INFORMATION TO USERS

This material was produced from a microfilm copy of the original document. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the original submitted.

The following explanation of techniques is provided to help you understand markings or patterns which may appear on this reproduction.

1. The sign or “target” for pages apparently lacking from the document photographed is “Missing Page(s)”. If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting thru an image and duplicating adjacent pages to insure you complete continuity.

2. When an image on the film is obliterated with a large round black mark, it is an indication that the photographer suspected that the copy may have moved during exposure and thus cause a blurred image. You will find a good image of the page in the adjacent frame.

3. When a map, drawing or chart, etc., was part of the material being photographed the photographer followed a definite method in “sectioning” the material. It is customary to begin photoing at the upper left hand corner of a large sheet and to continue photoing from left to right in equal sections with a small overlap. If necessary, sectioning is continued again — beginning below the first row and continuing on until complete.

4. The majority of users indicate that the textual content is of greatest value, however, a somewhat higher quality reproduction could be made from “photographs” if essential to the understanding of the dissertation. Silver prints of “photographs” may be ordered at additional charge by writing the Order Department, giving the catalog number, title, author and specific pages you wish reproduced.

5. PLEASE NOTE: Some pages may have indistinct print. Filmed as received.

University Microfilms International
300 North Zeeb Road
Ann Arbor, Michigan 48106 USA
St. John’s Road, Tyler’s Green
High Wycombe, Bucks, England HP10 8HR
KRAUSMAN, DENNIS

THEORY AND APPLICATIONS OF INTERVAL-AVERAGE SAMPLING

THE OHIO STATE UNIVERSITY, PH.D., 1976
THEORY AND APPLICATIONS OF INTERVAL-AVERAGE SAMPLING

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Dennis Krausman, B.E.E., M.Sc.

* * * * * *

The Ohio State University
1978

Reading Committee:
Robert B. Lackey
C. Earl Warren
Dean T. Davis

Approved By

Robert B. Lackey
Adviser
Department of Electrical Engineering
ACKNOWLEDGMENTS

The author's graduate study, including the performance of the research reported on in this dissertation, was sponsored by the Aeronautical Systems Division, Wright-Patterson Air Force Base, under a program of continuing education for scientists and engineers.

Many thanks are due to Professor Robert B. Lackey for his assistance and inspiration during the last several years.

Finally, completion of this work would not have been possible without the understanding and cooperation of my wife, Bonnie.
VITA

June 12, 1946 . . . . Born - Baltimore, Maryland
1975 . . . . . . . . . M.Sc., The Ohio State University, Columbus, Ohio
1969-Present . . . . Electronics Engineer, Aeronautical Systems Division Wright-Patterson Air Force Base

PUBLICATIONS

FIELDS OF STUDY

Major Field: Electrical Engineering

Studies in Digital Systems. Professors Robert B. Lackey and Kenneth J. Breeding

Studies in Communication Theory. Professors C. Earl Warren and Dean T. Davis

Studies in Computer Science. Professor B. Chandrasekaran

Studies in Statistics. Professor M. A. Fligner
TABLE OF CONTENTS

ACKNOWLEDGMENTS .................................................. ii
VITA ................................................................. iii
LIST OF FIGURES .................................................... vi

Chapter

1. INTRODUCTION ..................................................... 1
   1.1 Statement of the Problem ................................. 1
   1.2 Review of Prior Work .................................. 2
   1.3 Overview of Results .................................. 3

2. CONTRACTION MAPPING AND ENTIRE FUNCTION THEORY ......... 7
   2.1 Introduction .............................................. 7
   2.2 Principle of Contraction Mappings ..................... 8
   2.3 Applications of Contraction Mapping .................. 15
   2.4 Entire Functions ...................................... 18

3. SANDBERG'S THEOREM ............................................ 20

4. REPRESENTING WAVEFORMS WITH INTERVAL AVERAGE SAMPLES .... 25
   4.1 Interval-Average Sampling ............................. 25
   4.2 A Theorem By Duffin and Schaeffer ................... 27
   4.3 Application of Sandberg's Theorem .................... 30
   4.4 Recovery of a Waveform from Non-equally Spaced Point Samples ... 36
   4.5 Determination of $k_1/k_2$ ............................. 38

5. EXTENSION OF WILEY'S WIDEBAND FM DEMODULATION ............ 47
   5.1 FM Demodulation Theorem .............................. 47
   5.2 Demodulation of Signals in Additive Noise ............. 53

iv
6. APPLICATIONS OF INTERVAL-AVERAGE SAMPLING .. 55
7. CONCLUSIONS AND RECOMMENDATIONS .............. 58

APPENDIXES

<table>
<thead>
<tr>
<th>Appendix</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Minimum Spacings Between Interval-Average Samples</td>
<td>60</td>
</tr>
<tr>
<td>B</td>
<td>Recovery of Stochastic Processes from</td>
<td>65</td>
</tr>
<tr>
<td></td>
<td>Interval-Average Samples</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>Recovery of Distorted Bandlimited</td>
<td>74</td>
</tr>
<tr>
<td></td>
<td>Stochastic Processes</td>
<td></td>
</tr>
</tbody>
</table>

BIBLIOGRAPHY ................................................................. 80
LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Two fixed points in metric space $K$ with associated disjoint subspaces $S_1$ and $S_2$. For an arbitrary element $x_0$ in subspace $S_1$, the repeated application of a contraction operator $T$ will yield a sequence which converges to the fixed point $y_0$.</td>
<td>12</td>
</tr>
<tr>
<td>2.</td>
<td>Two fixed points in metric space $K$ with associated non-disjoint subspaces. An arbitrary point in the shaded region would converge to $y_0$ if $T$ is a contraction on $S_1$, and to $y_0^2$ if $T$ is a contraction on $S_2$. Therefore, $T$ cannot be a contraction on both $S_1$ and $S_2$ since a unique solution would no longer be assured.</td>
<td>13</td>
</tr>
<tr>
<td>3.</td>
<td>With $-1 &lt; T'x &lt; 1$ and initial point $x_0$ on the interval $[a,b]$ we obtain a convergent sequence using $y = Tx_n$ and $x_{n+1} = y_n$.</td>
<td>14</td>
</tr>
</tbody>
</table>
4. If $|T'x| < 1$ on the interval $[a,b]$, convergence to the fixed point is still possible, but, additional criteria must be applied to ensure uniqueness. In this example, convergence is unique on the interval $[a,b]$ but not on the interval $[a,b']$.

5. Bounds on $k_1$ and $k_2$. 

vii
CHAPTER 1
INTRODUCTION

1.1 Statement of the Problem

In processing continuous and sampled waveforms we frequently encounter situations in which a distortion is introduced. Obviously, we would like to recover the undistorted signal from the distorted form, using only limited a priori information. When the desired signal is known to belong to some general class of waveforms, and the known distortion process satisfies certain fairly general criteria, then the desired waveform can be recovered using a simple iterative procedure. This research deals with distortions known as contraction mappings and some associated applications such as the demodulation of very wideband FM and the reconstruction of waveforms from non-equally spaced interval-average samples. Some of the initial work in this area was done by Wiley [1] when he showed that a quite general theorem by Sandberg [2] provided useful results when applied to the FM and interval-average sample problems. The main objective of this research was to solve a number of unresolved problems cited by Wiley in his earlier work, and to provide additional
applications of Sandberg's theorem in the processing of
deterministic and stochastic waveforms.

1.2 Review of Prior Work

Beurling, Landau, Miranker, and Zames [2,3,4] provided
some of the earlier work in the use of contraction mapping
theory for the recovery of distorted waveforms. They were
specifically dealing with the situation in which a square-
integrable bandlimited signal is distorted by a monotonic
nonlinear device. Sandberg extended this work by providing
a quite general theorem which guarantees the existence and
uniqueness of a solution for the case where a bandlimited
square-integrable function is distorted by a function sat­
isfying two basic conditions. These conditions and the
proof of the theorem are presented in Chapter 3. Masry
[5] provided a non-constructive uniqueness proof that a
sample path through a stochastic process could be recov­
ered using only the knowledge of its zero-crossings. A
summary of Masry's results is contained in Appendix C as a
complement to the developments in Appendix B.

Wiley's objective [1,6] was to apply Sandberg's
theorem to the problem of demodulation of very wideband
FM. As a consequence of his findings, Wiley also provided
some new results in the recovery of waveforms from non-
equally spaced interval-average samples [1,7]. However,
in both of these cases, the particular proof which he used
imposed a number of unnecessary restrictions.
Here, a more general and less restrictive approach to the theory of non-equally spaced interval-average and point samples is taken. The removal of several restrictions allows the theory to be applied to a much broader class of problems, as will be illustrated in the following chapters.

1.3 Overview of Research Results

Chapter 2 is a summarization of necessary background mathematics in contraction mapping theory and entire function theory. This will form the basis for the presentation of Sandberg's theorem in Chapter 3 and the further developments in the later chapters.

Chapter 4 contains the prime development of the interval-average sampler and the technique to recover a waveform from its unequally spaced interval-average samples. In the development of Chapter 4 the averaging time is allowed to vary from sample-to-sample and is only bounded by the requirement not to exceed the total time between samples. Thus, the averaging interval can equal the time between samples. Initially, the averaging time is bounded above zero; but, later in the chapter it is allowed to become zero in the limit and the appropriate results are obtained. A theorem by Duffin and Schaeffer [8] is used to prove the applicability of Sandberg's theorem to this signal recovery problem. However, Duffin
and Schaeffer's theorem only provides a non-constructive proof of the existence of two constants which are necessary. Determining a bound on the larger of these constants is not too difficult but a method for bounding the other above zero has been extremely evasive. It is shown in Chapter 4 that a bound on the ratio of these two constants is easily computable. Fortunately, this ratio is a form which can be used in lieu of the individual constants.

The use of Sandberg's theorem requires the identification of two additional constants, a procedure which has been difficult in the past because these constants are functions of the two constants from Duffin and Schaeffer's theorem. The various bounds on these constants, $k_1$ and $k_2$, are graphically depicted and then the desired ratio of $k_1/k_2$ is determined. Wiley [1,6,7] had empirically selected this ratio equal to 1—a value which has not been supported by a proof relative to the metric space being considered.

Using the results of Chapter 4, the problem of demodulating very wideband FM was considered. In Chapter 5, the results presented include a doubling of the allowable modulating signal bandwidth, allowing the positive peak frequency deviation from the carrier frequency to be any finite value, and the proof that the restriction on phase deviation which Wiley had required is satisfied as a
consequence of the specific signals being processed. Also, it is shown that the troublesome coefficient \( k_1/k_2 \) is equal to the square of the ratio of minimum to maximum instantaneous frequencies. The action of FM demodulators in the presence of noise is a very difficult problem. In this case we cannot even utilize the "narrow-band assumption" of the signal to be processed. Because of these difficulties, no new results are presented on this matter. However, a discussion linking some prior work with the results of Appendix B is provided.

A number of areas for application of the interval-average sampling theory are presented in Chapter 6. These include two-dimensional processing, use with charge-coupled devices, and applicability to improve performance of other sampling devices.

