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A PEAK-LOAD DESIGN CRITERION WITH APPLICATION TO THE RUPTURE OF SOLID CIRCULAR CYLINDERS OF STRAIN-HARDENING MATERIALS SUBJECTED TO A COMBINATION OF TENSION, TORSION AND SMALL BENDING MOMENT.

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A PEAK-LOAD DESIGN CRITERION WITH APPLICATION TO THE RUPTURE OF SOLID CIRCULAR CYLINDERS OF STRAIN-HARDENING MATERIALS SUBJECTED TO A COMBINATION OF TENSION, TORSION AND SMALL BENDING MOMENT

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

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<td>$A_f$</td>
<td>fracture surface area</td>
</tr>
<tr>
<td>$A_0$</td>
<td>original cross-sectional area</td>
</tr>
<tr>
<td>$a$</td>
<td>radius of a solid circular cylinder</td>
</tr>
<tr>
<td>$B_v$</td>
<td>bulk modulus of elasticity</td>
</tr>
<tr>
<td>$b$</td>
<td>ratio of the normal strain at a critical point to the ultimate strain</td>
</tr>
<tr>
<td>$C_1, C_2$</td>
<td>integration constant</td>
</tr>
<tr>
<td>$c$</td>
<td>ratio of $\Delta \varepsilon_{zo}$ to $\varepsilon_{zo}$</td>
</tr>
<tr>
<td>$e$</td>
<td>eccentricity of the axial force</td>
</tr>
<tr>
<td>$e$</td>
<td>deviatoric strain</td>
</tr>
<tr>
<td>$F$</td>
<td>failure function defined as the sum of $(P/P_u)^2$ and $(T/T_u)^2$.</td>
</tr>
<tr>
<td>$G$</td>
<td>modulus of rigidity of a material</td>
</tr>
<tr>
<td>$L_0$</td>
<td>original gage length of a standard specimen</td>
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<tr>
<td>$\Delta L$</td>
<td>additional length</td>
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<tr>
<td>$M$</td>
<td>bending moment</td>
</tr>
<tr>
<td>$P$</td>
<td>axial load at failure in combined loading</td>
</tr>
<tr>
<td>$P_f$</td>
<td>load at failure in a standard specimen</td>
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<tr>
<td>$P_i$</td>
<td>axial load, not at failure</td>
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<tr>
<td>$P_u$</td>
<td>maximum attainable pure axial load of a combined load specimen</td>
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NOMENCLATURE (continued)

\( P_{uc} \) calculated \( P_u \) based on the ultimate strength of a material

\( r \) radial position of a small volume element

\( S \) deviatoric stress

\( S_f \) the stress at rupture of a standard specimen

\( S_u \) the engineering ultimate strength of a standard specimen

\( S_n \) engineering nominal stress obtained from a standard specimen

\( T \) torque at failure in combined loading

\( T_i \) torque, not at failure

\( T_u \) maximum attainable pure torque of a combined loading specimen

\( T_{uc} \) calculated \( T_u \) based on \( S_f \) of a material

\( \Delta u \) factor of proportionality

\( x \) x-axis of a cross section

\( y \) a distance from the x-axis on a cross section

\( z \) a distance along the longitudinal axis of a solid circular cylinder

\( \alpha \) the strain hardening coefficient of a material

\( \beta_i \) the ratio of normal strain to the effective strain, \( \bar{\varepsilon}_i \) at the outer fiber

\( x_i \)
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<td>shear strain</td>
</tr>
<tr>
<td>$\gamma_{\theta z F}$</td>
<td>shear strain at failure</td>
</tr>
<tr>
<td>$\varepsilon_{i}$</td>
<td>effective strain at the outer fiber of a cylinder</td>
</tr>
<tr>
<td>$\varepsilon_{r}$, $\varepsilon_{\theta}$, $\varepsilon_{z}$</td>
<td>true strains</td>
</tr>
<tr>
<td>$\varepsilon_{n}$</td>
<td>engineering strain obtained from a standard specimen</td>
</tr>
<tr>
<td>$\varepsilon_{u}$</td>
<td>ultimate strain, obtained from a standard specimen</td>
</tr>
<tr>
<td>$\varepsilon_{z F}$</td>
<td>normal strain at failure</td>
</tr>
<tr>
<td>$\Delta \lambda$</td>
<td>Poisson's ratio</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>effective stress</td>
</tr>
<tr>
<td>$\sigma_{i}$</td>
<td>effective stress at the outer fiber of a cylinder</td>
</tr>
<tr>
<td>$\sigma_{0}$</td>
<td>stress associated with zero strain, a material parameter</td>
</tr>
<tr>
<td>$\sigma_{u}$</td>
<td>maximum stress, a material parameter</td>
</tr>
<tr>
<td>$\tau_{r\theta}$, $\tau_{rz}$, $\tau_{\theta z}$</td>
<td>shear stress components</td>
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<td>$\tau_{\theta zi}$</td>
<td>shear stress</td>
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<tr>
<td>$\tau_{\theta z F}$</td>
<td>the shear stress associated with failure</td>
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<tr>
<td>$\omega$</td>
<td>constant, factor of proportionality</td>
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Chapter 1

INTRODUCTION

1.1 PEAK-LOAD DESIGN CRITERION

The general objective of this study was to develop a peak-load design criterion. The highest expected load is the failure-inducing agent and the manifestation of failure is a catastrophic rupture of the machine part.

The field of application of the peak-load design criterion would be for high-performance machine parts, where high-strength materials are usually employed.

Particular attention was given to combined loads which involved weak stress patterns. This approach was taken because failure generally starts from a situation where weak stress patterns exist.

As a start toward investigation in this general area, analytical and experimental studies were completed on the rupture-failure of a solid circular cylinder subjected to a combination of tension and torque. Additional analytical study was done for a radial cylinder subjected to a combination of tension, torque and bending moment. The bending moment was introduced by small eccentricity of the axial force.
1.1.1 The Need for a Peak-Load Design Criterion

Designers increasingly are challenged to design machines which will be operated at high speed and high temperature. Other design parameters such as light weight, small volume, long life, and low cost [1]* are also receiving more attention. Thus, the opposing demands of higher performance and smaller size are forcing the designer to efficiently utilize the strength of available materials.

The development of better design techniques will require emphasis on a number of factors. A more scientific form-synthesis approach [2] is indicated. Better knowledge of the properties of materials is required. The nature of the actual service conditions such as loads, temperature and environment, will have to be more accurately specified. In addition, a thorough understanding of the potential modes of mechanical failure is vital.

During its service life, a machine part may be subjected to a broad spectrum of loads. This is illustrated in Figure 1, where service loads are grouped into three categories: low load levels which are continually applied during the life of the part, intermediate load levels which occur frequently, and occasional peak loads.

In the design of high performance machine parts, it is common practice to base the analysis on a high-cycle fatigue criterion for the lower level cyclic loads, and on both a

*The numbers in brackets refer to entries in the list of references at the end of the dissertation.
Figure 1. Typical Load Spectrum
yielding criterion and a low-cycle fatigue criterion for the intermediate load levels. However, complete rupture or total collapse, based on an occasional peak load, is often neglected. Yet it is believed that a high percentage of parts used in high-performance machinery are subjected to such peak loads. Moreover, when properly taken into consideration, these loads can dominate and control the choice of dimensions. Very little attention has been given to this important fact.

Consider, for instance, the design of the mast of a helicopter. A tentative dimension of this part can be designed using a fatigue failure criterion by using the expected load spectrum. An analysis can be performed using a yielding failure criterion to come up with a proper dimension. Another synthesis based on rupture due to peak load can also be made. The peak load would be estimated based on one rough landing. The final result would be that the tentative dimension, obtained by using a peak-load design criterion, would be the most critical. Hence this mode of failure dominates the design of the helicopter mast.

Designers, in their continuing effort to obtain minimum weight systems, will have to guard against this occasional peak load. This is particularly true in situations where rupture of the part would lead to a disaster.

1.1.2 Mode of Mechanical Failure for a Peak Load

Mechanical failure may be defined as any change in size, shape, or material properties of a machine or machine
parts which renders it incapable of satisfactorily performing its intended function within a prescribed lifetime [1].

A systematic classification of failure modes has been proposed [2]. Three factors have been found which permit categorization of all possible modes. They are: manifestations of failure, general failure-inducing agents, and location of the actual failure.

The manifestation of failure may include excessive elastic deformation, plastic deformation, rupture or fracture, and material change. There are four failure-inducing agents: force, time, temperature and reactive environment. The location of actual failure in the machine part may be either on its surface or within the body of the part. A schematic representation of these parameters is shown in Figure 2.

A specific failure mode can be defined as a combination of one or more manifestations of failure, together with one or more failure-inducing agents, and a failure location. Obviously, hundreds of combinations can be systematically listed.

As an illustration, yielding failure occurs when the plastic deformation in a machine part, due to the imposed operational loads or motions, becomes great enough to interfere with the ability of the machine to satisfactorily perform its intended function.

As another illustration, fatigue failure is a general term given to the sudden and catastrophic separation of a machine part into two or more pieces due to the application of fluctuating loads or deformations over a period of time.
Figure 2. Modes of Mechanical Failure
In a similar way, the mode of mechanical failure for a peak-load design criterion can be defined. Here, failure is defined to occur when the deformation brought about by a maximum load causes the machine part to collapse or separate into two or more pieces.

The importance of the peak-load design criterion and its associated failure manifestation may not be overlooked. Consider, for example, the hinge of a flap near the fuselage of an aircraft. Suppose a high magnitude of aerodynamic force causes the hinge to yield. Although due to yielding the flap mechanism does not perform its intended function, the aircraft is still controllable, and the damage can be repaired once the aircraft lands. However, if the hinge is ruptured into two pieces, the broken part may hit the tail stabilizer, and the result would be a disaster.

In summary, the consideration of peak load in design of high-performance machinery is vital for two reasons. First, the consequence of the failure is a disaster, and second, the proper analysis of a part subjected to the high magnitude of load might dominate and control the design.

1.1.3 Types of Stress Patterns Induced in Load-Carrying Members

To simultaneously achieve compactness, minimum weight, and efficient utilization of the material, the form (shape) and the size of a load-carrying member should be scientifically synthesized. It is vital to induce as uniform a stress pattern as possible over as large a portion of the body as possible [2]. Such a distribution of stress is called a "strong" stress pattern.
Figure 3 shows a circular bar under a pure axial load. The normal stresses are distributed uniformly over a cross section. This is a strong stress pattern, i.e., a load-carrying member made of a particular material having such an induced stress pattern will, in general, be strong.

In Figure 4 the induced shear stress due to a transverse shear force is distributed nonuniformly over a cross section. However, for a large portion of the cross section the induced stress levels are high and distributed somewhat uniformly. Therefore, this is also a strong system.

In most machine parts, however, the loads and the configurations are such that an induced "weak" stress pattern is unavoidable. These weak stress patterns result from bending moments, torques, and Hertzian contact stresses. Also, various stress-concentration effects produce weak stress patterns.

The normal stress distribution on a circular cross section due to bending moment is shown in Figure 5. In this distribution a high stress level is acting only on a small percentage of the cross-sectional area. This is a weak stress pattern, i.e., a part having such an induced stress pattern will be weak in carrying load, for a given volume of material.

Figure 6 illustrates an induced weak stress pattern for a solid circular cross section due to torque.

Experienced designers have observed that weak stress patterns are usually the source of mechanical failure. Thus a study of the failure of mechanical parts having induced weakstress
Figure 3. Normal Stress Distribution due to Normal Force

Figure 4. Shear Stress Distribution due to Shear Force
Figure 5. Normal Stress Distribution due to Bending Moment

Figure 6. Shear Stress Distributions due to Torque
patterns, and subjected to an occasional peak load, is an investigation of vital concern to machine designers.

In the present investigation, a study was made of the rupture of solid cylindrical bars having circular cross sections, under a combination of axial and torsional load. This is an arbitrary selection of combined stress patterns which includes a weak stress pattern. Its choice was based on the fact that it is commonly encountered and is also one of the simplest cases of combined stresses to test in the laboratory.

1.2 METHOD OF APPROACH

1.2.1 Summary of the Results of a Literature Study.

Various methods of estimating the strength of load-carrying members were surveyed and the results are discussed in Chapter 2. Several conclusions can be presented.

The modulus of rupture in bending and the modulus of rupture in torsion are typical examples of extensions of linear elastic analysis, and are used to predict strength. Elastic analysis has also been used to estimate the bursting speed of rotating discs. The utilization of these "apparent stress" approaches provides simple procedures to estimate strength. They are reasonably accurate and have been used in engineering design. The disadvantage of the use of these apparent stress approaches is that they are limited to a particular material and a particular load configuration.
An elastic-plastic analysis of rupture of a load-carrying member is a more realistic approach. The bending problem can be handled using data obtained from a simple tension test. The torsion problem uses data from a torsion test on a tube. The results are generally satisfactory. However, the analysis associated with this approach involves considerable mathematical complexity.

Other types of analysis of rupture failure similar to the type of analysis for initial yielding failure, have been attempted. For ductile materials, either a maximum shear stress or a maximum normal stress criterion can be used with reasonable success in the tension-tension quadrant. In the tension-compression quadrant it has been suggested that a distortion energy criterion be used. Furthermore, when a ductile metal is loaded beyond its ultimate strength, the material which was originally isotropic becomes anisotropic. However, for brittle materials, fracture can be predicted with good accuracy using either the maximum-normal-stress or the maximum-shear-stress theories, for the tension-tension quadrant. In the tension-compression quadrant, a notch-modified distortion-energy criterion fits the experimental data closely. The magnitude of a suitable modification factor can be obtained as the ratio of ultimate strength in compression to ultimate strength in tension.

The occurrence of plastic instability is yet another method to estimate the peak load a part can withstand before collapse. For this method it is common to represent the stress-strain relation of the material by an exponential function. Generally the part is loaded in such a manner that a "strong" stress pattern is induced. This occurs, for example, in thin-walled tubes, diaphragms and spherical pressure vessels.
Limit design analysis has been widely used to estimate the load that will cause a plastic collapse of a structural member. This analysis applies to load-carrying members made from material which behaves as an "ideal plastic." The limit design concept brought about a shift from the safe-stress failure manifestation in elastic analysis to a safe-deformation failure manifestation.

Linear Elastic Fracture Mechanics (LEFM) is being used to estimate the maximum load a member with a given crack size can withstand. Conversely, for a given magnitude of load, the method can be employed to estimate an acceptable critical crack size. However, it should be pointed out that LEFM is valid only when the overall stress field induced by the load remains in the elastic range, and the plastic zone at the crack tip is relatively small compared to the size of the crack.

1.2.2 Description of the Present Investigation

As previously mentioned, the general objective of this study was to develop a peak-load design criterion. The highest expected load is the failure-inducing agent and the failure manifestation is a catastrophic rupture of the machine part.

This investigation dealt with a combination of axial and torsional loads on a solid circular cylinder. It is only one of many possible rupture investigations. Other investigations could be devised which induce other combinations of weak and strong stress patterns.

An associated analytical model was developed. This model involves multiaxial plastic analysis up to the point
of rupture. It uses basic material parameters obtained from a standard tension test.

The analytical work was based on the deformation theory of plasticity. Tensile and torsional loads were applied simultaneously, and both were increased monotonically.

The materials were assumed to be homogeneous and isotropic. The effect of previous strain history was excluded, hence the phenomena of cyclic hardening or cyclic softening did not enter into consideration. The effect of temperature and reactive environment were excluded by using ordinary ambient conditions.

Experimental work was performed to investigate the validity of the theoretical analysis. The materials used in this study were of the strain hardening types. They were Aluminum alloy 7075-T6, Elevated Temperature Drawing (e.t.d.)-150 steel, and Cold Drawn (CD) 1041 steel.

In the combined loading tests of specifically designed specimens, the rates of loading were made about the same as the rate of loading in the simple tension test.

An analytical study of a solid circular cylinder subjected to a combination of tension, torsion and bending moment has been attempted. In this study, the presence of a small bending moment was assumed to be caused by small eccentricity of the axial force.
1.3 PRINCIPAL RESULTS

Several results were obtained from this investigation. A failure envelope for the peak-load design criterion has been developed and verified experimentally. A method to estimate the failure envelope has been developed. It requires only the results of a standard tensile test on the material.

A failure theory was developed based on Hencky's total deformation theory of plasticity for a cylindrical body under a combination of tension and torsion. Specifically, the ultimate value of combined tension and torsion was calculated. The failure envelope for this case can be represented approximately by the expression,

\[(P/P_u)^2 + (T/T_u)^2 \approx 1\]

Here, \(P\) and \(T\) represent the magnitude of axial load and torque in combined load case, respectively. \(P_u\) represents the magnitude of pure axial load, the \(T_u\) represents the maximum torque.

The analytical work was verified experimentally by tests on three selected materials. The experimental results agreed with the analytical predictions, as can be seen from Figure 7. Experimental work was conducted using specimens made of these three materials. The configuration of the specimen is shown in Figure 8.

The degree of coincidence between analytical work and experimental results is shown in Table 1.
Figure 7. Experimental Results for the Three Materials Tested
Figure 8. Configuration of Specimen Used for Combined Load Tests
Table 1. The Average Value of Failure Function F for Aluminum 7075-T6, e.t.d.-150 Steel and CD-1041 Steel

<table>
<thead>
<tr>
<th>Material</th>
<th>Average F</th>
<th>Standard Deviation of F</th>
</tr>
</thead>
<tbody>
<tr>
<td>AL 7075-T6</td>
<td>.973</td>
<td>0.114</td>
</tr>
<tr>
<td>e.t.d.-150</td>
<td>.998</td>
<td>0.084</td>
</tr>
<tr>
<td>CD-1041</td>
<td>.971</td>
<td>0.044</td>
</tr>
</tbody>
</table>

F is defined as the sum of \((P/P_u)^2\) and \((T/T_u)^2\).

Based on this investigation, a failure theory for peak-load design can be stated. In words, the failure theory can be expressed as follows:

"Failure is predicted to occur for a solid cylindrical body subjected to a combination of tension P and torsion T, when the magnitude of the sum of the square of \(P/P_u\) and the square of \(T/T_u\) is equal to or exceeds unity."

It was discovered experimentally that the magnitude of the maximum pure axial load \(P_u\) and the maximum torque \(T_u\) can be estimated from standard tensile test results. The ultimate axial load \(P_u\) can be calculated by using the usual engineering ultimate stress. However, the ultimate torque \(T_u\) should be estimated based on the approximate "true" fracture stress.
Chapter 2

REVIEW OF THE LITERATURE ON ELASTIC-PLASTIC, RUPTURE STRENGTH AND FRACTURE INVESTIGATIONS

2.1 ELASTIC AND ELASTIC-PLASTIC RUPTURE INVESTIGATIONS

Treatments of the rupture phenomenon using elastic analyses have led to the use of an apparent strength called the modulus of rupture. Typical applications involve either the modulus of rupture in bending or the modulus of rupture in torsion, both primarily for parts having circular cross sections.

