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STANDARD COMPONENTS OF TYPE $M_{24}$ AND $\Omega^+(8,2)$

The Ohio State University

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STANDARD COMPONENTS OF TYPE $M_{24}$ AND $\Omega^+(8,2)$

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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* * * * *

The Ohio State University

1980

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"Association schemes of quadratic forms," to appear in J. Combinatorial Theory Ser. A

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CHAPTER I
INTRODUCTION

A subgroup $T$ of a finite group $G$ is said to be tightly embedded in $G$ if $T$ has even order and $T \cap T^g$ has odd order for $g \in G - N_G(T)$. A quasisimple subgroup $L$ of $G$ is said to be a standard component of $G$ if $T = C_G(L)$ is tightly embedded, $N_G(T) = N_G(L)$, and $[L, L^g] \neq 1$ for $g \in G$.

Since a fundamental result of M. Aschbacher [2] appeared, much work has been done toward the classification of groups $G$ with a standard component $L$ of known type. In particular, M. Aschbacher and G. Seitz [4] have treated the case where $L$ is an arbitrary known quasisimple group and $C_G(L)$ has 2-rank at least 2. Also, M. Aschbacher [1] has classified groups with a tightly embedded subgroup having a generalized quaternion Sylow 2-subgroup. Thus it remains to investigate the case where $C_G(L)$ has cyclic Sylow 2-subgroups.

In Chapter II of this dissertation, we study the case where $L \cong M_{24}$. In Chapter III, we treat the case where $L \cong \Omega^+(8,2)$. As both chapters contain their own introductory sections, we refer the reader to those sections for the precise statement of our result.
CHAPTER II

STANDARD COMPONENTS OF TYPE $M_{24}$

INTRODUCTION

In this chapter we study finite groups with a standard component isomorphic to $M_{24}$. (The notation used in this introductory section is the same as that used in the rest of this chapter, and is explained at the end of this section.)

In order to state our main theorem, we need the following hypothesis:

Hypothesis A.

Let $G$ be a finite group with $O(G) = 1$ having a standard component $L$ isomorphic to $A_8$ such that $C_G(L)$ has cyclic Sylow 2-subgroups. Then $E(G)$ is isomorphic to one of the following groups:

$A_8$, $A_8 \times A_8$, $A_{10}$, $SL(4,4)$ or $HS$,

where $HS$ denotes the Higman-Sims simple group of order $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$. 

2
Remark.

Hypothesis A is proved in R. Solomon [25] under some additional assumption concerning the U-conjecture. But the U-conjecture is not used in any other part of this chapter.

In this chapter, we prove

Theorem B.

Let $G$ be a finite group with $O(G) = 1$ having a standard component $L$ isomorphic to $M_{24}$ such that $C_L(G)$ has cyclic Sylow 2-subgroups. Suppose that Hypothesis A holds in every proper section of $G$. Then, either $E(G) \cong M_{24}$, or $E(G) \cong M_{24} \times M_{24}$.

In order to explain where Hypothesis A is used, we shall sketch the proof. Our proof follows the outline given by L. Finkelstein [7], and makes use of two 2-local subgroups, $N_L(A)$ and $N_L(B)$, of $L$, such that $N_L(A) \cong E_{64} \cdot 3:S_6$ and $N_L(B) \cong E_{16} \cdot A_8$. We let $z$ denote a generator of a Sylow 2-subgroup of $C_L(G)$. The case $|\langle z \rangle| \geq 4$ is easily ruled out (Lemma 2.1). Thus, assuming that $|\langle z \rangle| = 2$ and that $z \notin Z(G)$, we first prove $N_G(\langle z \rangle A)/C_G(\langle z \rangle A) \cong E_{64} \cdot 3:S_6$ (Lemmas 2.3 and 2.5). Using this, we prove $N_G(\langle z \rangle B)/C_G(\langle z \rangle B) \cong E_{16} \cdot A_8$ (Lemmas 2.4 and 2.9). Then we prove that $D = O_2(N_G(\langle z \rangle B)^{\infty})$ is isomorphic to $E_{256}$ (Lemmas 2.10, 2.11 and 2.12).
Using this, we prove that \( C = E_{4096} \) (Lemmas 2.6, 2.7, 2.8 and 2.13). We next prove \( N_G(C)^\infty \cong (E_{64} \cdot 3A_6) \times (E_{64} \cdot 3A_6) \) (Lemmas 2.14 and 2.15). Using this and Hypothesis A, we prove \( N_G(D)^\infty \cong (E_{16} \cdot A_8) \times (E_{16} \cdot A_8) \) (Lemmas 2.16 and 2.17). Finally, after observing some relation between \( N_G(C)^\infty \) and \( N_G(D)^\infty \) (Lemma 2.18), we apply a product fusion theorem of E. Shult [23] to obtain the desired conclusion (Lemma 2.22).

We remark that Proposition 1.4, which is used in the proof that \( C \cong E_{4096} \), may be of independent interest and is applicable to various cases.

Our notation is standard except possibly the following:

- \( E(X) \) the product of the quasisimple subnormal subgroup of \( X \),
- \( X^\infty \) the final term of the derived series of \( X \),
- \( X \setminus Y \) the wreath product of \( X \) by \( Y \),
- \( Z_n \) the cyclic group of order \( n \),
- \( E_n \) the elementary abelian group of order \( n \),
- \( A_n \) the alternating group of degree \( n \),
- \( S_n \) the symmetric group of degree \( n \),
- \( 3A_6 \) the triple cover of \( A_6 \),
- \( 3S_6 \) \( 3A_6 \) together with an automorphism of order 2 inducing \( S_6 \) on \( A_6 \) and inverting the
element of order 3 of the Schur multiplier of $A_6$.

$X = YZ$ means that $Y \triangleleft X$ and that $X = \langle Y, Z \rangle$. If $Y \cap Z = 1$ and if an emphasis is to be placed on that fact, then we write $X = Y \cdot Z$.

If $X$ is a 2-group, then by $J(X)$ we denote the usual Thompson subgroup generated by the abelian subgroups of maximal order, and by $ZJ(X)$ we denote the center $Z(J(X))$ of $J(X)$.

We use the "bar" and the "tilde" convention for homomorphic images. Thus, if $G$ is a group, $N$ is a normal subgroup and $\overline{G}$ (resp. $\widetilde{G}$) denotes the factor group $G/N$, then, for any subset $X$ of $G$, $\overline{X}$ (resp. $\widetilde{X}$) will denote the image of $X$ under the natural projection $G \rightarrow \overline{G}$ (resp. $G \rightarrow \widetilde{G}$).

In Section 2, we use symbols such as $N(X)$ and $C(X)$ to denote $N_G(X)$ and $C_G(X)$, respectively.
SECTION 1
PRELIMINARY RESULTS

In this section, we collect some helpful preliminary results.

Lemma 1.1.

Let $R$ be a Sylow 2-subgroup of a group $G$, $C$ be an elementary abelian subgroup of $R$ which is weakly closed in $R$ with respect to $G$. Let

$$\Gamma = \{E \leq R \mid E^g \leq C \text{ for some } g \in G, \ E \nsubseteq C\}.$$  

Then $|\langle C, x \rangle| = \max \{|CE/C| \mid E \in \Gamma\}$ for any involution $x$ of $R$ such that $\langle x \rangle \in \Gamma$.

Proof.

This is Corollary 4 (2) of D. Goldschmidt [10].

Lemma 1.2.

Let $E \cong E_{32}$ and $H \cong SL(4,2)$. Let $H$ act on $E$ so that $|C_E(H)| = 2$ and so that the action of $H$ on $E/C_E(H)$ is the same as that on a standard module. Then $H$ acts on $E$ decomposably.
Proof.

This is well-known and easy to verify. (For example, if we suppose the lemma is false and let $H$ act on
\[ X = \{ F | |F| = 16, \ F \cap C_E(H) = 1 \}, \]
then $X$ splits into two $H$-orbits whose lengths are both 8, and so it follows that a subgroup of index 8 of $H$ fixes two elements of $X$, which is absurd.)

Lemma 1.3.

Let $A \cong E_6$ and $K \cong 3S_6$, and let $K$ act faithfully on $A$. Then the following hold:

(i) $A^*$ splits into two $K$-classes, $\{ a^K \}$ and $\{ b^K \}$, of involutions, such that $C_K(a) \cong Z_2 \times S_4$ and $C_K(b) \cong S_5$.

(ii) For every involution $t$ of $K'$, $|[A,t]| = 4$.

(iii) For every involution $t$ of $K - K'$, $|[A,t]| = 8$.

(iv) If $a$ is as in (i), then,
\[ A = \langle x | \text{both } x \text{ and } ax \text{ are in } \{ a^K \} \rangle. \]

(v) For every involution $x$ of $A$, $C_A(C_K(x)) = \langle x \rangle$.

(vi) For every involution $x$ of $A$, $\{ x^K \} = \{ x^{K'} \}$. 
Proof.

Sylow 3-subgroups of \( \text{GL}(5,2) \) are abelian. Consequently, the condition that \( K \) acts faithfully on \( A \) implies that the action is irreducible. Therefore, 
\[
[A, O(K)] = A.
\]
Since each involution of \( K - K' \) inverts \( O(K) \), this implies (iii). Let \( t \) be an involution of \( K' \). Let \( r \) be a 5-element of \( K \) inverted by \( t \). Then, since 
\[
|[A, r]| = 16, \quad |[[A, r], t]| = 4.
\]
Since \( O(K) \) permutes the involutions of \( C_A(r) \) and since \( O(K) \) and \( t \) commute, \( t \) centralizes \( C_A(r) \). Thus (ii) holds. Let \( r \) be as above, and let \( P \cong Z_4 \) be a Sylow 2-subgroup of \( N_K(<r>) \). Then \( <r>P \) centralizes some involution \( b \) of \( C_A(r) \). Since \( C_K(b) \cap O(K) = 1 \), \( N_{C_K(b)}(<r>) = <r>P \). Since \( [A, O(K)] = A \) and since Sylow 3-subgroups of \( 3S_6 \) are non-abelian, the length of every \( K \)-orbit of \( A^# \) is divisible by 9. Hence, by Sylow's theorem, \( |C_K(b)| \) is \( 2^3 \cdot 5 \) or \( 2^3 \cdot 3 \cdot 5 \). But, since \( |\{b^K\}| \leq 63 \), \( |C_K(b)| \) cannot be \( 2^3 \cdot 5 \). Therefore, 
\[
|C_K(b)| = 2^3 \cdot 3 \cdot 5 \quad \text{and so} \quad C_K(b) \cong S_5.
\]
Since \( O(K) \) permutes the involutions of \( C_A(r) \), every involution of \( A \) centralized by a 5-element of \( K \) is in \( \{b^K\} \). Let \( a \) be an involution of \( A - \{b^K\} \). Then, 
\[
9 \cdot 5 \mid |\{a^K\}|.
\]
Since \( |A - \{b^K\}| = 45, \quad |\{a^K\}| = 45 \) and \( C_K(a) \cong Z_2 \times S_4 \). This proves (i). Since \( N_K(C_K(a)) = C_K(a) \) and \( N_K(C_K(b)) = C_K(b) \), (v) holds. Since \( C_K(a) \lhd K' \) and \( C_K(b) \lhd K' \),
(vi) holds. In order to prove (iv), let $V$ be a subgroup of order 32 of $A$ containing $a$. Take 5-elements $r_1, r_2, r_3$ of $K$ such that $C_A(r_1) \n C_A(r_2) \n C_A(r_3) \n C_A(r_1)$. Then, since $[C_A(r_1), C(K)] = C_A(r_1)$ for each $i$, $C_A(r_2) \n C_A(r_3) = C_A(r_3) \n C_A(r_1) = C_A(r_1) \n C_A(r_2) = 1$. Since $|C_A(r_i) \n V| \geq 2$ for each $i$, this implies that $V$ contains at least three involutions of $\{b^K\}$. Thus $A - V$ contains at least 17 involutions of $\{a^K\}$. Therefore, there is an involution $x$ of $A - V$ such that both $x$ and $ax$ are in $\{a^K\}$. Since $V$ was arbitrary, this proves (iv).

Proposition 1.4.

Let $K$ be a group satisfying the following:

(i) If $C = O_2(K)$, then $\Phi(\Phi(C)) = 1$.

(ii) $C/\Phi(C)$ is isomorphic to $\Phi(C)$ as a $K/C$-module and irreducible.

(iii) There exists a $K$-orbit $\{a^K\}$ of $\Phi(C)^#$ such that $C_{\Phi(C)}(C_K(a)) = \langle a \rangle$ and $\Phi(C) = \langle x \rangle$ both $x$ and $ax$ are in $\{a^K\}$. Then $C$ is homocyclic of exponent 4, and the correspondence which associates $x^2 \in \Phi(C)$ with $x\Phi(C) \in C/\Phi(C)$ is the unique isomorphism from $C/\Phi(C)$ onto $\Phi(C)$. 

Proof.

Let $\mathcal{C} = C/\Phi(C)$ and $\mathcal{K} = K/C$. By (ii), $\Phi(C) \subseteq Z(C)$. For each element $x$ of $\{a^K\}$, let $t(x)$ denote an element of $C - \Phi(C)$ such that $C_K(t(x)) = C_K(x)$. By our assumption, $t(x)$ is uniquely determined. Let

$$Y = \{t(x) | \text{ both } x \text{ and } a^x \text{ are in } \{a^K\}\}.$$ 

We have $\mathcal{C} = \langle Y \rangle$.

First suppose $t(a)^2 = 1$. Then, since $Z(C) \supseteq \Phi(C)$, $t(x)^2 = (t(x)t(a))^2 = 1$ for all $t(x) \in Y$. Hence, $[t(a), t(x)] = 1$ for all $t(x) \in Y$. Since $C = \langle Y, Z(C) \rangle$, this implies $t(a) \in Z(C)$. Since $K$ acts on $C$ irreducibly, this implies that $C$ is elementary abelian, which contradicts (ii).

Next suppose $t(a)^2 \neq 1$. Let $x$ be an arbitrary element of $\{a^K\}$. Then, since $Z(C) \supseteq \Phi(C)$, $C_K(t(x)^2) = C_K(t(x))$. Therefore, by our assumption, $t(x)^2 = x$. Hence, if $t(x) \in Y$, then $t(x)^2 = x$, $t(a)^2 = a$ and $(t(x)t(a))^2 = xa$. This again leads to $t(a) \in Z(C)$, and so the desired conclusion holds.

Lemma 1.5.

Let $N$ be $A_8$, $SU(4,2)$, $SL(5,2)$, $SU(5,2)$ or $Sp(4,4)$. Let $z$ be an involution of $\text{Aut}(N) - N$ such that $C_N(z) \geq S_6$. Then, there exists an involution $x$ of $C_N(z)$ such that $z^g = zx$ for some $g \in C_N(x)$.
Proof.

This is clear because $|Z_2 \times S_6|_2 \leq |\langle z \rangle N|_2$.

In the remainder of this section, we fix notation for $M_{24}$ and state some of its properties. For more detailed information, the reader is referred to U. Schoenwaelder [21].

Notation 1.6.

Let $L = M_{24}$. $L$ has an elementary abelian 2-subgroup $B$ of order 16 such that $N_L(B) = B \cdot H$ where $H$ is isomorphic to $SL(4,2) \cong A_8$. Let $B_1$ and $B_2$ be subgroups of $B$ of order 4 and 8, respectively, such that $B_1 \leq B_2$. Let $A = B_1 \cdot (C_H(B_1) \cap C_H(B/B_1))$. Then, $A \cong E_{64}$, and $N_L(A) = A \cdot K$ where $K \cong 3S_6$. Let $U$ be a Sylow 2-subgroup of $B \cdot H$ containing both $C_H(B_1) \cap C_H(B/B_1)$ and $C_H(B_2)$.

The following properties of $M_{24}$ are well-known.

Lemma 1.7.

The following hold:

(i) $\text{Aut}(L) = L$.

(ii) $U \in Syl_2(L)$. $B_2C_H(B_2) \cong E_{64}$. $A$ and $B_2C_H(B_2)$ are the only abelian subgroups of order 64 of $U$, and so
\[ J(U) = \langle A, B \rangle_{CH(B)} \].

(iii) \( ZJ(U) = J(U)' \cong E_{16} \).

(iv) \( J(U) \) is indecomposable.

(v) \( N_L(J(U))/J(U) \cong S_3 \times S_3 \) and \( N_L(J(U)) \) acts irreducibly on \( ZJ(U) \).

(vi) Every involution of \( L \) is conjugate to some involution of \( ZJ(U) \) in \( L \).

(vii) For every involution \( x \) of \( L \), \( x \in C_L(x)' \).

Lemma 1.8.

The following hold:

(i) \( N_{AK'}(B)/B \cong E_{16} \cdot (E_9 \cdot Z_2) \) and \( O(N_{AK'}(B)/B) = 1 \).

(ii) \( O^2(N_{BH}(A)/A) \cong E_4 \cdot E_9 \), \( [B_1, O^2(N_{BH}(A)/A)] = B_1 \), and \( B_1 \) is the unique minimal \( O^2(N_{BH}(A)/A) \)-invariant subgroup of \( A \).

Proof.

\( N_H(A) = N_H(B_1) \cong E_{16} \cdot (E_9 \cdot E_4) \) and \( O(N_H(A)) = 1 \).

(i) follows immediately from this. Since there exists a 3-element of \( N_H(A) \) whose action on \( B \) is fixed-point-free, \( O^2(N_{BH}(A)/A) \cong E_4 \cdot E_9 \) and \( [B_1, O^2(N_{BH}(A)/A)] = B_1 \).

For each element \( x \) of \( A - B_1 \), there exists an element \( t \) of \( B - B_1 \) such that \( 1 \not\in [x, t] \in B_1 \). Hence \( B_1 \) is the unique minimal \( O^2(N_{BH}(A)/A) \)-invariant subgroup of \( A \).
SECTION 2
PROOF OF THEOREM B

In the remainder of this chapter, we let $G$ denote a group which satisfies the hypotheses of Theorem B. We use the description of $L$ given in Notation 1.6. Let $z$ be a generator of a Sylow 2-subgroup of $C(L)$.

Lemma 2.1.

If $|\langle z \rangle| \geq 4$, then $\Omega_1(\langle z \rangle) \leq Z(G)$.

Proof.

Since $ZJ(\langle z \rangle U) = \langle z \rangle \times ZJ(U)$ and since $ZJ(U)$ is elementary abelian by Lemma 1.7 (iii), $\Omega_1(U^l(ZJ(\langle z \rangle U))) \leq \Omega_1(\langle z \rangle)$. Hence $\langle z \rangle U \in \text{Syl}_2(G)$. Since every involution of $\langle z \rangle U$ is conjugate to some involution of $ZJ(\langle z \rangle U)$ in $C(z)$ by Lemma 1.7 (vi) and since $N(J(\langle z \rangle U))$ controls the fusion of $ZJ(\langle z \rangle U)$, $\Omega_1(\langle z \rangle) \leq Z(G)$ by Glauberman's $Z^*$-theorem.

In the remainder of this chapter we assume $|\langle z \rangle| = 2$. We first determine $N(\langle z \rangle J(U)), N(\langle z \rangle A), N(\langle z \rangle B)$. 

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Lemma 2.2.

One of the following holds:

(i) \( N(\langle z \rangle J(U)) = N_{C(z)}(\langle z \rangle J(U)) \); or

(ii) \( [N(\langle z \rangle J(U)) : N_{C(z)}(\langle z \rangle J(U))] = 16 \).

Proof.

Let \( D = N_{C(z)}(\langle z \rangle J(U)) \) and \( N = N(\langle z \rangle J(U)) \), and let \( \overline{N} = N/C_N(\langle z \rangle ZJ(U)) \). From Lemma 1.7 (v), \( \overline{D} \cong S_3 \times S_3 \), and, under the action of \( D \), \( ZJ(U) \) splits into two \( D \)-classes, \( \{a^D\} \) and \( \{b^D\} \), of involutions, such that \( |\{a^D\}| = 9 \) and \( |\{b^D\}| = 6 \). By Lemma 1.7 (vii), \( a \cdot z \cdot b \) in \( G \). Hence one of the following holds:

1. \( \{z^N\} = \{z\} \);
2. \( \{z^N\} = \{z\} \cup \{(zb)^D\} \);
3. \( \{z^N\} = \{z\} \cup \{(za)^D\} \); or
4. \( \{z^N\} = \{z\} \cup \{(za)^D\} \cup \{(zb)^D\} \).

It is clear that (1) and (4) imply (i) and (ii), respectively. Thus suppose (2) or (3) occurs. Then \( |N| = 2^3 \cdot 3^2 \cdot 5 \) or \( 2^2 \cdot 3^2 \cdot 7 \). By K. Harada and H. Yamaki [15], \( N \) cannot act on \( \langle z \rangle ZJ(U) \) irreducibly. Since \( \langle z \rangle \) is not \( N \)-invariant, this means that \( ZJ(U) \) is \( N \)-invariant and that \( N \) acts on \( \langle z \rangle ZJ(U) \) indecomposably. Since \( C_N(ZJ(U)) \) centralizes both \( \langle\langle z \rangle Z(J(U))\rangle \)/\( ZJ(U) \) and \( ZJ(U) \), every element of odd order of \( C_N(ZJ(U)) \) centralizes \( \langle z \rangle ZJ(U) \). Hence \( C_N(ZJ(U)) \) is
a 2-group. Thus, again from [15], \( N/C_N(ZJ(U)) \cong A_5 \times Z_3 \) or \( A_6 \). But neither \( A_5 \times Z_3 \) nor \( A_6 \) contains a subgroup isomorphic to \( D \). Thus \( N \cong (A_5 \times Z_3) \times Z_2 \). But, then, \( \langle z \rangle ZJ(U) = [\langle z \rangle ZJ(U), O(N)] \times C_{\langle z \rangle ZJ(U)}(O(N)) \). Hence, \( N \) acts on \( \langle z \rangle ZJ(U) \) decomposably, which is absurd.

Let \( M = N(\langle z \rangle A) \).

Lemma 2.3.

Let \( M = M/(\langle z \rangle A) \). Then one of the following holds:

(i) \( M = N_C(z)(\langle z \rangle A) \); or

(ii) \( M = O(M) \times O_2(M) \cdot E \), \( [\langle z \rangle A, O(M)] = 1 \), \( O_2(M) \cong E_{64} \), the correspondence which associates \( [z, \overline{v}] \in A \) with \( \overline{v} \in O_2(M) \) is a \( K \)-isomorphism from \( O_2(M) \) onto \( A \).

Proof.

Let \( D = N_C(z)(\langle z \rangle A) \). By Lemma 1.3 (i), \( A \) splits into two \( D \)-classes, \( \{a^D\} \) and \( \{b^D\} \), of involutions, such that \( |\{a^D\}| = 45 \) and \( |\{b^D\}| = 18 \). By Lemma 1.7 (vii), \( a \neq z \neq b \). Since \( |\{z^M\}| \) must divide \( |GL(7,2)| \), this implies that either \( \{z^M\} = \{z\} \) or \( \{z^M\} = \{z\} \cup \{(za)^D\} \cup \{(zb)^D\} \). Note that \( C_N(\langle z \rangle A) = O(D) \) in any case.
If \( \{ z^M \} = z \), then (i) holds.

Assume \( \{ z^M \} = \{ z \} \cup \{ (za)^D \} \cup \{ (zb)^D \} \). Then
\[
|\mathbb{F}/O(\mathbb{D})| = 2^{10} \cdot 3^3 \cdot 5
\]
and \( A \) is \( M \)-invariant. For each element \( v \) of \( A \), if there exists an element of \( \mathbb{F}/O(\mathbb{D}) \) which centralizes \( A \) and sends \( z \) to \( zv \), then we let \( t(v) \) denote that element. From [15],
\[ C_{\mathbb{F}/O(\mathbb{D})}(A) \]
is nontrivial. Thus it contains \( t(v) \) for some \( v \in A^{##} \). Since \( D \) acts on \( A \) irreducibly,
\[ C_{\mathbb{F}/O(\mathbb{D})}(A) = \langle t(v) \rangle \quad v \in A \].
Hence, \( C_{\mathbb{F}/O(\mathbb{D})}(A) \cong E_{64} \),
\[
E/O(\mathbb{D}) = C_{\mathbb{F}/O(\mathbb{D})}(A) \cdot (E/O(\mathbb{D})),
\]
and the correspondence which associates \( \{ t(v), z \} = v \) with \( t(v) \) gives a \( K \)-isomorphism. Finally, since \( (O(\mathbb{D})C_{\mathbb{F}/O(\mathbb{D})})/O(\mathbb{D}) \cong E/O(\mathbb{D}) \), \( (O(\mathbb{D})C_{\mathbb{F}/O(\mathbb{D})})/O(\mathbb{D}) = \mathbb{F}/O(\mathbb{D}) \). Thus all the assertions in (ii) are proved.

Let \( N = N/(z)B \). The proof of the following lemma is similar to and easier than that of Lemma 2.3, and so it is omitted.

**Lemma 2.4.**

Let \( \bar{N} = N/(z)B \). Then one of the following holds:

(i) \( N = N_c(z)/(z)B \); or

(ii) \( N = O(\bar{N}) \times (O_2(\bar{N}) \cdot \bar{F}) \), \( \langle (z)B, O(\bar{N}) \rangle = 1 \), \( O_2(\bar{N}) \cong E_{16} \), the correspondence which associates \( [z, v] \in B \) with \( v \in O_2(\bar{N}) \) is an \( H \)-isomorphism from \( O_2(\bar{N}) \) onto \( B \).
Lemma 2.5.

Suppose Case (i) of Lemma 2.3 holds. Then \( z \in Z(G) \).

Proof.

First we prove \( \langle z \rangle U \in \text{Syl}_2(G) \). By way of contradiction, suppose \( \langle z \rangle U \notin \text{Syl}_2(G) \). Then, by Lemma 2.2, \([N(\langle z \rangle J(U)) : \langle z \rangle U]_2 = 16\). By Lemma 1.7 (ii), \( N(\langle z \rangle J(U)) \) acts on the set \( \{\langle z \rangle B_2 C_2(B_2), \langle z \rangle A\} \). Hence, \([N(\langle z \rangle J(U)) : \langle z \rangle U]_2 \geq 8\), which contradicts our assumption. Thus, \( \langle z \rangle U \in \text{Syl}_2(G) \), as desired. Then, by Lemma 2.2, \( N(\langle z \rangle J(U)) = N_{C(\langle z \rangle J(U))} \). By Lemma 2.7 (vi), every involution of \( \langle z \rangle U \) is conjugate to some involution of \( \langle z \rangle ZJ(U) \) in \( C(z) \). Since \( N(\langle z \rangle J(U)) \) controls the fusion of \( \langle z \rangle ZJ(U) \), Glauberman's Z*-theorem yields the desired conclusion.

From now on, we assume that Case (ii) of Lemma 2.3 holds. We shall determine a more detailed structure of \( O_2(M) \).

Lemma 2.6.

\( O_2(M/A) \) is elementary abelian.

Proof.

Since \( K \) acts on \( O_2(M/(\langle z \rangle A)) \) irreducibly, \( O_2(M/A) \) is either elementary abelian or extra special.
But, since $0^2(6,2)$ does not contain $3S_6$, $O_2(M/A)$ cannot be extra special.

Lemma 2.7.

$K$ acts on $O_2(M/A)$ decomposably.

Proof.

This is because

$$O_2(M/A) = C_{O_2(M/A)}(O(K)) \times [O_2(M/A),O(K)].$$

Let $C$ be the full inverse image of $[O_2(M/A),K]$.

Lemma 2.8.

One of the following holds:

(i) $C \cong E_{4096}$; or

(ii) $C$ is homocyclic of exponent 4, and the correspondence which associates $x^2 \in A$ with $xA \in C/A$ is the unique $K$-isomorphism from $C/A$ onto $A$.

Proof.

This follows from Lemma 1.3 (iv), (v) and Lemma 1.4.

Lemma 2.9

Case (ii) of Lemma 2.4 holds.
Proof.

Let $X = A[C,B]$. We shall prove that $X \leq N$, which implies $N \nmid C(z)$. Let $x$ be an arbitrary element of $B - A$. Then $[A,x] = [A,B]$, and so, by the last assertion of Lemma 2.3 (ii), $A[C,x] = X$. Let $y$ be an arbitrary element of $C$. Since $x$ is an involution and since $C$ is of exponent at most 4,

$$[y,x,x] = [y,x]^2.$$ 

Hence, $[y,x,x] \in [A,B]$, by Lemma 2.8 (i) or by the second assertion of Lemma 2.8 (ii).

Since $y$ was arbitrary, $[X,x] \in [A,x][C,x,x] \subseteq [A,B]$. Since $x$ was arbitrary, $[X,B] \subseteq [A,B]$. On the other hand, $[X,z] = [A,B]$ by the last assertion of Lemma 2.3 (ii). Therefore, $[z,B,X] \subseteq [z,X][B,X] \subseteq [A,B] \subseteq \langle z \rangle B$, as desired.

Now we determine $O_2(N)$.

Lemma 2.10.

$O_2(N/B)$ is elementary abelian.

Proof.

Since $H$ acts on $O_2(N/(\langle z \rangle B))$ irreducibly, $O_2(N/B)$ is either elementary abelian or extra special. But, since $O^*(4,2)$ does not contain $SL(4,2)$, $O_2(N/B)$ cannot be extra special.
Lemma 2.11.

\( H \) acts on \( O_2(N/B) \) decomposably.

Proof.

This is nothing more than Lemma 1.2.

Lemma 2.12.

\( D \cong E_{256} \).

Proof.

This follows immediately from Theorem 8.2 of G. Higman [17].

Lemma 2.13.

\( C \cong E_{4096} \).

Proof.

