n - n}}. \]

By combining (I.2.4.1) with (I.1.3.4), we see that

\[ (\text{I.2.4.2}) \quad x_{1,n+1} \leq \sqrt{n^2 + 2\lambda n - n}. \]

It is also known, see e.g., [7;1.18], that

\[ \lim_{|x| \to \infty} |\Gamma(\lambda + ix)| e^{(1/2)\pi |x|} |x|^{(1/2)-\lambda} = (2\pi)^{1/2} \]

which implies that

\[ |\Gamma(\lambda + ix)|^2 \approx \exp(-\pi |x| + (2\lambda - 1)\ln|x| + \ln(2\pi)) , \]

a result upon which our assumption (I.2.4.3) in the following theorem is based.

* Actually, it has been recently proved by Man-Yeki Goh that \( x_{1,n} \approx n \).

For the proof of this result and for more about Pollaczek polynomials and their asymptotics, see [20].
THEOREM I.2.4.1: Let \( f(z) \) be an entire function satisfying the condition

\[
(I.2.4.3) \quad \beta = \limsup_{R \to \infty} \frac{\max |\log|f(z)||}{\pi R - (2\lambda - 1) \ln R - \ln(2\pi)} < \rho,
\]

where \( \lambda > 0 \) and \( \rho (= 0.557737056) \) is the solution of the equation

\[
(I.2.4.4) \quad \frac{\pi x^2}{16} \exp\left(\frac{\pi x^2}{2} \left(\frac{1}{\lambda - \pi x^2}\right)\right) = 1.
\]

Then we have

\[
(I.2.4.5) \quad \limsup_{n \to \infty} \left| \int_{-\infty}^{\infty} f(x) w^{(\lambda)}(x) dx - Q_n(w^{(\lambda)}(x); f) \right|^{1/n} < 1.
\]

Proof: The proof of this theorem, as one might have expected by now, is similar to that of Theorem I.2.2.1, with the necessary modifications due to the change in the weight function.

We now outline the proof in the following steps.

From (I.1.3.22) and (I.2.4.1), we get

\[
\sum_{k=1}^{[n/2]} x_{k,n}^2 - \frac{2n^3 - 3n^2 + n}{2} - \frac{(2\lambda - 1)(n^2 - n)}{8}
\]

which implies that

\[
(I.2.4.6) \quad \sum_{k=1}^{[n/2]} x_{k,n}^2 + \sum_{k=1}^{[(n+1)/2]} x_{k,n+1}^2 = \frac{n^3}{6} \left(1 - \frac{3(2\lambda - 1)}{2n} + \frac{1}{2n^2}\right).
\]

From (I.2.4.1), we also obtain
(I.2.4.7) \[
\frac{1}{\gamma_n} = \frac{1}{\gamma_0} \frac{(n!)^2}{4^n} \frac{n}{\pi} \sum_{k=1}^{n} \left(1 + \frac{2\lambda - 1}{k}\right).
\]

Hence, it follows that

\[
|I_n| \leq \frac{1}{\gamma_0} \frac{(n!)^2}{4^n} \frac{n}{\pi} \sum_{k=1}^{n} \left(1 + \frac{2\lambda - 1}{k}\right) \frac{1}{R^{2n}} \cdot \exp\left(\left((\beta + \delta)(\pi R - (2\lambda - 1) \log R - \log(2\pi)) + \frac{n^3}{6\varepsilon R^2} \left(1 - \frac{3(2\lambda - 1)}{2n} + \frac{1}{2n^2}\right) \right) + \right)
\]

and

\[
\pi(\beta + \delta)R^3 - [(\beta + \delta)(2\lambda - 1) + 2n]R^2 - \frac{n^3}{3\varepsilon} \left(1 - \frac{3(2\lambda - 1)}{2n} + \frac{1}{2n^2}\right) = 0.
\]

By choosing \( R = R_n = \frac{2n + (2\lambda - 1)(\beta + \delta)}{\pi(\beta + \delta)} \) and \( \frac{\pi^2(\beta + \delta)^2}{4} = 1 - \varepsilon \), we obtain

\[
|I_n| \leq Kn \left(\frac{1 - \varepsilon}{4}\right)^n \frac{n}{\pi} \sum_{k=1}^{n} \left(1 + \frac{2\lambda - 1}{k}\right) \frac{1}{(1 + (2\lambda - 1)(1 - \varepsilon)^{1/2})^{2n}} \cdot \exp\left(\frac{1 - \varepsilon}{2\varepsilon} (1 + \varepsilon(n))\right)^n
\]

where \( K \) is a constant and \( \varepsilon(n) \to 0 \) \( (n \to \infty) \). Since

\[
\frac{n}{\pi} \sum_{k=1}^{n} \left(1 + \frac{2\lambda - 1}{k}\right) \leq e \sum_{k=1}^{n} \left(\frac{2\lambda - 1}{k}\right) \leq e(2\lambda - 1)(\log n + O(1)) = o(n^{2\lambda - 1})
\]
and \((1 + \frac{(2\lambda - 1)(1 - \epsilon)^{1/2}}{\pi n})2n \to e^{\frac{2(2\lambda-1)(1-\epsilon)^{1/2}}{\pi}} (n \to \infty)\), it follows that

\[|I_n| \leq K_1 n^{2\lambda} \left(\frac{1 - \epsilon}{4}\right)^n \left(\exp\left(\frac{1 - \epsilon}{2\epsilon} (1 + \epsilon(n))\right)\right)^n.\]

