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THE OHIO STATE UNIVERSITY, PH.D., 1975
AEROSEROVOELASTIC STABILITY ANALYSIS
OF AN AIRPLANE WITH A CONTROL
AUGMENTATION SYSTEM

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Robert Lee Moore, B. Aero. Astro. E., M.S.

* * * * *

The Ohio State University
1978

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The author wishes to express his appreciation to his advisor, Dr. B. E. Gatewood, for his advice and encouragement during the preparation of this work. The author is indebted to his coworkers at the Loads and Dynamics Branch of the Aeronautical Systems Division at Wright-Patterson Air Force Base for their support during the preparation of this work. Also the author wishes to thank the General Dynamics Corporation, Fort Worth Division, for use of the basic data for the airplane analyzed in the application case in this dissertation.

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LIST OF SYMBOLS

[A]  stability eigenvalue matrix defined by equations (A.5) and (A.6); also coefficient matrices of like powers of Laplace variable, s, defined by equation (A.1)

A_R  reference area

A_{ij}  generalized aerodynamic force for the i th mode due to resulting aerodynamic pressure of the j th mode

A_{ij}^c  generalized aerodynamic force for the i th mode due to resulting aerodynamic pressure of the motion of the j th control surface

A_{ij}(s)  Laplace transform of A_{ij} per unit normal coordinate \( \xi_j(s) \)

A_{ij}^c(s)  Laplace transform of A_{ij}^c per unit control surface coordinate \( \delta_j(s) \)

[A(s)]  Laplace transformed generalized unsteady aerodynamic force matrix, elements defined by equation (2.55) and (2.59)

[A_c(s)]  Laplace transformed generalized control surface unsteady aerodynamic force matrix, elements defined by equation (2.56)

A/P  airplane

a  nondimensional distance of airfoil strip from midchord to reference (elastic) axis

a_{0ij}, a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}  numerator coefficients of Laplace transformed generalized aerodynamic forces defined by equations (2.53) and (2.54)

\{a_R\}, \{a_i\}  coefficient arrays defined by equations (2.45), (2.46), and (2.57)

b  local semichord; also used as span in defining stability derivatives
reference semichord

denominator coefficients of Laplace transformed generalized aerodynamic forces defined by equation (2.40)

[C] generalized damping matrix

generalized damping of the i_th normal mode

Theodorsen function

rolling moment coefficient given by doublet-lattice program, equation (B.45)

rolling moment coefficient in stability and control notation

yawing moment coefficient in stability and control notation

yawing moment coefficient in stability and control notation

side force coefficient given by doublet-lattice program, equations (B.38) and (B.40)

side force coefficient in stability and control notation

vertical force coefficient given by doublet-lattice program, equations (B.37) and (B.39)

pitching moment coefficient given by doublet-lattice program, equations (B.41) and (B.43)

stability derivatives for rolling moment due to $\beta, p, r$, respectively

stability derivatives for rolling moment due to $\delta_A, \delta_{HT}, \delta_R$, respectively

stability derivatives for yawing moment due to $\beta, p, r$, respectively

stability derivatives for yawing moment due to $\delta_A, \delta_{HT}, \delta_R$, respectively

stability derivatives for side force due to $\beta, p, r$, respectively

stability derivatives for side force due to $\delta_A, \delta_{HT}, \delta_R$, respectively
\( C_{\delta j} \) hinge moment coefficient for the \( j \)th control surface

\( c \) nondimensional distance of unit span airfoil strip from midchord to control surface hinge line; also local chord in doublet-lattice aerodynamics

\( c_R \) reference chord (2b_R)

\( c_n \) local normal force coefficient

\( c_m \) local pitching moment coefficient

\( \text{c.g.} \) center of gravity

\( \text{c.p.} \) center of pressure

\( D \) matrix relating normalwash (or downwash) to lifting pressure for lifting surface elements

\( D_I \) matrix relating normalwash to lifting pressure for image elements

\( D \) matrix relating normalwash to lifting pressure for elements plus their images plus the contributions due to symmetry and ground effect

\( D_T \) partitioned matrix \(|D; E|\), relating normalwash to lifting pressures and doublet strengths

\( D_{2D} \) matrix relating the doublet strength to the local upwash (downwash) or sidewash using quasi-steady, two-dimensional slender body theory

\( \{d_R\}, \{d_I\} \) assays defined by equations (2.45) and (2.46)

\( E \) matrix relating normalwash to axial doublet strengths

\( E \) matrix relating normalwash to axial doublet strengths with the effects of symmetry and ground effect included

\( F_{2F_y} \) body forces in \( x \) and \( y \) directions, respectively

\( \mathbf{f}^M(x,y,z,t) \) force function vector due to the disturbed motion

\( \mathbf{f}^D(x,y,z,t) \) disturbance force function vector

\( \text{F.S.} \) fuselage station

\( f_i \) nondimensional deflection of \( i \)th mode normal to a lifting surface for doublet-lattice aerodynamics (using \( \phi_i/c_R \)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{z_1}, f_{y_1}$</td>
<td>nondimensional $z$ and $y$ deflections, respectively, of $i$th mode normal to a body for doublet-lattice aerodynamics</td>
</tr>
<tr>
<td>$[G(s)]$</td>
<td>feedback matrix of the transfer functions between the control surface rotations and the airframe motion at the sensor locations</td>
</tr>
<tr>
<td>$H. M.$</td>
<td>hinge moment</td>
</tr>
<tr>
<td>$h$</td>
<td>deflections normal to a lifting surface for doublet-lattice aerodynamics</td>
</tr>
<tr>
<td>$h_z, h_y$</td>
<td>$z$ and $y$ deflections, respectively, normal to a body for doublet-lattice aerodynamics</td>
</tr>
<tr>
<td>$\hat{H}(x,y,z,t)$</td>
<td>displacement function vector of the airplane</td>
</tr>
<tr>
<td>$h, \alpha, \beta$</td>
<td>vertical displacement, rotation, and control surface relative rotation, respectively, of unit span airfoil strip defined by Figure 4</td>
</tr>
<tr>
<td>$\Pi(s), \Pi(s), \Pi(s)$</td>
<td>Laplace transforms of vertical displacement, rotation, and control surface relative rotation, respectively, of unit span airfoil strip defined by Figure 4</td>
</tr>
<tr>
<td>$[I]$</td>
<td>identity matrix</td>
</tr>
<tr>
<td>$i$</td>
<td>$\sqrt{-1}$</td>
</tr>
<tr>
<td>$\hat{i}, \hat{j}, \hat{k}$</td>
<td>unit vectors along $x$, $y$, $z$ axes, respectively</td>
</tr>
<tr>
<td>$K$</td>
<td>velocity kernel function relating normal wash at one point to pressure induced by a doublet of unit strength at another point</td>
</tr>
<tr>
<td>$[K]$</td>
<td>generalized stiffness matrix</td>
</tr>
<tr>
<td>$K_i$</td>
<td>generalized stiffness of the $i$th normal mode</td>
</tr>
<tr>
<td>$K_{\delta_j}$</td>
<td>scalar modification factor for $j$th control surface aerodynamics</td>
</tr>
<tr>
<td>$K_{\delta_A}, K_{\delta_{HT}}, K_{\delta_R}$</td>
<td>scalar modification factors for aileron differential horizontal tail, and rudder control surfaces, respectively</td>
</tr>
<tr>
<td>$K_p, K_r, K_{Ny}$</td>
<td>gain values in roll rate, yaw rate, and lateral acceleration feedback loops, respectively</td>
</tr>
<tr>
<td>$k$</td>
<td>reduced frequency $\left(\frac{\omega}{V} \quad \text{or} \quad \frac{\omega_c}{2V}\right)$</td>
</tr>
</tbody>
</table>
the normalwash due to a potential doublet

lateral translation

unsteady unit span values of, respectively, lift, torsional moment about reference axis, and moment about control surface hinge line of airfoil strip in Figure 4

Laplace transformed unsteady unit span values, of respectively, lift, torsional moment about reference axis, and moment about control surface hinge line of airfoil strip in Figure 4

Laplace polynomial fractions defined by equation (2.13)

vertical and fore-and-aft distances, respectively, from airplane c.g. to lateral accelerometer

Mach number; also normalwash due to a point source

generalized mass matrix

generalized mass of the \( i \) th normal mode

generalized control surface inertia force matrix

generalized inertia force for the \( i \) th mode due to the motion of the \( j \) th control surface

aerodynamic rolling moment

terodynamic yawing moment

body moments about \( z \) and \( y \) axes, respectively

Laplace polynomial fractions defined by equation (2.13)
lateral acceleration sensed by lateral accelerometer

operator \( \frac{d}{dt} \)

orthogonal matrix in decomposition for eigenvalues

normalized form of Fourier transformed generalized aerodynamic force defined in equation (2.29) and (2.31)

Normalized form of Fourier transformed generalized control surface aerodynamic force defined in equations (2.30) and (2.32)

xvii
\( \tilde{q}_{ij}(s) \) normalized form of Laplace transformed generalized aero-
dynamic force defined in equations (2.55) and (2.59)

\( \tilde{q}_{ij}^c(s) \) normalized form of Laplace transformed generalized control
surface aerodynamic force defined in equation (2.56)

\( q_w \) free stream dynamic pressure

\( q_{s_i}(t) \) motion sensed by the \( i \) th sensor

\( \tilde{q}_{s_i}(s) \) Laplace transform of motion sensed by the \( i \) th sensor

\( \{\tilde{q}_s(s)\} \) array of transformed airframe motions sensed by the sensors

\( \{q(s)\} \) array of Laplace functions of the transformed normal
coordinates, defined by equations (A.2) and (A.6)

\( R \) upper triangular matrix in decomposition for eigenvalues

\( S \) surface integration area

\( \tilde{G}_d(x,y,z) \) control surface static unbalance function vector about the
hinge line for the \( j \) th control surface

\( s \) Laplace variable

\( s_R \) reference semispan (2b)

T.E. trailing edge

\( T_1,\ldots,T_{12} \) functions of geometry of unit span airfoil strip

\( \bar{T}_h, \bar{T}_a, \bar{T}_B \) Laplace polynomial fractions defined by equation (2.13)

\( t \) time

\( V \) airspeed of the airplane

\( [W] \) weighting matrix defined by equation (2.52)

\( w \) normalwash (or downwash) boundary values on lifting surfaces
and bodies; also normalwash boundary values on lifting
surfaces only

\( w_i \) normalwash boundary values on lifting surfaces and bodies
for the \( i \) th mode

\( w_i \) normalwash due to image lifting surface elements
normalwash due to interference doublet distribution

normalwash due to lifting surface elements

upwash (or downwash) boundary values for bodies

sidewash boundary values for bodies

$W_T = W - \Delta W$

coordinate about which aerodynamic moments are taken

Cartesian coordinates; also coordinates of receiving point in doublet lattice aerodynamics

dihedral angle: $\gamma_r$, receiving point, $\gamma_s$, sending point

lifting pressure coefficient

pressure difference function vector caused by disturbed motion

pressure difference function vector caused by motion in the $j$th mode

pressure difference caused by motion of the $j$th control surface

normalwash due to slender body elements

symmetry plane indication (1 symmetry, 0 no symmetry, -1 antisymmetry)

rotation of the $j$th control surface

Laplace transform of rotation of the $j$th control surface

array of transformed control surface rotations

ailerons, differential horizontal tail, and rudder rotations, respectively

feedback transfer function from $\dot{\phi}_G$ to $\delta_A$

feedback transfer function from $\dot{\psi}_G$ to $\delta_A$
\( \delta \text{A/Ny} \) feedback transfer function from Ny to \( \delta \text{A} \)

\( \delta \text{HT/} \phi \text{G} \) feedback transfer function from \( \phi \text{G} \) to \( \delta \text{HT} \)

\( \delta \text{HT/} \psi \text{G} \) feedback transfer function from \( \psi \text{G} \) to \( \delta \text{HT} \)

\( \delta \text{HT/Ny} \) feedback transfer function from Ny to \( \delta \text{HT} \)

\( \delta \text{R/} \phi \text{G} \) feedback transfer function from \( \phi \text{G} \) to \( \delta \text{R} \)

\( \delta \text{R/} \psi \text{G} \) feedback transfer function from \( \psi \text{G} \) to \( \delta \text{R} \)

\( \delta \text{R/Ny} \) feedback transfer function from Ny to \( \delta \text{R} \)

\( \varepsilon \) ground effect indication (\(-1\) ground effect, \(0\) no ground effect, \(1\) anti-ground effect)

\( \zeta \) damping coefficient of root

\( \zeta_i \) damping coefficient of the \( i \) th normal mode

\( n \) normal coordinate of airplane rigid body translation in the \( y \) direction

\( \bar{n} \) lateral coordinate in the plane of the lifting surface

\( \bar{u}_n \) doublet strength of interference body elements

\( \bar{u}_s \) doublet strength of slender body elements

\( \Xi_{M,i} \) component of the generalized force for the \( i \) th mode due to the disturbed motion

\( \Xi_{D,i} \) component of the generalized force for the \( i \) th mode due to the disturbing force

\( \xi, \zeta, n \) \( x, y, z \) coordinates, respectively, of sending point

\( \xi_i(t) \) \( i \) th generalized or normal mode coordinate

\( \xi_j(s) \) Laplace transform of \( j \) th normal mode coordinate
\{ \xi(s) \} \quad \text{array of transformed normal coordinates}

\sigma \quad \text{real part of root; also source strength in doublet-lattice aerodynamics}

\sigma_{\phi G}^{(i)} \quad \text{slope of the } i \text{th mode at the roll rate gyro location}

\sigma_{\psi G}^{(i)} \quad \text{slope of the } i \text{th mode at the yaw rate gyro location}

\tau \quad \text{lateral distance in plane of lifting surface}

\phi \quad \text{normal coordinate of airplane rigid body roll about c.g.}

\dot{\phi}_G \quad \text{roll rate sensed by the roll rate gyro}

\dot{\phi}_i(x,y,z) \quad i \text{th normal mode shape vector of the airplane}

\phi_{Ny}^{(i)} \quad \text{lateral displacement component of the } i \text{th mode at lateral accelerometer location}

\phi_{s_j}^{(i)}(x_i,y_i,z_i) \quad \text{mode shape component of the } j \text{th mode sensed by the } i \text{th sensor at location } (x_i,y_i,z_i)

[\Phi_s(s)] \quad \text{sensor matrix relating the transformed sensed motion by the sensor to the components of the transformed normal}

\psi \quad \text{normal coordinate of airplane rigid body yaw about c.g.}

\dot{\psi}_G \quad \text{yaw rate sensed by the yaw rate gyro}

\omega \quad \text{frequency}

\omega_d \quad \text{imaginary part root or damped frequency of root}

\omega_i \quad \text{frequency of the } i \text{th normal mode}

\omega_n \quad \text{undamped natural frequency of root}

[\omega_R],[\omega_T] \quad \text{matrices defined by equations (2.45), (2.46), and (2.58)}

\{ \} \quad \text{column matrix}

[ ] \quad \text{square or rectangular matrix}

[ ] \quad \text{diagonal matrix}
\[ [ ]^T \]
transpose of matrix

\[ [ ]^{-1} \]
inverse of matrix

Subscripts and Superscripts

A  aileron
B.  all bodies of airplane doublet-lattice aerodynamic idealization
C.S.  control surface
c  control surface
D  disturbance
H.T.  horizontal differential tail
I  imaginary part
i  coefficient in the \( i \)th equation; also normal coordinate
j  normal coordinate; also control surface
LL  lower left-hand quadrant of \( z-y \) plane
LR  lower right-hand quadrant of \( z-y \) plane
L.S.  lifting surfaces on total airplane
\( \varepsilon \)  roll moment
M  motion
n  number of normal modes; also yaw moment
na  number of discrete frequency fitting points
nc  number of control surfaces
ns  number of sensors
p  roll rate; also discrete frequency point
R  real part
R.S.  lifting surfaces on airplane right-hand side
r  yaw rate; also receiving point in doublet lattice aerodynamics

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sensor; also sending point in doublet lattice aerodynamics

UL  upper left-hand quadrant of z-y plane

UR  upper right-hand quadrant of z-y plane

y  side force

\( \beta \)  side slip angle
INTRODUCTION

The problem of interactions between an airplane's structural dynamics and its flight control system has gained recent attention due to innovations in the design of control augmentation systems, which have become powerful enough to influence the dynamic stability and structural response of the airframe. One consequence of high gain in the feedback loops is the potential for adverse interaction between the control system and the elastic structural dynamics. The control system transducers, usually rate gyros and accelerometers, sense not only rigid body motion but also elastic mode motion. Control surfaces, deflected in response to sensor signals from elastic structural modes, produce oscillatory aerodynamic and inertia forces which, depending on their phasing, tend to either damp or to amplify structural vibrations. A self-excited oscillation may result which in many respects is similar to flutter, but differs in that the energy to sustain the oscillations is supplied, in part, by the control system. The instability, therefore, is limited in amplitude by the capacity of the control system to supply energy at the frequency in question (rate limiting). This instability will be of limited amplitude only if the structural stress resulting from the oscillations is within the structural strength limits of the airframe. This type of instability involving the elastic structural dynamics is called an aeroservoelastic instability. A similar type of instability involving
the rigid airframe dynamics is referred to as an aeroservodynamic instability.

In two recent prototype fighter programs, unstable interactions between the airframe structural dynamics and the flight control augmentation system were encountered in flight (1,2). Even though interactions of this type had been experienced earlier, the criteria for controlling the undesired coupling evolved largely on an empirical basis.

As early as 1953, the British encountered unexpected oscillations of 3 Hz of quite alarming amplitude which they later attributed to a coupling between the fundamental bending mode of the aircraft and a gust alleviator installed on the nose of a Lancaster aircraft (3). Initially, the internal gain of the gust alleviator was set at 6 decibels below the servo instability boundary on the ground. The fix to the above in-flight instability was to reduce the internal gain of the gust alleviator servo by a further 3 decibels.

Also interaction problems on a research vehicle and on lifting bodies were encountered during flight (4,5) and in 1972 the NASA Flight Research Center attempted to develop empirical criteria based on ground test techniques (6) to insure a stable system during flight and to predict possible problems. For example, one of these criteria for precluding aeroservoelastic instabilities during flight consists of showing that at least a 6 decibel gain margin exists for servoelastic instabilities in ground tests at the highest operational control augmentation system gains. Unfortunately, these tests
(referred to as limit cycle and structural resonance) are of limited value in view of recent in-flight incidents involving prototype airplanes which satisfied the above NASA ground test criteria. Arthurs and Gallagher (1) point out that demonstration of substantial gain margins (6 decibels or more) in ground tests does not insure that the system is free from flight instabilities, involving elastic mode coupling, at operational gain levels. Analyses, complemented by ground tests, offer a more complete preflight confirmation of aero-servoelastic stability.

Thus, with the trend toward use of high gain, high response flight control augmentation systems, it has become clear that the interaction between structural, aerodynamic, and control system forces is such that each of the disciplines involved can no longer be treated completely independently. Until recently, the control system engineers in their design analysis efforts usually considered the control system and only rigid airplane modes (with no flexible airplane modes), and quasi-steady aerodynamics. Flexible effects were included to the extent of adjusting the control surface effectiveness to account for steady state aeroelastic effects of the deflected control surface. The structural dynamics engineers have traditionally considered a number of rigid and flexible airplane modes, including unsteady aerodynamics, but without control augmentation effects in their stability or flutter design analysis efforts. Since flight control and structural dynamic technologies have generally developed independently, the possible dynamic interaction effects between the flight control system and the structure have traditionally been inadequately
predicted and understood. The merging of these technologies to solve this problem has been hampered by the lack of common terminology, dissimilar analysis and test techniques, and a general lack of communications between the two disciplines. The occurrence of adverse, in-flight interaction between the structure and the flight control system in several recently tested aircraft attests to the need for improvements in this area. Barfield and Felt (7) discuss the merging of flight control and structural dynamics technologies needed to adequately predict and prevent these potentially destructive instabilities.

Stability analysis of an aeroservoelastic feedback loop system leads to mathematical formulation similar to that used in flutter analysis. However, due to the frequency dependent nature of the control system forces, the equation develops into a high-order eigenvalue problem, which is not readily solvable by classical flutter analysis techniques. It is for this reason that approximation techniques in this area have gained wide attention in the last few years. The following paragraphs summarize some of the analysis approaches used recently.

One analysis approach used by Roger (8) in the control configured vehicle (CCV) program was the application of the root locus method for determining stability. This method is widely used for stability and control analyses employing quasi-steady aerodynamics; however, when unsteady aerodynamic theory is employed, it is necessary to develop indicial functions or to make simplifying assumptions and use approximating functions in order to perform the Laplace transformation to
the aerodynamic equations. Simplifying assumptions were made by Roger in reference 8 and the method worked well, but Peloubet (9) used indicial functions approximations in transforming the unsteady aerodynamics to the Laplace domain and had trouble with extraneous positive real roots in the resulting stability analysis.

A second approach is the application of the polar plot of the open loop transfer function in conjunction with Nyquist's stability criteria (10). This approach determines stability in the frequency domain so that any oscillatory aerodynamic method can be employed directly. The equations of motion can be cast in the same form as commonly used for determining the flutter characteristics of an un-augmented system. The methods used in references 2, 9, 11 and 12 employ this method. However, no explicit information on the damping with each root or mode is yielded, and there are problems in analyzing systems with multiple feedback loops.

A third approach for analyzing system stability is the characteristic diagram technique used in the stability analysis of a recent prototype fighter (1). This method is a graphical method programmed on the digital computer for examining the closed loop stability characteristics of marginally damped system. This analysis approach, like the Nyquist polar plot method, determines the system stability in the frequency domain. One advantage over the Nyquist method is that close to an instability the eigenvalue can be estimated and so the damping for that mode can be extracted. Arthurs and Gallagher (1) found this method to be a highly effective approach to the
analysis of the prototype fighter stability involving control system coupling in that they were able to closely predict an aeroservoelastic instability involving a structural antisymmetric missile pitch mode coupling with the roll control augmentation system (CAS) before encountering it in flight. In addition, they were able to closely predict an aeroservodynamic instability involving rigid body roll coupling with the roll CAS at 3.3 Hz before a gradual loss of damping of the roll mode was evident from flight test results. A disadvantage of this method is that conclusive indication of an instability is obtained only if the poles introduced by the feedback transfer functions are first removed from the determinant for the eigenvalues.

In another approach to determine the stability of a closed loop system, Lotze and Sensburg (12) used an iterative technique, known as the p-k method (13). This analysis method proceeds by fixing the velocity and varying the reduced frequency aerodynamic parameter for each eigenvalue solution until the eigenvalue frequency matches the assumed transfer function frequency. This procedure lends itself well to stability solutions when the system includes frequency dependent feedback loops; however, like many of the previously discussed approaches, this method is a numerical, time consuming, but a viable approach because of today's high speed digital computers.

As discussed above, various approaches for predicting aeroaeroservoelastic stability have been conducted with various degrees of success and complexity. The need for compatible airplane mathematical models for stability and response performance analyses of an airplane
with a control augmentation system has been heightened by the recent trend towards use of high gain, high response flight control augmentation systems. As discussed earlier, recent incidents of interactions between the airplane structural dynamics, aerodynamics, and the control system indicate that each of the disciplines involved can no longer be treated completely independently. This work is intended to improve the analytical tools to predict aeroservoelastic instabilities and to help close the gap between aeroelasticity and control system engineering.

On the basis of the above discussion, an analytical method is presented which is useful and practical in mathematically modeling the airplane and its control augmentation system for use in stability and response performance. This approach can easily model systems with multiple feedback loops. In addition, this approach develops the equations, including the unsteady aerodynamics, in the Laplace domain, and uses the root locus method (10) for stability analysis. The root locus method is widely used in stability and control analyses and is used here because it is a convenient and useful method for determining the effects of varying the feedback loop gain on the system stability. System stability, as a function of feedback loop gain, can be easily evaluated from the root locus because explicit information on the damping of each root or mode is yielded.

It has recently been recognized (1,2,6,7) that the flight conditions, for which the total feedback loop gain (the product of the control surface effectiveness and the feedback loop gain) is a
maximum, are likely to be critical flight conditions for possible aeroservoelastic instabilities. To properly analyze these critical flight conditions for stability, realistic and accurate values of control surface effectiveness are needed. This effectiveness of the control surface is directly attributed to the accuracy of predicting the control surface aerodynamics. Therefore, this work recommends using an unsteady aerodynamic lifting surface theory for use in initial calculations and proposes two methods to modify the calculated unsteady control surface effectiveness based on steady state experimental data.

Chapter I of this work is devoted to development of the mathematical representation of the airframe and feedback control system in the Laplace domain. Chapter II contains a simple, but realistic and practical method for formulating the unsteady aerodynamics forces in the Laplace domain for use in the stability equations developed in Chapter I. In Chapter III, two methods are suggested to adjust the calculated unsteady aerodynamic control surface effectiveness based on available steady state experimental data. In Chapter IV, the above methods are applied to a stability analysis of a recent transonic prototype fighter for which some aeroservoelastic instabilities were encountered in-flight. The results of this analysis and comparison with flight test are given in Chapter V. A summary and conclusions of this work are given in Chapter VI.
CHAPTER I  DEVELOPMENT OF THE AEROSEROVOLASTIC
STABILITY EQUATIONS OF MOTION

The response of an airplane in flight to excitation produced by control surfaces is based on the modal approach where the response is assumed to consist of a superposition of a finite number of appropriate normal modes of the unrestrained airplane including both rigid body modes and elastic structural modes. This transformation is given by

\[ h(x,y,z,t) = \sum_{i=1}^{n} \dot{\phi}_i (x,y,z) \xi_i(t) \]  \hspace{1cm} (1.1)

where \( \dot{h}(x,y,z,t) \) is the displacement function vector of the airplane measured from an inertial reference frame, \( \dot{\phi}_i (x,y,z) \) is the \( i \) th normal mode shape vector of the airplane, and \( \xi_i(t) \) is the \( i \) th generalized or normal mode coordinate. This equation merely states that the airplane displacements are equal to the normal mode relative displacement shapes times the normal mode amplitudes as determined by the response calculations. Further, the above equation implies that the total airplane displacements are linear combinations of the chosen normal mode displacement shapes, each of which is associated with a time dependent normal mode coordinate \( \xi_i(t) \). These normal mode coordinates consist of an appropriate number of rigid body modes and a limited number of low frequency normal vibration modes of the structure. The determination of the mode shapes and frequencies of
the normal vibration modes of the structure is a major task which is assumed to be already done for purposes of this work. A finite element structural model is usually used to calculate the normal vibration modes of the structure. These normal mode shape $\phi_i(x,y,z)$ functions are sets of relative displacements at the structural reference points in the chosen modes. The reader is referred to references 14, 15, 16, and 17 for treatises on this subject.

There are actually an infinite number of normal vibration modes of the airframe structure. However, only a limited number of those at the low frequency end of the set need be retained since control system coupling with higher frequency structural vibration modes is unlikely because of the inherent decrease in control system gain with increasing frequency.

For derivation of the equations of motion, the airplane motion is assumed to consist of small perturbations of the normal mode displacements from an equilibrium flight conditions such as steady level flight. The inertial coordinate system from which the disturbed airplane motion is referenced is shown in Figure 1 and is assumed to have its origin at the center of gravity of the airplane in equilibrium flight and orientated along the airplane principle inertial axes. The linear differential equations of motion of the structure based on this inertial coordinate system translating at a constant speed, $V$, equal to the constant airspeed of the airplane, using normal modes as generalized coordinates, are derived from a consideration of the equilibrium of the generalized forces or by use of Lagrange's equations (15,16,18).
Figure 1. Coordinate System for Defining Airplane Motion and Control Surface Rotations
These dynamic equations are written as follows:

\[ M_i \ddot{\xi}_i + C_i \dot{\xi}_i + K_i \xi_i = \Xi_i^M + \Xi_i^D \]  
\[ (i = 1, 2, \ldots, n) \tag{1.2} \]

or

\[ M_i \ddot{\xi}_i + 2\zeta_i \omega_i M_i \dot{\xi}_i + \omega_i^2 M_i \xi_i = \Xi_i^M + \Xi_i^D \]  
\[ (i = 1, 2, \ldots, n) \tag{1.3} \]

where \( M_i = \iiint_V \phi_i(x,y,z)^2 \rho(x,y,z) dV \) is the generalized mass of the \( i \)th normal mode, \( \omega_i \) is the normal mode frequency of the corresponding mode \( \phi_i \) (for the rigid body modes \( \omega_i = 0 \) and for the elastic modes \( \omega_i \neq 0 \)), \( K_i = \omega_i^2 M_i \) is the generalized stiffness of the \( i \)th normal mode, \( C_i = 2\zeta_i \omega_i M_i \) is the damping for small structural damping values \( (\zeta_i \ll 1) \) of the \( i \)th normal mode, and where

\[ \Xi_i^M = \Xi_i^M (\xi_1, \ldots, \xi_n, \dot{\xi}_1, \ldots, \dot{\xi}_n, \ddot{\xi}_1, \ldots, \ddot{\xi}_n) \]
\[ = \int_S \bar{f}^M(x,y,z,t) \phi_i(x,y,z) dS \tag{1.4} \]

is the component of the generalized force due to the disturbed motion from which aerodynamic pressure differences are brought into play, and

\[ \Xi_i^D = \int_S \bar{f}^D(x,y,z,t) \phi_i(x,y,z) dS \tag{1.5} \]

is the component of the generalized force due to the disturbing force. It is noted that there are no cross or coupling terms between modes in the generalized mass, stiffness, or the assumed small
valued damping terms of equation (1.2) or (1.3) because of the orthogonality conditions for normal modes. However, it is evident that the terms introduce aerodynamic coupling among the normal coordinates.