Let \( y \) be a 3-element of \( N_H(C_H(B_1) \cap C_H(B/B_1)) \) such that \( C_B(y) = B_1 = A \cap B \). Then

\[
(5) \quad [(C_H(B_1) \cap C_H(B/B_1)),y] = C_H(B_1) \cap C_H(B/B_1).
\]

By the last assertion of Lemma 2.3 (ii),

\[
(6) \quad [C_B(y),z] = C_B(y) = B_1.
\]

We shall show \( [C_D(y),(C_H(B_1) \cap C_H(B/B_1))] = 1 \). Let \( x \) be an element of \( C_H(B_1) \cap C_H(B/B_1) \) such that \( [B,x] = B_1 \). Then, again by the last assertion of Lemma 2.3
(ii), $C_D(y) \leq B[D,x]$. Hence $[C_D(y), x] \leq [B, x][D, x, x] = B_1 \leq C_D(y)$. Thus $x$ normalizes $C_D(y)$. Since $x$ was arbitrary, this implies that $C_H(B_1) \cap C_H(B/B_1)$ normalizes $C_D(y)$. Therefore $[C_D(y), (C_H(B_1) \cap C_H(B/B_1)), y] = 1$. We also have $[C_D(y), y, (C_H(B_1) \cap C_H(B/B_1))]$. By (5) and the three-subgroup lemma, $[(C_H(B_1) \cap C_H(B/B_1)), C_D(y)] = [y, (C_H(B_1) \cap C_H(B/B_1))], C_D(y)] = 1$, as desired. Thus $(C_H(B_1) \cap C_H(B/B_1)), C_D(y)] \cong E_{256}$. (6) implies $(C_H(B_1) \cap C_H(B/B_1)), C_D(y)] \leq M$. Thus $C^\langle A \rangle$ contains an elementary abelian subgroup of order 256. This shows that Case (ii) of Lemma 3.8 does not occur.

We shall determine an approximate structure of $N(C)$ in the following two lemmas. Let $N(C) = N(C)/C$.

Lemma 2.14.

$G(C) \leq O(N(C))$, $N(C)^\infty \cong 3A_6 \times 3A_6$ where $z$ permutes the components, and $N(C)^\infty$ acts faithfully on $C$.

Proof.

All the involutions of $zC$ are conjugate to each other in $C(z)$. Therefore, $C_{N(C)}(z) = C_{N(C)}(z) = (z) \times O(C(z)) \times R$, where $R \cong 3S_6$. This already shows that $C_{N(C)}(z)$ is odd. Thus $G(C) \leq O(N(C))$.

By M. Harris and R. Solomon [16], $E(N(C)/O(N(C)))$ is
isomorphic to one of the following groups:

(7) $A_6$ or $A_6 \times A_6$;
(8) $A_8$, $SU(4,2)$, $SL(5,2)$, $SU(5,2)$ or $Sp(4,4)$;
(9) $P\Omega^-(6,3)$.

First suppose $E(\mathbb{N}(C)/O(\mathbb{N}(C))) \cong P\Omega^-(6,3)$. Since $C^\mathbb{N}(C)/O(C^\mathbb{N}(C)) \cong \mathbb{Z}_2 \times S_6$, $\mathbb{N}(C)/O(\mathbb{N}(C)) \cong PSO^-(6,3)$.

Let $R$ be a Sylow 2-subgroup of $\mathbb{N}(C)$ containing $z$. The 2-rank of $R$ is 4. Since $P\Omega^-(6,3)$ has only one class of involutions, $|[C,\tau]| = 64$ for every involution $\tau$ of $R \cap (\mathbb{N}(C))'$ by Lemma 1.3 (iii). Now let $\tau$ be an involution of $R - \mathbb{N}(C)'$. Since $\tau \notin Z(R)$, there is an element $g$ of $R$ such that $\tau \cdot g \notin 1$. Since $|[C,\Omega_1(\langle \tau, g \rangle)]| = 64$, $|[C,\tau]| \geq 8$. These show that $C$ is weakly closed in $R$. Thus $R \in Syl_2(G)$. Now let $t$ be an involution of $U - \mathbb{N}(A)'$. Then $t$ is conjugate to some involution of $A$ in $L$, and $|[C, t]| = 64$.

Since the 2-rank of $R$ is 4, this contradicts Lemma 1.1.

Next suppose $E(\mathbb{N}(C)/O(\mathbb{N}(C))) \cong A_6$. Let $R$ be a Sylow 2-subgroup of $\mathbb{N}(C)$ containing $z$. The 2-rank of $R \cap (\mathbb{N}(C))\langle z \rangle$ is 3, and, by Lemma 1.3 (ii), $|[C, \tau]| \geq 16$ for each involution $\tau$ of $R \cap (\mathbb{N}(C))\langle z \rangle$. By Lemma 1.3 (iii), $|[C, \tau]| = 64$ for every involution $\tau$ of $R - \mathbb{N}(C)\langle z \rangle$. These again show that $C$ is weakly closed in $R$, and so lead to the same contradiction as in the preceding paragraph.
We now suppose $E(N(C)/O(N(C))$ is isomorphic to one of the groups in (8). Let $\widetilde{N(C)} = N(C)/C(C)$. Let
\[
O(\widetilde{N(C)}) = \tilde{P}_0 \supset \tilde{P}_1 \supset \tilde{P}_2 \supset \cdots \supset \tilde{P}_k = 1
\]
be a characteristic composition series of $O(\widetilde{N(C)})$.

(Namely, $\tilde{P}_{i-1}$ is a minimal characteristic subgroup of $O(\widetilde{N(C)})$ properly containing $\tilde{P}_i$ for each $i$.) We claim that $\widetilde{N(C)}$ acts trivially on $\tilde{P}_i / \tilde{P}_i$ for every $i$. In general, if $p$ is an odd prime, then the 2-rank of $GL(n,p)$ is $n$ and every elementary abelian 2-subgroup of order $2^n$ of $GL(n,p)$ contains the involution of the center of $GL(n,p)$. This implies that if the rank of $\tilde{P}_i / \tilde{P}_i$ is less than or equal to the 2-rank of $\widetilde{N(C)}$, then the action of $\widetilde{N(C)}$ on $\tilde{P}_i / \tilde{P}_i$ is trivial. Since $\widetilde{N(C)}$ is isomorphic to a subgroup of $GL(12,2)$, this shows that we have only to consider the case in which $\widetilde{N(C)}/O(\widetilde{N(C)}) \cong A_8$ and $|\tilde{P}_i - \tilde{P}_i| = 3^5$ or $3^6$. But since every elementary abelian 2-subgroup of order 8 of $GL(6,3)$ contains involutions which are not conjugate to each other in $GL(6,3)$, and since $A_8$ possesses an elementary abelian 2-subgroup of order 8 all of whose involutions are conjugate to each other in $A_8$, the action of $\widetilde{N(C)}$ is trivial also in this case. Thus our claim is proved. Now, by the three-subgroup lemma,
Thus \( \overline{N(C)}^{\infty} \) is isomorphic to a perfect central extension of \( E(N(C)/O(N(C))) \). Since \( C_{\overline{N(C)}}(z) \cong \mathbb{Z}_2 \times 3S_6 \) and since the Schur multipliers of the groups in (8) are coprime to 3, this is absurd.

Finally, suppose \( E(N(C)/O(N(C))) \cong A_6 \times A_6 \). Then there exists an involution \( \overline{x} \) of \( C_{\overline{N(C)}}^{\infty}(\overline{z}) \) such that \( \overline{z} \overline{x} = \overline{zx} \) for some \( \overline{g} \in C_{\overline{N(C)}}^{\infty}(\overline{x}) \). Thus \( O(N(C)) = \langle C_{O(N(C))}(\overline{z}), C_{O(N(C))}(\overline{z}), C_{O(N(C))}(\overline{zx}) \rangle \). From the structure of \( C_{\overline{N(C)}}^{\infty}(\overline{z}) \), \( \overline{x} \) centralizes \( C_{O(N(C))}(\overline{z}) \). Since \( \overline{z} \) is conjugate to \( \overline{zx} \) in \( C_{\overline{N(C)}}^{\infty}(\overline{z}) \), \( \overline{x} \) centralizes also \( C_{O(N(C))}(\overline{zx}) \). Hence \( \overline{x} \in C_{\overline{N(C)}}^{\infty}(O(N(C))) \). Therefore \( C_{\overline{N(C)}}^{\infty}(O(N(C))) \supseteq N(C)^{\infty} \).

Thus \( N(C)^{\infty} \) is isomorphic to a perfect central extension of \( A_6 \times A_6 \). Since a Sylow 3-subgroup of \( GL(12,2) \) is isomorphic to \( (\mathbb{Z}_3 \wr \mathbb{Z}_3) \times (\mathbb{Z}_3 \wr \mathbb{Z}_3) \) and does not contain an extraspecial 3-subgroup of order 243, this implies that \( N(C)^{\infty} \cong 3A_6 \times 3A_6 \) and \( N(C)^{\infty} \) acts faithfully on \( C \).

Write \( N(C)^{\infty} = E \times (E)^{\overline{z}} \) with \( E \cong 3A_6 \), and let \( E \) denote the full inverse image of \( E \).

Lemma 2.15

\[ N(C)^{\infty} = E' \times (E^2)', \quad \text{and} \quad E' \cong A \cdot K'. \]
Proof.

First we shall show that \( C_{\mathbb{C}}(Z(\mathbb{E})) = C_{\mathbb{C}}(\mathbb{E}) \cong \mathbb{E}_6 \) and \( C_{\mathbb{C}}(\mathbb{E}) \cap C_{\mathbb{C}}(\mathbb{E}^2) = 1 \). Note that the action of \( C_{\mathbb{C}}(z(\mathbb{E} \times \mathbb{E}^2)) \) on \( C \) is fixed-point-free. Suppose \( C_{\mathbb{C}}(Z(\mathbb{E})) = 1 \). Then \( C_{\mathbb{C}}(Z(\mathbb{E})) = 1 \). On the other hand,

\[
C = \langle C_{\mathbb{C}}(\mathbb{E}) \mid 1 \neq x \in Z(\mathbb{E} \times \mathbb{E}^2) \rangle.
\]

Therefore, \( [(Z(\mathbb{E} \times \mathbb{E}^2)), \mathbb{E}] \) must centralize the entire \( C \), which contradicts the faithfulness of the action of \( E \times \mathbb{E}^2 \). Thus \( C_{\mathbb{C}}(Z(\mathbb{E})) \neq 1 \). Again since \( C_{\mathbb{C}}(Z(C_{\mathbb{N}}(C)(\mathbb{E}^2))) = 1 , \ C_{\mathbb{C}}(Z(\mathbb{E})) \cap C_{\mathbb{C}}(Z(\mathbb{E}^2)) = 1 \).

Therefore, \( |C_{\mathbb{C}}(Z(\mathbb{E}))| \leq 64 \), and \( \mathbb{E}^2 \) acts faithfully on \( C_{\mathbb{C}}(Z(\mathbb{E})) \). Since a Sylow 3-subgroup of \( GL(5,2) \) is abelian, \( GL(5,2) \) does not contain a subgroup isomorphic to \( \mathbb{E}^2 \). Hence \( |C_{\mathbb{C}}(Z(\mathbb{E}))| = 64 \). Since \( \mathbb{E}^2 \)

acts on \( C_{\mathbb{C}}(Z(\mathbb{E})) \) irreducibly, \( [C_{\mathbb{C}}(Z(\mathbb{E})), \mathbb{E}] = 1 \), as desired.

Next, let \( y \) be an element of order 15 of \( \mathbb{E}^2 \).

Then \( C_{\mathbb{C}}(y) = \langle C_{\mathbb{E}} \rangle \cong \mathbb{E}_6 \) and \( C_{\mathbb{N}}(\mathbb{C}^\infty(y)) = (y) \times \mathbb{E} \).

Hence \( \langle C_{\mathbb{C}}(\mathbb{E}), C_{\mathbb{N}}(\mathbb{C}^\infty(y)) \cap C \rangle = \langle C_{\mathbb{C}}(\mathbb{E}), C_{\mathbb{C}}(\mathbb{E}^2) \rangle = C \)

and \( C_{\mathbb{N}}(\mathbb{C}^\infty(y)) \cap C = C \). Therefore \( E' = \langle C_{\mathbb{N}}(\mathbb{C}^\infty(y)) \rangle \), \( C_{\mathbb{C}}(\mathbb{E})' = C_{\mathbb{N}}(\mathbb{C}^\infty(y)) \). Thus, \( E' \cap (\mathbb{E}^2)' \cap C = C_{\mathbb{C}}(\mathbb{E}^2)' \cap C_{\mathbb{C}}(\mathbb{E}) = 1 \), and so \( E' \cap (\mathbb{E}^2)' = 1 \). On the other hand, since \( E \not\subset \mathbb{N}(\mathbb{C}^\infty) \), \( E' \not\subset \mathbb{N}(\mathbb{C}^\infty) \). Therefore \( \mathbb{N}(\mathbb{C}^\infty) = E' \times (\mathbb{E}^2)' \). Finally, since \( E' \cong (\mathbb{E}^2)' \cong C_{\mathbb{E}^2} \times (\mathbb{E}^2)' \)(z),
E' is isomorphic to some subgroup of $N_C(z)(C_C(z)) = N_C(z)(A)$. Therefore, $E' \cong A \cdot K'$.

We next determine $N(D)$. Let $\overline{N(D)} = N(D)/D$.

Lemma 2.16.

$$\overline{C(D)} \leq \overline{O(N(D))}, \text{ and } \overline{N(D)}^c \cong A_6 \times A_8 \text{ where } z \text{ permutes the components.}$$

Proof.

All the involutions of $zD$ are conjugate in $D(z)$. Consequently, $\overline{C_{N(D)}(z)} = \overline{C_{N(D)}}(z) = \langle z \rangle \times \overline{O(C(z))} \times \overline{H} \cong A_8$. By Hypothesis A, $E(\overline{N(D)}/\overline{O(N(D))})$ is isomorphic to one of the following groups:

(10) $A_6 \times A_8$;

(11) $A_6$ or $L_4(4)$.

Now, in order to show that the groups in (11) do not occur, we first let $X$ denote the elementary abelian subgroup of order 16 of $E'$ such that $X \times X^2 \cong B$. Then $X \times X^2 \leq C_N(B)$, Hence $X \times X^2 = D$. Since $E' \cong A \cdot K'$, Lemma 1.8 (i) implies that $N_{E'}(X)/X \cong E_{16} \cdot (E_9 \cdot Z_2)$ and $O(N_{E'}(X)/X) = 1$. Therefore, $\overline{N(D)} \geq \overline{N_{E'}(X)} \times (\overline{N_{E'}(X)^2}) \cong (E_{16} \cdot (E_9 \cdot Z_2)) \times (E_{16} \cdot (E_9 \cdot Z_2))$ and $O(\overline{N_{E'}(X)} \times (\overline{N_{E'}(X)^2}) = 1$. This shows that $E(\overline{N(D)}/\overline{O(N(D))})$ is not isomorphic to any of the groups in (11). Now the argument used in the last paragraph of
the proof of Lemma 2.14 yields the desired conclusion.

The proof of the following lemma is similar to and easier than that of Lemma 2.15, and so it is omitted.

Lemma 2.17.

\( N(D)^{\circ} = F \times F^Z \), where \( F \not\subseteq B \cdot H \).

Recall that \( U \) is a Sylow 2-subgroup of \( L \). Let \( R \) be the Sylow 2-subgroup of \( F \) such that \( R \times R^Z \supseteq U \). Let \( V \) be the subgroup of order 4 of \( R \) such that \( V \times V^Z \supseteq B_1 \), and \( W \) be the subgroup of order 16 of \( R \) such that \( W \times W^Z \supseteq C_H(B_1) \cap C_H(B/B_1) \).

Lemma 2.18.

Either \( VW = C \cap E' \) or \( VW = C \cap (E^Z)' \).

Proof.

\((VW) \times (VW)^Z \subseteq C_M(A)\). Consequently, \((VW) \times (VW)^Z = C\). Let \( I = O^2(N_F(VW)) \). Then, since \( F \not\subseteq B \cdot H \), \( I/(VW) \cong E_4 \cdot E_9 \) and \( V \) is the unique minimal \( I \)-invariant subgroup of \( VW \) and \([V,I] = V\) by Lemma 1.8 (ii). Let \( X \) be an \((I \times I^Z)\)-invariant subgroup of \( C \) such that \( X \trianglelefteq VW \) and \( X \ntrianglelefteq (VW)^Z \). Then, \( 1 \trianglelefteq [X,I] \trianglelefteq VW \) and \( 1 \trianglelefteq [X,I^Z] \trianglelefteq (VW)^Z \). Since \( X \) was arbitrary, this means that \( VW \) and \((VW)^Z \) are the only
(I × I^2)-invariant direct factors of C. Since I × I^2 ≤ O^2(N(G)) = O(N(G))(N(G)^m) from Lemma 2.15, the desired conclusion holds.

We choose F so that Vw = C ∩ E'.

Lemma 2.19.

The following hold:
(i) J(R × R^2) = J(R) × J(R^2).
(ii) ZJ(R) = J(R)' ≅ E_{16}.
(iii) J(R) is indecomposable.

Proof.
Since R ≅ U, these assertions follow immediately from Lemma 1.7.

Lemma 2.20.

Every involution of R is conjugate to some involution of ZJ(R) in O^2(G). Every involution of R × R^2 is conjugate to some involution of ZJ(R) × ZJ(R^2) in O^2(G).

Proof.

By Lemma 2.18 and by our choice of F, each involution of R is conjugate to some involution of C ∩ E' in F. Since E' ≅ A⋅K' and since R ≅ U, each
involution of $C \cap E'$ is conjugate to some involution of $J(R)$ in $E'$ by Lemma 1.3 (vi) and Lemma 1.7 (ii), (vi). This proves the first assertion. Since $F$ centralizes $R^2$ and since $E'$ centralizes $C \cap (E^2)'$, the second assertion holds.

Lemma 2.21.

$$(R \times R^2)(z) \in \text{Syl}_2(G), \text{ and } R \times R^2 \in \text{Syl}_2(O^2(G)).$$

Proof.

By Lemma 2.19 (ii) and Lemma 2.20, $z$ cannot be fused to any involution of $R \times R^2$. On the other hand, each involution of the coset $z(R \times R^2)$ is conjugate to $z$ in $(R \times R^2)(z)$. Thus

$$[N((R \times R^2)(z)) : (R \times R^2)(z)] = [C_{N((R \times R^2)(z))}(z) : C_{(R \times R^2)(z)}(z)].$$

Since $C_{(R \times R^2)(z)}(z) \in \text{Syl}_2(C(z))$, this implies that $(R \times R^2)(z) \in \text{Syl}_2(G)$. Again by Lemma 2.19 (ii) and Lemma 2.21, $z \in O^2(G)$. Since $R \times R^2 \leq O^2(G)$, this means that $R \times R^2 \in \text{Syl}_2(O^2(G)).$

Lemma 2.22

$O^2(G) \cong M_{24} \times M_{24}.$
Proof.

By (ii) and (iii) of Lemma 2.19, Krull-Remak-Schmidt's theorem implies that each element of $N(J(R \times R^2))$ either fixes globally both $ZJ(R)$ and $ZJ(R^2)$ or interchanges $ZJ(R)$ and $ZJ(R^2)$. Since $R \times R^2 \cong \text{Syl}_2(O^2(G))$, this implies that $N_{O^2(G)}(J(R \times R^2))$ normalizes both $ZJ(R)$ and $ZJ(R^2)$. Thus, by Lemma 2.20, both $R$ and $R^2$ are strongly involution closed in $R \times R^2$ with respect to $O^2(G)$, namely, $\{t^{O^2(G)} \cap (R \times R^2) \leq R \}$ (resp. $R^2$) for each involution $t$ of $R$ (resp. $R^2$).

Therefore, by Corollary 2 of E. Shult [23], $[\langle R^{O^2(G)} \rangle, \langle R^{O^2(G)} \rangle^Z] = 1$. Since $C(z)/O(C(z)) \cong Z_2 \times M_{24}$, the desired conclusion follows immediately from this.

This completes the proof of Theorem B.
CHAPTER III
STANDARD COMPONENTS OF TYPE $\Omega^+(8,2)$

INTRODUCTION

In this chapter we study finite groups with a standard component isomorphic to $\Omega^+(8,2)$. (The notation used in this introductory section is the same as that used in the rest of this chapter, and is explained at the end of this section.)

In order to state our Main Theorem, we need the following hypotheses:

Hypothesis A.

Let $G$ be a finite group with $O(G) = 1$ having a standard component $L$ isomorphic to $A_8$, such that $C_G(L)$ has cyclic Sylow 2-subgroups. Then $E(G)$ is isomorphic to one of the following groups: $A_8, A_8 \times A_8, A_{10}, SL(4,4)$ or $HS$, where $HS$ denotes the Higman-Sims simple group of order $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$. 
Hypothesis B.

Let $G$ be a finite group with $O(G) = 1$ having a standard component $L$ isomorphic to $SU(4, 2)$ such that $C_G(L)$ has cyclic Sylow 2-subgroups. Then $E(G)$ is isomorphic to one of the following groups:

- $SU(4, 2)$
- $SU(4, 2) \times SU(4, 2)$
- $SL(4, 4)$
- $PSL(4, 3)$
- $PSU(4, 3)$
- $SL(5, 3)$
- $SU(5, 3)$
- $PSp(4, 9)$

Hypothesis C.

If $G$ is a finite group such that $O(G) = 1$, then

$$E(N(X)/O(N(X))) = (O(N(X))E(N(X))/O(N(X)))$$

for every 2-subgroup $X$ of $G$.

Remark.

Hypothesis A is proved in R. Solomon [25] under some additional assumption concerning the U-conjecture. Hypothesis B is almost proved in K. Gomi [11], and R. Foote has recently made an announcement which implies the completion of the proof of Hypothesis B. Hypothesis C is the well-known B-conjecture, and is also almost a theorem.
In this chapter, we prove the
Main Theorem.

Let $G$ be a finite group with $O(G) = 1$ having a standard component $L$ isomorphic to $\Omega^*(8,2)$ such that $C_G(L)$ has cyclic Sylow 2-subgroups. Suppose that Hypotheses A and B hold in every proper section of $G$ and that Hypothesis C holds in $G$. Then $E(G)$ is isomorphic to one of the following groups:

$\Omega^*(8,2)$, $\Omega^*(8,2) \times \Omega^*(8,2)$, $\Omega^*(8,4)$ or $M(22)$,
where $M(22)$ denotes Fischer's group of order $2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$.

The proof of the Main Theorem begins with a study of fusion of an involution $z$ of $C_G(L)$. Let $A_1$, $A_2$ and $A_3$ be the elementary abelian subgroup of order 64 of a Sylow 2-subgroup of $L$ such that $N_L(A_i)/A_i \cong \Omega^*(6,2)$ for each $i$, and let $B_i = \langle z, A_i \rangle$. In Lemma 3.2, we prove that $(N_G(B_i)/B_i)^\infty \cong \Omega^*(6,2)$, $E_{64} \cdot \Omega^*(6,2)$ or $Sp(6,2)$. In the rest of Section 3, we show that if $(N_G(B_i)/B_i)^\infty \cong \Omega^*(6,2)$ for all $i$, then $E(G) \cong \Omega^*(8,2)$.

The analysis of the case where $(N_G(B_i)/B_i)^\infty \cong Sp(6,2)$ (Section 4) follows the same line of argument as in S. Assa [5]. We take a 2-subgroup $C$ of $N_G(B_i)^\infty$ such that $C \cong Z_2 \times (D_8 \rtimes D_8 \rtimes D_8 \rtimes D_8)$ and determine $N_G(C)$. Mainly from the two facts that $N_G(C)/C_N_G(C)C//
Z(\(C\)) is isomorphic to a subgroup of \(O^+(8,2)\) and that it has an involution whose centralizer is isomorphic to \(\mathbb{Z}_2 \times \Sigma_6\), we conclude \(N_G(C)/C_{N_G}(C/(C/Z(C))) \cong \text{Aut}(\text{SU}(4,2))\).

Let \(e_2\) be the involution of \(C'\). We show that \(C\) is weakly closed in \(N_G(C)\) with respect to \(C_G(e_2)\). Thus \(N_G(C)\) contains a Sylow 2-subgroup of \(G\). A fusion argument shows \(z \notin \text{E}(G)\). Then we prove that \(C\) is strongly closed in a Sylow 2-subgroup of \(N_{E(G)}(C)\) with respect to \(C_{E(G)}(e_2)\). Thus \(C_{E(G)}(e_2) = O(C_{E(G)}(e_2))\).

Finally, we prove \(O(C_{E(G)}(e_2)) = 1\) appealing to D. Gorenstein and J. Walter [14], and conclude \(E(G) \cong M(22)\).

If \((N_G(B_1)/B_1)^\infty \cong E_{64}^*:\Omega^+(6,2)\), we construct a subgroup of \(G\) isomorphic to \(\Omega^+(8,4)\) or \(\Omega^+(8,2) \times \Omega^+(8,2)\) by using "BN-pair theory." We first prove that \(C_1 = O_2(N_G(B_1)^\infty)\) is abelian. In Section 5, we eliminate the possibility that \(C_1\) is homocyclic of exponent 4 by applying Hypothesis B to \(C_G(a)\), where \(a\) is an element of order 3 of \(L\) such that \(C_L(a) \cong \text{SU}(4,2)\). Thus \(C_1 \cong E_{4096}^*\). In Lemma 6.1, we prove that \((N_G(C_i)/C_i)^\infty \cong \Omega^+(6,4)\) or \(\Omega^+(6,2) \times \Omega^+(6,2)\) by Hypothesis A. Let \(C_i = O_2(N_G(B_i)^\infty)\) for each \(i \in \{2,3\}\). In the rest of Section 6, we prove that \(C_i \cong E_{4096}^*\) and \((N_G(C_i)/C_i)^\infty \cong (N_G(C_i)/C_i)^\infty\) for each \(i \in \{2,3\}\), and that there exists a \(z\)-invariant 2-subgroup \(S\) such that \(C_S(z) \in \text{Syl}_2(L)\).
and such that \( S \in \text{Syl}_2(N_G(C_i)) \) for each \( i \in \{1,2,3\} \).

(When \( (N_G(C_1)/C_1)^{\infty} \cong \Omega^+(6,4) \), this involves a computation in terms of a Singer cycle (Lemma 6.2).) Let \( G_0 = \langle N_G(C_i)^{\infty} \mid 1 \leq i \leq 3 \rangle \). Lemma 1.8 of K. Gomi [12, I] (which appears as Lemma 1.24 in this chapter) shows that \( G_0 \cong \Omega^+(8,4) \) or \( \Omega^+(8,2) \times \Omega^+(8,2) \). In order to complete the proof of the Main Theorem, we must, of course, show that \( G_0 \triangleleft G \). Fortunately, this problem has been treated by G. Seitz [22, III] in a more general context with the assumption that (some special case of) Hypothesis C holds. In fact, as remarked in the Introduction of [22, III], the argument used there applies to our case without any change once the above \( G_0 \) is constructed. So we simply appeal to the results of [22, III].

Our notation is standard except possibly the following:

\( E(X) \) the product of the quasisimple subnormal subgroup of \( X \),

\( X^{\infty} \) the final term of the derived series of \( X \),

\( X \wr Y \) the wreath product of \( X \) by \( Y \),

\( X \ast Y \) a central product of \( X \) and \( Y \),

\( Z_n \) the cyclic group of order \( n \),

\( E_n \) the elementary abelian group of order \( n \),

\( D_8 \) the dihedral group of order 8,
$Q_8$ the quaternion group,
$\Sigma_n$ the symmetric group of degree $n$,
$\Sigma'_n$ the alternating group of degree $n$,
$\Gamma L(2,4)$ $Z_3 \times \text{SL}(2,4)$ together with an automorphism of order 2 inverting $Z_3$ and inducing $\text{Aut}(\text{SL}(2,4))$ on $\text{SL}(2,4)$,
$\Gamma U(4,2)$ $Z_3 \times \text{SU}(4,2)$ together with an automorphism of order 2 inverting $Z_3$ and inducing $\text{Aut}(\text{SU}(4,2))$ on $\text{SU}(4,2)$,
$GF(q)$ the field of $q$ elements,
$M_n(q)$ the set of $n \times n$ matrices with entries in $GF(q)$.

$X = YZ$ means that $Y \triangleleft X$ and that $X = \langle Y, Z \rangle$. If $Y \cap Z = 1$ and if an emphasis is to be placed on that fact, then we write $X = Y \cdot Z$.

If $X$ is a 2-group, then by $J(X)$ we denote the usual Thompson subgroup generated by the abelian subgroups of maximal order.

Let $G$ be a group isomorphic to $\Sigma_5'$ (resp. $\Sigma_5$). Suppose $G$ acts on a group $V$ isomorphic to $E_{16}$. If the order of the centralizer in $V$ of an element of order 3 of $G$ is 4, then we refer to this action as the "standard action as $\Sigma_5^-(4,2)$ (resp. $O^-(4,2)$)." If the action of an element of order 3 of $G$ is fixed-point-free, then we refer to this action as the "standard action as $\text{SL}(2,4)$ (resp. $\text{Aut}(\text{SL}(2,4))$)."
We use the "bar" convention for homomorphic images. Thus, if \( G \) is a group, \( N \) is a normal subgroup and \( \overline{G} \) denotes the factor group \( G/N \), then, for any subset \( X \) of \( G \), \( \overline{X} \) will denote the image of \( X \) under the natural projection \( G \rightarrow \overline{G} \). Similarly, we use the "double bar" and the "tilde" convention.

In Sections 3 through 7, we let \( G \) denote a group which satisfies the hypothesis of the Main Theorem, and we use symbols such as \( N(X) \) and \( C(X) \) to denote \( N_G(X) \) and \( C_G(X) \), respectively, except in Theorems 4.29 through 4.41 where such symbols denote \( N_{E(G)}(X) \) and \( C_{E(G)}(X) \), respectively.
SECTION 1.
PRELIMINARY RESULTS

In this section, we collect a number of preliminary lemmas to be used in later sections.