In the last chapter the overall conclusions and recommendations for further work are presented.

A proof that the minimum spacing between interval-average sample points is greater than zero is included in Appendix A. This proof had to be developed in order to allow the sample averaging time to equal the total time between samples. The doubling of the bandwidth in the FM demodulation was a consequence of this proof.

Recovery of stochastic processes from non-equally spaced interval-average samples is shown to be possible using the same iterative process presented in Chapter 4.
The proof, developed in Appendix B, is intended to support the notion that by the proper choice of a signal set, and an appropriate metric, the theorem by Sandberg has tremendous potential in signal recovery problems. This appendix also supports the discussions in Chapter 5 on demodulation of an FM signal in noise.

Masry's results on recovery of stochastic processes from the zero-crossing times is presented in Appendix C, as a complement to the new results in Appendix B.

A point worth noting is that Landau, et. al. used monotonic distortion functions in their work. Both Wiley and Masry stated that their results were obtained with non-monotonic distortions. The clarifying point which the author will add is that the distortion need only be monotonic with respect to the metric used. Thus, if we are using integral-squared distance (difference energy) as a metric, the distortion of the signal amplitude can be very non-monotonic, as it is with the interval average sampler.
CHAPTER 2
CONTRACTION MAPPING AND ENTIRE FUNCTION THEORY

2.1 Introduction

This chapter is an overview of some results in the theory of contraction mapping and the theory of entire functions. The discussion on contraction mapping will include the associated existence and uniqueness proofs which support the development of results presented in Chapters 3 thru 6. The background on entire function theory leads to a key proof in Chapter 4 and Appendix A on the recoverability of waveforms from their non-uniform interval average samples.

Application of contraction mapping theory requires the identification of a metric space $K$ and a contraction operator which maps $K$ into itself. In signal processing the main difficulty is to identify the appropriate function set (i.e. the points in space $K$), a suitable measure of distance, and the form of the contraction operator for a given signal processing function. The signal spaces which are considered include: the set of square-integrable (finite energy) signals; the set of wide sense stationary
stochastic processes with ergodic mean and autocorrelation function; and the set of all continuous functions. The metrics which will be used with these signal spaces are:

\[ \rho_1(x,y) = \left[ \int_{-\infty}^{\infty} (x(t)-y(t))^2 dt \right]^{\frac{1}{2}} \]  

(2-1)

for the finite energy signal space;

\[ \rho_2(x,y) = \left[ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (x(t)-y(t))^2 dt \right]^{\frac{1}{2}} \]  

(2-2)

for the set of stochastic processes; and,

\[ \rho_3(x,y) = \max_t |x(t)-y(t)| \]  

(2-3)

for the set of continuous functions. A particular problem with the use of metrics \( \rho_2 \) and \( \rho_3 \) is that the results of Duffin and Schaeffer which deal with finite energy functions [8] are no longer applicable in developing proofs that a particular mapping of a metric space is a contraction. In Appendix B, where \( \rho_2(x,y) \) and sample paths of stochastic processes are considered, a substitute for Duffin and Schaeffer's inequality is provided.

2.2 Principle of Contraction Mappings

Definition. Let \( K \) be an arbitrary metric space. A mapping \( T \) of the space \( K \) into itself is a contraction if
there exists a number $\alpha$, $0<\alpha<1$, such that

$$\rho(Tx,Ty) \leq \alpha\rho(x,y) \quad (2-4)$$

for any two points $x, y \in K$.

We now consider the contraction mapping theorem from Kolmogorov and Fomin [9].

**Theorem 2-1: (Principle of Contraction Mappings).**

Every contraction mapping defined on a complete metric space, $K$, has one and only one fixed point (i.e. the equation $Tx=x$ has one and only one solution).

**Proof.** Let $x_0(t)$ be an arbitrary function in $K$.

Set $x_1(t)=Tx_0(t)$, $x_2(t)=Tx_1(t)=T^2x_0(t)$, and in general let $x_n(t)=Tx_{n-1}(t)=T^nx_0(t)$. We show that the sequence $\{x_n(t)\}$ is fundamental, i.e. $x_n(t) \to x(t)$ as $n \to \infty$. In fact,

$$\rho(x_n,x_m) = \rho(T^nx_0,T^mx_0) \leq \rho(x_0,x_{m-n})$$

$$\leq \alpha^n \rho(x_0,x_1) + \rho(x_1,x_2) + \cdots + \rho(x_{m-n-1},x_{m-n})$$

$$\leq \alpha^n \rho(x_0,x_1) \left[ 1 + \alpha + \alpha^2 + \cdots + \alpha^{m-n-1} \right]$$

$$\leq \frac{\alpha^n \rho(x_0,x_1)}{1-(1-\alpha)} \quad (2-5)$$

This quantity is arbitrarily small for sufficiently large $n$ since $\alpha<1$. Because $K$ is complete, $\lim_{n \to \infty} x_n$ exists. Let this value equal $x$. Then by virtue of the continuity of
the mapping $T$,

$$Tx = T \lim_{n \to \infty} x_n = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = x$$  \hspace{1cm} (2-6)$$

Therefore, given an initial point (or function), $x_0(t)$, the convergence to a fixed point, $x(t)$, is proven. In addition to the existence of a solution we must also consider the uniqueness of the resulting $x$. If $Tx=x_*$ and $Ty=y$, then

$$\rho(Tx,Ty) = \rho(x,y) \leq \alpha \rho(x,y)$$  \hspace{1cm} (2-7)$$

where $\alpha<1$; this implies that $\rho(x,y)=0$, i.e. $x=y$.

*Definition.* An open sphere, $S(y_0,r)$, in the metric space $K$ is the set of points $y \in K$ satisfying the inequality $\rho(y_0,y)<r$. The fixed point $y_0$ is called the center of the sphere, and the number $r$ is called its radius [10].

In Theorem 2-1, the center of the sphere is the fixed point $x$ and the points in the sphere form the subspace (of "distortions" of $x$) from which the arbitrary element $x_0$ can be selected. Once this arbitrary $x_0$ is chosen, repetitive applications of the contraction operator will result in a sequence of points (within the sphere) which converge to the fixed point. The uniqueness of the solution, and even the guaranteed existence, is contingent
upon the subspaces associated with the fixed points in some larger metric space being disjoint. This is illustrated in Figures 1 and 2. Note that if $T$ is a contraction on $S_1$ and on $S_2$, then $S_1$ and $S_2$ must be disjoint. If $T$ is a contraction on every subspace, $S_i$, in a set $\{S_i\}$, then the union of these subspaces forms a more general metric space in which $T$ is a contraction operator.

As an example in the use of Theorem 2-1, consider $y = Tx$ where $T$ is a function (possibly non-linear) defined on a closed interval $[a,b]$, satisfying the Lipschitz condition

$$|Tx_2 - Tx_1| \leq \alpha |x_2 - x_1|, \quad 0 < \alpha < 1 \quad (2-8)$$

and mapping the closed interval into itself. Then $T$ is a contraction mapping and according to Theorem 2-1 the sequence $\{x_n\}$ converges to the single root of the equation $Tx = x$. Specifically, the contraction condition is satisfied if

$$0 \leq |T'x| \leq \alpha < 1 \quad (2-9)$$

on the interval $[a,b]$. Figure 3 is a graphical illustration of the convergence. We have equation (2-9) satisfied. Start with an arbitrary point, $x_0$, where $Tx_0$
Figure 1. Two fixed points in metric space $K$ with associated disjoint subspaces $S_1$ and $S_2$. For an arbitrary element $y_0^1$ in subspace $S_1$, the repeated application of a contraction operator $T$ will yield a sequence which converges to the fixed point $y_0^1$. 
Figure 2. Two fixed points in metric space $K$ with associated non-disjoint subspaces. An arbitrary point in the shaded region would converge to $y_0^1$ if $T$ is a contraction on $S_1$, and to $y_0^2$ if $T$ is a contraction on $S_2$. Therefore, $T$ cannot be a contraction on both $S_1$ and $S_2$ since a unique solution would no longer be assured.
Figure 3. With \(-1 \leq T'x \leq 1\) and initial point \(x_0\) on the interval \([a,b]\) we obtain a convergent sequence using \(y = T x_n \) and \(x_{n+1} = y_n\).

Figure 4. If \(|T'x| > 1\) on the interval \([a,b]\), convergence to the fixed point is still possible, but, additional criteria must be applied to ensure uniqueness. In this example, convergence is unique on the interval \([a,b]\) but not on the interval \([a,b']\).
lies on the interval \([a, b]\). Using \(x_{n+1} = y_n = T x_n\) we obtain a set of successive approximations to the desired solution \(x\). To illustrate that \(|T'x|<1\) is not a necessary condition, we consider Figure 4 where the derivative of \(T\) is greater than 1 for some \(x\); but, there is only one root of \(T x = x\) on the interval \([a, b]\).

2.3 Application of Contraction Mappings

The most interesting applications of Theorem 2-1 are when the space \(K\) is a function space. For the case of differential equations we have Picard's Theorem [10].

**Theorem 2-2 (Picard):** Given a function \(f(t, x)\) defined and continuous on a plane domain, \(G\), containing the point \((t_0, x_0)\), suppose \(f\) satisfies a Lipschitz condition of the form

\[
|f(t, x_1) - f(t, x_2)| < M|x_1 - x_2| \quad (2-10)
\]

in the variable \(x\). Then there is an interval \(|t - t_0| \leq \delta\) in which the differential equation

\[
\frac{dx}{dt} = f(t, x) \quad (-11)
\]

has a unique solution, \(x = \phi(t)\), satisfying the initial condition

\[
\phi(t_0) = x_0 \quad (2-12)
\]
Proof. The differential equation (2-11) and the initial condition (2-12) are equivalent to the integral equation

\[ \phi(t) = x_0 + \int_{t_0}^{t} f(\tau, \phi(\tau)) d\tau. \]  

(2-13)

By the continuity of \( f \), we have

\[ |f(t, x)| \leq K \]  

(2-14)

in some domain \( G' \subset G \) containing the point \( (t_0, x_0) \).

Choose \( \delta > 0 \) such that

1) \( (t, x) \in G' \) if \( |t - t_0| \leq \delta, \ |x - x_0| \leq K\delta \)  

(2-15)

2) \( M\delta < 1 \)  

(2-16)

and let \( C^* \) be the space of continuous functions, \( \phi \), defined on the interval \( |t - t_0| \leq \delta \) and such that \( |\phi(t) - x_0| \leq K\delta \).

We will use the metric

\[ \rho(\phi_1, \phi_2) = \max_{t} |\phi_1(t) - \phi(t)|. \]  

(2-17)

The space \( C^* \) is complete since it is a closed subspace of all continuous functions on \([t_0 - \delta, t_0 + \delta]\). Consider the mapping \( \psi = \Phi \phi \) defined by the integral equation

\[ \psi(t) = x_0 + \int_{t_0}^{t} f(\tau, \psi(\tau)) d\tau. \]  

(2-18)
\[ \psi(t) = x_0 + \int_{t_0}^{t} f(\tau, \phi(\tau)) d\tau, \quad (|t-t_0| \leq \delta). \quad (2-18) \]

T is a contraction mapping, projecting \( C^* \) into itself.

