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DISSERTATION

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School of The Ohio State University

By
Cheng Hsiung Chiu, B.A., M.A.

The Ohio State University
1978

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PUBLICATIONS


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FIELDS OF STUDY

Major Field: Monetary Theory and Banking and Finance

Studies in Monetary Theory. Professors William G. Dewald and Karl Brunner

Studies in Banking and Finance. Professor Ernst Baltensperger
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<td>v, V, $z_1$, w</td>
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\[ \Sigma \] Dynamic covariance matrix of state variables when new information variable is not used to improve estimation of X

\[ \theta \] Innovation process generated by Kalman filtering process

\[ \alpha_i, \rho_i \] Constants

\[ \beta_i \] Constants. See K-M-W, p. 158

\[ \gamma_i \] Constants. See K-M-W, p. 160

\[ \delta_i \] Constants. See K-M-W, p. 160

\[ a_t = (a_t(1), a_t(2), a_t(3)) \] Transposed vector of serial correlated Gaussian disturbances. See K-M-W, p. 160

\[ u'_t = (u_t(1), u_t(2), u_t(3)) \] Transposed vector of White Gaussian disturbances. See K-M-W, p. 160

\[ \Delta' = (\delta_2, \delta_3, -\delta_3) \] See K-M-W, p. 160

\[ \Delta^t = (\rho_1 \delta_2, \rho_2 \delta_3, -\rho_3 \delta_3) \] See K-M-W, p. 160

\[ \Delta_i, i = 1, 2, 3 \] Variously defined variances

\[ r' = (\rho_{12}, \delta_3/\alpha_1, -\delta_3/\alpha_1) \] See K-M-W, p. 162

\[ r'_\rho = (\rho_1 \delta_2, \rho_2 \delta_3/\alpha_1, -\rho_3 \delta_3/\alpha_1) \] See K-M-W, p. 163

\[ r^* = (\beta_1 \delta_2, \delta_3/\alpha_1, 0) \] See K-M-W, p. 164

\[ r^*_{\rho} = (\beta_1 \delta_2, \rho_2 \delta_3/\alpha_1, 0) \] See K-M-W, p. 164

CHAPTER I
INTRODUCTION

In a world of uncertainty, due to continuous changes in structural relationships, random shocks to the economic system and information lags, the parameters of the economic system and the values of economic variables are often not precisely known to the policy maker. As a consequence, the channels through which the Federal Reserve actions are transmitted to financial markets and output markets are not clear. To improve the knowledge about economic activities and the results achieved by the past policy actions, several writers\(^1\) have stressed the importance of using monetary indicators and targets in the monetary control process.

A target variable should be related to monetary indicator on the one hand, and to the policy instruments such as the purchase or sale of open market securities, the discount rate, required reserve ratios

and repurchase agreements on securities on the other. The target is presumed to be affected by policy actions somewhat more rapidly and with less error than the indicator is affected. Furthermore, the target variable is presumed to be one whose past values are known with shorter lags than possible for indicators or ultimate objectives. A target is used to guide day to day monetary policy operations in the money markets. Free reserves, borrowing reserves, the three month treasury bill rates, the federal funds rate, base money and RPD's (i.e. volume of reserves held by banks against private deposits) are the possible target variables.

Why Use Control Theory Approach and the Kalman Filter?

A monetary indicator should have economic relationship with the ultimate policy variables such as price level, unemployment rate, balance of payments, on the one hand, and to the short-run target variables on the other. In short, a monetary indicator has two important properties:

1. A monetary indicator is a variable which contains information of the ultimate objective variables. Therefore it can be used as an auxiliary information variable to improve the estimation of the ultimate objective variables. The purpose of this dissertation is to develop an optimal stochastic monetary strategy, and to decompose the monetary strategy into two separate stages.

2. A monetary indicator can be used as a measure of thrusts of monetary policy. It is often used to indicate the current direction of monetary policy or the future effect of recent policies. This is
because indicators are presumed to be affected by policy instruments with minimal lags.