The first two lemmas are easy to verify and their proofs are omitted.

Lemma 1.1.

Let \( z \) be an involution acting on a 2-group \( Y \), and let \( a \) be an element of \( Y \) such that \([a,z]\) is an involution. Then \( z \) centralizes \([a,z]\).

Lemma 1.2.

Let \( x \) be an involution acting on an elementary abelian 2-group \( C \). Then the following hold:

1. \( |C_C(x)|^2 \geq |C| \).
2. If \( A \) is a subgroup of \( C_C(x) \), then \( |C_{C/A}(x)| \geq |[C,x]| \).
3. If \( A \) is an \( x \)-invariant subgroup of \( C \), then \( |C_C(x)| \geq |C_{C/A}(x)| \).
Lemma 1.3.

Let $R$ be a Sylow 2-subgroup of a group $G$, and let $Q$ be a subgroup of $R$ such that $[R : Q] = 2$. Let $z$ be an involution of $R - Q$ and suppose that each extremal conjugate of $z$ in $R$ is contained in $zQ$. Then $z \in G'$.

Proof.


Lemma 1.4.

Let $R$ be a Sylow 2-subgroup of a group $G$, and $C$ be an elementary abelian subgroup of $R$ which is weakly closed in $R$ with respect to $G$. Set

$R = R/C,$

$\Gamma = \{ E \subseteq R \mid E^g \subseteq C \text{ for some } g \in G, \ E \not\subseteq C \},$

$\Gamma' = \{ E \mid E \in \Gamma \},$

$\gamma = \max \{ |E| \mid E \in \Gamma \},$

$\gamma' = \max \{ |E| \mid E \in \Gamma' \}.$

Then the following hold:

1. If $E \in \Gamma$, then there exists $g \in G$ such that $E^g \subseteq C$ and $N_R(E)^g \subseteq R$.

2. If $x$ is an involution of $R$ such that $\langle x \rangle \in \Gamma$, then $|[C,x]| \leq \gamma'$. 
(3) If $E$ is an element of $\Gamma$ such that $|E| = \mathfrak{e}$, then $|C/(C \cap E)(x)| \leq |E|$ for every involution $x$ of $E - C$.

(4) If $E$ is an element of $\Gamma$ such that $|E| = \mathfrak{e}$, then $|C/(C \cap E)| \leq |E|^2$.

(5) If $E$ is an element of $\Gamma$ such that $|E| = \mathfrak{e}$, then $|[C, x]| \leq |E|$ for every involution $x$ of $E - C$.

(6) If $E$ is an element of $\Gamma$ such that $|E| = \mathfrak{e}$, then $|C/((C \cap E)[C, x])| \leq |E|$ for every involution $x$ of $E - C$.

Proof.

(1) is (9.3) of D. Goldschmidt [10].

(2) is Corollary 4 (2) of [10].

(3) follows from (1) and an equivalent statement may be found in the proof of Corollary 4 (1) of [10].

(4) follows from (3) and Lemma 1.2 (1), and is essentially the same as Corollary 4 (1) of [10].

(5) follows from (3) and Lemma 1.2 (2).

(6) follows from (3) and Lemma 1.2 (3).

Lemma 1.5.

Let $F$ be a special 2-group of order $2^{2n+1}$ with a subgroup $B$ such that $Z(F) = \Phi(F) \cong E_{2^n}$, $Z(F) \subseteq B \subseteq 2^{n+1}$.
\[ E_{2n+1} \text{ and } [B,F] = Z(F). \text{ Set } \widetilde{F} = F/Z(F), \text{ and } \overline{F} = F/B. \text{ Suppose that an involution } x \text{ acts on } F \text{ and that } B \text{ is } x\text{-invariant. Then } C^G_F(x) = C^G_{\overline{F}}(x). \]

Proof.

Since \([B,F] = Z(F)\), for each element \(y\) of \(Z(F)\), there exists an element \(t(y)\) of \(F\) such that \([B,t(y)] = \langle y \rangle\), and such an element is uniquely determined modulo \(B\). Note that the bijection between \(Z(F)\) and \(\overline{F}\) which associates the above \(t(y)\) with \(y\) is an \(x\)-isomorphism. Let \(z\) be an involution of \(B - Z(F)\). Now suppose the lemma is false. Then there is an element \(y\) of \(C_{Z(F)}(x)\) such that \([t(y),x] = z\). From Lemma 1.1, \(t(y) \in [\overline{F},x]\) and so \(t(y) \in [F,x]\). Hence \(y \in [Z(F),x]\). By a suitable choice of \(z\), we may assume \([t(y),x] = z\).

Then, again by Lemma 1.1, \([z,x] = 1\). Computing in the semi-direct product \(F\cdot\langle x \rangle\), we get \(1 = [(t(y)x)^2,t(y)x] = [t(y)^2z,t(y)x] \in y[Z(F),x]\). Since \(y \in [Z(F),x]\), this is a contradiction.

Lemma 1.6.

Let \(K\) be a group satisfying the following:

(i) If \(C = O_2(K)\), then \(\Phi(\Phi(C)) = 1\).
(ii) \( \mathbb{C}/\mathfrak{g}(\mathbb{C}) \) is isomorphic to \( \mathfrak{g}(\mathbb{C}) \) as a \( K/\mathbb{C} \)-module and irreducible.

(iii) There exists a \( K \)-orbit \( \{a^K\} \) of \( \mathfrak{g}(\mathbb{C}) \) such that \( \mathbb{C}_{\mathfrak{g}(\mathbb{C})}(C_K(a)) = \langle a \rangle \) and
\[
\mathfrak{g}(\mathbb{C}) = \langle x \mid \text{both } x \text{ and } ax \text{ are in } \{a^K\} \rangle.
\]
Then \( \mathbb{C} \) is homocyclic abelian of exponent 4, and the correspondence which associates \( x^2 \in \mathfrak{g}(\mathbb{C}) \) with \( x\mathfrak{g}(\mathbb{C}) \in \mathbb{C}/\mathfrak{g}(\mathbb{C}) \) is the unique \( K \)-isomorphism from \( \mathbb{C}/\mathfrak{g}(\mathbb{C}) \) onto \( \mathfrak{g}(\mathbb{C}) \).

Proof.

This is Proposition 1.4 of Chapter 2.

Lemma 1.7

Let \( A \) be a vector space of even dimension over \( GF(2) \) with a quadratic form.

(1) Assume either that the dimension is strictly greater than 4 or that the quadratic form is of minus type. Then, for each non-singular vector \( a \) of \( A \),
\[
A = \langle x \mid \text{both } a \text{ and } a + x \text{ are non-singular} \rangle.
\]

(2) Assume either that the dimension is strictly greater than 2 or that the quadratic form is of plus type. Then
Lemma 1.8.

Let \( G = \mathbb{Z}_2 \setminus \Sigma_n' \), \( n \geq 5 \). Let \( P = (G/Z(G))' \). 
(\( P \cong E_2^{2n-1} \Sigma_n' \) or \( E_2^{2n-2} \Sigma_n' \) for \( n \) odd or even, respectively.) Then there are one or two classes of complements to \( O_2(P) \) in \( P \), for \( n \) odd or even, respectively.

Proof.

See Lemma 11.3 of M. Aschbacher [1].

Lemma 1.9.

Let \( A = \langle e_i \mid 2 \leq i \leq 7 \rangle \cong E_{64} \) and regard \( A \) as a vector space over \( GF(2) \) with a quadratic form of plus type with associated alternating form \( f(\ , \) \) such that \( \langle e_i \mid 2 \leq i \leq 4 \rangle \) and \( \langle e_i \mid 5 \leq i \leq 7 \rangle \) are totally singular and \( f(e_i, e_9 - j) = \delta_{ij} \), where \( \delta_{ij} \) is Kronecker's delta. Let \( K \cong O^*(6,2) (\cong \Sigma_6) \) and let \( K \) act on \( A \) in a standard way. (Thus, an element \( x \) of \( A \) is "singular" (resp. "non-singular"), if \( C_K(x) \cong E_{16} \cdot (\Sigma_3 \setminus \mathbb{Z}_2) \) (resp. \( \mathbb{Z}_2 \times \Sigma_6 \cdot \)). For each non-singular element \( x \) of \( A \), let \( t(x) \) denote the orthogonal transvection of \( K \) such that
\[ [A, \tau(x)] = \langle x \rangle. \]

Then \( K = \langle t(e_2 e_7), t(e_4 e_5 e_7), t(e_2 e_3 e_7), t(e_3 e_6), t(e_4 e_5) \rangle \) and these are standard generators as \( g \). Let \( P = A \cdot K \) be a split extension of \( A \) by \( K \). Then the following hold:

1. There are exactly two conjugacy classes of complements to \( A \) in \( P \).
2. \( \langle t(e_2 e_7), e_2 e_7, t(e_4 e_5 e_7), t(e_2 e_3 e_7), t(e_3 e_6), t(e_4 e_5) \rangle \) is a complement and not conjugate to \( K \) in \( P \).

Proof.

(1) follows from Lemma 1.8. (2) can be verified by a direct computation.

Lemma 1.10.

Let \( F = \langle e_i | 1 \leq i \leq 7 \rangle \cong E_{128} \) and \( K = O^*(6, 2) \) or \( \Omega^*(6, 2) \). Let \( K \) act on \( F \) so that \( [e_1, K] = 1 \) and the action of \( K \) on \( F = F/\langle e_1 \rangle \) is the same as the one studied in Lemma 1.9. Furthermore, assume \( K \) acts on \( F \) indecomposably. Then the following hold:

1. The action of \( K \) on \( F \) is uniquely determined up to isomorphism. (If \( K \cong \Omega^*(6, 2) \), this action is the same as that of \( (Z_2 \setminus \Sigma_8') \)

\[ O_2((Z_2 \setminus \Sigma_8'), (Z_2 \setminus \Sigma_8')). \]

2. In particular, if \( x \) is a central involution of \( K \), and if \( y \) is an element of \( F \) such
that \( \overline{y} \) is singular in the sense of Lemma 1.9 and such that \( \overline{y} \in F(x) - [F, x], \) then \( [y, x] = e_1. \)

Proof.

(1) is essentially the same as (1) of Lemma 1.9.
(2) can be verified by a direct computation.

Lemma 1.11.

Let \( A, K \) be as in Lemma 1.9. Then the following hold:

1. By Lemma 1.7 (1), \( A = \langle x \rangle \) both \( x \) and \( xa \) are nonsingular element \( a \) of \( A. \)
2. For each element \( x \) of \( A^\# \), \( C_A(C_K(x)) = \langle x \rangle. \)
3. If \( V \) is a subgroup of order 16 of \( A, \) then \( |C_K(A)|_2 \leq 2 \) and \( |C_K(A)|_2 \leq 4. \)
4. If \( x \) is an involution of \( A \) and \( V \) is a subgroup of order 32 of \( A \) such that \( A = \langle x, V \rangle, \) then \( C_K(x) \cap N_K(V) \) is isomorphic to either \( O^+(4, 2) \) or \( O^-(4, 2). \)
5. If \( y \) is an involution of \( K', \) then \( |C_A(y)| = 16. \)
6. If \( E \) is an elementary abelian subgroup of order 8 of \( K \) which is not contained in \( K', \) then \( E \) contains an involution \( y \) such that
\[ |C_A(y)| = 8. \]

(7) If \( E \) is an elementary abelian subgroup of order 16 of \( K' \), then \( |C_A(E)| = 2 \).

Proof.

These properties are well-known or easy to verify by a matrix computation.

Lemma 1.12.

Let \( G \) be a group such that \( O_2(G) \cong E_{64} \), \( G/O_2(G) \cong Sp(6,2) \) and \( G/O_2(G) \) acts on \( O_2(G) \) in a standard way. Let \( H \) be a subgroup of \( G \) such that \( H \supseteq O_2(G) \) and \( H/O_2(G) \cong O^+(6,2) \). Assume \( G \) does not split over \( O_2(G) \). Then \( H \) does not split over \( O_2(G) \).

Proof.

Let \( P \) be the extra special group of plus type of order 128. (Thus \( P \cong D_8 \star D_8 \star D_8 \).) Let \( Q \) be the central product of \( Z_4 \) and \( P \) such that \( Z(Q) \cong Z_4 \).

Then

\[
\text{Inn}(Q) \cong E_{64},
\]

\[
C_{\text{Aut}(Q)}(Z(Q))/\text{Inn}(Q) \cong Sp(6,2),
\]

\[
(C_{\text{Aut}(Q)}(Z(Q)) \cap N_{\text{Aut}(Q)}(P))/\text{Inn}(Q) \cong O^+(6,2).
\]
By a direct computation, it follows that $\mathcal{C}_{\text{Aut}(Q)}(Z(Q)) \cap \mathcal{N}_{\text{Aut}(Q)}(P)$ does not split over $\text{Inn}(Q)$. Consequently, $\mathcal{C}_{\text{Aut}(Q)}(Z(Q))$ does not split over $\text{Inn}(Q)$. By U. Dempwolff [6], the nonsplit extension of $E_{64}$ by $\text{Sp}(6,2)$ with a standard action is uniquely determined up to equivalence of extensions. We also have that all the subgroups of $\text{Sp}(6,2)$ isomorphic to $O^{+}(6,2)$ are conjugate to each other. Thus, we may assume $G = \mathcal{C}_{\text{Aut}(Q)}(Z(Q))$ and $H = \mathcal{C}_{\text{Aut}(Q)}(Z(Q)) \cap \mathcal{N}_{\text{Aut}(Q)}(P)$, and hence the lemma follows immediately.

Lemma 1.13.

Let $C = E_{32}$ and $M = \text{SL}(4,2)$. Let $M$ act on $C$ so that $|\{C,M\}| = 16$ and so that the action of $M$ on $[C,M]$ is the same as that on a standard module. Then $M$ acts on $C$ decomposably.

Proof.

This lemma is essentially the same as Lemma 1.2 of Chapter II.

Lemma 1.14

Let $C = E_{16}$ and $M = \text{Sp}(4,2)$ and let $M$ act on $C$ in a standard way. Then the following hold for an
arbitrary elementary abelian subgroup $E$ of order 8 of $M$:

1. $|C_{C}(E)| \leq 4$.
2. $[C, x] \leq C_{C}(E)$ for every involution $x$ of $E$ such that $|[C, x]| = 2$.

Proof.

This is easily verified by a direct computation.

Lemma 1.15

Let $C = \langle e_{i}, f_{i} \mid 3 \leq i \leq 6 \rangle \cong E_{256}$. We regard $C$ as a vector space of dimension 4 over $GF(4)$ so that the action of an element $\alpha (a, 0, 1)$ of $GF(4)$ is given by $e_{i}^\alpha = f_{i}$ for each $i \in \{3, 4, 5, 6\}$. Furthermore, we introduce a Hermitian form $u$ on $C$ so that both $\langle e_{3}, f_{3}, e_{4}, f_{4} \rangle$ and $\langle e_{5}, f_{5}, e_{6}, f_{6} \rangle$ are totally singular and $u(e_{i}, e_{j} - f_{j}) = \delta_{i j}$ for each $i, j \in \{3, 4, 5, 6\}$. Let $M \cong Aut(SU(4, 2))$ and let $M$ act on $C$ in a standard way. Let $\alpha$ be an involution of $M$ which acts on $M'$ as a field automorphism. We assume $[f_{i}, \alpha] = e_{i}$ and $[e_{i}, \alpha] = 1$ for each $i$. Let $N = N_{M}(\langle e_{3}, f_{3}, e_{4}, f_{4} \rangle)$ and $S$ be a Sylow 2-subgroup of $N$ which contains $\alpha$. Then the following hold:
(1) $M$ has four classes of involutions and $|[C,x]| \geq 4$ for every involution $x$ of $M$.

(2) For every noncentral involution $x$ of $M$, $|[C,x]| = 16$.
For every central involution $x$ of $M$, $|[C,x]| = 4$.

(3) If $E$ is an elementary abelian 2-subgroup of order 4 of $M$, then $E$ contains at least one noncentral involution, and hence $|C_C(E)| \leq 16$ by (2).

(4) $O_2(N) = C_M(\langle e_3, f_3, e_4, f_4 \rangle) \cong E_{16}$. $N/O_2(N) \cong \Sigma_5$.

(5) $N/O_2(N)$ acts as $\text{Aut}(SL(2,4))$ in a standard way on both $\langle e_3, f_3, e_4, f_4 \rangle$ and $C/\langle e_3, f_3, e_4, f_4 \rangle$.

(6) $N/O_2(N)$ acts on $O_2(N)$ as $O^+(4,2)$ in a standard way. Hence, for every noncentral involution $b$ contained in $O_2(N)$, $O_2(N) = \langle x \in O_2(N) | \text{ both } x \text{ and } xb \text{ are noncentral} \rangle$ by Lemma 1.7 (1). In particular, $O_2(N) = \langle x \in O_2(N) | x \text{ is noncentral} \rangle$. Also, by Lemma 1.7 (2), $O_2(N) = \langle x \in O_2(N) | x \text{ is central} \rangle$.

(7) $O_2(N)$ is the unique elementary abelian subgroup of order 16 of $N'$.

(8) If $E$ is an elementary abelian subgroup of order 16 of $M$ which contains $a$, then $E \subseteq \langle a \rangle \times C_M'(a) \cong Z_2 \times Sp(4,2)$ and $C_C(E) \subseteq C_C(a) =$...
\[ \langle e_i \mid 3 \leq i \leq 6 \rangle, \text{ and hence, by Lemma 1.14,} \]
\[ |C_C(E)| \leq 4 \text{ and } |\langle C_C(E), [C, x] \rangle| \leq 8 \text{ for every central involution } x \text{ contained in } E. \]

(9). If \( E \) is an elementary abelian subgroup of order 16 of \( M \) which is not contained in \( M' \), then \( E \) contains an involution which is conjugate to \( a \) in \( M \), and hence, by (8),
\[ |C_C(E)| \leq 4 \text{ and } |\langle C_C(E), [C, x] \rangle| \leq 8 \text{ for every central involution } x \text{ contained in } E. \]

(10) For each element \( x \) of \( C - \langle e_3, f_3, e_4, f_4 \rangle \),
\[ |[x, O_2(N)]| = 8. \]

Proof.

These are easy to verify.

Lemma 1.16.

Let \( N \) be a group such that \( N/O(N) \cong PSL(3, 4) \).

Suppose \( N \) acts faithfully and irreducibly on an elementary abelian 2-group \( J \) of order at most 1024.

Then one of the following hold:

(1) \( |J| = 64 \) and \( N \cong SL(3, 4) \); or
(2) \( |J| = 512 \) and \( N \cong PSL(3, 4) \).

In each case, the action is uniquely determined up to isomorphism.
Proof.

(We argue as in the fourth paragraph of the proof of Lemma 2.14 of Chapter II.) Let

$$O(N) = P_0 \supset P_1 \supset P_2 \supset \cdots \supset P_k = 1$$

be a characteristic composition series of $O(N)$. (Namely, $P_i^{-1}$ is a minimal characteristic subgroup of $O(N)$ properly containing $P_i$ for each $i$.) In general, if $p$ is an odd prime, then the 2-rank of $GL(n,p)$ is $n$ and every elementary abelian 2-subgroup of order $2^n$ contains the involution of the center of $GL(n,p)$. On the other hand, the 2-rank of $PSL(3,4)$ is 4, and the rank of $P_i^{-1}/P_i$ is less than or equal to 4 for each $i$, for $N$ is isomorphic to a subgroup of $GL(10,2)$. Consequently, the action of $N^\infty$ on $P_i^{-1}/P_i$ is trivial. By the three-subgroup lemma,

$$1 = [O(N),N^\infty,\cdots,N^\infty,N^\infty]_k$$

$$= [O(N),N^\infty,\cdots,N^\infty]_k^{-1}$$

$$\cdots$$

$$= [O(N),N^\infty]$$

Thus $N^\infty$ is isomorphic to $PSL(3,4)$ or $SL(3,4)$, and $N$ is a central product of $N^\infty$ and $O(N)$. Now the desired conclusion follows immediately from the 2-modular character table of $SL(3,4)$ (G. James [19] and Appendix).
Lemma 1.17.

Let $N \cong PSL(3, 4)$ and let $N$ act faithfully on an elementary abelian 2-subgroup $J$ of order 512. Let $R \in Syl_2(N)$. Then the following hold:

1. $R$ contains exactly two elementary abelian subgroups $A, B$ of order 16.
3. The order of the centralizer in $J$ of one of $\{A, B\}$ is 16, while that of the other is 2.
4. We choose our notation so that $|C_J(A)| = 16$ and $|C_J(B)| = 2$.
5. $J$ splits into three $N$-classes of involutions the sizes of which are 21, 210, 280.
6. $N$ has exactly one class of involutions, and $|[J, x]| = 16$ for every involution $x$ of $N$.

Proof.

(1) is well-known. (2) follows from Lemma 1.16. Lemma 1.16 also asserts that the action is uniquely determined up to isomorphism. But this action is obtained in the following manner. We regard $SL(3, 4)$ as a matrix group. The additive group of all Hermitian matrices of $M_3(4)$ may be regarded as $E_{512}$, and if, for each element $x$ of this additive group and for each element $a$ of
SL(3,4), we let a act on x so that the image of x under the action of a is \( t(\alpha x) \) where \( t \) denotes "(transpose) \( x \) (field automorphism)," then \( Z(SL(3,4)) \) is in the kernel of this action. Now (3), (4), (5) can be verified directly by a matrix computation.

Lemma 1.18.

Let \( N \) be a group such that \( N/O(N) \cong M_{22} \). Suppose \( N \) acts faithfully on an elementary abelian 2-group \( J \) of order 1024. Then \( N \cong M_{22} \) and there are exactly two possibilities for the action, with orbits either

(1) 22, 231, and 770; or

(2) 77, 330 and 616.

Proof.

Arguing as in Lemma 1.16, we have \( N \cong M_{22} \) or \( 3M_{22} \). From the 2-modular character tables of \( M_{22} \) and \( 3M_{22} \) (G. James [19]), it follows that \( N \cong M_{22} \) and there are exactly two possibilities for the action. The orbit lengths are well-known. (See D. Hunt [18] and F. L. Smith [24].)

Lemma 1.19.

Let \( N \cong M_{22} \) and let \( N \) act faithfully on an elementary abelian 2-group \( J \) of order 1024. Suppose the action is the same as that described in Lemma 1.18.
(1). Let $R \in \text{Syl}_2(N)$. Then the following hold.

(1) $R$ contains exactly two elementary abelian subgroups $A,B$ of order 16. We choose our notation so that $N_N(A)/A = \Sigma_6^+$ and $N_N(B)/B = \Sigma_5$. $N$ contains exactly one class of involutions.

(2) Let $a,b$ and $c$ be elements of $J$ such that $|\{a^N\}| = 22$, $|\{b^N\}| = 231$ and $|\{c^N\}| = 770$, respectively. Then, $C_N(a) \cong \text{PSL}(3,4)$, $C_N(b) \cong E_{16} \cdot \Sigma_5$ and $C_N(c) \cong E_{16} \cdot (E_9 \cdot Z_4)$. We choose our notation so that $O_2(C_N(b)) = B$ and $O_2(C_N(c)) = A$.

(3) $|C_J(x)| = 64$ for any involution $x$ of $N$.

(4) $|C_J(A)| = 32$, $|C_J(A)/N_N(A)| = 16$ and $N_N(A)$ acts on $C_J(A)$ indecomposably. In particular, $[C_J(A),x] = 4$ for any involution $x$ of $N_N(A) - A$.

(5) All involutions of $[C_J(A),N_N(A)]$ are conjugate to $b$ under the action of $N$. Six involutions of $C_J(A) - [C_J(A),N_N(A)]$ are conjugate to $a$. The remaining ten involutions of $C_J(A) - [C_J(A),N_N(A)]$ are conjugate to $c$.

(6) Let $X$ be the set of pairs $(x,y)$ of involutions of $C_J(A)$ such that $x \in \{b^N\}$ and $y \in \{c^N\}$. Then $X$ splits into two $N_N(A)$-orbits.

(7) $|[J/C_J(A),A]| = |C_J/C_J(A)(A)| = 16$. 
Proof.

(1) and (2) are well-known. (3) follows from (2) by a counting argument. From (3), \( |C_J(A)| \leq 64 \). From (2), \( C_N(x) \) contains some conjugate of \( A \) for any involution \( x \) of \( J \) (See Lemma 1.17 (1)), and so \( C_J(A) \) contains some involution of each of the three orbits. In particular, it contains some involution of \( \{c^N\} \). Hence, \( |C_J(A)| \geq 32 \).

If \( |C_J(A)| = 64 \), then, from Lemma 1.8, at least one involution of \( C_J(A) \) must be centralized by \( N_N(A) \). This contradicts (2). Hence, \( |C_J(A)| = 32 \). The other assertions of (4) follow also from the fact that no involution of \( C_J(A) \) is centralized by \( N_N(A) \). Since all involutions of \( [C_J(A), N_N(A)] \) are centralized by a Sylow 2-subgroup of \( N_N(A) \), they belong to \( \{b^N\} \). Since \( C_J(A) \) splits into three orbits of involutions under the action of \( N_N(A) \), different orbits under this action must correspond to different \( N \)-orbits. Let \( y \) be any involution of \( C_J(A) - [C_J(A), N_N(A)] \) such that \( |\{y^N\}| = 6 \). Then, since \( 5 \nmid |C_{N_N(A)}(y)| \) and since \( y \) does not belong to \( \{b^N\} \), \( y \) must belong to \( \{a^N\} \). This also shows that any involution \( z \) of \( C_J(A) - [C_J(A), N_J(A)] \) such that \( |\{z^N\}| = 10 \) belongs to \( \{c^N\} \). Thus (5) is proved. Since \( [C_J(A), N_N(A)] \) splits into two orbits of involutions with lengths 6 and 9 under the action of \( C_{N_N(A)}(c) (= C_N(c)) \), (6) follows. If we choose the element \( a \) so that \( C_N(a) \) contains both \( A \) and \( B \) and consider the action of \( C_N(a) \)
on $J/\langle a \rangle$, then, by Lemma 1.17 (3), $A$ and $B$ of this lemma correspond to $A$ and $B$ of Lemma 1.17, respectively. Hence, (7) follows from Lemma 1.17 (4).

Lemma 1.20.

Let $L = \Omega^*(6,4)$. Let $\{t, u, v\}$ be a system of fundamental roots of type $(D_5)$, where the Dynkin diagram is as follows:

```
  o---o---o
 t   u   v
```

Figure 1. Dynkin diagram 1

For each $\rho \in \{t, -t, u, -u, v, -v\}$, let $X_\rho$ denote the corresponding root subgroup of $L$. Set $K = \langle X_t, X_{-t}, X_u, X_{-u} \rangle$. Thus, $K \cong SL(3,4)$. Let $\delta$ be an automorphism of $K$ such that $(X_\rho)^\delta = X_\rho$ for each $\rho \in \{t, -t, u, -u\}$. Then, there is an automorphism $\epsilon$ of $L$ such that $(X_\rho)^\epsilon = X_\rho$ for each $\rho \in \{t, -t, u, -u, v, -v\}$ and such that the restriction of $\epsilon$ to $K$ equals $\delta$. Moreover, if $\delta$ acts trivially on both $X_u$ and $X_{-u}$, then $\epsilon$ can be chosen so that $\epsilon$ may act trivially on both $X_v$ and $X_{-v}$.

Proof.

These assertions follow immediately, since, under the assumption of the lemma, $\delta$ can be expressed in the form of the (possibly non-commuting) product of a field automorphism and a diagonal automorphism of $K$. 

Lemma 1.21

Let $M = \Omega^+ (6,4)$ and $z$ be an involution of $\text{Aut}(M) - M$ such that $z$ acts on $M$ as the standard field automorphism. Let $S$ be a $z$-invariant Sylow 2-subgroup of $M$. Then the following hold.

1. Any involution $x$ of $\text{Aut}(M) - M$ such that $\text{CM}(x) = \Omega^+ (6,2)$ is conjugate to $z$ in $M < z >$.

2. Any involution of $xS$ is conjugate to $z$ in $S < z >$.

Proof.

These statements are easy to verify by a direct computation or a proof may be found in M. Archbacher and G. Seitz [3].

Lemma 1.22

Let $K = \Omega^+ (6,2)$, $U \in \text{Syl}_2 (K)$ and $(U,N)$ be a natural BN-pair of $K$. Let $\sigma$ be an automorphism of $K$ such that $U^\sigma = U$. Then, there is an element $g$ of $U$ such that $N^\sigma = N^g$.

Proof.

Since $\text{Out}(K) \cong \mathbb{Z}_2$ and since $\text{Aut}(K) - K$ contains an element $t$ such that $U^t = U$ and $N^t = N$, there is an element $g$ of $K$ such that $U^g = U^\sigma (=U)$ and $N^g = N^\sigma$. 
Since $N_K(U) = U$, $g \in U$.

Lemma 1.23.

Let $M$ be either $\Omega^+(6,2) \times \Omega^+(6,2)$ or $\Omega^+(6,4)$, and $z$ be an involutive automorphism of $M$ such that $K = C_M(z)$ is isomorphic to $\Omega^+(6,2)$. Let $(U,N)$ be a natural BN-pair of $K$, $S$ be a $z$-invariant Sylow 2-subgroup of $M$ such that $U = C_S(z)$, and let the distinguished generators of the Weyl group $W = N/(U \cap N)$ be labeled

$W = N/(U \cap N) \cong N$

be labeled

Figure 2. Dynkin diagram 2

Then there exists a natural BN-pair $(B_1, N_1)$ of $M$ satisfying the following:

1. $S \subseteq B_1$.
2. $z$ normalizes $B_1$ and $N_1$.
3. $C_{B_1}(z) = U$ and $C_{N_1}(z) = N$. 