The force due to the disturbed motion \( F^M \) in equation (1.4) represents the distribution of the pressure difference between the two sides of the surface caused by the motion of the air. Thus, in terms of pressures,

\[
F^M = \Delta p^M
\]  

As a consequence of the linearity of the applied forces, this aerodynamic pressure difference in equation (1.6) becomes

\[
\Delta p^M(x,y,z,t) = \sum_{j=1}^{n} \Delta p_j(x,y,z,t)
\]  

Equation (1.4) then becomes

\[
\Xi^M_i = \sum_{j=1}^{n} A_{ij}
\]  

where

\[
A_{ij} = \int \int \int S \Delta p_j(x,y,z,t) \hat{\phi}_i(x,y,z) dS
\]  

denotes the generalized aerodynamic force for the \( i \)th mode due to the resulting aerodynamic pressure of the motion of the \( j \)th mode.

Now in this work, the disturbing force is the result of control surface motion, activated in response to feedback signals from transducers located in the structure, which give rise to aerodynamic and
inertia forces. Thus, the disturbance force in equation (1.5) is

$$ F^D = \sum_{j=1}^{nc} \left[ \hat{S} \delta_j (x,y,z) \dot{\delta}_j (t) + \Delta p_c^j (x,y,z,t) \right] $$

(1.10)

where $\delta_j$ is the control surface rotation for the $j$th control surface, and $\hat{S} \delta_j (x,y,z)$ is the control surface static unbalance function vector about the hinge line for $j$th control surface. Using equation (1.10), equation (1.5) becomes

$$ \Xi_i^D = \sum_{j=1}^{nc} \int_{S} \left[ \hat{S} \delta_j (x,y,z) \dot{\delta}_j (t) + \Delta p_c^j (x,y,z,t) \right] \cdot \phi_i (x,y,z) dS $$

(1.11)

which can be rewritten as

$$ \Xi_i^D = \sum_{j=1}^{nc} M_{ij} \ddot{\delta}_j + \sum_{j=1}^{nc} A_{ij} $$

(1.12)

where

$$ M_{ij} = \int_{S} \hat{S} \delta_j (x,y,z) \cdot \phi_i (x,y,z) dS $$

(1.13)

denotes the generalized inertia force for the $i$th mode due to the motion of the $j$th control surface, and where

$$ A_{ij} = \int_{S} \Delta p_c^j (x,y,z,t) \cdot \phi_i (x,y,z) dS $$

(1.14)

denotes the generalized aerodynamic force for the $i$th mode due to the resulting aerodynamic pressure of the motion of the $j$th control surface.
By use of equations (1.8) and (1.12), the equations of motion (1.3), including the generalized forces contribution from control surface motion, can be written as

\[ M_i \ddot{\xi}_i + 2\zeta_i \omega_i M_i \dot{\xi}_i + \omega_i^2 M_i \xi_i - \sum_{j=1}^{n} A_{ij} = \sum_{j=1}^{n} M_{ij} \ddot{\delta}_j \]

\[ + \sum_{j=1}^{nc} A_{ij}^c \]

(1.15)

(\(i = 1, 2, \ldots, n\))

With the exception of the control system generalized forces on the right hand side, the above equations of motion are the same as those used in classical dynamic stability and flutter analysis for an unaugmented airplane.

When working with control systems, it is very useful to transform the equations of motion to the Laplace domain. Apply the Laplace transformation to the linear differential equations of (1.15) to get

\[ M_i s^2 \ddot{\xi}_i(s) + 2\zeta_i \omega_i M_i s \dot{\xi}_i(s) + \omega_i^2 M_i \xi_i(s) - \sum_{j=1}^{n} A_{ij}(s) \ddot{\xi}_j(s) \]

\[ = \sum_{j=1}^{nc} M_{ij} s^2 \ddot{\delta}_j(s) + \sum_{j=1}^{nc} A_{ij}^c(s) \ddot{\delta}_j(s) \]

(1.16)

(\(i = 1, 2, \ldots, n\))

where \(\ddot{\xi}_i(s) = \mathcal{L}[\ddot{\xi}_i(t)]\), \(\ddot{\delta}_j(s) = \mathcal{L}[\ddot{\delta}_j(t)]\), and where \(\ddot{\delta}_j(s)\) is the transform of the generalized aerodynamic force \(A_{ij}\) per unit normal coordinate \(\ddot{\xi}_i(s)\) and \(\ddot{\delta}_j(s)\) is the transform of the generalized aerodynamic force \(A_{ij}^c\) per unit control surface coordinate \(\ddot{\delta}_j(s)\).
The development of these generalized unsteady aerodynamic forces in the Laplace domain will be given in Chapter II. In general, these $\tilde{A}_{ij}(s)$ and $\tilde{A}_{ij}^c(s)$ are derived for a given flight condition (i.e., for a particular Mach number and altitude), will be applicable for all frequencies from zero to a maximum fit frequency, and will be most valid for Laplace arguments near the imaginary axis (small positive or negative damping) which is the region of the greatest physical interest for stability.

Equation (1.16) now represents a set of $n$ simultaneous linear algebraic equations in the unknown transformed normal coordinates $\bar{\xi}_1(s), \ldots, \bar{\xi}_n(s)$. Using the notation for damping and stiffness in equation (1.2) and putting the system of equations (1.16) into matrix form gives

$$\begin{bmatrix} s^2[M] + s[C] + [K] - [\tilde{A}(s)] \end{bmatrix} \{\bar{\xi}(s)\} = \begin{bmatrix} s^2[M^c] \\
+ [\tilde{A}^c(s)] \end{bmatrix} \{\bar{\delta}(s)\}$$

(1.17)

where $\{\bar{\xi}(s)\}$ is the $(nx1)$ column matrix of the transformed normal mode coordinates, $[s^2[M^c] + [\tilde{A}^c(s)]]$ is the $(nxnc)$ matrix of the inertia and aerodynamic forces produced by control surface motion, and $\{\bar{\delta}(s)\}$ is the $(ncx1)$ column matrix of the transformed control surface coordinates.

**Feedback and Sensor Matrix Equations**

A typical airplane control system feedback loop is shown in Figure 2. Control surfaces, deflected in response to sensor signals from the airframe response, produce aerodynamic and inertia forces
Figure 2. Typical Control System Feedback Block Diagram
which influence the dynamic stability and structural response of the airframe. In accordance with the block diagram of Figure 2, the relation between the servo induced control surface rotations and the airframe motion sensed by the sensors is given by the feedback matrix equation in the usual Laplace domain as

\[
\{\tilde{\delta}(s)\} = \begin{bmatrix} F(s)_{\text{servo}} & F(s)_{\text{CAS}} & F(s)_{\text{sensors}} \end{bmatrix} \{\tilde{q}_s(s)\} \tag{1.18}
\]

or

\[
\{\tilde{\delta}(s)\} = [G(s)]\{\tilde{q}_s(s)\} \tag{1.19}
\]

where \([G(s)] = \begin{bmatrix} F(s)_{\text{servo}} & F(s)_{\text{CAS}} & F(s)_{\text{sensors}} \end{bmatrix}\) is the \((\text{nc} \times \text{ns})\) matrix of transfer functions between the control surface rotations and the airframe motion at the sensor locations, and \(\{\tilde{q}_s(s)\}\) is the array of transformed airframe motions sensed by the sensors.

The motions sensed by the sensors can be expressed in terms of a linear combination of the products of the mode shape components sensed by the sensors at the sensor locations and the corresponding normal mode coordinates. These sensor equations are

\[
q_{s_i}(t) = \sum_{j=1}^{\text{ns}} \phi_{s_j}(x_i, y_i, z_i) \xi_j(t) \tag{1.20}
\]

where \(\text{ns}\) is the number of sensors, \(q_{s_i}(t)\) is the motion sensed by the \(i\) \text{th} sensor, \(\phi_{s_j}(x_i, y_i, z_i)\) is the mode shape component of the \(j\) \text{th} mode sensed by the \(i\) \text{th} sensor at location \((x_i, y_i, z_i)\), and \(\xi_j(t)\) is the \(j\) \text{th} normal mode coordinate (or one of its time
derivatives, e.g., for a rate gyro or an accelerometer). This system of sensor equations (1.20) when transformed to the Laplace domain and put into matrix form becomes

\[ \{\ddot{q}_s(s)\} = [\ddot{\phi}_s(s)]\{\ddot{\xi}(s)\} \] (1.21)

where \([\ddot{\phi}_s(s)]\) is the (nsxn) rectangular matrix of which the \(i^{th}\) column \((i = 1, \ldots, n)\) represents the ns components of the \(i^{th}\) mode sensed by the various sensors, and \(\{\ddot{\xi}(s)\}\) are the Laplace transformed normal mode coordinates. The \([\ddot{\phi}_s(s)]\) matrix is, in general, a function of the Laplace variable \(s\). This is because the \(\ddot{\xi}_i(t)\) coordinates in equation (1.20) can be time derivatives depending on the type of sensor (e.g., rate gyro or accelerometer) in which case the Laplace transform of these time derivatives will have powers of \(s\) premultiplying the transformed normal coordinates. These powers of \(s\) can be combined into the appropriate rows of \([\ddot{\phi}_s]\) to make \([\ddot{\phi}_s(s)]\).

For example, for the case of a symmetrical airplane (in mass, stiffness, and geometry about the xz plane), the control surface motions affecting the antisymmetric response of the airplane would be the rudder, aileron, and differential horizontal tail rotation. If the sensors are assumed to be a roll rate gyro, a yaw rate gyro, and a lateral accelerometer, then the feedback matrix equation of (1.19) becomes

\[
\begin{bmatrix}
\delta_A(s) \\
\delta_H(s) \\
\delta_R(s)
\end{bmatrix} = \begin{bmatrix}
\delta_A/\phi_G(s) & \delta_A/\phi_G(s) & \delta_A/Ny(s) \\
\delta_H/\phi_G(s) & \delta_H/\phi_G(s) & \delta_H/Ny(s) \\
\delta_R/\phi_G(s) & \delta_R/\phi_G(s) & \delta_R/Ny(s)
\end{bmatrix} \begin{bmatrix}
\ddot{\phi}_G(s) \\
\ddot{\psi}(s) \\
\ddot{Ny}(s)
\end{bmatrix}
\]
In addition, the corresponding sensor equations (1.20) for the airplane rigid body modes of roll, yaw, lateral translation, and the first \( nm \) antisymmetric normal modes of vibration, would be

\[
\dot{\phi}_G = \dot{\phi} + \sigma_{\phi G}^{(4)} \dot{\xi}_4 + \sigma_{\phi G}^{(5)} \dot{\xi}_5 + \ldots + \sigma_{\phi G}^{(n)} \dot{\xi}_n
\]

\[
\dot{\psi}_G = \dot{\psi} + \sigma_{\psi G}^{(4)} \dot{\xi}_4 + \sigma_{\psi G}^{(5)} \dot{\xi}_5 + \ldots + \sigma_{\psi G}^{(n)} \dot{\xi}_n
\]

\[
Ny = -\dot{z}_{Ny} \dot{\phi} + \dot{z}_{Ny} \dot{\psi} + \ddot{\eta} + \phi_{Ny} \dot{\xi}_4 + \phi_{Ny} \dot{\xi}_5 + \ldots + \phi_{Ny} \dot{\xi}_n
\]

where

\( n = 3 + nm \) is the total number of normal modes (rigid body plus elastic modes)

\( \sigma_{\phi G}^{(i)} \) = slope of the \( i \) th normal vibration mode at the roll rate gyro location for \( i = 4, 5, \ldots, n \)

\( \sigma_{\psi G}^{(i)} \) = slope of the \( i \) th normal vibration mode at the yaw rate gyro location for \( i = 4, 5, \ldots, n \)

\( \phi_{Ny}^{(i)} \) = lateral displacement component of the \( i \) th normal vibration mode at lateral accelerometer location for \( i = 4, 5, \ldots, n \)

\( \dot{\phi}_G \) = roll rate sensed by the roll rate gyro

\( \dot{\psi}_G \) = yaw rate sensed by the yaw rate gyro

\( Ny \) = lateral acceleration sensed by lateral accelerometer

\( \xi_1 \) = \( \phi \) is the normal coordinate of rigid body airplane roll about its c.g.

\( \xi_2 \) = \( \psi \) is the normal coordinate of rigid body airplane yaw about its c.g.
\[ \xi_3 = n \text{ is the normal coordinate of rigid body lateral translation in } y \text{ direction} \]

\[ \xi_{xNY} = \text{vertical distance from airplane c.g. to lateral accelerometer (+ up)} \]

\[ \xi_{xNY} = \text{fore-and-aft distance from airplane c.g. to lateral accelerometer (+ aft)} \]

\[ \xi_i = \text{normal coordinate of the } i\text{th normal mode of the airplane} \]

The above sensor equations when transformed into the Laplace domain and put into matrix form, become the sensor matrix equation of (1.21) as follows:

\[
\begin{bmatrix}
\ddot{\phi}_G(s) \\
\ddot{\psi}_G(s) \\
\ddot{\gamma}_y(s)
\end{bmatrix}
= 
\begin{bmatrix}
s & 0 & 0 \\
0 & s & 0 \\
-s^2 \xi_{xNY} & s^2 \xi_{xNY} & s^2
\end{bmatrix}
\begin{bmatrix}
\cdot \\
\cdot \\
\cdot
\end{bmatrix}
= 
\begin{bmatrix}
s_\Phi(4) & \ldots & s_\Phi(n) \\
s_\Psi(4) & \ldots & s_\Psi(n) \\
0 & & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{\phi} \\
\ddot{\psi} \\
\ddot{\gamma}_y
\end{bmatrix}
= 
\begin{bmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2 \\
\dot{\xi}_3 \\
\dot{\xi}_4 \\
\vdots \\
\dot{\xi}_n
\end{bmatrix}
\]

A corresponding set of feedback and sensor matrix equations can be written for the control surface motions affecting the symmetric response of the airplane. For example, the following could be considered typical: control surface motion - elevator rotation or horizontal tail pitch; sensors - pitch rate gyro and vertical accelerometer; and airplane normal modes - rigid body modes of pitch and vertical translation, and the first nm symmetric normal modes of vibration.
Aeroservoelastic Stability Matrix Equation

By combining the general form of the feedback equation (1.19) and the sensor equation (1.21), the control surface coordinates \( \{\delta(s)\} \) can be expressed in terms of the original normal coordinates \( \{\xi(s)\} \). Doing this and substituting the result into equation (1.17) gives the matrix equation

\[
\begin{bmatrix}
    s^2[M] + s[C] + [K] - [\tilde{A}(s)]
\end{bmatrix}
\{\xi(s)\}
= \begin{bmatrix}
    s^2[M_c] + [\tilde{A}_c(s)]
\end{bmatrix}
\begin{bmatrix}
    G(s)\[\dot{\phi}_s(s)\]
\end{bmatrix}
\{\xi(s)\}
\]  
(1.24)

or

\[
\begin{bmatrix}
    s^2[M] + s[C] + [K] - [\tilde{A}(s)] - [s^2[M_c] + [\tilde{A}_c(s)]]
\end{bmatrix}
\begin{bmatrix}
    G(s)\[\dot{\phi}_s(s)\]
\end{bmatrix}
\{\xi(s)\} = 0
\]  
(1.25)

If the right hand side of equation (1.24) is set to zero, the equations of motion reduce to those for the unaugmented airplane. The right hand side of equation (1.24) represents the effects of the control augmentation system on these equations of motion. Finally, the equation (1.25) represents the system of \( n \) equations of motion in normal mode coordinates of the airplane including the control augmentation system. For the assumed normal mode development, \([M], [C], \) and \([K]\) are diagonal matrices, but it is seen from equation (1.25) that the normal mode coordinates are coupled not only through the aerodynamic matrix \([\tilde{A}(s)]\) but also through the control augmentation system matrix \([s^2[M_c] + [\tilde{A}_c(s)]]\). 

The (nxn) coefficient matrix of equation (1.25) is set up for a given flight condition and a given set of control system feedback
loop gains. For nonzero solutions for the normal coordinates \(\{\tilde{\xi}\}\) in equation (1.25), the determinant formed by the coefficient matrix elements, which are polynomials in \(s\), must be equal to zero. Thus, the determinant of the \((nxn)\) coefficient matrix of equation (1.25) forms the characteristic equation of which the roots determine the stability of the airplane with the control augmentation system.

Instead of solving the stability determinant by direct expansion into the characteristic equation, which is a large polynomial in \(s\), and extracting the roots, a more accurate approach was taken as discussed in Appendix A. Briefly, this approach, also used in references 8 and 9, involves setting up equation (1.25) in terms of a large order eigenvalue problem and solving for the roots or eigenvalues, \(s\), by use of the QR transform method (19).

The following stability solution procedure is then used. The \((nxn)\) coefficient matrix of equation (1.25), of which the elements are polynomials in \(s\), is set up for a given flight condition and a given set of control feedback loop gains. The eigenvalue problem is formed, and the complex roots or eigenvalues are obtained. The above procedure is repeated for selected gain values of a given control feedback loop. The complex roots are then plotted on a root locus plot as a function of this gain variation. The feedback gain at which the real part of the root becomes zero is the gain that predicts that the augmented airplane system will become unstable. In general, the roots are complex, and in the notation of reference 10 are expressed as

\[
s = \sigma \pm i\omega_d = -\xi\omega_n \pm i\omega_n \sqrt{1 - \zeta^2}
\]

(1.26)
where $\sigma$ is the real part of the root, $\omega_d$ is the imaginary part of the root and is called the damped frequency of the root, $\omega_n$ is the undamped natural frequency of the root, and $\zeta$ is the damping coefficient associated with the root. Figure 3 shows these relations graphically, where the damping, $\zeta$, associated with each root is then given by

$$\zeta = \frac{-\sigma}{\omega_n} = \frac{-\sigma}{\sqrt{\sigma^2 + \omega_d^2}}$$

It is noted that the preceding equations and approach developed in this chapter could be extended to response analysis for various inputs. For example, if the servo induced control surface rotations, $\{\delta\}$, in equation (1.17) were replaced with the sum of servo induced and pilot induced control surface rotations, $\{\delta_s.I.\} + \{\delta_p.I.\}$, then the response due to a given pilot input could be set up and the solution obtained.
Figure 3. Typical Complex Root on a Root Locus Plot
Incompressible Strip Theory Formulation

To use the stability equations as developed in Chapter I, the unsteady aerodynamic forces must be formulated in the Laplace domain. This development is started by considering the two-dimensional, strip theory representations of a lifting surface with a control surface as shown in Figure 4. Assuming small disturbances, Theodorsen in reference 20 determined the forces and moments on an oscillating airfoil strip and control surface moving in an incompressible fluid. These forces and moments consist of the unsteady lift per unit span $L'$, the unsteady torsional moment about the reference axis per unit span $M'$, and the unsteady moment about the control surface hinge line per unit span $T'$. These expressions are

\[
L'(t) = \int_{-b}^{b} (p_L - p_u)dx = \int_{-b}^{b} \Delta p(x,t)dx
\]

\[
= \pi \rho b^3 \left\{ -\frac{h}{b} - \frac{2V}{b} C(k) \frac{h}{b} + a \ddot{\alpha} + \left[ 2(a - \frac{1}{2})C(k) - 1 \right] \frac{V}{b} \dot{\alpha} \right. \\
\left. - 2 \frac{V^2}{b^2} C(k) \alpha + \frac{T_1}{\pi} \ddot{\beta} + \left[ \frac{T_4}{\pi} - \frac{T_{11}}{\pi} C(k) \right] \frac{V}{b} \dot{\beta} \\
- \frac{2T_{10}}{\pi} C(k) \frac{V^2}{b^2} \dot{\beta} \right\}
\]

(2.1)
Figure 4. Geometry and Deflections for the Mean Line of a Chordwise-Rigid Airfoil with Trailing Edge Control Surface
\[ M'(t) = \pi \rho b^4 \left\{ \frac{a}{b} \ddot{h} + \frac{2(\frac{l_3}{2} + a)}{b} C(k) \frac{V}{b} \dot{h} - \left( \frac{1}{8} + a^2 \right) \ddot{a} \right. \\
+ \left[ a - \frac{1}{2} + 2(\frac{l_4}{2} - a^2) C(k) \right] \frac{V}{b} \dot{\alpha} + 2(\frac{l_2}{2} + a) C(k) \frac{V^2}{b^2} \alpha \\
+ \left[ \frac{T_7 + (c - a)T_1}{\pi} \right] \ddot{\beta} + \left[ \frac{T_1 - T_8 - (c - a) T_4 + \frac{1}{2}T_{11}}{\pi} \right] \dot{\beta} \\
+ (\frac{1}{2} + a) \frac{T_{11}}{\pi} C(k) \frac{V}{b} \ddot{\beta} + \left[ -\frac{T_4 + T_{10}}{\pi} \right] \dot{\beta} \\
+ 2(\frac{1}{2} + a) \frac{T_{10}}{\pi} C(k) \frac{V^2}{b^2} \beta \right\} \\
(2.2) \]

\[ T'(t) = \pi \rho b^4 \left\{ \frac{T_1}{\pi} \frac{\ddot{h}}{b} - \frac{T_{12}}{\pi} C(k) \frac{V}{b^2} \dot{h} + \left[ \frac{T_7 + (c - a)T_1}{\pi} \right] \ddot{h} \\
+ \left[ \frac{2T_9 + T_1 - (a - \frac{1}{2})T_4}{\pi} + (a - \frac{1}{2}) \frac{T_{12}}{\pi} C(k) \right] \frac{V}{b} \dot{\alpha} \\
- \frac{T_{12}}{\pi} C(k) \frac{V^2}{b^2} \alpha + \left[ \frac{T_4 T_{11}}{2\pi^2} + \frac{T_{11} T_{12}}{2\pi^2} C(k) \right] \frac{V}{b} \ddot{\beta} \\
+ \left[ -\frac{T_5 - T_4 T_{10}}{\pi^2} - \frac{T_{10} T_{12}}{\pi^2} C(k) \right] \frac{V^2}{b^2} \beta \right\} \]

where the \( T \) functions are functions of airfoil geometry, particularly \( c \) and \( a \); \( h \) is the vertical displacement of the reference axis, \( \alpha \) is the rotation of the unit span about the reference axis, \( \beta \) is the relative rotation of the attached control surface, and \( C(k) \) is the Theodorsen circulation function. \( C(k) \) is a complex value which is a function of the reduced frequency, \( k = \frac{b \omega}{V} \), and brings about a phase lag between the generated circulatory airloads and the instantaneous
velocity and effective angle of attack of the airfoil strip and control surface. Since the amount of energy extracted from or added to an oscillatory motion by the airstream is very sensitive to certain of these phase angles, it can be seen how important it is to include effects of unsteady flow in the prediction of flutter and aeroservoelasticity instabilities, even though the reduced frequency may be small.

Formulation with Approximate Theodorsen Function

As formulated for use on the electronic direct analog computer (21) to solve aeroelastic response problems, $C(k)$ can be replaced by $C(bp/V)$, where $p$ is the operator $d/dt$. In operator form, equations (2.1), (2.2), and (2.3) then become

$$L' = \pi \rho b^3 \left\{ - \frac{p^2 h}{b} - \frac{2v}{b} C\left(\frac{bp}{V}\right) \frac{ph}{b} + ap^2 \alpha ight\}$$

$$+ \left[ 2(a - \frac{1}{2}) C\left(\frac{bp}{V}\right) - 1 \right] \frac{V}{b} p\alpha - \frac{2v^2}{b^2} C\left(\frac{bp}{V}\right) \frac{a}{\alpha}$$

$$+ \frac{T_1}{\pi} p^2 \beta + \left[ \frac{T_4}{\pi} - \frac{T_11}{\pi} C\left(\frac{bp}{V}\right) \right] \frac{V}{b} \phi p - 2 \frac{T_{10}}{\pi} C\left(\frac{bp}{V}\right) \frac{v^2}{b^2} \beta \right\} (2.4)$$

$$M' = \pi \rho b^4 \left\{ \frac{a}{b} \frac{p^2 h}{b} + \frac{2(\frac{1}{2} + a)}{b} C\left(\frac{bp}{V}\right) \frac{V}{b} ph - \left( \frac{1}{8} + a^2 \right) p^2 \alpha ight\}$$

$$+ \left[ a - \frac{1}{2} + 2(\frac{1}{2} - a^2) C\left(\frac{bp}{V}\right) \right] \frac{V}{b} p\alpha + 2(\frac{1}{2} + a) C\left(\frac{bp}{V}\right) \frac{v^2}{b^2} \alpha$$

$$+ \left[ \frac{T_7}{\pi} + (c - a) \frac{T_1}{\pi} \right] p^2 \beta + \left[ \frac{T_1 - T_8 - (c - a) T_4 - \frac{1}{2} T_{11}}{\pi} \right]$$

$$+ (\frac{1}{2} + a) \frac{T_{11}}{\pi} C\left(\frac{bp}{V}\right) \frac{V}{b} \phi p + \left[ - \frac{T_4 + T_{10}}{\pi} \right]$$

$$+ 2(\frac{1}{2} + a) \frac{T_{10}}{\pi} C\left(\frac{bp}{V}\right) \frac{v^2}{b^2} \beta \right\} (2.5)$$
MacNeal in reference 21 indicates that Theodorsen's function $C\left(\frac{bp}{V}\right)$ may be accurately represented by an expression of the following type:

\[
C\left(\frac{bp}{V}\right) = \frac{K_1 \left(\frac{b}{V}\right) p + 1}{K_3 \left(\frac{b}{V}\right) p + 1} \frac{K_2 \left(\frac{b}{V}\right) p + 1}{K_4 \left(\frac{b}{V}\right) p + 1}
\]

(2.7)

This is specifically valid for sinusoidal motion where $p = i\omega$ and for the set of values: $K_1 = 10.61$, $K_2 = 1.774$, $K_3 = 13.51$, and $K_4 = 2.744$.

In this case equation (2.7) agrees within a few percent of the exact formulation of Theodorsen's function $C(k)$. Figure 5 shows this comparison.