First, a deterministic control stage in which the first property of the indicator, or equivalently the short run target, will be treated as if it were a policy instrument. And second, an estimation stage in which the second property of the monetary indicator will be analysed in a Kalman filter framework.

In order to link the monetary indicator analysis to the Kalman filtering technique of the optimal control theory, we assume that a policymaker is supposed to minimize an expected quadratic loss function subject to the constraint of an economic system as described by a version of the St. Louis model\(^2\) as follows:

\[
y_t = y_{t-1} + a_0 + \sum_{i=0}^{4} m_i \Delta M_{t-i} + \sum_{i=0}^{4} e_i \Delta E_{t-i}
\]

where \(a_0\): intercept.

\(m_i, e_i\): known coefficients

\(M\): money stock

\(E\): Federal budget expenditures

\(Y\): nominal GNP.

Let us also assume that at the beginning of the policy decision time, quarter \(t+1\), both \(Y_{t+1}\) and \(Y_t\) are unobservable. The policymaker is supposed to utilize all the information available to him to make a better estimate of \(Y_t\). This implies that in addition to the use of

equation (1.1) for the estimation purpose, he may also use a demand for money equation such as equation (1.2) below to improve the estimation of unknown GNP.

\[
M_t = h_1 Y_t + h_2 p_t + v_{2,t}
\]

where \(M_t, p_t, h_1\) and \(h_2\) are assumed to be known at the beginning of quarter \(t + 1\).

The monetary indicator \(M_t\) developed in the system of equations (1.1) and (1.2) resembles the Kalman filtering process. The Kalman filtering process is a technique to improve the estimation of unobservable variables in the form of equation (1.1), by filtering the auxiliary information contained in the form of equation (1.2). A specific form of the Kalman filter result is given below.

### Kalman Filter

The Kalman filtering technique gives the best estimation of state vector in the mean square sense when the system is given as follows.\(^\text{3}\)

\[
\begin{align*}
X_{t+1} &= FX_t + v_{1,t} & \text{state equation} \\
y_t &= HX_t + v_{2,t} & \text{measurement equation}
\end{align*}
\]

where \(X_t, v_{1,t}, y_t, v_{2,t}, Y_t\) : nxl vectors; \(F, H\) : nxn matrix; and \(H\) : mnxm matrix; the \(v_{1,t}, v_{2,t}\) sequences are zero mean Gaussian vectors, independent in time and independent of the Gaussian initial condition \(X_0, X_t + 1, X_t, v_{1,t}\) and \(v_{2,t}\) unknown; \(Y_t, F, H\) : known.

Further, let us denote

\[ E \nu_{1t} \nu_{1t}' = T_t \] known covariance matrix of state disturbances

\[ E \nu_{1t} \nu_{2t} = S_t \] known covariance matrix of state disturbances and measurement disturbances

\[ E \nu_{2t} \nu_{2t}' = J_t \] known covariance matrix of measurement errors

\[ E(X_0) = \mu_0 = \hat{X}_0 \] initial mean; a constant vector

\[ E(X_0 - \hat{X}_0)(X_0 - \hat{X}_0)' = P_0 \] initial covariance matrix of state variables

\[ \hat{X}_t = E(X_t/y_0,y_1,...,y_{t-1}) \] dynamic conditional mean of \( X_t \)

and

\[ P_t = E(X_t - \hat{X}_t)(X_t - \hat{X}_t)' \] dynamic covariance matrix of \( X_t \)

where, since \( \hat{X}_t \) is conditioned on all the information available up to time \( t-1 \), so does \( P_t \). Then \( \hat{X}_t + 1 \) and \( P_t + 1 \) obey:

\[ \begin{align*}
\hat{X}_t + 1 &= FX_t + K_t (y_t - HX_t) \\
K_t &= (FP_t H' + S)(HP_t H' + J)^{-1},
\end{align*} \]

\( K_t \) is the Kalman gain

\[ P_t + 1 = FP_t F' + T - (FP_t H' + S)(HP_t H' + J)^{-1}, \]

\( (HP_t H' + S) \) where \( P_t + 1 \) is known as the Riccati equation.