F. P. Cozzone [3] investigated the bending strength of prismatic bars loaded into the plastic region. He used the three types of cross sections most frequently encountered in aircraft practice. The original stress-strain curve up to the ultimate strength was replaced by an equivalent trapezoidal stress-strain curve. His investigations included experiments using 14-ST Aluminum forgings, Magnesium alloy forgings and 195-T6 Aluminum castings. The results showed fair correlation between theory and experiment.

It is common to assume that during bending plane sections remain plane, provided the shear stresses are small compared to the longitudinal bending stress. For further work and results on plastic bending, see appropriate textbooks and the technical literature [4-15,27].
C. W. MacGregor, et al. [16] verified that the actual maximum shear stress at fracture is lower than the computed modulus of rupture in torsion. They conducted tests on bars made of cast iron, SAE 1045 steel, SAE 1112 annealed steel, by running tension tests, double shear tests, and torsion tests. Most of their derivations for the plastic torsion case proceeded from the assumption that initially plane sections of the bar remain plane after the application of torque.

A. Nadai [14] extended the well-known membrane analogy for elastic torsion, by introducing a sand heap analogy for a perfectly plastic material. Mendelson [17] outlined a numerical method for handling the analysis of the elastic-plastic torsion of prismatic beams made from strain-hardening materials. For further results on plastic torsion, see references 12, 13, 27, and 88.

E. L. Robinson [18] showed that a comparison of tensile strength with the average "elastic" hoop stress in a rotating wheel gives a fairly reasonable estimate of the factor of safety against actual bursting. Bursting tests were conducted on model discs, with and without hole in the center of the discs. He tested SAE 4340 steel, SAE 1020 steel, and other metals. He found that regardless of whether or not a central hole was present, these wheels burst with almost the same average stress. This fact was verified by W. E. Skidmore [19] using 14-ST Aluminum alloy and SAE 4130 steel heat treated to several ultimate strengths. For further work on this subject, see references 20-24.

It is of great interest to designers that they should be able to determine or estimate the ultimate load a particular type of machine part can withstand. B. Crossland, et al. [25]
derived a formula to calculate the ultimate pressure of a thick-walled tube. Their theoretical results compared favorably with their experimental results.

The tensile strength of a material corresponds to the maximum load attained in a simple tension test. A similar type of maximum load condition occurs in thin-walled pipes and pressure vessels when subjected to a slowly increasing pressure. For design purposes it is of interest to determine what errors are involved if the maximum permissible pressures in these cases are calculated on the basis of the ultimate tensile strength alone. Results show that for thin-walled pipes or cylinders, depending somewhat upon the material, the maximum pressure calculated is underestimated by about 15 percent. However, for spherical shells, the maximum attainable pressure calculated on the basis of ultimate tensile strength is always overestimated as high as 22 percent [27,28].

2.2 RUPTURE UNDER COMBINED STRESSES

Current static failure theories assert that failure occurs when the first yielding of the material has been reached. However, it is believed that failure theories were originally intended to predict the final rupture of a part.

Most of the experimentation on failure under combined stresses has been performed using biaxial states of stress. Thin-walled tubes have been used extensively because of their simplicity. It is easy to obtain the desired two-dimensional combined stresses by applying various combinations of tension, torque and internal pressure. Such stresses can be calculated
from the applied loads by using force-equilibrium relationships. The three principal strains can be measured directly, independent of the loading. Also, both stress and strain are independent of material constants. In these tests the stress pattern induced is a strong stress pattern, i.e., the stress is approximately uniform through the thickness of the tube.

2.2.1 Rupture of Brittle Materials

Here the term brittle will be used exclusively in the sense that a brittle material can sustain without failure stresses of greater magnitude in compression than in tension [33].

W. R. Clough, et al. [29] ran combined stress tests on pearlitic nodular iron. They used thin-walled tubes with outside to inside diameter ratios of 1.09. They observed that the material fractured in a direction perpendicular to the greatest normal stress regardless of whether the other stress was tensile or compressive. This was the case for all specimens which did not buckle locally. The fracture data for their specimens may be interpreted as conforming to the maximum-normal stress theory of failure.

R. C. Grassi and I. Cornet [30] studied the fracture of gray cast iron using thin-walled tubes having diameter ratios of 1.12. They used various ratios of axial to tangential stress, ranging from pure tension to pure compression. Throughout their experiments, the loading on the specimen followed a schedule such that the desired ratio of axial to tangential stress was maintained constant. Their fracture data may be explained in part [tension-tension quadrant] by the maximum-normal stress theory.
L. F. Coffin [31] studied the fracture of cast iron in an attempt to discover a mechanism which would explain the fracture phenomenon in brittle materials. He wanted to propose a satisfactory law for use in engineering design. Tests were conducted wherein thin-walled tubes with a diameter ratio of 1.10 were subjected to two-dimensional states of combined stress. The experiments covered the complete two-dimensional stress field. He concluded that flow and fracture may be considered to be governed by the mechanisms for flow and fracture at the edge of the graphite flakes. He proposed the use of a modification of either the maximum shear-stress theory or the distortion-energy theory, by considering the random dispersion of cracks due to the graphite flakes in cast iron.

Based on the above mechanisms, I. Cornet, et al. [32] tested to fracture inoculated-iron specimens. They assumed that plates of friable graphite in the iron acted like solid iron with respect to compressive stresses, but as cavities producing stress concentration with respect to tensile stress. They proposed that the stress concentration factor due to ellipsoidal cavities was equal to the ratio of the fracture strength in pure compression to the fracture strength in tension. They showed that a distortion-energy criterion for fracture, modified by the stress concentration factor, agreed with their experimental data.

Later, I. Cornet, et al. [33] studied the fracture of brittle material such as high silicon cast iron, and also studied a relatively ductile material such as nodular cast iron. These materials, together with others listed below, were used to evaluate fracture theories.
Table 2. The Notch-Modifying Factor for Some Materials

<table>
<thead>
<tr>
<th>Material</th>
<th>Notch Modifying Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mild steel</td>
<td>1.0</td>
</tr>
<tr>
<td>Nodular iron</td>
<td>1.3</td>
</tr>
<tr>
<td>Gray cast iron</td>
<td>3.2</td>
</tr>
<tr>
<td>High silicon iron</td>
<td>7.0</td>
</tr>
<tr>
<td>Glass</td>
<td>8.0</td>
</tr>
<tr>
<td>Concrete</td>
<td>10.0</td>
</tr>
</tbody>
</table>

They concluded, for engineering design purposes, the notch-modified distortion-energy criterion can be applied to predict fracture under combined stress.

2.2.2 Rupture of Ductile Metals

E. A. Davis [34] tested annealed-copper specimens in the shape of hollow cylinders (diameter ratio 1.15) by subjecting them to axial tension combined with internal pressure. A few observations can be given on the types of fracture obtained for the various conditions of loading used. It appears as far as rupture is concerned, there is anisotropy in the material prior to fracture. The fracture appears to occur when the true principal normal stress reaches a certain value.

Later E. A. Davis [35] presented the results of combined stress tests on medium carbon steel, where particular attention was paid to the magnitude and the distribution of stresses and strains at the instant just preceding fracture. He concluded from his investigation that neither the maximum normal stress
nor the maximum normal strain can be used as a criterion for fracture. Before the ultimate strength was reached, the material behaved as though it were perfectly isotropic; however, beyond this point the material did not behave as an isotropic body. In most cases fracture occurred along planes of maximum shear stress.

J. Marin [36] presented test results for Alcoa 24ST specimens subjected to biaxial plastic stress, where the ratios of the principal stresses were either held constant or permitted to vary. He used thin-walled tubular specimens. The "biaxial nominal ultimate strength" based on the original dimension, and the "true fracture strength," based upon the dimension at fracture, for various biaxial-stress ratios, gave reasonably good agreement with the maximum-shear-stress and maximum-normal-stress theories.

2.3 PLASTIC INSTABILITY

Conditions similar to the plastic instability phenomenon in the simple tension test also occur for loading by complex stress states. In order to give general results, it is common practice to employ exponential equations for the strain-hardening characteristic. This was suggested by Swift [37]. P. B. Mellor [38] compared the analytical results with his experiments on balanced biaxial tension of copper, brass, aluminum and steels.

B. H. Jones, et al. [39] carried out their experiments with great care. They deformed nickel-chrome steel cylinders under a constant true stress condition. Instability occurred when either a maximum occurred in the total axial load or in the internal pressure [40]. A general treatment of cylinders under
axial load, internal pressure and twisting moment has been com-
pleted by several investigators [37,41-45].

Hill [46] advanced a theory for the instability of a
circular metal diaphragm made from material having linear
strain-hardening characteristics. His theoretical results
were in good agreement with experimental data on half-hard
aluminum [36]. Other work on this subject had been carried out
by several investigators [47-55].

Several investigators used a plastic instability approach
for predicting the bursting of rotating discs (see references
56-60).

2.4 THE LIMIT DESIGN APPROACH

The theory of limit analysis is used primarily in the design
of steel structures which are composed of various elements such
as beams, frames, girders, arches, etc. In this approach,
what is sought is the load-carrying capacity or limiting load
at which the structure will collapse.

The theory of elasticity leads to an allowable stress
concept as the basis of structural design [17,62]. The allowable
stress frequently has been the yield strength of the material,
and the design stress was some fraction of the yield stress,
depending upon the factor of safety used. However, for a
majority of problems, this type of approach is not realistic [17].
Since the important consideration in an engineering structure
is not whether the yield stress is exceeded at some point,
but whether the structure will carry the intended load or perform
its intended function. As a matter of fact, it is fairly
evident that in almost all structures, local plastic flow will occur at stress raisers and points of discontinuity in the geometry without producing functional failure. Also residual stress as high as half the yield strength may be retained in some rolled shapes as they come from the steel mills.

A basic assumption of the limit design approach is the approximate representation of the stress-strain curve of the material by an ideal-plastic stress-strain relationship. Plastic analysis based on this assumption is not restricted to the design of components for structures. The approach has been used for problems in metal forming. Also design problems involving cylindrical bodies, tubes and spheres under internal pressure, and rotating discs under the action of centrifugal forces have been solved using this technique. In fact, limit analysis constitutes a special case of a more general analysis wherein the strain-hardening characteristics of the material is accounted for. Further work in this area can be found in the following references: 12-15, 17, 27, 61 and 62-73.

2.5 THE FRACTURE MECHANICS APPROACH

In reality, the majority of structural components contain cracks. These are introduced during manufacture, or they may be initiated early in the service life of the structure. These cracks are frequently the primary source of mechanical failure.

The emergence of Fracture Mechanics has led to the development of a new design concept. It assumes the pre-existence of a significant crack-like defect. However, small cracks may be acceptable, provided the part can perform its intended function within a specified service life.
The fracture mechanics approach enables a quantitative relationship to be obtained relating the applied stress required to cause failure and the size of any defect or precrack that may be present in the component.

A. A. Griffith [74] studied the fracture phenomenon in cracked bodies using an energy-balance method. His premise was that the unstable propagation of a crack takes place if an increment of crack growth results in more stored-energy being released than is absorbed by the creation of the new crack surfaces.

An alternative interpretation of the fracture phenomenon, which focuses attention on the mechanical environment near the tip of a crack, was developed by G. R. Irwin [75,76]. This method is generally known as the Stress Intensity Factor approach. Thus the properties of a material in the presence of a crack can be measured in terms of certain stress intensity factors. This is similar to the conventional measurements of material properties in terms of stress.

In Fracture Mechanics, the fracture surface is assumed to be smooth and the material surrounding the crack is assumed to be a linearly elastic continuum. In reality, there exists a small plastic region at the tip of the crack. When the size of this plastic region is relatively small as compared to the size of the crack, the previous assumption applies. The Linear Elastic Fracture Mechanics approach is based upon the above assumptions.

Attempts have been made [77] to describe the fracture condition for a material which has low strength and high
toughness. For these materials the plastic zone at the crack tip may be so large as compared to the crack size that the LEFM assumptions do not apply [78]. Two approaches have been proposed, the crack-opening displacement criterion (COD) [79,80], and the J-contour integral [81].

Wells [79,80] assumed that crack extension takes place when the material at the crack tip has reached a certain maximum permissible plastic strain. This criterion has been proven to be equivalent to the stress-intensity factor criterion for those areas where the LEFM assumption applies.

As a method to circumvent the problem of fracture where the plastic region at the tip of the crack is not relatively small, Rice [81] introduced the J-contour integral. The J-integral method is a generalization of the energy release rate approach. While the energy release rate in the Griffith approach applies for LEFM, the J-integral is valid even if there is appreciable crack-tip plasticity. The path independence of the J-integral has been proven by using the deformation theory of plasticity [81,82].

2.6 BRIEF LITERATURE SURVEY FOR TENSION COMBINED WITH TORSION

In 1931 G. I. Taylor and H. Quinney [83] conducted investigations on the yielding of certain materials. They tested specimens made from copper and steel tubing. They carefully applied tension combined with torsion. Great care was taken to limit the degree of anisotropy. First, the tubular specimens were prestretched. Next, they were partially unloaded (reduced tension), and finally they were subjected to an increasing
torque while holding the reduced axial load constant. Thus the tubes were loaded such that the stress was distributed uniformly throughout the thickness of the metal. However, the history of loading was complex, the stress ratios were not held constant. Their data fit the von Mises (distortion energy) yielding criterion better than the Tresca (maximum shear stress) criterion.

J. Marin, et al. [84] conducted tests on thin-walled tubes made of aluminum alloy, 14ST6, under tension combined with torsion. The purpose of this investigation was to obtain a plastic stress-strain or flow relation for the material, when subjected to various combinations of biaxial stresses. Also, they attempted to determine the validity of a flow theory to predict plastic stress-strain relations for combined stresses. For their constant-stress-ratio tests, the plastic stress-strain agreed approximately with the incremental strain theory. It was also verified that the assumption of isotropic yielding was valid.

E. A. Davis [85] presents the results of an investigation on a series of tests involving a combination of tension, torsion, and internal pressure. In each of the tests a different amount of torsion was used. Specimens made of 1020 steel tubing were tested in such a manner that the directions of the incremental plastic strains were kept constant. The directions of the principal stresses were measured at each increment of strain and found to agree well with the resulting directions of the incremental strains.

In addition, when data points were plotted in an octahedral-shear-stress versus octahedral-shear-strain diagram, the data
points turn out to be clustered together along some characteristic curve.

In 1967 M. R. Shammamy and O. M. Sidebottom [86] presented results of an investigation on hollow and solid torsion-tension specimens. Three different materials were tested. They were: (1) a non-strain-hardening steel, annealed SAE 1035, with identical properties in tension and compression, (2) a strain-hardening steel, annealed rail steel, with identical properties in tension and compression, and (3) a strain-hardening aluminum alloy, 2024-T4, with different properties in tension and compression. In all cases average tension and compression stress-strain diagrams were approximated by two straight lines to represent the material properties.

Their incremental strain solutions were compared with solutions based on a total strain theory for the case of proportionate loading. The hollow tension-torsion specimens were subjected also to nonproportionate loading. Test data for proportionate loading were in excellent agreement with either the total strain theory or the incremental strain theory. Test data for the nonproportionate loading, in which one deformation was kept constant as the other was increased, fell between the two theories, but was closer to the predictions of the incremental strain theory.

In 1971 O. M. Sidebottom [87] investigated both an incremental strain theory and a total strain theory for solid circular torsion-tension specimens subjected to any prescribed loading path and the loads were such that it produced finite strains. Experimental data were obtained from test specimens
made of two metals, hot-rolled SAE 1045 steel and aluminum alloy 7075-T6. In all cases the resulting stress-strain diagrams were approximated by four straight lines.

They subjected one torsion-tension specimen of each metal to proportionate loading. Here excellent agreement was obtained between theory and experiment, and the difference between the two theories was found to be negligible. For further reference on torsion-tension, see references 88 and 89.

The results of this literature survey showed that the problem of tension combined with torsion for plastic deformation up to rupture had not been completed.
Chapter 3

BRIEF REVIEW OF THE THEORY OF
ELASTICITY AND THE THEORY OF
PLASTICITY

3.1 INTRODUCTION

The theory of elasticity and the theory of plasticity are used to describe the mechanics of deformation for most engineering materials. The common philosophy of both is to predict the behavior of a material in complex states of stress on the basis of certain experimental observations. These observations usually are simple and are performed under conditions easy to duplicate. The simplest experiment, as well as the most important, is the standard tensile test.

To obtain stress and strains distributions in load-carrying members, certain principles and hypotheses are used. These are:

a. equilibrium of forces for each small volume element in a continuum,
b. geometric compatibility conditions,
c. experimentally obtained stress-strain relations,
d. theories of elasticity and plasticity.

The force-equilibrium principle is generally expressed in terms of the stress components. The second condition is essentially a geometrical relationship among deformations and
strains to satisfy geometrical continuity. The third relationship, obtained by experiment for a particular material, together with the first two principles, is generally sufficient to describe stress and strain distributions in a body loaded in the linearly elastic range.

For multiaxial states-of-stress [9,88,90-97] several relationships between stress and strain have been proposed to generalize the experimentally obtained stress-strain relation. The initial yielding condition, for example, describes the circumstances in a multiaxial stress state for which an infinitesimal volume element begins to behave in an inelastic manner. A generalized stress-strain relationship covers both the elastic as well as the inelastic range [98-104].

In linear elastic problems, a generalized Hooke's law provides a complete relationship between stress components and strain components. For problems involving a post-yield condition, some additional hypotheses are required to describe the relations between the stress and the strain components. Several theories have been put forward. They are Hencky's total-strain theory, the Levy-von Mises incremental theory, and the Prandtl-Reuss incremental theory.

The existing theories on plasticity can be classified into three categories. They are:

a. The deformation, or total strain, theory of plasticity for strain-hardening materials. It is based on a hypothesis that the state of stress determines uniquely the state of strain, as long as plastic deformation continues,
b. The flow, or incremental, theories of plasticity. These hypothesize that the finite increment of strain is proportional to the current stress and the stress increment,

c. Batdorf and Budiansky have developed a theory in the plastic range based upon a physical mechanism of plastic deformation.

In the following review, the third category, c, above will be omitted.

3.2 A GENERALIZED HOOKE'S LAW

To show the development of the plastic stress-strain theories, it is first necessary to discuss a generalized Hooke's law in a different form. Hooke's law embodies three assumptions [35]:

a. Principal axes of stress and strain coincide.

b. Mohr's-circle diagrams for both the state of stress and the state of strain are similar.

c. Volume changes are proportional to the mean normal stress.

The first two assumptions are satisfied by the relationships below

$$\frac{e_x}{S_x} = \frac{e_y}{S_y} = \frac{e_z}{S_z} = \frac{\gamma_{xy}}{2\tau_{xy}} = \frac{\gamma_{yz}}{2\tau_{yz}} = \frac{\gamma_{zx}}{2\tau_{zx}} = \frac{1}{2G},$$  \hspace{1cm} (3.1)

where G is the modulus of rigidity, the e's represent the
deviatoric strains and the S quantities are the deviatoric stresses. The deviatoric stresses and strains are defined as follows

\[ e_x = \varepsilon_x - \frac{1}{3} (\varepsilon_x + \varepsilon_y + \varepsilon_z), \]
\[ S_x = \sigma_x - \frac{1}{3} (\sigma_x + \sigma_y + \sigma_z). \]  

(3.2)

The first three relationships in equation (3.1) are not independent since their sum forms an identity.