Now for a given flight condition at which the stability of system is to be investigated, the velocity $V$ and the air density $\rho$ are fixed. So using equation (2.7) for $C\left(\frac{bp}{V}\right)$ and transforming equations (2.4), (2.5), (2.6), and (2.7) into the Laplace domain, the following general form can be obtained for the sectional lift and moments:
Figure 5. A Complex Plot of the Exact Theodorsen Function $C(k)$ and the Approximate $C(k)$ Function

---

$\text{IMAG} = 0.4$

$\text{REAL} = 1.0$

$k = 1.0$, $k = 0.5$, $k = 0.2$, $k = 0.1$, $k = 0.05$, $k = 0.025$

EXACT $C(k)$

APPROXIMATE $C(k)$

OF EQ. (2.7) FOR $K_1 = 10.61$

$K_2 = 1.774$

$K_3 = 13.51$

$K_4 = 2.744$
\[ \ddot{\beta}'(s) = q_{\infty} A_{R \alpha} c_{R} \left\{ \left[ a_{1} h s^2 + a_{2} h s c \left( \frac{b s}{V} \right) \right] \ddot{h}(s) 
+ \left[ a_{1} \alpha s^2 + a_{2} \alpha s + a_{3} \alpha s c \left( \frac{b s}{V} \right) + a_{4} \alpha c \left( \frac{b s}{V} \right) \right] \ddot{a}(s) 
+ \left[ a_{1} \beta s^2 + a_{2} \beta s + a_{3} \beta s c \left( \frac{b s}{V} \right) + a_{4} \beta c \left( \frac{b s}{V} \right) \right] \ddot{\beta}(s) \right\} \] (2.8)

\[ \ddot{\gamma}'(s) = q_{\infty} A_{R \alpha} c_{R} \left\{ \left[ a_{1} \mu s^2 + a_{2} \mu s c \left( \frac{b s}{V} \right) \right] \ddot{h}(s) 
+ \left[ a_{1} \mu s^2 + a_{2} \mu s + a_{3} \mu s c \left( \frac{b s}{V} \right) + a_{4} \mu c \left( \frac{b s}{V} \right) \right] \ddot{a}(s) 
+ \left[ a_{1} \mu s^2 + a_{2} \mu s + a_{3} \mu s c \left( \frac{b s}{V} \right) + a_{4} \mu c \left( \frac{b s}{V} \right) \right] \ddot{\gamma}(s) \right\} \] (2.9)

\[ \ddot{\theta}'(s) = q_{\infty} A_{R \alpha} c_{R} \left\{ \left[ a_{1} \theta s^2 + a_{2} \theta s c \left( \frac{b s}{V} \right) \right] \ddot{h}(s) 
+ \left[ a_{1} \theta s^2 + a_{2} \theta s + a_{3} \theta s c \left( \frac{b s}{V} \right) + a_{4} \theta c \left( \frac{b s}{V} \right) \right] \ddot{a}(s) 
+ \left[ a_{1} \theta s^2 + a_{2} \theta s + a_{3} \theta s c \left( \frac{b s}{V} \right) + a_{4} \theta c \left( \frac{b s}{V} \right) \right] \ddot{\theta}(s) \right\} \] (2.10)

where

\[ C \left( \frac{b s}{V} \right) = \frac{(c_{1} s + 1)(c_{2} s + 1)}{(d_{1} s + 1)(d_{2} s + 1)} \] (2.11)

and where, for a given flight condition, the coefficients \( a_{ijkl} \)'s and \( c_{1}, c_{2}, d_{1}, d_{2} \) are constants, \( q_{\infty} \) is the free stream dynamic pressure,
\( A_R \) is a reference area, and \( c_R \) is a reference chord. Using equation (2.11) for \( C_{\text{fs}} \), the airfoil sectional force and moments of equations (2.8), (2.9), and (2.10) can be rewritten in the following general form:

\[
\begin{align*}
\tilde{L}'(s) &= q_\infty A_R c_R \left\{ \tilde{L}_h(s) \tilde{h}(s) + \tilde{L}_\alpha(s) \tilde{\alpha}(s) + \tilde{L}_\beta(s) \tilde{\beta}(s) \right\} \\
\tilde{M}'(s) &= q_\infty A_R c_R \left\{ \tilde{M}_h(s) \tilde{h}(s) + \tilde{M}_\alpha(s) \tilde{\alpha}(s) + \tilde{M}_\beta(s) \tilde{\beta}(s) \right\} \\
\tilde{T}'(s) &= q_\infty A_R c_R \left\{ \tilde{T}_h(s) \tilde{h}(s) + \tilde{T}_\alpha(s) \tilde{\alpha}(s) + \tilde{T}_\beta(s) \tilde{\beta}(s) \right\} 
\end{align*}
\]

(2.12)

where

\[
\begin{align*}
\tilde{L}_h(s) &= \frac{a'_{4L} s^4 + a'_{3L} s^3 + a'_{2L} s^2 + a'_{1L} s}{(s + b'_{1})(s + b'_{2})} \\
\tilde{L}_\alpha(s) &= \frac{a'_{4L} s^4 + a'_{3L} s^3 + a'_{2L} s^2 + a'_{1L} s + a'_{0L}}{(s + b'_{1})(s + b'_{2})} \\
\tilde{L}_\beta(s) &= \frac{a'_{4L} s^4 + a'_{3L} s^3 + a'_{2L} s^2 + a'_{1L} s + a'_{0L}}{(s + b'_{1})(s + b'_{2})} \\
\tilde{M}_h(s) &= \frac{a'_{4M} s^4 + a'_{3M} s^3 + a'_{2M} s^2 + a'_{1M} s}{(s + b'_{1})(s + b'_{2})} \\
\tilde{M}_\alpha(s) &= \frac{a'_{4M} s^4 + a'_{3M} s^3 + a'_{2M} s^2 + a'_{1M} s + a'_{0M}}{(s + b'_{1})(s + b'_{2})} \\
\tilde{M}_\beta(s) &= \frac{a'_{4M} s^4 + a'_{3M} s^3 + a'_{2M} s^2 + a'_{1M} s + a'_{0M}}{(s + b'_{1})(s + b'_{2})} 
\end{align*}
\]
and where the coefficients $a_{ijk}$'s, $b'_1$, $b'_2$ are constants for a given flight velocity $V$ and a given unit span airfoil geometry.

As developed in Chapter I, it is assumed that the displacements and rotations of the airfoil strip consist of a superposition of $n$ normal modes of the unrestrained airplane.

\[
\begin{align*}
\mathbf{h}(y,t) &= \sum_{i=1}^{n} \phi_{h_i}(y) \xi_i(t) \\
\mathbf{\alpha}(y,t) &= \sum_{i=1}^{n} \phi_{\alpha_i}(y) \xi_i(t) \\
\mathbf{\beta}(y,t) &= \sum_{i=1}^{n} \phi_{\beta_i}(y) \xi_i(t)
\end{align*}
\]

The quantities $\phi_{h_i}(y)$, $\phi_{\alpha_i}(y)$, $\phi_{\beta_i}(y)$ are the spanwise vertical deflection, airfoil rotation, and aileron rotation components, respectively, of the $i$th normal mode shape of the airfoil, and $\xi_i(t)$ is the $i$th normal mode coordinate.

From the normal mode development of the equations of motion in Chapter I, the generalized aerodynamic force $A_{ij}$ for the $i$th mode
due to the motion in the $j$th mode is

$$A_{ij} = \int_S \Delta \vec{p}_j(x,y,z,t) \cdot \vec{\phi}_i(x,y,z) dS$$  \hspace{1cm} (1.9)

Therefore, for the strip theory representation of an airfoil, the normal mode shape $\vec{\phi}_i$ for the $i$th mode will, in general, consist of the components $\phi_{h_i}(y)$, $\phi_{\alpha_i}(y)$, and $\phi_{\beta_i}(y)$ of equations (2.14) and the pressure vector $\Delta \vec{p}_j$ consists of the resulting sectional lift and moment components of $L_j'(y,t)$, $M_j(y,t)$, and $T_j'(y,t)$ due to motion in $h_j$, $\alpha_j$, and $\beta_j$ of the $j$th normal mode. Using this strip representation in equation (1.9), the generalized aerodynamic force $A_{ij}$ over the entire span $l$ of the airfoil is given by

$$A_{ij} = \int_0^l \left[ L_j'(y,t) \phi_{h_i}(y) + M_j'(y,t) \phi_{\alpha_i}(y) + T_j'(y,t) \phi_{\beta_i}(y) \right] dy \hspace{1cm} (2.15)$$

Now transforming equation (2.15) to the Laplace domain by use of the notation of (1.16), results in

$$\tilde{A}_{ij}(s) \tilde{\xi}_i(s) = \mathcal{L}[A_{ij}]$$

$$= \int_0^l \left[ \tilde{L}_j'(y,s) \phi_{h_i}(y) + \tilde{M}_j'(y,s) \phi_{\alpha_i}(y) + \tilde{T}_j'(y,s) \phi_{\beta_i}(y) \right] dy \hspace{1cm} (2.16)$$
Substituting equations (2.12) into the equation (2.16) above, gives

\[
\tilde{A}_{ij}(s)\tilde{\xi}_j(s) = q_{\infty}A_Rc_R \int_0^L \left[ \phi_{h_i}(y) \left\{ \tilde{\mathcal{L}}'_h(y,s)\tilde{h}_j(y,s) + \tilde{\mathcal{L}}'_\alpha(y,s)\tilde{a}_j(y,s) \\
+ \tilde{\mathcal{L}}'_\beta(y,s)\tilde{\alpha}_j(y,s) \right\} + \phi_{\alpha_j}(y) \left\{ \tilde{\mathcal{M}}'_h(y,s)\tilde{h}_j(y,s) \\
+ \tilde{\mathcal{M}}'_\alpha(y,s)\tilde{a}_j(y,s) + \tilde{\mathcal{M}}'_\beta(y,s)\tilde{\alpha}_j(y,s) \right\} \right] dy
\]

(2.17)

Transforming the j th mode components of equations (2.14) results in

\[
\tilde{h}_j(y,s) = \phi_{h_j}(y)\tilde{\xi}_j(s) \\
\tilde{a}_j(y,s) = \phi_{\alpha_j}(y)\tilde{\xi}_j(s) \\
\tilde{\alpha}_j(y,s) = \phi_{\beta_j}(y)\tilde{\xi}_j(s)
\]

Substituting these results in equation (2.17) gives

\[
\tilde{A}_{ij}(s)\tilde{\xi}_j(s) = q_{\infty}A_Rc_R \int_0^L \left[ \phi_{h_i}(y) \left\{ \tilde{\mathcal{L}}'_h(y,s)\phi_{h_j}(y) + \tilde{\mathcal{L}}'_\alpha(y,s)\phi_{\alpha_j}(y) \\
+ \tilde{\mathcal{L}}'_\beta(y,s)\phi_{\beta_j}(y) \right\} \right] dy\tilde{\xi}_j(s)
\]

(2.18)
Substituting equations (2.13) for the airfoil sectional lift and moments into equation (2.18) above, and equating coefficients of $\xi_j(s)$ on both sides, the general form for the transformed generalized aerodynamic force $\tilde{A}_{ij}$ is obtained as follows:

$$\tilde{A}_{ij}(s) = q_w A_R c_R \int_0^\infty \left[ \frac{a'_{ij}(y)s^4 + a^3_{ij}(y)s^3 + a^2_{2ij}(y)s^2 + a_{1ij}(y)s + a'_{ij}(y)}{(s + b'_1(y))(s + b'_2(y))} \right] dy$$

(2.19)

Now for the case of a constant chord airfoil along the span where $b'_1$ and $b'_2$ are constants and not functions of the span, equation (2.19) is of the form

$$\tilde{A}_{ij}(s) = q_w A_R c_R \left[ \frac{a^4_{ij}s^4 + a^3_{ij}s^3 + a^2_{2ij}s^2 + a_{1ij}s + a_{0ij}}{(s + b_1)(s + b_2)} \right]$$

(2.20)

where $b_1 = b'_1$ and $b_2 = b'_2$.

For the special case of pressure loads resulting from motion in a $j$th normal mode which involves only translational displacement $\phi_{hj}(y)$ with no effective angle of attack displacements $\phi_{aj}(y)$ or $\phi_{bj}(y)$, equation (2.18) along with Laplace expressions (2.13) for $L'_h$, $M'_h$, and $T'_h$ reduces to a subcase of equation (2.20). This special case for $\tilde{A}_{ij}(s)$, when $\phi_{hj}(y) \neq 0$ and $\phi_{aj}(y) = \phi_{bj}(y) = 0$, is

$$\tilde{A}_{ij}(s) = q_w A_R c_R \left[ \frac{a^4_{ij}s^4 + a^3_{ij}s^3 + a^2_{2ij}s^2 + a_{1ij}s}{(s + b_1)(s + b_2)} \right]$$

(2.21)

The form of this equation for $\tilde{A}_{ij}(s)$ will be used for $j$th pressure modes such as the normal modes of rigid airplane vertical translation,
lateral translation, and roll which involve only airfoil translation and no effective angle of attack displacements for the zero frequency case. For the zero frequency case where \( s = i\omega \), equation (2.21) will give zero for the generalized aerodynamic force which is what is physically desired for no effective angle of attack. If, however, the form of equations (2.20) is used instead of equation (2.21) for such pressure modes, extraneous roots with positive real parts can result in the subsequent root locus stability analysis. Such a problem with positive real roots did result in the root locus analysis of reference 9 which gave a false indication of instability.

In a similar manner as done in deriving the form of \( \tilde{A}_{ij}(s) \) in equation (2.20), the form of \( \tilde{A}^C_{ij}(s) \), the transformed generalized aerodynamic force for the \( i \) th mode due to motion of the \( j \) th control surface per unit \( j \) th control surface coordinate, is now considered. This is easily done by starting with equation (2.16) and replacing \( \tilde{A}_{ij}(s) \) with \( \tilde{A}^C_{ij}(s) \) and \( \tilde{\xi}_j(s) \) with \( \tilde{\xi}_j(s) \). In addition, the following holds: \( \tilde{h}_j(y, s) = \tilde{\alpha}_j(y, s) = 0 \) and \( \tilde{\beta}_j(s) = (1)\tilde{\xi}_j(s) \) for a \( j \) th pressure mode involving only pure rotation of the \( j \) th control surface. The resulting corresponding form of equation (2.20) for \( \tilde{A}^C_{ij}(s) \) reduces to

\[
\tilde{A}^C_{ij}(s) = q_\infty A^C_{R}\left[ \frac{a_{4ij}s^4 + a_{3ij}s^3 + a_{2ij}s^2 + a_{1ij}s + a_{0ij}}{(s + b_1)(s + b_2)} \right] 
\]  
(2.22)

which is the same form as equation (2.20).
For the general case where the chord of the airfoil is a function of the span, equations (2.20), (2.21), and (2.22) are not exact expressions for the transformed generalized aerodynamic forces $\tilde{A}_{ij}(s)$ and $\tilde{A}^C_{ij}(s)$. However, an accurate approximation for $\tilde{A}_{ij}(s)$ and $\tilde{A}^C_{ij}(s)$ can be provided by first fixing $b_1$ and $b_2$ based on a given flight velocity $V$ and reference or average semichord $b_R$ and by then determining the constants $a_{0ij}, a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}$ by fitting a series of calculated $\tilde{A}_{ij}(s)$ or $\tilde{A}^C_{ij}(s)$ for $s = i\omega$ at selected frequencies covering the desired frequency range of normal modes to the Laplace polynomial fraction form of equations (2.20), (2.21), and (2.22). This fitting procedure will be discussed in detail in subsequent paragraphs.

Extension to Doublet-Lattice Method

In the preceding development for a given flight condition, the form of equation (2.20) or the special case of (2.22), and equation (2.21) have been shown to provide an accurate representation of the generalized aerodynamic forces $\tilde{A}_{ij}(s)$ and $\tilde{A}^C_{ij}(s)$, respectively, using unsteady, two-dimensional, incompressible strip theory. Likewise, a logical extension follows that these same representations will provide an accurate approximation of the generalized aerodynamic forces $\tilde{A}_{ij}(s)$ and $\tilde{A}^C_{ij}(s)$ using unsteady, three-dimensional, compressible lifting surface aerodynamic theory such as the doublet-lattice method. Although only truly valid for sinusoidal motion, the doublet-lattice method for unsteady subsonic flow now has wide use in various forms in the aerospace industry. This is mainly because this method can be
used for complex configurations consisting of multiple, nonplanar interfering lifting surfaces and multiple bodies, is simple to apply, and gives accurate results for unsteady, subsonic, shock-free flow. A brief summary of the particular doublet-lattice method of reference 22, which is used in the application case of Chapter IV, is given in Appendix B.

Before beginning the development of the fitting procedure of the Laplace polynomial fractions of equations (2.20), (2.21) and (2.22), the form in which the generalized aerodynamic forces are usually calculated or given must be considered. From equations (1.16), (1.9), and (1.14), it follows that

\[ A_{ij}(s) \xi_j(s) = \mathcal{L}[A_{ij}] = \mathcal{L} \left[ \int_S \Delta p_j(x,y,z,t) \cdot \hat{\phi}_i(x,y,z) dS \right] \quad (2.23) \]

and

\[ A_{ij}^C(s) \xi_j(s) = \mathcal{L}[A_{ij}^C] = \mathcal{L} \left[ \int_S \Delta p_j^C(x,y,z,t) \cdot \hat{\phi}_i(x,y,z) dS \right] \quad (2.24) \]

Since the widely used unsteady lifting surface aerodynamic theories are only truly valid for steady oscillatory or sinusoidal motion, simple harmonic motion can be specified by setting \( s = i\omega \) in the above equations (2.23) and (2.24). This gives the following Fourier transformed forms:

\[ \tilde{A}_{ij}(i\omega) \tilde{\xi}_j(i\omega) = \int_S \Delta \tilde{p}_j(x,y,z,\omega) \cdot \hat{\phi}_i(x,y,z,\omega) dS \tilde{\xi}_j(i\omega) \quad (2.25) \]
and

$$A_{ij}^C(i\omega) \xi_j(i\omega) = \iint_S \Delta p_j(x,y,z,\omega) \cdot \phi_i(x,y,z) dS \xi_j(i\omega)$$  (2.26)

where \(\Delta p_j(x,y,z,\omega)\) is the pressure distribution due to sinusoidal motion of the \(j\)th normal mode per unit \(j\)th normal coordinate, and \(\Delta p_j^c(x,y,z,\omega)\) is the pressure distribution due to sinusoidal motion of the \(j\)th control surface per unit \(j\)th control surface coordinate.

Equate coefficients of \(\xi_j(i\omega)\) and \(\xi_j(i\omega)\) to get

$$\bar{A}_{ij}(i\omega) = \iint_S \Delta p_j(x,y,z,\omega) \cdot \phi_i(x,y,z) dS$$  (2.27)

and

$$\bar{A}_{ij}^C(i\omega) = \iint_S \Delta p_j^c(x,y,z,\omega) \cdot \phi_i(x,y,z) dS$$  (2.28)

These equations can be rewritten as

$$\bar{A}_{ij}(i\omega) = q_\omega A_{ij} R \bar{Q}_{ij}(i\omega)$$  (2.29)

and

$$\bar{A}_{ij}^C(i\omega) = q_\omega A_{ij} R \bar{Q}_{ij}^C(i\omega)$$  (2.30)

where

$$\bar{Q}_{ij}(i\omega) = \frac{1}{A_R} \iint_S \Delta \tilde{p}_j(x,y,z,\omega) \cdot \frac{\phi_i}{c_R} (x,y,z) dS$$  (2.31)

and

$$\bar{Q}_{ij}^C(i\omega) = \frac{1}{A_R} \iint_S \Delta \tilde{p}_j^c(x,y,z,\omega) \cdot \frac{\phi_i}{c_R} (x,y,z) dS$$  (2.32)
and where
\[ \tilde{\Delta}^{\text{p}}_{\text{pj}}(x,y,z,\omega) = \frac{\tilde{\Delta}^{\text{p}}_{\text{pj}}}{q_{\infty}}(x,y,z,\omega) \] (2.33)
and
\[ \tilde{\Delta}^{\text{C}}_{\text{pj}}(x,y,z,\omega) = \frac{\tilde{\Delta}^{\text{C}}_{\text{pj}}}{q_{\infty}}(x,y,z,\omega) \] (2.34)

The expressions \( \tilde{Q}_{ij}(i\omega) \) and \( \tilde{Q}_{ij}^{\text{C}}(i\omega) \) are the normalized form of the generalized aerodynamic forces and are the form given by the doublet-lattice lifting surface method of reference 22 for discrete frequencies. In addition, the pressure coefficient distribution \( \tilde{\Delta}^{\text{p}}_{\text{pj}}(x,y,z,\omega) \) is a function of Mach number, reduced frequency \( k = \frac{b_{R\omega}}{V} = \frac{b_{R\omega}}{2V} \), geometry of the airplane lifting surfaces and bodies, and the \( j \) th control surface deflection.

Therefore, for a given flight condition a calculated set of Fourier transformed generalized aerodynamic forces of equation (2.31) or (2.32) can be generated for the selected frequencies covering the desired range of normal modes being investigated for stability. For example, \( \tilde{Q}_{ij}(i\omega) \) can be calculated for \( p = 1, 2, \ldots, \text{na} \). Each of the functions \( \tilde{A}_{ij}(s) \) and \( \tilde{A}_{ij}^{\text{C}}(s) \), which are elements of the transformed generalized aerodynamic matrices \( [\tilde{A}(s)] \) and \( [\tilde{A}^{\text{C}}(s)] \), respectively, can be formed by fitting the following equation:

\[ \tilde{Q}_{ij}(i\omega) \approx \tilde{Q}_{ij}(s) = \frac{a_{4ij}s^4 + a_{3ij}s^3 + a_{2ij}s^2 + a_{1ij}s + a_{0ij}}{(s + b_1)(s + b_2)} \] (2.35)

and

\[ \tilde{Q}_{ij}^{\text{C}}(i\omega) \approx \tilde{Q}_{ij}^{\text{C}}(s) = \frac{a_{4ij}s^4 + a_{3ij}s^3 + a_{2ij}s^2 + a_{1ij}s + a_{0ij}}{(s + b_1)(s + b_2)} \] for \( s = i\omega \) (2.36)

\[ \tilde{Q}_{ij}(i\omega) \approx \tilde{Q}_{ij}(s) = \frac{a_{4ij}s^4 + a_{3ij}s^3 + a_{2ij}s^2 + a_{1ij}s + a_{0ij}}{(s + b_1)(s + b_2)} \] for \( s = i\omega \)
where
\[ \tilde{A}_{ij}(s) \equiv q_\infty A_R c_R \tilde{Q}_{ij}(s) \] (2.37)

and
\[ \tilde{A}_{ij}^c(s) \equiv q_\infty A_R c_R \tilde{Q}_{ij}^c(s) \] (2.38)

Likewise, for the special case where the pressure loads resulting from motion in the \( j \)th normal mode, which involves only translational displacement and no effective angle of attack, the following equation is used to fit the transformed generalized aerodynamic forces \( \tilde{A}_{ij}(s) \) of equation (2.37):
\[ \tilde{Q}_{ij}(i\omega) \equiv \tilde{Q}_{ij}(s) = \frac{a_{4ij}s^4 + a_{3ij}s^3 + a_{2ij}s^2 + a_{1ij} s}{(s + b_1)(s + b_2)} \] (2.39)

for \( s = i\omega \)

Fitting Procedure

The following fitting procedure is used to obtain the coefficients of the above Laplace polynomial fractions. The denominator constants \( b_1 \) and \( b_2 \) are not critical and are typically chosen as based on the poles of the Theodorsen function approximation of equation (2.7) for a reference semichord and the flight velocity. These constants are chosen as
\[ b_1 = \frac{V}{b_R K_3} \quad \text{for } K_3 = 13.51 \]

and
\[ b_2 = \frac{V}{b_R K_4} \quad \text{for } K_4 = 2.744 \] (2.40)
It is noted that these positive values for $b_1$ and $b_2$ provide poles for the Laplace polynomial fractions which lie on the negative real axis (stable half) of $s$-plane. The numerator coefficients of the Laplace polynomial fraction are obtained by a least squares fit for $s = i\omega$ to a series of calculated $\tilde{Q}_{ij}(i\omega_p)$ or $\tilde{Q}^c_{ij}(i\omega_p)$ which are in the form of a set of complex numbers $\tilde{Q}_{ijp} + \tilde{Q}^c_{ijp}$ corresponding to the discrete frequencies $\omega_p$ for $p = 1, 2, \ldots, na$.

For a given flight condition (Mach number and altitude) with $b_1$ and $b_2$ given by equation (2.40), set $s = i\omega$ in the right hand side of equation (2.35) to curve fit along the positive imaginary axis. Since the calculated generalized aerodynamic forces are truly valid only for sinusoidal motion, the following is obtained:

$$
\frac{a_4\omega^4 - ia_3\omega^3 - a_2\omega^2 + ia_1\omega + a_0}{-\omega^2 + i\omega(b_1 + b_2) + b_1b_2}
$$

(2.41)

where the subscripts $i$ and $j$ on numerator polynomial coefficients are dropped for notation simplification. Now for each discrete reduced frequency $k_p = \frac{\omega_p b_R}{V}$ or discrete frequency $\omega_p = \frac{k_p V}{b_R}$, it is desired to fit expression (2.41) to the calculated complex value $\tilde{Q}_{R_p} + i\tilde{Q}^c_{I_p}$. Therefore, the following desired expression can be written:

$$
\frac{a_4\omega_p^4 - ia_3\omega_p^3 - a_2\omega_p^2 + ia_1\omega_p + a_0}{-\omega_p^2 + i\omega_p(b_1 + b_2) + b_1b_2} = \tilde{Q}_{R_p} + i\tilde{Q}^c_{I_p}
$$

(2.42)

($p = 1, 2, \ldots, na$)
where $na$ is the number of discrete frequency fitting points spanning the desired frequency fit range, and where the subscripts $i$ and $j$ of the real and imaginary parts of the calculated generalized aero-
dynamic force $\tilde{Q}_R$ and $\tilde{Q}_I$, corresponding to the frequency $\omega_p$, have also been dropped for notation simplification.

Multiplying both sides of equation (2.42) by the denominator and equating the real and imaginary parts on each side of the resulting equation gives

(Real part)

$$-a_4\omega_p^2 + a_2\omega_p^2 - a_o = \tilde{Q}_R\omega_p^2 - (b_1 + b_2)\tilde{Q}_I\omega_p - b_1b_2\tilde{Q}_R \quad (2.43)$$

($p = 1, 2, \ldots, na$)

(Imaginary part)

$$a_3\omega_p^3 - a_1\omega_p = \tilde{Q}_I\omega_p^2 - (b_1 + b_2)\tilde{Q}_R\omega_p - b_1b_2\tilde{Q}_I \quad (2.44)$$

($p = 1, 2, \ldots, na$)

or in matrix form

$$\begin{bmatrix} \omega_R \end{bmatrix} \{ a_R \} = \{ d_R \} \quad (2.45)$$

and $$\begin{bmatrix} \omega_I \end{bmatrix} \{ a_I \} = \{ d_I \} \quad (2.46)$$

where

$$\begin{bmatrix} -\omega_1 & \omega_1 & -1 \\ -\omega_2 & \omega_2 & -1 \\ \vdots & \vdots & \vdots \\ -\omega_na & \omega_na & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \omega_1 \omega_1 \\ \omega_2 \omega_2 \\ \vdots \vdots \\ \omega_na \omega_na \end{bmatrix}$$
\[
\{a_R\} = \begin{bmatrix}
    a_4 \\
    a_2 \\
    a_0
\end{bmatrix} \quad \text{and} \quad \{a_I\} = \begin{bmatrix}
    a_3 \\
    a_1
\end{bmatrix}
\]

\[
\begin{align*}
\tilde{Q}_{R_1} \omega_1^2 + (b_1 + b_2) \tilde{Q}_{I_1} \omega_1 - b_1 b_2 \tilde{Q}_{R_1} \\
\tilde{Q}_{R_2} \omega_2^2 + (b_1 + b_2) \tilde{Q}_{I_2} \omega_2 - b_1 b_2 \tilde{Q}_{R_2} \\
\vdots & \quad \vdots & \quad \vdots \\
\tilde{Q}_{R_{na}} \omega_{na}^2 + (b_1 + b_2) \tilde{Q}_{I_{na}} \omega_{na} - b_1 b_2 \tilde{Q}_{R_{na}}
\end{align*}
\]

\[
\{d_R\} = \begin{bmatrix}
    \vdots \\
    \vdots \\
    \vdots \\
    \tilde{Q}_{R_{na}} \omega_{na}^2 + (b_1 + b_2) \tilde{Q}_{I_{na}} \omega_{na} - b_1 b_2 \tilde{Q}_{R_{na}}
\end{bmatrix}
\]

and

\[
\begin{align*}
\tilde{Q}_{I_1} \omega_1^2 - (b_1 + b_2) \tilde{Q}_{R_1} \omega_1 - b_1 b_2 \tilde{Q}_{I_1} \\
\tilde{Q}_{I_2} \omega_2^2 - (b_1 + b_2) \tilde{Q}_{R_2} \omega_2 - b_1 b_2 \tilde{Q}_{I_2} \\
\vdots & \quad \vdots & \quad \vdots \\
\tilde{Q}_{I_{na}} \omega_{na}^2 - (b_1 + b_2) \tilde{Q}_{R_{na}} \omega_{na} - b_1 b_2 \tilde{Q}_{I_{na}}
\end{align*}
\]

\[
\{d_I\} = \begin{bmatrix}
    \vdots \\
    \vdots \\
    \vdots \\
    \tilde{Q}_{I_{na}} \omega_{na}^2 - (b_1 + b_2) \tilde{Q}_{R_{na}} \omega_{na} - b_1 b_2 \tilde{Q}_{I_{na}}
\end{bmatrix}
\]

Now the matrices \([\omega_i]\) and \([d_R]\) are known in equation (2.45), as are the matrices \([\omega_i]\) and \([d_I]\) in equation (2.46), and the matrices \([a_R]\) and \([a_I]\) are the desired unknown coefficient matrices. Multiplying both sides of equations (2.45) and (2.46) by the transposed matrices \([\omega_R]^T\) and \([\omega_I]^T\), respectively, yields

\[
[\omega_R]^T [\omega_R] \{a_R\} = [\omega_R]^T \{d_R\} \quad (2.47)
\]

and

\[
[\omega_I]^T [\omega_I] \{a_I\} = [\omega_I]^T \{d_I\} \quad (2.48)
\]
The resulting square matrices $[[\omega_R]^T[\omega_R]]$ and $[[\omega_I]^T[\omega_I]]$ are invertible if they are non-singular. Therefore, solving equations (2.47) and (2.48) for the coefficient matrices $\{a_R\}$ and $\{a_I\}$, respectively, gives

$$\begin{bmatrix} a_4 \\ a_2 \\ a_0 \end{bmatrix} = \{a_R\} = [[\omega_R]^T[\omega_R]]^{-1}[\omega_R]^T\{d_R\}$$

(2.49)

and

$$\begin{bmatrix} a_3 \\ a_1 \end{bmatrix} = \{a_I\} = [[\omega_I]^T[\omega_I]]^{-1}[\omega_I]^T\{d_I\}$$

(2.50)

These equations are really a least squares fit over $na$ points giving the coefficients $a_{4ij}$, $a_{3ij}$, $a_{2ij}$, $a_{1ij}$, $a_{0ij}$, of the numerator of the Laplace polynomial fraction.