If \( \phi \in C^* \) and \( |t-t_0| \leq \delta \), then

\[ |\psi(t)-x_0| = \left| \int_{t_0}^{t} f(\tau, \phi(\tau)) d\tau \right| \]
\[ \leq \int_{t_0}^{t} |f(\tau, \phi(\tau))| d\tau \leq K|t-t_0| \leq K\delta \quad (2-19) \]

by (2-14); therefore, \( \psi = T\phi \) also belongs to \( C^* \). Also, we have

\[ |\psi_1(t)-\psi_2(t)| \leq \int_{t_0}^{t} \left| f(\tau, \phi_1(\tau)) - f(\tau, \phi_2(\tau)) \right| d\tau \]
\[ \leq M\delta \max_t |\phi_1(t) - \phi_2(t)| \quad (2-20) \]

and thus,

\[ \rho(\psi_1, \psi_2) \leq M\delta\rho(\phi_1, \phi_2) \quad . \quad (2-21) \]

But \( M\rho < 1 \), so that \( T \) is a contraction mapping. It follows from Theorem 2-1 that the integral equation (2-13) has a unique solution in the space \( C^* \) given by the convergent sequence

\[ \phi_{n+1}(t) = x_0(t) + \int_{t_0}^{t} f(\tau, \phi_n(\tau)) d\tau. \quad (2-22) \]
2.4 Entire Functions

**Definition.** An entire function is a function of a complex variable which is analytic throughout the entire (finite) complex plane. Thus an entire function is continuous and differentiable any number of times in the finite complex plane, and can be represented over the whole finite plane by its Taylor series

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \quad ; \quad a_n = \frac{f^{(n)}(0)}{n!} \quad (2-41) \]

where \( f^{(n)}(z) \) is the \( n \)th derivative of \( f(z) \) \([1]\).  

An entire function, \( f(z) \), which grows no faster than \( e^{\tau z} \) for some \( \tau \), is called an entire function of exponential type. Paley and Wiener provided a theorem \([1, 11, 12]\) stating that a necessary and sufficient condition for \( f(z) \) to be a square-integrable function of exponential type \( 2\pi f_B \) is that

\[ f_B \]

\[ f(z) = \int_{-f_B}^{f_B} F(f) e^{j2\pi fz} df , \quad (2-42) \]

where \( F(f) \) is a square-integrable function and

\[ F(f) = 0 , \quad |f| > f_B . \quad (2-43) \]

Worth noting, for future work on contraction mapping applications in the space of finite power signals.
and stochastic processes, is the results of Requicha [1,13] which shows that bounded bandlimited functions of finite power and nearly all sample functions of a wide-sense stationary bandlimited process with ergodic mean and autocorrelation function are entire functions of exponential type.
In this chapter some results of Sandberg [2] will be presented which provide a bridge between the theory of Chapter 2 and the new applications in Chapters 4-6 and Appendix B. Sandberg, using prior work by Beurling, Landau, Miranker, and Zames [3,4], developed a quite general theorem for use in the recovery of distorted waveforms. Specifically, distorted, bandlimited, square-integrable waveforms can be recovered using an iterative process if the particular distortion operator satisfies the conditions of the theorem. Landau and Miranker had considered the case where the distortion operator was in the form of an instantaneous compander. Wiley [1,6,7] applied Sandberg's theorem to wideband FM demodulation and to interval-average sample representation of waveforms. Extensions to both of these latter applications are presented in the next two chapters.

Let $H$ denote a real Hilbert space with $K$ being an arbitrary subspace of $H$. The transformation of an arbitrary element in $H$ into $K$ is accomplished using a projection operator, $P$. For the space of square-integrable functions
and a subspace of bandlimited square-integrable functions, P takes the form of a simple linear bandlimiting process. For real square-integrable functions we define an inner product and norm as

\[ (x_1, x_2) = \int_{-\infty}^{\infty} x_1(t)x_2(t) \, dt \]  \hspace{1cm} (3-1)

and

\[ ||x|| = (x, x)^{1/2}. \]  \hspace{1cm} (3-2)

This provides us with a measure of distance between \( x_1 \) and \( x_2 \) as

\[ ||x_1 - x_2|| = \left[ \int_{-\infty}^{\infty} (x_1 - x_2)^2 \, dt \right]^{1/2}. \]  \hspace{1cm} (3-3)

**Theorem 3-1 (Sandberg).** Let T be a mapping of \( K \) into \( H \) such that for all \( x_1, x_2 \epsilon K \):

\[ (Tx_1 - Tx_2, x_1 - x_2) \geq k_1 ||x_1 - x_2||^2 \]  \hspace{1cm} (3-4)

\[ ||PTx_1 - PTx_2||^2 \leq k_2 ||x_1 - x_2||^2 \]  \hspace{1cm} (3-5)

where \( k_1 \) and \( k_2 \) are positive constants. Then, for each \( h \epsilon K \), the equation
possesses a unique solution \((PT)^{-1}h \in K\) given by

\[
(PT)^{-1}h = \lim_{n \to \infty} x_n \quad (3-7)
\]

where

\[
x_{n+1} = \frac{k_1}{k_2} (h - PTx_n) + x_n \quad (3-8)
\]

and \(x_0\) is an arbitrary element of \(K\). Also,

\[
k_1^2 \|x_1 - x_2\|^2 \leq \|PTx_1 - PTx_2\|^2. \quad (3-9)
\]

Equation (3-4) ensures that the difference energy (or integral squared distance) between a distorted waveform and its undistorted version is always greater than zero unless the "distorted" and undistorted waveforms are identical. Equations (3-5) and (3-9) require that the difference energy between two distorted waveforms is bounded above and below by the difference energy in the undistorted waveforms multiplied by constant coefficients.

**Proof**. Note that

\[
(PTx_1 - PTx_2, x_1 - x_2) = (Tx_1 - Tx_2, Px_1 - Px_2)
\]

\[
= (Tx_1 - Tx_2, x_1 - x_2) \geq k_1 \|x_1 - x_2\|^2. \quad (3-10)
\]
for all \( x_1, x_2 \in K \). Using (3-6) we can write

\[
x = \overline{PT}x
\]  

(3-11)

where

\[
\overline{PT}x = c (h - PTx) + x
\]  

(3-12)

and \( c \) is any non-zero constant. From the previous chapter we know that (3-12) leads to the iterative equation

\[
x_{n+1} = \overline{PT}x_n = c (h - PTx_n) + x_n
\]  

(3-13)

when \( c \) is chosen such that \( \overline{PT} \) is a contraction operator. Using

\[
||\overline{PT}x_1 - PTx_2||^2 = ||x_1 - x_2 - cPTx_1 + cPTx_2||^2
\]

\[
= ||x_1 - x_2||^2 - 2c(PTx_1 - PTx_2) + c^2 ||PTx_1 - PTx_2||^2
\]

\[
\leq (1 - 2ck_1 + c^2k_2)||x_1 - x_2||^2
\]

(3-14)

for \( c > 0 \) we see that when

\[
k_1^2 < k_2
\]

(3-15)

we have
\[ \| P^T x_1 - P^T x_2 \| \leq \left( 1 - \frac{1}{k_2} \right)^{\frac{1}{2}} \| x_1 - x_2 \| \quad (3-16) \]

which satisfies (2-4) and Theorem 2-1 because

\[ 0 < \left( 1 - \frac{1}{k_2} \right)^{\frac{1}{2}} < 1 . \quad (3-17) \]

A second theorem by Sandberg states that if condition (3-4) is satisfied and if a solution exists, then that solution is unique. This allows us to be relatively unconstrained in the selection of $k_2$ such that rapid convergence of (3-8) will occur. Unfortunately, we have given up the guarantee of the existence of a solution.
4.1 Interval-Average Sampling

A method for recovering bandlimited signals from unequally spaced interval-average samples was described, by Wiley [1,7] as an extension of his work in the demodulation of wideband FM. Here, Wiley's interval-average sampling results are generalized in several important ways. In Chapter 5, when the wideband FM case is reconsidered, we see that these new results allow a considerable relaxation of prior restrictions. Also, the less restrictive conditions are shown to be limits based on more fundamental theory of the signal being processed.

Let $x(t)$ be a square-integrable function with Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad (4-1)$$

such that,

$$|F(\omega)| = 0 \text{ for } |\omega| > 2\pi f_B. \quad (4-2)$$
We define the sampling aperture as $e(n)$ and the set of sample times as $\{t_n\}$. It is important to note that $e$, the sampling aperture, need not be a constant and the sampling times need not be equally spaced. Let us restrict $e$ by

$$0 < e(n) \leq (t_n - t_{n-1}) \quad (4-3)$$

and let the sequence $\{t_n\}$ satisfy the following two conditions:

$$(t_n - t_{n-1}) \geq d > 0 \quad (4-4)$$

$$|t_n - \frac{n}{2f_B}| < \frac{L}{2f_B} \quad (4-5)$$

where $d$, $f_B$, and $L$ are positive constants.

Define $T$ as the interval-average sampling function,

$$T_x(t) = \sum_{n} \frac{u(t - t_n - e(n)) - u(t - t_n)}{e(n)} \int_{t_n - e(n)}^{t_n} x(t) dt \quad (4-6)$$

where $u(t)$ is the unit step function. Since $T$ is a non-linear process, the function $T_x$ is, in general, not band-limited. Therefore, we will obtain a bandlimited version by using a linear projection operator, $P$, which simply eliminates any components of $T_x$ outside the desired frequency band. Using $F[\cdot]$ to denote the Fourier transform operation, we have,
\[ F[PTx] = 0 \quad \text{for } |\omega| > 2\pi f_B. \quad (4-7) \]

Using the definitions and conditions cited above, we now state that the waveform \( x(t) \) can be completely recovered using the non-equally spaced interval-average samples from \( T x \) and the iterative procedure

\[
x_{n+1}(t) = x_n(t) + \frac{k_1}{k_2} (h - PTx_n(t)), \quad (4-8)
\]

where \( k_1 \) and \( k_2 \) are positive constants and \( h \) represents the bandlimited distorted version of \( x(t) \), i.e. \( h = PTx \).

To prove this we will use Sandberg's theorem in addition to the results of Duffin and Schaeffer [8] and Papoulis[14].

4.2 A Theorem By Duffin and Schaeffer

First we recall from Chapter 2 that a square-integrable function bandlimited to the frequencies

\[
|\omega| \leq 2\pi f_B \quad (4-9)
\]

is an entire function of exponential type. Next, we use the definition of Duffin and Schaeffer where a sequence \( \{t_n\} \) satisfying equations (4-4) and (4-5) is said to be of uniform density \( 2f_B \). From [8] we have the following theorem.