Another independent equation can be obtained from the compressibility condition based on the third assumption. This can be written as

\[ \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\sigma_x + \sigma_y + \sigma_z}{3B_v}, \]

(3.3)

where \( B_v = E/3(1 - 2v) \). Thus, only two independent material properties are needed, the modulus of elasticity \( E \) and Poisson's ratio \( v \).

Six independent equations result from the above formulation. They can be written in their usual cartesian form; that is:

\[ \varepsilon_x = \frac{1}{E} [\sigma_x - v(\sigma_y + \sigma_z)] \]
\[ \varepsilon_y = \frac{1}{E} [\sigma_y - v(\sigma_x + \sigma_z)] \]
3.3 HENCKY'S TOTAL STRAIN THEORY.

For a machine part which is loaded into the plastic region, Hencky's total strain theory provides a logical extension to Hooke's equations. Hencky's formulation embodies three assumptions. These are:

a. Principal axes of stress and strain coincide,

b. Mohr's circle diagrams of the state of strain and the state of stress are similar at any stage in the plastic deformation,

c. Volume changes are elastic.

The first two assumptions are satisfied by the following relationships,

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] , \\
\gamma_{xy} &= \frac{1}{G} \tau_{xy} , \\
\gamma_{yz} &= \frac{1}{G} \tau_{yz} , \\
\gamma_{zx} &= \frac{1}{G} \tau_{zx} .
\end{align*}
\]

\[3.4\]

\[
\begin{align*}
\frac{e_x}{S_x} &= \frac{e_y}{S_y} = \frac{e_z}{S_z} = \frac{\gamma_{xy}}{2\tau_{xy}} = \frac{\gamma_{yz}}{2\tau_{yz}} = \frac{\gamma_{zx}}{2\tau_{zx}} = \frac{\omega}{2G} .
\end{align*}
\]

\[3.5\]
Here again, the e's and the S's are the deviatoric strains and deviatoric stresses, respectively; and ω is a scalar factor of proportionality which depends upon the stage of inelastic deformation. The magnitude of ω at any stage can be obtained from the stress-strain relationship for the particular material. Observe that when ω = 1, the above expression is the same as in the elastic case.

The third assumption provides another independent relation,

\[ \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\sigma_x + \sigma_y + \sigma_z}{3B} \].

Next, it is convenient to resolve the total strain into elastic strain components and plastic strain components. It should also be noted that the plastic strain component of the total strain is incompressible. Thus, defining the relation between the plastic strain (\( e^p \)) components and the stresses gives,

\[ \frac{e^p_x}{S_x} = \frac{e^p_y}{S_y} = \frac{e^p_z}{S_z} = \frac{\gamma_{xy}}{2\tau_{xy}} = \frac{\gamma_{yz}}{2\tau_{yz}} = \frac{\gamma_{zx}}{2\tau_{zx}} = \frac{\phi}{2G} \].

Thus, ω and φ are related by,

\[ \omega = 1 + \phi \].

The six independent equations which result from the above formulations are,
\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] + \frac{\phi}{3G} [\sigma_x - \frac{1}{2} (\sigma_y + \sigma_z)] , \\
\varepsilon_y &= \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)] + \frac{\phi}{3G} [\sigma_y - \frac{1}{2} (\sigma_x + \sigma_z)] , \\
\varepsilon_z &= \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] + \frac{\phi}{3G} [\sigma_z - \frac{1}{2}(\sigma_x + \sigma_y)] , \\
(3.9)
\end{align*}
\]

\[
\gamma_{xy} = \frac{\omega}{G} \tau_{xy}
\]

\[
\gamma_{yz} = \frac{\omega}{G} \tau_{yz}
\]

\[
\gamma_{zx} = \frac{\omega}{G} \tau_{zx}
\]

A simplified expression can be obtained if the elastic strain part is neglected. For instance, this can be done for a case in which large strains are encountered.

3.4 THE LEVY-VON MISES INCREMENTAL THEORY

St. Venant in 1870 proposed that the principal axes of the strain increments might coincide with the axes of principal stress. Here strain increment is a finite increment, as opposed to a differential strain which is an infinitesimal increment. A general relation between strain increment and stress was introduced by Levy in 1871 and independently by von Mises in 1913.
The Levy-von Mises formulation embodies three assumptions,

a. Principal axes of stress and strain increment coincide,

b. Mohr's circle diagrams for the state of stress and the strain increment are similar at any stage of the plastic deformation,

c. The material is incompressible. Consequently, the elastic strain components can be neglected.

The first two assumptions are satisfied by the following relationship,

\[
\frac{\Delta e_x}{S_x} = \frac{\Delta e_y}{S_y} = \frac{\Delta e_z}{S_z} = \frac{\Delta \gamma_{xy}}{2\tau_{xy}} = \frac{\Delta \gamma_{yz}}{2\tau_{yz}} = \frac{\Delta \gamma_{zx}}{2\tau_{zx}} = \frac{\Delta u}{2G},
\]

where \( \Delta 's \) are finite increments. The quantity \( \Delta u \) is a scalar factor of proportionality which is dependent upon the stage of inelastic deformation.

The third assumption provides another independent equation,

\[
\Delta e_x + \Delta e_y + \Delta e_z = 0.
\]

The six equations representing the Levy-von Mises theory of plasticity can be rearranged to give

\[
\Delta e_x = \frac{\Delta u}{3G} [\sigma_x - \frac{1}{2} (\sigma_y + \sigma_z)]
\]
3.5 THE PRANDTL-REUSS INCREMENTAL THEORY

The stress and strain relations for an elastic-perfectly plastic solid body were first proposed by Prandtl (1924). But, the general form of the equations was later given by Reuss (1930). Recall that the Levy-von Mises equations neglected the elastic strain component. However, the Prandtl-Reuss formulation will include the elastic strain components.

The Prandtl-Reuss formulation embodies three assumptions. These are:

a. Principal axes of stress and plastic strain increment coincide,

b. Mohr's circle diagram for the state of stress and plastic strain increment are similar at any stage of

\[
\Delta \varepsilon_y = \frac{\Delta u}{3G} \left[ \sigma_y - \frac{1}{2} (\sigma_x + \sigma_z) \right],
\]

\[
\Delta \varepsilon_z = \frac{\Delta u}{3G} \left[ \sigma_z - \frac{1}{2} (\sigma_x + \sigma_y) \right],
\]

\[
\Delta \gamma_{xy} = \frac{\Delta u}{G} \tau_{xy},
\]

\[
\Delta \gamma_{yz} = \frac{\Delta u}{G} \tau_{yz},
\]

\[
\Delta \gamma_{zx} = \frac{\Delta u}{G} \tau_{zx}.
\]
plastic deformation,
c. Volume changes are elastic.

The first two assumptions are satisfied by the relationships below,

\[
\frac{\Delta \varepsilon^p_x}{S_x} = \frac{\Delta \varepsilon^p_y}{S_y} = \frac{\Delta \varepsilon^p_z}{S_z} = \frac{\Delta \gamma^p_{xy}}{2\tau_{xy}} = \frac{\Delta \gamma^p_{yz}}{2\tau_{yz}} = \frac{\Delta \gamma^p_{zx}}{2\tau_{zx}} = \Delta \lambda
\]  \hspace{1cm} (3.13)

Here the \(\Delta \varepsilon^p\)'s are the plastic components of the strain increments. Also, \(\Delta \lambda\) is a positive, scalar factor of proportionality which depends on the stage of plastic deformation.

The third assumption provides another independent relation,

\[
\Delta \varepsilon^e_x + \Delta \varepsilon^e_y + \Delta \varepsilon^e_z = \frac{\Delta \sigma_x + \Delta \sigma_y + \Delta \sigma_z}{3B_v}, \hspace{1cm} (3.14)
\]

where superscript "e" refers to elastic.

An alternate form of the Prandtl-Reuss equations is shown below, written in commonly used terms.

\[
\Delta \varepsilon_x = \frac{1}{E} \left[ \Delta \sigma_x - \nu (\Delta \sigma_y + \Delta \sigma_z) \right] + \frac{2}{3} \Delta \lambda \left[ \sigma_x - \frac{1}{2} \left( \sigma_y + \sigma_z \right) \right],
\]

\[
\Delta \varepsilon_y = \frac{1}{E} \left[ \Delta \sigma_y - \nu (\Delta \sigma_z + \Delta \sigma_x) \right] + \frac{2}{3} \Delta \lambda \left[ \sigma_y - \frac{1}{2} \left( \sigma_z + \sigma_x \right) \right],
\]
\[ \Delta \varepsilon_z = \frac{1}{E} [\Delta \sigma_z - \nu(\Delta \sigma_x + \Delta \sigma_y)] + \frac{2}{3} \Delta \lambda [\sigma_z - \frac{1}{2} (\sigma_x + \sigma_y)] , \]

(3.15)

\[
\Delta \varepsilon_{xy} = \frac{\Delta \tau_{xy}}{G} + \Delta \lambda \tau_{xy} ,
\]

\[
\Delta \varepsilon_{yz} = \frac{\Delta \tau_{yz}}{G} + \Delta \lambda \tau_{yz} ,
\]

\[
\Delta \varepsilon_{zx} = \frac{\Delta \tau_{zx}}{G} + \Delta \lambda \tau_{zx} .
\]

3.6 DISCUSSION OF THE VARIOUS PLASTICITY THEORIES

A unified approach has been presented to describe the various theories involving elastic and plastic deformation. All theories are based upon three assumptions; these are:

a. the coincidence between the axes of principal stress and principal strain,
b. the proportionality between stress and strain, and
c. the constancy of volume condition.

The first assumption is considered valid for a body loaded in the linearly elastic range. In the plastic region, experimental evidence shows that only in some cases is this assumption acceptable. In general, it is always possible to devise a loading path such that the assumption is violated. Pure torsion is a typical example. Observe the similarity
and the difference in this assumption for the elastic case, Hencky's theory, and the incremental theories.

The second assumption deals with the proportionality between stress and strain. In the plastic region this relation may not be proportional. However, if in the analysis the incremental change of strain is taken to be small, the approximation is generally sufficient for engineering purposes.

In the linearly elastic range, the modulus of elasticity and Poisson's ratio can be regarded as a measure of compressibility. The material is incompressible when the magnitude of Poisson's ratio is equal to one-half because the bulk modulus of elasticity $B_v$ tends to approach infinity.

In the plastic region the phenomenon is quite different. From early experience with metal-working processes, it was found that the volume remained essentially constant during plastic deformation. Thus, for large deformation cases, the constancy of volume condition can be expressed as,

$$
\varepsilon_x + \varepsilon_y + \varepsilon_z = 0. \quad (3.16)
$$

Actually, the volume of an infinitesimal element is not exactly constant during large deformation, but the changes in volume are small compared to the strains. Thus, one component of strain can be calculated from the other using the above relationship. And the accuracy is generally sufficient for practical engineering cases.
When the plastic part of the total strain is small compared to the total strain, the above constant-volume equation is incorrect. For this case, it is generally assumed that the sum of the plastic parts of the strain components is zero,

\[ \varepsilon^p_x + \varepsilon^p_y + \varepsilon^p_z = 0. \]  \hspace{1cm} (3.17)

This relation is used, for instance, in Hencky's theory when Hencky's theory is applied for large plastic deformation cases.

Another assumption frequently used is to set the sum of the finite increments of plastic strain components equal to zero; that is,

\[ \Delta \varepsilon^p_x + \Delta \varepsilon^p_y + \Delta \varepsilon^p_z = 0. \]  \hspace{1cm} (3.18)

This assumption is used in the incremental theories of Levy-von Mises and Prandtl-Reuss.

Hencky's total strain theory has the advantage of mathematical convenience. However, it does not conform very well to physical reality for some cases. Given the state of stress, the deformations predicted by the total strain theory are independent of the loading path. In the case of proportional loading, the deformation theory gives results which are comparable with incremental theory.
The Levy-von Mises incremental theory neglects the elastic part of the total strain components, and thus applies primarily to the case of large plastic deformation. This theory takes into account the effect of the loading path, and consequently describes more completely the physical behavior of the system.

The Prandtl-Reuss Incremental theory applies for cases of either small or large plastic deformations. This theory is believed to be the most accurate method of calculating the stress and strain distributions in a body when the loading induces plastic deformations. In general, numerical procedures employing digital computer are required.

Hencky's theory can be regarded as a special case of the Prandtl-Reuss theory, when the system is subjected to proportional loading. Also, the Levy-von Mises theory can be regarded as a special case of the Prandtl-Reuss theory when the elastic strain components can be neglected.
4.1 INTRODUCTION

In a standard tensile test the load is applied, in a continuous and increasing manner without any unloading, until rupture occurs. Ultimate strength is defined as the maximum load the specimen withstood divided by the original cross-sectional area. Rupture, or separation of the specimen into two pieces, occurs either at the maximum load, as, for example, in the case of brittle materials, or after attaining the maximum load, as, for example, occurs in testing ductile materials.

For the peak-load design criterion, the main interest will be the magnitude of the maximum load, or the maximum combined loads a machine part can withstand. Rupture will occur at the maximum load, or at a lower load level after the maximum load has occurred.

The author developed two analyses to the problem of estimating the strength of a solid cylinder. In the first analysis, the member was subjected to a combination of tension and torsion. In the second analysis, the member was subjected to a combination of tension, torsion and small bending moment. The bending moment was induced by a small eccentricity of the axial force.
Several methods in plasticity are available for solving these problems. They are Hencky's total strain theory, the Levy-von Mises incremental strain theory, and the Prandtl-Reuss incremental strain theory. The analysis can be carried out analytically, through a numerical approach using either the finite element method [110], the finite difference method [17,86] or the method of lines [111].

Hencky's total strain theory was used for the following reasons. First, the objective of the analysis was to estimate the rupture failure of a solid circular cylinder subjected to a monotonically increasing combined load. No unloading was permitted. Second, the tension and torsion loads were increased so that they induced proportional stressing for each small volume element. If this method of applying combined loads is used, then the solution using Hencky's total strain theory will be identical to the solution using incremental strain theories.

4.2 THE CASE OF A SOLID CIRCULAR CYLINDER SUBJECTED TO A COMBINATION OF TENSION AND TORSION

4.2.1 Introduction

In the following analysis, it was assumed that the material is isotropic and homogeneous. Furthermore, it was assumed that both tension and torsion were to be increased monotonically, with no unloading permitted.
In the following analysis, it is convenient to use a cylindrical coordinate system \((r, \theta, z)\). The z-axis was chosen to coincide with the longitudinal axis of the member.

It is not always mathematically possible to obtain an exact solution to a particular problem, and it is usual practice to obtain a solution by making some kind of an assumption about the compatibility condition [27]. Since the configuration of the specimen and the loading are axisymmetric, it was assumed that no warping would occur. That is, a plane section before deformation would remain a plane after loading, and a radial line would remain essentially a straight line.

At any stage of deformation, the axial strain \(\varepsilon_z\) would remain constant over a cross section. The shearing strain \(\gamma_{\theta z}\) would be distributed as a linear function of the radius. The other nonvanishing strain components would be \(\varepsilon_\theta\) and \(\varepsilon_r\).

Mathematically, the normal strain and the shear strain distribution can be written as

\[
\varepsilon_z = \text{constant} \tag{4.1}
\]

\[
\gamma_{\theta z} = \text{linear function of } r
\]

As presented in Chapter 3, the deformation theory can be expressed in terms of the cylindrical coordinate system as

\[
\frac{\varepsilon_r - \varepsilon_\theta}{\sigma_r - \sigma_\theta} = \frac{\varepsilon_\theta - \varepsilon_z}{\sigma_\theta - \sigma_z} = \frac{\varepsilon_z - \varepsilon_r}{\sigma_z - \sigma_r} = \frac{\gamma_{r\theta}}{2\tau_{r\theta}} = \frac{\gamma_{\theta z}}{2\tau_{\theta z}} = \frac{\gamma_{zr}}{2\tau_{zr}} = \frac{\omega}{2G} \tag{4.2}
\]
For a torsion-tension loaded member, where $\sigma_r = 0$ and $\sigma_\theta = 0$, this relation can be simplified to give the following expression,

$$\frac{\varepsilon_\theta - \varepsilon_z}{-\sigma_z} = \frac{\varepsilon_z - \varepsilon_r}{\sigma_z} = \frac{\gamma_{\theta z}}{2\tau_{\theta z}} .$$  \hspace{1cm} (4.3)

The incompressibility condition provides a relationship among the normal strains, that is,

$$\varepsilon_r + \varepsilon_\theta + \varepsilon_z = 0 .$$  \hspace{1cm} (4.4)

Equation 4.2 can be written in a form similar to equation 3.9, where the elastic strain portion is neglected. In cylindrical coordinate systems, the equations become

$$\varepsilon_r = \frac{\omega}{3\gamma} \left[ \sigma_r - \frac{1}{2}(\sigma_\theta + \sigma_z) \right] ,$$

$$\varepsilon_\theta = \frac{\omega}{3\gamma} \left[ \sigma_\theta - \frac{1}{2} (\sigma_r + \sigma_z) \right] ,$$

$$\varepsilon_z = \frac{\omega}{3\gamma} \left[ \sigma_z - \frac{1}{2} (\sigma_\theta + \sigma_r) \right] .$$  \hspace{1cm} (4.5)
Substituting $\sigma_r = \sigma_\theta = 0$ to the above equations will show that the strains in the radial and the circumferential directions are equal. Their magnitudes in terms of the axial strain $\varepsilon_z$, are

$$\varepsilon_r = \varepsilon_\theta = -\frac{\varepsilon_z}{2} \tag{4.6}$$

A relationship between the stress components and the strain components for a small volume element can be obtained by combining equation 4.3 and equation 4.6. Thus the result of Hencky's total strain theory is

$$\sigma_z = \varepsilon_z \frac{\tau_{\theta z}}{\gamma_{\theta z}} \tag{4.7}$$

It is the goal of this analysis to estimate the maximum load a member can withstand. Naturally, the magnitude of the plastic strain portion of the total strain will be large in comparison to the elastic strain portion, and hence the elastic strain component can be safely neglected.