Hence,

\[
\limsup_{n \to \infty} |\Delta_n|^{1/n} \leq \frac{1 - \epsilon}{4} \exp\left(\frac{1 - \epsilon}{2\epsilon}\right) < 1.
\]
PART II

ERROR ESTIMATES ON PADÉ APPROXIMANTS

OF STIELTJES SERIES
CHAPTER II.1

ON PADÉ APPROXIMANTS AND STIELTJES SERIES

§II.1.1. Introduction

The subject of Padé approximants is fairly old. It dates back to as early as Cauchy (1789-1857) and Jacobi (1804-1851) but was first treated in detail by Frobenius in 1881 [17]. Under the influence of Hermite (1822-1901) at the École Normale Supérieure, Padé, in his 1892 dissertation [27], classified these rational fraction approximants (now known as Padé approximants), arranged them in a table (which is now called the Padé table) and studied the structure of it in detail. In those early days, Padé approximants were considered as another form of continued fraction, which was already a well established subject. Indeed, if

\[ f(z) = b_0 + \frac{a_1 z}{1 + \frac{a_2 z}{1 + \frac{a_3 z}{1 + \ldots}}} = c_0 + c_1 z + c_2 z^2 + \ldots , \]

and \( P_0(z) = b_0 , \ P_1(z) = b_0 + a_1 z , \ P_2(z) = b_0 + \frac{a_1 z}{1 + a_2 z} , \ldots \)

are the truncations of the above continued fraction, then the even ones \( P_0 , P_2 , P_4 , \ldots \) occupy the main diagonal and the odd ones \( P_1 , P_3 , P_5 , \ldots \) march down the \([n,n+1]\) diagonal of the Padé table (to be defined later) of the formal power series \( c_0 + c_1 z + c_2 z^2 + \ldots . \)
References to the subject of Padé approximants and continued fractions can be found in the books Baker [2, 3], Luke [23, 24] and Wall [32].

During the early 1960's, physicists found that Padé approximants are very useful as a systematic method of extracting more information from formal power series expansions, and the method has been especially successful in critical phenomena. In scattering theory and quantum field theory where series of Stieltjes occur frequently, the approximants are known to converge to the expected solutions rapidly. The main contributors in this direction include the physicists or chemists Baker, Bessis, Chisholm, Fisher, Gammel, Nuttall, Wheeler and others. The interested reader should consult the articles listed in [6]. It should be noted, however, that hundreds of papers related to Padé approximants have been written during the past sixteen years.

In the next section of this chapter, we will discuss the different definitions of the Padé approximants and the Padé table. The third section will be devoted to the Stieltjes series and the Stieltjes transform.

§II.1.2. Padé approximants and Padé table for formal power series

The definition of Padé approximants is a natural extension of that of the Taylor series. Let

\[
(II.1.2.1) \quad f(z) = \sum_{i=0}^{\infty} a_i z^{-i}
\]
be a given formal power series with complex coefficients $a_i$. If $m$ and $n$ are natural numbers the $[n,m]$ Padé approximant of $f$ at the point infinity, denoted by $[n,m](f)$, is a rational function

$$\frac{P_m}{Q_n},$$

where $P_m(z) = \sum_{i=0}^{m} a_{i} z^{-i}$ and $Q_n(z) = \sum_{i=0}^{n} \beta_{i} z^{-i}$, $Q_n(z) \neq 0$,

and the coefficients $\alpha_i$ and $\beta_i$ are determined so that

$$(\text{II}.1.2.2) \quad f(z)Q_n(z) - P_m(z) = A z^{-(n+m+1)} + \ldots,$$

where $A$ is a constant (which may be zero). This means that the $n + m + 1$ coefficients in the power series expansion of the left hand side of (II.1.2.2) shall be zero, i.e., that the numbers $\alpha_i$ and $\beta_i$ shall satisfy the system of equations

$$(\text{II}.1.2.3) \quad \sum_{i=0}^{n} \alpha_{j-i} \beta_i = \begin{cases} \alpha_i, & 0 \leq j \leq m \\ 0, & m + 1 \leq j \leq n + m \end{cases}$$

(those $\alpha_k$ that may occur with negative indexes $k$ shall be replaced by 0). Since the last $n$ equations in (II.1.2.3) have $n + 1$ unknowns $\beta_i$ it is always possible to choose $\beta_i$ so that these $n$ equations are satisfied. After that, the numbers $\alpha_j$ are determined from the $m + 1$ first equations of (II.1.2.3).