**Theorem 4-1.** Let \( \{t_n\} \) be a sequence of uniform density
and let \( 0 < \gamma < 2\pi f_B \). If \( x(t) \) is an entire function of exponential type \( \gamma \) such that \( x(t) \) is a square-integrable function, then

\[
\sum_{n} \left| x(t_n) \right|^2 \leq B. \tag{4-10}
\]

The lengthy proof will not be included here. \( A \) and \( B \) are positive constants which depend exclusively on \( \gamma \) and \( \{t_n\} \).

Note that \( \gamma \) is the bandlimit of \( x(t) \) from (2-42).

Defining \( x(t^n_0) \) by

\[
x(t^n_0) = \frac{1}{\varepsilon(n)} \int_{t^n - \varepsilon(n)}^{t^n} x(t) \, dt, \tag{4-11}
\]

we know from the mean value theorem for integrals, \[15\], that \( x(t) \) equals \( x(t^n_0) \) at some \( t \) in the interval \([t^n - \varepsilon(n), t^n]\). Since \( x(t) \) may assume the value equal to \( x(t^n_0) \) at more than one point in the interval \([t^n - \varepsilon(n), t^n]\) we will use the convention that \( t^n_0 \) will represent the largest \( t \) in the interval \([t^n - \varepsilon(n), t^n]\) such that \( x(t^n_0) \) equals \( x(t) \). Clearly, if \( \{t^n\} \) satisfies (4-5), then \( \{t^n_0\} \) satisfies

\[
|t^n_0 - \frac{n}{2f_B}| < \frac{L}{2f_B} + \varepsilon(n). \tag{4-12}
\]

The maximum value of \( \varepsilon(n) \) from (4-3) provides
\[
\frac{L}{2f_B} + \varepsilon(n) \leq \frac{L}{2f_B} + \max_n (t_{n+1} - t_n) = \frac{L}{2f_B} \quad (4-13)
\]

Now, we must show that

\[
(t_n^0 - t_{n-1}^0) \geq \delta > 0. \quad (4-14)
\]

Since \( t_n^0 \) is a point in the closed interval \([t_n - \varepsilon(n), t_n]\) and \( t_{n-1}^0 \) is from \([t_{n-1} - \varepsilon(n), t_{n-1}]\), using (4-3) and (4-4) we find that the only possible point common to both intervals is \( t_{n-1} \). This can happen only when

\[
\varepsilon(n) = t_n - t_{n-1}^0. \quad (4-15)
\]

Therefore, assuming that \( t_{n-1}^0 = t_{n-1} \), i.e. the largest value of \( t \) in the \((n-1)\)th interval, we must prove that \( t_n^0 \neq t_{n-1} \). Obviously, if \( x(t_n^0) \neq x(t_{n-1}^0) \), we know that \( t_n^0 \neq t_{n-1} \). Where \( x(t_n^0) = x(t_{n-1}^0) \) there exists some \( \delta > 0 \) such that

\[
(t_n^0 - t_{n-1}^0) \geq \delta > 0. \quad (4-16)
\]

This is proven in Appendix A. From (4-12) and (4-16) we see that \( t_n^0 \) is a sequence of uniform density \( 2f_B \) and from Theorem 4-1 we obtain
4.3 Application of Sandberg's Theorem

Using the interval-average sampling operator, T, from (4-6) and the bandlimiting projection operator, P, from (4-7) we can restate the two inequalities in Sandberg's theorem:

\[
\int_{-\infty}^{\infty} (Tx_1 - Tx_2)(x_1 - x_2) dt \geq k_1 \int_{-\infty}^{\infty} (x_1 - x_2)^2 dt \quad (4-18)
\]

\[
\int_{-\infty}^{\infty} (PTx_1 - PTx_2)^2 dt \leq k_2 \int_{-\infty}^{\infty} (x_1 - x_2)^2 dt \quad (4-19)
\]

where \(x_1\) and \(x_2\) are any two real square-integrable functions bandlimited to band \(f_B\).

The inner product of (4-18) can be expanded to

\[
\int_{-\infty}^{\infty} (Tx_1 - Tx_2)(x_1 - x_2) dt =
\]

\[
\int_{-\infty}^{\infty} \left[ \sum_{n=-\infty}^{\infty} \frac{u(t-t_n + \epsilon(n)) - u(t-t_n)}{\epsilon(n)} \right] (x_1 - x_2) dt \cdot \int_{t_n - \epsilon(n)}^{t_n} (x_1 - x_2) dt . \quad (4-20)
\]

Removing the terms independent of \(t\) from under the
integral we obtain

\[
\int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})(x_1 - x_2) \, dt =
\]

\[
\sum_{n} \frac{1}{\epsilon(n)} \int_{t_n - \epsilon(n)}^{t_n} (x_1 - x_2) \, dt
\]

\[
\int_{-\infty}^{\infty} \left[ u(t - t_n + \epsilon(n)) - u(t - t_n) \right] (x_1 - x_2) \, dt.
\]

(4-21)

This can be further reduced using the sequence of steps:

\[
\int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})(x_1 - x_2) \, dt = \sum_{n} \frac{1}{\epsilon(n)} \int_{t_n - \epsilon(n)}^{t_n} (x_1 - x_2) \, dt
\]

\[
\int_{t_n - \epsilon(n)}^{t_n} (x_1 - x_2) \, dt.
\]

(4-22)

\[
\int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})(x_1 - x_2) \, dt = \sum_{n} \epsilon(n) \left[ \frac{1}{\epsilon(n)} \int_{t_n - \epsilon(n)}^{t_n} (x_1 - x_2) \, dt \right]^2
\]

(4-23)

\[
\int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})(x_1 - x_2) \, dt = \sum_{n} \epsilon(n) [x_1(t_n^0) - x_2(t_n^0)]^2.
\]

(4-24)

From (4-10) we have

\[
A \int_{-\infty}^{\infty} (x_1 - x_2)^2 \, dt \leq \sum_{n} [x_1(t_n^0) - x_2(t_n^0)]^2
\]

(4-25)

where the set \( \{t_n^0\} \) is the unequally spaced sample times.
as defined in (4-11). Rewriting (4-24),

\[ \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})(x_1 - x_2) \, dt \geq \sum_{m} \min_{n} [\varepsilon(m)] [x_1(t_n) - x_2(t_n)]^2. \]  

(4-26)

Thus, using

\[ \Delta t_{\text{min}} = \min_{n} [\varepsilon(n)] \]  

(4-27)

we obtain

\[ \frac{1}{\Delta t_{\text{min}}} \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})(x_1 - x_2) \, dt \geq A \sum_{n} [x_1(t_n) - x_2(t_n)]^2. \]  

(4-28)

From (4-25) and (4-28) it is apparent that

\[ \frac{1}{\Delta t_{\text{min}}} \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})(x_1 - x_2) \, dt \geq A \int_{-\infty}^{\infty} (x_1 - x_2)^2 \, dt \]  

(4-29)

and

\[ \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})(x_1 - x_2) \, dt \geq A \Delta t_{\text{min}} \int_{-\infty}^{\infty} (x_1 - x_2)^2 \, dt. \]  

(4-30)

In [8] and (4-3) it was shown that both A and \( \Delta t_{\text{min}} \) are non-zero positive constants, thus providing a value for \( k_1 \),

\[ k_1 \leq A \Delta t_{\text{min}}. \]  

(4-31)

and (4-18) is proven.
From Parseval's relation it is clear that

\[ \int_{-\infty}^{\infty} [P(Ty)]^2 dt \leq \int_{-\infty}^{\infty} [Ty]^2 dt. \quad (4-32) \]

Therefore,

\[ \int_{-\infty}^{\infty} (PTx_1 - PTx_2)^2 dt \leq \int_{-\infty}^{\infty} (Tx_1 - Tx_2)^2 dt. \quad (4-33) \]

To prove Sandberg's second inequality, (4-19), holds we can use (4-33) and show that

\[ \int_{-\infty}^{\infty} (Tx_1 - Tx_2)^2 dt \leq k_2 \int_{-\infty}^{\infty} (x_1 - x_2)^2 dt. \quad (4-34) \]

Expanding the left-hand side of (4-34),

\[ \int_{-\infty}^{\infty} (Tx_1 - Tx_2)^2 dt = \int_{-\infty}^{\infty} \left[ \sum_{n} \frac{u(t-t_n+\epsilon(n))-u(t-t_n)}{\epsilon(n)} \right]^2 dt. \quad (4-35) \]

Equivalently, we have

\[ \int_{-\infty}^{\infty} (Tx_1 - Tx_2)^2 dt = \int_{-\infty}^{\infty} \left[ \sum_{n} [u(t-t_n+\epsilon(n))-u(t-t_n)] [x_1(t_n^0)-x_2(t_n^0)] \right]^2 dt. \quad (4-36) \]
But the elements in the summation are orthogonal because of the non-overlapping unit pulse weightings. Thus all cross-products from the squaring of the summation are zero and we can take the following steps:

\[ \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})^2 dt = \]

\[ \int_{-\infty}^{\infty} \sum_{n} [u(t-t_n+\epsilon(n))-u(t-t_n)][x_1(t_n^0)-x_2(t_n^0)]^2 dt \quad (4-37) \]

\[ \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})^2 dt = \sum_{n} \int_{t_n-\epsilon(n)}^{t_n} [x_1(t_n^0)-x_2(t_n^0)]^2 dt \quad (4-38) \]

\[ \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})^2 dt = \sum_{n} \epsilon(n)[x_1(t_n^0)-x_2(t_n^0)]^2 . \quad (4-39) \]

Returning to (4-10) we note that

\[ \sum_{n} [x_1(t_n^0)-x_2(t_n^0)]^2 \leq B \int_{-\infty}^{\infty} (x_1-x_2)^2 dt . \quad (4-40) \]

Using (4-39) and the definition

\[ \Delta t_{\text{max}} = \max_{n} [\epsilon(n)] \quad (4-41) \]

we see that

\[ \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})^2 dt \leq \Delta t_{\text{max}} \sum_{n} [x_1(t_n^0)-x_2(t_n^0)]^2 \quad (4-42) \]
and thus

$$\frac{1}{\Delta t_{\text{max}}} \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})^2 dt \leq \sum_{n} [x_1(t_n^0) - x_2(t_n^0)]^2.$$  \hspace{1cm} (4-43)

Continuing, we have from (4-10) and (4-43)

$$\int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})^2 dt \leq B\Delta t_{\text{max}} \int_{-\infty}^{\infty} (x_1 - x_2)^2 dt.$$  \hspace{1cm} (4-44)

Refering back to (4-19) and (4-33) we see that Sandberg's second inequality is satisfied for

$$\int_{-\infty}^{\infty} (P T_{x_1} - P T_{x_2})^2 dt \leq B\Delta t_{\text{max}} \int_{-\infty}^{\infty} (x_1 - x_2)^2 dt$$  \hspace{1cm} (4-45)

with

$$k_2 \geq B\Delta t_{\text{max}}.$$  \hspace{1cm} (4-46)

The signal recovery procedure (4-8) can therefore be utilized to recover a waveform after it has been sampled by the interval-average sample operator, (4-6). Notice that from (4-3), (4-4) and (4-27)

$$0 < \Delta t_{\text{min}} \leq \min_{n} (t_n - t_{n-1}).$$  \hspace{1cm} (4-47)

Also,
It is rather interesting that the averaging intervals can vary from sample to sample.