(Under the condition (3), W is canonically embedded in the Weyl group
\[ W_1 = N_1/(B_1 \cap N_1), \]
and so we regard \( W \subseteq W_1 \). Also z acts on \( W_1 \) by the condition (2).)

(4) If
\[ M = \Omega^*(6,2) \times \Omega^*(6,2) \]
and if the distinguished generators of \( W_1 \) are labeled

![Figure 3. Dynkin diagram 3](image)

then either
\[ \{q, s, r\} = \{u_i v_i \mid i \in \{1,2,3\}\} \]
or
\[ \{q, s, r\} = \{u_i v_{4-i} \mid i \in \{1,2,3\}\}. \]

(5) If
\[ M = \Omega^*(6,4) \]
then \( q, s, r \) are the distinguished generators of \( W_1 \).
Proof.

Assume \( M \cong \Omega^+(6,2) \times \Omega^+(6,2) \). Then, \( M = M_1 \times M_2 \), \( M_2 = M_1^Z \cong \Omega^+(6,2) \), and the mapping \( x \rightarrow xx^Z \), \( x \in M_1 \), is an isomorphism from \( M_1 \) onto \( K \). Let \( N_0 \) be the inverse image of \( N \) under this isomorphism and define \( N_1 = N_0N_0Z \). Then \( (S,N_1) \) meets all the requirements.

Therefore, assume \( M \cong \Omega^+(6,4) \). Then, by Lemma 1.21, we may assume \( z \) acts on \( M \) as the standard field automorphism. Hence, there exists a BN-pair of \( K \) of the form \( (U,N^\ast) \) such that \( (S,N^\ast) \) is a BN-pair of \( M \). Therefore, from Lemma 1.22, we get the desired conclusion.

Lemma 1.24

Let \( G_0^\ast \) be a group with a BN-pair \( (B^\ast,N^\ast) \) such that \( B^\ast \cap N^\ast \) has a normal complement \( S^\ast \) in \( B^\ast \). Assume that the Weyl group \( W^\ast = N^\ast/(B^\ast \cap N^\ast) \) is isomorphic to the Weyl group \( W(\Sigma) \) of a root system \( \Sigma \) of rank \( k > 1 \) and that the isomorphism carries the distinguished generators of \( W^\ast \) onto the reflections \( w_1, \ldots, w_k \) associated with the fundamental roots \( \Pi = \{r_1, \ldots, r_k\} \). Identify \( W^\ast \) with \( W(\Sigma) \) and choose an element \( w_0 \in W^\ast \) which carries the set \( \Sigma^+ \) of positive roots onto the set of negative roots. For each \( w \in W^\ast \), choose a representative \( n(w) \) of \( w \) in \( N^\ast \), and define

\[
X^\ast_{r_i} = S^\ast \cap n(w_0w_i)^{-1}S^\ast n(w_0w_i).
\]

A root subgroup is a conjugate under \( N^\ast \) of \( X^\ast_{r_i}, 1 \leq i \leq k \).
Let \( r \to X_r^* \), \( r \in \Sigma \), be the one-to-one mapping of \( \Sigma \) onto the set of root subgroups such that
\[
n(w)X_r^* n(w)^{-1} = X_w^*(r)
\]
Furthermore, assume the following.

1. \( G_0^* = <X_r^* | r \in \Sigma > \) and \( G_0^* \) is quasisimple
2. for all independent roots \( r, s \in \Sigma \), it is possible to arrange the roots of the form \( \lambda r + \mu s, \lambda, \mu > 0 \), in some order so that
\[
[X_r^*, X_s^*] \subseteq \Pi X_{\lambda r + \mu s}^*
\]
3. if \( A_{ij} \) is the set of all roots of the form \( \lambda r_i + \mu r_j, i \neq j \), and \( A = A_{ij} \), then for each \( r \in A \), there exists a subset \( S \) of \( A_{ij} \) for some \( \{i, j\} \) such that \( r \in S \), \( w(S) \subseteq \Sigma^+ \) for some \( w \in W^* \), and \( X_r^* \subseteq <X_s^* | r \neq s \in S > \).

Now let \( G_0 \) be a group and suppose there exists subgroups \( X_r, r \in \pm \Pi \), satisfying the following conditions:

1. \( G_0 = <X_r | r \notin \pm \Pi > \) and \( G_0 \) is perfect
2. for each \( r \notin \pm \Pi \), there exists an isomorphism \( X_r^* \to X_r \)
3. for each \( \{i, j\}, i \neq j \), there exists an epimorphism \( <X_{\pm r_i}^*, X_{\pm r_j}^* > \to <X_{\pm r_i}^*, X_{\pm r_j}^* > \)
    which extends the isomorphisms given in (ii).

Then there exists an epimorphism \( \phi: G_0 \to G_0^*/Z(G_0^*) \) such that \( \ker \phi = Z(G_0) \) and \( (X_{\pm r_i}^*)^\phi = (X_{\pm r_i}^*) \), where is the natural epimorphism \( G_0^* \to G_0^*/Z(G_0^*) \).
Proof.

See Lemma 1.8 of K. Gomi [13,1].
SECTION 2

PROPERTIES OF $\Omega^+(8,2)$

In this section, we fix notation for $\Omega^+(8,2)$ and $O^+(8,2)$, and state some of their properties.

Notation 2.1.

Let $L = \Omega^+(8,2)$. We represent $L$ as a group of matrices in the following standard way. Let $V$ be a vector space of dimension 8 over $\text{GF}(2)$ with a quadratic form of plus type with associated alternating form $f(\cdot, \cdot)$. Let $\{v_i\}_{1 \leq i \leq 8}$ be a basis of $V$ such that $\{v_i\}_{1 \leq i \leq 4}$ and $\{v_i\}_{5 \leq i \leq 8}$ are totally singular and $f(v_i, v_j - j) = \delta_{ij}$, where $\delta_{ij}$ is Kronecker's delta. Finally, let $L$ act on $V$ so that $L$ leaves the above quadratic form invariant. Under this identification, let

$$U = \text{the subgroup of } L \text{ consisting of lower triangular matrices,}$$

$$p = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}$$
\[ q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]

\[ r = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \]

\[ s = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \]

\[ N = \langle p, q, r, s \rangle. \]

Then \((U, N)\) is a natural BN-pair, and the Dynkin diagram is as follows:

![Dynkin diagram 4](image)

That is to say, \(p, q, r, s\) satisfy the relations

\[(ps)^3 = (qs)^3 = (rs)^3 = (qr)^2 = (rp)^2 = (pq)^2 = 1.\]

Let \(w\) be the element of \(N\) such that \(U \cap U^w = 1\). Then
\[ w = \text{psqrsp \cdot qrqsrs} = \text{qsrpsq \cdot rpsrps} = \text{rpsqsr \cdot pqspqs}, \]

and, in terms of a matrix,

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

Also notice that

\[ \text{qrsqrs} = \text{sqrsqr}, \text{rpsrps} = \text{srpsrp}, \text{pqspqs} = \text{spqspq}. \]

Let

\[ P_1 = \langle U, q, r, s \rangle, \quad P_2 = \langle U, r, p, s \rangle, \quad P_3 = \langle U, p, q, s \rangle. \]

Then \( P_1 = C_L(v_1), \quad P_2 = N_L(\langle v_1, v_2, v_3, v_5 \rangle), \quad P_3 = N_L(\langle v_1, v_2, v_3, v_4 \rangle). \) Let \( A_i = O_2(P_i) \) for each \( i. \) Then \( P_i/A_i \cong \Sigma_8 \cong \Omega^+(6,2) \cong SL(4,2), \quad A_i \cong E_6(2), \) and \( P_i/A_i \) acts on \( A_i \) as \( \Omega^+(6,2) \) in a standard way. Let

\[
\begin{align*}
K_1 &= P_1 \cap P_1^{psqrsp}, \quad U_1 = U \cap U^{psqrsp}, \\
K_2 &= P_2 \cap P_2^{qsrpsq}, \quad U_2 = U \cap U^{qsrpsq}, \\
K_3 &= P_3 \cap P_3^{rpsqsr}, \quad U_3 = U \cap U^{rpsqsr}.
\end{align*}
\]

Then \( P_i = A_i \cdot K_i \) and \( U_i \in \text{Syl}_2(K_i). \)

Next let

\[
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\]

and let \( X = \langle L, v \rangle. \) Then \( X = O^+(8,2). \) Let

\[
\begin{align*}
Q &= \langle U, v \rangle, \quad P = \langle P_1, v \rangle, \quad K = \langle K_1, v \rangle.
\end{align*}
\]
Then \( Q \subseteq \text{Syl}_2(X) \), \( N_x(A_i) = P = A_1 \cdot K \) and \( N_x(A_i) = P_i \) for \( i \in \{2, 3\} \).

For \( 2 \leq i \leq 7 \), let \( e_i \) be the element of \( L \) such that \( [v_8, e_i] = v_1 \), \( [v_9 - i, e_i] = v_1 \), \( [v_k, e_i] = 1 \) for \( k \neq 8, 9 - i \).

For \( 3 \leq i \leq 6 \), let \( u_i \) be the element of \( L \) such that \( [v_7, u_i] = v_i \), \( [v_9 - i, u_i] = v_2 \), \( [v_k, u_i] = 1 \) for \( k \neq 7, 9 - i \).

For \( 4 \leq i \leq 5 \), let \( w_i \) be the element of \( L \) such that \( [v_6, w_i] = v_i \), \( [v_9 - i, w_i] = v_3 \), \( [v_k, w_i] = 1 \) for \( k \neq 6, 9 - i \).

Then \( U = \langle e_i, u_j, w_i \rangle \ 2 \leq i \leq 7, \ 3 \leq j \leq 6, \ 4 \leq l \leq 5 \rangle \) \( Q = \langle U, v \rangle \)

and these generators satisfy the following conditions.

\[
e_i^2 = u_j^2 = w_i^2 = v^2 = 1
\]

For \( 3 \leq i \leq 6 \), \( [e_7, u_i] = e_i \), \( [e_9 - i, u_i] = e_2 \)

For \( 4 \leq i \leq 5 \), \( [e_6, w_i] = e_i \), \( [e_9 - i, w_i] = e_3 \)

\( [u_6, w_i] = u_i \), \( [u_9 - i, w_i] = u_3 \).

\( [e_4, v] = [e_5, v] = e_4 e_5 \), \( [u_4, v] = [u_5, v] = u_4 u_5 \), \( [w_4, v] = [w_5, v] = w_4 w_5 \)

All unstated identities are trivial.

These give us the generators and relations for \( U \) and \( Q \).

In terms of these generators

\[
A_1 = \langle e_i \rangle \ 2 \leq i \leq 6
\]

\[
A_1 = \langle e_2, e_3, e_5, u_3, u_5, w_5 \rangle
\]

\[
A_3 = \langle e_2, e_3, e_4, u_3, u_4, w_4 \rangle
\]
Finally, let

\[ A_4 = \langle e_2, e_3, e_4 e_5, u_3, u_4 u_5, w_4 w_5 \rangle \]
\[ A_5 = \langle e_2, e_3, e_5, u_3 e_4, u_5 e_6, w_5 e_7 \rangle \]
\[ A_6 = \langle e_2, e_3, e_4, u_3 e_5, u_4 e_6, w_4 e_7 \rangle \]
\[ A_7 = \langle A_4, v \rangle \]
\[ P_i = N_L(A_i) \text{ for each } i \in \{4, 5, 6\} \]
\[ P_7 = N_x(A_7) \]

Then \( P_i/A_i \cong E_8 \cdot L_3(2) \) for each \( i \in \{4, 5, 6, 7\} \), where \( L_3(2) \) acts on \( E_8 \) in a standard way.

The following three lemma are well known.

Lemma 2.2

\[ \text{Out}(L) \cong \Sigma_3, \text{ and } \text{Out}(L) \text{ acts on } \{ \{ A_i^L \} | i \in \{1, 2, 3\} \} \text{ and } \{ \{ A_i^L \} | i \in \{4, 5, 6\} \} \text{ as a standard permutation group.} \]

Lemma 2.3

\( L \) has five classes of involutions and their representatives are \( e_2, e_2 e_7, e_2 w_4, e_2 w_5, e_2 w_4 w_5 \).
\[ C_L(e_2) \cong (D_8 \ast D_8 \ast D_8 \ast D_8) \cdot (\Sigma_3 \times \Sigma_3), \]
\[ C_X(e_2) \cong (D_8 \ast D_8 \ast D_8 \ast D_8) \cdot (\Sigma_3 \times (\Sigma_3 \setminus Z_2)) \]
\[ C_L(e_2 e_7) \cong C_L(e_2 w_4) = C_X(e_2 w_4) \cong C_L(e_2 w_5) = \]
\[ C_X(e_2 w_5) \cong E_{64} \cdot \Sigma_6, \quad C_X(e_2 e_7) \cong E_{64} \cdot (Z_2 \times \Sigma_6) \]
\[ C_L(e_2 w_4 w_5) \cong E_{64} \cdot (Z_2 \times \Sigma_4), \quad C_X(e_2 w_4 w_5) \cong \]
\[ E_{128} \cdot (Z_2 \times \Sigma_4) \]
In particular, the following hold for each involution \( t \) of \( L \).

1. \( C_L(t) \supseteq t \)
2. \( 0(C_L(t)) = 0(C_X(t)) = 1 \) and neither \( C_L(t) \)
or \( C_X(t) \) has a normal cyclic subgroup of
order 4.
3. \( \{t^L\} \cap A_i \neq \phi \) for some \( i \in \{1,2,3,4,5,6\} \).

Lemma 2.4

\( X \) has six classes of involutions and their repre-
sentatives are \( e_2, e_2e_7, e_2w_4, e_2w_4w_5, w, u_3v \). For each involu-
tion \( t \) of \( X \) such that \( t \notin \{e_2w_4\}^X \), \( \{t^X\} \cap A_7 \neq \phi \)

Lemma 2.5

1. \( Z(U) = \langle e_2 \rangle, \ Z_2(U) = \langle e_2, e_3 \rangle, \ Z_3(U) =\)
   \( \langle e_2, e_3, e_4, e_5, u_3 \rangle \)
2. If \( A \) is an elementary abelian subgroup of
   order 64 of \( U \), then \( A = A_i \) for some \( i \in \{1,2,3,4,5,6\} \), and so \( A \triangleleft U \).
3. Let \( \overline{U} = U/Z_2(U) \). Then
   \( A_1 \cap Z_3(U) = \langle e_4, e_5 \rangle \)
   \( A_2 \cap Z_3(U) = \langle e_5, u_3 \rangle \)
   \( A_3 \cap Z_3(U) = \langle e_4, u_3 \rangle \)
   \( A_4 \cap Z_3(U) = \langle e_4e_5, u_3 \rangle \)
   \( A_5 \cap Z_3(U) = \langle e_5, u_3e_4 \rangle \)
   \( A_6 \cap Z_3(U) = \langle e_4, u_3e_5 \rangle \)
(4) Let $\sigma$ be an automorphism of $U$ such that
\[ \sigma^{-2} \in \text{Inn}(U), \] then $A_i = A_i$ for some $i \in \{1,2,3\}$.

Proof.

We can easily verify (1), (2), and (3) using the generators and relations given in Notation 2.1. To prove (4), first note that
\[ \langle u_3^i e_4, u_3^i e_5 \rangle = \{ \overline{Y} | \overline{Y} \subseteq \underline{Z}_3(U), \ | \overline{Y} | = 4 \} - \{ \overline{A}_i \cap \underline{Z}_3(U) | i \leq i \leq 6 \} \]

Thus, from (2), $Z_2(U) \langle u_3^i e_4, u_3^i e_5 \rangle$ is characteristic in $U$. Hence, if we consider $\sigma$ to be an involutive automorphism of $\overline{Z}_3(U)$, $\langle u_3^i e_4, u_3^i e_5 \rangle$ is $\sigma$-invariant. Therefore $[\overline{Z}_3(U), \sigma] = 1$, $\langle u_3^i e_4 \rangle$, $\langle u_3^i e_5 \rangle$ or $\langle e_4^i e_5 \rangle$. If $[\overline{Z}_3(U), \sigma] = 1$ or $\langle u_3^i e_4 \rangle$, then $\overline{A}_3 \cap \underline{Z}_3(U)$ is $\sigma$-invariant, and hence $A_3$ is $\sigma$-invariant from (2) and (3). Similarly, if $[\overline{Z}_3(U), \sigma] = \langle u_3^i e_5 \rangle$ (resp. $\langle e_4^i e_5 \rangle$), then $A_2$ (resp. $A_1$) is $\sigma$-invariant. This proves (4).

Lemma 2.6

(1) $A_1$ is the unique self-centralizing elementary abelian normal subgroup of order 64 of $Q$.

(2) $A_7$ is the unique abelian subgroup of order 128 of $Q$, and so $A_7 = J(Q)$.  


Proof.

We can verify these statements from the generators and the relations of Notation 2.1.
SECTION 3
INITIAL REDUCTION

In the remainder of this chapter, we let $G$ denote a group which satisfies the hypotheses of the Main Theorem, and we use the description of $L$ given in Notation 2.1. Let $z$ denote a generator of a Sylow 2-subgroup of $C(L)$. If

$$[N(L) : LC(L)]_2 = 2,$$

then, by Lemma 2.2, we assume, without loss of generality, that

$$N_{N(L)}(A_1)/C_{N(L)}(A_1) \cong \Sigma_8.$$  

Thus, if

$$[N(L) : LC(L)]_2 = 2,$$

then there is an element $v'$ of $N_N(z) \langle A_1 \rangle$ such that

$$[e_2, v'] = [e_3, v'] = [e_6, v'] = [e_7, v'] = 1,$$

$$[e_4, v'] = [e_5, v'] = e_4 e_5,$$

$C_L(v') \cong Sp(6, 2)$, and

$v'^2 \in \langle z \rangle.$

Lemma 3.1.

If $|\langle z \rangle| \geq 4$, then $\Omega_1(\langle z \rangle) \subseteq Z(G)$.  

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Proof.

Let $x$ be the involution of $\langle z \rangle$, and let $N(L) = N(L)/C(L)$. Suppose the lemma is false. Then, by Theorem 1 of L. Finkelstein [8], $z^g \in N(L)$ for some $g \in G - N(L)$. Since $C(L)$ is tightly embedded in $G$, $\langle z \rangle^g \cap C(L) = 1$. Again, since $C(L)$ is tightly embedded in $G$, $C(L)^g \cap N(L)$ is tightly embedded in $N(L)$. Hence $C_{N(L)}(x^g)$ normalizes $C(L)^g \cap N(L)$. On the other hand, since $x^g \in \bigcup \langle z \rangle^g$ and $[N(L) : LC(L)]_2 \leq 2$, $x^g \in LC(L)$ and hence

$$C_{N(L)}(x^g) = C_{N(L)}(\overline{x^g}).$$

Thus $C(L)^g \cap N(L)$ is normalized by $C_{N(L)}(\overline{x^g})$. That is, to say,

$$C(L)^g \cap N(L) \cap C_{N(L)}(\overline{x^g}) \leq C_{N(L)}(\overline{x^g}).$$

This contradicts Lemma 2.3 (2).

In the remainder of this chapter, we assume $z^2 = 1$. Let $B_i = \langle z, A_i \rangle$, $D_i = N_{N(L)}(B_i)$ and $N_i = N(B_i)$ for each $i \in \{1, 2, 3, 4, 5, 6\}$.

Lemma 3.2.

Let $i$ denote one of $\{1, 2, 3\}$, and let $N_i = N_i/B_i$. Then $[B_i, O(N_i)] = 1$, and one of the following holds:
(1) $N_1 = D_1$

(2) $N_1 = 0_2(N_1)D_1$, $0_2(N_1) \cong E_{64}$, $[B_1, 0_2(N_1)] = A_1$, $[A_1, 0_2(N_1)] = 1$, $<0(N_1), 0_2(N_1), K_1> = 0(N_1) \times (0_2(N_1) \cdot K_1)$, $0_2(N_1)$ and $A_1$ are isomorphic as a $K_1$-module.

(3) $D_1 \cong 0(D_1) \times \Sigma_8$ and $N_1 = 0(N_1) \times S(6,2)$

Proof.

We prove this for $i = 1$. The orbits of $B_1$ under the action of $D_1$ are $\{z\}$, $\{e_2\}$, $\{(ze_2)^D\}$, $\{(e_2e_7)^D\}$ and $\{(ze_2e_7)^D\}$. Their lengths are $|\{z\}| = 1$, $|\{e_2\}| = |\{(ze_2)^D\}| = 35$, $|\{(e_2e_7)^D\}| = |\{(ze_2e_7)^D\}| = 28$. Note that the action of $K_1$ on $A_1$ is a standard action of $\Omega^+(6,2)$, and that, in this point of view, $e_2$ and $e_2e_7$ correspond to a singular and a non-singular vector, respectively.

From Lemma 2.3 (1), $e_2 \not\sim z \not\sim e_2e_7$. Hence $\{z\}$ is $\{z\}$, $\{z\} \cup \{(ze_2)^D\}$, $\{z\} \cup \{(ze_2)^D\} \cup \{(e_2e_7)^D\}$ or $\{z\} \cup \{(ze_2e_7)^D\}$. But since $|\{z\} \cup \{(ze_2e_7)^D\}| = 29$ and $29 \mid |L_7(2)|$, the last one is impossible. Note that $C_{\bar{N}_1}(B_1) = 0(D_1)$ in any case.

If $\{z\} = \{z\}$, (1) holds.

Assume $\{z\} = \{z\} \cup \{(ze_2)^D\}$. Then $|\bar{N}_1/0(D_1)| = 36$. $|\bar{D}_1/0(D_1)|$. First, suppose $\bar{N}_1$ acts on $B_1$ irreducibly. Then $e_2 \not\sim e_2e_7 \not\sim ze_2e_7$. But, since $|\{e_2\} \cup \{(e_2e_7)^D\} \cup \{(ze_2e_7)^D\}| = 91$ and $91$
\[ |L_7(2)|, \text{ this is impossible. Hence } A_1 \text{ is } \overline{N}_1 \text{-invariant.} \]

Since \( C_{\overline{N}_1/0(D_1)}(A_1) \) is a divisor of 36, \( C_{\overline{N}_1/0(D_1)}(A_1) \) is centralized by \( D_1/0(D_1) \). But the centralizer of \( D_1/0(D_1) \) in \( \text{Aut}(B_1) (\cong L_7(2)) \) is trivial. Therefore, \( C_{\overline{N}_1/0(D_1)}(A_1) = 1 \). From the list of irreducible subgroups of \( GL(6,2) \) (K. Harada and H. Yamaki 17), we conclude that \( D_1/0(D_1) \cong E_8 \) and \( \overline{N}_1/0(D_1) \cong S(6,2) \). Since \( (0(D_1)C_{\overline{N}_1}(0(D_1)))/0(D_1) \cong (0(D_1)K_1)/0(D_1) \), \( (0(D_1)C_{\overline{N}_1}(0(D_1)))/0(D_1) = \overline{N}_1/0(D_1) \). Since the odd share of the Schur multiplier of \( S(6,2) \) is trivial, this means that \( \overline{N}_1 \cong 0(\overline{N}_1) \times S(6,2) \). Thus all the assertions of (3) are proved.

Assume \( \{z\overline{N}_1\} = \{z\} \cup \{(z_2e_2)D_1\} \cup \{(z_2e_7)e_1\} \).

Then \( |\overline{N}_1/0(D_1)| = 64 |D_1/0(D_1)| \) and \( A_1 \) is \( \overline{N}_1 \)-invariant.

For any element \( x \) of \( A_1 \), if there is an element of \( \overline{N}_1/0(D_1) \) which centralizes \( A_1 \) and sends \( z \) to \( zx \), we denote that element by \( t(x) \). From [17], \( C_{\overline{N}_1/0(D_1)}(A_1) \) is non-trivial, so it contains \( t(x) \) for some \( x \in A_1 \).

Since \( D_1 \) acts on \( A_1 \) irreducibly, \( C_\overline{N}_1/0(D_1)(A_1) \supset \langle t(y) | y \in A_1 \rangle \). Hence \( C_{\overline{N}_1/0(D_1)}(A_1) \cong E_64 \), \( \overline{N}_1/0(D_1) = C_{\overline{N}_1/0(D_1)}(A_1) \cdot (D_1/0(D_1)) \), and the action of \( D_1/0(D_1) \) on \( C_{\overline{N}_1/0(D_1)}(A_1) \) is the same as that on \( A_1 \). Finally, since \( (0(D_1)C_{\overline{N}_1}(0(D_1)))/0(D_1) \cong (0(D_1)K_1)/0(D_1) \), \( (0(D_1)C_{\overline{N}_1}(0(D_1)))/0(D_1) \cong C_{\overline{N}_1/0(D_1)}(A_1) \cdot ((0(D_1)K_1)/0(D_1)) \).

This proves all the assertions of (2).
In the remainder of this section, we study several cases in which the conclusion is \( z \in Z(G) \). We first prove the following four preparatory lemmas.

Lemma 3.3.

Suppose \( [N(L) : LC(L)] = 2 \), \( v'^2 = 1 \) and \( \langle U, v', z \rangle \in \text{Syl}_2(G) \), where \( v' \) is the element defined in the first paragraph of this section. Then \( N(\langle B_4, v' \rangle) \subseteq N(L) \).

Proof.

First note that \( C(\langle B_4, v' \rangle) = C_{N(L)}(\langle B_4, v' \rangle) \). Set \( M = N(\langle B_4, v' \rangle) \), \( Y = N_{N(L)}(\langle B_4, v' \rangle) \) and \( \overline{M} = M/C(\langle B_4, v' \rangle) \). Then \( Y \cong E_8 \cdot \text{SL}(3,2) \). The orbits of \( \langle B_4, v' \rangle \) under the action of \( Y \) and their lengths are given by 
\[
|\{ z^y \}| = 1, 
|\{ e_2^y \}| = |\{ (ze_2)^y \}| = 7, 
|\{ (w_4 w_5)^y \}| = |\{ (zw_4 w_5)^y \}| = |\{ (e_2^2 w_4 w_5)^y \}| = |\{ (ze_2^2 w_4 w_5)^y \}| = 28, 
|\{ v'^y \}| = |\{ (z v')^y \}| = 8, 
|\{ (u_3 v')^y \}| = |\{ (zu_3 v')^y \}| = 56.
\]

By Lemma 1.3, there is a subgroup \( H \) of \( G \) such that \( [G : H] = 2 \) and \( z \notin H \). \( H \) contains either \( v' \) or \( zv' \). By a suitable choice of \( v' \), we may assume \( v' \in H \). Thus \( H \supseteq \langle A_4, v' \rangle \).

Hence \( z \) is not conjugate to any element of \( \langle A_4, v' \rangle \) in \( G \). On the other hand, since \( \langle U, v', z \rangle \) is a Sylow 2-subgroup of both \( Y \) and \( M \), \( |\{ z^M \}| \) is odd. Since a Sylow 2-subgroup of \( Y \) acts faithfully on \( \langle A_4, v' \rangle \), \( |C_Y(\langle A_4, v' \rangle)| \) is odd. Hence, \( C_M(\langle A_4, v' \rangle) \) centralizes \( \langle B_4, v' \rangle \), for \( C_M(\langle A_4, v' \rangle) \) centralizes both \( \langle A_4, v' \rangle \) and
\( \langle B_4, v' \rangle / \langle A_4, v' \rangle \). Therefore \( \overline{M} \) acts faithfully on \( \langle A_4, v' \rangle \), and so it is isomorphic to a subgroup of \( GL(7,2) \). In particular, if \( |z^M| \) divides \( |GL(7,2)| \).

Combining these results, we have that \( \{z^M\} \) is one of the following:

1. \( \{z\} \);
2. \( \{z\} \cup\{(zv')^Y\} \);
3. \( \{z\} \cup\{(zw_4w_5)^Y\} \cup\{(zv')^Y\} \cup\{(zu_3v')^Y\} \);
4. \( \{z\} \cup\{(ze_2w_4w_5)^Y\} \cup\{(zv')^Y\} \cup\{(zu_3v')^Y\} \).

Suppose (2), (3) or (4) occurs. Then \( |M| = |Y| \cdot |\{z^M\}| = 2^6 \cdot 3^3 \cdot 7 \) or \( 2^6 \cdot 3^2 \cdot 7 \cdot 31 \). From D. Gorenstein and K. Harada [13], we have \( \overline{M} = \Omega(\mathcal{N}) \times Y \). Since \( \langle z \rangle = C_{\langle B_4, v' \rangle}(Y) \), this implies that \( \langle z \rangle \) is \( M \)-invariant, which is absurd. Hence the lemma is proved.

Lemma 3.4.

Suppose \( N(L) \) contains a Sylow 2-subgroup of \( G \). Then \( N_i = D_i \) for each \( i \in \{1,2,3,4,5,6\} \). In particular, \( \langle z, U \rangle \in \text{Syl}_2(B_1(N_i')) \).

Proof.

First note that \( [N_i : D_i]_2 \leq 2 \). If \( i \in \{1,2,3\} \), the conclusion follows immediately from Lemma 3.2. Thus assume \( i \in \{4,5,6\} \). Then \( A_1 \) splits into three \( D_i \)-classes of involutions of lengths 7, 28, 28. From Lemma 2.3 (1), \( z \) cannot be conjugate to any element of \( A_1 \).
in $G$. Finally, $|\{z_1^{N_i}\}|$ divides $|\text{GL}(7,2)|$, and $|\{z_1^{N_i}\}|_2 \leq 2$. Combining these observations, we get $\{z_1^{N_i}\} = z$, and so the lemma is proved.

Lemma 3.5.

Suppose $N(L)$ contains a Sylow 2-subgroup of $G$.

Then, for each $i \in \{1,2,3,4,5,6\}$, every conjugate of $B_i$ that is contained in $N(L)$ is contained in $LC(L)$.

Proof.