A rigid-linear strain hardening characteristic will be used to represent the stress-strain relationship of the material, that is,

$$\bar{\sigma} = \sigma_0 + \alpha \varepsilon \tag{4.8}$$

Here, $\sigma_0$ represents the stress at zero true strain and $\alpha$ is a strain-hardening coefficient.
Since the analysis is being made for a multiaxial state-of-stress, an effective stress \( \bar{\sigma} \) will be used. This is defined as

\[
\bar{\sigma} = \frac{1}{\sqrt{2}} \left[ (\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 + 
6(\tau_{r\theta}^2 + \tau_{\theta z}^2 + \tau_{z r}^2) \right]^{\frac{1}{2}} .
\]  

(4.9)

Knowing that the nonzero stress components are \( \sigma_z \) and \( \tau_{\theta z} \), the effective stress expression becomes

\[
\bar{\sigma} = \left[ \frac{\sigma_z}{2} + 3\tau_{\theta z}^2 \right]^{\frac{1}{2}} .
\]  

(4.10)

The effective strain, \( \bar{\varepsilon} \), can be defined as

\[
\bar{\varepsilon} = \frac{\sqrt{2}}{3} \left[ (\varepsilon_r - \varepsilon_\theta)^2 + (\varepsilon_\theta - \varepsilon_z)^2 + (\varepsilon_r - \varepsilon_r)^2 + 
\frac{3}{2} (\gamma_{r\theta}^2 + \gamma_{\theta z}^2 + \gamma_{z r}^2) \right]^{\frac{1}{2}} .
\]  

(4.11)

The nonzero strain components are \( \varepsilon_r, \varepsilon_\theta, \varepsilon_z \), and \( \gamma_{\theta z} \). Using the relationship among the normal strain components described in equation 4.6, the expression for the effective strain becomes
In summary, for this analytical development, several results have been established. A compatibility condition has been assumed and expressed in equation 4.1. The total strain theory yields equation 4.7. The general relationships between stress and strain of a particular material has been defined in equation 4.8, equation 4.10 and equation 4.12.

These equations will be combined to obtain the normal stress and the shear stress distributions over the cross section. Finally, the magnitude of the axial force \( P \) and the torque \( T \) will be obtained by the process of integration, that is,

\[
P = \int_{0}^{a} \sigma z 2\pi rdr ,
\]

and

\[
T = \int_{0}^{a} \tau_{\theta z} 2\pi r^2 dr .
\]

4.2.2 Strain Distribution at Failure

During the application of an increasing axial load and an increasing torque, a small volume element at the outer surface of the member will experience straining in both an axial and
the tangential direction. Thus the magnitude of $\varepsilon_z$ and $\gamma_{\theta z}$ of the element is changed. The magnitude of the effection strain $\varepsilon |_{r=a}$ at every stage of deformation can be expressed using equation 4.12 as

$$\varepsilon |_{r=a} = \left[ \varepsilon_z^2 + \frac{1}{3} \gamma_{\theta z} \right]^{1/2} \quad (4.15)$$

Rupture failure is assumed to occur in a cylindrical body subjected to a combination of tension and torsion, when the effective strain at the outer fiber, $\varepsilon |_{r=a}$, becomes equal to or exceeds the effective strain at the time of failure, $\varepsilon_u$, in a simple test for a specimen of the same material.

Therefore, failure is assumed to occur when

$$\varepsilon |_{r=a} \geq \varepsilon_u \quad (4.16)$$

where

$$\varepsilon |_{r=a} = \left[ \varepsilon_z^2 + \frac{1}{3} \gamma_{\theta z} \right]^{1/2} \quad (4.17)$$

Here, $\varepsilon |_{r=a}$ = the effective strain at the outer fiber,
\[ \varepsilon_u = \text{the effective strain at failure of the specimen in a simple test,} \]
\[ \varepsilon_{ZF} = \text{the axial strain at failure for a combined load (constant with respect to } r), \]
and
\[ \gamma_{\theta ZF} \bigg|_{r=a} = \text{shear strain at the outer fiber at the time of failure for combined loadings.} \]

The axial strain at failure, \( \varepsilon_{ZF} \), is distributed uniformly over a cross section, as was assumed in equation 4.1.

For different combinations of axial force \( P \) and torque \( T \) at failure, the magnitude of the axial strain at failure, \( \varepsilon_{ZF} \) will be different. When the member is subjected to a pure torque, \( \varepsilon_{ZF} \) would be zero. When the member is subjected to pure tension, at failure \( \varepsilon_{ZF} = \varepsilon_u \).

The magnitude of shear strain at the outer fiber at failure can be expressed in terms of \( \varepsilon_u \) and \( \varepsilon_{ZF} \),

\[
\gamma_{\theta ZF} \bigg|_{r=a} = \varepsilon_u \left[ 3 \left( 1 - \left( \frac{\varepsilon_{ZF}}{\varepsilon_u} \right)^2 \right) \right]^\frac{3}{2} . \quad (4.18)
\]

Therefore, the shear strain distribution at the time of failure is

\[
\gamma_{\theta ZF} = \frac{r}{a} \varepsilon_u \left[ 3 \left( 1 - \left( \frac{\varepsilon_{ZF}}{\varepsilon_u} \right)^2 \right) \right]^\frac{3}{2} \quad (4.19)
\]
as prescribed by equation 4.1.
4.2.3 Stress Distributions at Failure

Knowing the distribution of axial strain and shear strain at any radius at the time of failure, the distributions for the normal stress and the shear stress over a cross section can be obtained as follows.

First, consider the normal stress components, $\sigma_z$. If the shear stress $\tau_{\theta z}$ is written in terms of $\sigma_z$ using equation 4.7 and it is substituted into equation 4.10, then the following is obtained,

$$\sigma = \sigma_z \left[ 1 + \frac{1}{3} \left( \gamma_{\theta z F}/\varepsilon_{z F} \right)^2 \right]^{\frac{1}{2}}. \quad (4.20)$$

Next if the shear strain $\gamma_{\theta z F}$ in equation 4.19 is substituted into equation 4.20, the result is equation 4.21 as follows,

$$\sigma = \sigma_z \left[ 1 + \left( \frac{r}{a} \right)^2 (\varepsilon_u/\varepsilon_{z F})^2 \left( 1 - (\varepsilon_{z F}/\varepsilon_u)^2 \right) \right]^{\frac{1}{2}}. \quad (4.21)$$

A rearrangement gives

$$\sigma_z = \sigma / \left[ 1 + \left( \frac{r}{a} \right)^2 (\varepsilon_u/\varepsilon_{z F})^2 \left( 1 - (\varepsilon_{z F}/\varepsilon_u)^2 \right) \right]^{\frac{1}{2}}. \quad (4.22)$$

If equation 4.8 is substituted into equation 4.22, the result is
\[ \sigma_z = \frac{\sigma_0 + \alpha \bar{\varepsilon}}{\left[1 + (r/a)^2 \left(\frac{\bar{\varepsilon}_u}{\varepsilon_{ZF}}\right)^2 \left(1 - \left(\frac{\varepsilon_{ZF}}{\bar{\varepsilon}_u}\right)^2\right)\right]^{1/2}}. \]  

(4.23)

Now, consider the term \( \bar{\varepsilon} \), the effective strain, in the above relationship. Substitution of equation 4.19 into equation 4.12 gives,

\[ \bar{\varepsilon} = \left[\varepsilon_{ZF}^2 + (r/a)^2 \bar{\varepsilon}_u^2 \left(1 - \left(\frac{\varepsilon_{ZF}}{\bar{\varepsilon}_u}\right)^2\right)\right]^{1/2}, \]  

(4.24)

or

\[ \bar{\varepsilon} = \varepsilon_{ZF} \left[1 + (r/a)^2 \left(\frac{\bar{\varepsilon}_u}{\varepsilon_{ZF}}\right)^2 \left(1 - \left(\frac{\varepsilon_{ZF}}{\bar{\varepsilon}_u}\right)^2\right)\right]^{1/2}. \]  

(4.25)

Next, if equation 4.25 is substituted into equation 4.23, the following result is obtained,

\[ \sigma_z = \sigma_0 \left(\frac{\varepsilon_{ZF}}{\bar{\varepsilon}_u}\right) \left[\left(\frac{\varepsilon_{ZF}}{\bar{\varepsilon}_u}\right)^2 + (r/a)^2 \left(1 - \left(\frac{\varepsilon_{ZF}}{\bar{\varepsilon}_u}\right)^2\right)\right]^{1/2} + \alpha \varepsilon_{ZF}. \]  

(4.26)

Now the distribution of shear stress can be calculated. This is accomplished by combining equation 4.26, equation 4.7, and equation 4.19. The result is
\[ \tau_{\theta z} = \frac{\sigma_0}{\sqrt{3}} \frac{(r/a)[1 - (\varepsilon_{zF}/\varepsilon_u)^2]}{1 + (\varepsilon_{zF}/\varepsilon_u)^2 + (r/a)^2 \left( 1 - (\varepsilon_{zF}/\varepsilon_u)^2 \right)^{1/2}} \]

Equation 4.27 and equation 4.26 represent the shear stress and the normal stress distribution over a cross section at the time of failure.

4.2.4 The Tension P and the Torque T at Failure

The magnitude of the tension P at failure for the combined loading condition can be calculated by an integration of the normal stress over the cross-sectional area. Thus,

\[ P = \int_0^a \sigma_z \cdot 2\pi r dr \quad (4.13) \]

If equation 4.26 is substituted for \( \sigma_z \) in equation 4.13 and the indicated integration is carried out, the following expression is obtained

\[ P = \pi a^2 \left[ \sigma_u (\varepsilon_{zF}/\varepsilon_u) + \sigma_0 (\varepsilon_{zF}/\varepsilon_u) (\varepsilon_u - \varepsilon_{zF}/(\varepsilon_u + \varepsilon_{zF})) \right] \quad (4.28) \]
In this expression, $\sigma_u$ represents the stress at failure, obtained from the characteristic stress-strain diagram for the material.

When $\varepsilon_{zF} = \varepsilon_u$, the axial force $P$ represents the magnitude of pure tension that produces failure. Denoting $P_u$ for the maximum attainable pure tension, a tension ratio $P/P_u$ can be defined by

$$P/P_u = (\varepsilon_{zF}/\varepsilon_u) + (\sigma_o/\sigma_u)(\varepsilon_{zF}/\varepsilon_u)(\varepsilon_u - \varepsilon_{zF})/(\varepsilon_u + \varepsilon_{zF}) .$$

(4.29)

Similarly, the torque $T$ can be calculated by an integration of the shear stress over the cross-sectional area. Thus,

$$T = \int_0^a \tau_{0z} 2\pi r^2 dr$$

When this integration is carried out, the result is

$$T = \pi a^3 \left[ 3\sigma_u \left( 1 - (\varepsilon_{zF}/\varepsilon_u)^2 \right)^2 + \sigma_o \left( 1 - 6(\varepsilon_{zF}/\varepsilon_u)^2 + 8(\varepsilon_{zF}/\varepsilon_u)^3 \right. \right. \right.$$

$$\left. \left. - 3(\varepsilon_{zF}/\varepsilon_u)^4 \right) \right] \left( 6\sqrt{3} \left[ 1 - (\varepsilon_{zF}/\varepsilon_u)^2 \right]^{3/2} \right)^{-1}$$
When $\varepsilon_{ZF} = 0$, the torque $T$ represents the magnitude of torque alone, $T_u$, that produces failure. Therefore, a torsion ratio $T/T_u$ can be defined,

$$T/T_u = \left[ 3\sigma_u \left( 1 - \frac{(\varepsilon_{ZF}/\varepsilon_u)^2}{1 - 6(\varepsilon_{ZF}/\varepsilon_u)^2 + 8(\varepsilon_{ZF}/\varepsilon_u)^3} \right) - 3(\varepsilon_{ZF}/\varepsilon_u)^4 \right] / \left[ (3\sigma_u + \sigma_0) \left( 1 + (\varepsilon_{ZF}/\varepsilon_u)^2 \right)^{3/2} \right]$$

(4.30)

Hypothetically, the variable $\varepsilon_{ZF}/\varepsilon_u$ can be eliminated from both equation 4.29 and equation 4.30 to give an expression involving tension ratio $P/P_u$ and $T/T_u$ for various values of $\sigma_o/\sigma_u$. This is a difficult mathematical manipulation and therefore was abandoned.

As an alternative, the author took several ratios of $\sigma_o$ and $\sigma_u$, and for each ratio, the variable $\varepsilon_{ZF}/\varepsilon_u$ was varied between 0 and 1. Then, the result of this calculation was plotted on a nondimensional coordinate system as shown in Figure 9 and Figure 10.

As an approximation, the analytical curve in these figures can be reasonably represented by a quarter of a circle. Therefore, the failure envelope can be written by the approximate expression

$$(P/P_u)^2 + (T/T_u)^2 \approx 1$$

(4.31)
Figure 9. Failure Envelope Obtained from Theoretical Analysis for Material having $\sigma_o/\sigma_u = 0.8$
Figure 10. Failure Envelope Obtained from Theoretical Analysis, for Materials having $\sigma_0/\sigma_u = 0.5$
4.3 RUPTURE ANALYSIS OF A SOLID CIRCULAR CYLINDER SUBJECTED TO A COMBINATION OF TENSION, TORQUE AND BENDING MOMENT, WHERE THE BENDING MOMENT IS INDUCED BY A SMALL ECCENTRICITY OF THE AXIAL FORCE

Besides a torsional moment combined with tension, a bending moment will be included in this analysis. However, the magnitude of the bending moment will be assumed to be small. This would be the case if the bending moment is induced by a small eccentricity of the axial force. This combination will be referred to as mixed mode of loading.

Since the order of magnitude of the bending moment is small compared with both the torsion and tension loads, and also because the member has a circular cross section, an analysis similar to the case of combined tension and torsion will be used. Thus the solution will be an approximation and only valid if the resulting radius of curvature associated with the bending action is very large.

4.3.1 Compatibility Condition

If the bending moment is relatively small, then the cylinder will remain essentially straight. This occurs, for example, in the body of a bolt when the nut is tightened and the two flanges being gripped have nonparallel faces; or, in the case where a slight eccentricity of the axial load exists. See Figure 11.

It will be assumed for this case that plane cross sections remain plane. The original plane will be deformed (moved) axially due to the tension, rotated due to the torque, and
Figure 11. Induced Bending Moment in a Nut-Bolt Assembly
inclined slightly due to the bending moment. For this case, the compatibility conditions can be expressed in terms of the strain components by the following relationships, Figure 12,

\[
\gamma_{\theta z} \bigg|_{r=a} = \frac{r}{a} \gamma_{\theta z} \bigg|_{r=a}, \quad (4.32)
\]

\[
\varepsilon_z \bigg|_{y} = \varepsilon_{z0} + \frac{y}{a} \Delta \varepsilon_{z0}. \quad (4.33)
\]

Here \( \gamma_{\theta z} = \) shear strain,
\( \varepsilon_z = \) normal strain,
\( \varepsilon_{z0} = \) normal strain at radius = 0,

and

\( \varepsilon_{z0} + \Delta \varepsilon_{z0} = \) normal strain at \( y = a \).

4.4.2 Material Characteristics

The material parameters for a particular metal would be obtained from an experiment. For rupture analysis, the stress-strain curve would be represented by an ideal rigid-linear strain hardening curve, as has been used in section 4.2.1. The curve can be expressed as

\[
\bar{\sigma} = \sigma_0 + \alpha \bar{\varepsilon}, \quad (4.8)
\]
Figure 12. Deformation of a Cross Section due to Tension, Torque, and Small Bending Moment
where \( \sigma = (\sigma_z^2 + 3\tau_{\theta z}^2)^{\frac{1}{2}} \)  \\
and \( \varepsilon = (\varepsilon_z^2 + \frac{1}{3}\gamma_{\theta z}^2)^{\frac{1}{2}} \).  

4.4.3 Hencky's Total Strain Theory

As stated previously, the bending moment is assumed to be relatively small. Consequently, the stress components would be approximately similar to the case of tension combined with torsion. Therefore, the stress components which are significant are \( \sigma_z \) and \( \tau_{\theta z} \).

A similar approximation can be made with regard to the strain components. The strain components which are significant are \( \varepsilon_z, \varepsilon_r, \varepsilon_\theta \) and \( \gamma_{\theta z} \).

Using Hencky's total strain theory, the resulting relationship between stress components and strain components would be similar to the case of combined tension and torsion. The relationship was derived in section 4.2. It was shown to be

\[\sigma_z \approx 3\varepsilon_z \cdot \frac{\tau_{\theta z}}{\gamma_{\theta z}}.\]  

4.4.4 The Rupture Failure Criterion

Rupture failure for a solid cylindrical body subjected to a combined mode of loading \((P,T,M)\) is assumed to occur
when a point at the outer fiber has an effective strain equal to or exceeding the effective strain at the time of failure of a specimen of the same material under a simple test.

The critical point in the mixed mode of loading is at $y = a$ (see Figure 13). Failure occurs at this point when

$$\varepsilon \bigg|_{y=a} \geq \varepsilon_u \ .$$  

The strain components at the critical points are interrelated in the following expression,

$$\varepsilon \bigg|_{y=a} = \left[ (\varepsilon_{zo} + \Delta \varepsilon_{zo})^2 + \frac{1}{3} \gamma_{\theta z} \right]^{\frac{1}{2}} \ .$$

Here $\varepsilon_{zo} = \text{the normal strain in the middle of the cylinder at the time of failure}$,

$\Delta \varepsilon_{zo} = \text{the difference between the normal strain at the critical point and the normal strain in the middle of the cylinder, at failure}$,

$\gamma_{\theta z} = \text{the shear strain at the critical points, at failure}$

and $\varepsilon \bigg|_{y=a} = \text{the effective strain at the critical point, at failure}$.
Figure 13. Critical Point on a Cross Section due to Tension, Torque, and Small Bending Moment
4.4.5 Strain Distributions at Failure

The distribution of the normal strain at failure, $\varepsilon_{ZF}$, can be obtained by using equation 4.33. Thus,

$$\varepsilon_{ZF} = \varepsilon_{zo} + \frac{y}{a} \Delta \varepsilon_{zo},$$  \hspace{1cm} (4.36)

where $y = r \sin \theta$.