It is noted that if weighting of certain frequency values in the above least squares fitting procedure is desired, then equations (2.45) and (2.46) become

$$[W][\omega_R]\{a_R\} = [W]\{d_R\}$$

(2.51)

and

$$[W][\omega_I]\{a_I\} = [W]\{d_I\}$$

(2.52)

where

$$[W] = \begin{bmatrix} W_1 & W_2 & 0 \\ 0 & \ldots & \ldots & \ldots \ldots \ldots \ldots \ldots & W_{na} \end{bmatrix}$$

and where $W_1, W_2, \ldots , W_{na}$ are the integer weighting values for the
desired weighted fit. This formulation now favors the frequency values which are weighted the most and results in the following least squares fit solution for the numerator coefficient matrices \( \{ a_R \} \) and \( \{ a_I \} \):

\[
\begin{align*}
\begin{bmatrix}
    a_4 \\ a_2 \\ a_0
\end{bmatrix} = \{ a_R \} &= \left[ [\omega_R]^T [W] [\omega_R] \right]^{-1} [\omega_R]^T [W] \{ d_R \} \\
\end{align*}
\tag{2.53}
\]

and

\[
\begin{align*}
\begin{bmatrix}
    a_3 \\ a_1
\end{bmatrix} = \{ a_I \} &= \left[ [\omega_I]^T [W] [\omega_I] \right]^{-1} [\omega_I]^T [W] \{ d_I \} \\
\end{align*}
\tag{2.54}
\]

where the known matrices on the left hand sides are defined in equations (2.45), (2.46), and (2.52). If all frequency values are weighted equally, equations (2.53) and (2.54) will, of course, reduce to equations (2.49) and (2.50).

The following Laplace polynomial fraction approximation then results for each transformed generalized force, corresponding to equations (2.20) and (2.22):

\[
\tilde{A}_{ij}(s) \equiv \omega_0 A_R C_R \tilde{Q}_{ij}(s) \\
\tag{2.55}
\]

where

\[
\tilde{Q}_{ij}(s) = \frac{a_{4ij}s^4 + a_{3ij}s^3 + a_{2ij}s^2 + a_{1ij}s + a_{0ij}}{(s + b_1)(s + b_2)}
\]

and

\[
\tilde{A}_{ij}^C(s) \equiv \omega_0 A_R C_R \tilde{Q}_{ij}^C(s) \\
\tag{2.56}
\]
where
\[ \tilde{Q}_{ij}(s) = \frac{a_{4ij}s^4 + a_{3ij}s^3 + a_{2ij}s^2 + a_{1ij}s + a_{0ij}}{(s + b_1)(s + b_2)} \]

and where the numerator coefficients are given by equations (2.53) and (2.54) and \( b_1 \) and \( b_2 \) in the denominator are given by equations (2.40).

The same fitting procedure for the special case of \( \tilde{A}_{ij}(s) \) where j th pressure mode involves only translational displacement and no effective angle of attack, results in the same general matrix equations as (2.53) and (2.54), except for the following changes since \( a_0 = 0 \):

\[ \{a_R\} = \begin{Bmatrix} a_4 \\ a_2 \end{Bmatrix} \]

and

\[ \begin{bmatrix} -\omega_1 & \omega_1 \\ -\omega_2 & \omega_2 \\ \cdots & \cdots \\ -\omega_{na} & \omega_{na} \end{bmatrix} \]

The following Laplace polynomial fraction approximation then results for each transformed aerodynamic force, corresponding to equation (2.21):

\[ \tilde{A}_{ij}(s) = q_\infty A_R c_R \tilde{Q}_{ij}(s) \]

where
\[ Q_{ij}(s) = \frac{a_{4ij}s^4 + a_{3ij}s^3 + a_{2ij}s^2 + a_{1ij}s}{(s + b_1)(s + b_2)} \]
where the \( j \)th pressure mode involves translation only and no effective angle of attack, and where the numerator coefficients are given by equations (2.53) and (2.54) with the changes given by equations (2.57) and (2.58), and \( b_1 \) and \( b_2 \) in the denominator are given by equations (2.40).

It is again reiterated that the values for \( b_1 \) and \( b_2 \) in the denominator of equations (2.55), (2.56), and (2.59) are not critical, but they remain the same values for all Laplace functions \( \bar{A}_{ij}(s) \) and \( \bar{A}^C_{ij}(s) \) in the matrices \([\bar{A}(s)]\) and \([\bar{A}^C(s)]\), respectively. They are typically chosen as based on the poles of the Theodorsen function approximation of equation (2.7) for a reference semichord \( b_R \) and the flight velocity \( V \). However, the set of coefficients in the numerator is different for each Laplace function \( \bar{A}_{ij}(s) \) or \( \bar{A}^C_{ij}(s) \) which, in turn, are elements of the matrices \([\bar{A}(s)]\) and \([\bar{A}^C(s)]\), respectively, in the matrix stability equation (1.25). These differences occur because the numerator coefficients for each \( \bar{A}_{ij}(s) \) or \( \bar{A}^C_{ij}(s) \) are obtained by a fit to a corresponding calculated set of discrete sinusoidal generalized aerodynamic forces \( \bar{Q}^i_{ij}(i\omega_p) \) or \( \bar{Q}^C_{ij}(i\omega_p) \) for \( p = 1, 2, \ldots, n_a \) covering the desired frequency range.

Figure 6 shows an example of a normalized generalized aerodynamic force matrix element \( \bar{Q}_{NN}(s) \), where both the displacement mode and the pressure mode are the same wing bending mode \( (i = j = N, \text{ the wing bending mode}) \), plotted for calculated discrete frequency values using strip theory and the corresponding fitted Laplace polynomial fraction of equation (2.55) for sinusoidal motion. Figures 7 and 8 show examples of two normalized generalized aerodynamic force matrix
Figure 6. Curve Fit for $s = i\omega$ of Normalized Generalized Aerodynamic Force $\bar{Q}_{NN}$, where $i = j = N$ the First Wing Bending Mode of a Typical Straight Wing Airplane, $V = 1000$ ft/sec, $b_R = 1.0$ ft
Figure 7. Curve Fit for $s = i\omega$ of Normalized Generalized Aerodynamic Force $\tilde{Q}_{11}$, where $i = j = 1$ the Rigid Airplane Roll Mode of Prototype Fighter with Wing Tip Missiles, $V = 933$ ft/sec, $b_R = 4.167$ ft
Figure 8. Curve Fit for $s = i \omega$ of Normalized Generalized Aerodynamic Force $\tilde{Q}_{11}$, where $i = 1$ the Rigid Airplane Roll Mode and $j = 1$ the Aileron Displacement Mode of Prototype Fighter with Wing Tip Missiles, $V = 933$ ft/sec, $b_R = 4.167$ ft
elements $\bar{Q}_{11}(s)$ and $\bar{Q}_{11}^C(s)$ for the application case of Chapter IV, plotted for calculated discrete frequency values using doublet-lattice theory and the corresponding fitted Laplace polynomial fraction of equations (2.59) and (2.56), respectively, for sinusoidal motion. Figure 7 is the plot of the normalized generalized aerodynamic force matrix element $\bar{Q}_{11}(i\omega)$ where both the displacement mode and the pressure mode are rigid airplane roll. Figure 8 is the plot of $\bar{Q}_{11}^C(i\omega)$ where the displacement mode is airplane roll ($i = 1$), and the pressure mode is aileron or flaperon displacement ($j = 1$). As can be seen, the fitting functions of equations (2.55), (2.56), and (2.59) give very good fits for all three cases involving not only strip theory but also doublet-lattice lifting surface theory. It is noted that Figures 7 and 8 reflect the worst fits to the doublet-lattice theory used in the application case of Chapter IV.

In this chapter, a simple, but realistic and practical method for formulating the unsteady aerodynamic forces in the Laplace domain has been developed. The fitted Laplace polynomial fractions $\bar{Q}_{ij}(s)$ and $\bar{Q}_{ij}^C(s)$ in equations (2.55), (2.56), and (2.59) for the transformed generalized aerodynamic forces $\bar{A}_{ij}(s)$ and $\bar{A}_{ij}^C(s)$, respectively, have been derived. These fitted Laplace polynomial fractions have been formulated for a given flight condition (i.e., for a particular Mach number and altitude), are applicable for all frequencies within the frequency fit range, and are most valid for Laplace arguments near the imaginary axis (small positive or negative damping) along which the fit is made and which is the region of the greatest physical interest for stability. The Laplace polynomial fractions are
realistic and physically realizable functions for the transformed generalized aerodynamic forces since their forms have been developed from unsteady aerodynamic strip theory, which assures an accurate fit for not only strip theory but also for unsteady, three-dimensional lifting surface aerodynamic theory as a logical extension.
It is emphasized in the introduction that the critical flight conditions for possible aeroservoelastic instabilities are likely to be those for which the total feedback loop gain (the product of the control surface effectiveness and the feedback loop gain) is a maximum. To properly analyze these likely flight conditions for stability, realistic and accurate values of control surface effectiveness are needed. This effectiveness of the control surface is directly attributed to the accuracy of predicting the control surface aero-dynamics.

Therefore, for initial stability calculations, this work proposes that the best available unsteady aerodynamic lifting surface theory, such as the subsonic flow, doublet-lattice method suggested in Chapter II and outlined in Appendix B, be used without modifications or empirical corrections. The theoretical unsteady aerodynamic forces due to control surface deflections will, in general, be greater than those encountered in flight and should lead to conservative results for establishing feedback loop gains for the desired airplane system stability and response in flight. Then as wind tunnel force and pressure model test data is gathered for a given design, these results may be used to modify the theoretically predicted unsteady aero-dynamics due to control surface deflections. Although very little
experimental unsteady aerodynamic data exists for control surface
deflections, and in very few cases would any attempt be made to gather
any on a specific configuration during design, steady state wind tun-
nel data is usually obtained for any new configuration. Two methods
are suggested in subsequent paragraphs to modify the calculated con-
trol surface generalized unsteady aerodynamic forces based on avail-
able steady state experimental data.

Modification Method Based on Hinge Moment Data

One of these methods is to modify or adjust the calculated gen-
eralized unsteady aerodynamic forces due to a given control surface
deflection by the ratio of the experimental to calculated values of
the static hinge moment for that control surface. By equations (2.24)
and (2.56), the transformed generalized aerodynamic forces for the
i th mode due to motion of the j th control surface, is given as

$$\bar{A}_{ij}^c(s)\delta_j(s) = \mathcal{L}\left[\int_0^\infty \Delta p_j^c(x,y,z,t)\cdot \tilde{\phi}_i(x,y,z)ds\right]$$

$$\equiv [q\omega ArcR\tilde{Q}_{ij}^c(s)]\delta_j(s) \quad (3.1)$$

where

$$\tilde{Q}_{ij}^c(s) = \frac{a_{4ij}s^4 + a_{3ij}s^3 + a_{2ij}s^2 + a_{1ij}s + a_{0ij}}{(s + b_1)(s + b_2)}$$

Then for the j th column of the matrix $[\bar{A}^c(s)]$ corresponding to the
control surface coordinate $\delta_j$, this procedure would be to multiply
all the Laplace function matrix elements $\bar{A}_{ij}^c(s)$ of that column
(i = 1,2, . . . ,n) by this real scalar before being used in the
stability calculations. So each control surface generalized force Laplace functions matrix element $\tilde{A}_{ij}^C(s)$ is replaced by

$$K_\delta \tilde{A}_{ij}^C(s)$$

(3.2)

where $K_\delta_{ij}$ is the ratio of the experimental to calculated values of the static hinge moment for the $j$th control surface. The resulting modified matrix $[\bar{A}^C(s)]$ is then

$$
\begin{bmatrix}
K_\delta_{11}^C(s) & K_\delta_{12}^C(s) & \cdots & K_\delta_{1n_c}^C(s) \\
K_\delta_{21}^C(s) & K_\delta_{22}^C(s) & \cdots & K_\delta_{2n_c}^C(s) \\
\vdots & \vdots & \ddots & \vdots \\
K_\delta_{n_1,1}^C(s) & K_\delta_{n_1,2}^C(s) & \cdots & K_\delta_{n_1,n_c}^C(s)
\end{bmatrix}
$$

(3.3)

The hinge moment experimental data is usually given in terms of aerodynamic hinge moment coefficients for selected flight conditions. The hinge moment per dynamic pressure $q_\infty$ and per unit control surface deflection $\delta_j$, is then given by the following expression which is based on the usual definition for hinge moments:

$$
\frac{H.M.}{q_\infty \delta_j} = A_R^C R C_{h\delta_j}
$$

(3.4)

where $A_R$ is a reference area, $c_R$ is a reference length, and $C_{h\delta_j}$ is the hinge moment coefficient for the $j$th control surface.
The corresponding calculated static hinge moment of equation (3.4) can be determined from the applicable unsteady aerodynamic theory used. For example, for the subsonic doublet-lattice method outlined in Appendix B, the corresponding hinge moment of equation (3.4) can be formed by using the data output generated by the computer program for a given input control surface deflection mode and for a reduced frequency of zero or nearly so (essentially a steady state condition). The theoretical hinge moment can then be calculated by use of the doublet-lattice computer program output data as follows:

\[
\frac{H.M. \text{ (Calc.)}}{q_\infty \delta_j} = \frac{1}{\delta_j} \sum_{i=1}^{N\text{BOX}} x_i^i \Delta C_{i,j} \Delta x_i \Delta y_i
\]  

(3.5)

where NBOX is the number of boxes used in the aerodynamic idealization of the j th control surface, \( x_i^i \) is the distance (perpendicular to the hinge line) from the hinge line to the midpoint of the quarter chord of the i th box, \( \Delta x_i \Delta y_i \) is the area of the i th box, and \( \Delta C_{i,j} \) is the pressure coefficient for the i th box on the j th control surface. The ratio of equation (3.4) to equation (3.5) is the hinge moment modification factor to the control surface generalized unsteady aerodynamic forces which are suggested in expressions (3.2) and (3.3).

**Modification Method Based on Stability Derivative Data**

A second method to modify the calculated generalized unsteady aerodynamic forces due to a given control surface deflection is to adjust them by the ratio of the experimental to calculated values of a selected static stability derivative, without aeroelastic effects, for that control surface. The experimental static stability
derivatives are usually obtained by wind tunnel force measurements. These derivatives are determined for selected Mach numbers and are usually obtained as rigid stability derivatives (without aeroelastic effects). A selected few of these wind tunnel based rigid stability derivatives are used to provide the experimental part of the modification ratios to the calculated control surface generalized unsteady aerodynamic forces. The stability derivatives without aeroelastic effects are desired for use in the modification ratios to the calculated control surface aerodynamics, since aeroelastic effects can be accounted for by including a sufficient number of flexible modes in the stability analysis. The method used to obtain the corresponding calculated doublet-lattice, stability derivatives due to control surface deflections, is discussed in Appendix C.

As an example of a selection of the stability derivatives to use in this modification procedure, the following is considered. The control surface motions affecting the antisymmetric response of the airplane would be rudder, aileron, and differential horizontal tail rotation. Therefore, an appropriate modification factor to the calculated generalized unsteady aerodynamic forces due to rudder rotation, would be the ratio of the experimental to calculated values of the rigid stability derivative for yaw moment due to rudder rotation $C_{n\delta_R}$. Likewise, an appropriate modification factor to the calculated generalized unsteady aerodynamic forces due to aileron rotation would be the ratio of the experimental to calculated values of the rigid stability derivative for roll moment due to aileron
rotation \( C_{\delta \delta A} \). Finally, an appropriate modification factor to the calculated generalized unsteady aerodynamic forces due to differential horizontal tail rotation would be the ratio of the experimental to calculated values of the rigid stability derivative for roll moment due to differential tail rotation \( C_{\delta \delta_{HT}} \). These modification factors for these control surface motions for antisymmetric airplane response would then be

\[
K_{\delta R} = \frac{C_{n \delta R}^{\text{Exp. rigid}}}{C_{n \delta R}^{\text{Calc. rigid}}}
\]  

(3.6)

\[
K_{\delta A} = \frac{C_{\delta \delta A}^{\text{Exp. rigid}}}{C_{\delta \delta A}^{\text{Calc. rigid}}}
\]  

(3.7)

\[
K_{\delta_{HT}} = \frac{C_{\delta \delta_{HT}}^{\text{Exp. rigid}}}{C_{\delta \delta_{HT}}^{\text{Calc. rigid}}}
\]  

(3.8)

which would be applied in the same way as the hinge moment modification factors in the expressions of (3.2) and (3.3).

In a similar manner, the control surface motion affecting the symmetric response of the airplane would be elevator rotation or horizontal tail pitch. Therefore, an appropriate modification factor to the calculated generalized unsteady aerodynamic forces due to elevator rotation, would be the ratio of the experimental to calculated
values of the rigid stability derivative for pitching moment due to elevator rotation or horizontal tail pitch $C_{m_{\delta_e}}$.

Comparison of the Two Methods

Based on the application case of Chapter IV, the following comparisons can be made of the two methods, suggested in the preceding paragraphs, to modify the calculated generalized unsteady aerodynamic forces due to control surface deflection. For trailing edge control surfaces, such as ailerons or a rudder, the hinge moment and the rigid stability derivative modification methods appear to give comparable results for the control surface unsteady aerodynamic modification ratios. However, for all movable control surfaces, such as an all movable horizontal tail, the rigid stability derivative modification method appears to give reasonable results while the hinge moment modification method does not. The reason for this is thought to be that the hinge moment for an all movable horizontal tail is working through a small moment arm, and thus a small difference in the theoretical compared to measured longitudinal location of the center of pressure can result in the calculated hinge moment being quite unrealistic compared to the measured value. But at the same time the calculated rigid stability derivative for roll moment due to differential horizontal tail rotation is a reasonable value because the moment arm for roll is much larger than that for the hinge moment. This moment arm difference between the roll and hinge moments results in a roll moment which is much less sensitive to small differences in the locations of the center of pressure.
The airplane analyzed in the application case is a modern prototype fighter. This particular prototype airplane is a good application case since this prototype unexpectedly encountered unstable interaction between the airframe structural dynamics and the flight control system during early flight tests. One instability occurred with the wing tip missiles installed, and involved coupling of the flight control system and the first antisymmetric, missile pitch mode of vibration at a frequency of 6.5 Hz. A second instability occurred without the wing tip missiles installed, and involved coupling of the flight control system with, primarily, rigid body roll motion of the airplane at a frequency of 3.5 Hz. The missiles-on instability occurred at a Mach number of 0.9 and an altitude of 20,000 feet, and the missiles-off instability occurred at the same Mach number and an altitude of 15,000 feet. These instabilities were eliminated by adding a notch filter and reducing the gain in the roll rate feedback loop of the control system.

A three view sketch of the airplane is shown in Figure 9. The basic configuration has an AIM-9 missile mounted on each wing tip launcher. Although the airplane is designed for high load factors, the wing is relatively flexible because of its low thickness to chord ratio.
Figure 9. Airplane 3-View
Flight Control System Data

The airplane design employs a completely fly-by-wire flight control system. Major components of the fly-by-wire system are shown in Figure 10. A schematic diagram of the airplane flight control system is shown in Figure 11. The basic flight control system has three channels referred to as the pitch, yaw, and roll channels. The pitch channel commands the all movable horizontal tail symmetrically. The yaw channel commands the rudder. The roll channel commands the aileron and differential horizontal tail deflections in a fixed ratio of 1.0 to 0.25, respectively. The sensor locations are given in Figures 9 and 10.

The roll and yaw channels of the flight control system are the ones which affect the antisymmetric response of the airplane. Since the flight instabilities were of antisymmetric response, these channels are the ones needed in an aeroservoelastic stability analysis to compare with flight test results. A block diagram of these roll and yaw channels is shown in Figure 12. Only roll rate is fed back through the roll channel. Roll rate, yaw rate, and lateral acceleration are fed back through the yaw channel by two paths, one directly and the other from the roll channel through the aileron-rudder interconnect. The blocks labeled $K_p$, $K_r$, and $K_{Ny}$ are the roll rate, yaw rate, and lateral acceleration feedback gains, respectively. These gains are varied separately in the stability analysis, and their nominal values are 0.2, 1.0, and 0.6, respectively. The blocks labeled $G_{AR1}$ and $G_a/57.3$ are variable gains that vary as a function
Figure 10. Fly-By-Wire Flight Control
Figure 11. Airplane Flight Control System
**Nominal Feedback Gain Values**

<table>
<thead>
<tr>
<th>Flight Condition</th>
<th>$K_p$</th>
<th>$K_f$</th>
<th>$K_{Ny}$</th>
<th>$F_B$</th>
<th>$G_{\omega}/57.3$</th>
<th>$G_{ARI}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M=0.9$ 20K ft</td>
<td>.2</td>
<td>1.0</td>
<td>.6</td>
<td>.5</td>
<td>-.0262</td>
<td>-.237</td>
</tr>
<tr>
<td>$M=0.9$ 15K ft</td>
<td>.2</td>
<td>1.0</td>
<td>.6</td>
<td>.5</td>
<td>-.02034</td>
<td>-.261</td>
</tr>
<tr>
<td>$M=0.9$ S.L.</td>
<td>.2</td>
<td>1.0</td>
<td>.6</td>
<td>.5</td>
<td>-.01047</td>
<td>-.305</td>
</tr>
</tbody>
</table>

**Figure 12. Airplane Roll and Yaw Channels of the Flight Control System**
of the angle of attack of the airplane. These gains are relatively small for the steady level flight conditions as indicated in Figure 12. The block indicated by F8 is a variable gain that is programmed as a function of flight condition. The transfer functions for the command servos and actuators (no load case) are shown in Figure 12. Although not shown in Figure 12, the sensor transfer function for the roll rate gyro, the yaw rate gyro, and the lateral accelerometer is

\[ F(s)_{\text{SENSOR}} = \frac{(314)^2}{s^2 + 2(0.6)(314)s + (314)^2} \]

Using the above sensor transfer function and the flight control system diagram of Figure 12, the elements of the feedback matrix \([G(s)]\) in equations (1.22) and (1.25) for antisymmetric response of this airplane are given in Table 1.

**Analysis Input Mode Data and Control Surface Deflection Data**

Although antisymmetric normal modes of vibration of the airplane had been computed using a finite element representation of the airplane structure and revised based on the ground vibration test (G.V.T.) results, the antisymmetric normal modes of vibration used in this analysis utilized G.V.T. measured mode shapes and frequencies, mainly because they were readily available and agreed well with corresponding analytical results. Three rigid body modes of airplane roll, yaw, and lateral translation and the first two antisymmetric flexible modes of missile pitch and wing first bending were included in the
<table>
<thead>
<tr>
<th>Element</th>
<th>Element Transfer Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{6A_s(s)}{2}$</td>
<td>$(-K_p) \left[ \frac{(314)^2}{s^2 + 2(314)s + (314)^2} \right] \left[ \frac{(52)^2}{s^2 + 2(52)s + (52)^2} \right] \left[ \frac{20}{s+20} \right]$</td>
</tr>
<tr>
<td>$\frac{6A_g(s)}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{6A_HG(s)}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{6HT_s(s)}{2}$</td>
<td>$(-0.25)(K_p) \left[ \frac{(314)^2}{s^2 + 2(314)s + (314)^2} \right] \left[ \frac{(52)^2}{s^2 + 2(52)s + (52)^2} \right] \left[ \frac{20}{s+20} \right]$</td>
</tr>
<tr>
<td>$\frac{6HT_g(s)}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{6HT_HG(s)}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{6R_s(s)}{2}$</td>
<td>$\left[ \frac{(314)^2}{s^2 + 2(314)s + (314)^2} \right] \left[ \frac{s^2(147)^2}{s^2 + 2(147)^2} \right] \left[ \frac{(52)^2}{s^2 + 2(52)s + (52)^2} \right] \left[ \frac{20}{s+20} \right] \left[ \frac{B_c}{s^2 + (55)^2} \right] \left[ \frac{5s^2 + 15}{s + 1} \right] \left[ \frac{5s^2 + 15}{s + 1} \right] \left[ \frac{5s^2 + 15}{s + 1} \right] \left[ \frac{5s^2 + 15}{s + 1} \right] \left[ \frac{5s^2 + 15}{s + 1} \right]$</td>
</tr>
<tr>
<td>$\frac{6R_g(s)}{2}$</td>
<td>$(K_p) \left( FB \right) \left[ \frac{1.5s^2 + 15}{s^2 + 15} \right] \left[ \frac{s^2(147)^2}{s^2 + 2(147)^2} \right] \left[ \frac{(52)^2}{s^2 + 2(52)s + (52)^2} \right] \left[ \frac{20}{s+20} \right]$</td>
</tr>
<tr>
<td>$\frac{6R_HG(s)}{2}$</td>
<td>$(-K_p) \left( FB \right) \left[ \frac{s^2(147)^2}{(s+147)^2} \right] \left[ \frac{(52)^2}{s^2 + 2(52)s + (52)^2} \right] \left[ \frac{20}{s+20} \right]$</td>
</tr>
</tbody>
</table>
antisymmetric missiles-on stability analysis. These modes are listed in Table 2, and the two flexible modes are shown in Figure 13. The same corresponding rigid body and flexible modes were included in the antisymmetric missiles-off stability analysis. These modes are listed in Table 3, and the two flexible modes are shown in Figure 14. It is noted that while the mode shapes of the missiles-off configuration are quite similar to the corresponding flexible mode shapes of the missiles-on configuration, the missiles-off modal frequencies have increased significantly (6.5 to 17.4 Hz and 8.0 to 10.9 Hz) because of the absence of the tip missiles. Although additional flexible modes could be included, only these two flexible modes were included in this stability analysis since the flight instabilities seemed to involve only these flexible modes and since control system coupling with higher frequency structural vibration modes was unlikely because of the inherent decrease in control system gain with increasing frequency.

For the five normal modes used in the stability analysis, the associated generalized masses, frequencies, and damping coefficients are given in Tables 4 and 5 for the missiles-on and missiles-off cases, respectively. Generalized mass values shown are for the whole airplane. The relative modal slopes and deflections for these modes at the sensor locations are given in Tables 4 and 5.