The important case which has been proven in this chapter is that interval-average samples with averaging intervals equal to the total time between samples can be used to uniquely reconstruct the sampled waveform.

4.4 Recovery of a Waveform from Non-equally Spaced Point Samples

One question that may arise is whether the iterative procedure in (4-8) can be used with non-equally spaced point samples. Wiley [7] conjectured that it could, but a notational difficulty precluded the proof. Using a slightly different representation, we can begin with the definition

\[ \sum \lim_{n \to 0} \frac{u(t-t_n+\varepsilon) - u(t-t_n-\varepsilon)}{2\varepsilon} \int_{t_n-\varepsilon}^{t_n+\varepsilon} x(t)dt = \sum \delta(t_n)x(t_n). \]  

(4-49)

Therefore the sampled waveform, as expected, is represented as

\[ Tx = \sum \delta(t_n)x(t_n). \]  

(4-50)

The left-hand side of (4-18) then becomes
\[ \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})(x_1 - x_2) \, dt = \int_{-\infty}^{\infty} \sum_{n} \delta(t_n)[x_1(t_n) - x_2(t_n)](x_1 - x_2) \, dt. \quad (4-51) \]

Continuing, we have

\[ \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})(x_1 - x_2) \, dt = \sum_{n} [x_1(t_n) - x_2(t_n)]^2. \quad (4-52) \]

Using (4-10) provides

\[ \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})(x_1 - x_2) \, dt \geq A \int_{-\infty}^{\infty} (x_1 - x_2)^2 \, dt \quad (4-53) \]

which is the proof that Sandberg's first inequality is satisfied for \( k_1 \) less than or equal to \( A \).

From equation (4-35) we have

\[ \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})^2 \, dt = \int_{-\infty}^{\infty} \sum_{n} \delta(t_n)[x_1(t_n) - x_2(t_n)]^2 \, dt \quad (4-54) \]

by use of the definition

\[ \left[ \sum_{n} \delta(t_n)x(t_n) \right]^2 = \sum_{n} \delta(t_n)[x(t_n)]^2. \quad (4-55) \]

Integrating the right-hand side of (4-54) we obtain

\[ \int_{-\infty}^{\infty} (T_{x_1} - T_{x_2})^2 \, dt = \sum_{n} [x_1(t_n) - x_2(t_n)]^2 \quad (4-56) \]
to use with (4-10) and (4-33) for

\[ \int_{-\infty}^{\infty} (PTx_1 - PTx_2)^2 dt \leq B \int_{-\infty}^{\infty} (x_1 - x_2)^2 dt. \quad (4-57) \]

When \( k_2 \) is greater than or equal to \( B \) the second inequality is satisfied. Thus the iterative equation, (4-8), can be used to recover a waveform from its non-equally spaced point samples.

4.5 Determination of \( \frac{k_1}{k_2} \)

Wiley [1,6] has shown empirical evidence that the iterative recovery procedure (4-8) has rapid convergence for

\[ \frac{k_1}{k_2} = 1. \quad (4-58) \]

The following development demonstrates how bounds can be placed on these constants. The main difficulty in obtaining good estimates for \( k_1 \) and \( k_2 \) is in determining non-trivial bounds on \( A \) and \( B \) for inequality (4-10). We will now consider the special case where the averaging period equals the entire interval between sample times. This will allow us to develop tighter bounds on \( k_1 \) and \( k_2 \) than those presented in [2] and elsewhere. Let us define \( \mu \) and \( \sigma \) such that
\[
\sum_{n} \int_{t_{n-1}}^{t_n} (x(t))^2 \, dt = \sum_{n} (t_n - t_{n-1}) \mu^2(n) + \sum_{n} (t_n - t_{n-1}) \sigma^2(n)
\]  
(4-59)

where \( \mu(n) \) is the average value of \( x(t) \) over the \( n \)th interval and \( \sigma^2(n) \) is the variance of \( x(t) \) over the \( n \)th interval. Clearly, \( \mu(n) \) represents the value obtained from the interval-average sampler when \( \varepsilon(n) \) equals the total time between samples.

Rewriting (4-59) and using (4-10) yields

\[
\frac{\Sigma_{\infty} \varepsilon(n)(\mu(n))^2}{\int_{-\infty}^{\infty} (x(t))^2 \, dt} = 1 \quad \frac{\Sigma_{\infty} \varepsilon(n)(\sigma(n))^2}{\int_{-\infty}^{\infty} (x(t))^2 \, dt} 
\]  
(4-60)

and

\[
1 - \frac{\Sigma_{\infty} \varepsilon(n)(\sigma(n))^2}{\int_{-\infty}^{\infty} (x(t))^2 \, dt} = \frac{\Sigma_{\infty} \varepsilon(n)(\mu(n))^2}{\int_{-\infty}^{\infty} (x(t))^2 \, dt} 
\]  
(4-61)

These equations can be further reduced to provide the inequalities

\[
\varepsilon_{\text{min}} \frac{\Sigma_{\infty} (\mu(n))^2}{\int_{-\infty}^{\infty} (x(t))^2 \, dt} \leq 1 - \frac{\Sigma_{\infty} \varepsilon(n)(\sigma(n))^2}{\int_{-\infty}^{\infty} (x(t))^2 \, dt} 
\]  
(4-62)

and

...
\[ 1 - \frac{\sum_{n} e(n)(\sigma(n))^2}{\int_{-\infty}^{\infty} (x(t))^2 \, dt} \leq \frac{\varepsilon_{\text{max}} \sum_{n} (\mu(n))^2}{\int_{-\infty}^{\infty} (x(t))^2 \, dt} \]  \hspace{2cm} (4-63)

Dividing (4-62) and (4-63) by \( \varepsilon_{\text{min}} \) and \( \varepsilon_{\text{max}} \), respectively, and using (4-31) and (4-46) we have

\[ k_1 = \frac{\varepsilon_{\text{min}}}{\varepsilon_{\text{max}}} \left( 1 - \frac{\sum_{n} e(n)(\sigma(n))^2}{\int_{-\infty}^{\infty} (x(t))^2 \, dt} \right) \]  \hspace{2cm} (4-64)

\[ k_2 = \frac{\varepsilon_{\text{max}}}{\varepsilon_{\text{min}}} \left( 1 - \frac{\sum_{n} e(n)(\sigma(n))^2}{\int_{-\infty}^{\infty} (x(t))^2 \, dt} \right) \]  \hspace{2cm} (4-65)

Before proceeding with (4-64) and (4-65) an examination of various bounds on \( k_1 \) and \( k_2 \) will be helpful.

These bounds are depicted in Figure 5. From (4-10), (4-31), and (4-46) we have

\[ A \leq B, \]  \hspace{2cm} (4-66)

\[ k_1 \leq \varepsilon_{\text{min}} A, \]  \hspace{2cm} (4-67)

\[ k_2 \geq \varepsilon_{\text{max}} B, \]  \hspace{2cm} (4-68)

and therefore \( k_1 \leq k_2 \) \hspace{2cm} (4-69)
Figure 5. Bounds on $k_1$ and $k_2$. 
with equality holding only for

\[ A \varepsilon_{\text{min}} = B \varepsilon_{\text{max}}. \]  

From [2] we have

\[ k_1^2 \leq k_2 \]  

as a condition for having a contraction mapping. Furthermore, we see from (4-64) and (4-65) that

\[ 0 \leq k_1 \leq 1 \]  

\[ 1 \leq k_2 \leq \frac{\varepsilon_{\text{max}}}{\varepsilon_{\text{min}}} = k_{2\text{max}}. \]  

Let \( \alpha \) be defined by

\[ \alpha = (1 - \frac{\sum_{n} \varepsilon(n)(\sigma(n))^2}{\int_{-\infty}^{\infty} x(t)^2 dt}) \]  

Then

\[ k_1 = \frac{\alpha}{k_{2\text{max}}} = \frac{\alpha^2}{k_2}. \]  

Dividing through by \( k_2 \), we obtain
Using (4-73) this becomes

\[
\frac{k_1}{k_2} = \frac{\alpha}{k_2k_{\max}} = \frac{\alpha}{\alpha k_2k_{\max} k_{\max}} = \frac{1}{k_{\max}}.
\]  

(4-76)

Equation (4-77) is a significant result considering the difficulties that have been encountered in attempting to obtain useful estimates of \( A \) in (4-10) and \( k_1 \) as used here. We not only have obtained a value for the coefficient in the iterative equation but also have found that it is independent of the characteristics of the signal being processed. Note that

\[
\frac{k_1}{k_2} = \frac{\varepsilon_{\min}^2}{\varepsilon_{\max}^2}.
\]

(4-77)

From (4-78) we can conclude that

\[
\frac{A}{B} = \frac{\varepsilon_{\min}}{\varepsilon_{\max}}.
\]

(4-79)

for the particular case where the sample values represent the average value of the waveform over the entire sample interval.
When the averaging intervals are less than the total time between samples we can show that the energy in the waveform, $x(t)$, is represented by

$$E = \sum_n e_n (\mu_1(n))^2 + \sum_n (\Delta t_n - e_n)(\mu_2(n))^2 + \sum_n \sigma_1^2(n) + \sum_n (\Delta t_n - e_n)(\sigma_2(n))^2 \quad (4-80)$$

where: $e_n$ is the averaging time for the $n^{th}$ interval, $[t_n - e_n, t_n]$,

$\mu_1$ is the interval-average of $x(t)$ during $e_n$,

$\Delta t_n$ is the time between the $(n-1)^{th}$ and the $n^{th}$ sample, $(t_n - t_{n-1})$,

$\mu_2$ is the interval-average during $[t_{n-1}, t_{n-} e_n]$,

$\sigma_1^2$ is the variance of $x(t)$ during $[t_{n-} e_n, t_n]$,

$\sigma_2^2$ is the variance of $x(t)$ during $[t_{n-1}, t_{n-} e_n]$.

Using (4-10) and (4-80) we obtain the previous results, (4-78), with $\alpha$ redefined as

$$\alpha = (1 - \frac{\sum_n (\Delta t_n - e_n)(\mu_2^2(n) + \sigma_2^2(n)) + \sum_n e_n \sigma_1^2(n)}{\int_{-\infty}^{\infty} (x(t))^2 dt}). \quad (4-81)$$

In treating the case of point samples we must define a term, $y(n)$, using

$$E_n - \Delta t_n x^2(t_n) = \Delta t_n y(n) \quad (4-82)$$
where \( E_n \) is the energy in the interval \([t_{n-1}, t_n]\), 
\( \Delta t_n \) is the time between the \((n-1)\)th and \(n\)th 
sample point, 
\( x(t_n) \) is the amplitude of \( x(t) \) at \( t=t_n \), 
\( y(n) \) is an unknown correction term.