The shear strain at the critical point, $\gamma_{\theta z}$, can be expressed in terms of $\varepsilon_u$ and $\varepsilon_{ZF}$, thus

$$\gamma_{\theta z} \bigg|_{y=a} = \varepsilon_u \left[ 3 \left( 1 - \left( \frac{\varepsilon_{zo} + \Delta \varepsilon_{zo}}{\varepsilon_u} \right)^2 \right) \right]^{\frac{1}{2}}.$$  \hspace{1cm} (4.37)

It was assumed that $\gamma_{\theta z}$ is only a function of radius (see equation 4.32). Therefore, the magnitude of shear strain at any radius $r$ is

$$\gamma_{\theta z} \bigg|_{r} = \frac{r}{a} \varepsilon_u \left[ 3 \left( 1 - \left( \frac{\varepsilon_{zo} + \Delta \varepsilon_{zo}}{\varepsilon_u} \right)^2 \right) \right]^{\frac{1}{2}}.$$  \hspace{1cm} (4.38)
4.4.6 Stress Distribution at Failure

First, consider the normal stress \( \sigma_z \). By using equation 4.7, the shear stress \( \tau_{0z} \) can be written in terms of \( \sigma_z \) as follows,

\[
\tau_{0z} = \sigma_z \cdot \frac{\gamma_{0ZF}}{3\varepsilon_{ZF}} \tag{4.39}
\]

Substitution of equation 4.36 for \( \varepsilon_{ZF} \) and equation 4.37 for \( \gamma_{0ZF} \) into equation 4.39 gives,

\[
\tau_{0z} = \sigma_z \cdot \frac{r}{a} \frac{\varepsilon_u}{\varepsilon_u} \left[ \frac{3 \left( 1 - \left( \frac{\varepsilon_{zo} + \Delta \varepsilon_{zo}}{\varepsilon_u} \right)^2 \right)}{3(\varepsilon_{zo} + \frac{y}{a} \Delta \varepsilon_{zo})} \right]^{\frac{1}{2}}. \tag{4.40}
\]

Next, if equation 4.40 is substituted into equation 4.10, the result is

\[
\overline{\sigma} = \sigma_z \left[ 1 + (r/a)^2 \left( 1 - \left( \frac{\varepsilon_{zo} + \Delta \varepsilon_{zo}}{\varepsilon_u} \right)^2 \right) \right]^{\frac{1}{2}} \left( \frac{\varepsilon_{zo} + (y/a) \Delta \varepsilon_{zo}}{\varepsilon_u} \right)^{\frac{1}{2}}. \tag{4.41}
\]

Therefore, by inversion \( \sigma_z \) can be written in terms of other variables.
The effective strain at any point can be written as follows,

$$\bar{e} = \left( \varepsilon_{z0} + \frac{y}{a} \Delta \varepsilon_{z0} \right) \left[ 1 + \left( \frac{r}{a} \right)^2 \left( 1 - \left( \frac{\varepsilon_{z0} + \Delta \varepsilon_{z0}}{\varepsilon_u} \right)^2 \right) \right]^{1/2} \left( \frac{\varepsilon_{z0} + (y/a) \Delta \varepsilon_{z0}}{\varepsilon_u} \right)^2 \right]^{1/2}.$$  

(4.43)

The final expression of normal stress can now be obtained. If equation 4.33 is substituted into equation 4.8, and the result is substituted into equation 4.42, then after simplification, the result is

$$\sigma_z = \frac{\sigma_0}{\left[ 1 + \left( \frac{r}{a} \right)^2 \left( 1 - \left( \frac{\varepsilon_{z0} + \Delta \varepsilon_{z0}}{\varepsilon_u} \right)^2 \right) \right]^{1/2} \left( \frac{\varepsilon_{z0} + (y/a) \Delta \varepsilon_{z0}}{\varepsilon_u} \right)^2} + \alpha \left( \varepsilon_{z0} + \frac{y}{a} \Delta \varepsilon_{z0} \right).$$  

(4.44)
The next step is to determine the shear stress distribution. This can be accomplished by combining equations 4.40 and 4.44. The result is

\[ \tau_{\theta z} = \frac{\sigma_0}{\sqrt{3}} \left( \frac{r}{a} \right) \left[ 1 - \left( \frac{\varepsilon_{zo} + \Delta \varepsilon_{zo}}{\varepsilon_u} \right)^2 \right]^{1/2} \left/ \left[ \left( \frac{\varepsilon_{zo} + (y/a) \Delta \varepsilon_{zo}}{\varepsilon_u} \right)^2 \right] + \right. \]

\[ \left. (r/a)^2 \left[ 1 - \left( \frac{\varepsilon_{zo} + \Delta \varepsilon_{zo}}{\varepsilon_u} \right)^2 \right] \right]^{1/2} + \frac{ar}{a^3} \left( \varepsilon_u^2 - (\varepsilon_{zo} + \Delta \varepsilon_{zo})^2 \right)^{1/2} \]

(4.45)

In both equations above, \( y = r \sin \theta \).

4.4.7 The Failure Envelope for a Circular Cylinder Subjected to the Mixed Mode of Loading

The magnitudes of the axial force \( P \), the torque \( T \) and the bending moment \( M \), which together produce rupture, can be obtained by integrations of the normal stress and shear stress over a cross section.

The integral equations are,

\[ P = \int_0^{2\pi} \int_0^a \sigma_z r dr d\theta \]  \hspace{1cm} (4.46)

\[ T = \int_0^{2\pi} \int_0^a \tau_{\theta z} r^2 dr d\theta \]  \hspace{1cm} (4.47)
\[ M = \int_{0}^{2\pi} \int_{0}^{a} \sigma_z r^2 \sin \theta \, dr \, d\theta \quad (4.48) \]

For convenience, the failure envelope will be plotted on dimensionless coordinates. If \( P_u \) is the magnitude of the axial force alone which produces rupture, then the strain components at the critical points are

\[ \Delta \varepsilon_{zo} = 0 \quad (4.49) \]

and

\[ \varepsilon_{zo} = \bar{\varepsilon}_u \quad (4.50) \]

The dimensionless form of the tension would be \( P/P_u \).

Also, if \( T_u \) is the magnitude of the torque which alone would produce failure, the associated normal strain components would be zero. The dimensionless form of the torque would be \( T/T_u \).

A dimensionless form for bending moment can be obtained by dividing the bending moment \( M \) by \( (aP) \). Here \( a \) is the radius of the cylinder and \( P \) is the axial force. In fact, this form, \( M/(aP) \), represents the dimensionless eccentricity, \( e/a \), where \( e \) is the amount of eccentricity.

A closed form for the integration of equations 4.46, 4.47 and 4.48 is not available. A numerical integration using Gaussian
Quadrature (15 points) was used to carry out the integration scheme.

The result of this numerical integration was shown in Figure 14. This figure shows the failure envelope for the case where

\[ \frac{\sigma_o}{\sigma_u} = 0.8 \]

\[ e/a = 0.01 \]

Also, the results are tabulated in Table 3.
Figure 14. Failure Envelope of a Cylinder with $\sigma_0/\sigma_u = 0.8$ and Eccentricity $e/a = 0.01$
Table 3. The Magnitudes of $P/P_u$ and $T/T_u$ for Eccentricity $e/a = 0.01$, for Material with $\sigma_0/\sigma_u = 0.8$

<table>
<thead>
<tr>
<th>$P/P_u$</th>
<th>$T/T_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>1.000</td>
</tr>
<tr>
<td>0.082</td>
<td>0.997</td>
</tr>
<tr>
<td>0.158</td>
<td>0.989</td>
</tr>
<tr>
<td>0.228</td>
<td>0.977</td>
</tr>
<tr>
<td>0.294</td>
<td>0.960</td>
</tr>
<tr>
<td>0.355</td>
<td>0.941</td>
</tr>
<tr>
<td>0.412</td>
<td>0.918</td>
</tr>
<tr>
<td>0.467</td>
<td>0.892</td>
</tr>
<tr>
<td>0.517</td>
<td>0.864</td>
</tr>
<tr>
<td>0.565</td>
<td>0.833</td>
</tr>
<tr>
<td>0.611</td>
<td>0.800</td>
</tr>
<tr>
<td>0.654</td>
<td>0.763</td>
</tr>
<tr>
<td>0.695</td>
<td>0.724</td>
</tr>
<tr>
<td>0.735</td>
<td>0.682</td>
</tr>
<tr>
<td>0.772</td>
<td>0.636</td>
</tr>
<tr>
<td>0.807</td>
<td>0.586</td>
</tr>
<tr>
<td>0.842</td>
<td>0.532</td>
</tr>
<tr>
<td>0.875</td>
<td>0.464</td>
</tr>
<tr>
<td>0.906</td>
<td>0.386</td>
</tr>
<tr>
<td>0.936</td>
<td>0.281</td>
</tr>
<tr>
<td>0.960</td>
<td>0.000</td>
</tr>
</tbody>
</table>
Chapter 5

EXPERIMENTAL METHOD

5.1 INTRODUCTION

As mentioned in Chapter 1, the general objective of the study was to develop a peak-load design criterion. The highest expected load constitutes the failure inducing agent, and total rupture is the manifestation of failure.

In this investigation a solid cylinder having a circular cross section was subjected to a loading combination of tension and torsion. In addition, an analysis was performed to include the effect of small eccentricity of the axial force. This bending analysis is included in Chapter 4.

The objective of the experimental program was to verify the failure envelope which resulted from the analysis in section 4.2. As previously mentioned, it was found that failure would occur in a solid circular cylinder subjected to a combination of axial load, \( P \), and torsion, \( T \), when the following relationship is satisfied,

\[
\left( \frac{P}{P_u} \right)^2 + \left( \frac{T}{T_u} \right)^2 \geq 1.
\]  

(5.1)
In this expression, $P_u$ constitutes pure axial load which would produce failure and $T_u$ represents the pure torque which would be required to produce failure.

5.2 DESCRIPTION OF THE EXPERIMENTAL SETUP

5.2.1 Materials and Specimens

Three materials were selected for the experimental program. These were: (1) aluminum alloy 7075-T6, (2) elevated-temperature-drawing e.t.d.-150 steel, and (3) cold-rolled 1041 steel. These materials were chosen because they are high strength materials having somewhat limited ductility, and typical of materials used in high performance machinery.

These materials were purchased as bar stock of 1-in diameter, 12 feet in length. The bar stock was cut into several pieces of different lengths. For control test specimens, three pieces 3-5/8 inches in length were used. They were cut one from each end of the bar, and one from the middle of the bar. The rest of the bar was cut into 4-5/8 inch lengths. These operations were performed for both the aluminum alloy and the e.t.d.-150 steel bars. However, for the CD-1041 steel, only two control specimens were fabricated.

Figure 8 shows the dimensions of the tension-torsion specimen. Its length, after machining, was 4.5 inches. Hexagonal heads at the two ends of the specimen were provided to permit applying a twisting moment. The shoulder of the hexagonal head was provided as a support in order to apply an
axial load. This specimen had a 3/16-inch diameter test section.

Figure 15 shows the dimensions of the control test specimen. It was decided to use a control test specimen with 1/4-inch diameter and 1-inch gage length. This decision was made because of the limited capacity of the Instron Universal testing machine which was used in the experimental program.

5.2.2 Test Apparatus

Figure 16 shows the arrangement of the testing apparatus. The system consisted of two parts, an Instron Universal Testing Machine which produced the required axial force, and a pulley-wire-rope assembly which produced the twisting moment. A thrust ball bearing was provided to permit application of a rotational displacement.

The Instron machine, Figure 17, has a 20,000 lb capacity. The load was applied to the specimen by the vertical movement of the machine's crosshead. Thus the magnitude of the load acting on the specimen depended upon the stiffness of the specimen. A load cell was attached to the Instron to measure the axial load. A universal joint was provided to reduce the possibility of inducing a bending moment.

A 1/16-inch wire rope was wound around a 14-inch diameter pulley, Figure 18. The pulley was supported by a journal bearing. By pulling on the wire rope, a twisting moment was transmitted to the main shaft and specimen from the pulley through a keyed joint. For the actual experiments it was
Figure 15. Configuration of Control Test Specimen
Figure 16. Experimental Set-Up
Figure 17. Instron Machine
Figure 18. Pulley-Wire Assembly
determined that the range of the torque loading required was less than 250 inch-pounds.

Figure 19 shows the specimen-gripping assembly. It consisted of a hexagonal socket, a pair of collets, and a nut. The socket was inside the main shaft and was connected to this shaft by a round pin transverse to the axis of the shaft.

At the bottom of a second shaft, a force transducer was placed in a series with this shaft. The force transducer (made of aluminum 2024-T4) was used to measure simultaneously and independently both the torque and the tension acting on the specimen.

By moving the crosshead of the Instron machine, and simultaneously loading or displacing the pulley-wire assembly, a combined tension and torsion was produced.

5.2.3 Instrumentation

Two methods of recording were used. The first recording system was used to give visualization of the instantaneous loading. This was done using the chart output of the Instron machine and a Gould strip-chart instrument. The chart of the Instron machine indicated the tensile load detected by the load cell. Due to friction between some of the parts in the test setup, this load cell reading did not represent the correct value of the axial load on the specimen. The Gould plotter was connected to the force transducer. Since this transducer was in series with the specimen, an accurate measurement of the load was recorded by it. However, the data from this
Figure 19. Gripping Assembly
chart was somewhat difficult to read. Therefore this plot was used only for visualization of the loading.

A second method of recording was used to detect both the amount of torque and the tension at any instant. A Fluke datalogger was connected to the force transducer. By setting its printer to "continuous," a continuous digital output from the force transducer was obtained. These records were subsequently used as the actual data. The datalogger was capable of printing digital output at about 4 data points per second.

Figure 20 shows the force transducer. Two sets of strain gages, one set to measure tension, and another set to measure torque, were arranged on the flat sides of the aluminum bar. Two strain gages of the first set were mounted parallel to the longitudinal axis, and the other two strain gages were mounted perpendicular to the longitudinal axis of the bar. This arrangement measured the axial load.

Another set of four strain gages was mounted on the bar to measure the torque only. This was done by mounting all the gages at 45 degrees with respect to the longitudinal axis of the bar.

5.3 EXPERIMENTAL PROCEDURE

The experimental procedure consisted of the following:
(1) calibration of the Instron Universal Testing Machine,
(2) calibration of the force transducer, (3) control tests on the standard tensile specimens of the selected materials, and
(4) the primary tests using the combined-loading specimens.
Figure 20. Configuration of the Force Transducer
Calibration of the Instron Machine

The Instron testing machine was calibrated using the procedure recommended by the manufacturer of this machine.

Calibration of the Force Transducer

The fabricated force transducer was calibrated so that any inaccuracy in the gage placements and connection effects of the wires would be compensated. It was calibrated in order to obtain the relation between the load and the voltage output. Possible cross effects between torque and axial load were studied.

Control Tests

Control tests were performed on specimens made from the three materials selected. These control tests were conducted using the Instron machine. A crosshead speed of 0.05 inch/minute was selected. The load was measured by the load cell. The strain on the specimen was measured using a strain transducer having a 1-inch gage length. The voltage output of the strain transducer was connected to a Gould strip chart.

After the standard control specimens were broken, measurements on the final dimensions of the necked region and the average diameter outside the necked region were taken.

Combined Load Tests

To established the failure envelope, three kinds of loading were used. First, several specimens were tested under
pure tension to obtain the average ultimate axial load $P_u$. Then, several specimens were tested under pure torque to determine the ultimate torque $T_u$. Finally, combinations of tension and torque were applied to the specimens to produce the required combined loading data.

With respect to the method of applying the combined load, three methods of loading were employed. In the first method, the axial load was obtained by manually controlling the movement of the crosshead. The axial load was increased in a stepwise fashion. The torque was given to the specimen by applying dead weights. This method of loading will be referred to as SP-ST loading (Step-Pull-Step-Twist).

In the second method, the movement of the crosshead was continuous. However, the twisting moment was still applied using pulleys and dead weights. Thus again the torque was increased in a stepwise fashion. This method of loading will be referred to as CP-ST (Continuous-Pull-Step-Twist) loading.

In the third method, both the axial load and the twisting moment were obtained through a continuous axial movement and a continuous rotational displacement. This continuous rotational movement was accomplished by wrapping a cable on a small drum which rotated at about 0.5 rpm. This loading will be referred to as CP-CT (Continuous-Pull-Continuous-Twist) loading.
Chapter 6

EXPERIMENTAL RESULTS

6.1 INTRODUCTION

Several important results were obtained from the experimental program. First, basic material parameters were determined from the results of the control tests. Second, the combined-load tests gave peak-load results for the three materials tested.

6.2 CALIBRATION OF FORCE TRANSUCER

The calibration of the tension part of the force-torque transducer showed a linear relationship between the output voltage and the applied tensile load. Also, the relation between output voltage and the torque load showed a linear correlation. See Figures 21 and 22.

It was also verified that the cross effects between the torque and axial force were negligible. Specifically, the axial force applied to the force transducer produced negligible voltage output on the torque measurement. Similarly, a pure twisting moment applied to the force transducer produced a very small voltage output on the axial force measurement.
Figure 21. Calibration of the Tension Part of the Force Transducer
Figure 22. Calibration of the Torque Part of the Force Transducer
6.3 RESULTS OF CONTROL TESTS

Basic material parameters for the three materials tested are shown in Table 4. The basic material parameters recorded are the ultimate strength $S_u$, the fracture stress $S_f$, and the percent reduction in the area associated with the necked region.

Figure 23 shows the engineering stress-strain diagrams for the three materials. The engineering stress, $S_n$, and the engineering strain, $\varepsilon_n$, were based on the original dimensions. Specifically, they were calculated using the following relations,

$$S_n = \frac{P}{A_0},$$

(6.1)

$$\varepsilon_n = \frac{\Delta L}{L_0}$$

where $P$ represents the load, $A_0$ and $L_0$ represent the original cross-sectional area and the original gage length of 1 inch. The additional length was measured by the strain transducer.

The true stress at fracture $S_f$ would be difficult to calculate, due to the triaxial state of stress occurring in the necked region. Herein, the fracture stress was approximated using the following relation,

$$S_f = \frac{P_f}{A_f},$$

(6.2)
Table 4. Results of Control Tests on the Three Materials

<table>
<thead>
<tr>
<th></th>
<th>Aluminum 7075-T6</th>
<th>Steel e.t.d.-150</th>
<th>Steel CD-1041</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ultimate Strength, $S_u$ (psi)</td>
<td>82,500</td>
<td>156,800</td>
<td>103,400</td>
</tr>
<tr>
<td>Stress at Rupture, $S_f$ (psi)</td>
<td>116,600</td>
<td>204,700</td>
<td>149,300</td>
</tr>
<tr>
<td>Reduction of Area (%)</td>
<td>29.5</td>
<td>36.8</td>
<td>40.6</td>
</tr>
</tbody>
</table>
Figure 23. Results of Control Tests for the Three Materials
where $P_f$ represents the load at fracture, and $A_f$ represents the fracture area.

6.4 RESULTS OF COMBINED-LOAD TESTS

**Aluminum 7075-T6 Specimens**

There were, altogether, 22 aluminum specimens tested. Among them, 6 specimens were tested under pure torque, and the rest were tested under combined axial load and twisting moment. All three methods of applying the combined load were used: SP-ST loading, CP-ST loading, and CP-CT loading.

The results of the test are shown in Table 5, and they are also shown graphically in Figure 24. Average pure axial load was obtained, $P_u = 2500$ lbf, and average pure torque was determined, $T_u = 105$ in.lbf. Based on these two values, a failure function "F" was calculated using the relationship below,

$$F = \left(\frac{P}{P_u}\right)^2 + \left(\frac{T}{T_u}\right)^2 . \quad (6.3)$$

The values for $F$ were tabulated in Table 5. It was found that the average was $F = 0.973$ with a standard deviation of 0.114.