The above results will be extensively used in Chapters II and III.

**Review of Literature**

Some writers have studied the availability of information flow along with the formulation of monetary strategy.⁴ Among them, the

---

works of Kareken-Muench-Wallace, Havrilesky and Tinsley are related to indicator and target framework outlined above.

Kareken-Muench-Wallace have shown how the central bank utilizes the incoming information flow in the monetary policy formulation process. However, there are two shortcomings in their paper. First, their analytical technique is limited to the analysis of single objective variable only and cannot be directly applied to the analysis of multiple-objective variable case.\(^5\) Second, although they have stressed the value of information in the policy decision process, they did not incorporate the value of information in the determination of the central bank’s reaction function.\(^6\)

Havrilesky tries to study the problem of choosing optimal monetary strategy within a stochastic IS-LM theoretical framework. The objective of policymaker is assumed to minimize a quadratic disutility function subject to a linear but stochastic Keynesian IS-LM economic

\(^5\)Kareken, Muench and Wallace, op. cit., p. 64. They write: "There are three assumptions, .... the first is that the central bank has but only one target variable." Notice that their target is equivalent to our objective variable.

\(^6\)See the first section of Chapter III.
system. Further he introduces a policy reaction function in which the policy instrument (base money) is not directly related to the goal variable (income) but related to some more readily available quantities: estimated values of intercepts of IS curve, demand for money function and supply of money function. Havrilesky's contribution is quite important. However, he does not specify "the constant term" in the optimal policy reaction function. Strictly speaking, since "the constant term" of the policy equation is unknown, the optimal monetary policy is not admissible, i.e., it cannot be implemented. Further, the roles that the target problem (a deterministic control) and the indicator problem (a Kalman filter estimation) play in the formulation of stochastic monetary strategy are not separately discussed. It is the purpose of this dissertation to generalize Kareken-Muench-Wallace model and Havrilesky model in the optimal control framework with the hope that the results in the control theory can become more useful to the analysis of monetary strategy.

Tinsley has presented an empirical "add factor" filter approach to the monetary indicator problem. He has shown that filtering the information contained in the money market model constructed by Parry, Pierce and Thomson has significantly improved the estimation of the variables of the MINNIE model.  

In Chapter IV we have applied both the Kalman filtering process and the "add factor" filtering process to the total spending equation of the St. Louis model and the demand for money equation of the MINNIE model. The result indicates that the Kalman filtering process performs much better than the "add factor" filtering process.

Our approach is to solve the stochastic monetary strategy in two stages. The first stage is to assume that there is only one target variable and then solve for the optimal deterministic reaction function. The solution procedure does not involve in the use of new information. The second stage is to solve the indicator problem (an estimation stage) by the use of new information. The overall optimal stochastic monetary strategy is obtained by cascading the deterministic reaction function with the indicator solution. This is essentially the result of the separation theorem and will be called "a two-stage decision rule for the conduct of monetary policy."

---

8To follow the approaches used in Kareken, Muench and Wallace, op. cit., and Havrilesky, op. cit., we shall not distinguish the target variable from the policy instrument in the following discussion.

9Since once we find the loss due to a particular target-indicator pair, we will be able to find the best pair by choosing the one which gives the lowest loss among all the possible target-indicator pairs. To concentrate on the issue of finding monetary strategy we shall confine our analyses to the cases where base money is considered as the only target variable.

10Definition of the separation theorem is given in n. 2 of Chapter III.
CHAPTER II

OPTIMAL OPEN MARKET STRATEGY: THE USE OF INFORMATION VARIABLES--COMMENTS ON KAREKEN, MUECH AND WALLACE'S PAPER

Section 1: Introduction

In their paper, Kareken, Muench and Wallace (K-M-W) provide a stochastic IS-LM model for the analysis of optimal monetary strategies with the use of information variables. They show how the central bank utilizes the incoming information flows to estimate the structural disturbances. Information flows are shown to be different for different information periods under consideration. Three different information periods are summarized in Table 1.