Proceeding as before, we have

\[
\frac{\sum_n \Delta t_n x^2(t_n)}{\int_{-\infty}^{\infty} x^2(t)dt} = 1 - \frac{\sum_n \Delta t_n y(n)}{\int_{-\infty}^{\infty} x^2(t)dt}. \tag{4-88}
\]

This leads to

\[
A = \frac{\alpha}{\Delta t_{\text{max}}} \tag{4-84}
\]

and

\[
B = \frac{\alpha}{\Delta t_{\text{min}}} \tag{4-85}
\]

where \( \alpha \) equals the right-hand side of (4-83). Examining 
this and the previous discussion of point sampling, we see 
that

\[
\frac{A}{B} = \frac{k_1}{k_2} = \frac{\Delta t_{\text{min}}}{\Delta t_{\text{max}}}. \tag{4-86}
\]

The results presented here should be of considerable 
interest relative to Duffin and Schaeffer's work on non­
harmonic Fourier series [8]. Caution must be exercised
however because (4-86) represents the ratio of the tightest bounds on $A$ and $B$, independent of the waveform. If $B$ is simply bounded by some value, $B_{\text{max}}$, then $A$ cannot be obtained from (4-86). The correct approach would be to estimate the value of the appropriate $\alpha$ and solve for $A$ using

\[ A = \frac{\alpha}{\Delta t_{\text{max}}} \quad (4-87) \]

or

\[ A = \frac{\epsilon_{\text{min}}}{\epsilon_{\text{max}}} \quad (4-89) \]

for point samples or interval-average samples respectively.
CHAPTER 5
EXTENSION OF WILEY'S WIDEBAND FM DEMODULATION

5.1 FM Demodulation Theorem

Wiley [1,6] proved that Sandberg's theorem can be applied to the demodulation of wideband FM in cases where conventional discriminators would result in substantial distortion. However, a number of restrictions were imposed as a result of the proof which Wiley offered. The three most significant restrictions were the limiting of the modulating signal bandwidth to less than one-half of the carrier frequency, limiting the positive peak frequency to less than twice the carrier frequency, and limiting the phase deviation of the modulated signal. This latter restriction is satisfied as a result of the signals being processed. In this chapter all of these restrictions are removed and other generalizations are made by using the results of Chapter 4 and Appendix A.

Theorem 5-1: Let a waveform, \( s(t) \), be represented by

\[
s(t) = A(t) \sin \left[ \Theta_0 + 2\pi f_c t + \int_{-\infty}^{t} x(\tau)d\tau \right]
\]  

(5-1)

where \( f_c \) is the carrier frequency in Hz.,
\( \theta_0 \) is an arbitrary phase angle,
\( \beta \) is an index of modulation,
\( x(t) \) is the modulating waveform which is a square-integrable function and
1) \( x(t) \) is bandlimited to frequencies less than the carrier frequency,
2) the amplitude of \( x(t) \) is bounded by

\[
-2\pi\frac{f_c}{\beta} < x(t) < 2\pi\frac{(f_c - f_{\text{min}})}{\beta}, \quad (5-2)
\]

with \( f_{\text{min}} \) bounded above zero and \( f_{\text{max}} \) bounded below infinity.

Then \( x(t) \) can be uniquely recovered (within a multiplicative constant) from knowledge of the zero-crossings of \( s(t) \). Furthermore, the iterative recovery process is given by

\[
x_{n+1}(t) = \frac{f_{\text{min}}^2}{f_{\text{max}}^2} (h - PT x_n(t)) + x_n(t) \quad (5-3)
\]

with

\[
\lim_{n \to \infty} x_n(t) = x(t), \quad (5-4)
\]

\( PT \) is as defined in (4-6) and (4-7) for \( \varepsilon(n) \) equal to the total time between samples, and \( h \) is a distorted estimate
of \( x \) obtained by the piece-wise linear (zero-order hold) process

\[
h(t) = \left[ \sum_{n} \frac{u(t-t_{n-1})-u(t-t_{n})}{2(t_{n}-t_{n-1})} \right]_{\text{a.c.}} \tag{5-5}
\]

In (5-5) the \([ \cdot ]_{\text{a.c.}}\) denotes the a.c. portion of the bracketed term, and \( \{t_{n}\} \) is the set of zero-crossing times of \( s(t) \).

**Proof.** The proof of Theorem 5-1 will be in three parts:

1) proof that the "interval-average samples" of \( x(t) \) can be obtained from the zero-crossing times of \( s(t) \),

2) proof that the assumptions on \( \{t_{n}\} \) contained in Duffin and Schaeffer's theorem are satisfied,

3) proof that in (4-8)

\[
\frac{k_1}{k_2} = \frac{f_{\text{min}}^2}{f_{\text{max}}^2} \tag{5-6}
\]

for the wideband FM demodulation case.

**Part I.** Clearly \( s(t) \) has \( \pi \) radians of phase change during the time between zero-crossings. Therefore the average frequency over the \( n^{th} \) interval is

\[
\frac{1}{2 \text{ rad/cy}} \frac{\pi \text{ rad}}{(t_{n}-t_{n-1}) \text{ sec}} = \frac{1}{2(t_{n}-t_{n-1})} \text{ Hz}. \tag{5-7}
\]
By removing the mean frequency term we have from (5-1) and (5-7)

\[ \frac{1}{2(t_n - t_{n-1})} - f_c = \frac{C}{t_n - t_{n-1}} \int_{t_n}^{t_{n-1}} x(t) \, dt \quad (5-8) \]

where \( C \) is a multiplicative constant. Recalling (4-11) we see that

\[ \frac{1}{C} x(t_n^0) = \frac{1}{2(t_n - t_{n-1})} - f_c . \quad (5-9) \]

Since we can only recover \( x(t) \) within a multiplicative constant, the \( 1/C \) term can be dropped from the remaining steps. Therefore, the distorted representation of \( x(t) \), using these interval-average samples, is

\[ h(t) = \sum_n [u(t-t_{n-1}) - u(t-t_n)]x(t_n^0) . \quad (5-10) \]

Note that (5-10) is in agreement with (5-5). Part 1 is therefore completed.

Part 2. From (4-4) and (4-5) we require the zero-crossing times of \( s(t) \), \( \{t_n\} \), to satisfy

\[ (t_n - t_{n-1}) \geq d > 0 \quad (5-11) \]

\[ |t_n - \frac{n}{2f_c}| < \frac{L}{2f_c} . \quad (5-12) \]
If \( \{t_n\} \) satisfies these inequalities, then \( \{t_n^0\} \) will also satisfy them, as proven in Chapter 4 and Appendix A. From the conditions on the maximum instantaneous frequency, we have

\[
\min_n (t_n - t_{n-1}) = \frac{1}{2f_{\text{max}}} > 0, \quad f_{\text{max}} < \infty. \quad (5-13)
\]

In order to prove (5-12) Wiley placed an upper bound on the phase deviation of \( s(t) \). Here, inequality (5-12) is shown to be satisfied by the characteristics of the square integrable modulating waveform. From (5-12) we see that the difference between the \( n\)th zero-crossing of \( s(t) \) and the \( n\)th zero-crossing of the unmodulated carrier is required to be bounded. Define a bi-polar function, \( g(t_n) \),

\[
g(t_n) = \left( t_n - \frac{n}{2f_c} \right). \quad (5-14)
\]

Now, during the entire time that the instantaneous frequency of \( s(t) \) is greater than \( f_c \), \( g(t_n) \) will be strictly decreasing. Where the instantaneous frequency of \( s(t) \) is less than \( f_c \), \( g(t_n) \) will be strictly increasing. Therefore an upper and lower bound on \( g(t_n) \) is equivalent to a bound on the integral of \( x(t) \) during intervals when \( x(t) \) is negative or positive, respectively. If \( x(t) \) is a square-integrable function, then the integrals of these positive
regions must be finite and likewise with the negative regions. It follows, therefore, that (5-12) is satisfied.

Part 3. With the interval-average samples of \( x(t) \) being obtained from the zero-crossing times of \( s(t) \), and these zero-crossings, \( \{ t_n \} \), being of uniform density, as defined by Duffin and Schaeffer, we have from (4-77) and (5-4) that

\[
\frac{k_1}{k_2} = \frac{\min \limits_n (t_n - t_{n-1})^2}{\max \limits_n (t_n - t_{n-1})^2} \quad (5-15)
\]

But

\[
\min \limits_n (t_n - t_{n-1}) = \frac{1}{2f_{\max}} \quad (5-16)
\]

and

\[
\max \limits_n (t_n - t_{n-1}) = \frac{1}{2f_{\min}} \quad (5-17)
\]

The simple substitution provides

\[
\frac{k_1}{k_2} = \frac{(2f_{\min})^2}{(2f_{\max})^2} = \frac{f_{\min}^2}{f_{\max}^2} \quad (5-18)
\]

Therefore, the iterative recovery procedure

\[
x_{n+1}(t) = \frac{f_{\min}^2}{f_{\max}^2} (h - PTx_n(t)) + x_n(t) \quad (5-19)
\]
is a consequence of Chapter 4 with \( h(t) \) as defined in (5-5), \( T \) as defined in (4-6), and \( P \) being a linear band-limiting function (projection operator) which removes all frequency components above \( f_c \).

It is important to note that if the bandwidth of \( x(t) \) exceeded \( f_c \), then the sampling rate based on the zero-crossings of \( s(t) \) would be insufficient to allow recovery of \( x(t) \).

5.2 Demodulation of Signals in Additive Noise

"The error analysis of the zero-crossing detector is extremely difficult, and no adequate theory is at present available."[16] When the transmitted bandwidth is narrow in comparison to the carrier frequency, and the carrier frequency is much larger than the cutoff of the post-detection filter, then the action of the zero-crossing detector approximates the performance of the limiter-discriminator.

Wiley [1] has shown, for square-integrable signals, that the demodulator output peak error is proportional to the square root of the noise energy contained in \( h^* \), which is the noise corrupted form of \( h \). Unfortunately, if the noise waveform is a sample path from a stochastic process, the energy is unbounded and we have rather disappointing results. In Appendix B it is shown that the recovery process works for stochastic processes which are wide-sense stationary with ergodic mean and
autocorrelation function. Using the metric for this set of waveforms would yield a peak output error proportional to the square root of the average power of the noise waveform. Thus, we know that the recovery procedure works for finite-energy signals alone, and for sample paths of stochastic processes alone. The appropriate extension to jointly include both of these function sets has not been developed yet.
CHAPTER 6
APPLICATIONS OF INTERVAL-AVERAGE SAMPLING

Many techniques for sampling a waveform do not actually provide point samples even though such an assumption is made when these approaches are employed. As long as the function being sampled varies only slightly over the actual sampling interval, or aperture time, then the resulting errors will be negligible. However, when the waveform changes rapidly during these sample windows, the techniques of Chapter 4 can be used to remove the resulting distortion.

The designer of high speed analog-to-digital converters encounters several conflicting design parameters such as the requirement for higher sample rates, the requirement for small aperture times to minimize sample error, and the requirement for larger aperture times to allow the sample-and-hold function to respond to the input waveform when a sample is to be obtained. Where the circumstances permit, high speed analog-to-digital converters could be designed with larger than normal aperture times, resulting in improved sample-and-hold operation. Since a sample-and-hold circuit can respond
to the average value of the sampled function over an interval, Sandberg's reconstruction procedure can be used to recover the original waveform from the distorted samples.