If $[N(L) : LC(L)]_2 = 1$, there is nothing to be proved. So we may assume $[N(L) : LC(L)]_2 = 2$. If $v'^2 = z$, then $N(L) - LC(L)$ contains no involution, and hence the desired conclusion follows immediately. Thus we may assume $v'^2 = 1$. Let $B_i^G$ be a conjugate of $B_i$ contained in $N(L)$. We may assume $B_i^G \leq \langle z,U,v' \rangle$. Let $Y = \{y \in B_i^G \mid \{y^{L,v'} \} \cap \langle B_4,v' \rangle = \emptyset \}$. Then, from Lemma 2.4, $|Y| = 127$ if $i = 1$ or 4, and $|Y| = 71$ if $i = 2, 3, 5$ or 6. In any case, $\langle Y \rangle = B_i$. Since $N(\langle B_4,v' \rangle)$ controls the fusion of $\langle B_4,v' \rangle$, Lemma 3.3 implies $Y^G \leq \langle z,U \rangle$. Hence $B_i^G \leq \langle z,U \rangle$ as desired.

Lemma 3.6.

Suppose $N(L)$ contains a Sylow 2-subgroup of $G$.

Then $\{z^G\} \cap \langle z,U \rangle = \{z\}$. 
Proof.

Let $x$ be an involution of $\langle z, U \rangle$ such that $x^g = z$ for some $g \in G$. By Lemma 2.3 (3), we may assume $x \in B_i$ for some $i \in \{1, 2, 3, 4, 5, 6\}$. By Lemma 3.5, we may assume $B_i \subseteq \langle z, U \rangle$. Hence, from Lemma 2.5 (2), $B_i^g = B_j$ for some $j \in \{1, 2, 3, 4, 5, 6\}$. From Lemma 3.4, $\langle z, U \rangle \in \text{Syl}_2(B_i(N_i))$, and so $\langle z, U \rangle \in \text{Syl}_2(B_j(N_j))$. Again from Lemma 3.4, $\langle z, U \rangle \in \text{Syl}_2(B_j(N_j))$. Therefore, there is an element $h$ of $N_j$ such that $\langle z, U \rangle^{gh} = \langle z, U \rangle$.

Since $h \in N_j = D_j \subseteq C(z)$ by Lemma 3.4, $x^g = z$. Thus we may assume $g \in N(\langle z, U \rangle)$. Since $z \in Z(\langle z, U \rangle)$, $x \in Z(\langle z, U \rangle) = \langle e_2, z \rangle$. Suppose $x \neq z$. Then, since $z$ can never be conjugate to $e_2$ by Lemma 2.3 (1), $z^g = ze_2$ and $(ze_2)^g = z$. Thus $g^2 \in C(z)$, and, by a suitable choice of $g$, we may assume $g$ is a 2-element. Then, $\langle z, U, g \rangle$ is a 2-group of order at least $2^{14}$. This implies that $[N(L) : LC(L)]_2 = 2$ and that both $\langle z, U, g \rangle$ and $\langle z, U, v' \rangle$ are Sylow 2-subgroups of $G$. Since $Z(\langle z, U, g \rangle) = \langle e_2 \rangle$ and $Z(\langle z, U, v' \rangle) = \langle z, e_2 \rangle$, this is a contradiction. Hence $x = z$ as desired.

Lemma 3.6.

If $[N(L) : LC(L)]_2 = 2$ and if $v'^2 = z$, then $z \in Z(G)$. 

Proof.

By Lemma 2.6 (2), \( \langle B_4, v' \rangle = J(\langle U, v' \rangle) \). Since \( \langle z \rangle = U^1(\langle B_4, v' \rangle) \), \( \langle z \rangle \) is a characteristic subgroup of \( \langle U, v' \rangle \), and so \( \langle U, v' \rangle \in \text{Syl}_2(G) \). Since \( \langle U, v' \rangle \subseteq \langle z, U \rangle \) contains no involution, \( \{z^G \} \cap \langle U, v' \rangle = \{z^G \} \cap \langle z, U \rangle = \{z\} \) from Lemma 3.6. Now Glauberman's \( Z^* \)-theorem yields the desired conclusion.

From now on, when \( [N(L) : L^C(L)] = 2 \), we assume \( v'^2 = 1 \), set \( v = v' \) and \( X = \langle L, v \rangle \), use the description of \( X \) given in Notation 2.1, and set \( B_7 = \langle z, A_7 \rangle \) and \( N_7 = N(B_7) \).

Lemma 3.8.

If \( [N(L) : L^C(L)] = 2 \) and if (1) of Lemma 3.2 holds for \( i = 1 \), then \( z \in Z(G) \).

Proof.

By Lemma 2.6 (1), \( \langle z, Q \rangle = \langle z, U, v \rangle \in \text{Syl}_2(G) \). We shall prove \( \{z^G \} \cap \langle z, Q \rangle = \{z\} \). First note that \( N_7 \) controls the fusion of \( B_7 \) as \( B_7 = J(\langle z, Q \rangle) \) by Lemma 2.6 (2). Let \( x \) be an involution of \( \langle z, Q \rangle \) such that \( x^g = z \) for some \( g \in G \). First suppose \( x \notin \langle z, U \rangle \).

Then, by Lemma 2.4, we may assume \( x \in B_7 \). Since \( N_7 \) controls the fusion of \( B_7 \), this contradicts Lemma 3.3.
Thus $x \in \langle z, U \rangle$. Then Lemma 3.6 implies $z = x$. Therefore, $\{z^G \} \cap \langle z, Q \rangle = \{z\}$, and so $z \in Z(G)$.

Lemma 3.9.

If $[N(L) : LC(L)]_2 = 1$ and if (1) of Lemma 3.2 holds for each $i \in \{1, 2, 3\}$, then $z \in Z(G)$.

Proof.

From Lemma 2.5 (4), $\langle z, U \rangle \in Syl_2(G)$. Then Lemma 3.6 implies that $\{z^G \} \cap \langle z, U \rangle = \{z\}$, and hence $z \in Z(G)$.

In Section 4 we shall treat the case (3) of Lemma 3.2 with $i = 1$, and in Sections 5 through 7 we shall treat the case (2) of Lemma 3.2 with $i = 1$, which will complete the proof of the Main Theorem.
SECTION 4

THE CASE IN WHICH $N_1/(B_1 O(N_1)) \cong \text{Sp}(6,2)$

In this section, we treat the case (3) of Lemma 3.2 with $i = 1$, and prove that $E(G) \cong M(22)$.

$N_1/A_1$ is isomorphic to either $O(N_1/A_1) \times \mathbb{Z}_2 \times \text{Sp}(6,2)$ or $O(N_1/A_1) \times \text{Sp}(6,2)$, where $\text{Sp}(6,2)$ denotes the double cover of $\text{Sp}(6,2)$. But, since $N_1/A_1$ contains $(\langle z \rangle \times P)/A_1 \cong \mathbb{Z}_2 \times \Sigma_8$, the latter is impossible. Hence $N_1/A_1 = O(N_1/A_1) \times (A_1 \langle z \rangle)/A_1 \times (A/A_1)$, where $A$ is such that $A \supset A_1$ and $A/A_1 \cong \text{Sp}(6,2)$. By a suitable choice of $v$, we may assume $P \subseteq A$.

Lemma 4.1.

Let $\overline{N(A_1)} = N(A_1)/A_1$. Then $\overline{N(A_1)} = O(\overline{N(A_1)})(\langle z \rangle \times \tilde{A})$ and $[A_1, O(\overline{N(A_1)})] = 1$.

Proof.

Let $\widetilde{C} = C_{\overline{N(A_1)}}(A_1)$. From the preceding paragraph, $C_{\overline{N(A_1)}}(z) = \overline{N_1} = 0(\overline{N_1}) \times \langle z \rangle \times \tilde{A}$. Hence $|C_{\overline{C}}(z)|_2 = |\langle z \rangle \times 0(\overline{N_1})|_2 = 2$ and so $\langle z \rangle \in \text{Syl}_2(C)$. Thus $\widetilde{C} = O(\overline{C})(z)$. By Frattini argument, $\overline{N(A_1)} = \overline{C_{\overline{N(A_1)}}}(z) = O(\overline{C})(O(\overline{N_1}) \times \langle z \rangle \times \tilde{A}) = O(\overline{N(A_1)})(\langle z \rangle \times \tilde{A})$, and $O(\overline{N(A_1)})$
Lemma 4.2.

A splits over $A_1$.

Proof.

Since $A \supseteq P$ and $P$ splits over $A_1$, the assertion follows from Lemma 1.12.

Let $Y$ be a complement to $A_1$ in $A$. It may be impossible to choose $Y$ so that $Y \supseteq K$. But, by Lemma 1.9 (2), we may assume $v \in Y$. We fix the following notation.

Notation 4.3.

Let $f_2$ be the element of $Y$ such that
\[ [e_7, f_2] = e_2, \quad [e_k, f_2] = 1 \text{ for } k \neq 7. \]
For $3 \leq i \leq 6$, let $f_i$ be the element of $Y$ such that
\[ [e_7, f_i] = e_i, \quad [e_9 - i, f_i] = e_2, \quad [e_k, f_i] = 1 \text{ for } k \neq 7, 9 - i. \]
Let $g_3$ be the element of $Y$ such that
\[ [e_6, g_3] = e_3, \quad [e_k, g_3] = 1 \text{ for } k \neq 6. \]
For $4 \leq i \leq 5$, let $g_i$ be the element of $Y$ such that
\[ [e_6, g_i] = e_i, \quad [e_9 - i, g_i] = e_3, \quad [e_k, g_i] = 1 \text{ for } k \neq 6, 9 - i. \]
Let $h$ be the element of $Y$ such that
\[ [e_5, h] = e_4, \quad [e_k, h] = 1 \text{ for } k \neq 5. \]

Let \( S = \langle f_i, e_j, h \mid 2 \leq i \leq 6, \ 3 \leq j \leq 5 \rangle. \)

Under this notation, we first prove the following lemma.

Lemma 4.4.

The following hold:

(1) \( S \in \text{Syl}_2(Y). \)

(2) \( f_i \in u_i C_{A_1}(u_i) \subseteq u_i \langle e_j \mid 2 \leq j \leq 6 \rangle \)
for \( 3 \leq i \leq 6. \)

\( g_i \in w_i C_{A_1}(w_i) \subseteq w_i \langle e_j \mid j = 2, 3, 4, 5, 7 \rangle \)
for \( 4 \leq i \leq 5. \)

(3) For every transvection \( a \) of \( Y \) and for every
element \( x \) of \( C_{A_1}(a) - [A_1,a] \), \( ax \) is
conjugate to \( f_2e_3 \) in \( A. \)

(4) Each involution of the coset \( f_3A_1 \) is conju-
gegate to either \( f_3 \) or \( f_3e_4 \) in \( A. \)

Proof.

(1) and (2) are clear from Notation 2.1 and Notation
4.3. For each transvection \( a \) of \( Y \), \( C_Y(a) (\cong E_{32}. \)
\( \text{Sp}(4,2)) \) acts transitively on \( C_{A_1}(a) - [A_1,a]. \) This
proves (3). Similarly, since \( C_Y(f_3) (\cong (E_4 \times (D_8 \times \text{D}_8)) \cdot (\Sigma_3 \times \Sigma_3)) \) acts transitively on \( C_{A_1}(f_3) - [A_1,f_3], \)
(4) holds.
Let $C_1 = \langle e_i, f_j \mid 2 \leq i \leq 6, \ 3 \leq j \leq 6 \rangle$ and $C = \langle C_1, f_2 \rangle$. Then $C_1 = \langle e_3, f_6 \rangle \ast \langle e_4, f_5 \rangle \ast \langle e_5, f_4 \rangle \ast \langle e_6, f_3 \rangle$, $C = \langle f_2 \rangle \ast C_1$, and $\langle e_i, f_9 - i \rangle \cong D_8$ for each $i \in \{3, 4, 5, 6\}$. We shall determine $N(C)$ in the following sequence of lemmas.

Lemma 4.5.

$$C_1 = \langle e_i, u_j \mid 2 \leq i \leq 6, \ 3 \leq j \leq 6 \rangle.$$  

Proof.

This is clear from Lemma 4.4 (2).

Lemma 4.6.

$$\langle C, z \rangle = \langle z, f_2 \rangle \ast C_1,$$ and $\langle z, f_2 \rangle \cong D_8$.

Proof.

Since $f_2^2 = 1$, $[B_1, f_2] \subseteq C_{A_1}(f_2)$. In particular, $[z, f_2] \subseteq \langle e_1 \rangle$. On the other hand, $[z, C_1] = 1$ from Lemma 4.5. Therefore $z$ normalizes $C$, and $[z, C]$ is a normal subgroup of order 2 of $C$. Hence, $[z, f_2] = e_2$ as desired.

Lemma 4.7.

The group generated by all the involutions of $\langle C, e_7 \rangle \ast C$ is $A_1$. 
Proof.

Let $e_7x$ be an involution of $\langle C,e_7 \rangle - C$. Since $C/\langle e_2 \rangle$ is elementary abelian, $x\langle e_2 \rangle \in C/\langle e_2 \rangle (e_7) = \langle e_i,f_2 | 2 \leq i \leq 6 $/\langle e_2 \rangle$. Hence $x \in \langle e_i,f_2 | 2 \leq i \leq 6 \rangle$. But $\langle e_i,f_2 | 2 \leq i \leq 6 \rangle$ is also elementary abelian, and so $x \in C \langle e_i,f_2 | 2 \leq i \leq 6 \rangle (e_7) = \langle e_i | 2 \leq i \leq 6 \rangle$. Hence $x \in A_1$. On the other hand, $A_1 = \langle e_7x | x \in \langle e_i | 2 \leq i \leq 6 \rangle \rangle$. So the lemma is proved.

Let $M = N(C)$, $D = M \cap N(A_1)$, $F = C_Y(\langle e_2,e_7 \rangle)$ and $\overline{M} = M/C$. Then $D = (C,0(D)) \langle e_7 \rangle = \langle e_7 \rangle \times F$ and $[C,0(D)] = 1$. But $0(D)\langle e_7 \rangle = C_D(C/Z(C)) \times D$ and so $D = 0(D)\langle e_7 \rangle \times F$. Note that $F \cong Sp(4,2) \cong \Sigma_6$.

Lemma 4.8.

$F'$ acts indecomposably on $B_1/\langle e_2,e_7 \rangle$.

Proof.

Regarding $A_1$ as a vector space over $GF(2)$, define an alternating form $f(,)$ on $A_1$ by $f(e_i,e_9 - j) = \delta_{ij}$ where $\delta_{ij}$ is Kronecker's delta. Then the action of $Y$ leaves $f(,)$ invariant. Now suppose the lemma is false. Then there is an element $x$ of $B_1 - A_1$ such that $C_Y(x) \not\cong F'$. $C_Y(x)$ is isomorphic to $O^+(6,2)$ or $O^-(6,2)$, and the action of $C_Y(x)$ on $A_1$ leaves invariant a certain quadratic form on $A_1$ whose associated
alternating form is $f(\ ,\ )$. Since the restriction of $f(\ ,\ )$ to $\langle e_2, e_7 \rangle$ is non-degenerate, $C_{G_Y(x)}(\langle e_2, e_7 \rangle) \cong O^+(4,2)$ or $O^-(4,2)$. This is a contradiction.

Lemma 4.9.

$$C_M(\overline{e}_7) = \overline{B}.$$  

Proof.

This follows immediately from Lemma 4.7.

Lemma 4.10.

$$|C_M(C/Z(C))|_2 = 2.$$  

Proof.

By Lemma 4.9, $|C_M(C/Z(C)) \cap C_M(Z(C)) \cap C_M(\overline{e}_7)|_2 = 1$. Hence $|C_M(C/Z(C)) \cap C_M(Z(C))|_2 = 1$. Since $[C_M(C/Z(C)) : C_M(C/Z(C)) \cap C_M(Z(C))] \leq 2$ and since $\overline{z} \in C_M(C/Z(C)) - (C_M(C/Z(C)) \cap C_M(Z(C)))$, the desired conclusion holds.

Let $\overline{M} = M/C_M(C/Z(C))$. Then $\overline{M}$ is isomorphic to a subgroup of $\text{Out}(C)/O_2(\text{Out}(C)) \cong O^+(8,2)$.

Lemma 4.11.

$$\overline{M}/O(\overline{M})$$ is isomorphic to either $Z_2 \times \Sigma_6$ or $\text{Aut}(\text{SU}(4,2))$.  

Proof.

From Lemma 4.10, \( [C_M(\overline{e_7}) : C_M(\overline{e_7})] \leq 2 \). From Lemma 4.9, \( C_M(\overline{e_7}) = \overline{D} = \langle \overline{e_7} \rangle \times \overline{F} \). But \( C_{\text{Out}(C)}/O_2(\text{Out}(C)) \langle \overline{e_7} \rangle \cong E_{64} \cdot \Sigma_6 \), where there is no \( \Sigma_6 \)-invariant subgroup of order 4 of \( E_{64} \). Hence \( C_M(\overline{e_7}) = \langle \overline{e_7} \rangle \times \overline{F} \). By M. Harris and R. Solomon [16], \( E(\overline{M}/O(\overline{M})) \) is isomorphic to one of the following groups:

(1) \( \Sigma_6 \) or \( \Sigma_6 \times \Sigma_6 \);

(2) \( \Sigma_8 \), \( \text{SU}(4,2) \), \( \text{SL}(5,2) \), \( \text{SU}(5,2) \) or \( \text{Sp}(4,4) \);

(3) \( \text{PSU}(4,3) \).

By considering the orders of these groups and \( \text{Out}^*(8,2) \), we can eliminate \( \text{SL}(5,2) \), \( \text{SU}(5,2) \), \( \text{Sp}(4,4) \) and \( \text{PSU}(4,3) \). In \( \text{Out}^*(8,2) \), no element of order 5 is centralized by a subgroup isomorphic to \( \Sigma_6 \). (See J. S. Frame [9].) Hence we can eliminate \( \Sigma_6 \times \Sigma_6 \). In order to eliminate \( \Sigma_8 \), we let \( C_M(C/Z(C)) = C_M(C/Z(C))/C_M(C) \).

Then \( C_M(C/Z(C)) = C(\overline{z}) \cong E_{512} \). We consider the action of \( \overline{M} \) on \( C(\overline{z}) \). Let \( \overline{C}_2 = \langle e_3, e_4, e_5, e_6 \rangle \). Note that \( \overline{D}' (= \overline{F}') \) acts decomposably on \( \overline{C}(\overline{z})/\overline{C}_2 \). That is to say, \( \overline{D}' \) normalizes \( \overline{C}(\overline{z})/\overline{C}_2 \) and \( \overline{C}/\overline{C}_2 \). Lemma 4.8 shows that \( \overline{D}' \) acts indecomposably on \( \overline{C}(\overline{z}) \). Hence there is no \( \overline{D}' \)-invariant complement to \( \overline{C} \) in \( \overline{C}(\overline{z}) \).

Now suppose \( E(\overline{M}/O(\overline{M})) \cong \Sigma_8 \). Since 7 divides \( |\Sigma_8| \) and \( |O(\overline{M})| \) divides \( 3^3 \cdot 5 \), \( (O(\overline{M})C_M(O(\overline{M}))/O(\overline{M})) \cong E(\overline{M}/O(\overline{M})) \).

From the class list of \( \text{Out}^*(8,2) \) ([9]), \( O(\overline{M}) = 1 \), and so
E(M) \equiv \Sigma_8'. So, from Lemma 1.13, there is an E(M)-invariant complement to \( \tilde{C} \) in \( \tilde{C}(z) \). Since \( E(M) \not\supset D' \), this is a contradiction.

Before examining the two cases of Lemma 4.11 separately, we prove the following three lemmas.

Lemma 4.12.

\[ Z((B, S)/\langle e_2 \rangle) = \langle e_2, e_3, f_2 \rangle/\langle e_2 \rangle, \]

where \( S \) is the Sylow 2-subgroup of \( Y \) defined in Notation 4.3.

Proof.

Let \( \overline{B}_1 S = (B, S)/\langle e_2 \rangle \). It is clear that \( \langle e_3, f_2 \rangle \subseteq Z(B, S) \). \( \overline{B}_1 S/\langle B, f_2 \rangle \) acts faithfully on \( \tilde{A}_1 \), and

\[ C_{\tilde{A}_1}(\overline{B}_1 S) = \langle \overline{e}_3 \rangle. \]

Hence, \( Z(\overline{B}_1 S) \subseteq \langle \overline{B}_1, f_2 \rangle \), and \( Z(\overline{B}_1 S) \cap \langle A_1 S \rangle = \langle e_3, f_2 \rangle \). Suppose \( Z(\overline{B}_1 S) \not= \langle e_3, f_2 \rangle \). Then there is an element \( x \) of \( B_1 - A_1 \) such that \( \tilde{x} \in Z(\overline{B}_1 S) \). So

\[ \|x^N(z)\|_2 = \|x^N\|_2 \leq 2. \]

This contradicts Lemma 3.2 (3). Thus \( Z(\overline{B}_1 S) = \langle e_3, f_2 \rangle \).

Lemma 4.13.

No involution of \( X \) is conjugate to \( z \) in \( G \).

Proof.

From Lemma 2.4, \( X \) has 6 classes of involutions. By Lemma 2.3 (1), no one of \( \{ e_2, e_2e_7, e_2w_4, e_2w_4w_5 \} \) is
conjugate to $z$. Let $\tilde{A} = A/A_1$. Then $u_3v$ is conjugate to $f_3h$. Since $|C_{A_1}(u_3v)| = 8$, this means that $u_3v$ is conjugate to $f_3h$ in $A$. Since $v \in Y$, $v$ is conjugate to $f_2$. Since both $C_A(f_3h)$ and $C_A(f_2)$ contain $\langle e_2, e_3, e_4, f_2, f_3, f_4, e_3, e_4, h \rangle \cong E_{512}$, $z \cdot f_3h$ and $z \cdot f_2$ from Lemma 2.6 (2). Hence $u_3v \cdot z \cdot v$, as desired.


Let $\tilde{A} = A/A_1$. Then every involution of $\tilde{A}$ is conjugate to some involution of $\tilde{P}$ in $\tilde{A}$.

Proof.

$\tilde{A}$ has four classes of involutions, and their representatives are $\tilde{v}, \tilde{u}_3, \tilde{u}_6v_6$ (these three are central) and $\tilde{u}_3v$, which are all contained in $\tilde{P}$.

From now on, up to the end of the proof of Lemma 4.41, we assume $\overline{\mathbb{M}}/\mathbb{O}(\overline{\mathbb{M}}) \cong \text{Aut}(SU(4,2))$.

Lemma 4.15.

$\overline{\mathbb{M}}$ is isomorphic to either $\text{Aut}(SU(4,2))$ or $\text{SU}(4,2)$ ($\cong (Z_3 \times SU(4,2)) \cdot Z_2$).

Proof.

Since 5 divides $|SU(4,2)|$ and $|O(\overline{M})|$ divides $3 \cdot 5 \cdot 7$, $(O(\overline{M})C(0(\overline{M}))/O(\overline{M}) \supseteq E(\overline{M}/O(\overline{M}))$. Now the lemma
follows from the class list of $O^*(8,2)$ ([9]).

Let $\mathcal{H} = O(C_0(C/Z(C)))$. Arguing as in the second paragraph of this section, we have $\mathcal{M}/\mathcal{H} = ((\mathcal{H}(\mathbb{R}))/\mathcal{H}) \times (\mathcal{M}_1/\mathcal{H})$, where $\mathcal{M}_1$ is such that $\mathcal{M}_1 \supset \mathcal{H}$ and $\mathcal{M}_1/\mathcal{H} \cong \mathcal{M}$. By a suitable choice of $\mathcal{M}_1$, we assume $\mathcal{M}_1 \supset 2 \langle \overline{e}_7 \rangle \times F (\cong \mathbb{Z}_2 \times \mathbb{Z}_6)$.

Lemma 4.16.

Every involution of $\mathcal{M}_1/\mathcal{H}$ is conjugate to some involution of $(\mathcal{H}(\langle \overline{e}_7 \rangle \times F))/\mathcal{H}$ in $\mathcal{M}_1/\mathcal{H}$.

Proof.

Let $\mathcal{M}_1 = \mathcal{M}_1/\mathcal{H}$. $\mathcal{M}_1$ has four classes of involutions, and their representatives are $\overline{e}_3$ (inner central), $\overline{e}_4$ (inner non-central), $e_7$ (field) and $\overline{e}_7 \overline{e}_3 \mathcal{H}$ ((field) $x$ (inner)), which are all contained in $\langle \overline{e}_7 \rangle \times \overline{F}$.

From the 2-modular character table of $SU(4,2)$ (Appendix), the action of $\mathcal{M}_1/\mathcal{H}$ on $C/Z(C)$ is uniquely determined. Specifically, the action is the same as that studied in Lemma 1.15. We fix the following notation.

Notation 4.17.

Let $M_1$ denote the full inverse image of $\mathcal{M}_1$. Let $x_1$ be an element of $M_1$ such that 

$[e_6, x_1] \in f_4 Z(C), \quad [f_6, x_1] \in e_4 f_4 Z(C), \quad [f_6, x_1] \in e_4 f_4 Z(C),$
Then \((Z(C)\langle e_3, f_3, e_4, f_4 \rangle)/Z(C)\) is a "totally singular subspace over GF(4)" in the sense of Lemma 1.15,

\[ CM/H((Z(C)\langle e_3, f_3, e_4, f_4 \rangle)/Z(C)). = (H\langle \overline{e_3}, \overline{e_4}, \overline{h}, \overline{x}_1 \rangle)/H, \]

\[ N_{M_1}/(Z(C)\langle e_3, f_3, e_4, f_4 \rangle) = N_{M_1}/((H\langle \overline{e_3}, \overline{e_4}, \overline{h}, \overline{x}_1 \rangle)/H), \]

\[ N_{M_1}/(Z(C)\langle e_3, f_3, e_4, f_4 \rangle)/C_{M_1}/((Z(C)\langle e_3, f_3, e_4, f_4 \rangle)/Z(C)) \]

is isomorphic to either \(\text{Aut}(\text{SL}(2,4))\) or \(\Gamma L(2,4) \equiv (Z_3 \times \text{SL}(2,4))\cdot Z_2\). This factor group acts both on \((Z(C)\langle e_3, f_3, e_4, f_4 \rangle)/Z(C)\) and on \(C/(Z(C)\langle e_3, f_3, e_4, f_4 \rangle)\) in a standard way as \(\text{Aut}(\text{SL}(2,4))\) or \(\Gamma L(2,4)\), whereas the action on \((H\langle \overline{e_3}, \overline{e_4}, \overline{h}, \overline{x}_1 \rangle)/H\) is the same as that of \(O^*(4,2)\) on a standard module. Since \(\overline{x}_1\) is conjugate to \(\overline{e_4}\) and since \(e_4\) is an involution, we may choose \(x_1\) as an involution. Let \(x_2\) be an element of \(M_1\) such that

\[ [e_4, x_2] \in e_3f_3Z(C), \quad [f_4, x_2] \in e_3f_3Z(C), \]

\[ [e_6, x_2] \in e_5f_5Z(C), \quad [f_6, x_2] \in e_5Z(C), \]

\[ \langle e_3, f_3, e_5, f_5 \rangle, x_2 \rangle \subseteq Z(C). \]

We choose \(x_2\) as an involution. Moreover, we choose \(x_1\) and \(x_2\) so that \(\langle A_1 S, x_1, x_2 \rangle\) is a \(z\)-invariant Sylow 2-subgroup of \(M_1\). Let \(R = \langle A_1 S, x_1, x_2 \rangle\).

We shall prove \(R\langle z \rangle \in Syl_2(G)\) in the following sequence of lemmas.
Lemma 4.18.

If $x$ is an element of $N_M(C(z))$ such that $x^2 \in \langle e_2 \rangle$ and $\overline{x}$ is a non-central involution of $E(M)$, then $C_{C(z)}(x)$ contains an abelian subgroup of order 64.

Proof.

By taking a suitable conjugate of $x$, we may assume $\overline{x} = \overline{w}_4 (= \overline{g}_4)$ or $\overline{zw}_4 (= \overline{zg}_4)$. (See Lemma 4.4 (2).) Since $x^2 \in \langle e_2 \rangle$, $x = g_4 y$ where $y \langle e_2 \rangle \in C(C(z))/\langle e_2 \rangle$. Consequently, $\langle e_2, e_3, e_4, f_3, f_4 \rangle \subseteq C_{C(z)}(x)$, and $C_{C(z)}(x)$ contains an abelian subgroup of order 4. Thus the lemma is proved.

Lemma 4.19.

$C(z)$ is weakly closed in $R(z)$ with respect to $C(e_2)$.

Proof.

Let $\overline{R(z)} = (R(z))/\langle C(z) \rangle$. By way of contradiction, let $C_2$ be a subgroup of $R(z)$ such that $C_2 \not\subseteq C(z)$, $C_2 \cong C(z)$ and $Z(C_2) = \langle e_2 \rangle$. Then $C_2/\langle e_2 \rangle \cong E_{1024}$. If $|\overline{C}_2| = 2$, then, by Lemma 1.15 (1), $|C_{C(z)}/\langle e_2 \rangle(\overline{C}_2)| \leq 4|C_{C/Z}(\overline{C}_2)| \leq 4 \cdot 64$. This means that $|C_2/\langle e_2 \rangle| \leq |\overline{C}_2| \cdot |C_{C(z)}/\langle e_2 \rangle(\overline{C}_2)| \leq 512$. This is a contradiction.
If $4 \leq |\tilde{C}_2| \leq 8$, then, by Lemma 1.15 (3),
$$|C(C\langle z\rangle)/\langle e_2 \rangle (\tilde{C}_2)| \leq 4 \cdot 16,$$
which leads to a similar contradiction. Hence $|\tilde{C}_2| = 16$. If $\tilde{C}_2 \neq \langle g_3, g_4, h, x_1 \rangle$, then, by Lemma 1.15 (9),
$$|C(C\langle z\rangle)/\langle e_2 \rangle (\tilde{C}_2)| \leq 4 \cdot 4,$$
which again leads to the same kind of contradiction. Hence $\tilde{C}_2 = \langle g_3, g_4, h, x_1 \rangle$. Then $|C_2 \cap (C\langle z\rangle)| = 2048/16 = 128$.
Since $|C(C\langle z\rangle)/\langle e_2 \rangle (g_4)| \leq 4 \cdot 16 = 64$, this means that $C_2 \ni g_4$ and $C_2/\langle e_2 \rangle \ni C(C\langle z\rangle)/\langle e_2 \rangle (g_4)$. But, by Lemma 4.18, the group generated by $g_4$ and the inverse image of $C(C\langle z\rangle)/\langle e_2 \rangle (g_4)$ contains an abelian subgroup of order 128. Since $C_2 \cong C\langle z\rangle \cong D_8 * D_8 * D_8 * D_8 * D_8$, this is impossible.