In spite of the fact that the numbers $\beta_i$ and $\alpha_i$ — and hence $P_m$ and $Q_n$ — are clearly not uniquely determined, the $[n,m]$ Padé approximant $P_m/Q_n$ is unique which is proved in the following way (Frobenius [17;p.2]): Let $P^*_m/Q^*_n$ be another $[n,m]$ Padé approximant
to $f$ and consider the expression

$$(fQ_n - P_m)Q_n^* - (fQ_n^* - P_m^*)Q_n = Q_n P_m^* - Q_n^* P_m.$$  

In the left hand side, all the coefficients of terms of degree at least $-(n + m)$ are zero by the definition of the $[n,m]$ Pade approximant. However, the right hand side is a polynomial in $z^{-1}$ of degree at most $n + m$ and consequently the expression vanishes identically which means that $P_m/Q_n = P_m^*/Q_n^*$, i.e., the $[n,m]$ Pade approximant to $f$ is unique.

In fact, the ratio $P_m(z)/Q_n(z)$ is given by Baker [2], namely,

$$\frac{P_m(z)}{Q_n(z)} = [n,m](f;z) = \frac{a_{m-n+1} a_{m-n+2} \cdots a_{m+1}}{z^{-n} z^{-n+1} \cdots 1},$$

when the determinant in the denominator is non-zero.

The Padé approximant can be defined in two other ways, which we state in the following two propositions (see [33]).
PROPOSITION II.1.2.1: Let $f$ be the formal power series (II.1.2.1). Let $P_m^*$ and $Q_n^*$, $Q_n^* \neq 0$, be polynomials in $z^{-1}$ of degree at most $m$ and $n$, respectively, determined such that

$$f(z)Q_n^*(z) - P_m^*(z) = Az^{-\tau} + ...,$$

where $A$ is a constant and the integer $\tau$ is as large as possible. (If no largest $\tau$ exists, we put $\tau = \infty$.) Then $P_m^*/Q_n^*$ is the $[n,m]$ Padé approximant of $f$.

PROPOSITION II.1.2.2: Let the polynomials in $z^{-1}$, $P_m^*$ and $Q_n^*$, $Q_n^*(z) \neq 0$, of degrees at most $m$ and $n$, be determined so that

$$f(z) - \frac{P_m^*(z)}{Q_n^*(z)} = Az^{-\tau} + ..., $$

where $A$ is a constant and the integer $\tau$ is as large as possible (or infinity). Then, $P_m^*/Q_n^*$ is the $[n,m]$ Padé approximant of $f$.

This means that $[n,m](f)$ is the unique rational function which has contact with $f$ of highest order at infinity.

Padé himself placed the approximants belonging to a given formal power series in a table of double entry, which is called the Padé table (see [27]) and Table 1 on the next page.
The sequence of squares in this table with $m = n + j$, for some fixed integer $j$, is called the $j$th diagonal. In particular, if $j = 0$, the sequence is called the main diagonal.

It is clear that the first row in the Padé table is actually the sequence of partial sums of the formal power series of $f$ itself.

We conclude this section by giving the following example.

**Example:** Let $f(z) = e^{z^{-1}} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \ldots$ then

$$[2,1](e^{z^{-1}}) = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{2}{3}z^{-1} + \frac{1}{6}z^{-2}}$$
because

\[(1 - \frac{2}{3}z^{-1} + \frac{1}{6}z^{-2})(1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \ldots) - (1 + \frac{1}{3}z^{-1})\]

\[= \frac{1}{72}z^{-4} + \ldots.\]

§II.1.3. The Stieltjes series and the Stieltjes transform

Our main results in the following chapter will be dealing with estimates of the error in approximating the Stieltjes transform by the Padé approximants of the associated Stieltjes series. In this section, we introduce some necessary definitions and properties concerning the Stieltjes transform.

Let \(\alpha\) be a non-decreasing bounded real-valued function on the interval \((a,b)\), with infinitely many points of increase. Then the measure \(d\alpha\) is positive on \((a,b)\). If we assume that all the moments

\[(II.1.3.1) \quad a_i = \int_a^b t^i d\alpha(t)\]

are finite, the formal power series

\[(II.1.3.2) \quad \sum_{i=0}^{\infty} a_i z^{-i}\]

is called a series of Stieltjes. In a natural way, this formal power series is associated with the function
The function $f(z)$ is called a Stieltjes transform of the function $\alpha(t)$. The function $f(z)$ is an analytic single-valued function of $z$ in the complex plane, cut along the interval $(a, b)$. For a proof of this last property of $f(z)$, see e.g., [32; p. 247].

If $a = -\infty$, $b = +\infty$, then the integral may represent two regular analytic functions (one in the upper half plane $\text{Im}(z) > 0$ and one in the lower half plane $\text{Im}(z) < 0$) which are not analytic continuations of each other.
§II.2.1 Introduction

The problem of convergence of a certain sequence of Padé approximants is in general a very difficult but certainly interesting and important one.

The only simple general class of formal power series whose "diagonal" Padé approximants are known to converge nicely is the class of series of Stieltjes associated with Stieltjes transform \( \int_0^\infty \frac{dx(t)}{1 - z^{-1}t} \).