1Kareken, Muench, and Wallace, op. cit. For ease of reference, K-M-W's notations will be used in this chapter, except the following:

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<table>
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<td>The central bank's asset portfolio</td>
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<td>Variance of disturbance</td>
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<td>Transformed innovation process</td>
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<td>The Riccati equation</td>
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TABLE 1
Available of Information Variable

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<th>Measurement variables</th>
<th>Information periods (I.P. = 1, 1=1, 2, 3.)</th>
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<tr>
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<td>I. P.=1 measurement variables of t=1 are available at t=1.</td>
</tr>
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<td>Past central bank asset portfolio, P_{t-1}b</td>
<td>0</td>
</tr>
<tr>
<td>i=1,2,...,t-1.</td>
<td>Past interest rates R_{t-1, i=1,2,...,t-1.}</td>
</tr>
<tr>
<td>Past money stocks M_{t-1,i=1,2,...,t-1.}</td>
<td>0</td>
</tr>
</tbody>
</table>

a. I.P. = 1 indicates "the first day of the first week," and will be called the first information period.

b. I.P. = 2 indicates "days of the first week," and will be called the second information period.

t stands for the day.
w stands for the 7th day of the week.

X indicates that measurement variables are available in an information period.

0 indicates that measurement variables are not available in an information period.

The table is read as follows: for example, I.P. = 2 and t=3, then P_{1}, P_{2} and R_{1} and R_{2} are measurement variables available to the central bank at the 3rd day of the first week.

b. We have included P_{t-1} as a measurement variable but K-M-W do not (See n. 6 below).
K-M-W do not provide a consistent treatment of their information extraction processes. They derive their results from the reduced form equations for the second information period—a single-information-source case, but derive the results from a transformation of the reduced form relationships for the third information period—a multiple-information-source case. To justify their approach, they write:

If P was the instrument variable for day t-1, then for day t the central banks information variables are R and M. To use equations similar to equation (17) to determine E_{t}^* a_{t-1}, it has to have information variables which are uncorrelated. Obviously, R_{t-1} and M_{t-1} are correlated. But there are transforms....

In our view, the need for making a transformation is not due to the fact that R_{t-1} and M_{t-1} are correlated but rather because of the fact that K-M-W's analytical technique can only deal with one information variable at each step of analysis, i.e., one information variable is analyzed at each step of a step-wise "regression" technique. The purpose of this chapter is to modify their analysis. Under a more general Kalman filter approach, to use equations similar to equation (K-17) in order to find the best estimation of a_{t-1}, it is not necessary that:

1. all information variables should be uncorrelated, and
2. there be an associated transformation.

\[\text{Ibid.}, \ p. \ 164. \ \text{Notice that K-M-W's P is our p. Emphasis is mine.}\]

\[\text{(K-17) stands for the equation (17) shown in Ibid.}\]
Finally, there are some unnecessarily restrictive assumptions given in the K-M-W work:

There are three assumptions which, because they are so unrealistic, we particularly regret having had to make. The first is that the central bank has but only one target variable. The second is that the economic structure, although stochastic, is linear. And the third is that the central bank knows with certainty the constants and coefficients of this structure.\(^4\)

The first assumption and the certainty (reduced form) constants assumption can be relaxed in our analysis.\(^5\) This is so, again, because K-M-W's approach is applicable to a single equation analysis only while a Kalman filter approach is capable of dealing with multiple-equation systems.

In Section 2, we shall reconstruct K-M-W's stochastic IS-LM structure in a Kalman filter format which allows all the information variables to be correlated. In Section 3 we shall show the relationship between our approach and K-M-W's transformation approach to the treatment of information processes in the third information period. In Section 4, we draw the conclusion that K-M-W's model can be effectively analyzed in a Kalman filter framework.

