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STUDIES IN THE GEOMETRY OF NUMBERS.
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STUDIES IN THE GEOMETRY OF NUMBERS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Liow-Jing L. Yang, B.A.

*****

The Ohio State University

1978

Reading Committee:

Alan C. Woods
John S. Hsia
Paul Ponomarev

Approved By

[Signature]

Adviser
Department of Mathematics
ACKNOWLEDGMENTS

It is a very pleasant experience to me having graduate study in the Ohio State University. As a foreign student, I feel quite at home in the Department of Mathematics. I am grateful to my teachers and classmates, who broaden my view of Mathematics and many other things. I owe a particular debt of gratitude to Professor A.C. Woods, who first introduced me to the Geometry of Numbers and has always given me helpful advice.
VITA

December 2, 1948 ......... Born - Taipei, Taiwan, Republic of China

1970 ..................... B.A., Department of Mathematics,
National Taiwan University,
Taipei, Taiwan, R.O.C.

1970-1971 .................. Instructor of Mathematics,
R.O.T.C. service, The First
Commissioned Officer Military
School, Chunli, Taiwan, R.O.C.

1971-1978 .................. Teaching Associate, Department
of Mathematics, The Ohio State
University, Columbus, Ohio

Fields of Study: Geometry of Numbers, Number Theory
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1. Lattices and Convex Sets

Let $V_i$ $(1 \leq i \leq n)$ be $n$ linearly independent points in the $n$-dimensional euclidean space $\mathbb{R}^n$. The set $\Lambda$ of all points $P = \sum_{1 \leq i \leq n} a_i V_i$ with integers $a_i$ $(1 \leq i \leq n)$ is called the lattice with basis $V_i$ $(1 \leq i \leq n)$, the lattice $\Lambda$ is also said to be generated by $V_i$ $(1 \leq i \leq n)$. The determinant of $\Lambda$ is defined by $d(\Lambda) = |\det(V_1, \ldots, V_n)|$, the volume of the parallelepipedon generated by $V_i$ $(1 \leq i \leq n)$. For a positive integer $m$, we denote by $\Lambda/m$ the lattice generated by $V_i/m$ $(1 \leq i \leq n)$. Let $K$ be a convex set in $\mathbb{R}^n$ and let $N_m$ be the number of lattice points of $\Lambda/m$ lying in the interior of $K$, then $\text{Vol}(K) = \lim_{m \to \infty} N_m d(\Lambda/m)$, where $\text{Vol}(K)$ is the (Lebesque) volume of $K$ in $\mathbb{R}^n$. First we state a simple principle.

**Dirichlet's pigeonhole principle.** Among $k+1$ elements from $k$ sets, there are two elements belonging to the same set.

Now we are in a good position to show Mordell's [28] proof of a fundamental theorem in the Geometry of Numbers.
Minkowski's convex body theorem. If \( K \) is convex and symmetric about the origin 0 in \( \mathbb{R}^n \) with \( \text{Vol}(K) > 2^n d(\Lambda) \), then \( K \) contains a point of \( \Lambda \) other than 0 in its interior.

Proof. Since \( \text{Vol}(K) = \lim_{m \to \infty} N_m d(\Lambda/m) > 2^n d(\Lambda) \) and \( d(\Lambda/m) = d(\Lambda)/m^n \), we have \( N_m > (2m)^n \) for sufficiently large \( m \). There are only \((2m)^n\) classes of points of \( \Lambda/m \) modulus 2\( \Lambda \). Then by Dirichlet's pigeonhole principle, there are two distinct points \( P_1 \) and \( P_2 \) among these \( N_m \) points in the interior of \( K \) such that \( P_1 - P_2 \) is a point of 2\( \Lambda \), i.e. \( (P_1 - P_2)/2 \) is a point of \( \Lambda \). By the symmetry of \( K \), \( -P_2 \) is in the interior of \( K \); then by the convexity of \( K \), \( (P_1 - P_2)/2 \) is in the interior of \( K \).

For a point set \( K \) in \( \mathbb{R}^n \), a lattice \( \Lambda \) is said to be \( K \)-admissible if there is no point of \( \Lambda \) except possibly 0 lying in the interior of \( K \). The infimum of \( d(\Lambda) \) taken over all \( K \)-admissible lattices \( \Lambda \) is called the critical determinant of \( K \) and denoted by \( \Delta(K) \). Next, we are going to prove a special case of Minkowski's successive minima theorem. In \( \Lambda - \{0\} \) we choose a point \( P_1 \) which has the minimal distance from 0, then choose successively \( P_i \) \((2 \leq i \leq n)\) such that \( P_i \) is linearly independent of \( P_1, \ldots, P_{i-1} \) and has the minimal distance from 0.

Minkowski's successive minima theorem. Let \( B_n \) be the n-dimensional unit ball \( \sum_{1 \leq i \leq n} x_i^2 \leq 1 \), then \( |P_1| \cdots |P_n| \Delta(B_n) \leq d(\Lambda) \).
Proof. Rotating the coordinate axes if necessary, we can assume the coordinates of $P_i$ ($1 \leq i \leq n$) are trianglized:

$P_1 = (p_{11}, 0, 0, \ldots, 0),$

$P_2 = (p_{21}, p_{22}, 0, \ldots, 0),$


\[ P_n = (p_{n1}, p_{n2}, p_{n3}, \ldots, p_{nn}). \]

Consider the n-dimensional ellipsoid

$E: \frac{x_1^2}{|P_1|^2} + \frac{x_2^2}{|P_2|^2} + \ldots + \frac{x_n^2}{|P_n|^2} \leq 1.$

It is easy to see that $\Lambda$ is $E$-admissible. Applying the linear transformation $T: x_i = x_i/|P_i|$ ($1 \leq i \leq n$) to $E$ and $\Lambda$, then $T(E) = B_n$ and $T(\Lambda)$ is $B_n$-admissible.

Thus

$\Delta(B_n) \leq d(T(\Lambda)) = \det(T) d(\Lambda) = d(\Lambda)/|P_1| \ldots |P_n|,$

so

$|P_1| \ldots |P_n| \Delta(B_n) \leq d(\Lambda).$

An infinite sequence of lattices $\Lambda_k$ ($1 \leq k \leq \infty$) is said to converge to a lattice $\Lambda_0$ if each $\Lambda_k$ ($0 \leq k \leq \infty$) has a basis $V_i^{(k)}$ ($1 \leq i \leq n$) such that

$\lim_{k \to \infty} V_i^{(k)} = V_i^{(0)}$ ($1 \leq i \leq n$).

A sequence of lattices $\Lambda_k$ ($1 \leq k \leq \infty$) is said to be bounded if there are two positive constants $C_1$ and $C_2$ independent of $k$ such that $d(\Lambda_k) < C_1$ for all $k$ and $|P| > C_2$ for all non-zero lattice points $P$ in $\Lambda_k$.

Mahler's compactness theorem. Among a bounded sequence of lattices, there is a subsequence converging to a lattice.
Proof. Let $\Lambda_k \ (1 \leq k < \infty)$ be the bounded sequence of lattices, then $d(\Lambda_k) \ (1 \leq k < \infty)$ are bounded below by Minkowski's convex body theorem. By Minkowski's successive minima theorem, for each $\Lambda_k$ we can choose $n$ linearly independent lattice points $P^{(k)}_i \ (1 \leq i \leq n)$ such that $|P^{(k)}_1| \ldots |P^{(k)}_n| \Delta(B_n) \leq d(\Lambda_k)$ and $|P^{(k)}_1| \leq \ldots \leq |P^{(k)}_n|$. Since $\Delta(B_n) \neq 0$ and all $|P^{(k)}_i|$ are bounded below and all $d(\Lambda_k)$ are bounded above, all $|P^{(k)}_i|$ are bounded above too. Then there exists a basis $V^{(k)}_i$ for each $\Lambda_k$ such that $P^{(k)}_i = \sum_{1 \leq j \leq n} c^{(k)}_{ij} V^{(k)}_j \ (1 \leq i \leq n)$, where all $c^{(k)}_{ij}$ are integers, $c^{(k)}_{ij} = 0$ if $j > i$, and $c^{(k)}_{ii} \geq 1$, and $|c^{(k)}_{ij}| \leq c^{(k)}_{ii} / 2$ when $j < i$. So $|V^{(k)}_i| \leq |P^{(k)}_i|$,

$$|V^{(k)}_i| \leq |P^{(k)}_n| + \sum_{1 \leq j < i - 1} |V^{(k)}_j| / 2 \quad (2 \leq i \leq n),$$

it follows that all $|V^{(k)}_i|$ are bounded above. Therefore, for $1 \leq i \leq n$ there is a convergent subsequence of $V^{(k)}_i \ (1 \leq k < \infty)$ with limit $V^{(0)}_i$, and $V^{(0)}_i \ (1 \leq i \leq n)$ do generate a lattice with positive determinant.

A lattice $\Lambda$ is called a critical lattice for $K$ if $\Lambda$ is $K$-admissible and $d(\Lambda) = \Delta(K)$. The determination of the values $\Delta(K)$ for various $K$ is an important problem in the Geometry of Numbers. The usual procedure to find $\Delta(K)$ is to study the properties of the critical lattices for $K$. In most cases, the existence of a critical lattice can be established by using Mahler's compactness theorem [23].
There is an interesting problem mentioned by Cassels [5]: Let $K_1$ and $K_2$ be convex symmetric (about the origin) sets in $\mathbb{R}^n_1$ and $\mathbb{R}^n_2$ respectively and let $K_1 \times K_2$ be the topological product of them in $\mathbb{R}^{n_1+n_2}$; is it always true that $\Delta(K_1 \times K_2) = \Delta(K_1) \Delta(K_2)$? When $n_2 = 1$, we can take $K_2$ to be the 1-dimensional interval $[-1,1]$; the conjecture that $\Delta(K_1 \times [-1,1]) = \Delta(K_1)$ is called the "cylinder conjecture".

In Chapter I, we shall verify this conjecture when $K_1$ is a 4-dimensional sphere by using Woods' "addition method" [37]. Following Bantegnie's work [2] on this problem, we found no particular difficulty in the remaining cases left by him except one knotty case (case 4 of section 6, Chapter I); sections 3 and 4 of Chapter I are prepared mainly for tackling this case. For other cases, it is just a matter of finding the right inequalities to work with.
2. Lattice Packings and Coverings of Spheres

Let \( \Lambda \) be an \( n \)-dimensional lattice and let \( K \) be the \( n \)-dimensional unit sphere centered at the origin \( 0 \) in \( \mathbb{R}^n \). Then we consider the system \( S = K + \Lambda \) of unit spheres centered at the lattice points of \( \Lambda \). \( S \) is called a lattice packing for \( \mathbb{R}^n \) if each point of \( \mathbb{R}^n \) lies in the interior of at most one sphere of \( S \), in other words, all the spheres of \( S \) are non-overlapping. It is clear that the density of \( S \) is \( \text{Vol}(K)/d(\Lambda) \). A lattice \( \Lambda \) gives a lattice packing of \( K \) for \( \mathbb{R}^n \) if and only if \( \Lambda \) is \( 2K \)-admissible; thus the question of determining \( \Lambda(2K) \) is equivalent to the one of finding the densest lattice packing of \( K \). A similar concept to packing is covering. \( S \) is called a lattice covering for \( \mathbb{R}^n \) if each point of \( \mathbb{R}^n \) lies in the interior or on the boundary of at least one sphere of \( S \).

The concepts of packing and covering can be considered more generally [32]; for examples, the unit sphere \( K \) can be replaced by another interesting set, or the lattice \( \Lambda \) can be replaced by a less regular set. Few [13] introduced the concepts of multiple packing and covering; instead of restricting each point in \( \mathbb{R}^n \) to be packed at most once or covered at least once, each point is packed at most \( k \) times in a \( k \)-fold packing and covered at least \( k \) times in a \( k \)-fold covering. In Chapter II, we shall discuss a problem in this connection and obtain some interesting results.
3. Binary Indefinite Quadratic Forms

Let \( f(x,y) = ax^2 + bxy + cy^2 \) be an indefinite binary form with real coefficients and \( f(x,y) \neq 0 \) for all integral \( (x,y) \neq (0,0) \), then \( f = 0 \) represents a pair of straight lines \( AB, CD \) (Figure 1), which divide the plane \( \mathbb{R}^2 \) into four parts. Take one of them, say \( BOC \); let \( M \) be the set of the integral points lying inside the angle \( BOC \) and let \( \Pi \) be the boundary of the convex hull of \( M \), then \( \Pi \) is a two-way infinite convex polygon. There are three similar polygons \( \Pi', -\Pi \) and \( -\Pi' \) in the three other angles. These four polygons are called the planar polygon associated with \( f \).

![Figure 1. The planar polygon](image-url)
The vertices and sides of the planar polygon are in a one-to-one correspondence, namely, each vertex \( P_i \) \((-\infty < i < \infty)\) corresponds to the side \( P_{i-1}P_{i+1} \) which is parallel to \( OP_i \). Let \( a_i \) be the ratio of the length of \( P_{i-1}P_{i+1} \) to the length of \( OP_i \). If \( P_0 = (p,q), P_{-1} = (p',q') \) and \((x',y') = xP_0 + yP_{-1} = (xp+yp', xq+yq')\); then the form \( f'(x,y) = f(x',y') = f(xp+yp', xq+yq') = a'x^2 + b'xy + c'y^2 \) is equivalent to \( f(x,y) \) and is Markov reduced, i.e. one root \( \alpha \) of \( f' = 0 \) is greater than 1 and the other root \( \beta \) with \(-1 < \beta < 0\). The vectors \( OP_0 \) and \( OP_{-1} \) are called the first and the second basic vectors of \( f' \). The linear transformation \( x = a_0 x' + y', y = x' \) will transform \( f' \) into the neighbouring form which has \( OP_1 \) and \( OP_0 \) as the first and the second basic vectors. In this way we have

\[
\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} \quad \text{and} \quad -\beta = [0, a_{-1}, a_{-2}, \ldots].
\]

Markoff [24] noted that

\[
\sqrt{b'^2 - 4a'c'} = \frac{\sqrt{b^2 - 4ac}}{|f(P_0)|} = \alpha - \beta = [a_0, a_1, a_2, \ldots] + [0, a_{-1}, a_{-2}, \ldots].
\]

The above polygon structure was given by Klein [20] for a geometrical interpretation of a simple continued fraction expansion of an irrational number. Venkov [35] found a similar polyhedron structure for the 3-dimensional case, i.e. for ternary indefinite quadratic forms. In Chapter III we shall use Venkov's method to compute the
polyhedra of the eleven forms obtained by him.

In Chapter IV we shall return to a problem on binary indefinite forms, it can be simply stated as following. Lagrange proved that every quadratic irrational number has a periodic simple continued fraction expansion from some point on [17], thus its partial quotients are bounded; now in a fixed quadratic field \( \mathbb{Q}(\sqrt{d}) \), can we find infinitely many nonequivalent irrational numbers such that the partial quotients of their simple continued fraction expansions are bounded uniformly? This question is to be answered affirmatively in Chapter IV. A computer was used to help find these numbers.

In the following the notations used in one chapter are independent of the ones in other chapters.
CHAPTER I

THE CRITICAL DETERMINANT OF A SPHERICAL CYLINDER IN $\mathbb{R}^5$

1. Introduction

Let $K$ be a point set in $\mathbb{R}^n$. The critical determinant $\Delta(K)$ of $K$ is defined as the infimum of the determinants of all $K$-admissible lattices $\Lambda$ in $\mathbb{R}^n$, i.e. all lattices $\Lambda$ which have no lattice point in the interior of $K$ except 0. A $K$-admissible lattice $\Lambda$ is said to be critical for $K$ if $d(\Lambda) = \Delta(K)$. Now let $K = K \times [-1,1]$, a cylinder in $\mathbb{R}^{n+1}$ with height 2 and cross-section $K$. Since any $K$-admissible lattice $\Lambda$ with basis $V_1, \ldots, V_n$ gives a $K$-admissible lattice $\Lambda$ with basis $(V_1,0), \ldots, (V_n,0), (0,1)$; we always have $\Delta(K) \leq \Delta(K)$. When $K$ is convex and symmetric about 0, it has been conjectured that $\Delta(K) = \Delta(K)$. If $K$ is 2-dimensional, the conjecture was proved to be true by Chalk and Rogers [6] and Yeh [39] independently. When $K$ is a 3-dimensional sphere, Woods [38] has shown that the equality holds. Bantegnie [2] has given some results when $K$ is a 4-dimensional sphere and we intend to complete this case.
Theorem. If $K$ is the 4-dimensional unit sphere centered at the origin, then $A(K)=A(K)$.

2. Some Known Results

Let $V_1,\ldots,V_n$ be $n$ linearly independent points of a lattice $\Lambda$ in $\mathbb{R}^n$. The convex hull $H$ of $\pm V_1,\ldots,\pm V_n$ is called an $n$-dimensional octahedron. $H$ is called a lattice octahedron if $H$ contains no point of $\Lambda$ except $\pm V_1,\ldots,\pm V_n$ and 0.

Lemma 1.1. If $H$ is a lattice octahedron with vertices $\pm V_1,\ldots,\pm V_n$ in $\Lambda$, renumbering $V_1,\ldots,V_n$ if necessary, then

(i) if $n=2$; $V_1$ and $V_2$ is a basis for $\Lambda$;
(ii) if $n=3$; $V_1,V_2$ and $Y$ is a basis for $\Lambda$, where

$Y=V_3$ or $(V_1+V_2+V_3)/2$;

(iii) if $n=4$; $V_1,V_2,V_3$ and $Y$ is a basis for $\Lambda$, where

$Y$ is one of $V_4$, $(V_2+V_3+V_4)/2$, $(V_1+V_2+V_3+V_4)/2$, $(\pm V_1\pm V_2\pm V_3\pm V_4)/3$, $(\pm 2V_1\pm V_2\pm V_3\pm V_4)/4$ and $(\pm 2V_1\pm 2V_2\pm V_3\pm V_4)/5$, the signs are independent of each other.