The three control surface deflections of the aileron rotation, differential horizontal tail rotation, and rudder rotation which were used in the representation of the control augmentation system in the stability analysis are listed in Table 6. The control surface
TABLE 2. ANALYSIS INPUT MODES, MISSILES-ON

<table>
<thead>
<tr>
<th>Mode</th>
<th>Normal Mode Coordinate</th>
<th>Mode Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A/P Roll, $\xi_1 = \phi$</td>
<td>$\hat{\phi}_1 = yk - zj$ (1.0 rad., Right wing up about x axis)</td>
</tr>
<tr>
<td>2</td>
<td>A/P Yaw, $\xi_2 = \psi$</td>
<td>$\hat{\phi}_2 = xj - yj$ (1.0 rad., Nose left about A/P c.g., c.g. at F.S. 320.5)</td>
</tr>
<tr>
<td>3</td>
<td>A/P Lateral Translation (L.T.), $\xi_3 = \eta$</td>
<td>$\hat{\phi}_3 = j$ (1.0 in., Along y axis)</td>
</tr>
<tr>
<td>4</td>
<td>1st Antisymmetric normal mode of vibration, $\xi_4$</td>
<td>$\hat{\phi}_4$ (see Figure 13, 6.5 Hz, mode shape displacement normalized to a missile launcher displ.)</td>
</tr>
<tr>
<td>5</td>
<td>2nd Antisymmetric normal mode of vibration, $\xi_5$</td>
<td>$\hat{\phi}_5$ (see Figure 13, 8.0 Hz mode, mode shape displacements normalized to a missile launcher displ.)</td>
</tr>
</tbody>
</table>
Figure 13. Missiles-On: Missile Pitch and First Wing Bending Modes
<table>
<thead>
<tr>
<th>Mode</th>
<th>Normal Mode Coordinate</th>
<th>Mode Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A/P Roll, $\xi_1 = \phi$</td>
<td>$\hat{\phi}_1 = y\hat{k} - z\hat{j}$ (1.0 rad., Right wing up about $x$ axis.)</td>
</tr>
<tr>
<td>2</td>
<td>A/P Yaw, $\xi_2 = \phi$</td>
<td>$\hat{\phi}_2 = x\hat{j} - y\hat{i}$ (1.0 rad., Nose left about A/P c.g., c.g. at F.S. 319.76)</td>
</tr>
<tr>
<td>3</td>
<td>A/P Lateral Translation (L.T.), $\xi_3 = n$</td>
<td>$\hat{\phi}_3 = \hat{j}$ (1.0 in., Along $y$ axis)</td>
</tr>
<tr>
<td>4</td>
<td>1st Antisymmetric normal mode of vibration, $\xi_4$</td>
<td>$\hat{\phi}_4$ (see Figure 14, 10.9 Hz mode, mode shape displacements normalized to a wing tip displ.)</td>
</tr>
<tr>
<td>5</td>
<td>2nd Antisymmetric normal mode of vibration, $\xi_5$</td>
<td>$\hat{\phi}_5$ (see Figure 14, 17.4 Hz mode, mode shape displacements normalized to a wing tip displ.)</td>
</tr>
</tbody>
</table>
Figure 14. Missiles-Off: Launcher Pitch and First Wing Bending Modes
<table>
<thead>
<tr>
<th>Mode</th>
<th>Modal Generalized Mass, $M_i$ (lb-in-sec²)</th>
<th>Modal Freq. (Hz)</th>
<th>Modal Damping Coef., $\xi$</th>
<th>Mode Slope at Roll Rate Gyro, $\sigma_{\phi G}^{(1)}$ (rad)</th>
<th>Mode Slope at Yaw Rate Gyro, $\sigma_{\psi G}^{(1)}$ (rad)</th>
<th>Mode Disp. at Lat. Accel, $\phi_{Ny}^{(1)}$ (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.95826 x 10⁵($I_{xx}$)</td>
<td>0</td>
<td>0</td>
<td>1.0</td>
<td>0</td>
<td>5.2(-1zNy)</td>
</tr>
<tr>
<td>2</td>
<td>.67645 x 10⁶($I_{zz}$)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1.0</td>
<td>-151.725(1xNy)</td>
</tr>
<tr>
<td>3</td>
<td>.52533 x 10²(1/A/P)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1.0</td>
</tr>
<tr>
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<td>.002</td>
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<td>-.00008772</td>
<td>-.006125</td>
</tr>
<tr>
<td>Mode,</td>
<td>Modal Generalized Mass, $M_i$ (lb-in-sec^2)</td>
<td>Modal Freq. (Hz)</td>
<td>Modal Damping Coef., $\zeta$</td>
<td>Mode Slope at Roll Rate Gyro, $\sigma_{\phi G} (i)$ (rad)</td>
<td>Mode Slope at Yaw Rate Gyro, $\sigma_{\psi G} (i)$ (rad)</td>
<td>Mode Displ. at Lat. Accel. $\phi_{Ny} (i)$ (in)</td>
</tr>
<tr>
<td>------</td>
<td>-----------------------------------------</td>
<td>------------------</td>
<td>---------------------</td>
<td>---------------------------------</td>
<td>---------------------------------</td>
<td>-----------------------------</td>
</tr>
<tr>
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<td>$.66511 \times 10^5 (I_{xx})$</td>
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<td>0</td>
<td>1.0</td>
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<td>0</td>
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<td>2</td>
<td>$.64545 \times 10^6 (I_{zz})$</td>
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<td>0</td>
<td>1.0</td>
<td>-150.99 (I_{xNy})</td>
</tr>
<tr>
<td>3</td>
<td>$.51660 \times 10^2 (M_A/p)</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1.0</td>
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<tr>
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<td>-.002772</td>
<td>-.0001053</td>
<td>.004425</td>
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<td>.0105</td>
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TABLE 5. MODAL DATA, MISSILES-OFF
<table>
<thead>
<tr>
<th>Control Surface</th>
<th>Control Surface Coordinate</th>
<th>Displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Aileron (Flaperon)</td>
<td>1.0 rad. (T.E. down on right side of A/P)</td>
</tr>
<tr>
<td></td>
<td>Rotation, $\delta_1 = \delta_A$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Differential Horizontal Tail Rotation, $\delta_2 = \delta_{HT}$</td>
<td>1.0 rad. (T.E. down on right side of A/P)</td>
</tr>
<tr>
<td>3</td>
<td>Rudder Rotation, $\delta_3 = \delta_R$</td>
<td>1.0 rad. (T.E. right)</td>
</tr>
</tbody>
</table>
generalized inertias for the five normal modes due to motions of each of the three control surfaces are given in Tables 7 and 8 for the missiles-on and missiles-off cases, respectively. These control surface generalized inertias were calculated by use of equations (1.13).

Using the modal data of Table 4 and the notation of equations (1.2) and (1.3), the generalized mass matrix [M], the generalized damping matrix [C], and the generalized stiffness matrix [K] in equation (1.25) are given in Tables 9, 10, and 11, respectively, for the five normal modes in the analysis for the missiles-on configuration. The corresponding matrices for the missiles-off configuration, using the modal data of Table 5, are given in Tables 12, 13, and 14.

Using the modal data of Table 4 and equation (1.23), the sensor matrix [\( \phi_s(s) \)] in equation (1.25) for the missiles-on configuration is given in Table 15. The corresponding sensor matrix for the missiles-off configuration, using the modal data of Table 5, is given in Table 16.

Using the data of Table 7, the control surface generalized inertia matrix [MC] in equation (1.25) for the missiles-on configuration is given in Table 17. The corresponding control surface inertia matrix for the missiles-off configuration, using the data of Table 8, is given in Table 18.

**Aerodynamic Model Data**

For use in the unsteady aerodynamic calculations and subsequent stability analysis, the flight condition data, reference chord, and
### Table 7. Control Surface Generalized Inertias, Missiles-On

<table>
<thead>
<tr>
<th>Mode, i</th>
<th>$M_{i,s}^{C} \delta_A$ (1b-in-sec^2)</th>
<th>$M_{i,s}^{C} \delta_{HT}$ (1b-in-sec^2)</th>
<th>$M_{i,s}^{C} \delta_R$ (1b-in-sec^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>.24478 x10^3</td>
<td>.14532 x10^2</td>
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<tr>
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<td>.12491 x10^3</td>
<td>-.43805 x10^2</td>
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<tr>
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### Table 8. Control Surface Generalized Inertias, Missiles-Off

<table>
<thead>
<tr>
<th>Mode, i</th>
<th>$M_{i,s}^{C} \delta_A$ (1b-in-sec^2)</th>
<th>$M_{i,s}^{C} \delta_{HT}$ (1b-in-sec^2)</th>
<th>$M_{i,s}^{C} \delta_R$ (1b-in-sec^2)</th>
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</thead>
<tbody>
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<td>.14532 x10^2</td>
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<td>.12534 x10^3</td>
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### TABLE 9. GENERALIZED MASS MATRIX, MISSILES-ON

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### TABLE 10. GENERALIZED DAMPING MATRIX, MISSILES-ON

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### TABLE 11. GENERALIZED STIFFNESS MATRIX, MISSILES-ON

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### TABLE 12. GENERALIZED MASS MATRIX, MISSILES-OFF

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### TABLE 13. GENERALIZED DAMPING MATRIX, MISSILE-OFF

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### TABLE 14. GENERALIZED STIFFNESS MATRIX, MISSILES-OFF

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### TABLE 15. SENSOR MATRIX, MISSILES-ON

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### TABLE 16. SENSOR MATRIX, MISSILES-OFF

<table>
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<td>0 s</td>
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</table>

**TABLE 18. CONTROL SURFACE GENERALIZED INERTIA MATRIX, MISSILES-OFF**

<table>
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<th>Mode</th>
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</table>
reference area for the missiles-on configuration are given in Table 19. The corresponding data for the missiles-off configuration are given in Table 20.

The finite element doublet-lattice, aerodynamic idealization used in the generation of the unsteady aerodynamics for the airplane configuration with missiles-on is shown in Figures 15 and 16. All lifting surfaces including wing, tail surfaces, control surfaces, ventral fin, missile launcher, and missile fins were modeled as panels and subdivided into boxes. The fuselage and missile body were modeled primarily with cylindrical elements for both interference and slender body effects. Aerodynamic interference between all lifting surfaces and bodies is inherent in the doublet-lattice method used of reference 22. The aerodynamic idealization for the missiles-off configuration is the same representation except that the missile body and associated fins are absent. The missile launcher, however, is still present even without the missile.

To check the aerodynamic modeling of the above airplane configuration, rigid stability derivatives for 0.9 Mach number were determined by the procedures of Appendix C from the calculated aerodynamic data of the doublet-lattice method. A comparison of these calculated derivatives with corresponding experimental derivatives obtained from wind tunnel tests is shown in Table 21. It is noted that there was no wind tunnel testing accomplished for the missiles-off configuration. The assumption was made that the airplane stability derivatives would be essentially the same with or without the tip missiles installed. The comparison using the calculated stability derivatives
### TABLE 19. FLIGHT CONDITION DATA AND REFERENCE CHORD AND AREA, MISSILES-ON

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mach No.</td>
<td>0.9</td>
</tr>
<tr>
<td>Altitude</td>
<td>20,000 ft.</td>
</tr>
<tr>
<td>$V$</td>
<td>933.24 ft/sec (TAS)</td>
</tr>
<tr>
<td>$q_{\infty}$</td>
<td>3.832 lb/in$^2$</td>
</tr>
<tr>
<td>$C_R$</td>
<td>100 in</td>
</tr>
<tr>
<td>$A_R$</td>
<td>280 ft$^2$</td>
</tr>
</tbody>
</table>

### TABLE 20. FLIGHT CONDITION DATA AND REFERENCE CHORD AND AREA, MISSILES-OFF

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mach No.</td>
<td>0.9</td>
</tr>
<tr>
<td>Altitude</td>
<td>15,000 ft.</td>
</tr>
<tr>
<td>$V$</td>
<td>951.62 ft/sec (TAS)</td>
</tr>
<tr>
<td>$q_{\infty}$</td>
<td>4.704 lb/in$^2$</td>
</tr>
<tr>
<td>$C_R$</td>
<td>100 in</td>
</tr>
<tr>
<td>$A_R$</td>
<td>280 ft$^2$</td>
</tr>
</tbody>
</table>
162 BOXES
20 INTERFERENCE ELEMENTS
23 SLENDER BODY ELEMENTS

Figure 15. Aerodynamic Idealization, Top and Side Views
Figure 16. Aerodynamic Idealization, End View
<table>
<thead>
<tr>
<th>Stability Derivative (per radian)</th>
<th>Doublet Lattice Model, Missiles-Off</th>
<th>Doublet Lattice Model, Missiles-On</th>
<th>Wind Tunnel Data, Missiles-On</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{z_B}$</td>
<td>-.0948</td>
<td>-.0944</td>
<td>-.1157</td>
</tr>
<tr>
<td>$C_{z_p}$</td>
<td>-.379</td>
<td>-.462</td>
<td>-.325</td>
</tr>
<tr>
<td>$C_{z_r}$</td>
<td>.111</td>
<td>.110</td>
<td>.171</td>
</tr>
<tr>
<td>$C_{z_{\delta A}}$</td>
<td>-.190</td>
<td>-.207</td>
<td>-.1184</td>
</tr>
<tr>
<td>$C_{z_{\delta HT}}$</td>
<td>-.0794</td>
<td>-.0796</td>
<td>-.062</td>
</tr>
<tr>
<td>$C_{n_{\delta R}}$</td>
<td>.0509</td>
<td>.0508</td>
<td>.0355</td>
</tr>
<tr>
<td>$C_{n_B}$</td>
<td>.437</td>
<td>.443</td>
<td>.260</td>
</tr>
<tr>
<td>$C_{n_p}$</td>
<td>.0406</td>
<td>.0334</td>
<td>-.004</td>
</tr>
<tr>
<td>$C_{n_r}$</td>
<td>-.524</td>
<td>-.524</td>
<td>-.456</td>
</tr>
<tr>
<td>$C_{n_{\delta A}}$</td>
<td>-.022</td>
<td>-.023</td>
<td>-.019</td>
</tr>
<tr>
<td>$C_{n_{\delta HT}}$</td>
<td>-.102</td>
<td>-.102</td>
<td>-.075</td>
</tr>
<tr>
<td>$C_{n_{\delta R}}$</td>
<td>-.172</td>
<td>-.172</td>
<td>-.1066</td>
</tr>
<tr>
<td>$C_{y_B}$</td>
<td>-1.052</td>
<td>-1.091</td>
<td>-1.243</td>
</tr>
<tr>
<td>$C_{y_p}$</td>
<td>-.0825</td>
<td>-.0678</td>
<td>.014</td>
</tr>
<tr>
<td>$C_{y_r}$</td>
<td>1.345</td>
<td>1.363</td>
<td>.90</td>
</tr>
<tr>
<td>$C_{y_{\delta A}}$</td>
<td>.0502</td>
<td>.0532</td>
<td>.0264</td>
</tr>
<tr>
<td>$C_{y_{\delta HT}}$</td>
<td>.207</td>
<td>.207</td>
<td>.1570</td>
</tr>
<tr>
<td>$C_{y_{\delta R}}$</td>
<td>.316</td>
<td>.316</td>
<td>.209</td>
</tr>
</tbody>
</table>
show this assumption to be generally correct except for the damping due to roll rate derivative $C_{\delta_p}$ which is about twenty percent less for the missiles-off configuration. Overall, the calculated values are in general agreement with the experimental data which validates the aerodynamic idealization.

Stability derivatives for airloads on the tip missile and launcher were also measured in wind tunnel tests. The lift coefficient and aerodynamic center data for the missile and launcher combination are very important from the point of view of predicting the antisymmetric missile pitch mode instability encountered during flight. A comparison of the doublet-lattice calculated lift coefficient and aerodynamic center with the corresponding values obtained from wind tunnel tests are shown in Table 22. This good comparison gives confidence that the analytic aerodynamic model for the missile and launcher combination is satisfactory for prediction purposes.

**Modification Factor Data for Control Surface Aerodynamics**

It is noted in Table 21 that the stability derivatives due to the control surface deflections are overpredicted. As discussed in Chapter III, one method to modify the calculated generalized unsteady aerodynamic forces due to a given control surface deflection is to scale them by the ratio of the experimental to calculated values of a selected rigid static stability derivative for that control surface. Using equations (3.7), (3.8), (3.6) and the data of Table 21, the modification factors for aileron rotation, differential horizontal tail rotation, and rudder rotation are given in Table 23. These
### TABLE 22. COMPARISON OF CALCULATED VALUES AND EXPERIMENTAL VALUES FOR MISSILE PLUS LAUNCHER AERODYNAMIC DATA, M=0.9

<table>
<thead>
<tr>
<th>Aerodynamic Parameter</th>
<th>Doublet Lattice Model</th>
<th>Wind Tunnel Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lift Coefficient, $C_L\alpha$ (per radian)</td>
<td>0.099</td>
<td>0.11</td>
</tr>
<tr>
<td>Aerodynamic Center Location</td>
<td>F.S. 380.21</td>
<td>F.S. 376.</td>
</tr>
</tbody>
</table>

### TABLE 23. MODIFICATION FACTORS TO THE CALCULATED CONTROL SURFACE GENERALIZED UNSTEADY AERODYNAMIC FORCES

<table>
<thead>
<tr>
<th>CONTROL SURFACE, $j$</th>
<th>MODIFICATION FACTOR, $K_{\delta_j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (aileron)</td>
<td>.5720</td>
</tr>
<tr>
<td>2 (HT)</td>
<td>.7789</td>
</tr>
<tr>
<td>3 (rudder)</td>
<td>.6198</td>
</tr>
</tbody>
</table>
factors can be used to scale the control surface generalized unsteady aerodynamic forces as suggested in expressions (3.2) and (3.3).

A comparison of calculated values and experimental values of control surface hinge moment data is given in Table 24. It is noted that although the hinge moment for the aileron and rudder are also overpredicted, the all movable horizontal tail hinge moment is a substantially underpredicted value. The reason for this inaccurate prediction is thought to be caused by the small moment arm for an all movable horizontal tail, and is discussed in more detail in the latter part of Chapter III. Also as discussed in Chapter III, another method to modify the calculated generalized unsteady aerodynamic forces due to a given control surface deflection is to scale them by the ratio of the experimental to calculated values of the hinge moment for that control surface. These ratios are given in Table 24. Except for the all movable horizontal tail, these hinge moment ratios for the aileron and rudder trailing edge control surfaces are reasonably comparable to the corresponding stability derivative ratios of Table 23. Therefore, to provide a reasonable and consistent set of modification factors to the calculated control surface generalized unsteady aerodynamic forces, the modification factors based on control surface stability derivatives given by Table 23 were chosen for this application case.

Rigid Body Mode Aerodynamic Modification Data

It is pointed out in reference 9 that the calculated, frequency domain, generalized unsteady aerodynamic forces for the rigid body
### TABLE 24. COMPARISON OF CALCULATED VALUES AND EXPERIMENTAL VALUES FOR CONTROL SURFACE HINGE MOMENT DATA, M=0.9

<table>
<thead>
<tr>
<th>Hinge Moment Parameter</th>
<th>Doublet Lattice Model</th>
<th>Wind Tunnel Data</th>
<th>Ratio of Experimental to Calculated</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{H.M.}{q_\infty \delta_A}$ (in$^3$/rad$^3$)</td>
<td>55,857.</td>
<td>31,354.</td>
<td>.561</td>
</tr>
<tr>
<td>$\frac{H.M.}{q_\infty \delta_{HT}}$ (in$^3$/rad$^3$)</td>
<td>17,547.</td>
<td>62,172.</td>
<td>3.68</td>
</tr>
<tr>
<td>$\frac{H.M.}{q_\infty \delta_R}$ (in$^3$/rad$^3$)</td>
<td>25,304.</td>
<td>11,536.</td>
<td>.456</td>
</tr>
</tbody>
</table>

### TABLE 25. MODIFICATION FACTORS FOR THE CALCULATED FREQUENCY DOMAIN, GENERALIZED UNSTEADY AERODYNAMIC FORCES FOR THE RIGID BODY MODES

<table>
<thead>
<tr>
<th>Mode</th>
<th>1 ($\phi$)</th>
<th>2 ($\psi$)</th>
<th>3 (LT)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>R</td>
<td>I</td>
<td>R</td>
</tr>
<tr>
<td>1 ($\phi$)</td>
<td>.7035</td>
<td>.7035</td>
<td>1.2256</td>
</tr>
<tr>
<td>2 ($\psi$)</td>
<td>-.1198</td>
<td>-.1198</td>
<td>.5869</td>
</tr>
<tr>
<td>3 (LT)</td>
<td>-.2065</td>
<td>-.2065</td>
<td>1.1393</td>
</tr>
</tbody>
</table>
modes can be modified by scaling them by the ratio of the experimental to the calculated values of rigid stability derivatives. The frequency domain, generalized aerodynamic terms and the corresponding stability derivative which can be deduced from each term, for a small value of reduced frequency in a like manner as developed in Appendix C, are as follows:

\[
\begin{bmatrix}
Q_\phi,\phi(i\omega) & Q_\phi,\psi(i\omega) & Q_\phi,n(i\omega) \\
Q_\psi,\phi(i\omega) & Q_\psi,\psi(i\omega) & Q_\psi,n(i\omega) \\
Q_n,\phi(i\omega) & Q_n,\psi(i\omega) & Q_n,n(i\omega)
\end{bmatrix}
\]

\[
\sim \begin{bmatrix}
(0 + iC_{\ell,p}) & (C_{\ell} + iC_{\ell,r}) & (0 + iC_{\ell}) \\
(0 + iC_{n,p}) & (C_{n} + iC_{n}) & (0 + iC_{n}) \\
(0 + iC_{y,p}) & (C_{y} + iC_{y}) & (0 + iC_{y})
\end{bmatrix}
\]

\[(4.2)\]

Thus, by forming the appropriate ratios of the missiles-on data of Table 21, the modification factors for both the real and imaginary parts of the frequency domain, unsteady generalized aerodynamic forces for the rigid body modes are given in Table 25.

Unsteady Aerodynamic Data

The mode shapes for the five input normal modes and three control surface deflections were carefully approximated by a least squares fit for the polynomial coefficients required for input to the doublet-lattice unsteady aerodynamics computer program calculations as discussed in Appendix B. The generalized aerodynamic forces \(\bar{Q}_{ij}(i\omega)\) and \(\bar{Q}^c_{ij}(i\omega)\) were then calculated for five values of reduced
frequency (corresponding to a frequency range of 0 to 15 Hz) for 0.9 Mach number. These generalized aerodynamic forces in the frequency domain were then fitted as developed in Chapter II to the Laplace domain. The fitted coefficients of the Laplace polynomial numerator for each element of the aerodynamic matrix \([\bar{A}(s)]\) are given in Table 26 for the missiles-on configuration. As discussed in Chapter II, the analysis input modes of airplane roll and lateral translation have no effective angle of attack at zero frequency, and so the numerator coefficient \(a_{0ij}\) is zero for any aerodynamic matrix element involving these pressure modes. The fitted coefficients of the Laplace polynomial numerator for each element of the control surface aerodynamic matrix \([\bar{A}_c(s)]\) are given in Table 27. In addition, the common denominator constants of equation (2.40) for each matrix are given in Tables 26 and 27 also. Although not shown, the corresponding fitted coefficients for the missiles-off configuration were generated in the same manner.

Quasi-Steady Aerodynamic Data

For comparison purposes, a stability analysis was performed with quasi-steady aerodynamics in addition to the stability analysis with the unsteady aerodynamics as discussed above. For this quasi-steady case, the aerodynamic matrix elements for the five analysis input modes are assumed to consist of a displacement term and a rate term, and the elements of the control surface aerodynamic matrix are assumed to consist of a displacement term only. Thus, instead of using the fitted Laplace polynomial fraction of equation (2.55) for
TABLE 26. FITTED COEFFICIENTS FOR THE LAPLACE POLYNOMIAL FRACTIONS OF EQUATION (2.55) FOR THE ELEMENTS OF THE AERODYNAMIC MATRIX \([\mathbf{A}(s)]\) FOR THE MISSILES-ON CONFIGURATION

<table>
<thead>
<tr>
<th>Mode 1</th>
<th>Mode 1</th>
<th>(a_{01j})</th>
<th>(a_{11j})</th>
<th>(a_{21j})</th>
<th>(a_{31j})</th>
<th>(a_{41j})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1</td>
<td>.0</td>
<td>-.35307x10^2</td>
<td>-.25597x10^1</td>
<td>-.36105x10^-1</td>
<td>-.24130x10^-4</td>
<td></td>
</tr>
<tr>
<td>1, 2</td>
<td>.44631x10^3</td>
<td>.41601x10^2</td>
<td>.10284x10^1</td>
<td>.72399x10^-2</td>
<td>.13504x10^-6</td>
<td></td>
</tr>
<tr>
<td>1, 3</td>
<td>.0</td>
<td>.37065x10^-1</td>
<td>.30639x10^-2</td>
<td>.33988x10^-4</td>
<td>.56298x10^-7</td>
<td></td>
</tr>
<tr>
<td>1, 4</td>
<td>-.12676x10^2</td>
<td>-.10007x10^-1</td>
<td>-.15461x10^-1</td>
<td>-.51594x10^-4</td>
<td>-.10969x10^-6</td>
<td></td>
</tr>
<tr>
<td>1, 5</td>
<td>-.13390x10^2</td>
<td>-.10352x10^-1</td>
<td>-.16095x10^-1</td>
<td>-.23013x10^-5</td>
<td>.20330x10^-6</td>
<td></td>
</tr>
<tr>
<td>2, 1</td>
<td>.0</td>
<td>.20379x10</td>
<td>.23338</td>
<td>.47801x10^-2</td>
<td>.13557x10^-6</td>
<td></td>
</tr>
<tr>
<td>2, 2</td>
<td>-.2069x10^4</td>
<td>-.19424x10^3</td>
<td>-.47552x10^-1</td>
<td>-.35087x10^-1</td>
<td>-.26791x10^-4</td>
<td></td>
</tr>
<tr>
<td>2, 3</td>
<td>.0</td>
<td>-.18942</td>
<td>-.13922x10^-1</td>
<td>-.17394x10^-3</td>
<td>-.21169x10^-6</td>
<td></td>
</tr>
<tr>
<td>2, 4</td>
<td>-.1079x10</td>
<td>-.54006x10^-1</td>
<td>.10599x10^-2</td>
<td>.11752x10^-4</td>
<td>.15055x10^-7</td>
<td></td>
</tr>
<tr>
<td>2, 5</td>
<td>-.32763x10</td>
<td>-.25675</td>
<td>-.40321x10^-2</td>
<td>-.19509x10^-4</td>
<td>.57217x10^-7</td>
<td></td>
</tr>
<tr>
<td>3, 1</td>
<td>.0</td>
<td>.12325x10^-1</td>
<td>.13577x10^-2</td>
<td>.27926x10^-4</td>
<td>.67424x10^-7</td>
<td></td>
</tr>
<tr>
<td>3, 2</td>
<td>-.14774x10^2</td>
<td>-.14042x10^-1</td>
<td>-.35655x10^-1</td>
<td>-.24527x10^-3</td>
<td>-.79932x10^-6</td>
<td></td>
</tr>
<tr>
<td>3, 3</td>
<td>.0</td>
<td>-.13463x10^-2</td>
<td>-.10001x10^-3</td>
<td>-.13611x10^-5</td>
<td>-.33781x10^-8</td>
<td></td>
</tr>
<tr>
<td>3, 4</td>
<td>-.64945x10^-2</td>
<td>-.31936x10^-3</td>
<td>.62245x10^-5</td>
<td>.56300x10^-7</td>
<td>.41345x10^-10</td>
<td></td>
</tr>
<tr>
<td>3, 5</td>
<td>-.19581x10^-1</td>
<td>-.14973x10^-2</td>
<td>-.21573x10^-4</td>
<td>-.95372x10^-7</td>
<td>.30094x10^-9</td>
<td></td>
</tr>
<tr>
<td>4, 1</td>
<td>.0</td>
<td>-.30070x10^-1</td>
<td>-.20495x10^-2</td>
<td>-.39152x10^-4</td>
<td>-.58145x10^-7</td>
<td></td>
</tr>
<tr>
<td>4, 2</td>
<td>.11771x10</td>
<td>.10691</td>
<td>.25740x10^-2</td>
<td>.16476x10^-6</td>
<td>.12476x10^-7</td>
<td></td>
</tr>
<tr>
<td>4, 3</td>
<td>.0</td>
<td>.99866x10^-4</td>
<td>.77472x10^-2</td>
<td>.83217x10^-7</td>
<td>.17611x10^-11</td>
<td></td>
</tr>
<tr>
<td>4, 4</td>
<td>-.53934x10^-2</td>
<td>-.52408x10^-3</td>
<td>-.14271x10^-4</td>
<td>-.95801x10^-7</td>
<td>-.14799x10^-9</td>
<td></td>
</tr>
<tr>
<td>4, 5</td>
<td>-.19796x10^-2</td>
<td>-.21294x10^-3</td>
<td>-.94772x10^-5</td>
<td>-.85978x10^-8</td>
<td>.44798x10^-9</td>
<td></td>
</tr>
<tr>
<td>5, 1</td>
<td>.0</td>
<td>-.85050x10^-2</td>
<td>-.69247x10^-3</td>
<td>-.11947x10^-4</td>
<td>-.75737x10^-7</td>
<td></td>
</tr>
<tr>
<td>5, 2</td>
<td>-.19334x10</td>
<td>-.17948</td>
<td>-.42422x10^-2</td>
<td>-.33904x10^-4</td>
<td>-.14128x10^-7</td>
<td></td>
</tr>
<tr>
<td>5, 3</td>
<td>.0</td>
<td>-.18757x10^-3</td>
<td>-.12536x10^-4</td>
<td>-.16079x10^-6</td>
<td>-.17160x10^-9</td>
<td></td>
</tr>
<tr>
<td>5, 4</td>
<td>-.35037x10^-1</td>
<td>-.26847x10^-2</td>
<td>-.36326x10^-4</td>
<td>-.77455x10^-7</td>
<td>.18675x10^-9</td>
<td></td>
</tr>
<tr>
<td>5, 5</td>
<td>-.33723x10^-1</td>
<td>-.29408x10^-2</td>
<td>-.61736x10^-4</td>
<td>-.45866x10^-6</td>
<td>-.69993x10^-9</td>
<td></td>
</tr>
</tbody>
</table>