A similar interval-averaging process occurs in the operation of charge-coupled devices. Here, depending on the particular clocking technique, the averaging time may be from one-fourth to one-half of the total time between samples. Again, the procedures developed in Chapter 4 will allow an expanded utilization of these devices.

Some of the state-of-the-art techniques for signal sampling involve electro-optic devices. One approach which uses an electron beam of a cathode ray tube and a linear diode array also provides interval-average samples as an output. The electron beam is z-axis modulated using the signal to be sampled. Each element of the diode array then receives an amount of charge proportional to the intensity of the beam as it is swept across the linear array. In this case the averaging or smoothing function is actually represented by the convolution (two-dimensional) of the diode surface area and a cross-section of the electron beam. The Fourier pair of this sampling technique can be implemented using a laser, a Bragg cell, and a diode array. When the signal to be sampled (at an appropriate intermediate frequency) is
applied to a Bragg cell it causes an angular deflection of a laser beam proportional to the frequency of the input waveform. Each element of the diode array then has an output proportional to the power present in a particular window of the spectrum. We are averaging over frequency instead of over time as with the previous approach. A key difference here, however, is that the original waveform is not recoverable because we have sampled the power spectrum and not the amplitude spectrum. The distortions in the power spectrum can be removed using the same procedure as with the other sampling approaches.

Another application area to be considered is the processing of two-dimensional images. While the averaging process resulting from image sensors can be treated as a two-dimensional extension of the other sampling approaches, an area of even more interest is in the general area of blurred images. Blurring frequently can be characterized as a convolution of the image with a spatially invariant point-spread function. It should not be too difficult to characterize these point-spread functions such that the iterative recovery procedure, expanded to two dimensions, could be used. Also the area of computational complexity would have to be addressed.
CHAPTER 7
CONCLUSIONS AND RECOMMENDATIONS

The prior work by Sandberg and Wiley has been used as a foundation for the new contributions developed in this work. The fact that a waveform can be recovered from non-equally spaced samples has been firmly established, for the cases where the samples represent point samples, where they represent interval-average samples over a part of the sample interval, or where they represent interval-average samples over the entire interval between samples. A key to establishing this latter case is included in the proof of Appendix A, where the finite distance between interval-average samples is presented.

Using the results of Chapter 4, several extensions to Wiley's wideband FM demodulation are developed. Included in these results is the doubling of the allowable bandwidth over previous restrictions, a considerable relaxation of the restriction on peak positive frequency deviation, and finally, the limitation on peak phase deviation from an unmodulated carrier was shown to be satisfied as a consequence of the class of signals being
processed. Again, the demodulation of very wideband FM is simply a special application of the recovery of a waveform from its interval-average samples where the averaging interval is the total time between samples.

The recovery of sample paths through a stochastic process are treated in Appendix B and in a summary of Masry's work in Appendix C. The results indicate that, within reasonable limits, these sample paths can be recovered using the same procedures that were used with deterministic waveforms. The area of processing stochastic processes is worthy of much more investigative work. Using the results of both Appendix B and Appendix C, it may be possible to establish a constructive proof on the recovery of stochastic processes from the zero-crossing times.
In this appendix, using the notation from Chapter 4 for interval-average samples, we show that

\[(t_n^0 - t_{n-1}^0) \geq \delta > 0.\]  \hspace{1cm} (A-1)

Recall from the discussion on entire functions in Chapter 1 that square-integrable functions which are bandlimited to band \(f_b\) are entire functions of exponential type \(2\pi f_b\). Therefore, these functions are continuous and differentiable. From Chapter 4 we note that if

\[(t_n - t_{n-1}) \geq \varepsilon(n) + \delta, \quad \delta > 0, \quad (A-2)\]

then equation (A-1) is satisfied. However, if

\[\varepsilon(n) = (t_n - t_{n-1}), \quad (A-3)\]

the point \(t_{n-1}\) becomes common to both the \((n-1)\)th and the \(n\)th averaging interval. Under condition (A-3) if
\( x(t_{n-1}) \), the average value over the \((n-1)\)th interval, does not equal \( x(t_n^0) \), then some minimum spacing must exist between \( t_n^0 \) and \( t_{n-1}^0 \). Using the results of Papoulis [14] we note that

\[
|f(t+\tau) - f(t)| < 2\pi f_B \sqrt{\frac{2E f_B}{3}}. \tag{A-4}
\]

With

\[
\tau = t_n^0 - t_{n-1}^0 \tag{A-5}
\]

we see that \( \tau \) must be greater than zero if

\[
x(t_n^0) \neq x(t_{n-1}^0). \tag{A-6}
\]

Now assume that

\[
x(t_n^0) = x(t_{n-1}^0). \tag{A-7}
\]

Further assume that

\[
t_{n-1}^0 = t_{n-1}^0. \tag{A-8}
\]

i.e., the largest value of \( t \) in the interval \([t_{n-2}, t_{n-1}]\) for which the average value of \( x \) equals the point value
of $x$. We must show under conditions (A-7) and (A-8), that

$$(t^0_n - t^0_{n-1}) > \delta > 0 . \quad (A-9)$$

There are three cases to consider. The first, fairly trivial one, is when

$$\frac{dx(t)}{dt} = 0 \quad (A-10)$$

over the entire closed interval $[t_{n-1}, t_n]$. Clearly, any point of $x$ on this interval is equal to the average value over the interval and, therefore, using the convention from Chapter 4 we choose $t^0_n$ to equal $t_n$. This provides a minimum spacing between $t^0_n$ and $t^0_{n-1}$ which is greater than zero.

The second case is when

$$\frac{dx(t)}{dt} \bigg|_{t=t_{n-1}} > 0 . \quad (A-11)$$

If the derivative of $x(t)$ at $t_{n-1}$ is greater than zero, then $x(t)$ must be greater than $x(t^0_{n-1})$ over some interval $(t_{n-1}, c)$. Since the average value of $x(t)$ over $[t_{n-1}, t_n]$ is equal to $x(t^0_{n-1})$, $x(t)$ must be less than $x(t^0_{n-1})$ over part of the interval $[t_{n-1}, t_n]$. Thus $x(t)$ must
cross the line at $x(t_{n-1}^0)$ somewhere in the interval $[c, t_n]$. Because of the continuity of $x$, $x(t)$ will assume the value of $x(t_{n-1}^0)$ at this crossing point. If multiple crossings occur we simply choose the last one as the time for $t_{n}^0$.

If only one crossing occurs, this will be at point $c$. The $n$th interval will then be segmented into two regions: a region where $x(t)$ is above the average; and, a region where $x(t)$ is below the average such that

$$
\int_{t_{n-1}}^{c} [x(t) - x(t_{n-1}^0)] dt = \int_{c}^{t_{n}} [x(t) - x(t_{n-1}^0)] dt.
$$

(A-12)

From Rolle's Theorem [15] we know that if

$$
x(t_{n-1}) = x(c) = x(t_{n-1}^0)
$$

(A-13)

there exists at least one point, $b$,

$$
t_{n-1} < b < c,
$$

(A-14)

where

$$
\left. \frac{dx(t)}{dt} \right|_{t=b} = 0.
$$

(A-15)

Therefore, we have
\[ t_n^0 = c \]  

and

\[ (t_n^0 - t_{n-1}) > b - t_{n-1} > 0. \]  

The third case is the mirror image of Case 2. We have

\[ \frac{dx(t)}{dt} \bigg|_{t=t_{n-1}} < 0, \]  

and begin from \( t_{n-1} \) to some point, \( c \), with \( x(t) \) less than the average. By an identical argument with Case 2, we see that equation (A-18) holds.

Thus, with \( x(t) \) being an entire function of exponential type, and using the convention in Chapter 4, we have proven that some \( t \) must exist such that the \( t_n^0 \) is a finite, non-zero, distance from \( t_{n-1}^0 \).
APPENDIX B
RECOVERY OF STOCHASTIC PROCESSES FROM
INTERVAL-AVERAGE SAMPLES

Since the interval-average sampling of finite energy signals (Chapter 4) and the wideband FM demodulation (Chapter 5) will, in general, be used when a corrupting noise signal is present, we must consider the action of the interval-average sampler (and the recovery process) on non-square-integrable waveforms. Specifically, stochastic processes will be considered, and it will be shown that sample paths through these processes can be recovered from their interval-average samples. Of course, we will still be unable to resolve multiples of a given waveform. This is equivalent to stating that any two sample paths with a cross-correlation other than $+1$ are distinguishable and will be recovered uniquely.

Theorem B-1: Let $X$ represent all wide-sense stationary bandlimited stochastic processes with zero mean, ergodic autocorrelation function, and variance $\sigma_x^2$. The elements of $X$ are bandlimited to $f_c$. If the sample times, $\{t_n\}$, have uniform density $2f_c$ as defined by Duffin and Schaeffer
[8], then the interval-average samples of a sample path, $x_1(t)$, through $X$ uniquely characterize $x_1(t)$ within a multiplicative constant and $x_1(t)$ can be recovered using the iterative formula

$$x_{1,n+1}(t) = x_{1,n}(t) + \frac{\Delta t^2_{\min}}{\Delta t^2_{\max}} (h - Pt_{x_1,n}(t)) \quad (B-1)$$

where $h$, $P$, and $T$ are as defined in Chapter 4.

**Proof:** The "set" of waveforms to be processed are sample paths of a bandlimited, zero mean, stochastic process with variance $\sigma^2_X$. An appropriate metric to use is obtained from the measure of average power, i.e.

$$\rho^2(x_1,x_2) = E[(x_1-x_2)^2] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (x_1(t)-x_2(t))^2 dt \quad (B-2)$$

where $x_1$ and $x_2$ are sample paths through any of the stochastic processes represented by $X$. The norm of $x_1$ is

$$\|x_1\|^2 = E[x_1^2(t)] = \sigma^2_X. \quad (B-3)$$

In order to be brief, the proof will only consider the case where the averaging interval represents the total time between samples. The extension to other cases
is relatively simple and would use the same methods employed in Chapter 4 when dealing with square-integrable functions. The proof will consist of five parts. The first will demonstrate that the interval-average sample points \( \{t_n^0\} \) have non-zero spacing, the second will prove the existence of Duffin and Schaeffer's constants, \( A \) and \( B \), the next two parts will show that Sandberg's inequalities are satisfied, and the last part will provide the coefficient \( k_1/k_2 \).