**E.t.d.-150 Steel**

The results for 22 e.t.d.-150 steel specimens are tabulated in Table 6, and are shown graphically in Figure 25. The average ultimate axial load was found to be $P_u = 4656$ lbf, and the average ultimate torque, $T_u = 204$ in.lbf.
Figure 24. Experimental Results for Al 7075-T6 Specimens Under Combined Loads
Table 5. Results of Combined Loading Tests of Aluminum 7075-T6

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Type of Loading</th>
<th>((P/P_u))</th>
<th>((T/T_u))</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>AL-02</td>
<td>---</td>
<td>0.977</td>
<td>0.977</td>
<td></td>
</tr>
<tr>
<td>AL-03</td>
<td>---</td>
<td>1.021</td>
<td>---</td>
<td>1.021</td>
</tr>
<tr>
<td>AL-04</td>
<td>---</td>
<td>1.021</td>
<td>---</td>
<td>1.021</td>
</tr>
<tr>
<td>AL-06</td>
<td>CP-ST</td>
<td>0.383</td>
<td>0.940</td>
<td>1.031</td>
</tr>
<tr>
<td>AL-07</td>
<td>SP-ST</td>
<td>0.392</td>
<td>0.862</td>
<td>0.896</td>
</tr>
<tr>
<td>AL-08</td>
<td>SP-ST</td>
<td>0.137</td>
<td>0.812</td>
<td>0.677</td>
</tr>
<tr>
<td>AL-09</td>
<td>CP-CT</td>
<td>0.712</td>
<td>0.780</td>
<td>1.121</td>
</tr>
<tr>
<td>AL-10</td>
<td>CP-ST</td>
<td>0.384</td>
<td>0.980</td>
<td>1.107</td>
</tr>
<tr>
<td>AL-11</td>
<td>CP-ST</td>
<td>0.400</td>
<td>0.974</td>
<td>1.108</td>
</tr>
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<td>AL-12</td>
<td>CP-ST</td>
<td>0.284</td>
<td>0.902</td>
<td>0.895</td>
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<td>AL-15</td>
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<td>0.989</td>
<td>---</td>
<td>0.989</td>
</tr>
<tr>
<td>AL-16</td>
<td>CP-ST</td>
<td>0.499</td>
<td>0.734</td>
<td>0.788</td>
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<tr>
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<td>CP-ST</td>
<td>0.838</td>
<td>0.565</td>
<td>1.021</td>
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<td>1.006</td>
<td>1.006</td>
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</tr>
<tr>
<td>AL-19</td>
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<td>0.966</td>
<td>0.966</td>
<td></td>
</tr>
<tr>
<td>AL-20</td>
<td>SP-ST</td>
<td>0.679</td>
<td>0.565</td>
<td>0.780</td>
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<tr>
<td>AL-22</td>
<td>ST-ST</td>
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<td>0.840</td>
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<td>AL-23</td>
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<td>0.987</td>
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<td>AL-24</td>
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<td>---</td>
<td>0.989</td>
</tr>
<tr>
<td>AL-26</td>
<td>CP-ST</td>
<td>0.703</td>
<td>0.783</td>
<td>1.107</td>
</tr>
<tr>
<td>AL-27</td>
<td>---</td>
<td>1.051</td>
<td>1.051</td>
<td></td>
</tr>
<tr>
<td>AL-28</td>
<td>---</td>
<td>0.980</td>
<td>---</td>
<td>0.980</td>
</tr>
</tbody>
</table>

average 0.973
standard deviation 0.114

\(P_u = 2500\) lb. \hspace{1cm} P = axial force
\(T_u = 105\) in.lb. \hspace{1cm} T = torque
Figure 25. Experimental Results for e.t.d.-150 Steel Specimens Under Combined Loads
Table 6. Results of Combined Loading Tests of Steel e.t.d.-150

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Type of Loading</th>
<th>((P/P_u))</th>
<th>((T/T_u))</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>ST-02</td>
<td>---</td>
<td>---</td>
<td>0.903</td>
<td>0.903</td>
</tr>
<tr>
<td>ST-04</td>
<td>---</td>
<td>1.031</td>
<td>---</td>
<td>1.031</td>
</tr>
<tr>
<td>ST-05</td>
<td>CP-ST</td>
<td>0.647</td>
<td>0.871</td>
<td>1.176</td>
</tr>
<tr>
<td>ST-06</td>
<td>CP-ST</td>
<td>0.756</td>
<td>0.641</td>
<td>0.982</td>
</tr>
<tr>
<td>ST-07</td>
<td>CP-ST</td>
<td>---</td>
<td>1.011</td>
<td>1.011</td>
</tr>
<tr>
<td>ST-08</td>
<td>---</td>
<td>0.999</td>
<td>---</td>
<td>0.999</td>
</tr>
<tr>
<td>ST-09</td>
<td>CP-CT</td>
<td>0.389</td>
<td>1.012</td>
<td>1.175</td>
</tr>
<tr>
<td>ST-10</td>
<td>---</td>
<td>---</td>
<td>1.011</td>
<td>1.011</td>
</tr>
<tr>
<td>ST-12</td>
<td>CP-ST</td>
<td>0.856</td>
<td>0.497</td>
<td>0.980</td>
</tr>
<tr>
<td>ST-13</td>
<td>SP-ST</td>
<td>0.439</td>
<td>0.888</td>
<td>0.982</td>
</tr>
<tr>
<td>ST-15</td>
<td>---</td>
<td>---</td>
<td>1.033</td>
<td>1.033</td>
</tr>
<tr>
<td>ST-16</td>
<td>CP-ST</td>
<td>0.660</td>
<td>0.759</td>
<td>1.012</td>
</tr>
<tr>
<td>ST-18</td>
<td>CP-ST</td>
<td>0.532</td>
<td>0.878</td>
<td>1.054</td>
</tr>
<tr>
<td>ST-19</td>
<td>---</td>
<td>0.989</td>
<td>---</td>
<td>0.989</td>
</tr>
<tr>
<td>ST-21</td>
<td>SP-ST</td>
<td>0.609</td>
<td>0.829</td>
<td>1.058</td>
</tr>
<tr>
<td>ST-22</td>
<td>CP-ST</td>
<td>0.341</td>
<td>0.917</td>
<td>0.957</td>
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<tr>
<td>ST-23</td>
<td>---</td>
<td>---</td>
<td>1.039</td>
<td>1.039</td>
</tr>
<tr>
<td>ST-24</td>
<td>CP-ST</td>
<td>0.325</td>
<td>0.917</td>
<td>0.947</td>
</tr>
<tr>
<td>ST-25</td>
<td>SP-ST</td>
<td>0.791</td>
<td>0.530</td>
<td>0.906</td>
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<td>ST-26</td>
<td>SP-ST</td>
<td>0.381</td>
<td>0.896</td>
<td>0.948</td>
</tr>
<tr>
<td>ST-27</td>
<td>CP-ST</td>
<td>0.428</td>
<td>0.770</td>
<td>0.776</td>
</tr>
<tr>
<td>ST-28</td>
<td>---</td>
<td>0.993</td>
<td>---</td>
<td>0.993</td>
</tr>
</tbody>
</table>

average \(0.998\)

standard deviation \(0.084\)

\(P_u = 4656\) lb.

\(T_u = 204\) in.\(\text{lb.}\)

\(P = \text{axial force}\)

\(T = \text{torque}\)
The average value of the failure function was $F = 0.998$ with a standard deviation of 0.084.

**CD-1041 Steel**

There were 14 specimens made from CD-1041 steel. Three specimens were broken under a pure torque, 2 specimens were broken under pure tension, and the rest of the specimens were tested under a combination of axial force and twisting moment. All combined-load tests were conducted by applying continuous increasing axial and rotational displacement.

It was found that the average ultimate axial load $P_u = 2890$ lbf, and the ultimate torque was $T_u = 140$ in.lbf. The average failure function was $F = 0.971$ with a standard deviation of 0.044.

The results were tabulated in Table 7. They are shown graphically in Figure 26.

### 6.5 OTHER OBSERVATIONS

Figure 27 shows the fracture surfaces of several specimens used in the experimental program. The typical fractured surface was almost flat when the specimen was subjected to pure torque. The fracture surface had a cup-and-cone appearance when the specimen was fractured under pure tension. When a combination of tension and torsion was applied, the fracture surface appearance was in between these extreme cases.
Figure 26. Experimental Results for CD-1041 Steel Specimens Under Combined Loads
Table 7. Results of Combined Loading Tests of Steel CD-1041

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Type of Loading</th>
<th>( \frac{P}{P_u} )</th>
<th>( \frac{T}{T_u} )</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>CD-01</td>
<td>---</td>
<td>1.004</td>
<td>---</td>
<td>1.004</td>
</tr>
<tr>
<td>CD-03</td>
<td>---</td>
<td>---</td>
<td>1.014</td>
<td>1.014</td>
</tr>
<tr>
<td>CD-04</td>
<td>---</td>
<td>---</td>
<td>0.982</td>
<td>0.982</td>
</tr>
<tr>
<td>CD-05</td>
<td>CP-CT</td>
<td>0.870</td>
<td>0.464</td>
<td>0.973</td>
</tr>
<tr>
<td>CD-06</td>
<td>CP-CT</td>
<td>0.649</td>
<td>0.724</td>
<td>0.945</td>
</tr>
<tr>
<td>CD-07</td>
<td>---</td>
<td>---</td>
<td>1.018</td>
<td>1.018</td>
</tr>
<tr>
<td>CD-08</td>
<td>CP-CT</td>
<td>0.994</td>
<td>---</td>
<td>0.994</td>
</tr>
<tr>
<td>CD-09</td>
<td>CP-CT</td>
<td>0.798</td>
<td>0.560</td>
<td>0.950</td>
</tr>
<tr>
<td>CD-10</td>
<td>CP-CT</td>
<td>0.789</td>
<td>0.606</td>
<td>0.990</td>
</tr>
<tr>
<td>CD-12</td>
<td>CP-CT</td>
<td>0.887</td>
<td>0.418</td>
<td>0.962</td>
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<tr>
<td>CD-13</td>
<td>CP-CT</td>
<td>0.812</td>
<td>0.453</td>
<td>0.865</td>
</tr>
<tr>
<td>CD-15</td>
<td>CP-CT</td>
<td>0.590</td>
<td>0.827</td>
<td>1.032</td>
</tr>
<tr>
<td>CD-16</td>
<td>CP-CT</td>
<td>0.719</td>
<td>0.639</td>
<td>0.925</td>
</tr>
<tr>
<td>CD-17</td>
<td>CP-CT</td>
<td>0.721</td>
<td>0.653</td>
<td>0.946</td>
</tr>
</tbody>
</table>

Average: 0.971
Standard deviation: 0.044

\( P_u = 2890 \text{ lb.} \)
\( T_u = 149 \text{ in. lb.} \)
P = axial load
T = torque
Figure 27. The Fracture Appearance due to (from left to right) Pure Torque, Combined Loads, and Pure Tension
It was observed during the test that when a specimen underwent plastic deformation, only a small length of the specimen was affected by the load. The effective length of the test section was only about 0.25 inch.

When a stepwise torque was applied in the combined load cases, an interesting phenomenon occurred. After certain deformation both in the axial and rotational directions, the axial load suddenly reduced. More straining was required to increase the axial load. A reduction of the axial load occurred several times before the specimen was broken. Figure 28 shows schematically the change of axial load and torque which occurred during this process.

Figure 29 shows a typical loading applied to a specimen when both the axial load and torque were obtained through continuous straining in the axial and rotational direction. For all cases tested, the maximum torque was always reached earlier than the maximum tension.
Figure 28. Variation of Loads in CP-ST Method
Figure 29. Variation of Loads in the CP-CT Method
Chapter 7

DISCUSSION OF THE RESULTS OF THE INVESTIGATION

7.1 INTRODUCTION

In this chapter, discussion of the correlations between experimental data and analytical results are presented. Results of the control tests are also given. Other observations on the experimental work are discussed briefly.

7.2 THE FAILURE FUNCTION

7.2.1 Peak Load Design Criterion

The main objective of this study is to develop a peak load design criterion. As a first step toward research in this area, analytical and experimental investigations were performed to estimate the strength of a cylindrical body subjected to a combination of tension and torsion.

The analytical derivation based on the deformation theory of plasticity lead to the expression,

\[ \left( \frac{P}{P_u} \right)^2 + \left( \frac{T}{T_u} \right)^2 = 1 \]  

(7.1)
In order to investigate the validity of the analytical result, the experimental investigation described in previous chapters was performed. For convenience in correlating theory and experiment, it was decided to use a failure function, $F$. $F$ is defined by the following relation,

$$F = \left( \frac{P}{P_u} \right)^2 + \left( \frac{T}{T_u} \right)^2 .$$  \hspace{1cm} (7.2)

If the variable $F$, obtained from experiment, turned out to be close to unity, then the analytical prediction would be in close agreement with experiment.

Test results using the three selected materials are shown in Figure 24, Figure 25 and Figure 26. For convenience, the numerical values of variable $F$ for each specimen are shown in Tables 8, 9 and 10. A study of these tables shows that the average values of the variable $F$ for each of these three materials, including all methods of applying combined loads, differed at the most 3% from the analytical estimates. The data scattered as much as 12%.

A discussion about the result of each method of applying combined loads will be presented in the next section.

7.2.2 Methods of Applying Combined Loads

Three methods of applying the combination of tension and torsion were used. They are designated as SP-ST (Step-Pull-Step-Twist), CP-ST (Continuous-Pull-Step-Twist) and CP-CT (Continuous-Pull-Continuous-Twist).
Table 8. Failure Function $F$ for Combined Loading Tests for Aluminum 7075-T6

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Type of Loading</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AL-02</td>
<td>pure torque</td>
<td>0.977</td>
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<tr>
<td>AL-03</td>
<td>pure tension</td>
<td>1.021</td>
</tr>
<tr>
<td>AL-04</td>
<td>pure tension</td>
<td>1.021</td>
</tr>
<tr>
<td>AL-06</td>
<td>CP-ST</td>
<td>1.031</td>
</tr>
<tr>
<td>AL-07</td>
<td>SP-ST</td>
<td>0.896</td>
</tr>
<tr>
<td>AL-08</td>
<td>SP-ST</td>
<td>0.677</td>
</tr>
<tr>
<td>AL-09</td>
<td>CP-CT</td>
<td>1.121</td>
</tr>
<tr>
<td>AL-10</td>
<td>CP-ST</td>
<td>1.107</td>
</tr>
<tr>
<td>AL-11</td>
<td>CP-ST</td>
<td>1.108</td>
</tr>
<tr>
<td>AL-12</td>
<td>CP-ST</td>
<td>0.895</td>
</tr>
<tr>
<td>AL-15</td>
<td>pure tension</td>
<td>0.989</td>
</tr>
<tr>
<td>AL-16</td>
<td>CP-ST</td>
<td>0.788</td>
</tr>
<tr>
<td>AL-17</td>
<td>CP-ST</td>
<td>1.021</td>
</tr>
<tr>
<td>AL-18</td>
<td>pure torque</td>
<td>1.006</td>
</tr>
<tr>
<td>AL-19</td>
<td>pure torque</td>
<td>0.966</td>
</tr>
<tr>
<td>AL-20</td>
<td>SP-ST</td>
<td>0.780</td>
</tr>
<tr>
<td>AL-21</td>
<td>SP-ST</td>
<td>0.893</td>
</tr>
<tr>
<td>AL-22</td>
<td>pure tension</td>
<td>0.987</td>
</tr>
<tr>
<td>AL-23</td>
<td>pure tension</td>
<td>0.989</td>
</tr>
<tr>
<td>AL-24</td>
<td>pure tension</td>
<td>0.989</td>
</tr>
<tr>
<td>AL-26</td>
<td>CP-ST</td>
<td>1.107</td>
</tr>
<tr>
<td>AL-27</td>
<td>pure torque</td>
<td>1.051</td>
</tr>
<tr>
<td>AL-28</td>
<td>pure tension</td>
<td>0.980</td>
</tr>
</tbody>
</table>

average                      0.973  
standard deviation of the scatter 0.114  

SP-ST: Step-Pull-Step-Twist  
CP-ST: Continuous-Pull-Step-Twist  
CP-CT: Continuous-Pull-Continuous-Twist
Table 9. Failure Function $F$ for Combined Loading Tests for Steel e.t.d.-150

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Type of Loading</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>ST-02</td>
<td>pure torque</td>
<td>0.903</td>
</tr>
<tr>
<td>ST-04</td>
<td>pure tension</td>
<td>1.031</td>
</tr>
<tr>
<td>ST-05</td>
<td>CP-ST</td>
<td>1.176</td>
</tr>
<tr>
<td>ST-06</td>
<td>CP-ST</td>
<td>0.982</td>
</tr>
<tr>
<td>ST-07</td>
<td>pure torque</td>
<td>1.011</td>
</tr>
<tr>
<td>ST-08</td>
<td>pure tension</td>
<td>0.999</td>
</tr>
<tr>
<td>ST-09</td>
<td>CP-CT</td>
<td>1.175</td>
</tr>
<tr>
<td>ST-10</td>
<td>pure torque</td>
<td>1.011</td>
</tr>
<tr>
<td>ST-12</td>
<td>CP-ST</td>
<td>0.980</td>
</tr>
<tr>
<td>ST-13</td>
<td>SP-ST</td>
<td>0.982</td>
</tr>
<tr>
<td>ST-15</td>
<td>pure torque</td>
<td>1.032</td>
</tr>
<tr>
<td>ST-16</td>
<td>CP-ST</td>
<td>1.012</td>
</tr>
<tr>
<td>ST-18</td>
<td>CP-ST</td>
<td>1.054</td>
</tr>
<tr>
<td>ST-19</td>
<td>pure tension</td>
<td>0.989</td>
</tr>
<tr>
<td>ST-21</td>
<td>SP-ST</td>
<td>1.058</td>
</tr>
<tr>
<td>ST-22</td>
<td>CP-ST</td>
<td>0.957</td>
</tr>
<tr>
<td>ST-23</td>
<td>pure torque</td>
<td>1.039</td>
</tr>
<tr>
<td>ST-24</td>
<td>SP-ST</td>
<td>0.947</td>
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<tr>
<td>ST-25</td>
<td>SP-ST</td>
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</tr>
<tr>
<td>ST-28</td>
<td>pure tension</td>
<td>0.993</td>
</tr>
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</table>

average standard deviation of the scatter 0.998 0.084

SP-ST: Step-Pull-Step-Twist
CP-ST: Continuous-Pull-Step-Twist
CP-CT: Continuous-Pull-Continuous Twist
Table 10. Failure Function $F$ for Combined Loading Tests for Steel CD-1041

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Type of Loading</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CD-01</td>
<td>CP-CT</td>
<td>1.004</td>
</tr>
<tr>
<td>CD-03</td>
<td>pure torque</td>
<td>1.014</td>
</tr>
<tr>
<td>CD-04</td>
<td>pure torque</td>
<td>0.982</td>
</tr>
<tr>
<td>CD-05</td>
<td>CP-CT</td>
<td>0.972</td>
</tr>
<tr>
<td>CD-06</td>
<td>CP-CT</td>
<td>0.945</td>
</tr>
<tr>
<td>CD-07</td>
<td>pure torque</td>
<td>1.018</td>
</tr>
<tr>
<td>CD-08</td>
<td>pure tension</td>
<td>0.994</td>
</tr>
<tr>
<td>CD-09</td>
<td>CP-CT</td>
<td>0.950</td>
</tr>
<tr>
<td>CD-10</td>
<td>CP-CT</td>
<td>0.990</td>
</tr>
<tr>
<td>CD-12</td>
<td>CP-CT</td>
<td>0.962</td>
</tr>
<tr>
<td>CD-13</td>
<td>CP-CT</td>
<td>0.865</td>
</tr>
<tr>
<td>CD-15</td>
<td>CP-CT</td>
<td>1.032</td>
</tr>
<tr>
<td>CD-16</td>
<td>CP-CT</td>
<td>0.925</td>
</tr>
<tr>
<td>CD-17</td>
<td>CP-CT</td>
<td>0.946</td>
</tr>
</tbody>
</table>

average: 0.971
standard deviation of the scatter: 0.044

CP-CT: Continuous-Pull-Continuous-Twist
Aluminum specimens and e.t.d.-150 steel specimens were subjected to each of these three methods of loading.