It was Stieltjes [29] who first proved that the moment problem on \([0,\infty)\) associated with the sequence \( \{a_n\} \), where \( a_n \) is as defined by (II.1.3.1), is determined if and only if the corresponding continued fraction expansion of \( f(z) = \sum_{i=0}^\infty a_i z^{-i} \) converges everywhere in the complex plane cut along the interval \([0,\infty)\). Since the truncations of the continued fraction expansion of \( f \), as mentioned in §II.1.1, are the \([n,n]\) or \([n,n+1]\) Padé approximants of \( f \), it follows that if, say,

\[
\sum_{i=0}^\infty a_i^{-1/2i} = \infty
\]
(see Carleman [4]; for instance, $a_1 = 0((2i)! R^{2i})$ for some $R > 1$), then each of the sequences $[[n,n](f)]$ and $[[n,n+1](f)]$ converges to the Stieltjes transform $\int_1^\infty \frac{d\alpha(t)}{1 - z^{-1}t}$ uniformly on every compact set in the complex plane which is disjoint from $[0,\infty)$.

In 1974, Freud in his paper [13] estimated the error in approximating

$$f(z) = \int_{-1}^1 \frac{d\alpha(t)}{z - t} \quad (z \notin [-1,1])$$

by the $[n,n]$ Padé approximants of the Stieltjes series associated with this integral. In his proof, Freud used the following form for the error term:

$$f(z) - [n,n](f;z) = \sum_{v=n}^{\infty} \frac{\gamma_{v+1}(d\alpha)}{\gamma_v(d\alpha)} \frac{1}{p_v(d\alpha;z)} \frac{1}{p_{v+1}(d\alpha;z)}$$

where $\gamma_v(d\alpha)$ and $p_v(d\alpha;z)$ are as defined in Chapter 0 of this thesis. Freud also used an estimate of $p_v^{-1}(d\alpha;z)$ using the Chebyshev polynomial of the first kind $T_v(z)$.

In 1975, Allen, Chui, Madych, Narcowich and Smith, gave in their joint paper [1], some useful explicit results that shows a natural connection between the Padé approximants of a series of Stieltjes and orthogonal polynomials. They also obtained the precise error formulas. Due to the importance of their results in our work, we summarize some of their results in the following theorem.
THEOREM II.2.1.1: Let \( f(z) = \sum_{i=0}^{\infty} a_i z^{-i} \) be a series of Stieltjes with measure \( \alpha \) as defined earlier and let \( j \geq -1 \) be an integer.

Suppose that

\[ \alpha_j(t) = t^{j+1} \alpha(t) \]

and that

\[ p_n(z) = p_n(\alpha_j; z) = \sum_{i=0}^{n} \gamma_i^{(n)}(\alpha_j) z^i \]

be the orthogonal polynomials generated by \( \alpha_j \) with leading coefficients \( \gamma_i^{(n)}(\alpha_j) = \gamma_i^{(n)} \), with zeros at \( \chi_{kn}(\alpha_j) = \chi_{kn} \) and Christoffel numbers \( \lambda_{kn}(\alpha_j) = \lambda_{kn} \), \( k = 1, 2, \ldots, n \).

Then the \([n,n+j] \) Padé approximant of \( f \) is given by

\[
\begin{align*}
[n,n+j](f;z) &= \sum_{i=0}^{j} a_i z^{-i} + \frac{p_{n+j}(z)}{Q_n(z)} \\
&= \sum_{i=0}^{j} a_i z^{-i} + z^{-j-1} \sum_{k=1}^{n} \lambda_{kn} (1 - z^{-1} \chi_{kn})^{-1} ;
\end{align*}
\]

where

\[ p_{n+j}(z) = \sum_{k=0}^{n-1} z^{-k} \sum_{j=n-k}^{n} \gamma_{2n-j-k}^{(n)} a_{n-j} \]

and

\[ Q_n(z) = z^{-n} p_n(z) = \sum_{k=0}^{n} \gamma_{n-k}^{(n)} z^{-k} . \]
The error is given by the following formulae

\[ E_{n,j}(z) = \int_0^\infty \frac{\alpha'(t)}{1 - z^{-1}t} - [n, n+j](f; z) \]

\[ = \frac{z^{-j}}{p_n(z)} \int_0^\infty \frac{p_n(t) \alpha_j'(t)}{z - t} \]

\[ = \frac{z^{-2n-j}}{z-n p_n(z)} \int_0^\infty \frac{p_n(t)t^n \alpha_j'(t)}{z - t} \]

\[ = \frac{z^{-j}}{p_n^2(z)} \int_0^\infty \frac{p_n^2(t) \alpha_j'(t)}{z - t} ; \quad z \notin [0, \infty) . \]

In the next section, we are going to estimate the error in approximating a Stieltjes transform of the form \( \int_a^b \frac{\alpha(t)}{1 - z^{-1}t} \), where \( a \) is finite and \( b \) is either finite or infinite, by the \([n, n+j]\) Padé approximant of the associated Stieltjes series. In doing so, we are going to use the error form given in Theorem II.2.1.1 above. Namely,

\[ (II.2.1.2) \quad \int_0^b \frac{\alpha(t)}{1 - z^{-1}t} - [n, n+j](f; z) = \frac{z^{-j}}{p_n^2(z)} \int_0^b \frac{p_n^2(t) \alpha_j'(t)}{z - t} \]

where \( z \in \mathbb{C} \setminus [0, b] \), \( \mathbb{C} = \) complex plane and \( \alpha \) is supported in the interval \([0, b]\).