The case $n=2$ is well known [17], Minkowski [26] proved the case $n=3$, Brunngraber [4] proved the case $n=4$ and Wolff [36] had a proof, Mordell [29] has given a simpler proof for all these cases.
From now on $X=(x_1, x_2, x_3, x_4)$, $Y=(y_1, y_2, y_3, y_4)$; $X_i, Y_i, P_i$ ($i=1, 2, 3, 4, 5$) denote points in $\mathbb{R}^4$; $X=(x, x_5)$, $O=(0, 0, 0, 0)$, $O=(0, 0)$, $XY=\sum_{1\leq i \leq 4} x_i y_i$, $x^2=xx$, $|x|=/x^2$, $K=\{X: |X| \leq 1\}$.

**Lemma 1.2.** For any positive integer $r \geq 2$,
\[ \sum_{1 \leq i < j \leq r} (x_i - x_j)^2 = \sum_{1 \leq i \leq r} x_i^2 - \left( \sum_{1 \leq i \leq r} x_i \right)^2. \]

**Lemma 1.3.** If $|x_i| < 1$ ($1 < i < r$), $r > 2$, then
\[ \sum_{1 \leq i < j \leq r} (x_i - x_j)^2 \leq r^2. \]

Lemma 1.3 is a corollary of Lemma 1.2, which is an identity.

Applying the general theory developed by Woods [37] to the 4-dimensional sphere $K$, we have the following three lemmas. We will only give the proofs of the first two.

**Lemma 1.4.** If a critical lattice $\Lambda$ of $K$ contains $(0, 1)$, then $\Delta(K) = \Delta(K)$.

Proof. Let $\ell$ be any straight line parallel to but different from the 5th-axis and passing through the interior of $K$, then $\ell$ contains no lattice point of $\Lambda$, since $\Lambda$ contains $(0, 1)$ and is $K$-admissible. Now project $\Lambda$ along the vector $(0, 1)$, i.e. the 5th-axis, onto the 4-dimensional space $x_5=0$, then we get a $K$-admissible 4-dimensional lattice $\Lambda$ and $d(\Lambda) = d(\Lambda)/|0, 1| = d(\Lambda)$. Thus $\Delta(K) \leq d(\Lambda) = d(\Lambda) = \Delta(K) \leq \Delta(K)$, then $\Delta(K) = \Delta(K)$. 

Lemma 1.5. If a critical lattice $\Lambda$ of $K$ contains only points with integral 5th-coordinates, then $\Delta(K) = \Delta(K)$.

Proof. Since the 5th-coordinates of the points of $\Lambda$ are integers, the intersection of $\Lambda$ and the 4-dimensional space $x_5 = 0$ is a 4-dimensional $K$-admissible lattice $\Lambda$. Since all other points of $\Lambda$ off the space $x_5 = 0$ have distance at least 1 from the space $x_5 = 0$, we have $d(\Lambda) < d(\Lambda)$. Thus $\Delta(K) = d(\Lambda) < d(\Lambda) = \Delta(K) < \Delta(K)$, then $\Delta(K) = \Delta(K)$.

Lemma 1.6. If $\Delta(K) < \Delta(K)$, then there exists a critical lattice $\Lambda$ of $K$ having five linearly independent points $X_i = (X_i, 1)$ (i = 1, 2, 3, 4, 5) on the top face $(K, 1)$ and at least two $X_i$ in the interior of $K$.

The following lemma is a result due to Cleaver [7], Woods [38] had a simpler proof for it.

Lemma 1.7. If $\Lambda$ is a lattice with $d(\Lambda) = 1$ in $\mathbb{R}^4$ and with no point in the interior of $K$ except 0, then any 4-dimensional sphere of radius 1 contains a point of $\Lambda$.

Consider a 4-dimensional regular simplex with side length 2 and the system of five 4-dimensional unit spheres centered at the vertices of the simplex. Let $\sigma_4$ denote the ratio of the volume of the part of the simplex covered by the spheres to the volume of the whole simplex. The following lemma is just a special case of Rogers' result [31] on packings of equal spheres.
Lemma 1.8. The density of a packing of equal 4-dimensional spheres in $\mathbb{R}^4$ is less than or equal to $\sigma_4$.

Schlafli [33] gave a method of computing the value of $\sigma_4$ and we can find $\sigma_4 = 3\pi\sqrt{5}(\arccos 4 - 2\pi/5)/2$ in Coxeter's paper [8].

Lemma 1.9. If $x_i$ $(i=1, 2, 3, 4, 5)$ are in $K$, let $H$ be the simplex (convex hull) generated by them, then $\text{Vol}(H) \leq 25\sqrt{5}/384$.

Proof. It is well known that among all triangles inside a 2-dimensional circle, the regular one inscribed has the maximum volume. Starting from this it is easy to see that among all the simplices inside $K$, the regular one has the maximum volume $25\sqrt{5}/384$.

3. Some Inequalities

Lemma 1.10. If $0 \leq r_{i} \leq 1$ $(1 \leq i \leq k)$, $k \geq 1$, then

$$r_1 + r_2 + ... + r_k \leq r_1 r_2 ... r_k + (k-1).$$

Proof. If $k=1$, $r_1 \leq r_1 + (1-1)$, true. If $k \geq 2$, then $(1-r_1)(1-r_k) \geq 0$, which implies $r_1 + r_k \leq (r_k r_1) + 1$. Then

$$r_1 + r_2 + ... + r_{k-1} + r_k \leq (r_k r_1) + r_2 + ... + r_{k-1} + 1$$

$$\leq (r_k r_1) r_2 ... r_{k-1} + (k-2) + 1,$$

by induction.

Lemma 1.11. If $0 \leq r_{i} \leq 1$ $(i=1, 2, 3, 4, 5)$,

$$2r_3 r_4 + r_5 = t, r_1 r_2 r_3 r_4 r_5 = s,$$ then $r_1 + r_2 \leq s/(t-2) + 1.$
Proof. By Lemma 1.10, \( r_3 r_4 r_5 \geq r_3 + r_4 + r_5 - 2 = t - 2 \), then \( r_1 r_2 \leq s/(t-2) \). By Lemma 1.10 again, \( r_1 + r_2 \leq r_1 r_2 + 1 \leq s/(t-2) + 1 \).

**Lemma 1.12.** If \( 0 < a < b < c < d < 1 \), \( abcd \leq 3/4 \) and \( b^2 c^2 \leq 1 - a/(3+4a) \), then \( f = a^2 + b^2 + c^2 + d^2 \) attains maximum only when \( d = 1 \).

Proof. Suppose \( d < 1 \) and we are to show that \( f \) gets greater value somewhere else. If \( a < b \), increasing \( d \) to \( d + \varepsilon \) and decreasing \( b \) to \( b - \varepsilon \) (\( \varepsilon \) is a sufficiently small positive number), we can get a greater value for \( f \). If \( a = b < c \), then \( a^2 c d \leq 3/4 \) and \( a^2 c^2 \leq 1 - a/(3+4a) \); increasing \( d \) to \( d + \varepsilon \) and decreasing \( c \) to \( c - \varepsilon \), we can get a greater \( f \). If \( 1/2 < a = b < c \), then \( a^3 d \leq 3/4 \) and \( a^4 + a/(3+4a) \leq 1 \); since \( a^4 + a/(3+4a) \) increases as \( a > 1/2 \), we can get a greater \( f \) by decreasing \( a = b = c \) to \( a - \varepsilon \) and increasing \( d \) to \( d + 3\varepsilon \). If \( a \leq 1/2 \), then \( f \leq 1/4 + 3 = 13/4 \); but \( f \) assumes a greater value \( 55/16 \) when \( a = 3/4 \), \( b^2 = 7/8 \), \( c = d = 1 \).

**Lemma 1.13.** If \( 0 < a < b < c < 1 \), \( abc \leq 3/4 \) and \( b^2 c^2 \leq 1 - a/(3+4a) \) then \( f = a^2 + b^2 + c^2 < 2.52 \).

Proof. If \( a \leq 1/2 \), then \( f \leq 1/4 + 2 \), done. If \( 1/2 < a \) and \( c = 1 \), then \( ab \geq 3/4 \) and \( b^2 \leq 1 - a/(3+4a) \); in case \( b^2 < 1 - a/(3+4a) \), we can increase \( b \) to \( b + \varepsilon \) and decrease \( a \) to \( a - \varepsilon \) to get a greater \( f \); now \( ab \leq 3/4 \) and \( b^2 = 1 - a/(3+4a) \), then \( a^2 (1 - a/(3+4a)) \leq 9/16 \), which
implies \( a < 0.804 \); since \( a^2 - a/(3+4a) \) increases as \( a > 1/2 \),
\[
f = a^2 + b^2 + c^2 \leq a^2 + (1-a/(3+4a)) + 1
\]
\[
< (0.804)^2 - 0.804/(3+4\cdot0.804) + 2 < 2.52.
\]
If \( 1/2 < a < b < c < 1 \), \( abc \leq 3/4 \) and \( b^2c^2 < 1-a/(3+4a) \), then we can increase \( c \) to \( c+\varepsilon \) and decrease \( a \) to \( a-\varepsilon \) to get a greater \( f \); now \( abc \leq 3/4 \) and \( b^2c^2 = 1-a/(3+4a) \), then
\[
a^2(1-a/(3+4a)) \leq 9/16,
\]
which implies \( a < 0.804 \); by Lemma 1.10,
\[
f = a^2 + (b^2 + c^2) \leq a^2 + ((1-a/(3+4a)) + 1) < 2.52.
\]

4. Lattice Points on \( K \times 1 \)

From now on we assume \( \Delta(K) < \Delta(K) \) and shall obtain a contradiction for each case in the following sections.

By Lemma 1.6, there exists a critical lattice \( \Lambda \) for \( K \) having five linearly independent points on \( K \times 1 \). By Lemma 1.4, \((0,1)\) is not a lattice point of \( \Lambda \). Consider all the lattice points \((X,1)\) of \( \Lambda \) on \( K \times 1 \). Choose \( Y_i = (Y_i,1) \) \((i=1,2,3,4)\) such that \( Y_1 \) has the least distance from \( 0 \) and for \( i \geq 2 \), \( Y_i \) is linearly independent of \( Y_1, \ldots, Y_{i-1} \) and has the least distance from \( 0 \) among all \( X \) which are linearly independent of \( Y_1, \ldots, Y_{i-1} \). It is clear that \( 0 \leq |Y_1| \leq |Y_2| \leq |Y_3| \leq |Y_4| \leq 1 \). \( K \) is so symmetric that we can assume the coordinates of \( Y_i \) \((i=1,2,3,4)\) are trianglized; i.e. \( Y_1 = (y_{11},0,0,0) \), \( Y_{11} > 0 \), \( Y_2 = (y_{21},y_{22},0,0) \), \( Y_3 = (y_{31},y_{32},y_{33},0) \), \( Y_4 = (y_{41},y_{42},y_{43},y_{44}) \).
Let $E$ denote the ellipsoid
$$x_1^2/|Y_1|^2 + x_2^2/|Y_2|^2 + x_3^2/|Y_3|^2 + x_4^2/|Y_4|^2 \leq 1 \text{ in } \mathbb{R}^4.$$ It is clear that $E$ is contained in $K$.

**Lemma 1.14.** There is no point $X=(X,1)$ of $\Lambda$ with $X$ in the interior of $E$.

**Proof.** By way of contradiction, suppose there is $X=(X,1)$ of $\Lambda$ with $X$ in the interior of $E$. Then
$$x_1^2/|Y_1|^2 + x_2^2/|Y_2|^2 + x_3^2/|Y_3|^2 + x_4^2/|Y_4|^2 < 1,$$ which implies $|X|^2/|Y_4|^2 < 1$, i.e. $|X|^2 < |Y_4|^2$. From the way we choose $Y_4$, $X$ must be linearly dependent of $Y_i$ ($i=1,2,3$), i.e. $x_4=0$. Now $x_1^2/|Y_1|^2 + x_2^2/|Y_2|^2 + x_3^2/|Y_3|^2 < 1$, which implies $|X|^2 < |Y_3|^2$, then $x_3=0$. Similarly, we can get $x_2=0$. Then $X=(x_1,0,0,0)$ and $x_1^2/|Y_1|^2 < 1$, which imply $|X|=|x_1|<|Y_1|$, a contradiction to the fact that $Y_1$ has the least distance from 0.

**Lemma 1.15.** If there are exactly $m$ hyperplanes parallel to $x_5=0$, with $0 \leq x_5 < 1$ and containing points of $\Lambda$, then $|Y_1||Y_2||Y_3||Y_4| \leq m/(m+1)$.

**Proof.** These $m$ hyperplanes are $x_5=i/m$ ($0 \leq i \leq m-1$). Since $\Lambda$ is $K$-admissible and $E$ is contained in $K$, the elliptic cylinder $E=E\times[-1-1/m,1+1/m]$ contains no point of $\Lambda$ except $0$ in its interior, i.e. $\Lambda$ is $E$-admissible. Apply the linear transformation $T$: $x_i'/x_i(|Y_i|(i=1,2,3,4)$, $x_5'=x_5/(1+1/m)$ to $E$ and $\Lambda$. Then the lattice $T(\Lambda)$ is admissible for $T(E)=K$. Therefore, we have
\[ \Delta(K) \leq d(T(A)) = \det(T) d(A) = \Delta(K)/(|Y_1| |Y_2| |Y_3| |Y_4| (1+1/m)). \]

Since \( \Delta(K) \neq 0 \), \( |Y_1| |Y_2| |Y_3| |Y_4| \leq 1/(1+1/m) = m/(m+1) \).

Now we consider all lattice points of \( A \) except \( Y_1 \) on the top face \( K \times 1 \). In the same way as we chose \( Y_1 \), we can choose \( P_i = (P_i,1) \) (i=1,2,3,4). Similarly, in \( \mathbb{R}^4 \) there is an ellipsoid \( E_1 \) contained in \( K \) with vertices \( \pm P_i \) (i=1,2,3,4) and there is no point \( X=(x_1,\ldots,x_5) \) of \( A \) except \( \pm Y_1,0 \) with \( X \) in the interior of \( E_1 \) and \( |x_5| \leq 1 \).

**Lemma 1.16.** Let \( T_2 \) be the linear transformation
\[ x_i' = x_i \quad (i=1,2,3,4), \quad x_5' = x_5 + x_1/(m(1+|Y_1|)) \]; then \( T_2(A) \) is admissible for the elliptic cylinder
\[ K' = E_1 \times [-1-|Y_1|/(m(1+|Y_1|)),1+|Y_1|/(m(1+|Y_1|))]. \]

**Proof.** By way of contradiction assume
\[ (X,x_5) = (x_1,x_2,x_3,x_4,x_5) \neq 0 \] is a lattice point of \( A \) and
\[ T_2((X,x_5)) = (x_1,x_5 + x_1/m(1+|Y_1|)) \] is in the interior of \( K' \), then \( X \) is in the interior of \( E_1 \) and
\[ |x_5 + x_1/(m(1+|Y_1|))| < 1+|Y_1|/(m(1+|Y_1|)). \] Since \( X \) is in the interior of \( E_1 \), either \( (X,x_5) = \pm Y_1 \) or \( |x_5| \leq 1 \). But
\[ Y_1 = (Y_1,1) = (y_{11},0,0,0,1), \quad y_{11} > 0; \]
then
\[ T_2(\pm Y_1) = \pm (y_{11},0,0,0,1+y_{11}/(m(1+|Y_1|))) = \pm (y_{11},1+|Y_1|/m(1+|Y_1|)) \]
which is not in the interior of \( K' \). Therefore \( |x_5| \geq 1 \), then \( |x_5| \geq 1 + 1/m \). Now \( X \) is in the interior of \( E_1 \),
\[ |x_5| \geq 1 + 1/m \] and
\[ |x_5 + x_1/m(1+|Y_1|)| < 1+|Y_1|/m(1+|Y_1|). \]
Since \( E_1 \) is contained in \( K \), \( |x_1| \leq |X| \leq 1 \), then
Lemma 1.17. \[ |P_1| |P_2| |P_3| |P_4| \leq (m+m|Y_1|)/(m+(m+1)|Y_1|). \]

Proof. The linear transformation defined by \( T_1(((P_i, 0)) = (P_i, 0)/|P_i| \) \((i=1,2,3,4), T_1((0,1))=(0,1) \) will transform \( E^4\times[-1,1] \) to \( K \). Let \( T_3 \) be the linear transformation \( x_i'=x_i \) \((i=1,2,3,4), x_5'=x_5/(1+|Y_1|/m(1+|Y_1|)) \).
It is clear that \( T_1(T_3(K')) = K \). By Lemma 1.16, \( T(A) \) is admissible for \( K' \), so \( T_1(T_3(T_2(A))) \) is admissible for \( K \).
Then \( \Delta(K) \leq d(T_1(T_3(T_2(A)))) = \det(T_1)\det(T_3)\det(T_2)d(A) \)
\[ = (|P_1| |P_2| |P_3| |P_4|)^{-1}(1+|Y_1|/m(1+|Y_1|))^{-1}.1.\Delta(K). \]
Therefore, \[ |P_1| |P_2| |P_3| |P_4| \leq m(1+|Y_1|)/(m+(m+1)|Y_1|). \]

In the process of choosing \( P_i \) \((i=1,2,3,4), \) we can have either \( P_1=Y_2, P_2=Y_3; \) or \( P_1=Y_2, P_2=c_1Y_1+c_2Y_2, P_3=Y_3, \)
where \( c_1, c_2 \) are real numbers and \( c_1\neq 0, |P_2|<|Y_3|; \)
or \( P_1=c_1Y_1, P_2=Y_2, P_3=Y_3, \) where \( c_1<0 \) and \( |P_1|<|Y_2|=|P_2|. \)
If \( P_1=Y_2, P_2=Y_3, \) then we set \( X_i=Y_i \) \((i=1,2,3,4), \)
by Lemmas 1.6 and 1.14, it is possible to choose \( X_5=(X_5,1) \)
on \( K\times1 \) such that \( X_5 \) is linearly independent of \( X_i \)
\((i=1,2,3,4) \) and the 4-dimensional convex hull \( H \) of \( X_i \)
\((i=1,2,3,4,5) \) contains no lattice point of \( A \) except \( X_i \)
\((i=1,2,3,4,5). \) Then by Lemmas 1.15 and 1.17 and the fact that \( 0<|Y_1|\leq|Y_2|\leq|Y_3|\leq|Y_4|\leq1 \) and
\[ 0<|P_1|\leq|P_2|\leq|P_3|\leq|P_4|\leq1, \] we have
\[ |X_1| |X_2| |X_3| |X_4| \leq m/(m+1) \] and
\[|X_2|^2|X_3|^2 \leq m(1+|X_1|)/(m+(m+1)|X_1|).\]

If \(P_1=Y_2, P_2=c_1Y_1+c_2Y_2\) (\(c_1\neq 0\)), \(P_3=Y_3\); then we set \(X_i=Y_i\) (\(i=1,2,3,4\)) and choose \(X_5\) as above. We have the same convex hull \(H\) and the same two inequalities.