Lemma 4.20.

$R\langle z\rangle \in \text{Syl}_2(G)$.

Proof.

Suppose the lemma is false, and let $x$ be a 2-element of $N(R\langle z\rangle) - R\langle z\rangle$. Since $\langle e_2 \rangle = Z(R\langle z\rangle)$, $e_2^x = e_2$. Hence, from Lemma 4.20, $(C\langle z\rangle)^x = C\langle z\rangle$. From Lemma 4.12, $\langle e_2, e_3, f_2 \rangle \supseteq Z_2(R\langle z\rangle) \supseteq \langle e_2, f_2 \rangle$. Hence $\langle e_2, f_2 \rangle^x \subseteq \langle e_2, e_3, f_2 \rangle$. Since $C = C_{C\langle z\rangle}(f_2)$, $\langle e_2, f_2 \rangle^x \supseteq \langle e_2, f_2 \rangle$.

First suppose $\langle e_2, f_2 \rangle^x = \langle e_2, e_3 \rangle$. Then $C^x = C(C\langle z\rangle)(e_3) \ni z$. Hence $z^{x^{-1}} \in C$. But Lemma 4.13 and
Lemma 4.14 imply that \( z \) can never be conjugate to any element of \( C \). This is a contradiction.

Next suppose \( \langle e_2, f_2 \rangle^X = \langle e_2, f_2 e_3 \rangle \). Then \( C^X = C \langle e_2 \rangle \langle f_2 e_3 \rangle \subsetneq z u_6 \). (See Lemma 4.4 (2).) Hence \( (z u_6)^x \in C \). But, since \( u_6 \) and \( e_2 \) are conjugate in \( X \), \( z u_6 \) and \( z e_2 \) are conjugate. Finally, \( z e_2 \) and \( z \) are conjugate in \( N_1 \). Therefore, \( z \) is fused into \( C \), which again contradicts Lemma 4.13 and Lemma 4.14.

Lemma 4.21.

\[ z \in E(G), \quad R \in \text{Syl}_2(E(G)). \]

Proof.

This follows immediately from Lemmas 4.13, 4.14 and 4.16.

We next determine \( J(R) \) and \( N_{E(G)}(J(R)) \). Set \( J = \langle e_i, f_i, g_3, g_4, h, x \mid 2 \leq i \leq 4 \rangle \). We prove \( J = J(R) \).

Lemma 4.22.

If \( x \) and \( y \) are involutions of \( R \) such that \( \overline{x}, \overline{y} \) and \( \overline{xy} \) are all non-central involutions of \( E(M_1) \) and such that \( \langle \overline{x}, \overline{y} \rangle \leq \langle g_3, g_4, h, x \rangle \), then \( \langle e_i, f_i, x, y \mid 2 \leq i \leq 4 \rangle \cong E_{256} \).
Proof.

By the action of $\mathbb{N}(\langle g_3, g_4, h, x_1 \rangle)$, we may assume $\bar{x} = g_3 g_4$ and $\bar{y} = g_4 h$. (See Notation 4.17.) Since $x$ and $g_3 g_4$ are both involutions and since $C_{C}(g_3 g_4)/\langle e_2 \rangle = \langle e_1, f_i \mid 2 \leq i \leq 4 \rangle/\langle e_2 \rangle = C_{C}/\langle e_2 \rangle(g_3 g_4)$, $x \in g_3 g_4 \langle e_i, f_i \mid 2 \leq i \leq 4 \rangle$. Similarly, $y \in g_4 h \langle e_i, f_i \mid 2 \leq i \leq 4 \rangle$. Hence $\langle e_i, f_i, x, y \mid 2 \leq i \leq 4 \rangle = \langle e_i, f_i, g_3 g_4, g_4 h \mid 2 \leq i \leq 4 \rangle$. Thus, by computing in $A$, we get the desired conclusion.

Lemma 4.23.

$J \cong E_{1024}$.

Proof.

Let $x$ be an involution of $\langle g_3, g_4, h \rangle$ such that both $\bar{x}$ and $\bar{x}_1 x$ are non-central involutions of $E(\mathbb{M})$. Then, by our choice of $x_1$ (Notation 4.17), $\langle e_i, f_i, x, x_1 \mid 2 \leq i \leq 4 \rangle \cong E_{256}$ from Lemma 4.22. Since $\langle g_3, g_4, h \rangle = \langle x \mid x \in \langle g_3, g_4, h \rangle$, both $\bar{x}$ and $\bar{x}_1 x$ are non-central $\rangle$ by Lemma 1.15 (6), this implies $x_1 \in Z(J)$. On the other hand, by computing in $A$, we have $\langle e_i, f_i, g_3, g_4, h \mid 2 \leq i \leq 4 \rangle \cong E_{512}$. Hence the desired conclusion holds.
Lemma 4.24.

Let $\widetilde{M}_1 = M_1 / Z(C)$. Then $\widetilde{I} \leq \widetilde{J}$ for every abelian subgroup $\widetilde{I}$ of $\widetilde{R}$ such that $\widetilde{I} = \widetilde{J} (= \langle g_3, g_4, h, x_1 \rangle)$.

Proof.

Let $\widetilde{I}$ be an arbitrary abelian subgroup of $\widetilde{R}$ such that $\widetilde{I} = \widetilde{J}$. First note that $\widetilde{I} \cap \widetilde{C} \subseteq C_\varphi(\langle g_3, g_4, h, x_1 \rangle) = \widetilde{C} \cap \widetilde{J}$. With each element $x$ of $\langle g_3, g_4, h, x_1 \rangle$, we associate an element $\varphi(x)$ of $C$ such that $x\varphi(x) \in \widetilde{I}$. Then, for any elements $x, y$ of $\langle g_3, g_4, h, x_1 \rangle$, $[\varphi(x), \varphi(y)] = [\varphi(y), x]$ as $[x\varphi(x), y\varphi(y)] = 1$. Now let $a$ be an element of $\langle g_3, g_4, h, x_1 \rangle$ such that $a$ is a central involution of $E(N)$. Suppose $\varphi(a) \in \widetilde{C} \cap \widetilde{J}$. Then, from Lemma 1.15 (10), $[[\varphi(a), \langle g_3, g_4, h, x_1 \rangle]] = 8$. Since $[\varphi(y), a] = [\varphi(a), \varphi(y)]$ must hold for every $y \in \langle g_3, g_4, h, x_1 \rangle$, $[[\varphi(a), \langle g_3, g_4, h, x_1 \rangle]]$. But, since $a$ is central, $|[\widetilde{C}, \widetilde{a}]| = 4$. from Lemma 1.15 (2). This is a contradiction. Hence $\varphi(a) \not\in \widetilde{J}$. Since $a$ was arbitrary, the desired conclusion follows from the fourth assertion of Lemma 1.15 (6).

Lemma 4.25.

$J = J(R)$. 
Proof.

Let $I$ be an abelian subgroup of order at least 1024 of $R$. We have only to show $I \subseteq J$. Arguing as in Lemma 4.19, we can easily show that $I = J$. Therefore $I \subseteq J$ from Lemma 4.24.

In order to determine $N_{E(G)}(J)$, we first prove that $E(G)$ has at most 3 classes of involutions.


Each involution of $C$ is conjugate to one of $e_2$, $f_2$, or $e_3 f_2$ in $E(G)$.

Proof.

Under the action of $M_1$, each involution of $C$ is conjugate to one of $e_2$, $f_2$, $e_3$, or $e_3 f_2$. But $e_3$ is conjugate to $e_2$ in $L$.

Lemma 4.27.

Each involution of $E(G)$ is conjugate to one of $e_2$, $f_2$, or $e_3 f_2$ in $E(G)$.

Proof.

By Lemmas 4.14 and 4.16, each involution of $E(G)$ is conjugate to some involution of $X$. Hence, by Lemma 2.4, we have only to show that each of $\{ e_2, e_2 e_7, e_2 w_4, ... \}$. 


$e_2 w_5, v, u_3 v$ is conjugate to one of $e_2, f_2, \text{ or } e_3 f_2$.

$e_2$ is certainly conjugate to $e_2$.

$e_2 e_7$ is conjugate to $e_2$ in $A$.

If we let $\tilde{A} = A/\Delta_1$, then $\tilde{e_2 w_4} = \tilde{e_4}$, and hence there is an element $x$ of $A$ such that $\tilde{e_2 w_4} x = \tilde{f_4}$.

Thus $(e_2 w_4)^x \in f_4 C A_1 (f_4) \subseteq C$. Hence the lemma is true for $e_2 w_4$ by Lemma 4.26.

Similarly, $\tilde{e_2 w_4 w_5} = \tilde{e_4} \tilde{e_5}$, and hence there is an element $x$ of $A$ such that $(e_2 w_4 w_5)^x \in f_4 f_5 C A_1 (f_4 f_5) \subseteq C$.

By the assumption that $v \in Y$, $v$ is conjugate to $f_2$ in $Y$.

$u_3 v$ is conjugate to $e_3 v$ in $X$. By Lemma 4.4 (3), $e_3 v$ is conjugate to $e_3 f_2$. Hence the lemma is true for $u_3 v$ by Lemma 4.26.

From now on, up to the end of the proof of Lemma 4.40, we adopt a convention that symbols such as $N(J)$ and $C(e_2)$ denote $N_E(G)(J)$ and $C_E(G)(e_2)$, respectively.

Lemma 4.28.

$N(J)/C(J) \cong M_{22}$ and the action is the same as that studied in Lemma 1.19.
Proof.

Let \( \overline{N(J)} = N(J)/C(J) \). \( N_M(J) \) contains a Sylow 2-subgroup of \( E(G) \). Hence \( N_M(J) \) contains a Sylow 2-subgroup of \( N(J) \). \( O_2(N_{M_1}(J)) = \langle e_5, e_6, f_5, f_6 \rangle \), or \( \Gamma L(2,4) \), where the action on \( O_2(N_{M_1}(J)) \) is the same as that on a standard module. Hence \( \overline{N(J)} \) is of \( M_{22} \)-type. Now the desired conclusion follows from D. Gorenstein and K. Harada [13] and from Lemmas 4.27, 4.25, 1.16, 1.17 (5) and 1.18.

Remark.

By Lemma 1.19 (2), this conclusion, in turn, implies \( N_{M_1}(J)/O_2(N_{M_1}(J)) \cong \text{Aut}(SL(2,4)) \) and hence \( N_{M_1}/H \cong \text{Aut}(SU(4,2)) \).

Lemma 4.29.

\( E(G) \) has exactly three classes of involutions with representatives \( e_2, f_2 \) and \( e_3 f_2 \).

Proof.

This is clear from Lemma 4.27 and Lemma 4.28.

We next determine \( C(e_2) \).
Lemma 4.30.

\[ C(e_2) = C(e_2)N(C). \]

Proof.

Let \( \widehat{C(e_2)} = C(e_2)/\langle e_2 \rangle \). We show \( \widehat{C} \) is strongly closed in \( \tilde{R} \) with respect to \( \widehat{C(e_2)} \). Arguing as in Lemma 4.19, we can easily show that \( \widehat{C} \) is weakly closed. Let \( \Gamma, \Gamma', \delta, \vartheta' \) be as in Lemma 1.4. Let \( \tilde{E} \) be an element of \( \Gamma \) such that \( |\tilde{E}| = 6 \). First suppose \( |\tilde{E}| = 2 \), and let \( \tilde{x} \) be the involution of \( \tilde{E} \). Then, from Lemma 1.15 (1), \( |[\tilde{C}, \tilde{x}]| \geq 4 \), which contradicts Lemma 1.4 (5). Next suppose \( 8 \geq |\tilde{E}| \geq 4 \). Then, from Lemma 1.15 (3), there is an involution \( \tilde{x} \) of \( \tilde{E} \) such that \( |[\tilde{C}, \tilde{x}]| = 16 \). This again contradicts Lemma 1.4 (5). Hence \( |\tilde{E}| = 16 \). Suppose \( \tilde{E} \not\subseteq M_1' \). Then, by Lemma 1.15 (9), there is an involution \( \tilde{x} \) of \( \tilde{E} \) such that \( |(\tilde{C}(\tilde{E}), [\tilde{C}, \tilde{x}])| \leq 2 \cdot 8 = 16 \), which contradicts Lemma 1.4 (6). Hence \( \tilde{E} \subseteq M_1' \). By Lemma 1.15 (7) and Notation 4.17, \( \tilde{E} = \langle 3_3, 3_4, h, x_1 \rangle \). If \( |\tilde{C} \cap \tilde{E}| = 2 \), then, by Lemma 1.15 (2), \( |(\tilde{C} \cap \tilde{E}), [\tilde{C}, \tilde{x}]| \leq 16 \) for every central involution \( \tilde{x} \) contained in \( \tilde{E} \), which contradicts Lemma 1.4 (6). Hence, \( |\tilde{C} \cap \tilde{E}| \geq 4 \). Therefore, \( |\tilde{E}| \geq 128 \), where \( \tilde{E} \) denotes the full inverse image of \( E \). But, by Lemma 4.24, \( E \subseteq J(R) \), and so \( E \) is elementary abelian.
Since $C$ does not contain any elementary abelian subgroup of order 128, this is a contradiction. Hence $\tilde{C}$ is strongly closed in $\tilde{R}$ with respect to $C(e_2)$, and so the Main Theorem of D. Goldschmidt \cite{10} yields the desired conclusion.

In order to prove $O(C(e_2)) = 1$, we have to determine $C(e_2 f_2)$ and $C(f_2)$. Before this is done, we need some more preparation.

Let

$I = \langle e_2, f_2, e_3, f_3, e_3 \rangle$.

Lemma 4.31.

Let

$\overline{N(J)} = N(J)/C(J)$.

Then the following hold:

(1) $\tilde{R}$ contains exactly two elementary abelian subgroups

$\langle e_5, f_5, e_6, f_6 \rangle,$
$\langle g_5, x_2, e_5, f_5 \rangle$

of order 16.

(2) $\overline{N(J)}(\langle e_5, f_5, e_6, f_6 \rangle) \cong E_{16} \cdot \text{Aut}(SL(2, 4)),$
$\overline{N(J)}(\langle g_5, x_2, e_5, f_5 \rangle) \cong E_{16} \cdot (\text{Sp}(4, 2))',$
where the actions are standard.
Proof.

(1) follows from Lemma 1.19 (1) and Notation 4.17. As noted immediately after Lemma 4.28, \( N_{M_1}(J)/\langle e_5, f_5, e_6, f_6 \rangle \cong \mathcal{S}_5 \). Hence (2) follows from Lemma 1.19 (1). \( C_J(\langle f_5, e_5 \rangle) = \langle e_2, f_2, f_3, f_3, g_3 \rangle \). On the other hand, by Lemma 1.19 (4), \( |C_J(\langle g_5, x_2, e_5, f_5 \rangle)| = 32 \). Hence (3) holds. Note that \( e_2 g_3 \in \langle (e_3 f_2)^E(G) \rangle \), for \( e_2 g_3 \) is conjugate to \( e_3 f_2 \) in \( A \) by Lemma 4.4 (3). But, from Lemma 4.30, \( e_3 f_2 \) and \( e_2 g_3 \) are not conjugate in \( C(e_2) \). Therefore, (4) follows from Lemma 1.19 (6). (5) follows from Lemma 1.19 (2). (6) follows from (3) and (5). For every element \( x \) of \( J(\langle g_5, x_2, e_5, f_5 \rangle - I, [\langle x \rangle, J(\langle g_5, x_2, e_5, f_5 \rangle] \neq 1. \) Since
I = Z(J\langle g_5, x_2, e_5, f_5 \rangle) from (3) and since \langle x \rangle, J\langle g_5, x_2, e_5, f_5 \rangle \triangleleft J\langle g_5, x_2, f_3, e_3 \rangle, this proves (7).

Lemma 4.32.

\[ O(C(\langle e_2, e_3 f_2 \rangle)) \triangleleft C(\langle e_2, e_3 f_2 \rangle) \]

Proof.

Let \( \overline{C(e_2)} = C(e_2)/(O(C(e_2))Z(C)) \). Clearly \( \overline{C(\langle e_2, e_3 f_2 \rangle)} \subseteq \overline{C(e_2)}(\overline{e_3}) \). In the sense of Lemma 1.15, \( \langle e_3, f_3 \rangle \) is the "1-dimensional subspace of \( \overline{C} \) over \( GF(4) \) spanned by \( \overline{e_3} \)." Hence \( \langle e_3, f_3 \rangle \triangleleft C(\langle e_2, e_3 f_2 \rangle) \). Since \( \overline{g_3} \) is the "transvection with respect to \( \langle e_3, f_3 \rangle \)," \( \overline{g_3} \triangleleft C(\langle e_2, e_3 f_2 \rangle) \). Suppose, by way of contradiction, \( \overline{I} \nsubseteq \overline{C(\langle e_2, e_3 f_2 \rangle)} \). Then there is an element \( x \) of \( C(\langle e_2, e_3 f_2 \rangle) \) such that \( g_3^x = g_3 a \) where \( a \in C \) and \( \overline{a} \in C(\langle e_2, e_3 f_2 \rangle) \). Since the full inverse image of \( C(\langle e_2, e_3 f_2 \rangle) = \langle \overline{e_i}, \overline{f_i} \mid 2 \leq i \leq 5 \rangle \) in \( C(e_2)/O(C(e_2)) \) is centralized by \( g_3 O(C(e_2)) \) and since \( g_3 \) and \( g_3 a \) are both involutions, \( a O(C(e_2)) \) is an involution. So there is an element \( g \) of \( C(\langle e_2, e_3 f_2 \rangle) \) such that \( g \overline{a} \langle \overline{e_i}, \overline{f_i} \mid 2 \leq i \leq 4 \rangle \). Therefore this fusion must occur in \( N_C(\langle e_2, e_3 f_2 \rangle) \). This contradicts Lemma 4.31 (5).

The proof of Lemma 4.33 is similar to and easier than that of Lemma 4.32, and so it is omitted.

Lemma 4.33

\[ O(C(\langle e_2, e_2 g_3 \rangle)) \triangleleft C(\langle e_2, e_2 g_3 \rangle) \]
Lemma 4.34.

If $x, y$ are elements of $I$ such that $x \in \{e_2^E(G)\}$ and $y \in \{(f_2e_3)^E(G)\}$, then

$$0(C(\langle x, y \rangle )) I \triangleleft C(\langle x, y \rangle )$$

Proof.

This is clear from Lemma 4.31 (4), 4.32, and 4.33.

Now we can determine the approximate structure of $C(e_3f_2)$

Lemma 4.35.

$$0(C(e_3f_2)) I \triangleleft C(e_3f_2)$$

Proof.

Note that $C_N(J)(e_3f_2)/C(J) \cong E_{16} \cdot (E_9 \cdot Z_4)$ and $I \triangleleft C_N(J)(e_3f_2)$. (See Lemma 1.19 (2) and 4.31 (5), (6).) We wish to show $I$ is strongly closed in $C(e_3f_2)$ with respect to $C(e_3f_2)$. By way of contradiction, let $x$ be an element of $C(e_3f_2)$ such that $x \notin I$ and $x^g \in I$ for some $g \in C(e_3f_2)$. Since $N_{C(e_3f_2)}(J)$ controls the fusion of $J$ in $C(e_3f_2)$, $x \notin J$. (See Lemma 4.31 (6).) Thus we
may assume $C_{R}(e_{3}f_{2})(x)^{g} \subseteq C_{R}(e_{3}f_{2})$. First suppose $x \in J \langle g_{5}, x_{2}, e_{5}, f_{5} \rangle$. By Lemma 1.19 (5), $I$ contains a subgroup $I_{1}$ of order 16 all of whose involutions are conjugate to $e_{2}$ in $E(G)$. Let $y$ be an involution of $I_{1}$. Suppose $y^{g} \in I$. Then, since $N_{C}(e_{3}f_{2})(J)$ controls the fusion, there is an element $g_{1}$ of $C_{N}(J)(f_{2}e_{3})$ such that $(y^{g})^{g_{1}} = y$. By Lemma 4.31 (6), $x^{g_{1}} \in I$. This contradicts Lemma 4.34. Hence $y^{g} \in I$. Again since $N_{C}(e_{3}f_{2})(J)$ controls the fusion, this means that $y^{g} \in J$. Since $y$ was arbitrary, $I_{1}^{g} \cap J = 1$. Since $I_{1} \subseteq C_{R}(e_{3}f_{2})(x)$, $I_{1}^{g} \subseteq C_{R}(e_{3}f_{2})$. Hence, $JI_{1}^{g} = J \langle g_{5}, x_{2}, f_{5}, e_{5} \rangle$.

From Lemma 1.19 (3), $|C_{J}(x)| = 64$. $(C_{J}(x))^{g} \subseteq C_{R}(e_{3}f_{2})$ $(I_{1}^{g}) = C_{J}(I_{1}^{g}). I_{1}^{g} = C_{J}(\langle g_{5}, x_{2}, e_{5}, f_{5} \rangle). I_{1}^{g} = I \cdot I_{1}^{g}$.

Hence there is an element $y$ of $C_{J}(x) - I$ such that $y^{g} \in I$. This again contradicts the fact that $N_{C}(e_{3}f_{2})(J)$ controls the fusion of $J$. Therefore, no element of $J \langle g_{5}, x_{2}, e_{5}, f_{5} \rangle - I$ is fused into $I$. This, in particular, implies that $I$ is weakly closed in $C_{R}(e_{3}f_{2})$. Consequently, if we define $\gamma^{\circ}$ as in Lemma 1.4, then $\gamma^{\circ} = 1$. This contradicts Lemma 1.4 (2). Hence $I$ is strongly closed and the lemma follows from the main theorem of [11].

Lemma 4.36

$C(e_{3}f_{2})$ is 2-constrained.
Proof.

Set $\overline{C(e_3f_2)} = C(e_3f_2)/0(C(e_3f_2))$. By Lemma 4.35, $\overline{1} \triangleleft \overline{C(e_3f_2)}$, $\overline{J\langle g_5, x_2, e_5, f_5 \rangle \in Syl_2(C(e_3f_2)(\overline{1}))}$. By way of contradiction, suppose $E(C(e_3f_2)) \neq 1$ and let $\overline{W} \in Syl_2(E(C(e_3f_2)))$. We may assume $\overline{W} \subseteq \overline{J\langle g_5, x_2, e_5, f_5 \rangle}$. But then, by Lemma 4.31 (7), $[\overline{W}, \overline{J\langle g_5, x_2, e_5, f_5 \rangle}] \cap \overline{1} \neq 1$. This is absurd.

Lemma 4.37.

Let $\overline{C(f_2)} = C(f_2)/\langle f_2 \rangle$. Then $\overline{C(f_2)}$ has three classes of involutions with representatives $\overline{e_2}$, $\overline{e_3}$ and $\overline{f_4g_3}$.

Proof.

From Lemma 1.19 (2), $\overline{N_{C(f_2)}(\overline{J})/C(f_2)(\overline{J})} \cong PSL(4,3)$. Hence, by Lemma 1.17 (5), the involutions of $\overline{J}$ split into three classes under the action of $\overline{N_{C(f_2)}(\overline{J})}$. Let $\overline{x}$ be an involution of $\overline{N_{C(f_2)}(\overline{J})}$. We prove $\overline{x}$ is fused into $\overline{J}$. By Lemma 1.17 (6), we may assume $\overline{xJ} = \overline{e_5J}$. From Notation 4.3, $[g_3, e_5] = 1$. Since $J \subseteq N(C)$ and $e_5 \in C$ and $g_3 \notin C$, $g_3 \notin [J, e_5]$. That is, $g_3 \in C_j(e_5) - [J, e_5]$. Hence, by Lemma 1.17 (6), $\overline{x}$ is conjugate to either $\overline{e_5}$ or $\overline{e_5g_3}$. Recall that we assumed $v \in Y$. (See the paragraph
immediately before Notation 4.3.) Hence, from the action of \( v \) on \( A_1 \) (See the generators and relations given in Notation 2.1.), \( f_2^V = f_2, \ e_5^V = e_4 \) and \( g_3^V = g_3 \). Consequently, \( e_5 \) and \( e_5g_3 \) are conjugate to \( e_4 \) and \( e_4g_3 \), respectively, in \( C_A(f_2) \). Therefore, every involution of \( N_{C(f_2)}(J) - J \) is fused into \( J \). Hence \( C(f_2) \) has at most three classes of involutions.

Next consider the coset \( f_4g_3\langle f_2 \rangle \). In \( A/A_1 \), both \( f_4g_3A_1 \) and \( f_4g_3f_2A_1 \) are non-central involutions, and hence conjugate to \( u_3vA_1 \). Since \( |[A_1, u_3v]| = 8 \), this means that \( f_4g_3 \) and \( f_4g_3f_2 \) are conjugate to \( u_3v \) in \( A_1 \). Hence, from the proof of Lemma 4.27, \( f_4g_3 \) and \( f_4g_3f_2 \) are conjugate to \( e_3f_2 \) in \( E(G) \). On the other hand, one element of the coset \( e_3\langle f_2 \rangle \) is conjugate to \( e_2 \) and the other is (conjugate to) \( e_3f_2 \) in \( E(G) \), and one element of the coset \( e_2\langle f_2 \rangle \) is \( e_2 \) and the other is conjugate to \( f_2 \) in \( E(G) \). Therefore, \( \tilde{e}_2 \not\sim f_4g_3 \) in \( C(f_2) \) and the lemma is proved.

Lemma 4.38.

\[ C(f_2) \not\leq C(e_2). \]
Proof.

Since $C_{N(J)}(f_2)/C(J) \cong \text{PSL}(3,4)$ by Lemma 1.19 (2), the lemma follows immediately.

Lemma 4.39.

$C(f_2)/\langle f_2 \rangle \cong \text{PSU}(6,2)$.

Proof.

Let $C(f_2) = C(f_2)/\langle f_2 \rangle$. By Lemma 4.37, $C(f_2)$ contains three classes of involutions. Let $\bar{x}$ be an involution which is conjugate either $e_3$ or $\overline{f_4g_3}$. Then, from the second paragraph of the proof of Lemma 4.37, we may assume $x$ is conjugate to $e_3f_2$ in $E(G)$. Hence $C(x)$ is 2-constrained by Lemma 4.36, and so $C(x)(f_2)$ is 2-constrained. Since $[C_{C(x)}(f_2):C(f_2)(\bar{x})] \leq 2$, this means that $\overline{C(f_2)(\bar{x})}$ is 2-constrained. Next consider $\overline{C(f_2)(\bar{e}_2)}$. Since $e_2 \not\sim f_2 \sim e_2f_2$ in $E(G)$, $\overline{C(f_2)(\bar{e}_2)} = \overline{C(f_2)(e_2)}$. From Lemma 4.30, $\overline{C(f_2)(e_2)/O(C(f_2)(e_2))}$ is isomorphic to an extension of $D_8 \ast D_8 \ast D_8 \ast D_8$ by $SU(4,2)$, where the action of $SU(4,2)$ on $(D_8 \ast D_8 \ast D_8 \ast D_8)/Z(D_8 \ast D_8 \ast D_8 \ast D_8)$ is irreducible.

Hence the centralizer of each involution of $\overline{C(f_2)}$ is
2-constrained. Since \( \overline{C(f_2)} \) is connected in the sense of D. Gorenstein and J. Walter [14], \( O(\overline{C(f_2)}(e_2)) = 1 \) by Theorem B of [14]. Lemma 4.38 shows that \( e_2 \) is not isolated, and so, by a result of D. Parrot [20], \( \overline{C(f_2)} \cong PSU(6,2) \).

Lemma 4.39.

\[ E(G) \cong M(22) \]

Proof.

Since \( PSU(6,2) \) is 2-generated and 2-balanced, Theorem A of [14] implies \( O(C(x)) = 1 \) for every involution \( x \) of \( E(G) \). Hence, by a result of D. Hunt [18] or D. Parrot [20], \( E(G) \cong M(22) \).

In the remainder of this section, we assume \( \overline{M/0(M)} \cong \mathbb{Z}_2 \times \Sigma_6 \) and derive a contradiction.

Lemma 4.40.

\[ A_1 S(z) \in \text{Syl}_2(G). \]

Proof.

We can prove this arguing the same way as in the proofs of Lemmas 4.19 and 4.20.
Lemma 4.41.

\[ z \notin E(G), \ A_1 S \in \text{Syl}_2(E(G)). \]

Proof.

This follows immediately from Lemmas 4.13 and 4.14.

Lemma 4.42.

\[ J(A_1 S) = \langle e_2, e_3, e_4 \rangle C_Y(\langle e_2, e_3, e_4 \rangle) = \langle e_i, f_i, e_3, e_4, h \mid 2 \leq i \leq 4 \rangle \cong E_{512}. \]

Proof.

This can be proved by a direct computation. (See Notation 4.3.)

Lemma 4.43.

\[ N_{E(G)}(J(A_1 S)) = O(N_{E(G)}(J(A_1 S))) N_A(J(A_1 S)). \]

Proof.