* We can assume, without loss of generality, that \( z = 0 \).
We are also going to apply the same technique that Freud [14] applied in estimating \( p_n^{-1}(z) \) by using the Chebyshev polynomial \( T_n(z) \).

§II.2.2. Main results

Our main results in this chapter are the following:

**THEOREM II.2.2.1:** Let \( \alpha \) be a measure with support in \([0,1]\) and let \( f(z) \) be the Stieltjes series \( \sum_{i=0}^{\infty} a_i z^{-i} \), \( a_i = \int_0^1 t^i \alpha(t) \).

Then for any positive integer \( n \) (\( n \geq 2 \)) and any integer \( j \) (\( j \geq -1 \)), we have

\[
(II.2.2.1) \quad \left| \int_0^1 \frac{\alpha(t)}{1 - z^{-1}t} - [n,n+j](f;z) \right| \leq \frac{a_{j+1}}{\Delta^3 |z|^j (R^{n-1} - R^{-n+1})^2}
\]

where \( \Delta = d(z;[0,1]) \) = the distance between \( z \) and the interval \([0,1] \), \( R = |w| > 1 \), \( w = 2z - 1 - 2\sqrt{z(z-1)} \) and \( z \in \mathbb{C} \setminus [0,1] \).

**THEOREM II.2.2.2:** Let \( \alpha \) be a measure with support contained in \([0,\infty) \) and \( \alpha(t) \leq e^{-t}dt \). Let \( f(z) = \sum_{i=0}^{\infty} a_i z^{-i} \), \( a_i = \int_0^{\infty} t^i \alpha(t) \).

Then for every positive integer \( n \) and every fixed integer \( j \) (\( j \geq -1 \)), we have
(II.2.2.2) \[ \int_0^\infty \frac{d\alpha(t)}{1 - z^{-1}t} \leq [n+1,n+j+1](f;z) \]

\[ \leq \frac{4\pi|z|^{3/2}}{\Delta^{n^3+(1/2)}} \frac{\gamma_n^2(\alpha_j)}{\gamma_{n+1}^2(\alpha_j)} \frac{1 + \epsilon_n}{\exp(\text{Re}(z))[\exp(4\text{Re}(\sqrt{-z}))]^\frac{1}{n}} \]

where \( \Delta = d(z;[0,\infty)) \), \( \alpha_j(t) = t^{j+1}\alpha(t) \), \( \epsilon_n \to 0 (n \to \infty) \), \( z \in \mathbb{C} \setminus [0,\infty) \), \( \sqrt{-z} \) must be taken real and positive if \( z < 0 \), \( \text{Re}(\sqrt{z}) = \text{the real part of } \sqrt{z} \).

Moreover, the bound for the remainder in the right hand side of (II.2.2.2) holds uniformly in every closed domain with no points in common with \( z \geq 0 \).

THEOREM II.2.2.3: Let \( Q(t) \), \( 0 \leq t < \infty \), be a convex differentiable function for \( t > 0 \), with \( Q'(t) > 0 \) and non-decreasing for \( t > 0 \). Let \( \{q_n\} \) be the sequence defined by (I.1.2.1) and \( w_Q(t) = \exp(-2Q(t)) \). Let

(II.2.2.3) \[ \int_0^\infty \exp((1 - \frac{a}{n})t - 2Q(t))dt < \infty \]

for some positive constant \( a \) and \( n = 1,2,3,\ldots \). Then

(II.2.2.4) \[ \left| \int_0^\infty \frac{\exp(-2Q(t))dt}{1 - z^{-1}t} \right| - [n+1,n](f;z) \leq \]

\[ \leq \frac{K|z|^{3/2}n^{1/2}q_n^2}{\Delta^{n^3\exp(\text{Re}(z))}} \frac{1 + \epsilon_n}{[\exp(4\text{Re}(\sqrt{-z}))]^\frac{1}{n}} \]
where $\Delta = d(z;[0,\infty))$, $\varepsilon_n \to 0 \ (n \to \infty)$, $z \in \mathbb{C} \setminus [0,\infty)$, $\sqrt{-z}$ must be taken real and positive if $z < 0$, $\Re(\sqrt{z}) = \text{the real part of } \sqrt{z}$.

Moreover, the bound for the remainder in the right hand side of (II.2.2.4) holds uniformly in every closed domain with no points in common with $z > 0$.

§II.2.3. Auxiliary results

To prove the theorems of the previous section, we are going to use the following two lemmas whose proofs, apart from some slight modifications to suit our purpose, are in Freud [9;§III.7].

**Lemma II.2.3.1:** Let $T_n(z)$ be the Chebyshev polynomial of the first kind of degree $n$. Then

$$
(\text{II.2.3.1}) \quad \frac{1}{2}(R^n - R^{-n}) \leq \left| T_n(2z - 1) \right| \leq \frac{1}{2}(R^n - R^{-n})
$$

where $z \in \mathbb{C} \setminus [0,1]$, $R = |w| > 1$ and

$$
(\text{II.2.3.2}) \quad w = 2z - 1 + 2\sqrt{z(z - 1)}.
$$

Proof: By $\sqrt{z(z - 1)}$, we denote the branch of this function in $\mathbb{C} \setminus [0,1]$ which takes positive values for real $z > 1$. If we let $2z - 1 = \cos \theta$, $0 \leq \theta \leq \pi$, then it follows that the function
\[ h(z) = 2z - 1 - 2\sqrt{z(z - 1)} = \cos \theta - i \sin \theta = e^{-i\theta} \]

is analytic, single-valued in \( z \in \mathbb{C} \setminus [0,1] \).