If \(P_1=c_1Y_1\) (\(c_1<0\)), \(P_2=Y_2, P_3=Y_3\) and \(|P_1|<|Y_2|=|P_2|\), then we set \(X_i=Y_i\) (\(i=1,2,3,4\)), and take \(X_5=P_1=(P_1,1)\). It is easy to see that

\[|X_1|/|X_5|/|X_3|/|X_4| \leq m/(m+1)\]

and

\[|X_5|^2/|X_3|^2 \leq m(1+|X_1|)/(m+(m+1)|X_1|).\]

Now consider the convex hull \(H\) of \(X_i\) (\(i=1,2,3,4,5\)). Claim: \(H\) contains no lattice point of \(\Lambda\) except \(X_i\) (\(i=1,2,3,4,5\)). Suppose \(P\) is a lattice point of \(\Lambda\) in \(H\), then \(P=(P,1)\) and \(P=\sum_{1\leq i\leq 5} b_iX_i\), where \(b_i\geq 0\) and \(\sum_{1\leq i\leq 5} b_i=1\). If \(b_5=0\), then \(P\) must be one of \(X_i\) (\(i=1,2,3,4\)) by Lemma 1.14. If \(b_5\neq 0\) and \(b_4\neq 0\), then

\[\sum_{1\leq i\leq 5} b_i|X_i|=|X_4|\leq|P|=|\sum_{1\leq i\leq 5} b_iX_i|\]

by the definition of \(X_4\).

Then \(\sum_{1\leq i\leq 5} b_i|X_4|\leq\sum_{1\leq i\leq 4} b_i|X_4|+b_5|X_5|\), which implies \(b_5|X_4|\leq b_5|X_5|\), then \(|Y_4|=|X_4|\leq|X_5|=|P_1|<|Y_2|\leq|Y_4|\), a contradiction. Therefore, when \(b_5\neq 0\), we have \(b_4=0\); similarly, we have \(b_3=b_2=0\). Now \(b_5\neq 0\), \(b_4=b_3=b_2=0\), i.e. \(P=b_1X_1+b_5X_5=(b_1+c_1b_5)X_1, c_1<0, b_1+b_5=1\); since \(\Lambda\) is \(K\)-admissible and \((0,1)\) is not in \(\Lambda\), we have \(b_1=0\), i.e. \(P=X_5\). Thus \(H\) contains no lattice point of \(\Lambda\) except \(X_i\) (\(i=1,2,3,4,5\)).
Summing up the above discussion, we have the following lemma.

**Lemma 1.18.** On the top face $K \times 1$ of $K$, there are five linearly independent points $X_i = (X_i^1, 1)$ of $\Lambda$ such that the convex hull $H$ of them contains no further point of $\Lambda$ and among the five values $|X_i^1|$ ($i=1,2,3,4,5$), there are four of them, say $a,b,c,d$, satisfying $0 < a < b < c < d < 1$, $abcd < m/(m+1)$ and $b^2 c^2 < 1 - a/(m+(m+1)a)$.

From now on we shall work on the set $\{X_i : i=1,2,3,4,5\}$ the order is not important.

**Lemma 1.19.** Let $H$ be the convex hull of $\pm X_i$ ($i=1,2,3,4,5$), then $H$ is a lattice octahedron for $\Lambda$.

**Proof.** By way of contradiction, suppose there is $X=(X,X_5)$ of $\Lambda$ in $H$ and $X \neq 0$, $\pm X_i$ ($i=1,2,3,4,5$). It is clear that $H$ is contained in $K$ and by Lemma 1.18, $\pm (H \times 1)$ contain no point of $\Lambda$ except $\pm X_i$ ($i=1,2,3,4,5$). Then $X$ must lie on the boundary of $K$ and $|x_5|<1$. Since $K$ is strictly convex, $(X, \pm 1)$ must be two lattice points among $\pm X_i$ ($i=1,2,3,4,5$). Then $(0, \pm 1-x_5)$ are lattice points of $\Lambda$. But this contradiction to the fact that $\Lambda$ is $K$-admissible and $(0,1)$ is not in $\Lambda$.

**Lemma 1.20.** $\sum_{1 \leq i \leq 5} |X_i| \leq 4+m/(m+1) < 5$, and

$\sum_{1 \leq i \leq 5} X_i^2 \leq 4+(m/(m+1))^2 < 5$. 
Proof. By Lemmas 1.10 and 1.18,
\[ \sum_{1 \leq i \leq 5} |X_i| \leq a+b+c+d+1 \leq abcd+3+1 \leq \frac{m}{(m+1)+4}. \]
Similarly,
\[ \sum_{1 \leq i \leq 5} X_i^2 \leq a^2+b^2+c^2+d^2+1 \leq a^2b^2c^2d^2+3+1 \]
\[ \leq \left( \frac{m}{(m+1)} \right)^2+4. \]

Lemma 1.21. If \( m=3 \), then \[ \sum_{1 \leq i \leq 5} X_i^2 < 4.52. \]
Proof. By Lemma 1.18,
\[ \sum_{1 \leq i \leq 5} X_i^2 \leq a^2+b^2+c^2+d^2+1, \]
\[ 0<a<b<c<d<1, \quad abcd \leq 3/4 \quad \text{and} \quad b^2c^2 \leq 1-a/(3+4a). \]
Then by Lemmas 1.12 and 1.13,
\[ a^2+b^2+c^2+d^2+1 < 2.52+1+1 = 4.52. \]

Let \( \Lambda' \) be the sublattice of \( \Lambda \) generated by \( X_i \)
\( (i=1,2,3,4,5) \). It is clear that \( d(\Lambda')/d(\Lambda)=h \cdot m \), where \( h \) is a positive integer. By Lemma 1.9,
\[ d(\Lambda') < 4! \cdot 25\sqrt{5}/384 = 25\sqrt{5}/16. \]
The following lemma is a special case of Bantegnie's results [2].

Lemma 1.22. \( d(\Lambda')/d(\Lambda) = h \cdot m \leq 7. \)
Proof. There are \( m \) layers of lattice points parallel to the hyperplane \( x_5=0 \) and with \( -1/2<x_5<1/2 \).
Project these lattice points along the \( x_5 \)-axis on the hyperplane \( x_5=0 \). Since \( \Lambda \) contains no point of the type \( (0,x_5) \) with \( 0<|x_5|<1 \), this projection is one-one. Let \( S \)
denote the image of this projection. Since \( \Lambda \) is \( K \)-admissible, \( K/2+\Lambda \) forms a lattice packing for \( R^5 \), then
\( \frac{K}{2} + S \) forms a packing (not necessarily a lattice packing) on the hyperplane \( x_5 = 0 \), i.e. the space \( \mathbb{R}^4 \). The density of this packing of equal 4-dimensional spheres in \( \mathbb{R}^4 \) is 
\[
\frac{\text{Vol}(K/2) \cdot h \cdot m / d(\Lambda')}{\sigma_4},
\]
which is less than
\[
\sigma_4 = \frac{3\pi\sqrt{5}}{2} (\text{arcsec} 4 - 2\pi/5)/2,
\]
by Lemma 1.8. Therefore,
\[
h \cdot m \leq \frac{d(\Lambda')}{\sigma_4} \frac{\text{Vol}(K/2)}{\text{Vol}(K/2)}
\]
\[
< \frac{25\sqrt{5}}{16} \cdot \frac{3\pi\sqrt{5}}{2} (\text{arcsec} 4 - 2\pi/5) \frac{2^h}{\pi^2/2} < 7.4.
\]

By Lemma 1.22, we only need to consider the following cases: \( m = 1; m = 2; h = 1, m = 3; h = 1, m = 4; h = 1, m = 5; m = 1, m = 6; h = 2, m = 3; h = 1, m = 7. \)

5. \( m = 1; m = 2; h = 1, m = 5 \)

If \( m = 1 \), then \( \Lambda \) contains only points with integral 5th-coordinates, a contradiction to Lemma 1.5.

If \( m = 2 \), let \( \Lambda' \) denote the 4-dimensional lattice of all the lattice points of \( \Lambda \) on the hyperplane \( x_5 = 0 \), then
\[
d(\Lambda') = 2d(\Lambda) = 2\Delta(K) < 2\Delta(K) = 2 \cdot 1/2 = 1,
\]
where the value \( \Delta(K) = 1/2 \) was a result by Korkine and Zolotareff [21]. Then by Lemma 1.7, \( K \times 1/2 \) contains a lattice point of \( \Lambda \) in its interior, a contradiction to the fact that \( \Lambda \) is \( K \)-admissible.

The case \( h = 1, m = 5 \) was settled by Bantegnie [2].
6. $h=1, m=3$

Since $d(\Lambda')/d(\Lambda) = h \cdot m = 3$ and $m \neq 1$, by renumbering $X_i$ (i=1,2,3,4) if necessary, we may take $X_i$ (i=1,2,3,4) and $Y$ as a basis for $\Lambda$, where $Y$ is one of the following points: $X_5/3$, $(\pm X_1 + X_5)/3$, $(\pm X_1 + X_2 + X_5)/3$, $(\pm X_1 + X_2 + X_3 + X_5)/3$, $(-X_1 - X_2 + X_3 + X_5)/3$, $(\pm X_1 + X_2 + X_3 + X_4 + X_5)/3$, $(-X_1 - X_2 + X_3 + X_4 + X_5)/3$.

By Lemma 1.19, the convex hull $H$ of $\pm X_i$ (i=1,2,3,4,5) is a lattice octahedron for $\Lambda$, then the 4-dimensional convex hull of $\pm X_i$ (i=1,2,3,5) contains no point of $\Lambda$ except 0 and $\pm X_i$ (i=1,2,3,5). Then by Lemma 1.1, $Y$ can not be one of $X_5/3$, $(\pm X_1 + X_5)/3$, $(\pm X_1 + X_2 + X_5)/3$.

If $Y$ is one of $(-X_1 - X_2 + X_3 + X_5)/3$ and $(-X_1 + X_2 + X_3 + X_4 + X_5)/3$, which have 5th-coordinates 0 and 1, then $m=1$, done.

Therefore, $Y$ can only be one of the following four points: $(\pm X_1 + X_2 + X_3 + X_5)/3$, $(X_1 + X_2 + X_3 + X_4 + X_5)/3$, $(-X_1 - X_2 + X_3 + X_4 + X_5)/3$.

1) $Y = (X_1 + X_2 + X_3 + X_5)/3$.

Consider lattice points $Y - X_i$ (i=1,2,3,5), which have the same 5th-coordinate 1/3, then $Y - X_i$ (i=1,2,3,5) are not in the interior of $K$, i.e. $9(Y - X_i)^2 \geq 9$ (i=1,2,3,5). Adding these four inequalities together, we obtain $\sum_{i<j} (X_i - X_j)^2 \geq 36$. By $i,j=1,2,3,5$
Lemmas 1.2 and 1.3, \[ \sum_{i<j} (X_i - X_j)^2 \leq 16 \] equality holds only when \( X_1 + X_2 + X_3 + X_5 = 0 \). So
\[ 4 \geq (x_1^2 + x_2^2 + x_3^2 + x_5^2) \geq 4, \]
then the equality holds and \( X_1 + X_2 + X_3 + X_5 = 0 \), i.e. \( Y = (0, 4/3) \). By Lemma 1.6, there exists a lattice point \( X = (X, 1) \) of \( \mathbb{A} \) with \( |X| < 1 \), then \( X - Y = (X, -1/3) \) is in the interior of \( K \), a contradiction.

2) \( Y = (-X_1 + X_2 + X_3 + X_5)/3 \).

Consider lattice points \( Y \) and \( Y - X_i \) (i=2,3,5), which have 5th-coordinates 2/3 and -1/3, then \( 9Y^2 \geq 9 \) and
\[ 3(Y-X_i)^2 \geq 3 \] (i=2,3,5). Adding these four inequalities together, we obtain
\[ 2X_1^2 + 3(x_2^2 + x_3^2 + x_5^2) - 2(x_1x_2 + x_1x_3 + x_1x_5) \geq 18. \]
Since \( 2X_1^2 - 2(x_1x_2 + x_1x_3 + x_1x_5) \leq 2 + 2 \cdot 3 = 8 \), we have
\[ 3(x_2^2 + x_3^2 + x_5^2) \geq 10, \] then \( x_2^2 + x_3^2 + x_5^2 \geq 10/3 > 3 \), a contradiction.

3) \( Y = (X_1 + X_2 + X_3 + X_4 + X_5)/3 \).

Consider the lattice point \( Y - X_i \) (i=1,2,3,4,5), which have the same 5th-coordinate 2/3, then
\[ 9(Y-X_i)^2 \geq 9 \] (i=1,2,3,4,5). Adding these five inequalities together, we obtain
\[ 4 \sum_{1 \leq i < 5} x_i^2 + \sum_{1 \leq i < j \leq 5} (X_i - X_j)^2 \geq 45. \]
By Lemma 1.3, \( \sum_{1 \leq i < j \leq 5} (X_i - X_j)^2 \leq 25 \), then \( 4 \sum_{1 \leq i \leq 5} x_i^2 \leq 20 \),
\[ 4 \sum_{1 \leq i \leq 5} x_i^2 \geq 20, \]
a contradiction to Lemma 1.20.
4) \( \mathbf{y} = (-\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4 + \mathbf{x}_5)/3. \)

Since \( \mathbf{y} \) has the 5th-coordinate \( 1/3 \), \( 9\mathbf{y}^2 \geq 9 \), i.e.

\[
\sum_{1 \leq i \leq 5} \mathbf{x}_i^2 + 2 \sum_{3 \leq i < j \leq 5} \mathbf{x}_i \mathbf{x}_j + 2(\mathbf{x}_1 + \mathbf{x}_2)(\mathbf{x}_3 + \mathbf{x}_4 + \mathbf{x}_5) + 2\mathbf{x}_1 \mathbf{x}_2 \geq 9.
\]

Consider the lattice points \( \mathbf{y} - \mathbf{x}_i \) \( (i=3,4,5) \), which have the same 5th-coordinate \( -2/3 \), then \( 3(\mathbf{y} - \mathbf{x}_i)^2 \geq 3 \) \( (i=3,4,5) \). Adding these three inequalities together, we obtain

\[
2(4+9/16)-2 \sum_{3 \leq i < j \leq 5} \mathbf{x}_i \mathbf{x}_j \geq 9 + (\mathbf{x}_1 - \mathbf{x}_2)^2.
\]

By Lemma 1.20 and the fact that \( \mathbf{x}_1 - \mathbf{x}_2 \) is a lattice point with the 5th-coordinate 0, we obtain from (1.2)

\[
2(4+9/16)-2 \sum_{3 \leq i < j \leq 5} \mathbf{x}_i \mathbf{x}_j \geq 9 + 1,
\]

i.e.

\[
\prod_{3 \leq i < j \leq 5} \mathbf{x}_i \mathbf{x}_j \geq 7/16.
\]

Also from (1.2), \( 4 \sum_{1 \leq i \leq 5} \mathbf{x}_i \geq 20 + 4 \sum_{3 \leq i < j \leq 5} \mathbf{x}_i \mathbf{x}_j \), add this to (1.1), we obtain

\[
6 \sum_{1 \leq i \leq 5} \mathbf{x}_i^2 + 4\mathbf{x}_1 \mathbf{x}_2 \geq 29 + (\prod_{1 \leq i \leq 5} \mathbf{x}_i)^2.
\]

Since \( (\mathbf{x}_1 + \mathbf{x}_2)^2 + (\mathbf{x}_1 - \mathbf{x}_2)^2 = 2(\mathbf{x}_1^2 + \mathbf{x}_2^2) \leq 4 \) and \( (\mathbf{x}_1 - \mathbf{x}_2)^2 \geq 1 \), we have \( (\mathbf{x}_1 + \mathbf{x}_2)^2 \leq 3 \);
then \( 4\mathbf{x}_1 \mathbf{x}_2 = (\mathbf{x}_1 + \mathbf{x}_2)^2 - (\mathbf{x}_1 - \mathbf{x}_2)^2 \leq 3 - 1 = 2 \).