**Part 1.** Requicha [1,13] showed that sample paths of the stochastic processes specified are entire functions of exponential type. Therefore, these sample paths are continuous and differentiable. Thus, the results of Appendix A are valid for establishing that

\[
(t_n^0 - t_{n-1}^0) > d > 0. \quad (B-4)
\]

**Part 2.** From the property of interval-average samples we know that \( x_1(t_n^0) \) represents the mean value of \( x_1 \) over the interval \([t_{n-1}, t_n]\). Let the variance of \( x_1 \) over this same interval be represented by \( \sigma_1(n) \). Then

\[
E[x_1^2] = \lim_{N \to \infty} \left( \sum_{m=-N+1}^{N} \Delta t_m \right)^{-1} \sum_{n=-N+1}^{N} \Delta t_n [x_1^2(t_n^0) + \sigma_1^2(n)]. \quad (B-5)
\]
Let

\[ T_n = \sum_{m=-N+1}^{N} (t_m - t_{m-1}). \]  

(B-6)

Rewriting (B-5) we have

\[ \lim_{N \to \infty} \frac{1}{T_n} \sum_{n=1}^{N} \Delta t_n x_1^2(t_n^n) \geq \frac{1}{E[x_1^2]} \lim_{N \to \infty} \frac{1}{T_n} \sum_{n=1}^{N} \Delta t_n \sigma_1^2(n) \]

(B-7)

This conveniently leads to

\[ A = \frac{\alpha}{\Delta t_{max}} \leq \frac{1}{E[x_1^2]} \lim_{N \to \infty} \frac{1}{T_n} \sum_{n=1}^{N} x_1^2(t_n^n) \leq \frac{\alpha}{\Delta t_{min}} = B \]  

(B-8)

for

\[ \alpha = 1 - \frac{1}{E[x_1^2]} \lim_{N \to \infty} \frac{1}{T_n} \sum_{n=1}^{N} \Delta t_n \sigma_1^2(n) \]  

(B-9)

Thus, (B-8), which is a new form of Duffin and Schaeffer's equation for interval-average samples of finite power waveforms, provides the necessary constants, A and B.

Part 3. With use of metric (B-2) we write the equivalent of Sandberg's first inequality,
\[ E^2[(T_1 - T_2)(x_1 - x_2)] \geq k_1 E^2[(x_1 - x_2)^2]. \] (B-10)

The left-hand side of (B-10) can be expanded as

\[ E^2[(T_1 - T_2)(x_1 - x_2)] = E^2[x_1 T_1 + x_2 T_2 - x_1 T_2 - x_2 T_1]. \] (B-11)

Examine \( x_1 T_1 \) and note that

\[ E[x_1 T_1] = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} x_1(t) \sum_{n} [u(t-t_{n-1}) - u(t-t_{n})]x_1(t_{n}^{0})dt. \] (B-12)

Continuing, we have

\[ E[x_1 T_1] = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \sum_{n} [u(t-t_{n-1}) - u(t-t_{n})]x_1(t)x_1(t_{n}^{0})dt, \] (B-13)

and

\[ E[x_1 T_1] = \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{n=-a}^{a} \frac{\Delta t_{n}}{\Delta t_{n}} \int_{t_{n-1}}^{t_{n}} x_1(t)x_1(t_{n}^{0})dt, \] (B-14)

\[ E[x_1 T_1] = \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{n=-a}^{a} \Delta t_{n} x_1^2(t_{n}^{0}) = E[x_1^2(t_{n}^{0})]. \] (B-15)
We can proceed using (B-15) and rewriting (B-11),

\[
E^2[(T_{x_1}-T_{x_2})(x_1-x_2)] = \\
E^2[x_1^2(t^0) + x_2^2(t^0) - 2x_1(t^0)x_2(t^0)].
\] (B-16)

From (B-8) we use

\[
E^2[(x_1(t^0)-x_2(t^0))^2] \geq A^2E^2[x_1^2(t)].
\] (B-17)

Thus \( k_1 \) equals \( A^2 \). Note that the writing of (B-16) and (B-17) was simplified by use of a notational convenience defined in (B-15).

Part 4. Sandberg's second inequality is

\[
E^2[(PT_{x_1}-PT_{x_2})^2] \leq k_2E^2[(x_1-x_2)^2].
\] (B-18)

From Parseval we have

\[
E^2[(PT_{x_1}-PT_{x_2})^2] \leq E^2[(T_{x_1}-T_{x_2})^2].
\] (B-19)

Rewriting the right-hand side of (B-19) yields

\[
E^2[(T_{x_1}-T_{x_2})^2] = E^2[x_1^2(t^0) + x_2^2(t^0) - 2x_1(t^0)x_2(t^0)].
\] (B-20)

Equation (B-8) is then used and
\[ E^2[x_1^2(t^0)+x_2^2(t^0)-2x_1(t^0)x_2(t^0)] \leq B^2E^2[(x_1-x_2)^2] \]

is obtained. From (B-19), (B-20), and (B-21) we show that

\[ E^2[(PTx_1-PTx_2)^2] \leq B^2E^2[(x_1-x_2)^2] \]

satisfies (B-18) where \( k_2 \) equals \( B^2 \).

Part 5. From the previous four parts, we know that the necessary conditions for Sandberg's theorem are satisfied. Therefore, the iterative equation for recovery of the sample path through \( X \) using the interval-average samples is

\[ x_{1,n+1}(t) = x_{1,n}(t) + \frac{k_1}{k_2}(h-PTx_{1,n}(t)) \]

where:

- \( x_{1,n} \) is the \( n^{th} \) approximation to the sample path, \( x_1(t) \),
- \( h \) is the distorted form of \( x_1 \) using piece-wise linear approximation based on the interval-average values,
- \( P \) is the bandlimiting projection operator,
- \( T \) is the interval-average sampler using \( \{t_n\} \),

and

\[ \frac{k_1}{k_2} = \frac{A^2}{B^2} = \frac{\Delta t_{\text{min}}^2}{\Delta t_{\text{max}}^2}. \]
The important point to note is that the same iterative recovery process is used (with the same value for $k_1/k_2$) for the two separate metric spaces: 1) square-integrable functions with integral-squared difference as the squared metric; and, 2) stochastic processes belonging to $X$ with expected squared distance as the squared metric.

Although the process works with the two classes of signals individually, we have not shown that a linear combination of a finite energy signal and a sample path through a stochastic process will be recovered. If the combined waveform falls into one of the two classes considered, then it will be recoverable. If it falls outside these two classes, then one must treat a much broader class of waveforms to demonstrate the applicability of Sandberg's theorem.

Masry [5] has treated the problem of recovery of a distorted bandlimited stochastic process from the zero-crossings of a sample path. In view of the results contained in this chapter, it may be possible to extend Masry's work such that a constructive proof of the recovery process will be available using the zero-crossing times. To encourage additional work in this area, Masry's non-constructive proof is contained in Appendix C. In effect he has proven that $k_1$ is greater than zero, however, we still need to determine $k_1/k_2$ in order to use
Sandberg's iterative reconstruction procedure.
APPENDIX C

RECOVERY OF DISTORTED BANDLIMITED STOCHASTIC PROCESSES

The recovery of distorted bandlimited stochastic processes has been investigated by Masry [5] with some surprising results. Masry's first theorem is reproduced here to compliment the development in Appendix B. Some assumptions must first be made which will allow proof of the theorem. Let the input process, $x(t)$, be a real second-order mean-square continuous stationary bandlimited stochastic process. Let $S(\omega)$ be the spectral density of the process $x(t)$ and $C(\tau)$ its covariance function,

$$C(\tau) = \int_{-W}^{W} S(\omega) e^{i\omega \tau} d\omega \quad (C-1)$$

where $(-W, W)$ is the bandwidth of the process and where we have assumed the process is normalized to have zero mean and unit variance. Assume that two such processes, $x_1(t)$ and $x_2(t)$ are jointly stationary. Then $x=(x_1, x_2)$ is a two-dimensional stationary bandlimited process since

$$|S_{x_i x_j}(\omega)|^2 \leq S_{x_i x_i}(\omega) S_{x_j x_j}(\omega); \ i, j = 1, 2. (C-2)$$

74
Denote the set containing all processes satisfying the above conditions as $\mathcal{X}$. Next, let the process, $y(t)$, be defined by

$$y(t) = A[x(t)]. \quad (C-3)$$

The process $y(t)$ is stationary. We shall assume that

1) $E[A^2(x(t))] < \infty$ for all $t$, \hspace{1cm} (C-4)
2) $y(t)$ is mean-square continuous
3) $E[x(t)y(t)] \neq 0$. \hspace{1cm} (C-5)

Let $A$ denote the class of admissible functions, $A(x)$, satisfying these three conditions.

If the function $A$ represents a hard-limiter

$$A(x) = \text{sgn} \ x \quad (C-6)$$

and $x \in \mathcal{X}$ is Gaussian, then

$$R_{yy}(\tau) = \left( \frac{2}{\pi} \right) \sin^{-1} C_{xx}(\tau) \quad (C-7)$$

so that $y(t) = \text{sgn}[x(t)]$ is a stationary second-order mean-square continuous process.
The linear system \( L \) is an ideal low-pass filter with transfer function

\[
H(\omega) = \begin{cases} 
1, & |\omega| \leq W \\
0, & |\omega| > W 
\end{cases} \quad (C-8)
\]

Thus,

\[
z(t) = L(A(x(t))) = T(x(t)) \quad (C-9)
\]

is a second-order mean-square continuous stationary band-limited process.

**Theorem C-1:** Let \( x_1(t) \) and \( x_2(t) \) be two jointly Gaussian processes in \( X \). Let

\[
z_i(t) = [L(A(x_i(t)))] \quad i=1,2 \quad (C-10)
\]

where \( A \in A \). If

\[
z_1(t) = z_2(t) \text{ almost surely for some } t=t_0 \quad (C-11)
\]

then

\[
x_1(t) = x_2(t) \text{ almost surely for all } t. \quad (C-12)
\]

**Proof:** Consider the functional \( J \),
which is equivalent to

$$J = \alpha \mathbb{E}[(x_1(t) - x_2(t))^2], \text{ for all } t. \quad (C-20)$$

And, since by (C-13), J equals zero and \(\alpha \neq 0\) by (C-18) and (C-5), we have from (C-20) that

$$x_1(t) = x_2(t) \text{ almost surely for all } t \quad (C-21)$$

so that the two processes \(x_1(t)\) and \(x_2(t)\) are equivalent.

At this point Masry correctly pointed out the surprising result that the companding function does not have to be monotonic. He continued by stating that this result had no counterpart for deterministic bandlimited functions, thus, the non-monotonic companding functions treated in this thesis are significant in illustrating such a relationship.

The non-monotonic function which Masry used for stochastic processes is the two-level quantizer

$$A[x(t)] = \text{sgn } x(t) \quad (C-22)$$

which has been shown to be an admissible companding function. From Theorem C-1 we have
\[ L[\text{sgn } x_1(t)] = L[\text{sgn } x_2(t)] \text{ almost surely for all } t \] (C-23)

implies

\[ x_1(t) = bx_2(t) \text{ almost surely for all } t \] (C-24)

where \( b \) is a real constant. Thus no two processes in \( X \) can have the same zero-crossings unless one is a constant multiple of the other. It is important to note that the proof in this appendix is non-constructive. Insight into reconstruction techniques may be built on the hypothesis of Chapters 4 and 5.
BIBLIOGRAPHY