We can easily see that \( h(z) \) tends to 0 when \( |z| \to \infty \), and it tends to \( e^{-i\theta} \) or to \( e^{i\theta} \) when \( z \) tends to the point \( \cos \theta \), \( 0 \leq \theta \leq \pi \), of the upper or, respectively, of the lower shore of the cut \([0,1]\).

Therefore, it follows from the maximum principle that,
\[ |h(z)| < 1 \quad \text{for} \quad z \in \mathbb{C} \setminus [0,1] \].

By combining this last inequality with the identity
\[ (2z - 1 - 2\sqrt{z(z - 1)}) (2z - 1 + 2\sqrt{z(z - 1)}) = 1, \]
we obtain that \( R = |w| > 1 \), where \( w = 2z - 1 + 2\sqrt{z(z - 1)} \).

From the well known formula
\[ T_n(z) = \frac{1}{2}((z + \sqrt{z^2 - 1})^n + (z - \sqrt{z^2 - 1})^n) \]
we get
\[ T_n(2z - 1) = \frac{1}{2}((2z - 1 + 2\sqrt{z(z - 1)})^n + (2z - 1 - 2\sqrt{z(z - 1)})^n) \]
\[ = \frac{1}{2}(w^n + w^{-n}) \]

which, by taking absolute values, implies (II.2.3.1).

This completes the proof of the Lemma.
LEMMA II.2.3.2: Let \( \mathbb{d}_1 \) and \( \mathbb{d}_2 \) be two arbitrary measures with supports contained in \([a, b]\), \(-\infty < a < b < \infty\). Then

\[
(\text{II.2.3.3}) \quad \left| \frac{p_{n-1}(\mathbb{d}_2;z)}{p_n(\mathbb{d}_1;z)} \right| \leq \frac{1}{\Delta} \frac{\gamma_{n-1}(\mathbb{d}_1)}{\gamma_n(\mathbb{d}_1)} \left( \int_a^b p_{n-1}^2(\mathbb{d}_2;t) d\mathbb{B}_1(t) \right)^{1/2}
\]

where \( z \in \mathbb{C} \setminus [a, b] \) and \( \Delta = d(z;[a, b]) \).

\textbf{Proof:} We express \( p_{n-1}(\mathbb{d}_2;z) \) by the aid of the Lagrange interpolation formula over the zeros \( x_{kn} \) (\( k = 1, 2, \ldots, n \)) of \( p_n(\mathbb{d}_1) \). From [9; formula (6.3)§III.6], we deduce

\[
p_{n-1}(\mathbb{d}_2;z) = \frac{\gamma_{n-1}(\mathbb{d}_1)}{\gamma_n(\mathbb{d}_1)} \sum_{k=1}^n \left[ \lambda_n(\mathbb{d}_1;x_{kn}) \frac{p_{n-1}(\mathbb{d}_1;x_{kn}) p_{n-1}(\mathbb{d}_2;x_{kn})}{z - x_{kn}} \right] p_n(\mathbb{d}_1;z)
\]

where \( \lambda_n(\mathbb{d}_1;x_{kn}) \) are the Christoffel numbers.

Therefore,

\[
\frac{p_{n-1}(\mathbb{d}_2;z)}{p_n(\mathbb{d}_1;z)} \leq \frac{1}{\Delta} \frac{\gamma_{n-1}(\mathbb{d}_1)}{\gamma_n(\mathbb{d}_1)} \sum_{k=1}^n \left| \lambda_n(\mathbb{d}_1;x_{kn}) \right| \left| p_{n-1}(\mathbb{d}_1;x_{kn}) \right| \left| p_{n-1}(\mathbb{d}_2;x_{kn}) \right|
\]

\[
\leq \frac{1}{\Delta} \frac{\gamma_{n-1}(\mathbb{d}_1)}{\gamma_n(\mathbb{d}_1)} \left( \sum_{k=1}^n \lambda_n(\mathbb{d}_1;x_{kn}) p_{n-1}^2(\mathbb{d}_1;x_{kn}) \right)^{1/2}
\]

\[
\cdot \left( \sum_{k=1}^n \lambda_n(\mathbb{d}_1;x_{kn}) p_{n-1}^2(\mathbb{d}_2;x_{kn}) \right)^{1/2}
\]

and in view of the quadrature formula, we obtain (II.2.3.6).

This completes the proof of the Lemma.
In addition to the above two lemmas, we are going to use the following two theorems.