Section 2: K-M-W's Model in a Kalman Filter Format

One important issue of monetary control is that the Federal Open Market Committee does not know the actual values of its current

\(^4\)Ibid., p. 156.

\(^5\)Unknown reduced form coefficients have been discussed in Havrilesky, op. cit. Further discussion on this subject is given in Chapter III.
objective variables such as total spending, GNP, when it executes open market policy. However this by no means implies that the monetary authority does not have any idea about the possible value of GNP. It can predict GNP by large econometric models or small models that relate it to some readily observable current monetary aggregates. Perhaps the most promising approach would be a mixture of the above two approaches since it allows the monetary authority to update its estimates from econometric models with current observations. In terms of K-M-W's model, the last approach can be decomposed into two parts:

1. A reduced form derived from the theoretical model provides an estimate of GNP, Y, as

\[ Y_t = \delta_0 + \delta_1 p_t + \Delta' a_{t-1} + \Delta' u_t \]

where \( Y_t \), \( a_{t-1} \), and \( u_t \) are unknown magnitudes, and the constant coefficients and \( p_t \) are known magnitudes.

2. Two measurement variables, \( R_{t-1} \) and \( M_{t-1} \), provide information about \( a_{t-1} \) which, in turn, provides information about \( Y_t \).

\[ R_{t-1} = \gamma_0 + \frac{\delta_1}{\alpha_1} p_{t-1} + \Gamma' a_{t-1} \]

\[ M_{t-1} = \alpha_5 + (\alpha_7 + \frac{\delta_1}{\alpha_1}) p_{t-1} + \alpha_6 \Gamma' a_{t-1} + a_{t-1} \]

where \( M_{t-1} \) and \( R_{t-1} \) are observable at day t.

The first part says that a better estimation of \( a_{t-1} \) gives a better estimation of GNP. The second part says that although \( a_{t-1} \) is not observable, it is directly related to two known magnitudes, the rate
of interest and money stock, at \( t-1 \), so the relationship between 
\( R_{t-1} \) and \( M_{t-1} \) and the estimate of \( a_{t-1} \) may be used to update the estimation of GNP at day \( t \).

The function of monetary indicators developed so far fits into the Kalman filter framework very well. We therefore rewrite the K-M-W model as a system consisting of a state equation (2.1) and a measurement equation (2.2) below. Since the "current" information at day \( t-i \) is used to estimate the state vector at the same day, the system will be called the estimation model.

**Estimation model in the third information period.**

The state is constructed from equation (K-5)

\[
(2.1) \quad a_{t-i} = \rho a_{t-i-1} + u_{t-i}.
\]

The measurement equation is derived from equation (K-20) and equation (K-35)

\[
(2.2) \quad y_{t-i} = c p a_{t-i-1} + c u_{t-i}.
\]

where

\[
y_{t-i} = \begin{pmatrix}
R_{t-i} - \gamma_0 - \left(\frac{\delta}{\alpha_1}\right) p_{t-i} \\
M_{t-i} - a_5 - \left(a_7 + a_6 \frac{\delta}{\alpha_1}\right) p_{t-i}
\end{pmatrix},
\]

\[
a_{t-i} = \begin{pmatrix}
a_{t-i}(1) \\
a_{t-i}(2) \\
a_{t-i}(3)
\end{pmatrix}, \quad u_{t-i} = \begin{pmatrix}
u_{t-i}(1) \\
u_{t-i}(2)
\end{pmatrix}.
\]
In the above specification, constants and the policy instrument have been put at the left hand side of equation (2.2), because we have to subtract these magnitudes from $R_{t-1}$ and $M_{t-1}$ in order to provide information about $a_{t-1}$. Note that we treat $p_{t-1}$ either as a constant or as a policy instrument that reacts to the past known magnitudes. In either case, $p_{t-1}$ is known at $t-1$ so it is a component of the measurement vector.\(^6\)

Due to the known coefficients structure of the model, the current debate that whether a measurement variable plays the role, first, as an indicator reflecting credit demands in the economy or,

\[^6\text{For the treatment of } p_{t-1}, \text{ K-M-W state that}
\]

At the beginning of day $t$, where $2 \leq t \leq w$, the central bank gets observations on $P_{t-1}$ and $R_{t-1}$.