Set \( W = N_{E(G)}(J(A_1 S)), \ \tilde{W} = W/C_{E(G)}(J(A_1 S)), \)

\( B = \langle e_5, e_6, e_7 \rangle, \ W_1 = N_Y(\langle e_2, e_3, e_4 \rangle) \cap N_Y(B). \) Then

\[ \tilde{N}_A(J(A_1 S)) = \tilde{B} \cdot \tilde{W}_1 \cong E_8 \cdot \text{SL}(3,2) \text{ and } \tilde{A}_1 S \in \text{Syl}_2(\tilde{B} \cdot \tilde{W}_1). \]

We shall prove \( \tilde{B} \) is strongly closed in \( \tilde{A}_1 S \) with respect to \( \tilde{W} \). Define \( \Gamma \) and \( \Phi \) as in Lemma 1.4 (Note that we do not yet prove even that \( E_1 \) is weakly closed.), and, by way of contradiction, suppose \( \Gamma \nparallel \Phi \).
Let $\widetilde{E}$ be an arbitrary element of $\Gamma$. Then, since

$$[J(A_1S),\tilde{\gamma}] = \langle e_2, e_3, e_4 \rangle$$

for every involution $\tilde{\gamma}$ of $\widetilde{E}$,

$$[J(A_1S),\tilde{\gamma}_1] = [J(A_1S),\tilde{\gamma}_2]$$

for any involutions $\tilde{\gamma}_1, \tilde{\gamma}_2$ of $\widetilde{E}$. But $|[J(A_1S)\langle e_2, e_3, e_4 \rangle, \tilde{\gamma}]| = 4$ for each involution $\tilde{\gamma}$ of $\widetilde{A_1S} - \widetilde{E}$, and $[J(A_1S)\langle e_2, e_3, e_4 \rangle, \tilde{\gamma}_1] \neq [J(A_1S)\langle e_2, e_3, e_4 \rangle, \tilde{\gamma}_2]$ if $\gamma_1B \neq \gamma_2B$. Hence $|\widetilde{E}| = 2$.

Since $\widetilde{E}$ was arbitrary, $\gamma = 1$. In particular, $\widetilde{E}$ is weakly closed. These contradict Lemma 1.4 (6). Thus $\widetilde{B}$ is strongly closed. Hence $\tilde{W} = O(\tilde{W})(\tilde{B}W_1)$ by D. Goldschmidt [10]. We next prove $O(\tilde{W}) = 1$. Arguing as in Lemma 1.16, we can easily show that $O(\tilde{W})$ centralizes $\tilde{B}W_1$. But, by inspection, it follows that $\langle e_2, e_3, e_4 \rangle$ is the unique minimal $\tilde{B}W_1$-invariant subgroup of $J(A_1S)$ and that $\langle f_3, f_4, e_2, e_3, e_4 \rangle/\langle e_2, e_3, e_4 \rangle$ is the unique minimal $\tilde{B}W_1$-invariant subgroup of $J(A_1S)/\langle e_2, e_3, e_4 \rangle$. Therefore $O(\tilde{W})$ centralizes $J(A_1S)$, and so $O(\tilde{W}) = 1$, as desired.

Now we can derive a contradiction. From Lemma 4.4 (2), (4), either $f_3$ or $e_4 f_3$ is conjugate to $u_3$ in $A$. But $u_3$ is conjugate to $e_3$ in $L$. Hence either $f_3$ or $e_4 f_3$ is fused to $e_3$ in $E(G)$. Since $f_3, e_4 f_3$ and $e_3$ are all in $J(A_1S)$, this fusion must take place in $N_{E(G)}(J(A_1S))$. This contradicts Lemma 4.43.

This contradiction shows that the case $\tilde{W}/O(\tilde{M}) \cong \mathbb{Z}_2 \times \Sigma_6$ in Lemma 4.11 does not occur, and hence $E(G) \cong$
M(22) as proved in Lemmas 4.15 through 4.39.
SECTION 5
FURTHER 2-LOCAL ANALYSIS I

In this section we treat the case (2) of Lemma 3.2 with $i = 1$ and continue our 2-local analysis. Let $D_1 = N_{N(L)}(B_1)$ as in Section 3. Let $F = O_2(N_1)$. For an element $x$ of $A_1$, let $t(x)$ denote an element of $F$ such that $[t(x), z] = x$. Then the bijection from $A_1$ onto $F/B_1$ which associates $t(x)B_1$ with $x$ may be considered to be a $K_1$ module isomorphism between $A_1$ and $F/B_1$. First we shall determine a more detailed structure of $N_1$.

Lemma 5.1

Let $\tilde{N}_1 = N_1/A_1$. Then $\tilde{F} \cong E_{128}$.

Proof.

Suppose $\tilde{F} \not\cong E_{128}$. Then, by the action of $K_1$ on $\tilde{F}$, $\tilde{F}$ is an extra-special 2-group of plus-type such that $Z(\tilde{F}) = \langle \tilde{z} \rangle$, $\tilde{t}(x)$ is of order 2 if $x \in \{ e_2 \}^{K_1}$, and $\tilde{t}(x)$ is of order 4 if $x \in \{ e_2 e_7 \}^{K_1}$. (See the first
paragraph of the proof of Lemma 3.2). Therefore, $t(e_2 e_7)^2 \in B_1 - A_1$ and $C_F t(e_2 e_7)^2 \not\subseteq B_1$. On the other hand, since $|\{z^F\}| = |F/B_1|$, $C_F(x)$ must be equal to $B_1$ for all $x \in B_1 - A_1$. This is a contradiction.

Lemma 5.2

Let $\tilde{N}_1 = N_1/A_1$. Then $N_1$ acts on $\tilde{F}$ decomposably.

Proof.

This follows immediately from Lemma 1.5 and Lemma 1.10 (2).

By Lemma 5.2, $F$ contains a subgroup $C_1$ of order 4096 such that $A_1 \subseteq C_1$ and $C_1 \subseteq N_1$. We assume $t(x) \in C_1$ for all $x \in A_1$.

Lemma 5.3

One of the following holds.

1. $C_1$ is an elementary abelian 2-group of order 4096.
2. $C_1$ is a homocyclic abelian 2-group of exponent 4 of order 4096.
Proof.

This follows immediately from Lemma 1.6 and Lemma 1.11 (1), (2).

Let $R = \langle F, U_1 \rangle$ or $\langle F, U_1, v \rangle$ according as $|\left[ N(L) : C(L)L \right]|_2 = 1$ or $2$, where $U_1$ is the Sylow 2-subgroup of the complement $K$ of $A_1$ in $P_1$ which is defined in Notation 2.1. Then $R \in Syl_2(N_1)$.

Lemma 5.4

If $C$ is an abelian subgroup of order at least 2048 of $R$, then $C \subseteq C_1$ or the following hold:

1. $|\left[ N(L) : C(L)L \right]|_2 = 2$
2. $|C| = 2048$, $|C/F| = |C/\langle F, U_1 \rangle| = 2$
   $|C \cap A_1| = |(C \cap C_1)/A_1| = 32$

Proof.

Let $C$ be an abelian subgroup of order at least 2048 not contained in $C_1$. Let $\bar{R} = R/F$. Suppose $(C_1C)/C_1 \neq C_1$. Then, since $C_{C_1}(z) = A_1$, $|(C_1C)/C_1| = |C|/|C \cap C_1| \geq |C|/|C_{C_1}(z)| \geq 32$. Therefore $|\overline{C}| = 16$. Then, by Lemma 1.11 (3), $|C_{C_1}^C(C)| = |C_{A_1}^C(C)| \leq 8$, which implies $|C| \leq |(C_1C)/C_1| \cdot |C_{C_1}^C(C)| = 512$. This is absurd. Thus $(C_1C)/C_1$
$\varphi z_{C_1/C_1}$, and so $|C| = |\overline{C}| \cdot |C \cap C_1|$. Suppose $|\overline{C}| \geq 8$, then by Lemma 1.11 (3), $|C_{A_1}(\overline{C})| = |C_{C_1/A_1}(\overline{C})| \leq 8$. Therefore, $|C_{C_1}(\overline{C})| \leq 64$, which implies that $|C| \leq 64 \cdot 16 = 1024$. This is absurd. If $|\overline{C}| = 4$, then, by Lemma 1.11 (5), $|C_{A_1}(\overline{C})| = |C_{C_1/A_1}(\overline{C})| \leq 16$, which again leads to the same kind of contradiction. If $|\overline{C}| = 2$, then, by Lemma 1.11 (5), $\overline{C} \not\subseteq U_1$, and $|C_{A_1}(\overline{C})| = |C_{C_1/A_1}(\overline{C})| = 32$. This proves the lemma.

In the remainder of this section we assume that the case (2) of Lemma 5.3 holds and derive a contradiction. (The case (1) will be treated in Section 6 and Section 7.

Lemma 5.5

If $x$ is an element of order 3 of $K_1$ such that $|C_{A_1}(x)| = 16$, then $C(x)/O(C(x)) \cong Aut(SU(5,3))$. Moreover, $|[N(L):LC(L)]|_2 = 2$.

Proof.

Set $C = C(x), Y = C_L(x), \tilde{C} = C/\langle x \rangle$. Note that $Y \cong \Omega^-(6,2) \cong U_4(2)$. Since $C(z,x) = C_{\tilde{C}}(\tilde{z})$, $\tilde{Y}$ is a standard component of $\tilde{C}$ with $\langle \tilde{z} \rangle \in Syl_2(C_{\tilde{C}}(\tilde{Y}))$. $|[N_{\tilde{C}}(\tilde{Y})]: \tilde{Y}C_{\tilde{C}}(\tilde{Y})|_2 = 1$ or 2 according as $|[N(L):LC(L)]|_2 = 1$ or 2. (See the assumption we made in the first paragraph
of Section 3). Now that we are assuming that (2) of Lemma 5.3 holds, \( O_2(N_C(\langle \tilde{z}, \tilde{C}_{A_1}(x) \rangle )) \) contains a homocyclic abelian group of exponent 4 of order 256. Therefore, by Hypothesis A, \( \tilde{C}/O(\tilde{C}) \cong \mathrm{Aut}(SU(5,3)) \) and \( [N_C(\tilde{y}) : \tilde{y}C_C(\tilde{y})]_2 = 2 \). This proves the lemma.

We shall prove \( R \in \mathrm{Syl}_2(G) \) in the following sequence of lemmas.

**Lemma 5.6**

\[ N(F) = N_1 \]

**Proof.**

Let \( N(A_1) = N(A_1)/A_1 \). Since \( F^* = A_1 \), \( N(F) \subseteq N(A_1) \). Note that \( C_{N(A_1)}(\tilde{z}) = N(\langle A_1, z \rangle) = \tilde{N}_1 \). Thus, in determining \( N_{N(A_1)}(\tilde{F}) \), we can argue exactly as in the proof of Lemma 3.2. Specifically, one of the following holds.

1. \( N_{N(A_1)}(\tilde{F}) \approx \tilde{N}_1 \)
2. \( (N_{N(A_1)}(\tilde{F})/\tilde{F})^\infty \approx E_{64} \cdot \tilde{E}_8 \)
3. \( (N_{N(A_1)}(\tilde{F})/\tilde{F})^\infty \approx S_p(6, 2) \)

But if (2) or (3) occurs, \( |C_{N(F)}(x)|_2 \geq 8192 \) for any element \( x \) of order 3 of \( K_1 \) such that \( |C_{A_1}(x)|_2 = 16 \). This contradicts Lemma 5.5.
Lemma 5.7

F char R

Proof.

By Lemma 5.4, C_1 char R. Since R/F acts faithfully on \( \Omega_1(C_1)(=A_1) \), F = C_R(\Omega_1(C_1)), which is characteristic in R.

Lemma 5.8

R \in Syl_2(G)

Proof.

This is clear from Lemma 5.6 and 5.7.

Let a be an element of order 3 of \( K_1 \) such that \( C_{A_1}(a) = \langle e_2 e_6 e_7, e_3 e_6, e_4, e_5 \rangle \). Then, by Lemma 5.5, either v or vz is contained in \( (C(a))^\circ \). By a suitable choice of v, we may assume v \in (C(a))^\circ.

Lemma 5.10

\[ C_{C_1}(v) \cong (\mathbb{Z}_4)^5 \]
Proof.

Let \( \tilde{N}_1 = N_1/A_1 \). Since \( v \in (C(a))^\infty \), from the structure of \( U_5(3) \), the centralizer of \( v \) in \( C_{C_1}(a) \) is isomorphic to \( (Z_4)^3 \). In particular, \( C_{C_1}(v) \supseteq \langle t(e_2e_6e_7), t(e_3e_6), t(e_4e_5) \rangle \). Since \( C_{C_1}(v) \) is \( C_{K_1}(v) \)-invariant, this implies that \( C_{C_1}(v) = C_{C_1}(v) \) as desired.

Lemma 5.11

\( v \) is fused into \( C_1 \) in \( G \).

Proof.

From the structure of \( U_5(3) \), \( v \) is fused into \( C_{C_1}(a) \) in \( (C(a))^\infty \).

Now we are in a position to derive a final contradiction. By Lemma 5.10, \( v^g \in C_1 \) for some \( g \in G \). Since \( |C_{N_1}(x)|_2 \geq 2^{18} \) for every involution \( x \) of \( C_1 \), and since \( |C_{N_1}(x)|_2 \leq 2^{18} \) for every involution \( x \) of \( N_1 - C_1 \), we may assume \( v^g \) is extremal in \( R \). Thus we can choose \( g \) so that \( (C_R(v))^g \subseteq C_R(v^g) \). From Lemma 5.9, \( \langle C_{C_1}(v), v \rangle \cong Z_2 \times (Z_4)^5 \). Therefore, from Lemma 5.4, either \( (C_{C_1}(v))^g \subseteq C_1 \) or (2) of the same lemma holds with \( C = \langle C_{C_1}(v)^g, v^g \rangle \). Suppose the latter occurs, and
let $x$ be an element of $C_{C_1}(v)^g - C_1$. Then, since

$$|C_L(C_{C_1}(v))^g/C_1| = 2, \quad x^2 \in C_1.$$ Since $C_{C_1}(v)$ is of

exponent 4, $x^2 \in A_1$. Therefore, $(A_1(C_{C_1}(v))^g)/A_1 \cong E_6$. But, since $C_{C_1}(v) \cong (Z_4)^5$, this is impossible.

Hence $(C_{C_1}(v))^g \leq C_1$ and so $A_1 = \langle C_{A_1}(v)^g, v^g \rangle$. Note that $C_{U_1}(v)$ is of $E_6$-type and act on $C_{A_1}(v)$ faithfully.

Since $F$ centralizes $A_1$ and $C_{A_1}(v)^g \leq A_1$, $C_{U_1}(v)^g \cap F = 1$.

Thus $C_{U_1}(v)^g \cong (F \cdot (C_{U_1}(v))^g)/F$. Clearly $(F \cdot (C_{U_1}(v))^g)/F \leq C_{R/F}(v^g) \cap N_{R/F}(C_{A_1}(v)^g)$. But by Lemma 1.11 (4),

$$|C_{R/F}(v^g) \cap N_{R/F}(C_{A_1}(v)^g)| \leq 8.$$ This contradiction proves that (2) of Lemma 5.3 does not occur.
SECTION 6
FURTHER 2-LOCAL ANALYSIS II

In this section we treat the case (1) of Lemma 5.3, and continue our 2-local analysis. Let \( M_1 = (N(C_1))^\circ \) and 
\[ N(C_1) = N(C_1)/C_1. \]

Lemma 6.1

\( \overline{M}_1 \) acts on \( C_1 \) faithfully, and one of the following holds.

1. \( \overline{M}_1 = \Omega^+(6,2) \times \Omega^+(6,2) \) and \( \overline{z} \) permutes the components.
2. \( \overline{M}_1 = \Omega^+(6,4) \) and \( \overline{z} \) acts as a field automorphism.

Proof.

Since \( [C_1, z] = A_1 \), every involution of \( \langle C_1, z \rangle - C_1 \) is conjugate to \( z \) in \( C_1 \langle z \rangle \). Hence 
\[ \overline{C}_{N(C_1)}(\overline{z}) = \frac{N_{C(z)}(C_1)}{N_{C(z)}(\langle z \rangle)) \cap N_{C(z)}(C_1)} = \overline{C}_1 = \overline{N}_1. \] From the structure of \( \overline{N}_1 \) (See Lemma 3.2), \( \overline{M}_1 (\equiv \Omega^+(6,2)) \) is a
standard component of $\overline{N(C_1)}$ with $\langle \overline{z} \rangle \in \text{Syl}_2 (C_{\overline{N(C_1)}}(K_1))$, where $\overline{K_1}$ is the complement of $\overline{A_1}$ in $\overline{F_1}$ defined in Notation 2.1. Hence, if we let $N(C_1) = \overline{N(C_1)}/O(N(C_1))$, then, by Hypothesis B, one of the following holds.

\begin{enumerate}
\item[(i)] $E(N(C_1)) \cong \Omega^+(6,2)$
\item[(ii)] $E(N(C_1)) \cong \Omega^+(6,2) \times \Omega^+(6,2)$ and $z$ permutes the component.
\item[(iii)] $E(N(C_1)) = \Omega^+(6,4)$ and $\overline{z}$ acts as a field automorphism.
\item[(iv)] $E(N(C_1)) \cong F_{10}$
\item[(v)] $E(N(C_1)) \cong HS$
\end{enumerate}

First suppose (i) or (ii) occurs. Then a Sylow 2-subgroup of $\overline{N_1}$ is a Sylow 2-subgroup of $\overline{N(C_1)}$. Let $R$ be the Sylow 2-subgroup of $N_1$ defined in Section 5. By Lemma 5.4, $C_1$ is the unique abelian subgroup of order 4096 of $R$. Hence $R \in \text{Syl}_2(G)$ and $C_1$ is weakly closed in $R$. We shall apply Lemma 1.4 with $C = C_1$. Let $\Gamma$, $\Gamma^\gamma$, $\gamma$, $\gamma^\gamma$ be as in Lemma 1.4. Since the 2-rank of $\overline{R}$ is 5, $\gamma^\gamma \leq 32$. Hence $\overline{z} \not\in \Gamma^\gamma$ by Lemma 1.4 (2). Thus $\gamma^\gamma \leq 16$. Let $E$ be an element of $\Gamma$ such that $|E| = \gamma$. First suppose $|E| = 2$, and let $x$ be an element of $E - C_1$. Then, since $\overline{x} \neq \overline{z}$, $|[C_1/A_1, \overline{x}]| = |[A_1, \overline{x}]| \geq 2$. Hence $|[C_1, x]| \geq 4$, which contradicts Lemma 1.4 (5). Next suppose $8 \geq |E| \geq 4$. Again since $\overline{z} \not\in \overline{E}$, there is an element $x$ of $E - C_1$ such that $|[C_1/A_1, \overline{x}]| = |[A_1, \overline{x}]| = 4$ by Lemma
1.11 (5). Then $|C_1x| > 16$, which contradicts Lemma 1.4 (5). Thus $|E| = 16$. Suppose $E \leq \langle z, u \rangle$. Then, since $z \notin E$, $|C_{C_1/A_1}(E)| = |C_{A_1}(E)| = 2$ by Lemma 1.11 (7), which contradicts Lemma 1.4 (3). Thus $E \not\leq \langle z, u \rangle$. Hence, by Lemma 1.11 (6), there is an element of $x$ of $E - \langle C_1, z, u \rangle$ such that $|C_{C_1/A_1}(x)| = |A_1x| = 8$. Then $|C_1x| = 64$, which contradicts Lemma 1.4 (5). This proves that $C_1$ is strongly closed in $R$, which contradicts the fusion in $L$. Thus neither (i) nor (iv) can occur.

Next suppose (v) occurs. Since $K_1\langle z \rangle$ acts on $C_1$ faithfully, the kernel of the action of $N(C_1)$ on $C_1$ is contained in $O(N(C_1))$. But a Sylow 5-subgroup of HS is an extraspecial group of order 125, while a Sylow 5-subgroup of $\text{Aut}(C_1)(\cong L_{12}(2))$ is an elementary abelian group of order 125. This contradiction shows that (v) cannot occur.

Thus (ii) or (iii) occurs. Let $x$ be any involution of $K_1$. Then, from the action of $z$ on $E(N(C_1))$, there is an involution $\tilde{y}$ of $E(N(C_1))$ such that $[\tilde{y}, z] = \tilde{x}$ and $[\tilde{x}, \tilde{y}] = 1$. We can choose $\tilde{y}$ so that $[\tilde{y}, z] = \tilde{x}$ and $[\tilde{x}, \tilde{y}] = 1$. Since $\langle \tilde{x}, \tilde{z} \rangle \cong E_4$, $O(N(C_1)) = \langle C_0(N(C_1)) \tilde{z} \rangle$, $C_0(N(C_1))(\tilde{x})$, $C_0(N(C_1))(\tilde{zx})$. On the other hand, from the structure of $D_l$, $C_0(N(C_1))(\overline{z}) \subseteq C_0(N(C_1))(\overline{x})$. Since $\overline{z\tilde{y}} = \overline{zx}$ and $\overline{x\tilde{y}} = \overline{x}$, we also have $C_0(N(C_1))(\overline{zx}) \subseteq C_0(N(C_1))(\overline{x})$. Thus $O(N(C_1)) = C_0(N(C_1))(\overline{x})$. That is to say, $\overline{x} \in \overline{x}$.
$C_{N(C_1)}^{O(N(C_1))}$. Since $C_{N(C_1)}^{O(N(C_1))} \supseteq N(C_1)$, this implies that $C_{N(C_1)}^{O(N(C_1))} \supseteq E(N(C_1))$. Since the odd shares of the Schur multipliers of $\Omega^+(6,2) \times \Omega^+(6,2)$ and $\Omega^+(6,4)$ are both trivial, we get the desired conclusion.

Note that, in each case of Lemma 6.1, $C_{M_1}^1(\overline{A}_2) \cong C_{M_1}^1(\overline{A}_3) = E_64$ and $|\langle C_{M_1}^1(\overline{A}_2), C_{M_1}^1(\overline{A}_3) \rangle| = 1024$. Let $S$ be a $z$-invariant Sylow 2-subgroup of $M_1$ such that $S$ contains $\langle C_{M_1}^1(\overline{A}_2), C_{M_1}^1(\overline{A}_3) \rangle$ and $C_S(z) = U$.

Lemma 6.2

If (2) of Lemma 6.2 occurs, then the following hold for each $i \in \{1,2,3\}$.

1. The case (2) of Lemma 3.2 occurs, and so $N_i$ contains a unique elementary abelian 2-subgroup $C_i$ of order 4096.
2. $N(C_i)^\theta/C_i \cong \Omega^+(6,4)$ and acts on $N(C_i)^\theta/C_i$ as a field automorphism.
3. $S \in Syl_2(M_i)$

Proof.

We prove this for $i = 2$.

From the modular character degrees of $\Omega^+(6,4)$ (R. Steinberg [25]), the action of $\overline{M}$ on $C_1$ is uniquely
determined. Specifically, this action is the same as the natural action of \( \mathfrak{S}^+(6,4) \) on a 6-dimensional vector space over \( \text{GF}(4) \). Let \( Y \) be a subgroup of order 64 of \( C_1 \) spanned by \( A_2 \cap C_1 = A_2 \cap A_1 = E_8 \) as a vector space over \( \text{GF}(4) \) in the above sense. Let \( \tilde{N}_{M_1}(Y) = N_{M_1}(Y) \)

\[ C_{M_1}(Y) \] 

Then \( \tilde{C}_{M_1}(Y) \cong E_64, \tilde{N}_{M_1}(Y) \cong GL(3,4), \tilde{A}_2 = \tilde{C}_{K_1}(A_1 \cap A_2) = C_{M_1}(Y)(z) \cong E_8, \tilde{N}_{K_1}(A_1 \cap A_2) = \tilde{C}_{N_{M_1}(Y)}(\bar{z}) \cong L_3(2) \). Let \( \tilde{N}_{M_1}(Y) = N_{M_1}(Y)/Y \), and consider the action of \( \tilde{N}_{M_1}(Y) \) on \( \tilde{C}_{M_1}(Y) \) is the same as that on \( \tilde{C}_1 \). Since \( \tilde{N}_{M_1}(Y) \) acts on \( \tilde{C}_{M_1}(Y) \) and \( \tilde{C}_1 \) transitively, and since \( C_{M_1}(Y) = C_1 \supseteq \tilde{A}_2 \) and \( \tilde{A}_2 \cong E_8, C_{M_1}(Y) \) is elementary abelian. Let \( \tilde{C}_2 = [\tilde{A}_2, Z(\tilde{N}_{M_1}(Y))] \). Then clearly \( \tilde{C}_2 = E_{64} \) and \( \langle \tilde{C}_2, \tilde{C}_1 \rangle = C_{M_1}(Y) \). Since \( \tilde{N}_{K_1}(A_1 \cap A_2) \langle \bar{z} \rangle \) normalizes \( Z(\tilde{N}_{M_1}(Y)) \), and since \( \tilde{A}_2 \) is \( \tilde{N}_{K_1}(A_1 \cap A_2) \langle \bar{z} \rangle \)-invariant, \( \tilde{C}_2 \) is \( Z(\tilde{N}_{M_1}(Y))\tilde{N}_{K_1}(A_1 \cap A_2) \langle \bar{z} \rangle \)-invariant. Our first aim is to show that \( \tilde{C}_2 \) is \( \tilde{N}_{M_1}(Y) \)-invariant.

Let \( \tilde{C}_2 \) denote the full inverse image of \( \tilde{C}_2 \). In this paragraph we shall prove that \( \tilde{C}_2 \) is elementary abelian. Since \( A_2 \subseteq C(Y) \), and since both \( Y \) and \( A_2 \) are elementary abelian, \(YA_2 \) is elementary abelian. On the other
hand, \([\tilde{C}_2, \tilde{z}] \subseteq \tilde{A}_2\). That is, \([C_2, z] \subseteq YA_2\). Then by Lemma 1.1, \([C_2, z] \subseteq C_{YA_2}(z) = A_2\). Thus \([C_2, z] = A_2\). In particular, \(A_2 \triangleleft C_2\). Now let \(\overline{y}\) be an element of \(Z(N_{M_1}(Y))^\#\).

Then, since the full inverse image \(YA_2\) of \(\tilde{A}_2\) is elementary abelian, the full inverse image \(A\) of \(\tilde{A}_2\) is also elementary abelian. Therefore, since \(C_2/A_2 = A_2 A_2 = A/A \cap A_2\), \(C_2/A_2\) is elementary abelian. Hence \(C_2 \subseteq A_2\).

But, since \(Z(N_{M_1}(Y))(N_{K_1}(A_1 \cap A_2))\) acts irreducibly on \(\tilde{C}_2\) and \(Y\), \(\tilde{C}_2\) is either \(Y\) or \(1\). Thus \(\tilde{C}_2 = 1\) and \(C_2\) is elementary abelian as desired.

Let \(\overline{J}\) be a Singer cycle of \(N_{K_1}(A_1 \cap A_2)\). So \(\overline{J} \approx Z_7\). Set \(\overline{H} = Z(N_{M_1}(Y))(N_{K_1}(A_1 \cap A_2))\) and \(\overline{H} = Z(N_{M_1}(Y))\overline{J}\).

Note that \(\overline{H}_1\) acts on \(\tilde{C}_1\) and \(\tilde{C}_2\) irreducibly.

Now set

\[\overline{\varphi} = \{\tilde{W} | \tilde{W}\text{ is a }\overline{H}\text{-invariant complement of }\tilde{C}_1\text{ in }C_{M_1}(Y)\}\]

\[\overline{\varphi}' = \{\tilde{W} | \tilde{W}\text{ is a }\overline{H}_1\text{-invariant complement of }\tilde{C}_1\text{ in }C_{M_1}(Y)\text{ such that the inverse image of }\tilde{W}\text{ is elementary abelian}\}\].
Note that $\tilde{C}_2 \in \Phi$ and $\tilde{C}_2 \in \Phi'$. We shall prove $\Phi = \Phi'$.

Since $|C_{\text{Aut}}(C_1)(\overline{H})| = |Z(N_{M_1}(Y))| = 3$, $|\Phi| = 3 + 1 = 4$.

Let us pick up one element $a$ of $\tilde{A}_2$ and fix it. Since $|C_{\overline{H}}(\tilde{C}_1(\tilde{a}))| = 4$, and since $|\Phi| = 4$, and since $\overline{H}$ acts on $\tilde{C}_1$ and $\tilde{C}_2$ irreducibly,

$$\Phi = \{ \langle ax \overline{H} \rangle | x \in C_{\tilde{C}_1}(C_{\overline{H}}(\tilde{a})) \}$$

We do not yet know whether $(N_{M_1}(Y))'$ acts on $C_{M_1}(Y)$ decomposably or not. But we have already shown that the structure of $C_{M_1}(Y)$ and the action of $\overline{H}$ on $C_{M_1}(Y)$ are the same as those when $M_1$ splits over $C_1$. Therefore, we may let $B$ be another group isomorphic to $SL(3,4)$ and let $B$ act on $C_M(Y)$ so that $\tilde{C}_1$ and $\tilde{C}_2$ are $B$-invariant. Moreover, we may regard $\overline{H}$ as a subgroup of $B$ so that the restriction of the action of $B$ on $C_{M_1}(Y)$ to $\overline{H}$ is the same as the original action of $\overline{H}$. Then, since $C_{\text{Aut}}(B) = C_{\text{Aut}}(\overline{H})$ and $C_{\overline{H}}(B(a)) = C_{\tilde{C}_1}(C_{\overline{H}}(\tilde{a}))$, the same argument as above leads to

$$\Phi = \{ \langle axB \rangle | x \in C_{\tilde{C}_1}(C_B(\tilde{a})) \}$$

$$= \{ \tilde{W} | \tilde{W} \text{ is a } B\text{-invariant complement of } \tilde{C}_1 \text{ in } C_{M_1}(Y) \}.$$ 

Let $W$ be the full inverse image of an arbitrary element $\tilde{W}$ of $\Phi$. Thus $\tilde{W} = \langle axB \rangle$ for some $x \in C_{\tilde{C}_1}(C_B(\tilde{a}))$. 