From (1.4) we obtain

\[
6 \prod_{1 \leq i \leq 5} \mathbf{x}_i \geq 27 + (\prod_{1 \leq i \leq 5} \mathbf{x}_i)^2.
\]

Now we set \( (\prod_{1 \leq i \leq 5} \mathbf{x}_i)^2 = e^2 \) \( (e \geq 0) \), \( t = \mathbf{x}_3^2 + \mathbf{x}_4^2 + \mathbf{x}_5^2 \) and \( s = \mathbf{x}_1^2 \mathbf{x}_2^2 \mathbf{x}_3^2 \mathbf{x}_4^2 \mathbf{x}_5^2 \), then
\[ |X_3 + X_4 + X_5| = \left| (-X_1 - X_2 + X_3 + X_4 + X_5) + \sum_{1 \leq i \leq 5} X_i \right| / 2 \]
\[ = |3Y + \sum_{1 \leq i \leq 5} X_i| / 2 \geq 3|Y|/2 - e/2 \geq (3-e)/2. \]

From (1.3), \[ |X_3 + X_4 + X_5| = \sqrt{(X_3^2 + X_4^2 + X_5^2 + 2 \sum_{3 \leq i < j \leq 5} X_i X_j)} \]
\[ \leq \sqrt{(t-7/8)}, \text{ then} \]

(1.6) \[ \sqrt{(t-7/8)} \geq (3-e)/2. \]

If \[ t = \sum_{3 \leq i \leq 5} X_i \leq 2, \text{ then} \sum_{1 \leq i \leq 5} X_i \leq 4, \text{ a contradiction to (1.5); now } t > 2, \text{ then by Lemmas 1.11 and 1.18, we have} \]

(1.7) \[ X_1^2 + X_2^2 \leq s/(t-2) + 1 \leq 9/16(t-2) + 1. \]

Since \( (X_1 - X_2)^2 \geq 1, \)

(1.8) \[ 2X_1 X_2 \leq X_1^2 + X_2^2 - 1 \leq 9/16(t-2). \]

From (1.4), (1.7) and (1.8) we obtain

\[ 6(t + 9/16(t-2) + 1) + 18/16(t-2) \geq 29 + e^2, \]

then \[ 12t^2 - (70 + 2e^2)t + (101 + 4e^2) \geq 0; \text{ since } t \leq 3, \]

(1.9) \[ t \leq (35 + e^2 - \sqrt{(13 + 22e^2 + e^4)})/12. \]

From (1.6) and (1.9),

\[ (35 + e^2 - \sqrt{(13 + 22e^2 + e^4)})/12 \geq 7/8 + ((3-e)/2)^2, \text{ then} \]

\[ e \geq (5 + 4e^2 + 2(13 + 22e^2 + e^4))/36, \text{ which implies } e > 0.35. \]

From (1.5), \[ \sum_{1 \leq i \leq 5} X_i^2 \geq 4.5 + e^2 / 6 > 4.52, \text{ a contradiction to Lemma 1.21.} \]

7. \( h=1, m=4 \)

Since \( d(A') / d(A) = 4 \) and \( m \neq 1, 2, \) renumbering
X_i (i=1,2,3,4,5) if necessary, we may take X_i (i=1,2,3,4) and \( \mathbf{y} \) as a basis for \( \Lambda \), where \( \mathbf{y} \) is one of the following points:

\[
(a_1 x_1 + a_2 x_2 + x_3 )/4, \quad a_i \text{ are integers, } |a_i| \leq 2 \quad (i=1,2); \\
(\pm x_1 \pm x_2 + x_3 + x_5 )/4, \quad (2x_1 + \pm x_2 + x_3 + x_5 )/4, \quad (2x_1 + 2x_2 \pm x_3 + x_5 )/4, \\
(2x_1 + 2x_2 + 2x_3 + x_5 )/4, \quad (\pm x_1 + x_2 + x_3 + x_4 + x_5 )/4, \\
(-x_1 - x_2 + x_3 + x_4 + x_5 )/4, \quad (2x_1 \pm x_2 + x_3 + x_4 + x_5 )/4, \\
(2x_1 + 2x_2 \pm x_3 + x_4 + x_5 )/4, \quad (2x_1 + 2x_2 + 2x_3 \pm x_4 + x_5 )/4, \\
(2x_1 + 2x_2 + 2x_3 + 2x_4 + x_5 )/4.
\]

By Lemma 1.1, \( \mathbf{y} \) can not be one of \((a_1 x_1 + a_2 x_2 + x_3 )/4, \quad a_i \text{ are integers, } |a_i| \leq 2 \quad (i=1,2); \)

\[
(\pm x_1 \pm x_2 + x_3 + x_5 )/4, \quad (2x_1 + \pm x_2 + x_3 + x_5 )/4, \\
(2x_1 + 2x_2 \pm x_3 + x_5 )/4, \quad (2x_1 + 2x_2 + 2x_3 + x_5 )/4.
\]

If \( \mathbf{y} \) is one of \((2x_1 \pm x_2 + x_3 + x_4 + x_5 )/4 \) and \((2x_1 + 2x_2 + 2x_3 + x_4 + x_5 )/4, \) which have multiples of 1/2 as their 5th-coordinates, then \( m=1 \) or 2, done.

Therefore, \( \mathbf{y} \) can only be one of the following eight points:

\[
(2x_1 \pm x_2 + x_3 + x_5 )/4, \quad (\pm x_1 + x_2 + x_3 + x_4 + x_5 )/4, \quad (-x_1 - x_2 + x_3 + x_4 + x_5 )/4, \\
(2x_1 + 2x_2 \pm x_3 + x_4 + x_5 )/4, \quad (2x_1 + 2x_2 + 2x_3 + x_4 + x_5 )/4.
\]

1) \( \mathbf{y} = (2x_1 + x_2 + x_3 + x_5 )/4. \)

Consider the lattice points \( 2y_1 - x_i \quad (i=2,3,5) \), which have the same 5th-coordinate 1/2, then

\[
4(2y_1 - x_i)^2 \geq 4 \quad (i=2,3,5). \]

Adding these three inequalities together, we obtain
(x_2^2 + x_3^2 + x_5^2) + \sum_{i<j} (x_i - x_j)^2 \geq 12.

i, j = 2, 3, 5

By Lemmas 1.2 and 1.3, \[ \sum_{i<j} (x_i - x_j)^2 \leq 9 \] and the equality holds only when \( x_2 + x_3 + x_5 = 0 \); then
\[ 3 \geq x_2^2 + x_3^2 + x_5^2 \geq 3, \] so the equality holds and \( x_2 + x_3 + x_5 = 0 \).

Then \( Y = (x_1/2, 5/4) \), so \( Y - X = (-x_1/2, 1/4) \) is in the interior of \( K \), a contradiction.

2) \( Y = (2x_1 - x_2 + x_3 + x_5)/4 \).

Consider the lattice points \( Y \) and \( Y - X \), which have 5th-coordinates \( 3/4 \) and \(-1/4 \), then \( 16Y^2 \geq 16 \) and \( 16(Y - X_1)^2 \geq 16 \). Adding these two inequalities together, we obtain
\[ 8x_1^2 + 2(x_2^2 + x_3^2 + x_5^2) - 4x_2(x_3 + x_5) + 4x_3x_5 \geq 32; \] then
\[ 8 + 2 \cdot 3 + 4 \cdot 2 + 4 \geq 32, \] i.e. \( 26 \geq 32 \), a contradiction.

3) \( Y = (x_1 + x_2 + x_3 + x_4 + x_5)/4 \).

Consider the lattice points \( Y - X_i \) \( (i = 1, 2, 3, 4, 5) \), which have the same 5th-coordinate \( 1/4 \), then
\[ 16(Y - X_i)^2 \geq 16 \] \( (i = 1, 2, 3, 4, 5) \). Adding these five inequalities together, we obtain
\[ \sum_{1 \leq i \leq 5} x_i^2 + 3 \sum_{1 \leq i < j \leq 5} (x_i - x_j)^2 \geq 80. \]

By Lemma 1.3, \( \sum_{1 \leq i < j \leq 5} (x_i - x_j)^2 \leq 25, \) then
\[ \sum_{1 \leq i \leq 5} x_i^2 \geq 80 - 3 \cdot 25 = 5, \] a contradiction to Lemma 1.20.
4) \( Y = \left( -X_1 + X_2 + X_3 + X_4 + X_5 \right)/4. \)

Consider the lattice points \( Y \) and \( Y - X_i \) (i=2,3,4,5), which have 5th-coordinates 3/4 and -1/4, then \( 8Y^2 \geq 8 \) and \( 2(Y - X_i)^2 \geq 2 \) (i=2,3,4,5). Adding these five inequalities together, we obtain

\[
\sum_{1 \leq i \leq 5} X_i^2 + \sum_{2 \leq i \leq 5} X_i^2 - X_1 \geq 16; \quad \text{then}
\]

\[
\sum_{1 \leq i \leq 5} X_i^2 + 4 + 4 \geq 16, \quad \text{i.e.} \quad \sum_{1 \leq i \leq 5} X_i^2 \geq 8, \quad \text{a contradiction.}
\]

5) \( Y = \left( -X_1 - X_2 + X_3 + X_4 + X_5 \right)/4. \)

Consider the lattice points \( Y \) and \( Y - X_i \) (i=3,4,5), then \( 4Y^2 \geq 4 \) and \( 4(Y - X_i)^2 \geq 4 \) (i=3,4,5). Adding these four inequalities together, we obtain

\[
\sum_{1 \leq i \leq 5} X_i^2 + 2X_1X_2 + \sum_{3 \leq i < j \leq 5} (X_i - X_j)^2 \geq 16. \quad \text{By Lemma 1.3,}
\]

\[
\sum_{3 \leq i < j \leq 5} (X_i - X_j)^2 \leq 9, \quad \text{then} \quad \sum_{1 \leq i \leq 5} X_i^2 + 2 + 9 \geq 16,
\]

i.e. \( \sum_{1 \leq i \leq 5} X_i^2 \geq 5, \quad \text{a contradiction to Lemma 1.20.} \)

6) \( Y = \left( 2X_1 + 2X_2 + X_3 + X_4 + X_5 \right)/4. \)

Consider the lattice points \( 2Y - X_1 - X_2 - X_1 \) (i=3,4,5), then we have \( 4(2Y - X_1 - X_2 - X_1)^2 \geq 4 \) (i=3,4,5). Adding these three inequalities together, we obtain

\[
\sum_{3 \leq i \leq 5} X_i^2 + \sum_{3 \leq i < j \leq 5} (X_i - X_j)^2 \geq 12. \quad \text{In the same way as in 1),}
\]

we can get \( X_3 + X_4 + X_5 = 0. \) Then \( Y = ((X_1 + X_2)/2, 7/4), \)

\( Y - X_2 = ((X_1 - X_2)/2, 3/4) \) and \( Y - X_1 - X_2 = ((X_1 - X_2)/2, -1/4). \)
So we have \((X_1 - X_2)^2 \geq 4\) and \((X_1 + X_2)^2 \geq 4\), which imply \((X_1 - X_2)^2 + (X_1 + X_2)^2 \geq 8\), i.e. \(X_1^2 + X_2^2 \geq 4\), a contradiction.

7) \(Y = (2X_1 + 2X_2 - X_3 + X_4 + X_5)/4\).

Consider the lattice points \(2Y - X_1 - X_2\) and \(2Y - X_1 - X_2 - X_i\) \((i=4,5)\), then \(4(2Y - X_1 - X_2)^2 \geq 4\) and \(4(2Y - X_1 - X_2 - X_i)^2 \geq 4\) \((i=4,5)\). Adding these three inequalities together, we obtain
\[
\sum_{3 \leq i \leq 5} X_i^2 + \sum_{3 \leq i < j \leq 5} (X_i - X_j)^2 \geq 12.
\]
In the same way as in 1) we can get \(X_1^2 + X_2^2 + X_3^2 = 0\), then \(Y = ((X_1 + X_2 - X_3)/2, 5/4)\).

Consider the lattice points \(Y - X_1 = ((-X_1 + X_2 - X_3)/2, 1/4)\), \(Y - X_2 = ((X_1 - X_2 - X_3)/2, 1/4)\), \(Y - X_1 - X_2 = ((-X_1 - X_2 - X_3)/2, -3/4)\) and \(Y - X_1 - X_2 + X_3 = ((-X_1 - X_2 + X_3)/2, 1/4)\); then
\[
(-X_1 + X_2 - X_3)^2 \geq 4, \quad (X_1 - X_2 - X_3)^2 \geq 4, \quad (-X_1 - X_2 - X_3)^2 \geq 4
\]
and \((-X_1 - X_2 + X_3)^2 \geq 4\). Adding these four inequalities together, we obtain \(4(X_1^2 + X_2^2 + X_3^2) \geq 16\), i.e. \(X_1^2 + X_2^2 + X_3^2 \geq 4\), a contradiction.

8) \(Y = (2X_1 + 2X_2 + 2X_3 + 2X_4 + X_5)/4\).

The lattice point \(2Y - X_1 - X_2 - X_3 - X_4 = X_5/2\) is in the interior of \(K\), a contradiction.

8. \(h=1, m=6; h=2, m=3\)

Since \(d(\Lambda')/d(\Lambda)=6\) and \(m \neq 1,2\), renumbering \(X_i\) \((i=1,2,3,4,5)\) if necessary, we may take a basis for
A as one of the following three types.

I. $X_i$ (i=1,2,3,4) and $Y$, where

$$Y = \left( \sum_{1 \leq i \leq 4} a_i X_i + X_5 \right) / 6; \quad a_i \text{ are integers and } |a_i| \leq 3 \quad (i=1,2,3,4).$$

II. $X_i$ (i=1,2,3) and $Y_j$ (j=1,2), where

$$Y_1 = \left( \sum_{1 \leq i \leq 3} a_i X_i + X_4 \right) / u \quad \text{and} \quad a_i, b_i \text{ are integers and } |a_i| \leq 1,$$

$$|b_i| \leq 1 \quad (i=1,2,3), \quad u=2 \text{ or } 3.$$

III. $X_i$ (i=1,2,3) and $Y_j$ (j=1,2), where

$$Y_1 = \left( \sum_{1 \leq i \leq 3} a_i X_i + X_4 \right) / 2 \quad \text{and} \quad a_i, b_i \text{ are integers and } |a_i| \leq 1,$$

$$|b_i| \leq 1 \quad (i=1,2,3).$$

By Lemma 1.1, we know $a_i \neq 0$ for all $i=1,2,3,4$.

If $|a_i|=1$ for all $i=1,2,3,4$, then the 5th-coordinate of $Y$ has absolute value less than 1 and

$$|Y| = \left| \pm X_1 \pm X_2 \pm X_3 \pm X_4 \pm X_5 \right|/6 < 1, \quad \text{i.e. } Y \text{ is in the interior of } K, \text{ a contradiction. Therefore, one of } a_i, \text{ say } a_4, \text{ must be } \pm 3 \text{ or } \pm 2.$$

If $a_4=\pm 3$, then

$$2Y = Y_1 = \left( \sum_{1 \leq i \leq 3} a_i' X_i + X_5 \right) / 3 \quad \text{modulus } A',$$
where \(a_i\) are integers and \(|a_i| \leq 1\) (i=1,2,3). By Lemma 1.1, 
\[ Y_1 = (\pm X_1 + X_2 + X_3 + X_5)/3 \]  
renumbering \(X_i\) (i=1,2,3,5) or change \(Y\) to \(-Y\) if necessary, we may take \(Y_1\) to be one of 
\[ \pm (X_1 + X_2 + X_3 + X_5)/3 \]  
and \(\pm (X_1 - X_2 + X_3 + X_5)/3\). If  
\[ Y_1 = (X_1 - X_2 + X_3 + X_5)/3, \text{ then } m=1 \text{ or } 2, \text{ done.} \]  
If  
\[ Y_1 = (X_1 + X_2 + X_3 + X_5)/3, \text{ we can get a contradiction in the} \]  
same way as in cases 1) and 2) of section 6.

Now \(a_4 = \pm 2\) and \(a_i \neq \pm 3\) (i=1,2,3), then 
\[ 3Y = Y = (\sum_{1 \leq i \leq 3} a_i X_i)/2 \mod \Lambda'; \]  
where \(a_i\) are integers and \(|a_i| \leq 1\) (i=1,2,3). By Lemma 1.1, 
\[ Y_1 = (X_1 + X_2 + X_5)/2 \]  
or \( (X_1 + X_2 + X_3 + X_5)/2 \) after renumbering \(X_i\) (i=1,2,3), If  
\[ Y_1 = (X_1 + X_2 + X_5)/2, \text{ we can get a} \]  
contradiction in the same way as in case 1) of section 7.

If  
\[ Y_1 = (X_1 + X_2 + X_3 + X_5)/2, \text{ then } Y = (\pm X_1 + X_2 + X_3 + 2 X_4 + X_5)/6 \]  
since \(a_4 = \pm 2\), \(|a_i| \leq 3\) and \(a_i \neq \pm 3\) (i=1,2,3). If  
\[ Y = (X_1 + X_2 + X_3 + 2 X_4 + X_5)/6, \text{ then } m=1, \text{ done.} \]  
For all other cases, by Lemma 1.20, \(Y\) is in the interior of \(K\),  
a contradiction.

II. \( Y_1 = (\sum_{1 \leq i \leq 3} a_i X_i + X_4)/u, \) 
\[ Y_2 = (\sum_{1 \leq i \leq 3} b_i X_i + Y_1 + X_5)/6; \]  
\(u=2\) or 3.

In this case we shall take either all upper signs or all lower signs. Consider the unimodular transformation 
\[ Z_1 = \pm Y_1 + 6 Y_2/u, \]  
\[ Z_2 = Y_1 u Y_2, \]  
then \(X_5 = - (\sum_{1 \leq i \leq 3} b_i X_i + Z_1), \)
\[ X_4 = - \sum_{1 \leq i \leq 3} a_i x_i + u^2 z_1 + 6 z_2. \]

So the lattice \( \Lambda \) is generated by \( X_1, X_2, X_3, X_5 \) and \( \pm z_2 \), where \( \pm z_2 \) is of the type 
\((c_1 x_1 + c_2 x_2 + c_3 x_3 + c_5 x_5 + x_4)/6, c_i \) are integers \((i=1,2,3,5)\). But this type has been discussed in case I.

III. \( Y_1 = (\sum_{1 \leq i \leq 3} a_i x_i + x_4)/2, a_i \) are integers and \(|a_i| \leq 1;\)
\[ Y_2 = (\sum_{1 \leq i \leq 3} b_i x_i + x_5)/3, b_i \) are integers and \(|b_i| \leq 1.\)

\( Y_1 \) has a multiple of 1/2 for its 5th-coordinate.

Renumbering \( x_i \) \((i=1,2,3,5)\) or change \( Y_2 \) to \(-Y_2\) if necessary, we may take \( Y_2 \) to be one of \((\pm x_1 + x_2 + x_3 + x_5)/3\)
and \((-x_1 - x_2 + x_3 + x_5)/3\). If \( Y_2 = (-x_1 - x_2 + x_3 + x_5)/3\),
then \( m=1 \) or \(2, \) done. If \( Y_2 = (\pm x_1 + x_2 + x_3 + x_5)/3, \) we can get a contradiction as in cases 1) and 2) of section 6.