**THEOREM II.2.3.3**: Let \( r \) be an arbitrary real number. Then the generalized Laguerre polynomials \( L_n^{(r)} \) satisfy

\[
L_n^{(r)}(z) = \frac{1}{2^n} \frac{1}{(-r/2) - (1/4)} \cdot e^{-a/2} \cdot \exp\left(2(-n)^{1/2} \sum_{v=0}^{p-1} c_v(z) n^{-p/2} + o(n^{-p/2})\right),
\]

where \( z \in \mathbb{C} \setminus [0, \infty) \), \( c_0(z) = 1 \), \( c_v(z) \) is independent of \( n \); it is regular in \( \mathbb{C} \setminus [0, \infty) \), \( -z^{(-r/2)-(1/4)} \) and \( (-z)^{1/2} \) must be taken real and positive if \( z < 0 \). The bound for the remainder holds uniformly in every closed subset of \( \mathbb{C} \setminus [0, \infty) \).

**Proof**: See [30; Theorem 8.22.3].

Let us now denote by \( \Phi \) the class of non-negative functions \( \varphi(x) \) which satisfy the following three conditions:

1. \( \varphi(x) \) is differentiable for \( x \neq 0 \),
2. \( \varphi'(x) > 0 \) for \( x > 0 \),

and

3. \( \varphi'(x) \) is non-decreasing for \( x > 0 \).

**Example**: For \( 0 \leq \rho \leq 1 \), we have \( \varphi_\rho(x) = |x|^\rho \in \Phi \).
THEOREM II.2.3.4: Let $Q^*(x) = x^{r/2} [\varphi(x^{1/2})]^{-1} e^{x/2}$ where
\[ \varphi \in \mathcal{S} \text{ and } r \text{ is a non-positive integer.} \]
If for a polynomial $P_n(x)$ of degree not greater than $n$ and an $\varepsilon > 0$, we have
\[ |P_n(x)| \leq Q^*(x), \quad 0 \leq x \leq (1 + \varepsilon)n, \]
then there exists two constants $c_\varepsilon$ (depending on $\varepsilon$ at most) and $a$ such that
\[ |P_n(x)| \leq c_\varepsilon Q^*(x) e^{-ax/n}, \quad x > 0. \]

Proof: See [10;Theorem III].

We are now ready to prove the main results stated in the previous section.

§II.2.4. Proofs of main results

Proof of Theorem II.2.2.1: From (II.2.1.2) and the assumption that $\alpha^j$ is supported in $[0,1]$, we get

\[ (II.2.4.1) \int_0^1 \frac{d\alpha(t)}{1 - z^{-1} t} - [n,n+j](f;z) = \frac{1}{z^j n^2 p_n(\alpha_j; z)} \int_0^1 \frac{p_n^2(\alpha_j; t) \alpha_j^j(t)}{z - t} \]  

where $z \in \mathbb{C} \setminus [0,1]$ and $j \geq -1$. 
From (II.2.4.1) and the fact that \( \int_0^1 p_n^2(\alpha_j'z)dz = 1 \), it follows

\[
\int_0^1 \frac{\alpha(t)}{1-z^{-1}t - [n,n+j]}(f;z) \leq \frac{1}{\Delta|z|^j|p_n^2(\alpha_j'z)|}.
\]

Therefore, the proof of the theorem will be complete if we find an inequality for \(|p_n(\alpha_j'z)|^{-2}\) in (II.2.4.2) above.

From (II.2.3.3) with \( a = 0 \), \( b = 1 \), \( d\beta_2(t) = (t(1-t))^{-1/2} \)
and \( d\beta_1(t) = \alpha_j(t) = t^{j+1}\alpha(t) \), and the known facts

\[
p_n([t(1-t)]^{-1/2};t) = \sqrt{2} T_n(2t - 1) \quad \text{(see e.g., [30;4.1.2) and (2.3.4)]},
\]

and

\[
\frac{\gamma_{n-1}(\alpha_j)}{\gamma_n(\alpha_j)} \leq \frac{1}{2} \quad \text{(see e.g., [9;81.7, Lemma 7.2])},
\]

we get

\[
\frac{1}{|p_n(\alpha_j'z)|^2} \leq \frac{1}{4\Delta^2} \frac{a_{j+1}}{|T_{n-1}(2z - 1)|^2}; \quad a_j = \int_0^1 t^j\alpha(t).
\]

From (II.2.3.1), we obtain

\[
\left| \frac{1}{T_{n-1}(2z - 1)} \right| \leq \frac{2}{R^{n-1} - R^{-n+1}}
\]

which when combined with the previous inequality and (II.2.4.2) implies (II.2.2.1).

This proves the theorem.
Proof of Theorem II.2.2.2: From (II.2.1.2), the assumption that \( \alpha \) is supported in \([0, \infty)\), and the fact that \( \int_0^\infty p_n^2(\alpha_J)\,d\alpha_J = 1 \), it follows that

\[
(II.2.4.3) \quad \left| \int_0^\infty \frac{d\alpha(t)}{1 - z^{-1}t} - [n+1,n+j+1](f;z) \right| \leq \frac{1}{\Delta|z|^j|p_{n+1}(\alpha_J;z)|^2};
\]

\( j \geq -1 \) and \( z \in \mathbb{C} \setminus [0, \infty) \).