Comparison of the results obtained using these loadings reveals that the CP-CT loading gave the highest failure function \( F \). The CP-ST method yielded higher values than the SP-ST method. Table 11 shows the average values for aluminum and e.t.d.-150 specimens when they were tested under these three methods of loading. Also, the CP-CT loading gave less scatter. This fact might be explained from the possible slight impact effect induced by the stepwise loading.

<table>
<thead>
<tr>
<th></th>
<th>SP-ST (F)</th>
<th>CP-ST (F)</th>
<th>CP-CT (F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al. 7075-T6</td>
<td>0.812</td>
<td>1.008</td>
<td>1.121*</td>
</tr>
<tr>
<td>e.t.d.-150</td>
<td>0.974</td>
<td>1.015</td>
<td>1.175*</td>
</tr>
</tbody>
</table>

*Based on one specimen

7.3 CONTROL TESTS

The fundamental problem of the designer is to use simple test data and relate them to the expected strength of a machine part subjected to complex states of stress or loading situations. For this purpose, simple tension control tests were performed for the three materials.
One of the two material characteristics required for the peak load design criterion for a solid cylindrical body subjected to tension combined with torsion, is the maximum tension $P_u$. Comparisons were made between the actual maximum load $P_u$ obtained from tests of combined-load specimens and the calculated maximum tensile load $P_{uc}$ based on results obtained in a standard tension test. The calculated value $P_{uc}$ is estimated by using the engineering ultimate strength. Table 12 shows the degree of agreement between calculated values and actual values. Calculated values are generally lower than the actual data.

<table>
<thead>
<tr>
<th>Material</th>
<th>$S_u$ (psi)</th>
<th>$P_{uc}$ (lb)</th>
<th>$P_u$ (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al.7075-T6</td>
<td>82,500.</td>
<td>2280.</td>
<td>2500.</td>
</tr>
<tr>
<td>e.t.d.-150</td>
<td>156,800.</td>
<td>4330.</td>
<td>4650.</td>
</tr>
<tr>
<td>CD-1041</td>
<td>103,400.</td>
<td>2860.</td>
<td>2880.</td>
</tr>
</tbody>
</table>

While the magnitude of $P_u$ can be estimated by using the ultimate strength of the material obtained from a standard tension test, the maximum torque $T_u$, in general, cannot be estimated using this data. This is due to the fact that in a standard tension test, necking occurs beyond the ultimate strength and rupture occurs after appreciable change in diameter. For a solid cylindrical bar subjected to a pure torque, the change in diameter at rupture is insignificant.
A method is proposed to estimate $T_u$. In this method, a material characteristic called fracture stress $S_f$ will be used. Fracture stress is defined as the load at fracture divided by the fracture area. This is not the "true" fracture stress, because, beyond the ultimate strength, the stress state in the necking region is no longer uniaxial.

In the calculation of $T_u$, it is assumed that the shear stress is distributed uniformly over a cross section, as shown in Figure 30. This is similar to the assumption of fully plastic state analysis, where the stress-strain diagram is represented by a rigid-perfectly plastic curve.

Using the definition of effective stress, the magnitude of the shear stress is given by

$$\tau_{\theta z} = \frac{S_f}{\sqrt{3}} \quad (7.3)$$

Therefore, the maximum torque $T_u$ can be calculated by an integration,

$$T_u = \int_{0}^{a} \tau_{\theta z} \cdot 2\pi r^2 dr \quad (7.4)$$

$$T_u = \frac{2}{3} \sqrt{3} \pi a^3 S_f \quad (7.5)$$
Figure 30. Assumed Shear Stress Distribution for the Estimation of $T_u$
Comparisons were made between the calculated maximum torque $T_{uc}$ and the actual maximum torque obtained from experiment. This is shown in Table 13. In general, the agreement is good.

Table 13. Comparison Between Calculated $T_{uc}$ and Actual Maximum Torque $T_u$

<table>
<thead>
<tr>
<th>$S_f$ (psi)</th>
<th>$T_{uc}$ (in.lb.)</th>
<th>$T_u$ (in.lb.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al.7075-T6</td>
<td>116,600.</td>
<td>111.</td>
</tr>
<tr>
<td>e.t.d.-150</td>
<td>204,700.</td>
<td>204.</td>
</tr>
<tr>
<td>CD-1041</td>
<td>149,300.</td>
<td>149.</td>
</tr>
</tbody>
</table>

7.4 OTHER OBSERVATIONS

7.4.1 Observations on Continuous Straining

All CD-1041 steel specimens were tested in combined loading where the axial and the rotational displacements were increased continuously. Only one specimen made from aluminum and one specimen made from e.t.d.-150 steel were tested using this method of loading.

Figure 29 shows typically how the axial force and torque changed with time during the experiment. It was observed that the maximum torque reached its maximum value earlier than the maximum tension. Subsequent calculations showed that the value
of variables associated with the maximum torque, $F_t$, was close to the failure value $F$. This is shown in Table 14.

The author has attempted to explain this phenomenon. First, it has been stated before that the change of diameter of a cylindrical bar does not depend upon the amount of torque, but it depends upon the axial force. The relation between the strain components is

$$\varepsilon_r = \varepsilon_\theta = -\frac{\varepsilon_z}{2}. \quad (7.6)$$

Second, it has been derived that the distribution of shear stress and normal stress as functions of radius are given by the following,

$$\sigma_z = \sigma_0 \frac{\varepsilon_z F}{\varepsilon_u} \left[ \left( \frac{\varepsilon_z F}{\varepsilon_u} \right)^2 + \left( \frac{r}{a} \right)^2 \left[ 1 - \left( \frac{\varepsilon_z F}{\varepsilon_u} \right) \right] \right]^{\frac{1}{2}} + a \varepsilon_z F \quad (7.7)$$

$$\tau_{\theta z} = 0.577 \sigma_0 \left( \frac{r}{a} \right) \left[ 1 - \left( \frac{\varepsilon_z F}{\varepsilon_u} \right)^2 \right]^{\frac{1}{2}} \left[ \left( \frac{\varepsilon_z F}{\varepsilon_u} \right)^2 + \left( \frac{r}{a} \right)^2 \left[ 1 - \left( \frac{\varepsilon_z F}{\varepsilon_u} \right)^2 \right]^{\frac{1}{2}} + 0.577 a \varepsilon_z F \left( \frac{r}{a} \right) \left[ \varepsilon_u^2 - \varepsilon_z^2 \right]^{\frac{1}{2}} \right. \quad (7.8)$$
Table 14. The Magnitude of the Failure Function Associated with the Maximum Torque in Combined Load Tests

<table>
<thead>
<tr>
<th>Specimen</th>
<th>$F_t$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CD-05</td>
<td>0.920</td>
<td>0.973</td>
</tr>
<tr>
<td>CD-06</td>
<td>0.859</td>
<td>0.945</td>
</tr>
<tr>
<td>CD-09</td>
<td>0.827</td>
<td>0.950</td>
</tr>
<tr>
<td>CD-10</td>
<td>0.784</td>
<td>0.990</td>
</tr>
<tr>
<td>CD-12</td>
<td>0.808</td>
<td>0.962</td>
</tr>
<tr>
<td>CD-13</td>
<td>0.608</td>
<td>0.865</td>
</tr>
<tr>
<td>CD-15</td>
<td>0.984</td>
<td>1.031</td>
</tr>
<tr>
<td>CD-16</td>
<td>0.829</td>
<td>0.925</td>
</tr>
<tr>
<td>CD-17</td>
<td>0.846</td>
<td>0.945</td>
</tr>
<tr>
<td>average</td>
<td>0.829</td>
<td>0.954</td>
</tr>
</tbody>
</table>

$F_t$: failure function $F$ associated with maximum attainable torque in combined load tests

$F$: failure function
Figure 31 shows schematically the distribution of the shear stress and normal stress. The magnitude of axial load $P$ and torque $T$ can be obtained from the following integral equation,

$$P = \int_{0}^{a} \sigma_{z} \cdot 2\pi r dr$$  \hspace{1cm} (7.9)

$$T = \int_{0}^{a} \tau_{\theta z} \cdot 2\pi r^2 dr$$  \hspace{1cm} (7.10)

When the specimen was subjected to both tension and torsion, as the axial strain increased, the diameter of the bar decreased. Due to the shape of the stress distributions, the change in diameter might be expected to affect the torque $T$ more than the axial force $P$. As the axial strain and shear strain increased, both the torque $T$ and tension $P$ were also increased, until the torque $T$ reached its maximum value. Additional axial straining would reduce the diameter and thus also the torque.
Figure 31. Schematic of Normal Stress and Shear Stress Distribution
Chapter 8

SUMMARY AND CONCLUSIONS OF THIS STUDY

8.1 INTRODUCTION

In this chapter, a summary of the results of the study is presented. This includes the results of both the analytical and experimental investigation.

The principal conclusions of the investigation are also presented.

8.2 PEAK-LOAD DESIGN CRITERION

The main objective of the study was to develop a peak-load design criterion. The maximum combined loads a particular machine member can support without rupture were taken as the failure criterion.

In this study, the type of loading of greatest concern was that loading which induced a weak stress pattern. Particularly, a combination of tension and torque applied to a solid cylindrical body was chosen to be investigated analytically and experimentally.

An analytical investigation was also performed on a solid circular cylinder subjected to a combination of tension,
torsion and small bending moment. The bending moment was
induced by a small eccentricity of the axial force.

8.3 ANALYTICAL WORK

Two analyses have been completed. In the first analysis,
a solid circular cylinder was subjected to a combination of
tension, $P$, and torsion, $T$. In the second analysis, bending
moment due to a small eccentricity of the axial forces was
included.

It was assumed in both analyses that the material is
isotropic and homogeneous. The material characteristics were
represented by a rigid-linear strain hardening stress-strain
relationship up to rupture.

The combined loads were assumed to increase monotonically,
without unloading.

The relationships between stress components and strain
components were obtained through the use of Hencky's total
strain theory of plasticity.

A compatibility condition is assumed. Here, a plane
cross section before deformation was assumed to remain a plane
after deformation occurred.

Failure was assumed to occur when the effective strain
of a small volume element at a critical point became equal to
the effective strain at failure of the same material in a simple
test.
8.3.1 The Case of Tension Combined with Torsion

It was found that the failure envelope of a solid circular cylinder subjected to a combination of tension $P$ and torque $T$ could be approximated by the following relationships,

$$(P/P_u)^2 + (T/T_u)^2 = 1.$$  

Here, $P_u$ represents the pure axial force that would produce failure. Similarly, $T_u$ represents the magnitude of pure torque associated with failure.

The magnitude of $P_u$ and $T_u$ for a particular size of cylinder and for a certain material can be estimated. This will be presented in section 8.4.

Subsequent analysis showed that proportional stressing of a small volume element of the cylinder can be maintained provided the ratio of axial force $P$ and the torque $T$ during the monotonically increasing loads is almost constant.

8.3.2 The Effect of Small Eccentricity of Axial Load

In a real application, such as in the case of torquing a high-strength bolt, the presence of some amount of bending moment is unavoidable. This bending moment is usually induced by a slight eccentricity of the axial force. It might also be induced by lack of parallelism between the plane of the bolt head and the plane of the face of the nut.

Analysis of a solid circular cylinder subjected to a combination of tension, torsion and small bending moment has been
completed. The analysis was similar to the case when bending moment was not present. A slight variation was made with regard to the compatibility condition.

It was assumed that any cross section would remain plane during the application of the combined loads. The plane would be deformed (moved) axially due to tension, rotated due to torque, and inclined slightly due to bending moment.

It was found that the failure envelope for a solid circular cylinder subjected to a combination of tension, torque and small bending moment was slightly different than the failure envelope for the case when bending moment was not present. The strength of the member is reduced in the tension side.

8.4 THE EXPERIMENTAL WORK

Experimental work was performed in order to determine the validity of the analytical result. The activity included the design of experimental setup, the selection of specimen configuration and running the tests.

The experimental setup consisted of a pulley-wire assembly and an Instron machine. The pulley-wire assembly provided the torque to the specimen, while the Instron machine subjected the specimen to tension.

A unique configuration for the specimen was invented which permitted both tension and torsion to be applied simultaneously to the specimen. The specimen was designed specifically for the purpose of rupture testing.
A force transducer was designed, fabricated and used to measure the tension and torsion simultaneously and independently.

8.4.1 Experimental Description

Control tests were performed on standard tensile specimens (1 inch gage length) using the Instron machine. Two material parameters were required for the peak-load design criterion. They were the ultimate strength and the stress at rupture for the material.

Three methods of applying the combined tension and torsion were used. In the first method, SP-ST, the tension and torque were increased in a stepwise fashion. In the second method, CP-ST, the tension was increased continuously and a stepwise torque was imposed. In the CP-CT method, both tension and torque were increased in a continuous manner.

8.4.2 Experimental Results

Three materials were used in the experiment. They were Aluminum alloy 7075-T6, e.t.d.-150 steel and CD-1041 steel. These materials were chosen because they have relatively high yield strength and exhibit strain hardening characteristics.

Control tests for the three selected materials were completed. The two material characteristics required for peak-load design criterion were recorded. They were the engineering ultimate strength, $S_u$, and the stress at rupture, $S_f$. The stress at rupture was defined as the load at rupture divided by the rupture area.
Combined load tests for the three selected materials have been performed. The magnitude of tension and torque during the tests were recorded up to rupture. These data were used to determine the combined load at failure.

A failure function $F$ was defined. The use of this variable enabled a comparison to be made between experimental data and theoretical development.

The average values of failure function for each of these materials showed a close agreement with analytical estimates.

8.5 PRINCIPAL CONCLUSIONS

The peak-load design criterion for a solid circular cylinder subjected to a combination of tension and torsion has been derived analytically, and verified experimentally.

The failure envelope can be simplified to the following relationship

$$(\frac{P}{P_u})^2 + (\frac{T}{T_u})^2 = 1.$$  

The magnitude of $P_u$ and $T_u$ can be estimated by using the following relationships,

$$P_u = S_u \cdot \pi \cdot a^2$$

$$T_u = S_f \cdot \frac{2\pi}{3\sqrt{3}} a^3$$
Here, $S_u$ and $S_f$ refer to the engineering ultimate strength and the stress at rupture, respectively. The radius of the cylinder is designated by "a." These two material parameters can be obtained from a standard tensile test.

The author proposed a peak-load failure criterion with regard to this investigation. The criterion can be stated as follows:

"Failure is predicted to occur for a solid cylindrical body subjected to a combination of tension $P$ and torsion $T$, when the magnitude of the sum of the square of $P/P_u$ and the square of $T/T_u$ is equal to or exceeds unity."

An application of this investigation would be in the case of torquing high-strength bolts.
Chapter 9

FUTURE STUDIES

In this investigation the author studied both analytically and experimentally the strength of a solid circular cylinder subjected to a combination of tension and torque. This selection can be viewed as the first thrust of a general investigation which would include combinations of all stress patterns, particularly combinations of the weak stress patterns.

The author has also studied analytically the effect of small eccentricity of the axial load. It is suggested for future study to verify experimentally the effect of small bending moment superimposed on a combination of tension and torsion. This situation occurs in torquing high-strength bolts, where bending moment is induced by some eccentricity of the axial force or by nonparallel faces of bolt head and nut.

It is also suggested that studies be completed on the effect of cracks on the solid circular cylinder subjected to a combination of tension, torsion and small bending moment.

Experimental studies should be made on other high-strength materials, particularly on rather brittle materials.

With respect to material parameters, it is suggested to also record the stress at rupture for engineering materials,
particularly materials with high yield strength. This material parameter, together with the ultimate strength, is necessary when the peak-load failure criterion is to be applied.
Appendix A

THE RELATIONSHIP BETWEEN AXIAL FORCE AND TORQUE REQUIRED TO PRODUCE PROPORTIONAL STRESSING IN A SMALL VOLUME ELEMENT

In order to apply Hencky's theory it is necessary that proportional loading of a small volume element be maintained. Therefore, the axial force $P$ and the torque $T$ should be increased in such a manner that proportional stressing is produced in a small element of volume.

A volume element is said to be stressed proportionally if the ratio of principle stresses remains constant as the stresses are changed under the application of the external loads.

Consider a volume element at the surface of the cylinder. Here, the nonzero stress components are $\sigma_z$ and $\tau_{\theta z}$. The principal stresses can be calculated using

$$
\sigma_1 = \frac{\sigma_z}{2} + \sqrt{\frac{\sigma_z^2}{4} + \tau_{\theta z}^2},
$$

(A.1)

and

$$
\sigma_2 = \frac{\sigma_z}{2} - \sqrt{\frac{\sigma_z^2}{4} + \tau_{\theta z}^2}.
$$

(A.2)
Therefore, the ratio of the principal stresses is

$$\frac{\sigma_1}{\sigma_2} = \frac{\sigma_z}{2} + \sqrt{\frac{\sigma_z^2}{4} + \tau_{\theta z}^2}$$

Suppose that during the application of the increasing external loads, the normal and shear stresses are changed but they are always proportional, that is

$$\tau_{\theta z} = c \sigma_z \quad (A.4)$$

Then the ratio of the principal stresses during the loading becomes

$$\frac{\sigma_1}{\sigma_2} = \frac{1}{2} + \sqrt{\frac{1}{4} + c^2} = \text{constant} \quad (A.5)$$

Therefore, proportional stressing on a small volume element can be achieved provided the shear stress is always proportional to the normal stress.
The question is, how to achieve the proportionality between shear stress and normal stress for the case of a solid circular cylinder subjected to tension combined with torsion. In other words, what should the ratio be between the increasing axial force $P$ and the increasing torque $T$, such that they always produce proportional stressing on a small volume element?

Referring to Figure 30, assume that the combined loads produce an effective strain $\varepsilon_i$ at the outer fiber. The magnitudes of the axial force and torque at this stage are $P_i$ and $T_i$. The normal strain $\varepsilon_{zi}$ and shear strain $\gamma_{\theta zi}$ are interrelated by the following relationship,

$$\sqrt{\varepsilon_{zi}^2 + \frac{1}{3} \gamma_{\theta zi}^2} \bigg|_{r=a} = \varepsilon_i$$

For mathematical convenience, the axial strain $\varepsilon_{zi}$ will be written in terms of the effective strain of the outer fiber,

$$\varepsilon_{zi} = \beta_i \varepsilon_i$$

Following the same analysis as in Chapter 4, the stress distributions over a cross section due to the axial force $P_i$ and torque $T_i$ are
Observation of the two equations above shows that the ratio of shear stress $\tau_{0z_1}$ and normal stress $\sigma_{z1}$ depend upon the position of the small volume element, $r$, and the variable $\beta_i$. The ratio is

$$\frac{\tau_{0z_1}}{\sigma_{z1}} = \frac{r\sqrt{1 - \beta_i^2}}{a\beta_i \sqrt{3}}.$$ (A.10)

Therefore, it can be concluded that for a small volume element to be subjected to a proportional stressing during the increase of combined loads, the variable $\beta_i$ should be maintained constant.

It is of interest to see how the axial force $P_i$ and torque $T_i$ should be increased in order to hold the variable $\beta_i$ constant.