9. \( h=1, m=7 \)

Since \( d(\Lambda')/d(\Lambda) = 7 \) and \( m \neq 1, \) renumbering \( x_i \)
\((i=1,2,3,4)\) if necessary, we may take \( x_i \) \((i=1,2,3,4)\) and \( y \) as a basis for \( \Lambda, \) where \( Y = (\sum_{1 \leq i \leq 5} a_i x_i)/7, \)
\( a_i \) are integers, \(|a_i| \leq 3 \) \((i=1,2,3,4)\) and \( a_5 = 1.\)

By Lemma 1.1, we know all \( a_i \neq 0 \) \((i=1,2,3,4).\)

If \( \sum_{1 \leq i \leq 5} |a_i| = 7 = \sum_{1 \leq i \leq 5} a_i, \) then \( m=1, \) done.

If \( \sum_{1 \leq i \leq 5} |a_i| \leq 7 \neq \sum_{1 \leq i \leq 5} a_i, \) then \( Y \) is in the interior of \( K, \) a contradiction.
If necessary, replacing \( Y \) by a point in \( \pm kY + \mathbf{A}' \) 
\((k=1,2,3)\) and renumbering \( X_i \) \((i=1,2,3,4,5)\); we can have 
\(|a_1|, |a_2|, |a_3|, |a_4|, a_5\rangle = (3,3,1,1,1) \) or \((3,2,1,1,1)\) or 
\((3,2,2,1,1)\), and \( Y \) is one of the following 23 points:

\[
\begin{align*}
(3X_1 \pm 3X_2 + X_3 + X_4 + X_5)/7, & \quad (-3X_1 - 3X_2 + X_3 + X_4 + X_5)/7, \\
(3X_1 \pm 3X_2 - X_3 + X_4 + X_5)/7, & \quad (3X_1 + 2X_2 + X_3 + X_4 + X_5)/7, \\
(\pm 3X_1 - 2X_2 + X_3 + X_4 + X_5)/7, & \quad (\pm 3X_1 - 2X_2 - X_3 + X_4 + X_5)/7, \\
(-3X_1 + 2X_2 + 2X_3 + X_4 + X_5)/7, & \quad (-3X_1 + 2X_2 - 2X_3 + X_4 + X_5)/7, \\
(3X_1 + 2X_2 - 2X_3 + X_4 + X_5)/7, & \quad (3X_1 - 2X_2 - 2X_3 - X_4 + X_5)/7.
\end{align*}
\]

1) \( Y = (3X_1 + 3X_2 + X_3 + X_4 + X_5)/7 \).

Consider the lattice points \( Y - X_1 - X_2 \) and \( Y - X_1 \) 
\((i=1,2)\), which have 5th-coordinates \(-5/7\) and \(2/7\); then 
\(49(Y - X_1 - X_2)^2 \geq 49\) and \(49(Y - X_1)^2 \geq 49\) \((i=1,2)\). Adding 
these three inequalities together, we obtain

\[
21(X_1^2 + X_2^2) - 6X_1X_2 + 5 \sum_{1 \leq i < j \leq 5} (X_i - X_j)^2 \\
\geq 147 + 8 \sum_{3 \leq i < j \leq 5} (X_i - X_j)^2 + \sum_{3 \leq i < 5} X_i^2.
\]

By Lemmas 1.2 and 1.20, 
\(\sum_{1 \leq i < j \leq 5} (X_i - X_j)^2 \leq 5(4 + 49/64)\); 
then 
\(21 \cdot 2 + 6 + 5 \cdot 5(4 + 49/64) > 147 + 8 \cdot 3\), i.e. 
\(167 + 9/64 > 171\), a contradiction.

2) \( Y = (3X_1 - 3X_2 + X_3 + X_4 + X_5)/7 \).

Consider the lattice points \( Y \) and \( Y - X_1 \), then 
\(28Y^2 \geq 28\) and \(21(Y - X_1)^2 \geq 21\). Adding these two
inequalities together, we obtain
\[11X_1^2+8X_2^2+\sum_{1\leq i\leq 5} X_i^2-6X_2 \sum_{3\leq i\leq 5} X_i+2 \sum_{3\leq i<j\leq 5} X_i X_j \geq 49.\]

Then \[11+8+5+6\cdot 3+2\cdot 3 > 49, \text{ i.e. } 48 > 49,\]
a contradiction.

3) \[Y = (-3X_1-3X_2+X_3+X_4+X_5)/7.\]

Consider the lattice points \(Y\) and \(Y+X_i\) (i=1,2), then \[147Y^2 \geq 147 \text{ and } 49(Y+X_i)^2 \geq 49 \text{ (i=1,2).}\] Adding these three inequalities together, we obtain
\[40(X_1^2+X_2^2)+15 \sum_{1\leq i\leq 5} X_i^2-16(X_1+X_2) \sum_{3\leq i\leq 5} X_i \geq 245+5 \sum_{3\leq i<j\leq 5} (X_i-X_j)^2+3(X_i-X_j)^2.\]

Then \[40\cdot 2+15\cdot 5+16\cdot 2\cdot 3 > 245+5\cdot 3+3, \text{ i.e. } 251 > 263,\]
a contradiction.

4) \[Y = (-3X_1-3X_2-X_3+X_4+X_5)/7.\]

The same argument in 3) can be applied here, just replacing \(X_3\) by \(-X_3\) there.

5) \[Y = (3X_1+3X_2-X_3+X_4+X_5)/7.\]

Since \(Y\) has the 5th-coordinate 1, \(m=1,\) done.

6) \[Y = (3X_1-3X_2-X_3+X_4+X_5)/7.\]

The same argument in 2) can be applied here, just replacing \(X_3\) by \(-X_3\) there.

7) \[Y = (3X_1+2X_2+X_3+X_4+X_5)/7.\]

Consider the lattice points \(Y-X_i\) (i=1,2), \[2Y-X_1-X_2\]
and \(2Y-X_1-X_2-X_j\) \((j=3,4,5)\); then \(3(Y-X_1)^2 \geq 3\) \((i=1,2)\), \(2(2Y-X_1-X_2)^2 \geq 2\) and \(7(2Y-X_1-X_2-X_j)^2 \geq 7\) \((j=3,4,5)\).

Adding these six inequalities together, we obtain

\[
x_1^2+5x_2^2+\sum_{1 \leq i \leq 5} x_i^2 + 2 \sum_{3 \leq i < j \leq 5} (x_i-x_j)^2 \geq 29.
\]

Then \(1+5+5+2 \cdot 9 > 29\), i.e. \(29 > 29\), a contradiction.

8) \(Y = \frac{(3X_1+2X_2-X_3+X_4+X_5)}{7}\).

Consider the lattice points \(Y\) and \(Y-X_1\), then

\[
49Y^2 \geq 49 \quad \text{and} \quad 49(Y-X_1)^2 \geq 49.
\]

Adding these two inequalities together, we obtain

\[
23X_1^2+6X_2^2+2 \sum_{1 \leq i \leq 5} x_i^2 - 4X_1X_2 - 2X_1(X_4+X_5-X_3)
\]

\[
+8X_2(X_4+X_5-X_3)+4(X_4X_5-X_3X_4-X_3X_5) \geq 98.
\]

Then \(23+6+2 \cdot 5+4+2 \cdot 3+8 \cdot 3+4 \cdot 3 > 98\), i.e. \(85 > 98\), a contradiction.

9) \(Y = \frac{(3X_1-2X_2+X_3+X_4+X_5)}{7}\).

The same argument in 8) can be applied here, just replacing \(X_1\) by \(-X_1\) \((i=2,3)\) there.

10) \(Y = \frac{(-3X_1-2X_2+X_3+X_4+X_5)}{7}\).

The same argument in 8) can be applied here, just replacing \(X_1\) by \(-X_1\) \((i=1,2,3)\) there.

11) \(Y = \frac{(3X_1-2X_2-X_3+X_4+X_5)}{7}\).

The same argument in 8) can be applied here, just replacing \(X_2\) by \(-X_2\) there.
12) \( Y = \frac{-3X_1 - 2X_2 - X_3 + X_4 + X_5}{7} \).

The same argument in 8) can be applied here, just replacing \( X_i \) by \(-X_i \) \((i=1,2)\) there.

13) \( Y = \frac{-3X_1 + 2X_2 + X_3 + X_4 + X_5}{7} \).

Consider the lattice points \( Y \) and \( 2Y + X_1 - X_2 \), then
\[ 21Y^2 \geq 21 \quad \text{and} \quad 7(2Y + X_1 - X_2)^2 \geq 7. \]
Adding these two inequalities together, we obtain
\[ X_1^2 + 3 \sum_{1 \leq i \leq 5} X_i^2 - 6X_1X_2 - 2X_1 \sum_{3 \leq i \leq 5} X_i \geq 28 + \sum_{3 \leq i < j \leq 5} (X_i - X_j)^2. \]
Then \( 1 + 3 \cdot 5 + 6 + 2 \cdot 3 > 28 + 3 \), i.e. \( 28 > 31 \), a contradiction.

14) \( Y = \frac{-3X_1 + 2X_2 - X_3 + X_4 + X_5}{7} \).

Since \( Y \) has the 5th-coordinate 0, \( m=1 \), done.

15) \( Y = \frac{3X_1 + 2X_2 + 2X_3 + X_4 + X_5}{7} \).

Consider the lattice points \( 2Y - X_1 - X_2 - X_3 \), \( 3Y - X_1 - X_2 - X_3 \), \( 3Y - X_1 - X_2 - X_3 - X_4 \) \((i=4,5)\) and \( 2Y - X_1 - X_j \) \((j=2,3)\); then \( 98(2Y - X_1 - X_2 - X_3)^2 \geq 98 \),
\[ 49(3Y - X_1 - X_2 - X_3)^2 \geq 49, \quad 49(3Y - X_1 - X_2 - X_3 - X_4)^2 \geq 49 \quad (i=4,5) \]
and \( 49(2Y - X_1 - X_j)^2 \geq 49 \) \((j=2,3)\). Adding these six inequalities together, we obtain
\[ 16X_1^2 + 46(X_2^2 + X_3^2) + 50(X_4^2 + X_5^2) - 2X_1(X_2 + X_3) - 8X_1(X_4 + X_5) - 6X_2X_3 + 2X_4X_5 - 24(X_2 + X_3)(X_4 + X_5) \geq 343. \]
Then \( 16 + 46 \cdot 2 + 50 \cdot 2 + 2 \cdot 2 + 8 \cdot 2 + 6 + 2 + 24 \cdot 2 \cdot 2 > 343 \),
i.e. \( 332 > 343 \), a contradiction.
16) \( Y = \frac{(3X_1+2X_2-2X_3+X_4+X_5)}{7} \).

Consider the lattice point \( Y \) and \( Y-X_1 \), then
\[ 49Y^2 \geq 49 \quad \text{and} \quad 49(Y-X_1)^2 \geq 49. \]
Adding these two inequalities together, we obtain
\[ 18X_1^2+13X_2^2+13X_3^2+13 \sum_{i=1}^{5} X_i^2 \geq 98+2(X_1-X_3)^2+2(X_4-X_5)^2+4(X_2-X_4)^2 \]
\[ +4(X_2-X_5)^2+2(X_1+X_2)^2+(X_1+X_4)^2 \]
\[ +(X_1+X_5)^2+8(X_2+X_3)^2+4(X_3+X_4)^2 \]
\[ +4(X_3+X_5)^2. \]
Then \( 18+13+13+13 \cdot 5 > 98+2(X_1-X_3)^2+2(X_4-X_5)^2+4(X_2-X_4)^2 \]
\[ +4(X_2-X_5)^2, \]
so \( 109 > 98+2+2+4+4 = 110 \), a contradiction.

17) \( Y = \frac{(-3X_1-2X_2-2X_3+X_4+X_5)}{7} \).

The argument in 16) can be applied here after replacing \( X_2 \) by \(-X_2\) there. But this time we obtain
\[ 109 > 98+2(X_1-X_3)^2+2(X_4-X_5)^2+2(X_1-X_4)^2+8(X_2-X_3)^2 \]
\[ \geq 98+2+2+8 = 112, \]
a contradiction.

18) \( Y = \frac{(-3X_1-2X_2-2X_3+X_4+X_5)}{7} \).

The argument in 16) can be applied here after replacing \( X_i \) by \(-X_i\) \((i=1,2)\) there. But this time we obtain
\[ 109 > 98+2(X_4-X_5)^2+(X_1-X_4)^2+(X_1-X_5)^2+8(X_2-X_3)^2 \]
\[ \geq 98+2+1+1+8 = 110, \]
a contradiction.
19) \( Y = (-3X_1+2X_2+2X_3+X_4+X_5)/7 \).

Consider the lattice points \( Y, 2Y+X_1-X_2-X_3 \) and 
\( 2Y+X_1-X_1 \) \((i=2,3)\), then \( 14Y^2 \geq 14, 7(2Y+X_1-X_2-X_3)^2 \geq 7 \) and \( 7(2Y+X_1-X_1)^2 \geq 7 \) \((i=2,3)\). Adding these four inequalities together, we obtain 
\[ X_1^2 + 4(X_2^2 + X_3^2) + 2 \sum_{1 \leq i \leq 5} X_i^2 - 4X_1(X_2 + X_3) - 2X_2X_3 + 4X_4X_5 \geq 35. \]
Then \( 1 + 4 + 2 + 5 + 4 + 2 + 4 > 35 \), i.e. \( 33 > 35 \), a contradiction.

20) \( Y = (-3X_1+2X_2-2X_3+X_4+X_5)/7 \).

The argument in 16) can be applied here after replacing \( X_1 \) by \(-X_1\) there. This time we obtain
\[ 109 > 98 + 2(X_4 - X_5)^2 + 4(X_2 - X_4)^2 + 4(X_2 - X_5)^2 + 2(X_1 - X_2)^2 \]
\[ + (X_1 - X_4)^2 + (X_1 - X_5)^2 \geq 98 + 2 + 4 + 1 + 1 = 112, \]
a contradiction.

21) \( Y = (3X_1+2X_2+2X_3-X_4+X_5)/7 \).

Since \( Y \) has the 5th-coordinate 1, \( m=1 \), done.

22) \( Y = (3X_1+2X_2-2X_3-X_4+X_5)/7 \).

The argument in 16) can be applied here after replacing \( X_4 \) by \(-X_4\) there. This time we obtain
\[ 109 > 98 + 2(X_1 - X_3)^2 + 4(X_2 - X_5)^2 + (X_1 - X_4)^2 + 4(X_3 - X_4)^2 \]
\[ \geq 98 + 4 + 4 + 1 + 4 = 109, \]
a contradiction.
23) \( Y = \frac{(3X_1 - 2X_2 - 2X_3 - 4X_4 + X_5)}{7} \).

The same argument in 19) can be applied here after replacing \( X_i \) by \(-X_i\) \((i=1,2,3,4)\) there.

Now all possible cases have been discussed and in each case we got a contradiction which proves the main theorem in this chapter. It is well known [5,21] that \( \Delta(K) = 1/2 \), then \( \Delta(K) = 1/2 \) by the theorem we just proved.

In conclusion of this chapter, I should point out that Lemmas 1.14,1.15,1.17, and 1.20 are directly from Professor A.C. Woods' unpublished notes, which are essential for the knotty case 4) of section 6.
CHAPTER II

MULTIPLE LATTICE PACKINGS AND COVERINGS OF SPHERES

1. Introduction

Let \( \Lambda \) be an \( n \)-dimensional lattice and \( K_n \) be the \( n \)-dimensional unit sphere centered at the origin in \( \mathbb{R}^n \).

Consider the system \( S = K_n + \Lambda \) of unit spheres centered at \( \Lambda \).

S is called a **k-fold lattice packing** if each point of \( \mathbb{R}^n \) lies in the interior of at most \( k \) spheres of \( S \). S is called a **k-fold lattice covering** if each point of \( \mathbb{R}^n \) lies inside or on at least \( k \) spheres of \( S \).

Let \( d^*_k \) denote the **density** of a closest \( k \)-fold lattice packing of \( \mathbb{R}^n \), and let \( D^*_k \) denote the thinnest \( k \)-fold lattice covering of \( \mathbb{R}^n \). If \( K_n + \Lambda \) is an \( m \)-fold packing, then \( K_n + \bigcup_{0 \leq i \leq k-1} (\Lambda + iP/k) \) gives us a \( mk \)-fold packing, where \( P \) is a primitive point of \( \Lambda \). Thus it is trivial that \( d^*_m k \geq kd^*_m \); similarly, \( D^*_m k \leq KD^*_m \).

Given a lattice \( \Lambda \), as \( m \) increases to infinity, \( K_n + \Lambda/m \) will give us a \( k \)-fold packing, where \( k \) is approximately the number of points of \( \Lambda/m \) inside \( K_n \), then \( k \) is about \( m^n \text{Vol}(K_n)/d(\Lambda) \), which is the density of \( K_n + \Lambda/m \).

Thus it is clear that \( \lim_{k \to \infty} d^*_k/k = 1 \); similarly, \( \lim_{k \to \infty} D^*_k/k = 1 \).
Everything is trivial in case $n=1$ and we have $d_k^1 = D_k^1 = k$. If $n \geq 2$, then $d_k^n < 1$ and $D_k^n > 1$; the two limits above imply that $d_k^n > k d_1^n$ and $D_k^n < k D_1^n$ for sufficiently large $k$. Thus it is desirable to find the exact values for $d_k^n$ and $D_k^n$ or simply to find when the two inequalities hold. Such problems have been studied by Few [14,15], Blundon [3], Heppes [18], Danzer [9], G. Fejes Tóth and F. Florian [12], Few and Kanagasabapathy [16]. Some of their results are:

- $d_k^2 = k d_1^2$ (for $k=2,3,4,$), $d_k^2 > k d_1^2$ (for $k \geq 5$) [14,18];
- $d_k^n > 2d_1^n$ (for $n \geq 3$) [14,16];
- $d_k^n > k d_1^n$ for all $(n,k)$ (for $n \geq 3$, $k \geq 2$) except $(3,k),(4,k)$ and $(5,3)$ (for $k=3,5,7,9,11$) [12];
- $D_k^2 = 2D_1^2$, $D_k^3 = 2,84\ldots D_1^3$, $D_k^4 = 3,60\ldots D_1^4$ [3];
- $D_k^3 = 1,90\ldots D_1^3$ [15]; $D_k^5 \leq 4,48\ldots D_1^5$ [9];
- $D_k^2 < k D_1^2$ (for $k \geq 3$), $D_k^3 < k D_1^3$ (for $k \geq 21$) [12].