\(b_1 = 16.579\), \(b_2 = 81.624\)
### TABLE 27. FITTED COEFFICIENTS FOR THE LAPLACE POLYNOMIAL FRACTIONS OF EQUATION (2.56) FOR THE ELEMENTS OF THE AERODYNAMIC MATRIX $\tilde{A}(s)$ FOR THE MISSILES-ON CONFIGURATION

<table>
<thead>
<tr>
<th>Mode</th>
<th>C.S.</th>
<th>$a_{01j}$</th>
<th>$a_{11j}$</th>
<th>$a_{21j}$</th>
<th>$a_{31j}$</th>
<th>$a_{41j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1</td>
<td></td>
<td>0.97554x10³</td>
<td>0.72399x10²</td>
<td>0.79573</td>
<td>-0.18215x10⁻²</td>
<td>0.24764x10⁻⁴</td>
</tr>
<tr>
<td>1, 2</td>
<td></td>
<td>0.37518x10³</td>
<td>0.26682x10²</td>
<td>0.42076</td>
<td>0.13523x10⁻²</td>
<td>-0.33935x10⁻⁵</td>
</tr>
<tr>
<td>1, 3</td>
<td></td>
<td>0.23855x10³</td>
<td>0.17656x10²</td>
<td>0.17614</td>
<td>0.16639x10⁻³</td>
<td>-0.26245x10⁻⁵</td>
</tr>
<tr>
<td>2, 1</td>
<td></td>
<td>0.10824x10³</td>
<td>0.68983x10¹</td>
<td>0.36264x10⁻¹</td>
<td>0.36569x10⁻⁴</td>
<td>-0.53142x10⁻⁵</td>
</tr>
<tr>
<td>2, 2</td>
<td></td>
<td>0.48141x10³</td>
<td>0.34176x10²</td>
<td>0.31832</td>
<td>0.88041x10⁻³</td>
<td>0.42179x10⁻⁵</td>
</tr>
<tr>
<td>2, 3</td>
<td></td>
<td>-0.80843x10³</td>
<td>-0.57424x10²</td>
<td>-0.47425</td>
<td>0.59846x10⁻⁴</td>
<td>-0.71977x10⁻⁵</td>
</tr>
<tr>
<td>3, 1</td>
<td></td>
<td>0.71980</td>
<td>0.44182x10⁻¹</td>
<td>0.13199x10⁻³</td>
<td>0.18936x10⁻⁶</td>
<td>-0.25630x10⁻⁷</td>
</tr>
<tr>
<td>3, 2</td>
<td></td>
<td>0.27999x10</td>
<td>0.19472</td>
<td>0.16274x10⁻²</td>
<td>0.48683x10⁻⁵</td>
<td>-0.28191x10⁻⁷</td>
</tr>
<tr>
<td>3, 3</td>
<td></td>
<td>-0.42756x10</td>
<td>-0.29729</td>
<td>-0.22104x10⁻²</td>
<td>-0.74946x10⁻⁷</td>
<td>-0.34713x10⁻⁷</td>
</tr>
<tr>
<td>4, 1</td>
<td></td>
<td>0.15402x10</td>
<td>0.11453</td>
<td>0.14814x10⁻²</td>
<td>-0.39358x10⁻⁵</td>
<td>0.21713x10⁻⁷</td>
</tr>
<tr>
<td>4, 2</td>
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<td>0.14506</td>
<td>0.11429x10⁻¹</td>
<td>0.19438x10⁻³</td>
<td>0.11306x10⁻⁵</td>
<td>0.62575x10⁻⁸</td>
</tr>
<tr>
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<td>-0.87822x10⁻¹</td>
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<td>-0.63178x10⁻⁶</td>
<td>-0.22813x10⁻³</td>
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</table>

$b_1 = 16.579, b_2 = 81.624$
each element of the transformed aero matrix \([\bar{A}(s)]\), the following is used:

\[
\bar{A}_{ij}(s) \equiv q_{\infty}A_R c_R \bar{Q}_{ij}(s) \tag{4.3}
\]

where \(\bar{Q}_{ij}(s) = a_{0ij} + a_{1ij}s\)

The coefficients are obtained as follows from the real and imaginary parts of the generalized aerodynamic force in the frequency domain for a small value of reduced frequency:

\[
a_{0ij} = \tilde{Q}_{Rij}(i\omega_p) \tag{4.4}
\]

and

\[
a_{1ij} = \frac{\tilde{Q}_{Iij}(i\omega_p)}{\omega_p} \tag{4.5}
\]

where \(\tilde{Q}_{Rij}(i\omega_p)\) and \(\tilde{Q}_{Iij}(i\omega_p)\) are the real and imaginary parts of \(\tilde{Q}_{ij}(i\omega_p)\), respectively, and where the frequency \(\omega_p\) corresponds to the small value of reduced frequency used in the calculation of \(\tilde{Q}_{ij}(i\omega_p)\).

Again as for the pure unsteady aerodynamics, the analysis input modes of airplane roll and lateral translation have no effective angle of attack at zero frequency, and so the coefficient \(a_{0ij}\) is zero for any aerodynamic matrix element involving these pressure modes. Likewise instead of using the fitted Laplace polynomial fraction of equation (2.56) for each element of the control surface transformed aerodynamic matrix \([\bar{A}^c(s)]\), the following is used:

\[
\bar{A}^c_{ij}(s) \equiv q_{\infty}A_R c_R \bar{Q}^c_{ij}(s) \tag{4.6}
\]
The coefficient $a_{0ij}$ is obtained as follows from the real part of the control surface generalized aerodynamic force in the frequency domain for a small value of reduced frequency:

$$a_{0ij} = \Re \{ \bar{Q}_{ij}^c (s) \}$$

where $\Re \{ \bar{Q}_{ij}^c (s) \}$ is the real part of $\bar{Q}_{ij}^c (s)$, and where the frequency $\omega_p$ corresponds to the small value of reduced frequency used in the calculation of $\bar{Q}_{ij}^c (i\omega_p)$. Thus, for the rigid body modes and the control surface deflections, this quasi-steady aerodynamic representation conforms to the methods used in stability and control using stability derivatives. However, this method does not account for phase lags present in pure unsteady aerodynamics. As discussed in Chapter II, this quasi-steady method is analogous to neglecting the Theodorsen lag function $C(k)$ in strip theory aerodynamics.

The coefficients of the quasi-steady aerodynamic representation of equation (4.3) for each element of the aerodynamic matrix $[\bar{A}(s)]$ are given in Table 28 for the missiles-on configuration. Likewise, the coefficients of the quasi-steady aerodynamic representation of equation (4.6) for each element of the matrix $[\bar{A}^c(s)]$ are given in Table 29. Although not shown, the corresponding quasi-steady aerodynamic coefficients for the missiles-off were generated in the same manner. It is noted that while these quasi-steady aerodynamic matrix terms for the rigid body and control surface pressure modes could be


<table>
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<tr>
<th>Mode 1</th>
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<th>$a_{01}$</th>
<th>$a_{11}$</th>
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<tr>
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<td>.32846</td>
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<td>0</td>
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<tr>
<td>5, 5</td>
<td></td>
<td>-2.4912x10^{-4}</td>
<td>-3.6189x10^{-6}</td>
</tr>
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</table>
TABLE 29. QUASI-STEADY COEFFICIENTS FOR THE LAPLACE POLYNOMIAL OF EQUATION (4.6) FOR THE ELEMENTS OF THE AERODYNAMIC MATRIX $|A^c(s)|$ FOR THE MISSILES-ON CONFIGURATION

<table>
<thead>
<tr>
<th>Mode C.S. $i, j$</th>
<th>$a_{0ij}$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.71953</td>
</tr>
<tr>
<td>1, 2</td>
<td>0.27704</td>
</tr>
<tr>
<td>1, 3</td>
<td>0.17674</td>
</tr>
<tr>
<td>2, 1</td>
<td>$8.0450 \times 10^{-1}$</td>
</tr>
<tr>
<td>2, 2</td>
<td>0.35536</td>
</tr>
<tr>
<td>2, 3</td>
<td>-0.59747</td>
</tr>
<tr>
<td>3, 1</td>
<td>$5.3158 \times 10^{-3}$</td>
</tr>
<tr>
<td>3, 2</td>
<td>$2.0659 \times 10^{-2}$</td>
</tr>
<tr>
<td>3, 3</td>
<td>$-3.1584 \times 10^{-2}$</td>
</tr>
<tr>
<td>4, 1</td>
<td>$1.1349 \times 10^{-2}$</td>
</tr>
<tr>
<td>4, 2</td>
<td>$1.0776 \times 10^{-3}$</td>
</tr>
<tr>
<td>4, 3</td>
<td>$4.4745 \times 10^{-3}$</td>
</tr>
<tr>
<td>5, 1</td>
<td>$1.2124 \times 10^{-2}$</td>
</tr>
<tr>
<td>5, 2</td>
<td>$-1.1375 \times 10^{-2}$</td>
</tr>
<tr>
<td>5, 3</td>
<td>$-8.9578 \times 10^{-3}$</td>
</tr>
</tbody>
</table>
generated by use of the associated experimental stability derivatives, the quasi-steady aerodynamic terms for the flexible airplane pressure modes would still need to be generated by use of an unsteady aerodynamic theory, such as the doublet-lattice theory, for a small value of reduced frequency.
CHAPTER V RESULTS

Using the data of the previous chapter, the matrix equation of motion (1.25) was set up for a given set of control system feedback loop gains and the given critical flight conditions for both the missiles-on and the missiles-off configurations. The elements of this matrix were multiplied by a common denominator formed by the denominator terms of the control system and the denominator of the Laplace polynomial fraction aerodynamic matrix elements. After the elements of the coefficient matrix of equations (1.25) were multiplied by the common denominator, the matrix equation of motion was formulated in terms of an eigenvalue problem and solved for the roots or eigenvalues by use of the QR transform method as discussed in Appendix A.

The common denominator for the yaw channel is seen to be the common denominator for the combined roll and yaw channels of the control system. This common denominator of the control system produces eleven roots. The common denominator of the Laplace unsteady aerodynamic fraction produces two additional roots, and the second order system for each normal mode produces two more roots. Therefore, the total number of roots for each normal mode included in the stability analysis of the augmented airplane was 15 roots. So when all five analysis modes were included in the analysis, the total number of roots for the augmented airplane system was 75. A sample computer
program output of the complex roots extracted by the stability analysis root solution is shown in Table 30 for one set of gains for the case plotted in Figure 22. As can be seen, the number of roots can become quite large when a large number of modes are included in the analysis. However, the QR transform method for eigenvalue extraction is accurate for large order systems and has been used to accurately extract roots from systems with as many as 200 roots (8).

As discussed in Chapter I and Appendix A, the stability eigenvalue problem was formed for a given airplane configuration, a given flight condition, and a given set of control system feedback loop gains. The complex roots or eigenvalues were obtained as shown in the example of Table 30, and then the procedure was repeated for selected gain values of a given control feedback loop. The significant roots of the aeroservoelastic airplane system were then plotted on a root locus plot as a function of this gain variation.

Using the data of Chapter IV, the two airplane configurations of missiles-on and missiles-off were analyzed for one flight condition each. Root locus plots were made for, primarily, a roll rate feedback gain variation, but also in some cases for a yaw rate feedback gain variation and a lateral acceleration feedback gain variation. As discussed in Chapter IV, these stability analyses used various aerodynamic representations, including unmodified and modified control surface unsteady aerodynamics, unmodified and modified rigid body aerodynamics, and unsteady and quasi-steady aerodynamics.
### TABLE 30. SAMPLE COMPUTER PROGRAM OUTPUT FOR THE STABILITY ANALYSIS

#### ROOT SOLUTION

**Stability Analysis**

<table>
<thead>
<tr>
<th>Adjustable Roll Rate Feedback Gain</th>
<th>Contribution of Roll Rate Gain to Flap Gain</th>
<th>Contribution of Roll Rate Gain to H.T.</th>
<th>Overall Gain from Lat. Accel. to Rate</th>
<th>Overall Gain from Yaw Rate Gain to Rudder</th>
</tr>
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<tbody>
<tr>
<td>0.300</td>
<td>1.200</td>
<td>0.250</td>
<td>0.300</td>
<td>0.518</td>
</tr>
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</table>

**Degrees of Freedom Included in Analysis**

- Roll
- Yaw
- Lat
- Flex1
- Flex2

**IER = 0**

**The Roots Are**

<table>
<thead>
<tr>
<th>Root</th>
<th>Real</th>
<th>Imaginary</th>
<th>Damping (Zeta)</th>
<th>Freq, Damped (Hz)</th>
<th>Neutrally Stable</th>
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</thead>
<tbody>
<tr>
<td>1</td>
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<td>0.0</td>
<td>-1</td>
<td>0.000</td>
<td>Stable</td>
</tr>
<tr>
<td>2</td>
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<td>0.0</td>
<td>-1</td>
<td>0.000</td>
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</tr>
<tr>
<td>3</td>
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<td>0.25119 E+03</td>
<td>0.59911</td>
<td>40.640</td>
<td>Stable</td>
</tr>
<tr>
<td>4</td>
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<td>0.25119 E+03</td>
<td>0.59911</td>
<td>40.640</td>
<td>Stable</td>
</tr>
</tbody>
</table>

**Roots are stable.**
| X   | Y   | A       | B       | C       | D       | E       | F       | G       | H       | I       | J       | K       | L       | M       | N       | O       | P       | Q       | R       | S       | T       | U       | V       | W       | X       | Y       | Z       |
|-----|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.0 | 0.0 | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   |
| 0.1 | 0.1 | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   | 0.100   |
| 0.2 | 0.2 | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   | 0.200   |
| 0.3 | 0.3 | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   | 0.300   |
| 0.4 | 0.4 | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   | 0.400   |
| 0.5 | 0.5 | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   | 0.500   |
| 0.6 | 0.6 | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   | 0.600   |
| 0.7 | 0.7 | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   | 0.700   |
| 0.8 | 0.8 | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   | 0.800   |
| 0.9 | 0.9 | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   | 0.900   |

**Legend:**
- **A:** X
- **B:** Y
- **C:** A
- **D:** B
- **E:** C
- **F:** D
- **G:** E
- **H:** F
- **I:** G
- **J:** H
- **K:** I
- **L:** J
- **M:** K
- **N:** L
- **O:** M
- **P:** N
- **Q:** O
- **R:** P
- **S:** Q
- **T:** R
- **U:** S
- **V:** T
- **W:** U
- **X:** V
- **Y:** W
- **Z:** X
Analysis Using Unmodified Aerodynamics

Figure 17 shows the roots for the missiles-on configuration with unmodified unsteady aerodynamics for all feedback gains off and on at nominal values. As can be seen the missile pitch flexible mode (6.7 Hz) is stable for the control augmentation system off, but unstable when the control system is on. The wing bending mode (8.4 Hz) is stabilized slightly with the control system on, and as usually desired, the Dutch roll airplane mode is stabilized with the control augmentation system on. Mode and feedback loop deletion studies show the two roots between the missile pitch flexible mode and the Dutch roll mode to be, predominantly, a rigid body yaw mode and a rigid body roll mode which have coupled with the various components of the control system. The rigid body roll root appears to be a coupling of the un-augmented airplane roll convergence root and the aileron and horizontal tail actuators through the roll rate feedback loop. The rigid body yaw root appears to result for the analysis input airplane rigid body yaw mode coupling with the rudder actuator through the yaw rate feedback loop.

Figure 18 shows the root locus solution for the missiles-on configuration for a roll rate feedback gain variation, the other gains nominal, and with unmodified unsteady aerodynamics. The missile pitch flexible mode (6.7 Hz) is predicted to be unstable at a gain of half the value for which this instability was encountered in flight testing. Thus the missile pitch instability prediction is conservative using unmodified unsteady aerodynamics. The rigid body roll root can be
Figure 17. Root Locus Solution of Missiles-On Configuration Including Five Modes with all Feedback Gains at Both Nominal and Zero Values, Unmodified Aerodynamics, $M=0.9, 20,000$ ft
Figure 18. Root Locus Solution of Missiles-On Configuration
Including Five Modes with Roll Rate Feedback Gain Variation, Other Gains Nominal, Unmodified Aerodynamics, $M=0.9$, 20,000 ft
driven unstable, but it would take double the nominal gain to do so. The root locus solution of Figure 19 is for the same missiles-on case as Figure 18 discussed above, except that the stability analysis included only the airplane roll mode and the missile pitch flexible mode. It is noticed that the missile pitch instability is predicted at only a slightly increased gain as when all five modes are included in the stability analysis.

The root locus solution for the missiles-off configuration with unmodified aerodynamics and for a roll rate feedback gain variation is shown in Figure 20. The two flexible modes are stable, but the rigid body roll mode, although coupled with the rigid body yaw mode in the root locus plot, is now nearly unstable at the nominal roll rate gain of 0.2. This prediction for the rigid body roll instability is at the same gain as the instability encountered in flight testing but the predicted frequency (4.6 Hz) is considerably higher than the flight instability frequency (3.5 Hz). For a given roll rate gain, the rigid body roll mode is more critical for the missiles-off configuration compared to the missiles-on configuration because the missiles-off configuration is effectively operating at a higher roll rate gain since the airplane roll inertia is substantially less (31 percent less) without the tip missiles. The solution of Figure 21 is for the same missiles-off case as for Figure 20 discussed above, but the stability analysis included the airplane roll mode only. The root locus solution is plotted versus roll rate feedback gain and clearly shows the rigid body roll root to be a coupling of the
Figure 19. Root Locus Solution of Missiles-On Configuration Including only the Airplane Roll Mode and the First Flexible Mode with Roll Rate Feedback Gain Variation, Other Gains Nominal, Unmodified Aerodynamics, \( M=0.9 \), 20,000 ft
Figure 20. Root Locus Solution of Missiles-Off Configuration Including Five Modes with Roll Rate Feedback Gain Variation, Other Gains Nominal, Unmodified Aerodynamics, M=0.9, 15,000 ft
Figure 21. Root Locus Solution of Missiles-Off Configuration Including only the Airplane Roll Mode with Roll Rate Feedback Gain Variation, Other Gains Nominal, Unmodified Aerodynamics, M=0.9, 15,000 ft
unaugmented airplane roll convergence root and the aileron and horizontal tail actuators through the roll rate feedback loop. It is noticed that this rigid body roll instability is predicted at the same gain value when the stability analysis includes either the airplane roll mode only or all five analysis input modes.

Analysis Using Modified Control Surface Aerodynamics

As pointed out in the Introduction and Chapter III, the control surface effectiveness (along with the feedback loop gain) is a direct multiplicative factor in determining the total feedback loop gain. So to properly analyze flight conditions of stability, realistic and accurate values of control surface effectiveness are needed. Therefore, in addition to using an accurate unsteady aerodynamic lifting surface theory to calculate the control surface aerodynamics, these calculated control surface aerodynamics can be modified so the zero frequency case of the aerodynamics matches the steady state experimental data. In an effort to improve the instability predictions compared with flight test results, multiplicative modification factors based on the selected control surface rigid stability derivative ratios given in Table 23 were applied to the magnitude of the calculated control surface aerodynamics in the remainder of the root locus solutions shown in this chapter.

Figure 22 shows the root locus solution for the missiles-on configuration for a roll rate feedback gain variation and with modification factors applied, as discussed above, to the control surface aerodynamics. For a given roll rate gain the overall effect to the root locus solution is to effectively reduce the total feedback loop
Figure 22. Root Locus Solution of Missiles-On Configuration Including Five Modes with Roll Rate Feedback Gain Variation, Other Gains Nominal, Modified Magnitude of Control Surface Aerodynamics, $M=0.9$, 20,000 ft
Figure 23. Root Locus Solution of Missiles-Off Configuration Including Five Modes with Roll Rate Feedback Gain Variation, Other Gains Nominal, Modified Magnitude of Control Surface Aerodynamics, M=0.9, 15,000 ft
gain because of the reduced control surface effectiveness. Thus, this solution of Figure 22 shows the missile pitch mode (6.7 Hz) to go unstable at a roll rate feedback gain of 0.2 instead of 0.1 for the solution using the unmodified control surface aerodynamics. This prediction using the modified control surface aerodynamics matches the flight test gain setting and frequency for the missile pitch instability. Figure 23 shows the corresponding root locus solution for the missiles-off configuration with the modification factors applied to the control surface aerodynamics. Again as for the missiles-on configuration, the effect of the reduced control surface effectiveness increases the roll rate feedback gain necessary to cause an instability. This solution shows the rigid body roll root to go unstable at a roll rate feedback gain of 0.4 instead of 0.2 for the solution of Figure 20 using unmodified control surface aerodynamics. Thus, the prediction of roll rate feedback gain required for instability using unmodified control surface aerodynamics more closely matches the gain (0.2) for which the flight test instability occurred. However, the prediction of the frequency of instability (4.3 Hz) for this rigid body roll root using modified control surface aerodynamics is a somewhat better match with the flight test instability frequency (3.5 Hz) than the prediction (4.6 Hz) using unmodified control surface aerodynamics.

**Effect of Modified Rigid Body Mode Aerodynamics**

In an effort to explain possible causes of the poor correlation between the predicted and flight test results for this rigid body roll
instability, modifications to the rigid body aerodynamics and additional phase lag in the control surface aerodynamics were investigated. Figure 24 shows the root locus solution for the missiles-off configuration for a roll rate feedback gain variation using both unmodified rigid body aerodynamics and rigid body aerodynamics modified by the factors of Table 25. It is seen that very little difference results in modifying the rigid body aerodynamics except that the rigid body roll root is very slightly destabilized compared to results using unmodified rigid body aerodynamics. This difference is most probably because of the modification factor applied based on the ratio of damping due to roll rate derivatives. This roll rate damping derivative for the airplane was overpredicted compared to experimental data. Thus, the modification factors applied to the rigid body aerodynamics would account for this reduction in roll damping and would, in turn, be destabilizing in roll.

Effect of Additional Phase Lag in Control Surface Aerodynamics

For the effect of additional phase lag, Figure 25 shows the root locus solution for the missiles-off configuration for a roll rate feedback gain variation, with and without an additional 20 degrees of phase lag on the control surface aerodynamics. For clarity only the two critical roots of rigid body roll and yaw are plotted. It is seen that this extra 20 degree phase lag in the control surface aerodynamics, in addition to applying the modification factors to the magnitude of the control surface aerodynamics, predicts a rigid body roll instability at 3.6 Hz and for a roll rate feedback gain of about
Figure 24. Root Locus Solution of Missiles-Off Configuration Including Five Modes with Roll Rate Feedback Gain Variation, Other Gains Nominal, Modified Magnitude of Control Surface Aerodynamics, Modified Rigid Body Mode Aerodynamics, M=0.9, 15,000 ft
Figure 25. Root Locus Solution of Missiles-Off Configuration Including Five Modes with Roll Rate Feedback Gain Variation, Other Gains Nominal, Modified Magnitude and Additional 20 Degree Phase Lag in Control Surface Aerodynamics, $M=0.9$, 15,000 ft
0.25. This prediction is close to the flight results for this rigid body roll instability. This additional phase lag in the control surface aerodynamics could, possibly, result from shock formations due to the oscillatory control surface motion in the 0.9 Mach number, high subsonic speed regime which is the analysis and test flight condition. The doublet-lattice, unsteady aerodynamics lifting surface theory used for analysis prediction accounts for compressibility effects but assumes the flow to be shock-free. Tijdeman and Bergh (23) have shown experimentally that such shock effects in the high subsonic flow regime can cause more phase lag than that predicted by lifting surface theory for an oscillating control surface on a wing.

In addition to possible extra phase lag beyond that predicted by theoretical control surface unsteady aerodynamics, possible inadequacies in the application of modification factors to control surface unsteady aerodynamics based on steady state experimental data, have been recently pointed out. Giesing, Kalman, and Rodden (24) have shown that as frequency is increased, the experimental data approach the unmodified theory. They attribute this fact to viscous effects being reduced as the frequency of oscillation is increased. They indicate that on the average the static modification factors for control surface unsteady aerodynamics are useful up to about a reduced frequency of 0.1. Beyond this point it is better to use the original, unmodified theory. Probably a combination of both an additional phase lag in the control surface aerodynamics and the inadequacy of the static modification factors for the control surface unsteady aerodynamics, are involved in causing the poor correlation between the
predicted and flight test results for this rigid body roll instability.

For the effect of phase lag on the missiles-on configuration, Figure 26 shows the root locus solution for the missiles-on configuration for a roll rate feedback gain variation, with and without the additional 20 degrees of phase lag on the control surface aerodynamics. For clarity only the critical missile pitch mode is plotted. It is seen that although the extra phase lag is very slightly destabilizing, the roll rate feedback gain and frequency at which the missile pitch mode goes unstable is essentially unchanged as compared to the solution without the extra phase lag.

**Yaw Rate Feedback Gain Variation**

For completeness, root locus solutions were calculated for the missiles-on and missiles-off configurations for a yaw rate feedback gain variation, other gains nominal, and with modification factors applied to the control surface aerodynamics. Figure 27 shows the root locus solution for the missiles-on configuration. The two flexible modes are not at all affected by the yaw rate gain variation, and the rigid body roll root is affected very little. As expected, the Dutch roll mode is stabilized with increased yaw rate gain. The rigid body yaw root can be driven unstable at a frequency of 4.1 Hz and at a yaw rate gain of about 2.5 times the nominal value. The corresponding root locus solution of Figure 28 for the missiles-off configuration shows the same general results where the rigid body yaw root can now be driven unstable at a yaw rate gain of about 2.0 times the nominal value.
Figure 26. Root Locus Solution of Missiles-On Configuration Including Five Modes with Roll Rate Feedback Gain Variation, Other Gains Nominal, Modified Magnitude and Additional 20 Degree Phase Lag in Control Surface Aerodynamics, M=0.9, 20,000 ft
Figure 27. Root Locus Solution of Missiles-On Configuration Including Five Modes with Yaw Rate Feedback Gain Variation, Other Gains Nominal, Modified Magnitude of Control Surface Aerodynamics, $M=0.9$, 20,000 ft
Figure 28. Root Locus Solution of Missiles-Off Configuration Including Five Modes with Yaw Rate Feedback Gain Variation, Other Gains Nominal, Modified Magnitude of Control Surface Aerodynamics, M=0.9, 15,000 ft
Lateral Acceleration Feedback Gain Variation

In addition, root locus solutions were calculated for the missiles-on and missiles-off configurations for a lateral acceleration feedback gain variation, other gains nominal, and with modification factors applied to the control surface aerodynamics. Figure 29 shows the root locus solution for the missiles-on configuration. The two flexible modes, the rigid body roll root, and the Dutch roll mode are affected very little by the lateral acceleration gain variation. Even though the rigid body yaw root is affected significantly with increasing the lateral acceleration gain, it is not destabilized. The corresponding root locus solution of Figure 30 for the missiles-off configuration shows the same general results.

Effect of Quasi-Steady Aerodynamics

For comparison purposes, a stability analysis, as discussed in Chapter IV, was performed using quasi-steady aerodynamics for the missiles-on and missiles-off configurations for a roll rate feedback gain variation, other gains nominal, and with modification factors applied to the control surface aerodynamics. Figure 31 shows the root locus solution for the missiles-on configuration using both unsteady aerodynamics and quasi-steady aerodynamics. Figure 32 shows the corresponding comparison for the missiles-off configuration. Surprisingly for these flight conditions, the solution using the quasi-steady aerodynamics differs very little from the solution using the pure unsteady aerodynamics except for modes with frequencies above about 7 Hz where the results start to diverge due to unsteady effects. One reason for this can be seen in Figure 8 where a displacement term
Figure 29. Root Locus Solution of Missiles-On Configuration Including Five Modes with Lateral Acceleration Feedback Gain Variation, Other Gains Nominal, Modified Magnitude of Control Surface Aerodynamics, $M=0.9$, 20,000 ft
Figure 30. Root Locus Solution of Missiles-Off Configuration Including Five Modes with Lateral Acceleration Feedback Gain Variation, Other Gains Nominal, Modified Magnitude of Control Surface Aerodynamics, $M=0.9$, 15,000 ft
Figure 31. Root Locus Solution of Missiles-On Configuration Including Five Modes with Roll Rate Feedback Gain Variation, Other Gains Nominal, Quasi-Steady Aerodynamics and Modified Magnitude of Control Surface Aerodynamics, $M=0.9, 20,000\ ft$
Figure 32. Root Locus Solution of Missiles-Off Configuration Including Five Modes with Roll Rate Feedback Gain Variation, Other Gains Nominal, Quasi-Steady Aerodynamics and Modified Magnitude of Control Surface Aerodynamics, $M=0.9$, 15,000 ft
only, which is used in the quasi-steady, control surface aero­
dynamics representation of equation (4.6), is a good approximation
up to about a frequency of 7 Hz.