By an analysis similar to the analysis in section 4.3, the magnitude of axial force $P_i$ is
Similarly the torque $T_i$ is

$$T_i = \frac{\pi a^2}{6\sqrt{3}} \frac{3\overline{\sigma}_i(1 - \beta_i^2)^2 + \sigma_0(1 - 6\beta_i^2 + 8\beta_i^3 - 3\beta_i^4)}{(1 - \beta_i^2)^{3/2}}.$$  \hspace{1cm} (A.12)

By dividing the above relationships respectively by $P_u$ and $T_u$, the following relationships are obtained,

$$\frac{P_i}{P_u} = \frac{\sigma_i}{\sigma_u} \beta_i + \beta_i \frac{1 - \beta_i}{1 + \beta_i} \frac{\sigma_0}{\sigma_u}$$ \hspace{1cm} (A.13)

$$\frac{T_i}{T_u} = \frac{\frac{\sigma_i}{\sigma_u}(1 - \beta_i^2)^2 + \frac{\sigma_0}{\sigma_u}(1 - 6\beta_i^2 + 8\beta_i^3 - 3\beta_i^4)}{(3 + \frac{\sigma_0}{\sigma_u})(1 - \beta_i^2)^{3/2}}.$$ \hspace{1cm} (A.14)

Table 14 shows the variation of the ratio between $P_i/P_u$ and $T_i/T_u$ for $\sigma_0/\sigma_u = 0.8$ and by holding the variable $\beta_i$ at a constant value 0.5. The table shows that the ratio changes slightly as the loads are increased. The plots in Figure 31 show that for practical purposes, the ratio between $P_i/P_u$ and $T_i/T_u$ should be almost constant in order to produce proportional stressing in a volume element.
Figure 32. Loading Path to Produce Proportional Stressing in a Small Volume Element.
Therefore, for a circular cylinder subjected to increasing tension and torsion, the member can be analyzed using Hencky's deformation theory when the ratio of the increasing axial force $P$ and the increasing torque $T$ is constant.

Table 15. The Variation of Axial Force $P_i$ and Torque $T_i$ During Loading, by Holding the Variable $\beta_i = 0.5$, and $\frac{\sigma_i}{\sigma_u} = 0.8$

<table>
<thead>
<tr>
<th>$\frac{\sigma_i}{\sigma_u}$</th>
<th>$\frac{P_i}{P_u}$</th>
<th>$\frac{T_i}{T_u}$</th>
<th>$\frac{P_i T_i}{P_u T_u}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.82</td>
<td>0.543</td>
<td>0.662</td>
<td>0.820</td>
</tr>
<tr>
<td>0.84</td>
<td>0.553</td>
<td>0.676</td>
<td>0.818</td>
</tr>
<tr>
<td>0.86</td>
<td>0.563</td>
<td>0.689</td>
<td>0.817</td>
</tr>
<tr>
<td>0.88</td>
<td>0.573</td>
<td>0.703</td>
<td>0.815</td>
</tr>
<tr>
<td>0.90</td>
<td>0.583</td>
<td>0.717</td>
<td>0.813</td>
</tr>
<tr>
<td>0.92</td>
<td>0.593</td>
<td>0.730</td>
<td>0.812</td>
</tr>
<tr>
<td>0.94</td>
<td>0.603</td>
<td>0.744</td>
<td>0.810</td>
</tr>
<tr>
<td>0.96</td>
<td>0.613</td>
<td>0.758</td>
<td>0.809</td>
</tr>
<tr>
<td>0.98</td>
<td>0.623</td>
<td>0.771</td>
<td>0.808</td>
</tr>
<tr>
<td>1.00</td>
<td>0.633</td>
<td>0.785</td>
<td>0.806</td>
</tr>
</tbody>
</table>

$P_i =$ the current axial force

$T_i =$ the current torque

$P_u =$ the magnitude of axial force alone to produce failure

$T_u =$ the magnitude of torque alone to produce failure

$\sigma_i =$ the effective stress associated with $P_i$ and $T_i$
Appendix B

DISTRIBUTION OF THE STRESS COMPONENTS OF A SOLID CIRCULAR CYLINDER UNDER TENSION, TORSION, AND SMALL BENDING

In the analysis of a solid cylinder, having a circular cross section, subjected to a mixed mode of loading where the bending moment was induced by a small eccentricity of the axial force, it was assumed that the stress components which were significant were axial stress \( \sigma_z \) and shear stress \( \tau_{\theta z} \).

In the following section, an analysis is presented of the consequence of the small bending moment assumption on the induced stress components.

B.1 EQUILIBRIUM OF A SMALL VOLUME ELEMENT

The differential equations of equilibrium in cylindrical coordinates are

\[
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{B.1}
\]

\[
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0 \tag{B.2}
\]
In the following analysis, several assumptions are made with regard to the actual stress system.

First, it will be assumed that the stress components do not vary in the longitudinal (z) directions. Therefore, all

$$\frac{\partial \sigma_r}{\partial z} = 0 . \quad (B.4)$$

Second, it will be assumed that the normal stress in both the radial and tangential directions are zero. That is,

$$\sigma_r = 0 \quad , \quad (B.5)$$

$$\sigma_\theta = 0 \quad . \quad (B.6)$$

Therefore, the equilibrium equations B.1, B.2 and B.3 become

$$0 + 1 \frac{\partial \sigma_r}{\partial \theta} + 0 + 0 = 0 \quad (B.7)$$

$$\frac{\partial \tau_r \theta}{\partial r} + 0 + 0 + \frac{2 \tau_r \theta}{r} = 0 \quad (B.8)$$

$$\frac{\partial \tau_r z}{\partial r} + 1 \frac{\partial \tau_\theta z}{\partial \theta} + 0 + \frac{\tau_r z}{r} = 0 \quad (B.9)$$
First, consider equation B.8,

$$\frac{2 \tau_r \theta}{a} + 2 \frac{\tau_r \theta}{r} = 0 . \quad (B.8)$$

This equation can be written as

$$\frac{a}{r} (r^2 \tau_r \theta) = 0 ,$$

If it is integrated, the result is

$$\tau_r \theta = \frac{C_1(\theta)}{r^2} , \quad (B.10)$$

where $C_1(\theta)$ is a function of $\theta$.

Now, consider equation (B.7),

$$\frac{1}{r} \frac{\partial \tau_r \theta}{\partial \theta} = 0 , \quad (B.7)$$

If equation (B.10) is substituted into equation B.7, the result is

$$\frac{1}{r} \frac{a}{\theta} \left( \frac{C_1(\theta)}{r^2} \right) = 0 ,$$

or

$$\frac{1}{r^3} \frac{\partial C_1}{\theta} = 0 .$$
Therefore, \( C_1 = \text{constant} \). \hfill (B.11)

Now, equation B.10 becomes,

\[
\tau = \frac{C_1}{r^2} . \hfill (B.12)
\]

Applying the boundary condition, that \( \tau r^0 = 0 \) at \( r = a \), and using equation B.12, the results is that

\[
\tau r^0 = 0 . \hfill (B.13)
\]

Next consider equation B.9.

\[
\frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} = 0 , \hfill (B.9)
\]

which can be written as

\[
\frac{\partial}{\partial r} (r \tau_{rz}) = -\frac{\partial \tau_{\theta z}}{\partial \theta} . \hfill (B.14)
\]

By the process of integration, equation (B.14) becomes
In the analysis in chapter 4, it was found that $\tau_{rz}$ is a function of $r$ and $\theta$. Therefore, $\tau_{rz}$ exists. The question was, whether the magnitude of $\tau_{rz}$ is insignificant relative to $\sigma_z$ and $\tau_{\theta z}$, such that it can be safely neglected.

### B.2 Magnitude of the Shear Stress, $\tau_{rz}$

If equation (B.14) is integrated, using limits integration between $r=a$ to $r=r$,

$$
\left[ \tau_{rz} \right]_r - \left[ \tau_{rz} \right]_{r=a} = - \int_a^r \frac{\partial \tau_{\theta z}}{\partial \theta} \, dr . \quad \text{(B.16)}
$$

Since at the surface $r = a$, $\tau_{rz} = 0$, thus

$$
\tau_{rz} = \int_a^r \frac{\partial \tau_{\theta z}}{\partial \theta} \, dr . \quad \text{(B.17)}
$$

The distribution of shear stress $\tau_{\theta z}$ over a cross section has been derived in chapter 4. For mathematical convenience, a group of variables will be represented by the following symbol

$$
b = \frac{\epsilon_{zo} + \Delta \epsilon_{zo}}{\epsilon_u} ,
$$
and

\[ c = \frac{\Delta \varepsilon_{z0}}{\varepsilon_{z0}} \]

In the above expression,

- \( \varepsilon_u \) = effective strain at rupture of the material in a simple test,
- \( \varepsilon_{z0} \) = normal strain at radius = 0 at the time of failure under mixed mode of loadings,

and

- \( \Delta \varepsilon_{z0} \) = the difference between normal strain at the critical point and the normal strain at radius = 0.

Using these symbols, the shear stress \( \tau_{\theta z} \) can be written as follows,

\[
\tau_{\theta z} = \frac{\sigma_0}{r^3} (1 - b^2)^{\frac{1}{2}} (r^2)(1 + c)/\left((1 + \frac{r}{a} c \sin \theta)^2 b^2 (1 + \frac{r}{a} c \sin \theta)^2 \right) + \frac{\Delta \varepsilon_u}{\sqrt{3}} (r^2)(1 - b^2)^{\frac{1}{2}}
\]

(B.18)

Substitution of \( \tau_{\theta z} \) in equation B.18 into equation B.17 gives

\[
\frac{r}{a} \tau_{rz} = -0.577 \sigma_0 \int_{r/a}^1 \frac{(1 - b^2)^{\frac{1}{2}}(r^2)^2(1 + c)(1 + \frac{r}{a} c \sin \theta)(c \cos \theta)}{\left((1 + \frac{r}{a} c \sin \theta)^2 b^2 + (\frac{r}{a})^2 (1 + c)^2(1 - b^2)^2 \right)^{1.5}} d\left(\frac{r}{a}\right). \]

(B.19)
The distribution of shear stress $\tau_{rz}$ over a cross section can be obtained by numerical integration of equation B.19. Using Gaussian Quadrature, shear stress distribution was found, Figure 32.

Observation of the stress distributions reveals that in the inner section the $\tau_{rz}$ component is significant relative to $\tau_{\theta z}$. However, for a large portion of the cross section, especially toward the outer surface, the assumption that $\tau_{rz}$ can be neglected is justified.
Figure 32. The Stress Distributions in a Cylinder Subjected to Tension, Torsion and Small Bending Moment.
C.1 INTRODUCTION

In this section, a sample calculation for a specimen made of e.t.d.-150 steel is demonstrated. Actual data used were recorded using a Fluke datalogger. The data consist of digital millivolt outputs, representing the axial force and the torque applied to the specimen. The inputs to the datalogger were supplied by the force and torque transducer.

When both axial force and torque were zero, outputs from the force transducer were recorded. These were used as reference points.

C.2 CALCULATION OF AXIAL FORCE P

A calibration was performed to obtain the static sensitivity of the force transducer. It was found that the static sensitivity for axial force was

\[ k_a = 862.9 \text{ lb/mV} \]

with a standard deviation of 1.17 lb/mV. The linearity of the system is shown in Figure 21.
Column 1 of Table 16 shows the output from the force transducer representing the axial force. The axial force acting on the specimen was calculated using the following relationship

\[ P = (mV_a - mV_{ao})xk_a \]  

(C.1)

where \( P \) represents the axial force, and \( mV_a \) and \( mV_{ao} \) represent the output voltages at a certain load level, and at the reference point, respectively.

The results of the these axial load calculations were tabulated in column 3 of Table 16.

C.3 CALCULATION OF TORQUE \( T \)

The static sensitivity of the torque part of the force transducer was

\[ k_T = 80.1 \text{ in.lbf./mV} \]

with a standard deviation of 2.22 in.lbf./mV. Figure 22 shows the linearity of the system.

Column 2 of Table 16 shows the outputs from this test which represent the torque. The magnitude of the twisting moment at any stage of loading was calculated using the relation
\[ T = (mV_{T0} - mV_T)xk_T \]  

Here \( T \) represents the torque, \( mV_T \) represents the output at a certain level of loading, and \( mV_{T0} \) represents the reference point for torque.

The results are tabulated in column 4 of Table 16.

C.4 THE FAILURE FUNCTION \( F \)

The next step is to obtain the maximum of the combination of axial force and torque. In order to do this, a failure function \( F \) was introduced, defined as

\[ F = \left( \frac{P}{P_u} \right)^2 + \left( \frac{T}{T_u} \right)^2 \]  

In this expression, \( P_u \) represents the maximum axial load, and \( P/P_u \) is defined as a tension ratio. Similarly, \( T_u \) is the maximum torque and \( T/T_u \) is called the torque ratio. It was found that the average values for \( P_u \) and \( T_u \) for e.t.d.-150 steel were

\[ P_u = 4650 \text{ lb.} \]

\[ T_u = 203 \text{ in.lb.} \]
Column 5 and column 6 show the values of tension ratios and torque ratios for each level of combined loading. The failure function $F$ associated with each load level is shown in column 7.

By observation of the output from the computer program (Table 16), it was found that the maximum value of $F$ was

$$F = 1.175.$$ 

The loading path is plotted in Figure 33.
Figure 33. Determination of Failure Function $F$
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<th>MVA</th>
<th>MVT</th>
<th>PULL</th>
<th>TWIST</th>
<th>P/PU</th>
<th>T/TU</th>
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TABLE 16 (Continue)

THE VALUE OF TENSION RATIO, TORQUE RATIO AND VARIABLE F, FOR E.T.D.-150 STEEL, SPECIMEN NUMBER 9

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<th>MVT</th>
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### TABLE 16 (Continue)

**THE VALUE OF TENSION RATIO, TORQUE RATIO AND VARIABLE F, FOR E.T.D.-150 STEEL, SPECIMEN NUMBER 9**

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C.5 UNCERTAINTY ANALYSIS

As presented earlier, the failure function $F$ was based on several variables. These variables were the axial force $P$, the torque $T$, both evaluated at failure. Also required are the maximum axial force $P_u$ and the maximum torque $T_u$. Each of these variables has some inaccuracy. The overall inaccuracy which results from the combined effect of each of the individual inaccuracies of these variables will be estimated.

The relationship used in the calculation of the failure function $F$ was

$$F = \left(\frac{P}{P_u}\right)^2 + \left(\frac{T}{T_u}\right)^2.$$  \hspace{1cm} (C.3)

If each of the independent variables has an uncertainty expressed as a standard deviation, then the variance of the failure function $F$, $(\Delta F)^2$, may be estimated using the following form [112],

$$(\Delta F)^2 = \left[ \frac{\partial F}{\partial P} \Delta P \right]^2 + \left[ \frac{\partial F}{\partial P_u} \Delta P_u \right]^2 + \left[ \frac{\partial F}{\partial T} \Delta T \right]^2 + \left[ \frac{\partial F}{\partial T_u} \Delta T_u \right]^2,$$

\hspace{1cm} (C.4)

where, $\Delta F = \text{estimated standard deviation of } F$, $\Delta P = \text{estimated standard deviation of } P$, $\Delta P_u = \text{standard deviation of } P_u$, $\Delta T = \text{estimated standard deviation of } T$, and $\Delta T_u = \text{standard deviation of } T_u$. 


Each of the partial derivatives in equation C.4 can be derived. For instance, for variable P,

\[ \frac{\partial F}{\partial P} = \frac{3}{P} \left[ \left( \frac{P}{P_u} \right)^2 + \left( \frac{T}{T_u} \right)^2 \right], \quad (C.5) \]

\[ \frac{\partial F}{\partial P} = \frac{2P}{(P_u)^2}. \quad (C.6) \]

Therefore, the variance of F can be represented in the following form,

\[ (\Delta F)^2 = \left[ \frac{2P\Delta P}{P_u^2} \right]^2 + \left[ \frac{2P^2\Delta P_u}{P_u^3} \right]^2 + \left[ \frac{2T\Delta T}{T_u^2} \right]^2 + \left[ \frac{2T^2\Delta T_u}{T_u^3} \right]^2. \quad (C.7) \]

For e.t.d. steel specimen number 9, the required variables were calculated. They are

- \( P = 1811.4 \) lb,
- \( P_u = 4650.1 \) lb,
- \( T = 205.6 \) in.lb,
- \( T_u = 203.1 \) in.lb.

The standard deviation of \( P_u \) and \( T_u \) were estimated using the following form [112],

\[ \Delta P_u = \sqrt{\frac{\sum (P - P_u, av)^2}{N-1}} \quad (C.8) \]
where $P_{u,av}$ is the average value, $N$ is the number of data points.

It was found that

$$\Delta P_u = 77.1\text{ lb},$$
$$\Delta T_u = 9.\text{ in lb},$$

Next, the standard deviation of the axial force $P$ and the torque $T$ have to be determined. The magnitude of $P$ was calculated using the following equation,

$$P = k_a(mV_a - mV_{ao}), \quad (C.1)$$

where $k_a$ is the static sensitivity of the axial force part of the force transducer, and the $mV$'s are the output of the force transducer.

The variance of $P$ can be calculated as follows,

$$(\Delta P)^2 = \left( \frac{\partial P}{\partial k} \Delta k_a \right)^2 + \left( \frac{\partial P}{\partial mV} \Delta mV \right)^2, \quad (C.9)$$

or $$(\Delta P)^2 = (mV\Delta k_a)^2 + (k_a\Delta mV)^2. \quad (C.10)$$

From the calibration of the force transducer, the magnitude of $\Delta k_a$ was known to be

$$\Delta k_a = 1.17 \text{ lb/mVolt},$$
and the error of the output signal was estimated as

\[ \Delta mV = 0.01 \].

Substituting these values into equation C.10 and using

\[ k_a = 862.9 \text{ lb/mV} \] \quad \text{and} \quad \left( mV_a - mV_{ao} \right) = 2.099 \text{ mV},

the standard deviation of \( P \) was obtained,

\[ \Delta P = 8.97 \text{ lb} \].

Using a similar procedure, the standard deviation of \( T \) was calculated. It was

\[ \Delta T = 5.7 \text{ in. lb}. \]

Finally, the overall error of the computed value of \( F \), the failure function, can be estimated by substituting all the variables into equation C.4,

\[
\begin{align*}
(\Delta F)^2 &= \left[ \frac{(2)(1811.4)(8.97)}{4650^2} \right]^2 + \left[ \frac{(2)(1811.4)^2(77)}{4650^3} \right]^2 + \\
&\quad \left[ \frac{(2)(205.6)(5.7)}{203^2} \right]^2 + \left[ \frac{(2)(205.6)^2(9)}{203^3} \right]^2
\end{align*}
\]

\[ \Delta F = 0.10, \text{ or } 10\%. \]

By studying the relative value of the standard deviation of each variable, and their influence on the total inaccuracy of the
computed $F$, it was found that the error of $T_u$ was the dominant factor.
BIBLIOGRAPHY


100. R. Hill, "The Mathematical Theory of Plasticity," Chapter 12, O.U.P.