Next, we are going to find an inequality for \( |p_n(\alpha_J;z)|^{-1} \). Using (II.2.3.3) with \( a = 0 \), \( b = \infty \), \( d\alpha(t) = t^j e^{-t} dt \) and \( d\beta(t) = \alpha(t) = t^j e^t \), and the fact that \( p_n(t^j e^{-t};t) = \mathcal{L}_n^{(j+1)}(t) \); \( \mathcal{L}_n^{(j+1)} \) is the nth generalized Laguerre polynomial, and the assumption \( \alpha(t) \leq e^{-t} \) \( dt \), we get

\[
\frac{1}{|p_{n+1}(\alpha_J;z)|^2} \leq \frac{1}{\Delta^2} \frac{\gamma_n^2(\alpha_J)}{\gamma_{n+1}^2(\alpha_J)} \frac{\int_0^\infty [\mathcal{L}_n^{(j+1)}(t)]^2 \, d\alpha_J(t)}{[\mathcal{L}_n^{(j+1)}(z)]^2} \leq \frac{1}{\Delta^2} \frac{\gamma_n^2(\alpha_J)}{\gamma_{n+1}^2(\alpha_J)} \frac{\int_0^\infty [\mathcal{L}_n^{(j+1)}(t)]^2 t^j e^{-t} \, dt}{[\mathcal{L}_n^{(j+1)}(z)]^2} \leq \frac{1}{\Delta^2} \frac{\gamma_n^2(\alpha_J)}{\gamma_{n+1}^2(\alpha_J)} \frac{1}{[\mathcal{L}_n^{(j+1)}(z)]^2} .
\]

By using the main term of (II.2.3.4) in this inequality with \( r = j+1 \),
we obtain,

\[
\frac{1}{|p_{n+1}(\alpha_j;z)|^2} \leq \frac{4\pi}{\Delta^2} \frac{\gamma_n^2(\alpha_j)}{\gamma_{n+1}^2(\alpha_j)} \frac{|z|^{j+(3/2)}}{n^{j+(1/2)} \exp(\text{Re}(z))} \cdot \frac{1 + \epsilon_n}{[\exp(4\text{Re}(\sqrt{-z}))]^{1/n}}
\]

where, as in Theorem II.2.3.3, \( z \in \mathbb{C} \setminus [0,\infty) \), \( \sqrt{-z} \) must be taken real and positive if \( z < 0 \), and the bound for the remainder holds uniformly in every closed subset of \( \mathbb{C} \setminus [0,\infty) \).

Using this inequality in (II.2.4.3), we get (II.2.2.2) and the proof of the theorem is completed.

**Proof of Theorem II.2.2.3:** By letting \( j = -1 \) and \( \alpha(t) = e^{-2Q(t)} dt \) in (II.2.4.3), we obtain

\[
(\text{II.2.4.5}) \quad \left| \int_0^\infty \frac{e^{-2Q(t)} dt}{1 - z^{-1}t} - [n+1,n](f;z) \right| \leq \frac{|z|}{\Delta |p_{n+1}(e^{-2Q};z)|^2};
\]

where \( z \in \mathbb{C} \setminus [0,\infty) \).

Using (II.2.3.3) with \( a = 0 \), \( b = \infty \), \( d\beta_2(t) = e^{-t} dt \) and \( d\beta_1(t) = e^{-2Q(t)} dt \), and the fact that

\[ p_n(e^{-t};z) = L_n^{(0)}(t) = L_n(t), \]

we get
From (II.2.4.6), we deduce that

\[
\frac{1}{|p_{n+1}(e^{-2Q};z)|^2} \leq \frac{\gamma_n^2(e^{-2Q}) \int_0^\infty \left| \mathcal{E}_n(t)e^{-2Q(t)} \right| dt}{\gamma_{n+1}(e^{-2Q}) \Delta^2 \left| \mathcal{L}_n(z) \right|^2}.
\]

We now use (II.2.4.8) and Theorem II.2.3.4 with $p_n(t) = \mathcal{E}_n(t)$ and $Q^*(t) = e^{t/2}$ to conclude,

\[
\left| \mathcal{E}_n(t) \right| \leq c_{1} e^{(t/2)-(a_1t/n)}, \quad c_{1} \text{ and } a_1 \text{ are constants.}
\]

Hence,

\[
\left| \mathcal{E}_n(t) \right|^2 \leq c e^{t-(at/n)}, \quad c \text{ and } a \text{ are constants.}
\]

(This constant $a$ must be the one to choose in our assumption (II.2.2.3).)

Combining this inequality with (II.2.4.7) and (II.2.2.3) into (II.2.4.6), we obtain

\[
\frac{1}{|p_{n+1}(e^{-2Q};z)|^2} \leq \frac{c d_{n+1}^2 \int_0^\infty e^{\exp[(1 - \frac{a}{n})t - 2Q(t)]} dt}{\Delta^2 \left| \mathcal{L}_n(z) \right|^2} \leq \frac{K_{1} q_{n+1}}{\Delta^2 \left| \mathcal{L}_n(z) \right|^2}, \quad K_{1} \text{ is a constant.}
\]
We can now easily see that (II.2.2.4) follows from this inequality, (II.2.4.5), and the main term of (II.2.3.4) when \( r = 0 \).

This completes the proof of the theorem.
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