**Effect of Notch Filter**

As discussed in the beginning of Chapter IV, the prototype air­
plane used as the stability analysis application case unexpectedly
encountered instabilities involving this missile pitch flexible mode
and the rigid body roll mode during early flight tests. These in­
stabilities were eliminated by adding a notch filter and reducing
the gain in the roll rate feedback loop of the flight control system.
The notch filter characteristics of the added notch filter are shown
in Figure 33. It is noted that while the effective gain of the lower
frequencies (2 to 8 Hz) is reduced, the effective gain of the higher
frequencies (above 8 Hz) is increased. To show the effect of adding
this notch filter to the flight control system, a stability analysis
was performed with the notch filter for the missiles-on and missiles­
off configurations for a roll rate feedback gain variation, other
gains nominal, and with modification factors applied to the control
surface aerodynamics. Figure 34 shows the root locus solution for
the missiles-on configuration with and without the notch filter.
As can be seen the missile pitch flexible mode (6.7 Hz) is stabilized
with the notch filter by not only a reduction in the roll rate feed­
back gain at that frequency but also by a significant stabilizing
phase shift. Figure 35 shows the corresponding comparison for the
missiles-off configuration. With the notch filter in the flight
Figure 33. Notch Filter Characteristics

FILTER TRANSFER FUNCTION
\[
\frac{4s^2 + 64s + 6400}{s^2 + 80s + 6400}
\]
Figure 34. Root Locus Solution of Missiles-On Configuration Including Five Modes with Notch Filter and Roll Rate Feedback Gain Variation, Other Gains Nominal, Modified Magnitude of Control Surface Aerodynamics, M=0.9, 20,000 ft
Figure 35. Root Locus Solution of Missiles-Off Configuration Including Five Modes with Notch Filter and Roll Rate Gain Feedback Variation, Other Gains Nominal, Modified Magnitude of Control Surface Aerodynamics, M=0.9, 15,000 ft
control system, the rigid roll root is stabilized compared to the results with no filter. It is also noted that with the notch filter included, the 11.2 Hz flexible mode is affected significantly by increasing the roll rate feedback gain, but it is not destabilized. Thus, the stability analysis does predict that the addition of the notch filter to the roll rate feedback loop of the flight control system will stabilize the unstable roots, which agrees with the flight test results with the notch filter installed.

Root Locus Solution Summary

A summary for the root locus solutions for the missiles-on and missiles-off configurations including five analysis modes and a roll rate gain variation, is given in Table 31. As discussed earlier, the missile pitch flexible mode instability encountered in flight is predicted well by the root locus solution including modification factors applied to the control surface aerodynamics. It is noted that further changes such as modification to the rigid body aerodynamics, small additional phase lags in the control surface aerodynamics, or the use of quasi-steady aerodynamics do not significantly change this prediction. The rigid body roll mode instability encountered in flight is best predicted by the root locus solution including modification factors applied to the control surface aerodynamics plus an additional 20 degree phase lag in control surface aerodynamics.
### TABLE 31. SUMMARY OF FLIGHT TEST RESULTS AND ROOT LOCUS SOLUTIONS INCLUDING FIVE ANALYSIS MODES AND A ROLL RATE FEEDBACK GAIN VARIATION

<table>
<thead>
<tr>
<th>Analysis Case or Flight Test Result</th>
<th>Antisym Missile Pitch Mode Instab., Missiles-on, 0.9 M, 20K ft</th>
<th>Rigid Body Roll Root Instab. Missiles-off, 0.9 M, 15K ft</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Roll Rate Gain at Instab., ( K_p )</td>
<td>Freq. at Instab., (Hz)</td>
</tr>
<tr>
<td>Flight test</td>
<td>.2</td>
<td>6.5</td>
</tr>
<tr>
<td>Root Locus with Unmod. Calc. Aero</td>
<td>.1</td>
<td>6.7</td>
</tr>
<tr>
<td>Root Locus with Mod. C.S. Aero</td>
<td>.2</td>
<td>6.7</td>
</tr>
<tr>
<td>Root Locus with Mod. C.S. Aero + Mod. R.B. Aero</td>
<td>.2</td>
<td>6.7</td>
</tr>
<tr>
<td>Root Locus with Mod. C.S. Aero + 20° phase lag</td>
<td>.17</td>
<td>6.6</td>
</tr>
<tr>
<td>Root Locus with Quasi-Steady Aero + Mod. C.S. Aero</td>
<td>.2</td>
<td>6.7</td>
</tr>
<tr>
<td>Root Locus with notch filter + Mod. C.S. Aero</td>
<td>STABLE</td>
<td>-</td>
</tr>
</tbody>
</table>
CHAPTER VI SUMMARY AND CONCLUSIONS

From the results presented in Chapter V, several conclusions can be reached. One is that for an accurate aeroservoelastic stability analysis, a complete and accurate knowledge is needed of the flight control system and of the airplane rigid body and flexible modes which are to be included in the analysis. All significant feedback loops in the flight control system should be included, and the analysis input modes of the airplane should span the frequency response range of the flight control system and include enough flexible modes to provide control surface aeroelastic effects. Secondly, and most important, however, is that an accurate representation of the unsteady aerodynamics is needed for not only the analysis modes but, particularly, for the control surface deflections.

For accuracy, this author recommends that the best available unsteady aerodynamic lifting surface theory, such as the doublet-lattice method for subsonic flow, be used for the aerodynamic representation. Since the approach of this work was to develop the equations of motion in the Laplace domain and to use the root locus method for stability analysis, these unsteady aerodynamics had to be transformed from the commonly used frequency domain into the Laplace domain. The most important part of this work is this formulation, and a simple, but realistic and practical method for formulating the unsteady aero- dynamic forces in the Laplace domain has been developed. The fitted
Laplace polynomial fractions for the Laplace transformed generalized aerodynamic forces have been derived. These fitted Laplace polynomial fractions have been formulated for a given flight condition (i.e., a particular Mach number and altitude), are applicable for all frequencies within the desired frequency fit range, and are most valid for Laplace arguments near the imaginary axis (small positive or negative damping) along which the fit is made and which is the region of the greatest physical interest for stability. These Laplace polynomial fractions are realistic and physically realizable functions for the transformed generalized aerodynamic forces since their forms have been developed from unsteady aerodynamic strip theory, which assures an accurate fit for not only strip theory but also for unsteady, three-dimensional lifting surface aerodynamic theory as a logical extension. The validity of the fit for unsteady aerodynamic lifting surface theory was shown in Chapter II to be good, and its subsequent use in the stability analysis of the application case of Chapter V produced results which compared reasonably well with flight test results.

The accuracy of the control surface unsteady aerodynamics is important because the control surface effectiveness is a direct multiplier factor in the total feedback loop gain. Since it is known that the flight conditions for which the total feedback gain (the product of the control surface effectiveness and the feedback loop gain) is a maximum are likely to be critical flight conditions for possible aeroservoelasticity instabilities, realistic and accurate values of
control surface effectiveness are needed to properly analyze these likely flight conditions for stability. This effectiveness of the control surface is directly attributed to the accuracy of predicting the control surface aerodynamics. For initial stability calculations, this author recommends that the unsteady aerodynamic lifting surface theory be used without modifications or empirical corrections for generation of the control surface unsteady aerodynamics. These theoretical unsteady aerodynamic forces due to control surface deflections will, in general, be greater than those encountered in flight and should lead to conservative results for establishing feedback loop gains for the desired airplane system stability and response in flight. Then as wind tunnel and pressure model test data is gathered for a given design, these results may be used to modify the theoretically predicted unsteady aerodynamics due to control surface deflections.

One of these modification methods is to scale the calculated generalized unsteady aerodynamic forces due to a given control surface deflection by the ratio of the experimental to calculated values of the static hinge moment for that control surface. The second of these modification methods is to scale the calculated aerodynamic forces due to a given control surface deflection by the ratio of the experimental to calculated values of a selected static stability derivative, without aeroelastic effects, for that control surface. As demonstrated in the application case, the second approach appears to work well for both trailing edge and all movable control surfaces,
while the first approach appears to only work well for trailing edge control surfaces.

Because of the need for modification factors for the calculated control surface generalized unsteady aerodynamic forces and possible additional phase lags due to shock formation, continued research is recommended to develop improved analytical methods for computing unsteady aerodynamics. Particular emphasis should be placed on oscillatory control surfaces in the high subsonic and transonic flow regime.

Also the results of Chapter V show that for a refined aerodynamic idealization, which gives reasonable correlation with experimentally determined rigid body stability derivatives, modification factors do not need to be applied to the rigid body unsteady aerodynamics. In addition, the results of Chapter V show that in the lower frequency range the quasi-steady aerodynamics representation may be adequate for aeroservoelastic stability analyses. A comparison analysis using unsteady and quasi-steady aerodynamics should be done to determine the valid frequency range for stability analysis using quasi-steady aerodynamics.

This work presents an analytical method which is useful and practical in mathematically modeling the airplane and its control augmentation system for use in stability and response analyses. This approach can easily model systems with multiple feedback loops. This work develops the equations, including the unsteady aerodynamics, in the Laplace domain and uses the root locus method for stability analysis. The root locus method is widely used in stability and control
analyses and is used here because it is a convenient and useful method for determining the effects of varying the feedback loop gain on the system stability. System stability, as a function of feedback loop gain, can be easily evaluated from the root locus because explicit information on the damping of each root or mode is yielded. However, because of the frequency dependent nature of the control system and the unsteady aerodynamics, the root locus solution approach results in a large order stability problem when many modes are included in the analysis. But when this stability problem is formulated in terms of a large order eigenvalue problem, it can be accurately solved for the roots or eigenvalues by use of the QR transform method. The QR transform method is very stable numerically because of the use of orthogonal transformations throughout the decomposition solution for the eigenvalues. Such a numerically stable method is then inherently accurate even for the computation of eigenvalues of large order matrices. Thus, use of the QR transform method in the root locus solution makes the root locus method a practical approach for aeroservoelastic stability analyses of large order systems.

In summary, using the capability of today's large computers, the analysis approach developed in this work appears to be an accurate and practical means of predicting the stability characteristics of an airplane and its control augmentation system.
APPENDIX A

SOLUTION OF STABILITY DETERMINANT

As developed in Chapter I, the matrix equation of motion (1.25) for the augmented airplane is set up for a given flight condition and a given set of control system feedback loop gains. Since the elements of the (nxn) matrix of coefficients are polynomials in the Laplace variable s, equation (1.25) can be reduced to the following form:

\[ \begin{bmatrix} s^m[A_m] + s^{m-1}[A_{m-1}] + \ldots + s[A_1] + [A_0] \end{bmatrix} \{ \ddot{y}(s) \} = 0 \]

(A.1)

where \( m \) is the maximum order of the Laplace variable \( s \) of the elements of the (nxn) coefficient matrix of equation (1.25), and \( n \) is the number of normal mode coordinates and the order of the square matrix of coefficients of equation (1.25) and the above equation. To form an eigenvalue problem, of order \( k = mn \), the following first order systems of equations in \( s \) are defined:

\[
\begin{align*}
\{ \ddot{q}_1 \} &= \{ \ddot{x} \} \\
\{ \ddot{q}_2 \} &= s\{ \ddot{q}_1 \} = s\{ \ddot{x} \} \\
\{ \ddot{q}_3 \} &= s\{ \ddot{q}_2 \} = s^2\{ \ddot{x} \} \\
&\vdots \\
\{ \ddot{q}_m \} &= s\{ \ddot{q}_{m-1} \} = s^{m-1}\{ \ddot{x} \}
\end{align*}
\]
Then equation (A.1) can be written as

\[
[A_m]s\bar{q}_m + [A_{m-1}]s\bar{q}_{m-1} + \ldots + [A_1]s\bar{q}_2 + [A_0]s\bar{q}_1 = 0
\]

(A.3)

and solving for \( s\bar{q}_m \) results in

\[
s\bar{q}_m = -[A_m]^{-1}[A_0]s\bar{q}_1 - [A_m]^{-1}[A_1]s\bar{q}_2 - \ldots - [A_m]^{-1}[A_{m-1}]s\bar{q}_m
\]

(A.4)

Equations (A.2) and (A.4) can be combined to form the following set of simultaneous equations of order \( k \):

\[
\begin{bmatrix}
s\bar{q}_1 \\
s\bar{q}_2 \\
\vdots \\
s\bar{q}_{m-1} \\
s\bar{q}_m
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
-A_m^{-1}[A_1] - A_m^{-1}[A_2] - \ldots - A_m^{-1}[A_{m-1}]
\end{bmatrix}
\begin{bmatrix}
s\bar{q}_1 \\
s\bar{q}_2 \\
\vdots \\
s\bar{q}_{m-1} \\
s\bar{q}_m
\end{bmatrix}
\]

(A.5)

Equation (A.5) can be written in more compact form as follows:

\[
s(\bar{q}) = [A](\bar{q})
\]

or

\[
[[A] - s[I]](\bar{q}) = 0
\]

(A.6)

This equation (A.7) is now in the form of an eigenvalue problem, of which the eigenvalues or the complex roots determine the stability of the augmented airplane at a given flight condition and a given set of
control system feedback gains. It is noted that the order of this
eigenvalue problem can be large; however, the Q-R transform method
used in the solution, appears to be much more accurate than trying
to extract the roots of the corresponding polynomial characteristic
equation of the same large order, k.

The QR transformation for calculation of eigenvalues is based on
a decomposition of an arbitrary real matrix \( A \) into a product \( QR \) where
\( Q \) is an orthogonal matrix and \( R \) is an upper triangular matrix. This
is accomplished by a series of successive premultiplications of the
matrix \( A \) and the resulting products by plane rotation (orthogonal)
matrices \( S_{ji} \). By this means all elements below the diagonal can be
zeroed, and the final result is an upper triangular matrix. This
process is expressed as

\[
R = \begin{pmatrix}
I & S_{jl} \\
i=1 & j=i+1
\end{pmatrix} A = Q^T A \tag{A.8}
\]

where it is noted that the notation brackets indicating a matrix have
been dropped in this discussion for simplicity. Thus, the desired
decomposition is obtained

\[
A = A_1 = Q_1 R_1 \tag{A.9}
\]

where \( Q_1 \) is an orthogonal matrix and \( R_1 \) is an upper triangular matrix.

If then \( Q_1 \) and \( R_1 \) are multiplied in reverse order, the resulting
matrix is called \( A_2 \), which in turn can be decomposed into the product
\( Q_2 R_2 \).

\[
R_1 Q_1 = A_2 = Q_2 R_2 \tag{A.10}
\]
So a sequence of matrices can be defined as follows:

\[ A_n = Q_n R_n = R_{n-1} Q_{n-1} \quad n = 1, 2, \ldots \quad (A.11) \]

where \( A_1 = A \), and where \( Q_n \) is an orthogonal matrix, and \( R_n \) is an upper triangular matrix. This process of deriving the sequence of matrices \( \{A_n\} \) from the matrix \( A \) by successive orthogonal decompositions is called the QR transformation.

Using equation (A.11), it is seen that

\[ A_n = R_{n-1} Q_{n-1} = Q_{n-1}^T A_{n-1} Q_{n-1} \quad (A.12) \]

where \( R_i = Q_i^T A_i \) and \( Q_i^{-1} = Q_i^T \). Therefore, all the matrices \( A_n \) are similar, and have the same eigenvalues. Based on the equation (A.12), it follows that

\[
A_{n+1} = Q_n^T A_n Q_n \\
= Q_n^T (Q_{n-1}^T A_{n-1} Q_{n-1}) Q_n \\
= S_n^T A_n S_n
\]

where the matrix product \( S_n = Q_1 Q_2 \ldots Q_n \) is an orthogonal matrix also.

The convergence of the sequence of matrices \( \{A_n\} \) depends upon the convergence of \( S_n \). If the sequence of matrix products \( \{S_n\} \) converges to a nonsingular matrix \( S_n \) as \( n \to \infty \), then limit \( A_n \) exists and is an
upper triangular matrix. This is seen by the following:

\[ S_n = Q_1 \cdots Q_{n-1} Q_n = S_{n-1} Q_n \]  
(A.14)

then

\[ Q_\infty = \lim_{n \to \infty} Q_n = \lim_{n \to \infty} S_{n-1}^T S_n = I \]  
(A.15)

Therefore,

\[ A_\infty = \lim_{n \to \infty} A_n = \lim_{n \to \infty} Q_n R_n = Q_\infty R_\infty \]

\[ = I R_\infty \]

\[ = R_\infty \]  
(A.16)

So \( A_\infty = \lim_{n \to \infty} A_n \) exists and is upper triangular. Since \( A_\infty = R_\infty \) is upper triangular, the diagonal contains the eigenvalues, and these are the eigenvalues of \( A_1 = A \) since the matrix \( A_\infty = R_\infty \) is similar to \( A_1 \).

The QR decomposition is very stable numerically because of the use of plane rotation (orthogonal) matrices throughout. Because of the use of such orthogonal transformations, the QR transform method of calculating the eigenvalues of a general nonsymmetric real matrix \( A \) is very stable numerically. Such a numerically stable method is then inherently accurate even for the computation of eigenvalues of large order matrices.

The computer program used in this work, first reduces the matrix \( A \) to upper Hessenberg form by use of elementary similarity transformations, and then uses the QR transform method on the reduced matrix to
calculate the eigenvalues. The reason that the matrix A is first reduced to upper Hessenberg form (almost upper triangular form, i.e., a matrix with one extra sloping line below the diagonal) is that once reduced to that form, the decomposition to the matrix product QR involves less computational time. Time is saved since the required number of operations for the decomposition of Hessenberg matrix is on the order of \((n^2)\) compared to \((n^3)\) for the unreduced matrix (25). In addition, the QR transform method has the desirable property that all successive \(A_n\) have the same form as the original matrix A. Therefore, once the matrix A is reduced to Hessenberg form, computation time is saved for each of the successive decompositions of \(A_n\) into a product \(Q_nR_n\) because every \(A_n\) is also of upper Hessenberg form.
APPENDIX 8

THE DOUBLET-LATTICE METHOD FOR
THE UNSTEADY AERODYNAMIC REPRESENTATION

It is recommended in Chapter II that the fitted transformed
gerenalized aerodynamic forces be generated from an oscillatory,
three-dimensional, subsonic compressible lifting surface aerodynamic
teachory such as the doublet-lattice method of reference 22. The referenced doublet-lattice method is versatile because it can accommodate multiple, nonplanar, interfering lifting surfaces of any planform and multiple bodies in subsonic flow. The doublet-lattice method is a finite-element lifting surface theory that reduces to the vortex-lattice method in steady flow. Application of the doublet-lattice method is simple since no pressure loading functions are required as in the kernel function method. An early development of the doublet-lattice method was performed by Albano and Rodden (26) with improvements and extension to include bodies performed by Giesing, Kalman, and Rodden (22). The reader is referred to references 26 and 27 for further details on the doublet-lattice method and to reference 22 for details on the specific method and computer program applied in the application case of Chapter IV. Only an outline of the specific method of reference 22 used in this work is given here for completeness.

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The problem in oscillatory lifting surface theory is to find a distribution of lifting pressure coefficient $\Delta C_p$ that will generate the proper normal wash at all points on the lifting surfaces. The normal wash or velocity normal to an oscillatory surface or body, $W = V \Re(e^{i\omega t})$ is related to the lifting pressure $\Delta p = q_0 \Re(e^{i\omega t})$ by the following integral equation:

$$w_S(x, y, z, \gamma_r) = \frac{1}{8\pi} \iint K(x - \xi, y - \eta, z - \zeta, \gamma_s, \gamma_r, k, M)\Delta C_p(\xi, \eta, \zeta) dS_{L.S.} \tag{B.1}$$

where $L.S.$ indicates integration over all lifting surfaces. The quantities $\gamma_s, \gamma_r$ are the sending and receiving surface dihedral angles. The kernel $K$ is the normal wash at a point $x,y,z$ induced by a pressure doublet of unit strength located at $\xi,\eta,\zeta$, and $k$ and $M$ are the reduced frequency and Mach number, respectively. Equation (B.1) is essentially the familiar integral equation of lifting surface theory for surfaces alone in subsonic, oscillatory flow formulated by Küssner (28) in 1940.

If a body is introduced, there are additional contributions to the normal velocity. The first contribution may be called the slender body term and represents the flow field generated by bodies without considerations of interference.

$$\Delta W(x, y, z, \gamma_r) = \frac{1}{8\pi} \int B L(x - \xi, y - \eta_a, z - \zeta_a, \gamma_s, \gamma_r, k, M)_{\mu}(\xi) d\xi$$

$$+ \frac{1}{4\pi} \int B M(x - \xi, y - \eta_a, z - \zeta_a, \gamma_r, M)_{\sigma}(\xi) d\xi \tag{B.2}$$
The limit $B_n$ indicates integration over all bodies. The subscript $a$ on $n$ and $\zeta$ indicate the location of body axes. Here the term $\tilde{\mu}_s(\xi)$ represents an axial multipole distribution (dipole, quadrupole, etc.) whose orientation is given by $\gamma_s$. The second integral exists only in steady flow and is a source distribution used to represent the body volume effects. The slender body terms are known since $\tilde{\mu}_s(\xi)$ and $\sigma(\xi)$ are determined using the slender body theory of Miles (29).

A second contribution to the normalwash flow field caused by the introduction of bodies into the flow field arises from "image" lifting surface elements. These are placed within the body to help divert the flow around a body in the presence of a lifting surface. The strength of the image elements is the same as the external elements.

$$w_I(x,y,z,\gamma_r) = \frac{1}{8\pi} \iint_{L.S.} \mathcal{K}(x-\xi_1, y-\eta_I, z-\zeta_I, \gamma_{SI}, \gamma_r, k, M) \Delta \mathcal{C}_p(\xi, \eta, \zeta) dS$$

(B.3)

The subscript $I$ on $n$, $\zeta$ and $\gamma_s$ indicates the image position on the image surface.

The image system is not completely effective in diverting the flow around the bodies, however, and a residual flow must be added. This residual flow gives rise to a third contribution to the normalwash flow field caused by the bodies which is generated by an interference doublet distribution $\tilde{\mu}_{n}(\xi)$.

$$w_n(x,y,z,\gamma_r) = \frac{1}{8\pi} \int_B \mathcal{L}(x-\xi_1, y-\eta_n, z-\zeta_n, \gamma_s, \gamma_r, k, M) \tilde{\mu}_n(\xi) d\xi$$

(B.4)
The form of this equation is identical to the first integral of equation (B.2). The reason it is not combined with \( \tilde{\nu}_s \) is the fact that \( \tilde{\nu}_s \) is known while \( \tilde{\nu}_n \), like \( \Delta C_p \), is unknown.

In the direct problem, the normalwash boundary conditions are specified and the lifting pressure, \( \Delta C_p \), and body interference distribution, \( \tilde{\nu}_n \), are solved for

\[
\mathbf{w} = \mathbf{w}_S + \Delta \mathbf{w} + \mathbf{w}_I + \mathbf{w}_n \quad \text{(B.5)}
\]

Here \( \mathbf{w} \) is the prescribed normalwash on lifting surfaces and bodies. If the known quantities are placed on the left-hand side while the unknown quantities are placed on the right, then the following equation results:

\[
\mathbf{w} - \left( \frac{1}{8\pi} \int_{B.} \mathbf{L}\tilde{\nu}_s d\xi + \frac{1}{4\pi} \int_{B.} \mathbf{M}d\xi \right) = \frac{1}{8\pi} \int_{L.S.} \mathbf{K} \Delta C_p dS + \frac{1}{8\pi} \int_{L.S.} \mathbf{K}_I \Delta C_p dS + \frac{1}{8\pi} \int_{B.} \mathbf{L}\tilde{\nu}_n d\xi \quad \text{(B.6)}
\]

Here the subscript \( I \) on \( \mathbf{K} \) indicates \( \mathbf{K}(x - \xi, y - \eta, z - \zeta, \gamma_s, \gamma_r, k, M) \).

\[
\mathbf{w} - \Delta \mathbf{w} = \frac{1}{8\pi} \int_{L.S.} (\mathbf{K} + \mathbf{K}_I) \Delta C_p dS + \frac{1}{8\pi} \int_{B.} \mathbf{L}\tilde{\nu}_n d\xi \quad \text{(B.7)}
\]

There are still further contributions to the normalwash and these arise from planes of symmetry and ground effect. If the assumption is made that the right half of the aircraft lies in the upper right-hand quadrant of the \( z-y \) plane, then a subscript UR may be applied to contributions made from lifting surfaces in this quadrant. Similarly, UL
indicates upper left which contains the contribution from the left side of the aircraft. The subscript LR indicates lower right and, in this quadrant, the ground effect of the right side is contained. The subscript LL indicates lower left and this quadrant contains the contribution of the ground effect of the left half of the aircraft. Equation (B.7) may be expanded to include these contributions as follows:

\[ w = (\Delta w_{UR} + \delta \Delta w_{UL} + \epsilon \Delta w_{LR} + \epsilon \delta \Delta w_{LL}) \]

\[ = \frac{1}{8\pi} \int \int_{L.S.} \left\{ (K + K_1)_{UR} + \delta (K + K_1)_{UL} + \epsilon (K + K_1)_{LR} + \epsilon \delta (K + K_1)_{LL} \right\} \]

\[ \times \Delta z dS + \frac{1}{8\pi} \int \int_B \left\{ L_{UR} + \delta L_{UL} + \epsilon L_{LR} + \epsilon \delta L_{LL} \right\} \sigma n d\xi \quad (B.8) \]

The quantities \( \delta \) and \( \epsilon \) are the symmetry and ground effect indicators. For instance, \( \delta = 1,0,-1 \) indicates symmetry, no symmetry, and antisymmetry, respectively. Similarly, \( \epsilon = -1,0,1 \) indicates ground effect, no ground effect, and anti-ground effect, respectively. The changes to the argument lists denoted by these subscripts are as follows:

- **UR:** \( n = n, \quad \zeta = \zeta, \quad \gamma_S = \gamma_S \)
- **UL:** \( n = -n, \quad \zeta = \zeta, \quad \gamma_S = -\gamma_S \)
- **LR:** \( n = n, \quad \zeta = -\zeta, \quad \gamma_S = -\gamma_S \)
- **LL:** \( n = -n, \quad \zeta = -\zeta, \quad \gamma_S = \gamma_S \)

The basic method of solution of equation (B.8) is to idealize the lifting surfaces into small boxes and the bodies into small axial elements. The number of boxes or elements necessary for convergence
depends on the reduced frequency $k$. Figure 36 shows a typical idealization. The lifting surfaces are idealized by dividing the surfaces into small trapezoidal elements or boxes arranged in strips parallel to the freestream so that surface edges, fold lines, and hinge lines lie on box boundaries as shown in Figures 36 and 37. Then, to represent the steady-flow effects, a horseshoe vortex is placed on each of the boxes such that the bound vortex of the horseshoe system coincides with the quarter-chord line of the box. To represent the oscillatory increment, a distribution of acceleration potential doublets (which have the steady-flow doublet strength subtracted) of uniform strength is superimposed on the bound vortex. The surface boundary condition is a prescribed normalwash applied at the control point of each box. The control point is centered spanwise on the three-quarter-chord line of the box as shown in Figure 37. The influences of all vortices and doublets are summed for each control point to obtain the total dimensionless normalwash, $w$, at the control point. The unknowns are assumed constant over these box and body elements and the normalwash boundary condition is applied to each box and body element. This forms as many equations as unknowns and the system may be solved. Equation (B.8) becomes

$$ w_{Tr} = \sum_{s=1}^{N1} \Delta C_{ps} \int \int_{\text{ELEMENT}_s} K_{Trs} \, dS + \sum_{s=1}^{N2} \mu n_{is} \int_{\text{ELEMENT}_s} L_{Trs} \, d\xi $$

where $s$ and $r$ indicate sending and receiving points, respectively, and
Figure 36. Typical Aerodynamic Idealization for a Configuration
Figure 37. Surface Idealization into Boxes and Location of Vortices, Doublets and Collocation Points
\[ w_T = w - \Delta w_T \]
\[
\Delta w_T = (\Delta w_{UR} + \delta \Delta w_{UL} + \varepsilon \Delta w_{LR} + \varepsilon \delta \Delta w_{LL})
\]
\[
K_T = \frac{1}{8\pi} \left\{ (K + K_1)_{UR} + \delta(K + K_1)_{UL} + \varepsilon(K + K_1)_{LR} + \varepsilon \delta(K + K_1)_{LL} \right\}
\]
\[
L_T = \frac{1}{8\pi} \left\{ l_{UR} + \delta l_{UL} + \varepsilon l_{LR} + \varepsilon \delta l_{LL} \right\}
\]

\[ N_1 = \text{number of lifting surface boxes for all surfaces} \]
\[ N_2 = \text{number of axial body elements for all bodies} \]

In matrix notation:

\[
\{ w_T \} = [D_T] \left\{ \begin{array}{c} -\Delta C \varepsilon \\ \bar{\mu} \end{array} \right\}
\]

\[ \text{(B.10)} \]

where

\[ [D_T] = [\bar{D}; \bar{E}] \]

\[ \text{(B.11)} \]

in which

\[ \bar{S} = \left\{ (D + D_1)_{UR} + \delta(D + D_1)_{UL} + \varepsilon(D + D_1)_{LR} + \varepsilon \delta(D + D_1)_{LL} \right\} \]

\[ \text{(B.12)} \]

Here

\[ D_{RS} = \iint_{\text{ELEMENT}} \frac{1}{8\pi} K dS \]

\[ D_{I,RS} = \iint_{\text{ELEMENT}} \frac{1}{8\pi} K_I dS \]

\[ = \iint_{\text{ELEMENT}} \frac{1}{8\pi} K(x - \xi, y - \eta, z - \zeta, \gamma_r, \gamma_{s_I}, k, M) dS \]

\[ \text{(B.13)} \]
The matrix elements $D_{rs}$ have the subscript $r$ on the receiving point quantities, $x, y, z, \gamma_r$, and the subscript $s$ on the sending element quantities, $\xi, \eta, \zeta, \gamma_s$. The matrix partition $E$ is

$$E_{rs} = \frac{1}{4\pi} \left\{ E_{UR} + \delta E_{UL} + \varepsilon E_{LR} + \varepsilon \delta E_{LL} \right\}$$

where

$$E = \int_{\text{BODY ELEMENT}} \text{L} d\xi,$$  \hspace{1cm} (B.14)

Once $\mathbf{w}_T$ is known, $\Delta C_p$ and $\hat{\mu}_n$ can be found, and these can be used to find the loads on the lifting surfaces and bodies.