We intend to show that $d_k^3 > k d_1^3$, $d_k^4 > k d_1^4$, $d_k^5 > k d_1^5$ and $D_k^3 < k D_1^3$ for all odd $k \geq 3$. From now on $k$ denotes a positive odd integer greater than 1.

2. $d_k^3 > k d_1^3$

Let $\Lambda$ be the 3-dimensional lattice generated by $(0,0,2)$, $(0,2,0)$ and $(\sqrt{2},1,1)$; then $K_3 + \Lambda$ is a closest lattice packing of unit spheres in $\mathbb{R}^3$ with density $d_1^3$ [5].
Let $\Lambda_k$ be the lattice generated by $(0,0,2),(0,2,0)$ and $(\sqrt{2}/k,1,1)$; then $\Lambda_k = \bigcup_{0 \leq i \leq k-1} (\Lambda + (2i\sqrt{2}/k,0,0))$, since $k$ is odd. Thus $K_3 + \Lambda_k$ is a $k$-fold lattice packing with density $kd_1^3$. Now consider the lattice $\Lambda'_k$ generated by $(0,0,2-2\epsilon), (0,2-2\epsilon,0)$ and $((1+\epsilon')\sqrt{2}/k,1-\epsilon,1-\epsilon)$, where $\epsilon$ is a sufficiently small number and $\epsilon' = \sqrt{1+2\epsilon-\epsilon^2}-1$.

**Lemma 2.1.** $K_3 + \Lambda'_k$ is a $k$-fold lattice packing with density greater than $kd_1^3$.

**Proof.** Since $\Lambda'_k$ is a neighboring lattice of $\Lambda_k$ and $K_3 + \Lambda_k$ is a $k$-fold lattice packing, all points in $\mathbb{R}^3$ except possibly certain small neighborhoods of $P + \Lambda'_k$ will be covered at most $k$ times by $K_3 + \Lambda'_k$, where $P$ is one of the following 12 points:

$(0, \pm (1-\epsilon), 0), (0,0, \pm (1-\epsilon))$, 
$(\pm (1+\epsilon')/2, \pm (1-\epsilon)/2, \pm (1-\epsilon)/2)$. 

By symmetry it suffices to consider the two neighborhoods around $(0,0,1-\epsilon)$ and $((1+\epsilon')\sqrt{2}/2, (1-\epsilon)/2, (1-\epsilon)/2)$. 

Let $X = x(0,0,2-2\epsilon) + y(0,2-2\epsilon,0) + z((1+\epsilon')\sqrt{2}/k,1-\epsilon,1-\epsilon)$ be any lattice point of $\Lambda'_k$, where $x, y$ and $z$ are integers. The square of the distance between $X$ and $(0,0,1-\epsilon)$ is

$$f(X) = (1-\epsilon)^2 (2x+z-1)^2 + (1-\epsilon)^2 (2y+z)^2 + (1+\epsilon')^2 2z^2/k^2.$$ 

If $z \neq 0$, then one of $2x+z-1$ and $2y+z$ must be odd; it follows that $f(X) > 1$. 

If \( z=0 \), then \( f(X) = (1-\epsilon)^2(2x-1)^2 + (1-\epsilon)^2(2y)^2 \); so \( f(X) \leq 1 \) only when \( y=0 \) and \( x=0 \) or \( 1 \).

Thus in \( \Lambda_k' \) there are only two lattice points \((0,0,0)\) and \((0,0,2-2\epsilon)\) with distance less than 1 from \((0,0,1-\epsilon)\) and all other lattice points have distance greater than 1 from \((0,0,1-\epsilon)\).

The square of the distance between \( X \) and 
\[
((1+\epsilon')\sqrt{2}/2,(1-\epsilon)/2,(1-\epsilon)/2)
\]
is 
\[
g(X) = (2x+z-1/2)^2(1-\epsilon)^2 + (2y+z-1/2)^2(1-\epsilon)^2 + 2(z/k-1/2)^2(1+\epsilon')^2.
\]

Since \( x, y, z \) are integers and \( k \) is odd, \( g(X) \leq 1 \) only when \( 2x+z=2y+z=0 \) or 1, and \( |z/k-1/2| \leq 1/2 \).

If \( 2x+z=2y+z=0 \), then \( z \) is even; \( g(X) < 1 \) when \( z=2i \) \((1 \leq i \leq (k-1)/2)\), and \( g(X)=1 \) when \( z=0 \) since 
\[
(1-\epsilon)^2/4 + (1-\epsilon)^2/4 + (1+\epsilon')^2/2 = 1.
\]

If \( 2x+z=2y+z=1 \), then \( z \) is odd; \( g(X) < 1 \) when \( z=2i-1 \) \((1 \leq i \leq (k-1)/2)\), and \( g(X)=1 \) when \( z=k \).

Thus in \( \Lambda_k' \) there are only \( k-1 \) points
\[
((1+\epsilon')2i\sqrt{2}/k,0,0)
\]
and
\[
((1+\epsilon')(2i-1)\sqrt{2}/k,1-\epsilon,1-\epsilon) \quad (1 \leq i \leq (k-1)/2)
\]
with distance less than 1 from
\[
((1+\epsilon')\sqrt{2}/2,(1-\epsilon)/2,(1-\epsilon)/2);
\]
there are two lattice points.
and all other lattice points have distance greater than 1 from it. Thus $K_3 + \Lambda_k'$ is a k-fold packing for $\mathbb{R}^3$. It is clear that $K_3 + \Lambda_k'$ has a density equal to $kd_1^3/(1-\epsilon)^2(1-\epsilon') = kd_1^3/(1-\epsilon)^2\sqrt{(1+2\epsilon-\epsilon^2)} > kd_1^3$.

3. $d_k^4 > kd_1^4$

Let $\Lambda$ be the 4-dimensional lattice generated by $(0,0,0,0), (0,0,2,0), (0,2,0,0)$ and $(1,1,1,1)$, then $K_4 + \Lambda$ is a closest lattice packing of unit spheres in $\mathbb{R}^4$ with density $d_1^4$ [21]. Let $\Lambda_k$ be the lattice generated by $(0,0,0,2), (0,0,2,0), (0,2,0,0)$ and $(1/k,1,1,1)$; then $\Lambda_k = \bigcup_{0 \leq i \leq k-1} ((2i/k,0,0,0) + \Lambda)$, since $k$ is odd. Thus $K_4 + \Lambda_k$ is a k-fold lattice packing with density $kd_1^4$. Now consider the lattice $\Lambda_k'$ generated by $(0,0,0,2-2\epsilon), (0,0,2-2\epsilon,0)$, $(0,2-2\epsilon,0,0)$ and $((1+\epsilon')/k,1-\epsilon,1-\epsilon,1-\epsilon)$, where $\epsilon$ is a sufficiently small positive number and $\epsilon' = \sqrt{(1+6\epsilon-3\epsilon^2)} - 1$.

**Lemma 2.2.** $K_4 + \Lambda_k'$ is a k-fold lattice packing with density greater than $kd_1^4$.

**Proof.** Since $\Lambda_k'$ is a neighboring lattice of $\Lambda_k$ and $K_4 + \Lambda_k$ is a k-fold lattice packing, all points in $\mathbb{R}^4$ except possibly certain small neighborhoods of $P + \Lambda_k'$ will be covered at most k times by $K_4 + \Lambda_k'$, where $P$ is one of
the following 24 points:

\((0,0,0,\pm(1-\varepsilon)), (0,0,\pm(1-\varepsilon),0), (0,\pm(1-\varepsilon),0,0),\\
(\pm(1+\varepsilon'),0,0,0), (\pm(1+\varepsilon')/2,\pm(1-\varepsilon)/2,\pm(1-\varepsilon)/2)\).

By symmetry it suffices to consider the three neighborhoods around \((0,0,0,1-\varepsilon),(1+\varepsilon',0,0,0)\) and \(((1+\varepsilon')/2,(1-\varepsilon)/2,(1-\varepsilon)/2,(1-\varepsilon)/2)\). It is easy to see that in \(\Lambda_k\), there are only two lattice points \((0,0,0,0)\) and \((0,0,0,2-2\varepsilon)\) with distance less than 1 from \((0,0,0,1-\varepsilon)\); and all other lattice points have distance greater than 1 from \((0,0,0,1-\varepsilon)\). In \(\Lambda_k\), there are only \(k-1\) lattice points \(((1+\varepsilon')2i/k,0,0,0)\) \((1\leq i\leq k-1)\) with distance less than 1 from \((1+\varepsilon',0,0,0)\); and all other lattice points have distance greater than 1 from \((1+\varepsilon',0,0,0)\). In \(\Lambda_k\), there are only \(k-1\) lattice points \(((1+\varepsilon')2i/k,0,0,0)\) and \(((1+\varepsilon')(2i-1)/k,1-\varepsilon,1-\varepsilon,1-\varepsilon)\) \((1\leq i\leq (k-1)/2)\) with distance less than 1 from \(((1+\varepsilon')/2,(1-\varepsilon)/2,(1-\varepsilon)/2,(1-\varepsilon)/2)\); there are two lattice point \((0,0,0,0)\) and \((1+\varepsilon',1-\varepsilon,1-\varepsilon,1-\varepsilon)\) with distance equal to 1 from \(((1+\varepsilon')/2,(1-\varepsilon)/2,(1-\varepsilon)/2,(1-\varepsilon)/2)\), since \(\varepsilon'=(\sqrt{1+6\varepsilon-3\varepsilon^2})-1\); and all other lattice points have distance greater than 1 from it. Thus \(K_4+\Lambda_k^\prime\) is a \(k\)-fold lattice packing for \(\mathbb{R}^4\). It is clear that \(K_4+\Lambda_k^\prime\) has a density equal to

\[kd_1^4 /((1-\varepsilon)^3(1+\varepsilon')) = kd_1^4 /((1-\varepsilon)^3\sqrt{1+6\varepsilon-3\varepsilon^2}) > kd_1^4.\]
Let \( \Lambda \) be the 5-dimensional lattice generated by 
\[
(\sqrt{2}, 0, 0, 0, 0, \sqrt{2}), (\sqrt{2}, 0, 0, \sqrt{2}, 0), (\sqrt{2}, \sqrt{2}, 0, 0, 0, 0) \quad \text{and} \quad (2\sqrt{2}, 0, 0, 0, 0, 0),
\]
then \( K_5 + \Lambda \) is a closest lattice packing of unit spheres in \( \mathbb{R}^5 \) with density \( d_1^5 \) [22]. Let \( \Lambda_k \) be the lattice generated by 
\[
(\sqrt{2}, 0, 0, 0, \sqrt{2}), (\sqrt{2}, 0, 0, \sqrt{2}, 0), (\sqrt{2}, \sqrt{2}, 0, 0, 0, 0) \quad \text{and} \quad (2\sqrt{2}, 0, 0, 0, 0, 0),
\]
then 
\[
\Lambda_k = \bigcup_{0 \leq i \leq k-1} ((2i\sqrt{2}/k, 0, 0, 0, 0) + \Lambda).
\]
Thus \( K_5 + \Lambda_k \) is a \( k \)-fold lattice packing with density \( kd_1^5 \). Now consider the lattice \( \Lambda'_k \) generated by 
\[
((1+\varepsilon')\sqrt{2}, 0, 0, 0, 0, 0), ((1-\varepsilon)\sqrt{2}),
((1+\varepsilon')\sqrt{2}, 0, 0, 0, 0, 0), ((1-\varepsilon)\sqrt{2}, 0, 0, 0, 0, 0),
((1+\varepsilon')\sqrt{2}, (1-\varepsilon)\sqrt{2}, 0, 0, 0, 0) \quad \text{and} \quad ((1+\varepsilon')2\sqrt{2}/k, 0, 0, 0, 0, 0),
\]
where \( \varepsilon \) is a sufficiently small positive number and 
\( \varepsilon' = \sqrt{(1+2\varepsilon-\varepsilon^2)}-1 \).

**Lemma 2.3.** \( K_5 + \Lambda'_k \) is a \( k \)-fold lattice packing with density greater than \( kd_1^5 \).

**Proof.** Since \( \Lambda'_k \) is a neighboring lattice of \( \Lambda_k \) and \( K_5 + \Lambda_k \) is a \( k \)-fold lattice packing, all points in \( \mathbb{R}^5 \) except possibly certain small neighborhoods of \( P + \Lambda'_k \), where \( P \) is one of the following 40 points:
\[
(\pm (1+\varepsilon')\sqrt{2}/2, \pm (1-\varepsilon)\sqrt{2}/2, 0, 0, 0),
(\pm (1+\varepsilon')\sqrt{2}/2, 0, \pm (1-\varepsilon)\sqrt{2}/2, 0, 0),
(\pm (1+\varepsilon')\sqrt{2}/2, 0, 0, \pm (1-\varepsilon)\sqrt{2}/2, 0),
\]
will be covered at most \( k \) times by \( K_5 + \Lambda'_k \),
\((\pm (1+\epsilon'))\sqrt{2}/2, 0, 0, 0, \pm (1-\epsilon)\sqrt{2}/2)\),
\((0, \pm (1-\epsilon)\sqrt{2}/2, \pm (1-\epsilon)\sqrt{2}/2, 0, 0)\),
\((0, \pm (1-\epsilon)\sqrt{2}/2, 0, 0, \pm (1-\epsilon)\sqrt{2}/2)\),
\((0, 0, \pm (1-\epsilon)\sqrt{2}/2, \pm (1-\epsilon)\sqrt{2}/2, 0)\),
\((0, 0, \pm (1-\epsilon)\sqrt{2}/2, 0, \pm (1-\epsilon)\sqrt{2}/2)\).

By symmetry it suffices to consider the two neighborhoods around
\(((1+\epsilon')\sqrt{2}/2, (1-\epsilon)\sqrt{2}/2, 0, 0, 0)\) and
\((0, 0, (1-\epsilon)\sqrt{2}/2, (1-\epsilon)\sqrt{2}/2)\). It is easy to see that in \(\Lambda_k^1\)
there are only \(k-1\) lattice points \(((1+\epsilon')2i\sqrt{2}/k, 0, 0, 0, 0)\)
and \(((1+\epsilon')(2i-1)\sqrt{2}/k, (1-\epsilon)\sqrt{2}/2, 0, 0, 0)\) \((1\leq i \leq (k-1)/2)\) with
distance less than 1 from \(((1+\epsilon')\sqrt{2}/2, (1-\epsilon)\sqrt{2}/2, 0, 0, 0)\); there are two lattice points \((0, 0, 0, 0, 0)\) and
\(((1+\epsilon')\sqrt{2}, (1-\epsilon)\sqrt{2}, 0, 0, 0)\) with distance equal to 1 from
\(((1+\epsilon')\sqrt{2}/2, (1-\epsilon)\sqrt{2}/2, 0, 0, 0)\), since \(\epsilon' = \sqrt{(1+2\epsilon-\epsilon^2)} - 1\);
and all other lattice points have distance greater than 1 from it. In \(\Lambda_k^1\) there are only two lattice points
\((0, 0, 0, 0, 0)\) and \((0, 0, 0, (1-\epsilon)\sqrt{2}, (1-\epsilon)\sqrt{2})\) with distance
less than 1 from \((0, 0, 0, (1-\epsilon)\sqrt{2}/2, (1-\epsilon)\sqrt{2}/2)\), and all
other lattice points have distance greater than 1 from it.
Thus \(K_5^5 + \Lambda_k^1\) is a k-fold lattice packing for \(R^5\). It is
clear that \(K_5^5 + \Lambda_k^1\) has a density equal to
\(kd_1^5 / ((1-\epsilon)^4(1+\epsilon')) = kd_1^5 / ((1-\epsilon)^4\sqrt{(1+2\epsilon-\epsilon^2)}) > kd_1^5\).
Let $\Lambda$ be the 3-dimensional lattice generated by $(0,0,4/\sqrt{5}), (0,4/\sqrt{5},0)$ and $(2/\sqrt{5}, 2/\sqrt{5}, 2/\sqrt{5})$, then $K_3 + \Lambda$ is a thinnest lattice covering of unit spheres in $\mathbb{R}^3$ with density $D_1^3 [1]$. Let $\Lambda_k$ be the lattice generated by $(0,0,4/\sqrt{5}), (0,4/\sqrt{5},0)$ and $(2/\sqrt{5}, 2/\sqrt{5}, 2/\sqrt{5})$; then

$$\Lambda_k = \bigcup_{0 \leq i \leq k-1} ((4i/\sqrt{5}), 0, 0) + \Lambda,$$

since $k$ is odd. Thus $K_3 + \Lambda_k$ is a k-fold lattice covering with density $kD_1^3$.

Now consider the lattice $\Lambda'_k$ generated by

$(0,0,(1+\varepsilon)4/\sqrt{5}), (0,(1+\varepsilon)4/\sqrt{5},0)$ and

$((1-\varepsilon')2/\sqrt{5}, (1+\varepsilon)2/\sqrt{5}, (1+\varepsilon)2/\sqrt{5})$, where $\varepsilon$ is a sufficiently small positive number and

$$\varepsilon' = \sqrt{(1+\varepsilon)}(1-8\varepsilon-4\varepsilon^2)-1 \approx 1.5\varepsilon.$$

Lemma 2.4. $K_3 + \Lambda'_k$ is a k-fold lattice covering with density less than $kD_1^3$.