Since \( C_1 \left( C_B(\tilde{a}) \right) = C_1(\tilde{a}) \), \((\alpha x)^2 = 1\). Since \( B \) acts transitively on \( \tilde{W} \), this implies that \( W \) is elementary abelian and so \( \tilde{W} \in \phi' \). Since \( W \) was arbitrary, \( \phi \subseteq \phi' \).

Again since \( C_1 \left( C_H(\tilde{a}) \right) = C_1(\tilde{a}) \), \( \phi' \subseteq \phi \). Thus \( \phi = \phi' \) as desired.

Let \( \overline{I} \) be a Singer cycle of \( \overline{N_M(Y)} \) which contains \( \overline{H_1} \). So \( \overline{I} = \mathbb{Z}_{63} \). Since \( \overline{I} \) centralizes \( \overline{H_1} \), and since \( \phi = \phi' \), \( \overline{I} \) acts on \( \phi \). Since \( [\overline{I} : \overline{H_1}] = 3 \) and \( |\phi| = 4 \), \( \overline{I} \) fixes at least one element \( \tilde{C} \) of \( \phi \). Since \( \langle \overline{I}, \overline{H} \rangle = \overline{N_M(Y)} \), this \( \tilde{C} \) is \( \overline{N_M(Y)} \)-invariant. Then, since

\[
C_{Aut} \tilde{C}_1(\overline{N_{M_1}(Y)}) = C_{Aut} \tilde{C}_1(\overline{H}) \quad \text{and} \quad C_{\overline{C}_1(\overline{N_{M_1}(Y)},a)} = C_{\overline{C}_1(\overline{N_{M_1}(Y)})},
\]

the same argument as above leads to

\[
\phi = \left\{ \tilde{W} | \tilde{W} \text{ is an } \overline{N_{M_1}(Y)} \text{-invariant complement of } \tilde{C}_2 \text{ in } C_Y \right\}
\]

In particular, since \( \tilde{C}_2 \in \phi \), \( \tilde{C}_2 \) is \( \overline{N_{M_1}(Y)} \)-invariant as desired.

Recall that \( \tilde{C}_2 \) is also \( \overline{z} \)-invariant. As before, let \( C_2 \) denote the full inverse image of \( \tilde{C}_2 \). Thus \( C_2 \cong E_{4096} \). In the third paragraph of this proof, it is shown that \([C_2,z] = A_2\). This means \( C_2 \subseteq N(\langle z, A_2 \rangle) \). From Lemma 3.2, the case (2) of the same lemma occurs for \( i = 2 \). Then, from an analogue of Lemma 5.4 for \( i = 2 \), \( C_2 \) is the
unique abelian group of order 4096 of $N_2$. From what we have shown, $N(2)^{\infty}/C_2 \leq N_{M_1}^{\infty}(C_2)/C_2 \leq N_{M_1}^{\infty}(Y)/C_2 = E_64\cdot \text{SL}(3,4)$. Since the action of $\mathbb{Z}((N_{M_1}^{\infty}(Y)/C_2)/(C_2C_1/C_2))$ on $C_2C_1/C_2$ is fixed point free, $N_{M_1}^{\infty}(Y)/C_2$ splits over $C_2C_1/C_2$ by the Frattini argument. Thus a Sylow 2-subgroup of $N_{M_1}^{\infty}(Y)/C_2$ is isomorphic to a Sylow 2-subgroup of $\Omega^+(6,4)$. Since a Sylow 2-subgroup of $\Omega^+(6,2) \times \Omega^+(6,2)$ is not isomorphic to a Sylow 2-subgroup of $\Omega^+(6,4)$, $(N(C_2))^{\infty}/C_2 = \Omega^+(6,4)$ from an analogue of Lemma 6.1 for $i = 2$.

Since $C_2 = C_{M_1}(Y) = C_{M_1}(\bar{A}_2)$, $S \in \text{Syl}_2(N_{M_1}^{\infty}(Y))$. Hence $S \in \text{Syl}_2(N(C_2)^{\infty})$ and the lemma is proved.

In order to establish a similar result for the case (1) of Lemma 6.2, we need the following two preparatory lemmas.

Lemma 6.3

Suppose (1) of Lemma 6.1 occurs, and set $\bar{M}_1 = \bar{Y} \times \bar{Z}$ with $\bar{Y} \times \bar{Z} = \bar{Z} \equiv \Omega^+(6,2)$. Then $C_1 = [C_1, \bar{Y}] \times [C_1, \bar{Z}]$, $\bar{Y}(\text{resp.} \bar{Z})$ acts on $[C_1, \bar{Y}]$ (resp. $[C_1, \bar{Z}]$) in the standard way, and $\bar{Y}(\text{resp.} \bar{Z})$ acts on $[C_1, \bar{Z}]$ (resp. $[C_1, \bar{Y}]$) trivially.
Proof.

This is clear from the 2-modular character table of \( \Omega^+(6,2) \setminus \mathbb{Z}_2 \), which can easily be obtained from that of \( \Omega^+(6,2) \).

Lemma 6.4

Under the assumption and the notations of Lemma 6.3, let \( Y \) (resp. \( Z \)) denote the full inverse image of \( \bar{Y} \) (resp. \( \bar{Z} \)). Then \( M_1 = Y \times Z \), \( (Y')^Z = Z \), \( Y \cap C_1 = [C_1, Y'] \), \( Z \cap C_1 = [C_1, Z'] \), \( Y' \) (resp. \( Z' \)) splits over \( Y \cap C_1 \) (resp. \( Z \cap C_1 \)).

Proof.

Let \( y \) be an element of order 7 of \( Y' \). Then \( C_{C_1}(y) = [C_1, \bar{Z}] \) and \( \overline{C_{M_1}(y)} = \langle \bar{y} \rangle \times \bar{Z} \). Thus \( O_2(C_{M_1}(y)) \cong E_{64} \) and \( C_{M_1}(y)/O_2(C_{M_1}(y)) \cong \mathbb{Z}_7 \times \mathbb{Z}^+(6,2) \). (We do not yet know whether this extension splits or not), and \( C_{M_1}(y) = Z' \cap M_1 \). Since \( (Y')^Z = Z' \) and \( Y' \cap Z' = 1 \), we conclude that \( M_1 = Y' \times Z' \). Finally, since \( Y' \cong Z' \cong C_{Y'} \times Z' \), \( P_1 = \mathbb{P}_1 \), the fact that \( P_1 \) splits over \( A_1 \) implies that \( Y' \).
Lemma 6.5

If (1) of Lemma 6.2 occurs, then the following hold for each $i \in \{1,2,3\}$.

1. The case (2) of Lemma 3.2 occurs, and so $N_i$ contains a unique elementary abelian 2-subgroup $C_i$ of order 4096.
2. $N(C_i^\omega/C_i) \cong \Omega^+(6,2) \times \Omega^+(6,2)$ and $zC_i/C_i$ permutes the components.
3. $S \subseteq \text{Syl}_2(M_i)$

Proof.

We prove this for $i = 2$.

We use the notation of Lemma 6.4. Set $C_2 = \langle \langle x, x^z | x \in Y', xxz \in A_2 \rangle \rangle$. Then $C_2 \cong E_{4096}$. Note that $C_2 = C_{M_1}(A_1)$. Since $[C_2, z] = A_2$, $C_2 \subseteq N(\langle z, A_2 \rangle)$. Thus, from Lemma 3.2, the case (2) of the same lemma occurs for $i = 2$, $C_2$ is the unique abelian subgroup of order 4096 of $N_2$. From Lemma 6.4, $N_{M_1}(C_2) = \langle x, x^z | x \in Y', xxz \in N_p(A_2) \rangle$. Thus $N_{M_1}(C_2)/C_2 \cong E_8 \cdot L_3(2) \times E_8 \cdot L_3(2)$. In particular, a Sylow 2-subgroup of $N_{M_1}(C_2)/C_2$ is isomorphic
to a Sylow 2-subgroup of $\Omega^+(6,2) \times \Omega^+(6,2)$. Since $N_{M_1}(C_2) = N_{M_1}(C_2)^\prime$, $N_{M_1}(C_2) \subseteq N(C_2)^\infty$. Since a Sylow 2-subgroup of $\Omega^+(6,2) \times \Omega^+(6,2)$, we get (2) from an analogue of Lemma 6.1 for $i = 2$. Since $C_2 = C_{M_1}(A_1)^\infty$, $N_{M_1}(C_2) \supseteq S$ by our choice of $S$. This proves (3).
SECTION 7
BN-PAIR

In this section we continue with the notation and the hypotheses of Section 6, and construct a subgroup $G_0$ of $G$ such that $G_0 = \Omega^+(8,2) \times \Omega^+(8,2)$ or $\Omega^+(8,4)$.

Notation 7.1

For each $i$, let $C_i$ be the unique elementary abelian 2-subgroup of order 4096 of $N_i$, and set $M_i = N(C_i)^{\circ}$ and $\overline{N(C_i)} = N(C_i)/C_i$. Let

\[ L_1 = M_1 \cap M_1^{psqrsp}, \quad L_2 = M_2 \cap M_2^{qsrsr}, \quad L_3 = M_3 \cap M_3^{qsrsr} \]

\[ S_1 = S \cap S^{psqrsp}, \quad S_2 = S \cap S^{qsrsr}, \quad S_3 = S \cap S^{qsrsr} \]

Thus $K_i \subseteq L_i$ and $U_i \subseteq S_i \subseteq L_i$ for $i \in \{1,2,3\}$. Let

\[ x_s = S_1 \cap S_1^{qsrsr}, \quad x_q = S_1 \cap S_1^{qsrsr} \]

\[ x_r = S_1 \cap S_1^{qsrsr}, \quad x_p = S_2 \cap S_2^{qsrsr} \]

For $t \in \{p,q,r,s\}$, let

\[ x_{-t} = (x_t)^t, \quad y_t = \langle x_t, x_{-t} \rangle \]

Finally, we set
\[ G_0 = \langle Y_p, Y_q, Y_r, Y_s \rangle \].

From Lemma 1.23, we know that, for each \( i \in \{1,2,3\} \), there exists a natural BN-pair \((B_i, N_i)\) of \( N_i \) such that \( S \subseteq B_i \) and such that either

(1) \( k_i \cong \Omega^*+(6,2) \times \Omega^*(6,2) \), and if the distinguished generators of the Weyl group \( W_i = N_i / (B_i \cap N_i) \) are labeled

\[ \begin{array}{c}
\bullet u_{i1} \quad \bullet u_{i2} \quad \bullet u_{i3} \\
\bullet v_{i1} \quad \bullet v_{i2} \quad \bullet v_{i3}
\end{array} \]

Figure 5. Dynkin diagram 5

and if we set \( I = \{ u_{ij} v_{ij}, j \in \{1,2,3\} \} \) and \( J = \{ u_{ij} v_{i4} - j \}, j \in \{1,2,3\} \), then

\[ \{ \overline{u}, \overline{v}, \overline{r} \} = I \) or \( J \) if \( i = 1 \),
\[ \{ \overline{r}, \overline{v}, \overline{r} \} = I \) or \( J \) if \( i = 2 \),
\[ \{ \overline{r}, \overline{v}, \overline{r} \} = I \) or \( J \) if \( i = 3 \); or

(2) \( M_i \cong \Omega^*(6,4) \), and

\[ \{ \overline{u}, \overline{v}, \overline{r} \} \) are the distinguished generators of \( W_i \) if \( i = 1 \),
\[ \{ \overline{r}, \overline{v}, \overline{r} \} \) are the distinguished generators of \( W_i \) if \( i = 2 \),
\[ \{ \overline{r}, \overline{v}, \overline{r} \} \) are the distinguished generators of \( W_i \) if \( i = 3 \).
Lemma 7.2.

\[ [S : S \cap S^p] = 4. \]

Proof.

Arguing in \( M_2 \), we have \( [S : S \cap S^p] = 4 \) from the second paragraph of Notation 7.1. This proves the lemma.

Arguing similarly in \( M_1 \), we can prove the following three lemmas.

Lemma 7.3.

\[ [S : S \cap S^q] = 4. \]

Lemma 7.4.

\[ [S : S \cap S^r] = 4. \]

Lemma 7.5.

\[ [S : S \cap S^s] = 4. \]

Lemma 7.6.

For each \( i \in \{1,2,3\} \), we have \( M_i = C_i \cdot L_i \) and \( S_i = S \cap L_i \in \text{Syl}_2(L_i) \).

Proof.

First assume \( i = 1 \). Consider the following chain of conjugates of \( S \):
The orders of the intersections of two consecutive conjugates are known by Lemmas 7.2, 7.3, 7.4 and 7.5, and hence we can estimate the order of the intersection of all conjugates. As \(|S| = 2^{24}\), we have \(|S_1| \geq 4096\).

Now \(C_{M_1}(z) = P_1\) and so
\[
C_1 \cap L_1(z) \subseteq C_1 \cap M_1, \quad p \quad \text{psqrs} \cap C(z) = A_1 \cap P_1, \quad \text{psqrs} = A_1 \cap K_1 = 1.
\]

Hence \(C_1 \cap L_1 = 1\). Then, since \(S_{1} \subseteq S\) and \(|S_{1}| = |S_1| \geq 4096 = |S|\), it follows that \(S = S_{1}\). Since \(K_1 \subseteq L_1\), \(L_1 = \langle S_{1}K_1 \rangle \subseteq L_1 \subseteq \overline{L}_1\) and so \(L_1 = \overline{L}_1\). Thus the lemma is proved for \(i = 1\). The proofs for \(i = 2, 3\) are exactly the same as above.

Lemma 7.7.

The following hold:

1. \(S \cap S^{w} = 1\)
2. \(X_p = S \cap S^{wp} = S_{3} \cap S_{3}^{spqspq}\)
3. \(X_q = S \cap S^{wq} = S_{3} \cap S_{3}^{spqsp}\)
4. \(X_r = S \cap S^{wr} = S_{2} \cap S_{2}^{srpsp}\)
5. \(X_s = S \cap S^{ws} = S_{2} \cap S_{2}^{rpsrp} = S_{3} \cap S_{3}^{pqspq}\)

Proof.

As \(S \cap S^{w} \cap C(z) = U \cap U^{w} = 1\), \(1\) holds. By definition \(X_p \subseteq S \cap S^{qrsqrsqrsqrs} = S \cap S^{wp}\). Now \(S_{2} \subseteq L_{2}\)
and \( p, r, s \in K_2 \subseteq L_2 \). Hence, computing in \( L_2 \), we have \( |X_p| = 4 \) by the definition of \( X_p \). Also, since \( S^P \cap S^{WP} = 1 \) and \([S : S \cap S^P] = 4\), it follows that \( |S \cap S^{WP}| \leq 4\). Comparing the order, we have \( X_p = S \cap S^{WP} \). If we argue similarly using \( L_3 \) in place of \( L_2 \), we have \( S_3 \cap S_3^{SPQSQ} = S \cap S^{WP} \). Thus (2) is proved. (3), (4) and (5) can be proved in a similar manner.

Lemma 7.8.

If \( \Pi_i \cong \Omega^*(6,4) \) for each \( i \in \{1,2,3\} \), then \( G_0 \cong \Omega^*(8,4) \).

Proof.

Let \( G_0^* = \Omega^*(8,4) \) and let \( B^*, N^*, S^* \) and \( W^* \) be as in Lemma 1.24. Then \( W^* \) is isomorphic to the Weyl group of a root system of type \((D_4)\). Let \( \Pi \) be a system of fundamental roots and choose our notation so that the labeling of the Dynkin diagram is as follows:

![Figure 6. Dynkin diagram 6](image)

For \( t \in \Pi \), define \( X_t^* \) as in Lemma 1.24 and set \( Y_t^* = \{X_t^*, X_{-t}^*\} \). Then the second paragraph of Notation 7.1 and
Lemma 7.6 show that there exists an isomorphism
\[ \alpha : \langle Y^*_q, Y^*_s, Y^*_r \rangle \rightarrow \langle Y_q, Y_s, Y_r \rangle \]
such that \( (X^*_t)^\alpha = X_t \) for each \( t \in \{q, -q, s, -s, r, -r\} \).
Also it follows from Notation 7.1 and Lemma 7.5 and (4), (5) of Lemma 7.6 that there exists an isomorphism
\[ \beta : \langle Y^*_r, Y^*_s, Y^*_p \rangle \rightarrow \langle Y_r, Y_s, Y_p \rangle \]
such that \( (X^*_t)^\beta = X_t \) for each \( t \in \{r, -r, s, -s, p, -p\} \).
Letting \( \delta \) be the restriction of \( \beta^{-1}\alpha \) to \( \langle Y_r, Y_s \rangle \), we apply Lemma 1.20 to \( \langle Y_r, Y_s, Y_p \rangle (\cong SL(4,4)) \) and \( \langle Y_r, Y_s \rangle (\cong SL(3,4)) \). Then we obtain an automorphism \( \xi \) of \( \langle Y_r, Y_s, Y_p \rangle \) such that the restriction of \( \xi \) to \( \langle Y_r, Y_s \rangle \) is \( \delta \) and \( X_r^\xi = X_r \) and \( X_{-r}^\xi = X_{-r} \). Thus, replacing \( \beta \) by \( \beta\xi \), we may assume \( \alpha \) and \( \beta \) are equal on \( \langle Y_r, Y_s \rangle \). Also, from Notation 7.1, Lemma 7.5 and (2), (3), (5) of Lemma 7.6, it follows that there exists an isomorphism:
\[ \gamma : \langle Y_p, Y_s, Y_q \rangle \rightarrow \langle Y_p, Y_s, Y_q \rangle \]

such that \((X_t^*)^\gamma = X_t^\gamma \) for each \( t \in \{p, -p, s, -s, q, -q\} \).

Applying Lemma 1.20 to \( \langle Y_p, Y_s, Y_q \rangle \) and \( \langle Y_s, Y_q \rangle \), we may assume \( \alpha \) and \( \gamma \) are equal on \( \langle Y_p, Y_s \rangle \). After doing that, we again apply Lemma 1.20 to \( \langle Y_p, Y_s, Y_q \rangle \) and

\( \langle Y_p, Y_s \rangle \). Thus we may also assume that \( \beta \) and \( \gamma \) are equal on \( \langle Y_p, Y_s \rangle \). Now \( \langle Y_q, Y_s, Y_r \rangle \), \( \langle Y_r, Y_s, Y_p \rangle \) and

\( \langle Y_p, Y_s, Y_q \rangle \) are all perfect. Therefore, \( G_0 \) is perfect.

We can now apply Lemma 1.24 to prove that \( G_0/Z(G_0) = G_0^* \).

Since \( G_0 \) is \( z \)-invariant, \( |Z(G_0)| = 1 \) from the structure of \( C(z) \). Since the odd share of the Schur multiplier of \( G_0^* \) is trivial, \( G_0 \cong G_0^* = \Omega^+(8, 4) \) as desired.

**Lemma 7.9**

If \( \bar{M}_1 \cong \Omega^+(6, 2) \times \Omega^+(6, 2) \) for each \( i \in \{1, 2, 3\} \), then \( G_0 \cong \Omega^+(8, 2) \times \Omega^+(8, 2) \) and \( z \) interchanges the components.

**Proof.**

Let \( G_0^* \cong \Omega^+(8, 2) \times \Omega^+(8, 2) \) and \( z^* \) be an automorphism of \( G_0^* \) which interchanges the component. Let \( B^*, N^*, S^* \) and \( W^* \) be as in Lemma 1.24. (Note that the assumption (3) of the lemma fails to hold in this case.) Then
\( W^* \) is isomorphic to the direct product of two copies of the Weyl group of a root system of type \( D_4 \). Let \( \Pi^* \) be a system of fundamental roots and choose notations so that the labeling of the Dynkin diagram is as follows:

\[
\begin{array}{c}
\begin{array}{c}
\text{Figure 7. Dynkin diagram 7}
\end{array}
\end{array}
\]

For \( u \in \Pi^* \) define \( X_u^* \) as in Lemma 1.24, and set \( Y_u^* = \langle X_u^*, X_{-u}^* \rangle \). Then there is an isomorphism

\[
\alpha : \langle Y_{q_i}^*, Y_{s_i}^*, Y_{r_i}^* \mid i = 1, 2 \rangle \rightarrow \langle Y_q^*, Y_s^*, Y_r^* \rangle
\]

such that \( (X_{t_1}^* X_{t_2}^*)^\alpha = X_t^* \) and \( (X_{-t_1}^* X_{-t_2}^*)^\alpha = X_{-t}^* \) for each \( t \in \{q, s, r\} \). Also there is an isomorphism

\[
\beta : \langle Y_{r_i}^*, Y_{s_i}^*, Y_{p_i}^* \mid i = 1, 2 \rangle \rightarrow \langle Y_r^*, Y_s^*, Y_p^* \rangle
\]

such that \( (X_{t_1}^* X_{t_2}^*)^\beta = X_t^* \) and \( (X_{-t_1}^* X_{-t_2}^*)^\beta = X_{-t}^* \) for each \( t \in \{r, s, p\} \). Then both \( \alpha \) and \( \beta \) are isomorphisms from \( Y_{r_1}^* \times Y_{r_2}^* \) onto \( Y_r^* \) and from \( Y_{s_1}^* \times Y_{s_2}^* \) onto \( Y_s^* \). Thus each of \( Y_r^* \) and \( Y_s^* \), in two different ways, as a direct product of two subgroups each isomorphic to \( L_2(2) \). But
Krull-Schmidt theorem shows that such direct product decomposition is unique up to permutations of direct factors. Hence either \((Y^*_r)^\alpha = (Y^*_r)^\beta\) and \((Y^*_s)^\alpha = (Y^*_s)^\beta\) for each \(i\) or \((Y^*_r)^\beta = (Y^*_r)^\alpha\) and \((Y^*_s)^\alpha = (Y^*_s)^\beta\) for each \(i\). In the latter case, replace \(\beta\) by \(\delta\beta\), where \(\delta\) is an isomorphism of \(\langle Y^*_r, Y^*_s, Y^*_p | i = 1,2 \rangle\) such that \((X^*_{t_i})^\delta = X^*_{t_3 - i}\) and \((X^*_{-t_i})^\delta = (X^*_{-t_3 - i})\) for each \(t \in \{r, s, p\}\) and for each \(i \in \{1, 2\}\). Thus we may choose \(\alpha\) and \(\beta\) so that \((Y^*_r)^\alpha = (Y^*_r)^\beta\) and \((Y^*_s)^\alpha = (Y^*_s)^\beta\) for each \(i\). Then, since 

\[X^*_{t_1} = Y^*_{t_1} \cap (X^*_{t_1} X^*_{t_2})\] \[X^*_{-t_1} = Y^*_{t_1} \cap (X^*_{-t_1} X^*_{-t_2})\]

for each \(t \in \{r, s\}\) and for each \(i \in \{1, 2\}\), it follows that \((X^*_{t_1})^\alpha = (X^*_{t_1})^\beta\) and \((X^*_{-t_1})^\alpha = (X^*_{-t_1})^\beta\) for each \(t\) and for each \(i\). Since \(|X^*_{t_1}| = |X^*_{-t_1}| = 2\), this implies that \(\alpha = \beta\) on \(\langle Y^*_r, Y^*_s, Y^*_p | i = 1,2 \rangle\). Again, there is an isomorphism 

\[\gamma : \langle Y^*_p, Y^*_s, Y^*_q | i = 1,2 \rangle \rightarrow \langle Y^*_p, Y^*_s, Y^*_q \rangle\]

such that \((X^*_{t_1} X^*_{t_2})^\gamma = X_t\) and \((X^*_{-t_1} X^*_{-t_2})^\gamma = X_{-t}\) for each
t ∈ {p, s, q}. Arguing as above, we may choose \( γ \) so that \( α = γ \) on \( \langle Y_{s_i}^*, Y_{q_i}^* \rangle_{i=1,2} \). After choosing \( γ \) in this manner, we then argue as above using the fact that \( β = γ \) on \( \langle Y_{s_i}^* \rangle_{i=1,2} \). Then we can prove that \( β = γ \) on \( \langle Y_{r_i}^*, Y_{s_i}^* \rangle_{i=1,2} \). This implies that there exists direct product decompositions

\[
Y_p = Y_{p_1} \times Y_{p_2}, \quad Y_q = Y_{q_1} \times Y_{q_2}
\]

\[
Y_r = Y_{r_1} \times Y_{r_2}, \quad Y_s = Y_{s_1} \times Y_{s_2}
\]

such that \( (Y_{t_i}^*)^X = Y_{t_i} \) for each \( t ∈ \{p, q, r, s\} \) and for each \( i ∈ \{1, 2\} \) and for each \( x ∈ \{α, β, γ\} \) when \( (Y_{t_i}^*)^X \) is meaningful. Then \( (Y_{t_i}^*)^Z = Y_t \) for each \( t ∈ \{p, q, r, s\} \), and \( [Y_{t_i}, Y_{t_{i'}}] = 1 \) for each \( t, t' ∈ \{p, q, r, s\} \) and for each \( i ∈ \{1, 2\} \). Hence, if we set

\[
G_1 = \langle Y_{p_1}, Y_{q_1}, Y_{r_1}, Y_{s_1} \rangle
\]

then \( G_1 \) is a central product of \( G_1 \) and \( G_1^Z \). Moreover, we can apply Lemma 1.24 to \( \langle Y_{p_1}^*, Y_{q_1}^*, Y_{r_1}^*, Y_{s_1}^* \rangle \) and \( G_1 \) to conclude that \( G_1/Z(G_1) \cong \Omega^+(8, 2) \). From the structure of \( C(z) \), \( |Z(G_1)|_2 = 1 \). Since the odd share of the Schur
multiplier of $\Omega^+(8,2)$ is trivial, $G_1 = \Omega^+(8,2)$. And hence, $G_0 = \Omega^+(8,2) \times \Omega^+(8,2)$ as desired.

As remarked in the Introduction of this chapter, this completes the proof of the Main Theorem.
APPENDIX

2-MODULAR CHARACTER TABLES (COMPUTED BY H. SUZUKI)

Table 1. The 2-modular irreducible characters of $\text{SL}(3,4)$ faithful on the center

<table>
<thead>
<tr>
<th>Element</th>
<th>1</th>
<th>3</th>
<th>5A</th>
<th>5B</th>
<th>7A</th>
<th>7B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Centralizer in $\text{PSL}(3,4)$</td>
<td>$2^6 \cdot 3^2 \cdot 5 \cdot 7$</td>
<td>9</td>
<td>5</td>
<td>5</td>
<td>7</td>
<td>7</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>0</th>
<th>$\tau$</th>
<th>$\tau'$</th>
<th>$\alpha$</th>
<th>$\bar{\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>0</td>
<td>$\tau'$</td>
<td>$\tau$</td>
<td>$\alpha$</td>
<td>$\bar{\alpha}$</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>$\alpha$</td>
<td>$\bar{\alpha}$</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>$\bar{\alpha}$</td>
<td>$\alpha$</td>
</tr>
</tbody>
</table>

$\tau = \frac{1 + \sqrt{5}}{2}$  $\tau' = \frac{1 - \sqrt{5}}{2}$  $\alpha = \frac{-1 + \sqrt{7}}{2}$  $\bar{\alpha} = \frac{-1 - \sqrt{7}}{2}$
Table 2. The 2-modular irreducible characters of $SU(4,2)$

<table>
<thead>
<tr>
<th>Element</th>
<th>1</th>
<th>3A</th>
<th>3B</th>
<th>3C</th>
<th>3D</th>
<th>5</th>
<th>9A</th>
<th>9B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Centralizer</td>
<td>$2^6 \cdot 3^4 \cdot 5$</td>
<td>108</td>
<td>54</td>
<td>648</td>
<td>648</td>
<td>5</td>
<td>9</td>
<td>9</td>
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</table>

<table>
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<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$-2$</td>
<td>1</td>
<td>$\alpha$</td>
<td>$\bar{\alpha}$</td>
<td>-1</td>
<td>$\omega$</td>
<td>$\bar{\omega}$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$-2$</td>
<td>1</td>
<td>$\alpha$</td>
<td>$\bar{\alpha}$</td>
<td>-1</td>
<td>$\omega$</td>
<td>$\bar{\omega}$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>0</td>
<td>-3</td>
<td>-3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
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<tr>
<td>14</td>
<td>2</td>
<td>-1</td>
<td>5</td>
<td>5</td>
<td>-1</td>
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</tr>
<tr>
<td>20</td>
<td>-4</td>
<td>-1</td>
<td>$\bar{\delta}$</td>
<td>$\delta$</td>
<td>0</td>
<td>$\bar{\delta}$</td>
<td>$\delta$</td>
<td></td>
</tr>
<tr>
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<td>-4</td>
<td>-1</td>
<td>$\bar{\delta}$</td>
<td>$\delta$</td>
<td>0</td>
<td>$\bar{\delta}$</td>
<td>$\delta$</td>
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</tr>
<tr>
<td>64</td>
<td>4</td>
<td>2</td>
<td>-8</td>
<td>-8</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

$$\alpha = \frac{-1 + 3\sqrt{3}}{2}, \quad \bar{\alpha} = \frac{-1 - 3\sqrt{3}}{2}$$
$$\gamma = 2 + 3\sqrt{3}, \quad \bar{\gamma} = 2 - 3\sqrt{3}$$
$$\delta = \frac{1 + \sqrt{3}}{2}, \quad \bar{\delta} = \frac{1 - \sqrt{3}}{2}$$
$$\omega = \frac{-1 + \sqrt{3}}{2}, \quad \bar{\omega} = \frac{-1 - \sqrt{3}}{2}$$
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