The calculation of the flow field due to the slender body terms, i.e., $\hat{\mu}_S$ and $\sigma$, is performed using the same discretization technique.

$$\Delta \mathbf{w}_T = \frac{1}{8\pi} \sum_{s=1}^{N3} \hat{\mu}_s \int_{\text{BODY ELEMENT}_s} L_{rs} d\xi + \frac{1}{4\pi} \sum_{k=1}^{N3} \sigma_s \int_{\text{BODY ELEMENT}_s} M_{rs} d\xi$$ \hspace{1cm} (B.15)

where $N3$ is the number of slender body elements. If symmetry planes and ground effect are accounted for and matrix notation is introduced, then equation (B.15) becomes

$$\{ \mathbf{w}_T \} = [L_T] \{ \hat{\mu}_S \} + [M_T] \{ \sigma \}$$ \hspace{1cm} (B.16)

where

$$M_T = \frac{1}{4\pi} \left\{ M_{UR} + \delta M_{UL} + \varepsilon M_{LR} + \varepsilon \delta M_{LL} \right\}$$
Slender body theory states that $\tilde{u}$ is directly proportional to the local velocity normal to the body axis (the direction of $\tilde{u}_S$ is parallel to this velocity).

$$\tilde{u}_S = wD^2D$$  \hspace{1cm} (B.17)

where $D^2D$ is the proportionality constant which is dependent on local body cross section. The values of $w$ which act normal to the body axis of equation (B.17) are part of the larger set that acts normal to all surfaces and bodies.

Normalized Boundary Conditions

Before the equation (B.9) or (B.10) can be solved for the unknowns $\Delta C_p$ and $\tilde{u}_n$, the normalwash $w$ must be determined at each lifting surface element or box and at each axial body element in both the $z$ and $y$ directions. The normalwash boundary conditions are obtained by taking the substantial derivative of the total deflection distribution which is assumed to consist of a superposition of modal deflections. There are various methods of describing these modes but only the polynomial approach is used in the computer program of reference 22. The polynomial approach lends itself to scientific investigation where the modes are simple. However, when the modes become complicated such as with airplane normal modes of vibration, it is desirable to incorporate other more practical modal input methods such as supplying deflections and streamwise slopes for each element directly.

For the computer program of reference 22, the total deflection distribution of a lifting surface normal to itself is made up of a set of modes, $f_i$. 
where $\xi_i$ is the $i$th generalized coordinate and NM is the number of modes. It is noted that this equation is the same form as equations (1.1) for the modal deflection $c_R f_i = \phi_i$. The total normalwash is likewise

$$w = \frac{W}{U_\infty} = \sum_{i=1}^{NM} \xi_i w_i$$  \hspace{1cm} (B.19)

where

$$w_i = \left\{ \frac{df_i}{d(x/c_R)} + i \frac{\omega c_R}{V} f_i \right\}$$  \hspace{1cm} (B.20)

or

$$w_i = \left\{ \frac{df_i}{d(x/c_R)} + ik 2f_i \right\}$$  \hspace{1cm} (B.21)

and where

$$k = \frac{\omega c_R}{2V}$$

For lifting surfaces the modes may be approximated by a least squares fit for the polynomial coefficients $a_{inm}$, of the form

$$f_i = \sum_{n=0}^{\leq 5} \sum_{m=0}^{\leq 5} a_{inm} \left( \frac{x}{c_R} \right)^n \left( \frac{\tau}{c_R} \right)^m$$  \hspace{1cm} (B.22)

$$\frac{df_i}{d(x/c_R)} = \sum_{n=0}^{\leq 5} \sum_{m=0}^{\leq 5} na_{inm} \left( \frac{x}{c_R} \right)^{n-1} \left( \frac{\tau}{c_R} \right)^m$$  \hspace{1cm} (B.23)
where \( \tau \) is the lateral distance in the plane of the lifting surface. Equations (B.22) and (B.23) represent up to fifth-degree polynomials in both the lateral and longitudinal directions.

In the same manner as for the lifting surfaces, the total deflection distribution of a body normal to itself is made up of a set of modes, \( f_{z_i} \) and \( f_{y_i} \).

\[
h_z = c_R \sum_{i=1}^{NM} f_{z_i} \bar{\xi}_i
\]

\[
h_y = c_R \sum_{i=1}^{NM} f_{y_i} \bar{\xi}_i
\]

(B.24)

(B.25)

Then realizing that bodies have no lateral coordinate, the mode shapes may be approximated by a least squares fit for the polynomial coefficients \( az_{in} \) and \( ay_{in} \), of the form

\[
f_{z_i} = \sum_{n=0}^{\leq 5} az_{in} \left( \frac{x}{c_R} \right)^n
\]

(B.26)

\[
f_{y_i} = \sum_{n=0}^{\leq 5} ay_{in} \left( \frac{x}{c_R} \right)^n
\]

(B.27)

\[
\frac{df_{z_i}}{d(x/c_R)} = \sum_{n=0}^{\leq 5} naz_{in} \left( \frac{x}{c_R} \right)^{n-1}
\]

(B.28)

\[
\frac{df_{y_i}}{d(x/c_R)} = \sum_{n=0}^{\leq 5} nay_{in} \left( \frac{x}{c_R} \right)^{n-1}
\]

(B.29)
The normalwash for bodies is then given by a form similar to equation (B.21) for lifting surfaces.

\[
\begin{align*}
{w}_z & = \left\{ \frac{df_{z_i}}{d(x/c_R)} + ik2f_{z_i} \right\} \\
{w}_y & = \left\{ \frac{df_{y_i}}{d(x/c_R)} + ik2f_{y_i} \right\} 
\end{align*}
\] (B.30)

and

Thus, the normalwash \( \{w\}_i \) arrays for use in equation (B.9) or (B.10) can be determined for each mode \( i \). The \( \{w\}_i \) arrays have the following order: first, all of the lifting surface normalwash values \( w \) are determined, then the \( z \) or upwash values \( w_z \), and then the \( y \) or sidewash values \( w_y \).

\[
\{w\}_i = \left\{ \begin{array}{c}
w \\
w_z \\
w_y 
\end{array} \right\}
\] (B.32)

Generalized Forces

Once \( \Delta C_p \) and \( \hat{u}_n \) arrays have been found by solving equation (B.10), the loads on the lifting surfaces and bodies can be determined. These pressures, forces, and moments along with their appropriate displacements, are outlined as follows:

\[
\begin{align*}
\Delta C_p, h & \quad \text{(normal to lifting surfaces)} \\
\alpha(F_z/q_{\infty})/\alpha x, h_z & \quad \text{(z-direction on bodies)} \\
\alpha(F_y/q_{\infty})/\alpha x, h_y & \quad \text{(y-direction on bodies)} \\
\alpha(M_z/q_{\infty})/\alpha x, dh_z/dx & \quad \text{(z-direction on bodies)} \\
\alpha(M_y/q_{\infty})/\alpha x, dh_y/dx & \quad \text{(y-direction on bodies)}
\end{align*}
\]
For the displacements expressed in terms of generalized modal coordinates as in equations (B.18), (B.24) and (B.25), the generalized aerodynamic forces can be determined. The method used in reference 22 expresses these generalized aerodynamic forces in the same normalized form of equations (2.33) or (2.34) for the mode shape \( f_i = \phi_i/c_R \).

In particular, the computer program used gives these as

\[
\bar{Q}_{ij} = \frac{1}{A_R} \left[ G \int_\text{R.S.} \Delta C_{p j} f_i \, dS + g \left\{ \int_B \frac{a(F_{z j}/q_\infty)}{\alpha \xi} f_{z i} \, d\xi + \int_B \frac{a(F_{y j}/q_\infty)}{\alpha \xi} f_{y i} \, d\xi \right\} 
+ \int_B \frac{a(M_{z j}/q_\infty)}{\alpha \xi} \frac{d f_{z i}}{d \xi} \, d\xi + \int_B \frac{a(M_{y j}/q_\infty)}{\alpha \xi} \frac{d f_{y i}}{d \xi} \, d\xi \right] \tag{B.33}
\]

where \( q_\infty \) is the free-stream dynamic pressure, \( c_R \) the reference chord length, and \( A_R \) the total reference area. The integration limit R.S. indicates that only the lifting surfaces on the right-hand side of the aircraft are considered.

\[ G = \begin{cases} 
1 & \text{if lifting surface lies in plane of symmetry (e.g., vertical fin)} \\
2 & \text{otherwise}
\end{cases} \]

The integration limit B. indicates all bodies lying on the right-hand side of the aircraft.

\[ g = \begin{cases} 
1 & \text{if body lies on plane of symmetry (e.g., a fuselage)} \\
2 & \text{otherwise (e.g., a nacelle)}
\end{cases} \]
The integral over the lifting surface R.S. represents a series of integrals over each of the lifting surfaces which go to make up the total configuration. The value of G for each of these surfaces may be different. A similar argument is valid for the integral over the bodies B., thus the value of g may vary from body to body.

**Calculated Aerodynamic Data**

It is desirable and sometimes necessary to generate conventional aerodynamic data. Such data, in addition to being useful in itself, provides an excellent check for the modeling of the specific configuration to be run. This aerodynamic data is computed as part of the computer program of reference 22. This computed data was used in the application case of Chapter IV to check aerodynamic idealization and was used to, optionally, provide a means to adjust the generalized aerodynamic forces of the matrix \([\bar{A}(s)]\) for control surface deflection pressure modes. The aerodynamic data given by the computer program is the following.

The local normal force coefficient and pitching moment coefficient about the local \(\frac{1}{4}\)-chord point are

\[
c_n = \frac{1}{c} \int_{\text{chord}} \Delta C_p \, d\xi \tag{B.34}
\]

\[
c_m = -\frac{1}{c^2} \int_{\text{chord}} \Delta C_p (\xi - \xi_{\frac{1}{4}}) \, d\xi \quad \text{(nose up)} \tag{B.35}
\]

where \(c\) is the local chord length. The local center of pressure is

\[
c.p.\text{Re} = \frac{-\text{Re}(c_m)}{\text{Re}(c_n)} + 0.25
\]
\[ c.p.\text{Im} = \frac{-\text{Im}(c_m)}{\text{Im}(c_n)} + 0.25 \]  

The total vertical and side-force coefficients on lifting surfaces are

\[ C_z = \frac{(1 + \delta)}{A_R} \int_{\text{R.S.}} cc_n d\xi \]  

\[ C_y = \frac{(1 - \delta)}{A_R} \left( \frac{G}{2} \right) \int_{\text{R.S.}} cc_n d\xi \]  

The total vertical and side-force coefficients on bodies are

\[ C_{z_b} = \frac{(1 + \delta)}{A_R} \left( \frac{g}{2} \right) \int_{\text{B.}} \frac{\partial (F_z/q_\infty)}{\partial \xi} d\xi \]  

\[ C_{y_b} = \frac{(1 - \delta)}{A_R} \left( \frac{g}{2} \right) \int_{\text{B.}} \frac{\partial (F_y/q_\infty)}{\partial \xi} d\xi \]  

The pitching and yawing moment coefficients on lifting surfaces taken about the point XM are

\[ C_N = \frac{(1 + \delta)}{A_R c_R} \int_{\text{R.S.}} \left\{ c^2 c_m - cc_n (\xi_{1a} - XM) \right\} d\eta \]  

(nose up)  

\[ C_N = \frac{(1 - \delta)}{A_R c_R} \left( \frac{G}{2} \right) \int_{\text{R.S.}} \left\{ c^2 c_m - cc_n (\xi_{1a} - XM) \right\} d\xi \]  

(nose right)
For bodies

\[ C_{M_b} = \frac{(1 + \delta)}{A_{RC}} \left( \frac{g}{2} \right) \int_B \left\{ \frac{\partial (M_z/q_\infty)}{\partial \xi} - (\xi - XM) \frac{\partial (F_z/q_\infty)}{\partial \xi} \right\} d\xi \] (nose up) \hspace{1cm} (B.43)

\[ C_{N_b} = \frac{(1 - \delta)}{A_{RC}} \left( \frac{g}{2} \right) \int_B \left\{ \frac{\partial (M_y/q_\infty)}{\partial \xi} - (\xi - XM) \frac{\partial (F_y/q_\infty)}{\partial \xi} \right\} d\xi \] (nose right) \hspace{1cm} (B.44)

The total rolling moment coefficient for the aircraft is

\[ C_\xi = \frac{-(1 - \delta)}{2s_R^2A} \left\{ \int_{R.S.} c_n \eta d\eta + \frac{G}{2} \int_{R.S.} c_n \xi d\xi + \frac{g}{2} \int_B \frac{\partial (F_z/q_\infty)}{\partial \xi} \eta d\xi \\ + \frac{g}{2} \int_B \frac{\partial (F_y/q_\infty)}{\partial \xi} \xi d\xi \right\} \] (right wing down) \hspace{1cm} (B.45)

where \( s_R \) is the reference semispan, and \( \eta_a, \xi_a \) are the body axis coordinates.
APPENDIX C

DERIVATION OF RIGID STABILITY DERIVATIVES FROM THE CALCULATED AERODYNAMIC DATA OF THE DOUBLET-LATTICE METHOD

The aerodynamic data of equations (B.37) through (B.45) are part of the calculated output of the doublet-lattice method computer program of reference 22. These aerodynamic data, calculated for a small value of reduced frequency \( k = \varepsilon \), can be used to obtain airplane rigid body static and dynamic stability derivatives, and stability derivatives due to control surface deflection. Rodden and Giesing (30) describe a method for determining these derivatives. These calculated derivatives can then be compared to corresponding experimental derivatives obtained from wind tunnel tests. On the basis of these comparisons, changes to the doublet-lattice aerodynamic idealization can then be made or adjustments can be made to the calculated generalized aerodynamic forces due to a given control surface deflection, based on available steady state experimental data.

To illustrate the determination of the stability derivatives from this aerodynamic data, the following is considered. For rolling moment due to rigid airplane roll \( \phi \), the series expansion of the roll moment coefficient in terms of the rolling lateral stability derivatives can be expressed as

\[
C_\ell = C_{\ell \phi}^0 + C_{\ell \phi} \left( \frac{\dot{\phi} b}{2V} \right) + C_{\ell p} \left( \frac{\ddot{\phi} b^2}{4V^2} \right) + \ldots \quad (C.1)
\]
where $b$ is the reference span and $V$ is the flight speed. Now for sinusoidal motion $\phi = \phi_0 R(e^{i\omega t})$, the rolling moment coefficient of equation (C.1) is

$$C_\ell = R(k C_\ell e^{i\omega t}) \quad \text{(C.2)}$$

The following complex amplitude of the coefficient is then:

$$C_\ell = \phi_0 \left[ ik \left( \frac{2s_R}{c_R} \right) C_{\ell R} - k^2 \left( \frac{2s_R}{c_R} \right)^2 C_{\ell p} + \ldots \right] \quad \text{(C.3)}$$

where $s_R = \frac{b}{2}$ is the reference semi-span, and $k = \frac{\omega c_R}{2V}$ the reduced frequency. For small values of reduced frequency $k$, equation (C.3) can be truncated with the velocity terms, which gives

$$C_\ell = \phi_0 \left[ ik \left( \frac{2s_R}{c_R} \right) C_{\ell R} \right] \quad \text{(C.4)}$$

Now the rolling moment in the usual stability and control notation is

$$M_\phi = q_{\infty} bA_R C_\ell \quad \text{(C.5)}$$

and in the computer program of reference 22, the rolling moment is

$$M_\phi = q_{\infty} bA_R C_\ell \quad \text{(C.6)}$$

where $C_\ell$ is the complex value of the airplane rolling moment coefficient given by the computer program. So equating the rolling moments of equations (C.5) and (C.6), gives

$$\bar{C}_\ell = C_\ell = C_\ell^R + iC_\ell^I \quad \text{(C.7)}$$
Equating the imaginary parts of equations (C.4) and (C.7) gives

\[ \phi_0 k \left( \frac{2s_R}{c_R} \right) C_\ell P = C_\ell Im(\phi) \]

or

\[ C_\ell P = \frac{1}{\phi_0} \frac{1}{k} \left( \frac{c_R}{2s_R} \right) C_\ell Im(\phi) \]  \hspace{1cm} (C.8)

where \( C_\ell Im(\phi) \) is the imaginary part of equation (B.45) given by the doublet-lattice computer program for the rolling moment coefficient due to roll, \( \phi \) is the amplitude of the input rigid body airplane roll mode, and \( k \) is the small input value of reduced frequency.

Likewise for rolling moment due to aileron deflection \( \delta_A \), the series expansion of the rolling moment coefficient in terms of lateral stability derivatives can be expressed as

\[ C_\ell = C_\ell \delta_A + C_\ell \delta_A \left( \frac{\delta A^b}{2V} \right) + \ldots \]  \hspace{1cm} (C.9)

Now for sinusoidal motion \( \delta_A = \delta_A^0 \mathcal{R}(e^{i\omega t}) \), the rolling moment coefficient is given by equation (C.2). The following complex amplitude of the coefficient is then:

\[ \tilde{C}_\ell = \delta_A^0 \left[ C_\ell \delta_A + ik \left( \frac{2s_R}{c_R} \right) C_\ell \delta_A + \ldots \right] \]  \hspace{1cm} (C.10)

Again for small values of the reduced frequency \( k \), equation (C.10) can be truncated with the displacement terms, which gives

\[ \tilde{C}_\ell = C_\ell \delta_A \delta_A^0 \]  \hspace{1cm} (C.11)
Equating the real parts of equations (C.11) and C.7) gives

\[ C_2 \delta_A = C_2^{\text{RL}}(\delta_A) \]

or

\[ C_2 \delta_A = \frac{1}{\delta_{A_0}} C_2^{\text{RL}}(\delta_A) \]  

(C.12)

where \( C_2^{\text{RL}}(\delta_A) \) is the real part of equation (B.45) given by the doublet-lattice computer program for the rolling moment coefficient due to aileron deflection, and \( \delta_{A_0} \) is the amplitude input of the aileron deflection mode.

In a like manner, for yawing moment due to the rigid airplane yaw \( \psi \) (see Figure 1), the series expansion of the yawing moment coefficient in terms the lateral stability derivatives can be expressed as

\[ C_n = C_n^\psi + (C_n^\psi - C_n^\eta \frac{\dot{\psi}b}{2V}) + (C_n^\psi - C_n^\eta \frac{\ddot{\psi}b}{4V^2}) \ldots \]  

(C.13)

where airplane yaw \( \psi \) used here is a combination of both yaw and side slip as these motions are usually comprehended in stability and control analysis. Now for sinusoidal motion \( \psi = \psi_0 e^{i\omega t} \), the yawing moment coefficient of equation (C.13) is

\[ C_n = \text{RL}(C_n e^{i\omega t}) \]  

(C.14)

The following complex amplitude of the coefficient of equation (C.13) is then
It can be observed that because of the stability combination in the velocity term, the stability derivative for yaw moment due to yaw rate $C_{n_r}$ cannot be determined uniquely from this equation above. Therefore, it becomes necessary to also consider the yawing moment due to rigid airplane lateral translation $n$ along the $y$ axis (see Figure 1). The series expansion of the yawing moment coefficient in terms of the lateral stability derivatives due to lateral translation is

$$C_n = C_n \left( \frac{\dot{n}}{V} \right) + C_{n_{\dot{B}}} \left[ \frac{(\dot{V})_b}{2V} \right] + \ldots \ldots \quad (C.16)$$

Now for sinusoidal motion $\frac{\dot{n}}{V} = \frac{\Pi_0}{V} \cos(i\omega t)$, the yawing moment coefficient is given by equation $(C.14)$, and the following complex amplitude of the coefficient of equation $(C.16)$ is

$$\bar{C}_n = \frac{ik2\Pi_0}{c_R} \left[ C_{n_{\dot{B}}} + i\left( \frac{2sR}{c_R} \right) C_{n_{\dot{B}}} + \ldots \ldots \right] \quad (C.17)$$

For small values of the reduced frequency $k$, equation $(C.15)$ and $(C.17)$ can be truncated with the velocity terms, which gives, respectively

$$\bar{C}_n = \psi_o \left[ C_{n_{\dot{B}}} + ik \left( \frac{2sR}{c_R} \right) \left( C_{n_{\dot{B}}} - C_{n_r} \right) \right] \quad (C.18)$$

and

$$\bar{C}_n = \frac{2\Pi_0}{c_R} \left[ -k^2 \left( \frac{2sR}{c_R} \right) C_{n_{\dot{B}}} + ikC_{n_{\dot{B}}} \right] \quad (C.19)$$
Now the yawing moment in the usual stability and control notation is

\[ M_n = q_\infty b A R C_n \]  

(C.20)

and in the computer program of reference 22, the yawing moment is

\[ M_n = q_\infty c R A R C_n \]  

(C.21)

where \( C_n \) is the airplane yawing moment coefficient given by the doublet-lattice computer program. So equating the yawing moment of equations (C.20) and (C.21), gives

\[ C_n = \left( \frac{c R}{2s_R} \right) C_n = \left( \frac{c R}{2s_R} \right) (c_n R + iC_n I m) \]  

(C.22)

where \( b = 2s_R \). Equating the real parts of equations (C.18) and (C.22) for yawing moment coefficient due to airplane yaw, gives

\[ \psi_0 C_n B = \left( \frac{c R}{2s_R} \right) C_n R E (\psi) \]

or

\[ C_n B = \frac{1}{\psi_0} \left( \frac{c R}{2s_R} \right) C_n R E (\psi) \]  

(C.23)

where \( C_n R E (\psi) \) is the real part of equations (B.42) and (B.44) given by the doublet-lattice computer program for the yawing moment coefficient due to yaw, and \( \psi_0 \) is the amplitude of the input rigid body airplane yaw mode. It is noted that \( C_n B \) can also be evaluated by equating the imaginary part of equations (C.19) and (C.32) for yawing
moment coefficient due to airplane lateral translation, which gives

\[
\left( \frac{2\eta_0}{c_R} \right) k C_{n_B} = \left( \frac{c_R}{2s_R} \right)
\]

or

\[
C_{n_B} = \frac{1}{k} \left( \frac{c_R}{2\eta_0} \right) \left( \frac{c_R}{2s_R} \right) C_n^{\text{Im}(n)} \tag{C.24}
\]

where \( C_n^{\text{Im}(n)} \) is the imaginary part given by the doublet-lattice computer program for the yawing moment coefficient due to airplane lateral translation, \( \eta_0 \) is the amplitude of the input rigid body airplane lateral translation mode, and \( k \) is the small input value of reduced frequency.

Equating the real parts of equations (C.19) and C.22) for yawing moment coefficients due to airplane lateral translation, gives

\[
-k^2 \left( \frac{2\eta_0}{c_R} \right) \left( \frac{2sR}{c_R} \right) C_{n_B} = \left( \frac{c_R}{2s_R} \right) C_n^{\text{Re}(n)}
\]

or

\[
C_{n_B} = -\frac{1}{k^2} \left( \frac{c_R}{2\eta_0} \right) \left( \frac{c_R}{2s_R} \right)^2 C_n^{\text{Re}(n)} \tag{C.25}
\]

where \( C_n^{\text{Re}(n)} \) is the real part of equations (B.42) and (B.44) given by the doublet-lattice computer program for the yawing moment coefficient due to lateral translation, \( \eta_0 \) is the amplitude of the input rigid body airplane lateral translation mode, and \( k \) is the small input value of reduced frequency. Finally, equating the imaginary parts of (C.18)
and (C.22) for yawing moment coefficient due to airplane yaw, gives

\[ \psi k \left( \frac{2s_R}{c_R} \right) (c_n x - c_n r) = \left( \frac{c_R}{2s_R} \right) c_n \text{Im}(\psi) \]

or

\[ c_n r = -\frac{1}{\psi} \frac{1}{k} \left( \frac{c_R}{2s_R} \right)^2 c_n \text{Im}(\psi) + c_n x \]

By use of equation (C.25) the above equation becomes

\[ c_n r = -\frac{1}{\psi} \frac{1}{k} \left( \frac{c_R}{2s_R} \right)^2 c_n \text{Im}(\psi) - \frac{1}{k^2} \left( \frac{c_R}{2\psi} \right) \left( \frac{c_R}{2s_R} \right) c_n \text{Re}(n) \]  \hspace{1cm} (C.26)

where \( c_n \text{Im}(\psi) \) is the imaginary part of equations (B.42) and (B.44) given by the doublet-lattice computer program for the yawing moment coefficient due to yaw, and \( c_n \text{Re}(n) \) is the real part of the yawing moment coefficient due to lateral translation.

Similarly, for yawing moment due to rudder rotation \( \delta_R \), the series expansion of the yawing moment coefficient in terms of the rudder lateral stability derivatives is

\[ c_n = c_n r \delta_R + c_n r \left( \delta_R \frac{b}{2v} \right) + \ldots \]  \hspace{1cm} (C.27)

Now for sinusoidal motion \( \delta_R = \delta_R \chi(t, e^{i\omega t}) \), the yawing moment coefficient of equation (C.27) is given by equation (C.14). The following complex amplitude of the coefficient is then:
\[
\bar{C}_n = \delta_{R_0} \left[ C_n \delta_R + i k \left( \frac{2s_R}{c_R} \right) C_n \delta_R + \ldots \right] \tag{C.28}
\]

Again for small values of the reduced frequency \( k \), equation (C.28) can be truncated with the displacement terms, which gives

\[
\bar{C}_n = C_n \delta_{R_0} \delta_R \tag{C.29}
\]

Equating the real parts of equations (C.29) and (C.22), gives

\[
C_n \delta_{R_0} \delta_R = \left( \frac{c_R}{2s_R} \right) C_n^{\mathcal{R}}(\delta_R)
\]

or

\[
C_n \delta_{R_0} = \frac{1}{\delta_{R_0}} \left( \frac{c_R}{2s_R} \right) C_n^{\mathcal{R}}(\delta_R) \tag{C.30}
\]

where \( C_n^{\mathcal{R}}(\delta_R) \) is the real part given by the doublet-lattice computer program for the yawing moment coefficient due to rudder rotation, and \( \delta_{R_0} \) is the amplitude of the input rudder rotation mode.

In a like manner, the stability derivatives for side force can also be obtained. The results for the stability derivatives given by equations (C.8), (C.12), (C.23), (C.26), and (C.30) are examples of how the stability derivatives can be obtained from the doublet-lattice method for rigid body airplane input modes or control surface input modes, and for a small value of reduced frequency.
LIST OF REFERENCES