Proof. Since $\Lambda'_k$ is a neighboring lattice of $\Lambda_k$, and $K_3 + \Lambda'_k$ is a k-fold lattice covering, all points in $\mathbb{R}^3$ except possibly certain small neighborhoods near $P + \Lambda'_k$ will be covered at least k times by $K_3 + \Lambda'_k$, where $P$ is one of the following 24 points:

$(\pm 2/\sqrt{5}, \pm 1/\sqrt{5}, 0), (\pm 2/\sqrt{5}, 0, \pm 1/\sqrt{5}), (0, \pm 2/\sqrt{5}, \pm 1/\sqrt{5}),$

$(0, \pm 1/\sqrt{5}, \pm 2/\sqrt{5}), (\pm 1/\sqrt{5}, 0, \pm 2/\sqrt{5}), (\pm 1/\sqrt{5}, \pm 2/\sqrt{5}, 0).$

Because of our choice for $\varepsilon'$, every point in $\mathbb{R}$ near the first sixteen points above will be covered at least $k$ times.
by $K_3^+\Lambda_k^1$. By symmetry it suffices to consider the neighborhood near $(1/\sqrt{5}, 0, 2/\sqrt{5})$. In $\Lambda_k^1$ there are $3(k-1)/2$ lattice points $((1-\varepsilon')4i/\sqrt{5}k, 0, 0)$, $((1-\varepsilon')(2i-1)2/\sqrt{5}k, \pm(1+\varepsilon)2/\sqrt{5}, (1+\varepsilon)2/\sqrt{5})$ $(1\le i\le (k-1)/2)$ with distance less than $c<1$ from $(1/\sqrt{5}, 0, 2/\sqrt{5})$, where $c$ is a positive number dependent on $k$ but independent on $\varepsilon$. Thus $K_3^+\Lambda_k^1$ is a $k$-fold lattice covering. It is clear that $K_3^+\Lambda_k^1$ has a density equal to $kD_1^3/((1+\varepsilon)^2(1-\varepsilon')) < kD_1^3/((1+\varepsilon)^2(1-1.6\varepsilon)) < kD_1^3$.

Combining all the known results listed in section 1 with the four lemmas 2.1-2.4, we have the following theorem.

**Theorem.** The strict inequality $d_k^n > k d_1^n$ holds for all $(n,k)$ $(n\ge 2, k\ge 2)$ except $(2,2), (2,3), (2,4)$. The strict inequality $D_k^3 < kD_1^3$ holds for all $k\ge 2$. 
CHAPTER III

ON VENKOV'S METHOD OF FINDING THE MINIMUM OF A RATIONAL INDEFINITE TERNARY QUADRATIC FORM

1. Introduction

Let \( f(x, y, z) = ax^2 + ay^2 + a''z^2 + 2b''xy + 2b'xz + 2byz \) be an indefinite form with real coefficients and positive

\[
\begin{vmatrix}
    a & b'' & b' \\
    b'' & a' & b \\
    b' & b & a
\end{vmatrix}
\]

let \( \min(f) = \inf |f(x, y, z)| \) for all integers \((x, y, z) \neq (0, 0, 0)\).

Markoff [25] obtained the first three largest values for \( \min(f) / \sqrt{\text{det}(f)} \) and we can find a complete proof of this due to Oppenheim in Dickson's book [10], where the fourth value was also obtained. So far the best result on this was given by Venkov [35], who obtained the first eleven values and showed that there is essentially one form corresponding to each of these eleven values and all these eleven forms have rational coefficients. Venkov had a method of finding the minimum of a given rational indefinite ternary quadratic form which does not represent
zero. In this chapter we shall sketch his method and apply it to his eleven forms. This method is also used to show the equivalence of two special forms.

2. The Structure of Integral Points
Near the Double Cone \( f=0 \)

It is clear that in \( \mathbb{R}^3 \), \( f=0 \) defines two convex cones \( \pm K \) with the origin 0 as a common vertex. Assuming \( f \) does not represent zero, then there is no integral point on \( \pm K \). Let \( M \) be the set of integral points lying inside \( K \), and let \( \Pi(f) \) be the convex hull of \( M \). \( \pm \Pi(f) \) is called the polyhedron associated with \( f \), and its vertices are called the interior vertices of the cone \( f=0 \) or the form \( f \).

A plane \( px+qy+rz=Epx=s \) with rational coefficients is called hyperbolic if the intersection of \( Epx=0 \) and \( f=0 \) is a pair of straight lines. On a hyperbolic plane \( Epx=0 \), the integral points near the pair of straight lines will give us a four-piece polygonal structure like the planar polygon we mentioned in Introduction 3. An integral point \( P \) outside the cone \( \pm K \) will be called an exterior vertex of \( f \) if for every rational hyperbolic plane passing through \( OP \), \( P \) is a vertex of the associated planar polygon in this plane. From Venkov's work \([34,35]\), we have the following four lemmas.
Lemma 3.1. An integral point P is a vertex (interior or exterior) of f if and only if P is a vertex of the planar polygons of all rational hyperbolic planes passing through OP.

Lemma 3.2. If P is an exterior vertex of f, then there exists a two-way infinite belt (sequence) of faces of II(f) which are connected side by side and all the connected sides are parallel to OP.

Lemma 3.3. Assume min(f)=1. If P is an interior vertex of f, then $1 \leq f(P) \leq \sqrt[3]{144 \det(f)/\pi^2}$; if P is an exterior vertex of f, then $-4 \sqrt[3]{\det(f)} \leq f(P) \leq -1$.

Lemma 3.4. Assume min(f)=1 and let P be an integral point. If $1 \leq f(P) < 2$, then P is an interior vertex; if $-2 < f(P) \leq -1$, then P is an exterior vertex.

Now let f have rational coefficients. A linear transformation $x'=a_1x+b_1y+c_1z, y'=a_2x+b_2y+c_2z, z'=a_3x+b_3y+c_3z$ with integral coefficients and determinant ±1 and transforming f to itself is called an automorphism of f. Two integral points $(x,y,z),(x',y',z')$ connected by an automorphism are said to be equivalent and denoted by $(x,y,z)\simeq(x',y',z')$. Let us denote by m and $-m'$ the positive and negative minima of f over all integral points except 0, where m and m' are positive numbers and $\min(f)=\min(m,m')$. 
If \( P \) is an integral point with \( f(P)=m \) or \( m' \), then by Lemma 3.1, \( P \) is an interior or exterior vertex of \( f \). Venkov [34] showed that there exists only a finite number of non-equivalent interior vertices, then on \( \Pi(f) \) there are only a finite number of non-equivalent faces. It follows from Lemma 3.2 that there are only a finite number of non-equivalent exterior vertices. Thus it is possible to calculate \( \Pi(f) \) and obtain a complete finite system of non-equivalent vertices, sides and faces of \( \Pi(f) \). Then it is easy to find all non-equivalent face-belts mentioned in Lemma 3.2 and obtain all non-equivalent interior and exterior vertices. An example of such calculation is given in the next section.

3. Numerical Examples

Let \( f(x,y,z)=-7x^2-7y^2+17z^2-6xy+2yz+18yz \). It is convenient to denote \( f \) by

\[
\begin{bmatrix}
-7 & -3 & 1 \\
-3 & -7 & 9 \\
1 & 9 & 17
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
-7 & -3 & 1 \\
-7 & 9 \\
17
\end{bmatrix}.
\]

Use a computer to calculate the values of \( f(x,y,z) \) for all integral points \((x,y,z)\) with \(|x| \leq 30, |y| \leq 30, 0 \leq z \leq 30\). Among these values we find the absolute minimum 7. It is reasonable to guess \( \min(f) \) to be 7. Now we consider the form \( g=f/7 \) and run the computer again. This time we use
Lemmas 3.3 and 3.4 to select the possible interior vertices. Then we choose a convenient plane for projection, e.g. xy-plane is a good one for \( g \); project these possible interior vertices on the xy-plane and label their z-coordinates (Figure 2). It is easy to connect them to form a possible portion of \( \Pi(g) \). Since the equations of all these possible faces in Figure 2 are of the type \( px+qy+rz=l \), \( (p,q,r)=l \); these faces are true faces of \( \Pi(g) \). Then from equivalent triples of interior vertices we can obtain automorphisms (Figure 2). From Figure 2 and Lemma 3.2, it is clear that \( g \) has two non-equivalent interior vertices \( A, C \) and two non-equivalent exterior vertices \( A-D, (A-C)/2 \). But \( g(A)=8/7, g(C)=12/7, g(A-D)=-8/7 \) and \( g((A-C)/2)=-1 \); thus \( m=8/7, m'=1 \) and \( \min(g)=1, \min(f)=7 \), as we guessed.

In fact, \( g \) is the fifth form obtained by Venkov. In the following we include the graphs (Figures 3-12) of the polyhedra of the other ten forms \( f_i (1 \leq i \leq 11, i \neq 5) \) obtained by Venkov [35], who only gave the graph for \( f_8 \).
\[
g = f_5 = \begin{bmatrix}
-1 & -3/7 & 1/7 \\
-1 & 9/7 \\
17/7
\end{bmatrix}
\]

\[
\det = \frac{1200}{7^3} = 3,498...
\]

\[m = \frac{8}{7}, \quad m' = 1\]

\[
A(-1,0,1) \approx H(1,2,1) \quad x' = -y + z
\]

\[
D(-2,1,1) \approx G(0,3,1) \quad y' = -x + z
\]

\[
F(-2,3,1) \approx F(-2,3,1) \quad z' = z
\]

\[
D(-2,1,1) \approx A(-1,0,1) \quad x' = y - 2z
\]

\[
F(-2,3,1) \approx C(1,0,1) \quad y' = x + 2z
\]

\[
G(0,3,1) \approx H(1,2,1) \quad z' = z
\]

\[
D(-2,1,1) \approx H(1,2,1) \quad x' = 2x + 5z
\]

\[
F(-2,3,1) \approx C(1,0,1) \quad y' = -y + 3z
\]

\[
G(0,3,1) \approx J(5,0,3) \quad z' = x + 3z
\]

Figure 2. The polyhedron of \(g\)
\[ f_1 = \begin{bmatrix} -1 & -1/2 & 0 \\ -1 & 0 \\ 2 \end{bmatrix} \]

\[ \text{det} = \frac{3}{2} = 1.5 \]

\[ m = 1, \ m' = 1 \]

\[
A(1,0,1) = D(1,2,2) \quad x' = x \\
B(0,1,1) = B(0,1,1) \quad y' = -2x - 3y + 4z \\
C(-1,1,1) = E(-1,3,2) \quad z' = -x - 2y + 3z
\]

\[
A = B = C = D = E
\]

Figure 3. The polyhedron of \( f_1 \)
$$f_2 = \begin{bmatrix} 1 & 1/2 & 0 \\ -1 & 0 \\ -2 \end{bmatrix}$$

$$\text{det} = 5/2 = 2.5$$

$$m = 1, m' = 1$$

Figure 4. The polyhedron of \( f_2 \)
$$f_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det = 3$$

$m = 1, m' = 1$

$x' = 2x + 3y + 6z$

$y' = 3x + 2y + 6z$

$z' = 2x + 2y + 5z$

$A = B = C = D = E = F = G$

![Figure 5. The polyhedron of $f_3$](image-url)
\[
\begin{bmatrix}
-1 & -1/2 & 0 \\
0 & 1 & 0 \\
-5/2 & & 
\end{bmatrix}
\]

\[f_4 = \begin{bmatrix}
-1 & -1/2 & 0 \\
0 & 1 & 0 \\
-5/2 & & 
\end{bmatrix}\]

\[
\det = \frac{25}{8} = 3.125
\]

\[m = 1, \ m' = 1\]

\[C(0, 2, 1) \Rightarrow G(2, 4, 1) \quad x' = -2x + y \]
\[A(0, 1, 0) \Rightarrow B(1, 2, 0) \quad y' = -3x + 2y \]
\[B(1, 2, 0) \Rightarrow A(0, 1, 0) \quad z' = z \]

\[J(2, 5, 2) \Rightarrow A(0, 1, 0) \quad x' = -3x + 2y - 2z \]
\[G(2, 4, 1) \Rightarrow C(0, 2, 1) \quad y' = -6x + 5y - 6z \]
\[B(1, 2, 0) \Rightarrow H(1, 4, 2) \quad z' = -2x + 2y - 3z \]

\[I(-1, 3, 2) \Rightarrow H(1, 4, 2) \quad x' = -x \]
\[C(0, 2, 1) \Rightarrow C(0, 2, 1) \quad y' = -x + y \]
\[E(-1, 1, 0) \Rightarrow B(1, 2, 0) \quad z' = z \]
\[C(0, 2, 1) \Rightarrow D(0, 2, -1) \quad x' = x \]
\[A(0, 1, 0) \Rightarrow A(0, 1, 0) \quad y' = y \]
\[B(1, 2, 0) \Rightarrow B(1, 2, 0) \quad z' = -z \]

\[A = B = E = H = I \]

\[C = D \]

Figure 6. The polyhedron of \(f_4\)
\[ f_6 = \begin{bmatrix} -1 & 0 & -1/2 \\ -1 & -1/2 & 3 \end{bmatrix} \]

\[ \det = \frac{7}{2} = 3.5 \]

\[ m = 1, m' = 1 \]

\[ C(1,-1,1) \approx D(2,1,2) \quad x' = -4x - y + 5z \]
\[ A(1,0,1) \approx A(1,0,1) \quad y' = -y \]
\[ B(0,1,1) \approx E(4,-1,3) \quad z' = -3x - y + 4z \]

\[ A(1,0,1) = B(0,1,1) \quad x' = y \]
\[ B(0,1,1) = A(1,0,1) \quad y' = x \]
\[ F(-1,1,1) = C(1,-1,1) \quad z' = z \]

\[ C(1,-1,1) = B(0,1,1) \quad x' = -y - z \]
\[ A(1,0,1) = F(-1,1,1) \quad y' = x \]
\[ B(0,1,1) = G(-2,0,1) \quad z' = z \]

Figure 7. The polyhedron of \( f_6 \)
The polyhedron of $f_7$. 

Figure 8.
\[
\begin{bmatrix}
-7/5 & 1 & 9/10 \\
-11/5 & 1/10 & 1
\end{bmatrix}
\]

\( f_8 = \begin{bmatrix} -7/5 & 1 & 9/10 \end{bmatrix} \)

\( \det = 3 \cdot 13^{2/5^3} = 4,056 \)

\( m=1, \ m'=1 \)

\( A(0,-1,2) = B(1,0,1) = C(1,1,1) = P(0,0,1) = F(-1,-1,3) \)

\( = x'=-y \)

\( y'=-x \)

\( z'=x+y+z \)

\( A(0,-1,2) = D(0,2,3) = G(1,-1,2) = I(-1,3,6) \)

\( = x'=2x+3y+2z \)

\( y'=2x+y+z \)

\( z'=x+y+z \)

\( A(0,-1,2) = N(6,-8,13) \)

\( x'=-3x+14y+10z \)

\( G(1,-1,2) = L(3,-5,8) \)

\( y'=3x-22y-15z \)

\( z'=-5x+35y+24z \)

\( A=B=C=D=E=F \)

\( P=K=H \)

\( G=L=M \)

\( \begin{figure}
\text{Figure 9. The polyhedron of } f_8
\end{figure} \)
\[
\begin{bmatrix}
-1 & 1/3 & 0 \\
-1 & -1 & 4/3 \\
& & 8/3
\end{bmatrix}
\]

\[f_9 = \begin{bmatrix}
-1 & 1/3 & 0 \\
-1 & -1 & 4/3 \\
& & 8/3
\end{bmatrix}
\]

\[
\text{det}=7 \cdot 2^{4/3} \\
=4,148...
\]

\[m=5/3, \ m'=1\]

\[F(2,3,1) = C(-1,0,1) \quad x'=y+2z\]

\[D(2,1,1) = A(1,0,1) \quad y'=-x+2z\]

\[A(1,0,1) = D(2,1,1) \quad z'=z\]

\[E(3,0,2) = D(2,1,1) \quad x'=2y+z\]

\[A(1,0,1) = A(1,0,1) \quad y'=x-y-z\]

\[B(1,-2,3) = C(-1,0,1) \quad z'=-x+2y+2z\]

\[D(2,1,1) = G(-2,-1,2) \quad x'=-x\]

\[A(1,0,1) = C(-1,0,1) \quad y'=-y\]

\[E(3,0,2) = H(-3,0,2) \quad z'=y+z\]

\[A=B=C=D=E\]

---

**Figure 10. The polyhedron of \(f_9\)**
\[ f_{10} = \begin{bmatrix} -1 & 1/2 & 0 \\ -1 & 3/2 \\ 21/8 \end{bmatrix} \]

\[ \det = \frac{125}{2^5} = 4,218 \ldots \]

\[ m = 3/2, \ m' = 1 \]

\begin{align*}
A(-1,0,1) & \approx B(1,0,1) & x' &= x - y + 2z \\
B(1,0,1) & \approx C(3,2,1) & y' &= x + z \\
C(3,2,1) & \approx D(3,4,1) & z' &= z \\
B(1,0,1) & \approx H(8,3,3) & x' &= -10x + 9y + 18z \\
C(3,2,1) & \approx I(6,1,3) & y' &= -3x + 2y + 6z \\
G(6,3,2) & \approx J(3,0,2) & z' &= -4x + 4y + 7z \\
B(1,0,1) & \approx B(1,0,1) & x' &= -x + 2z \\
J(3,0,2) & \approx L(1,-1,2) & y' &= x + y + z \\
K(3,-1,3) & \approx K(3,-1,3) & z' &= z \\
B(1,0,1) & \approx K(3,-1,3) & x' &= -5x + 8y + 8z \\
J(3,0,2) & \approx L(1,-1,2) & y' &= x - 3y - 2z \\
K(3,-1,3) & \approx B(1,0,1) & z' &= -4 + 8y + 7z
\end{align*}

Figure 11. The polyhedron of \( f_{10} \)
$f_{11} = \begin{bmatrix} -1 & 1/2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

$\det = 9/2 = 4.5$

$m = 2, m' = 1$

$C(0, 2, 1) = F(2, 4, 1) \quad x' = y$

$A(0, 0, 1) = C(0, 2, 1) \quad y' = -x + y + 2z$

$B(2, 0, 1) = A(0, 0, 1) \quad z' = z$

$C(0, 2, 1) = E(0, -2, 3) \quad x' = -x$

$A(0, 0, 1) = A(0, 0, 1) \quad y' = -y$

$B(2, 0, 1) = D(-2, 0, 3) \quad z' = x + y + z$

Figure 12. The polyhedron of $f_{11}$
4. Another Application

As Mordell [27] pointed out, in Dickson's book [10] there are two forms whose equivalence was not decided. These two forms are \( g_1 = x^2 - 3y^2 - 2yz - 23z^2 \), and \( g_2 = x^2 - 7y^2 - 6yz - 11z^2 \). In Figure 13 we have two small portions of \( \Pi(g_1) \) and \( \Pi(g_2) \); the triple \((A,B,C)\) is transformed to \((D,E,F)\) by the unimodular transformation \( x' = 25x - 28y - 100z, \ y' = 4x - 4y - 17z, \ z' = -8x + 9y + 32z \). Then it is easy to see that \( g_1 \) is transformed to \( g_2 \) by this transformation, so they are equivalent.