One of these observations provides no information about $a_{t-1}$, though, since the central bank must have used either $P$ or $R$ as its instrument variable for day $t-1$. If it used $P$, then.... (K-M-W, op. cit., p. 162, emphasis mine).

This quotation should be interpreted carefully, since, as shown in equation (2.2), $p_{t-1}$ appears as a component of $y_{t-1}$; $p_{t-1}$ does indirectly provide information about $a_{t-1}$.
second, as an indicator reflecting the monetary authority's actions does not create problems in our model. Equation (2.2) has explicitly taken care of the function of the first indicator because its left hand side components are known and provide information about $a_{t-1}$. The function of the second indicator can be inferred from equation (2.2), too.\footnote{An indicator of the monetary authority's action can be formulated by interchanging the policy instrument $p_{t-1}$ and state vector $a_{t-1}$ in equation (2.2) as}

\begin{equation}
\tilde{y}_{t-1} = \tilde{c} p_{t-1} + c u_{t-1}
\end{equation}

where

\[ \tilde{y}_{t-1} = \begin{pmatrix} R_{t-1} - \gamma_0 \\ M_{t-1} - \gamma_5 \end{pmatrix} - c p a_{t-1-1}, \text{ and} \]

\[ \gamma = \begin{pmatrix} \delta_1 \\ \alpha_1 \\ \alpha_7 + \alpha_6 \frac{\delta_1}{\alpha_1} \end{pmatrix} \]

since $a_{t-1}$ on the left hand side of equation (2.2-a) is an unknown vector and has to be estimated in order to provide the policy indicator. Thus both the indicator of reflecting economic activity (2.2) and the indicator of policy action (2.2-a) require the estimation of the state variable $a_{t-1}$. According to this analytical framework, the policymaker has to utilize all the available information $R$ and $M$ to construct the policy action indicator, $\tilde{c} p_{t-1} + c u_{t-1}$. D.A. Bowers and L.E. Duro have used a method similar to equation (2.2-a) for the empirical study of a "Neutralized Money Stock," but they use only one information $M$, rather than all the information, $R$ and $M$. See D.A. Bowers and L.E. Duro, "An Alternate Estimation of the Neutralized Money Stock," \textit{Journal of Finance}, Vol. 27, No. 1 (March, 1972), 61-64.
The updated estimation.

With the use of measurement equation to update the estimation from the state equation, the Kalman filter technique gives the best estimation of $\hat{a}_{t-1}$, $\hat{a}_{t-1}$, as follows:

(2.3) \[ \hat{a}_{t-1} = \rho \hat{a}_{t-1} + K_{t-1} \theta_{t-1} \]

where \[ \theta_{t-1} = y_{t-1} - E(y_{t-1}) = y_{t-1} - \rho \hat{a}_{t-1} \]

and

(2.4) \[ K_{t-1} = (\rho \pi_{t-1} (cp)' + S) (\rho \pi_{t-1} (cp)' + J)^{-1} \]

with a dynamic covariance matrix of the state vector, or the so-called the discrete Riccati equation, as

\[ \pi_{t-1} = E(a_{t-1} - \hat{a}_{t-1}) (a_{t-1} - \hat{a}_{t-1})' = \rho \pi_{t-1} \rho' + \phi_{t-1} \]

\[ - [\rho \pi_{t-1} (cp)' + S] [(cp) \pi_{t-1} (cp)' + J]^{-1} \]

\[ [\rho \pi_{t-1} (cp)' + S]' . \]

\[ S = E (u_{t-i} u_{t-i}') c' = \phi c' . \]

\[ J = c E (u_{t-i} u_{t-i}') c' = c \phi c' . \]