A(5,0,1)\( \equiv \) D(25,3,-8) \( x' = 25x - 28y - 100z \)
B(1,0,0)\( \equiv \) E(25,4,-8) \( y' = 4x - 4y - 17z \)
C(2,1,0)\( \equiv \) F(22,4,-7) \( z' = -8x + 9y + 32z \)

Figure 13. The equivalence of \( g_1 \) and \( g_2 \)
ON THE RESTRICTED MARKOFF SPECTRA
OF QUADRATIC FIELDS

1. Let $f(x,y)=ax^2+bxy+cy^2=a(x-\alpha y)(x-\beta y)$ be a real binary quadratic form with positive discriminant $\text{disc}(f)=b^2-4ac=a^2(\alpha-\beta)^2$, and let $\min(f)=\inf |f(x,y)|$, for integers $(x,y)\neq (0,0)$.

Markoff [24] discovered that there are only a countable number of values greater than $1/3$ for $\min(f)/\sqrt{\text{disc}(f)}$. The structure of these values less than $1/3$ are more complicated. The set of all the values $\min(f)/\sqrt{\text{disc}(f)}$ is called the Markoff spectrum.

Now we restrict $\alpha,\beta$ to be in a real quadratic field $\mathbb{Q}(\sqrt{d})$, where $d$ is a positive non-square integer. We shall show that the restricted spectrum has a non-zero limit point for every $\mathbb{Q}(\sqrt{d})$.

2. We denote a finite continued fraction

$$\frac{1}{a_0+\frac{1}{a_1+\frac{1}{a_2+\ldots+\frac{1}{a_m}}}}$$

by $[a_0,a_1,a_2,\ldots,a_m]$. 69
where \( a_0 \) is a real number and \( a_i \) \((1 \leq i \leq m)\) are positive numbers. \( a_i \) \((0 \leq i \leq m)\) are called the partial quotients of the continued fraction. From Hardy and Wright's book [17] we have the following two lemmas.

**Lemma 4.1.** If \( p_i \) and \( q_i \) are defined by
\[
p_0 = a_0, \quad p_i = a_i p_{i-1} + p_{i-2}, \quad q_i = a_i q_{i-1} + q_{i-2}, \quad (2 < i < m);
\]
then
\[
[a_0, a_1, a_2, \ldots, a_m] = \frac{p_m}{q_m}.
\]

**Lemma 4.2.** \( p_m q_{m-1} - p_{m-1} q_m = (-1)^{m-1}. \)

The continued fraction \([a_0, a_1, a_2, \ldots; a_m]\) is said to be **simple** if \( a_0 \) is an integer and \( a_i \) \((1 \leq i \leq m)\) are positive integers. An **infinite** simple continued fraction \([a_0, a_1, a_2, \ldots; a_m, \ldots]\) is the limit of its \( m \)-th convergent \([a_0, a_1, a_2, \ldots, a_m]\), as \( m \to \infty \). A **periodic** continued fraction \([a_0, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_k]\) is an infinite continued fraction with repeated partial quotients \( b_1, b_2, \ldots, b_k \). It is well known [17] that a simple continued fraction is periodic if and only if it is a quadratic irrational number. In Perron's book [30] we can find the following lemma due to Galois.

**Lemma 4.3.** If \( \alpha = [a_0, a_1, a_2, \ldots; a_m] \) with positive integers \( a_i \) \((0 \leq i \leq m)\), then the conjugate \( \beta \) of \( \alpha \) lies between \(-1\) and \( 0 \) and \( -\beta = [0, a_m, a_{m-1}, \ldots, a_2, a_1, a_0] \). Conversely, if \( \alpha > 1 \) is a quadratic irrational number and
its conjugate $\beta$ lies between $-1$ and 0, then the simple continued fraction of $\alpha$ is of the type $[a_0, a_1, a_2, \ldots, a_m]$.

3. Given two positive integers $m$ and $n$, let $x_m$ be the continued fraction
\[
[2n+2,1,2n,\ldots,1,2n,1,2n+1,1,2n,\ldots,1,2n,1,2n+1,2n+3].
\]
We are going to show that $x_m$ is in the field $\mathbb{Q}(\sqrt{n^2+2n})$.

**Lemma 4.4.** Let $[1,2n,\ldots,1,2n] = \frac{p_{2m-1}}{q_{2m-1}}^{2m}$ and $[1,2n,\ldots,1,2n,1] = \frac{p_{2m}}{q_{2m}}^{2m}$, then

\begin{align*}
(4.1) \quad [1,2n,\ldots,1,2n,1,2n+1] &= \frac{p_{2m+2}}{p_{2m+1}}^{2m} , \\
(4.2) \quad [1,2n,\ldots,1,2n,1,2n+1,2n-1] &= \frac{2np_{2m+2} - p_{2m+1}}{4np_{2m+1} - 2np_{2m+2}} \\
(4.3) \quad [1,2n,\ldots,1,2n,1,2n+1,2n+3] &= \frac{(2n+4)p_{2m+2} - p_{2m+1}}{(4n+4)p_{2m+1} - 2np_{2m+2}}.
\end{align*}

**Proof.** By Lemma 4.1 and induction, it is easy to see that $p_{2m+1} = 2np_{2m} + p_{2m-1}$, $p_{2m} = p_{2m-1} + p_{2m-2}$; $q_{2m+1} = 2nq_{2m} + q_{2m-1}$, $q_{2m} = q_{2m-1} + q_{2m-2}$; and $q_{2m-1} = 2np_{2m-2}$, $q_{2m} = p_{2m-1}$.

Then $[1,2n,\ldots,1,2n,1,2n+1] = \frac{[1,2n,\ldots,1,2n,\frac{2n+2}{2n+1}]}{2m}$.
\[
\frac{(2n+2)P_{2m-1} + (2n+1)P_{2m-2}}{(2n+2)q_{2m-1} + (2n+1)q_{2m-2}} = \frac{P_{2m-1} + (2n+1)P_{2m}}{q_{2m-1} + (2n+1)q_{2m}}
\]

\[
\frac{P_{2m+1} + P_{2m}}{q_{2m+1} + q_{2m}} = \frac{P_{2m+2}}{q_{2m+2}} = \frac{P_{2m+2}}{p_{2m+1}}.
\]

Similarly, \([1,2n, \ldots, 1,2n,1,2n+1,2n-1]\)

\[
\frac{(2n-1)P_{2m+2} + P_{2m}}{(2n-1)P_{2m+1} + q_{2m}} = \frac{2np_{2m+2} - p_{2m+1}}{4np_{2m+1} - 2np_{2m+2}},
\]

and \([1,2n, \ldots, 1,2n,1,2n+1,2n+3]\)

\[
\frac{(2n+3)P_{2m+2} + P_{2m}}{(2n+3)P_{2m+1} + q_{2m}} = \frac{(2n+4)p_{2m+2} - p_{2m+1}}{(4n+4)p_{2m+1} - 2np_{2m+2}}.
\]

Lemma 4.5. Let \(A,B,C,D\) be positive integers with \((A,B)=1\), \((C,D)=1\) and

\[
A = \frac{[1,2n, \ldots, 1,2n,1,2n+1,2n-1,1,2n, \ldots, 1,2n,1,2n+1,2n+3]}{2m},
\]

\[
B = \frac{[1,2n, \ldots, 1,2n,1,2n+1,2n-1,1,2n, \ldots, 1,2n,1,2n+1]}{2m};
\]

\[
C = \frac{[1,2n, \ldots, 1,2n,1,2n+1,2n-1,1,2n, \ldots, 1,2n,1,2n+1]}{2m};
\]

\[
D = \frac{[1,2n, \ldots, 1,2n,1,2n+1,2n-1,1,2n, \ldots, 1,2n,1,2n+1]}{2m};
\]

\[
A = (4n^2 + 6n)p_{2m+2}^2 + p_{2m+1}^2,
\]

\[
B = 4p_{2m+1}^2 + (8n^2 + 16n)p_{2m+2}^2 - (4n^2 + 8n)p_{2m+2}^2,
\]

\[
C = 2np_{2m+2},
\]

\[
D = p_{2m+1}^2 + 4np_{2m+1}p_{2m+2} - 2np_{2m+2}^2;
\]

and

\[
AD - BC = 1.
\]
Proof. From (4.3) and (4.1) we have

\[
\begin{align*}
A &= \left[ 1, 2n, \ldots, 1, 2n, 1, 2n+1, 2n-1, \ldots, 1, 2n, 1, 2n+1, 2n-1, \ldots, 1, 2n, 1, 2n+1 \right] \quad \text{and} \\
&= \frac{(2n+4)p_{2m+2} - p_{2m+1}}{(4n+4)p_{2m+1} - 2np_{2m+2}} \\
B &= \left[ 1, 2n, \ldots, 1, 2n, 1, 2n+1, 2n-1, \ldots, 1, 2n, 1, 2n+1 \right] \\
&= \frac{p_{2m+2}}{p_{2m+1}}.
\end{align*}
\]

Then by Lemmas 4.1, 4.2 and (4.1), (4.2); we have

\[
A = (2np_{2m+2} - p_{2m+1})((2n+4)p_{2m+2} - p_{2m+1}) \\
+ p_{2m+2}((4n+4)p_{2m+1} - 2np_{2m+2}) \\
= (4n^2 + 6n)p_{2m+2}^2 + 2p_{2m+1}.
\]

\[
B = (4np_{2m+1} - 2np_{2m+2})((2n+4)p_{2m+2} - p_{2m+1}) \\
+ p_{2m+1}((4n+4)p_{2m+1} - 2np_{2m+2}) \\
= 4p_{2m+1}^2 + (8n^2 + 16n)p_{2m+2}^2 - (4n^2 + 8n)p_{2m+2}^2.
\]

\[
C = (2np_{2m+2} - p_{2m+1})p_{2m+2} + p_{2m+2}p_{2m+1} = 2np_{2m+2}.
\]

\[
D = (4np_{2m+1} - 2np_{2m+2})p_{2m+2} + p_{2m+2}p_{2m+1}p_{2m+1} \\
= p_{2m+1}^2 + 4np_{2m+1}p_{2m+2} - 2np_{2m+2}.
\]

By Lemma 4.2, \( AD - BC = (-1)^{4m+4} = 1. \)

Lemma 4.6. \( x_m \) satisfies the equation

\[
2np_{2m+2}^2 x_m^2 - 4np_{2m+2}^2 (np_{2m+2} + p_{2m+1}) x_m - 4(n^2p_{2m+2}^2 + p_{2m+1}^2) = 0.
\]

Proof. From the definition of \( x_m \), we have

\[
\begin{align*}
x_m - (2n+2) \\
&= \left[ 1, 2n, \ldots, 1, 2n, 1, 2n+1, 2n-1, \ldots, 1, 2n, 1, 2n+1, 2n-1, \ldots, 1, 2n, 1, 2n+1 \right] \\
&= \frac{A+C(x_m - (2n+2))}{B+D(x_m - (2n+2))}.
\end{align*}
\]
Then \(C(x_m-(2n+2))^2+(A-D)(x_m-(2n+2))-B = 0\), or
\[C x_m^2 + (A-D-(4n+4)C)x_m + ((2n+2)^2C+(2n+2)(D-A)-B) = 0.
\]
But by Lemma 4.5,
\[C = \frac{2n^2}{2m+2},
\]
\[A-D-(4n+4)C
\]
\[= \frac{((4n^2+6n)p_{2m+2}^2+p_{2m+1})-\left(p_{2m+1}^2+4np_{2m+1}p_{2m+2}-2np_{2m+2}^2\right)}{(2n+2)^2C+(2n+2)(D-A)-B}
\]
\[= -4np_{2m+2}^2\left(n^2+2n\right)^2.
\]

**Corollary.** \(x_m\) is in the field \(\mathbb{Q}(\sqrt{n^2+2n})\).

**Proof.** The discriminant of the equation in Lemma 4.6 is
\[
(4np_{2m+2})^2(n^2+p_{2m+1})^2+32np_{2m+2}^2(n^2+p_{2m+1}^2)
\]
\[= (4n_{2m+2}^2(n^2+2n))^2.
\]

4. Now for a given positive integer \(n\), let \(f_m(x,y)\)
\[= 2np_{2m+2}^2x^2-4np_{2m+2}^2(n^2+p_{2m+2}^2){x}{y}-4(n^2+p_{2m+2}^2)^2{y}^2.
\]
We shall use a method originated from Euler [11] to find \(\min(f_m)\), this method was mentioned by Hightower [19].
Lemma 4.7. Let \( \alpha = [a_0, a_1, \ldots, a_h, b_1, b_2, \ldots, b_k] \) and \( f(\alpha, 1) = 0 \), where \( f(x, y) = ax^2 + bxy + cy^2 \); then there is an integer \( i \) with \( 1 \leq i \leq k \) such that the convergent \( s/t = [a_0, a_1, \ldots, a_h, b_1, \ldots, b_i] \) makes \( |f(s, t)| = \min(f) \). Furthermore, \( b_{i+1} \geq \max(b_1, \ldots, b_k) - 1 \) with the convention \( b_{k+1} = b_1 \).

Proof. For each \( i \) (\( 1 \leq i \leq k \)), the convergent \( s/t = [a_0, \ldots, a_h, b_1, \ldots, b_i] \) gives a vertex \((s, t)\) for the planar polygon associated with \( f \) (see Introduction 3.) and \( \min(f) \) attains at one of these vertices. Let \( \alpha' = [b_{i+1}, b_{i+2}, \ldots, b_k, b_1, b_2, \ldots, b_i] \) and let \( \beta' \) be the conjugate of \( \alpha' \); then by Lemma 4.3,
\[
-\beta' = [0, b_i, b_{i-1}, \ldots, b_2, b_1, b_k, b_{k-1}, \ldots, b_{i+2}, b_{i+1}].
\]
The second statement of the lemma follows immediately from the fact that
\[
\sqrt{(b^2 - 4ac)|f(s, t)|} = \alpha' - \beta' = [b_{i+1}, b_{i+2}, \ldots, b_k, b_1, b_2, \ldots, b_i] + [0, b_i, b_{i-1}, \ldots, b_2, b_1, b_k, b_{k-1}, \ldots, b_{i+1}].
\]

Lemma 4.8. \( \min(f_m) = 2np_{2m+2}^2 \).

Proof. By Lemmas 4.5, 4.6 and 4.7, the convergent
\[
[2n+2, 1, 2n, \ldots, 1, 2n, 1, 2n+1, 2n-1, 1, 2n, \ldots, 1, 2n, 1, 2n+1]
\]
\[
= [2n+2, C/D] = ((2n+2)C+D)/C \quad \text{gives us} \quad \min(f_m)
\]
\[
= |C((2n+2)C+D-(2n+2)C)^2+(A-D)((2n+2)C+D-(2n+2)C)C-BC^2|
\]
\[
= |C(AD-BC)| = C = 2np_{2m+2}^2.
\]
Theorem. For a positive non-square integer \( d \), let

\[
\begin{align*}
M_d = \{ \frac{\min(f)}{\sqrt{\text{disc}(f)}} : f(x,y) &= a(x-\alpha y)(x-\beta y), \ a \neq 0, \ a \neq \beta; \ \alpha, \beta \in \mathbb{Q}(\sqrt{d}) \}; \\
\end{align*}
\]

then \( M_d \) has a non-zero limit point.

Proof. It is well known [17] that the Pell equation \( x^2 - dy^2 = 1 \) has infinitely many solutions. In particular, there are positive integers \( n \) and \( y \) such that \( dy^2 = (n+1)^2 - 1 = n^2 + 2n \). By the Corollary of Lemma 4.6, \( \min(f_m)/\sqrt{\text{disc}(f_m)} \) are in \( M_d \) for all positive integers \( m \).

By Lemma 4.8,

\[
\min(f_m)/\sqrt{\text{disc}(f_m)} = \frac{2n p_{2m+2}^2}{4p_{2m+2}(np_{2m+2} + p_{2m+1})\sqrt{(n^2 + 2n)}}
\]

\[
= \frac{1}{2(1+p_{2m+1}/np_{2m+2})\sqrt{(n^2 + 2n)}}
\]

\[
= \frac{1}{2(1+q_{2m+2}/np_{2m+2})\sqrt{(n^2 + 2n)}}
\]

\[
\rightarrow \frac{1}{2(1+1/n[1,2n])\sqrt{(n^2 + 2n)}} \quad \text{(as } m \rightarrow \infty) \]

\[
= \frac{1}{2(n+2)}
\]

5. Since all the forms we got have rational coefficients, we can restrict \( \alpha \) and \( \beta \) to be conjugates in \( \mathbb{Q}(\sqrt{d}) \) for the definition of \( M_d \) and have the same result.

The problem of deciding the structure of \( M_d \) was proposed by Professor A.C. Woods, who showed to his class
that $M_5$ has a non-zero limit point.

Another related problem, due to late Professor B. Diviš, is that whether every real real quadratic field contains a number whose simple continued fraction expansion is made up only of 1's and 2's.
LIST OF REFERENCES