---

\(^8\)General formulas for the Kalman gain and the discrete Riccati equation are given in Chapter I.
In the analysis shown above, the correlated relationship between

\[
\Phi = E (u_{t-1} u'_{t-1}) = \begin{pmatrix}
\sigma^2_{u(1)} & 0 & 0 \\
0 & \sigma^2_{u(2)} & 0 \\
0 & 0 & \sigma^2_{u(3)}
\end{pmatrix}
\]

and known initial conditions \( ^9 \)

\[
\hat{a}_0 = a_0 \\
\hat{\pi}_0 = 0.
\]

In the analysis shown above, the correlated relationship between \( R_{t-1} \) and \( M_{t-1} \) is given in two places:

1. The covariance matrix of the measurement disturbances, \( J \), which has nonzero values for the off-diagonal components, and

2. The covariance matrix of the first term on the right hand side of the measurement equation, i.e.,

\[
\text{Covariance (} c_p a_{t-1-1}, c_p a_{t-1-1}') = c_p \pi_{t-1-1} (c_p)',
\]

has nonzero values for the off-diagonal components.

Finally, it is worthwhile to note that the Kalman gain is in fact a time-varying partial regression coefficient matrix which solves the state estimation in the mean-square sense for linear systems. The resulting best estimation of the state vector is given in a recursive manner as follows:

The state estimation consists of three stages:

1. The predict stage at \( t-1 = 1 \). The prediction of \( a_1 \) is based on the given initial mean \( a_0 \) as \( \hat{a}_1 = \rho \hat{a}_0 \)

\( ^9 \) The initial conditions are given in K-M-W, \textit{op. cit.}, p. 162. Notice that K-M-W's initial state variance \( \pi_1 \) is redefined as \( \pi_0 \) in this chapter.
where $\hat{a}_1$ denotes the prediction of $a_1$ without using measurements at $t-i$, $i = 1, 2, \ldots, t-1$.

2. The update stage at $t-1 = 1$. The measurement vector $y_1$ is available. The policymaker has to
   i) construct an innovation process $\theta_1$ as $\theta_1 = y_1 - E(y_1)$ such that $\theta_1$ is independent of the predict stage $\hat{a}_1 = \hat{a}_0$.
   ii) use the initial state covariance, together with the covariance matrices of the model disturbances $\Phi$, $S$ and $J$ to find the Kalman gain $K_1$.
   iii) combine the results given in i) and ii) above to form the Kalman filter for $t-1 = 1$ as $K_1 \theta_1$. This magnitude can be used to improve the result in the predict-stage.

3. The estimation stage. The policymaker combines the predict- and update-stages to form the best estimation of $a_1$ as $\hat{a}_1 = \hat{a}_0 + K_1 \theta_1$, and the updated state covariance $\pi_1 = E(\hat{a}_1 - \hat{a}_1)(\hat{a}_1 - \hat{a}_1)'$.

In general, given $\Phi$, $S$, and $J$, the best estimation of $a_{t-i}$ and the updated covariance matrix $\pi_{t-i}$ can be obtained recursively based on the known $\hat{a}_{t-i-1}$; $\pi_{t-i-1}$ and the innovation process $\theta$ at $t-i$, for $i = 1, 2, \ldots, t-1$. The results are shown in equations (2.3) through (2.5).

Notice that the time-varying Kalman gain at time $t-i$ is independent of the innovation process at $t-i$. The implication of the Kalman gain in the optimal monetary control will be discussed in detail in Chapter III.
Section 3. Transformation of the Estimation Model

The relationship between the original estimation model and K-M-W's transformed model.

In the previous section, we do not make any transformation to eliminate the correlated information variables of the estimation model and yet we obtain the best estimation of $a_{t-1}^i$, $i = 1, 2, \ldots, t-1$. The purpose of this section is to establish the relationship between our estimation model and the K-M-W's transformed model.

The K-M-W's transformed model is derived by applying a Z-transform to both matrix $c$ and the measurement vector $y$. The transformed magnitudes $y^*$ and $c^*$ are related to the original magnitudes $y$ and $c$ according to the relationships:

\begin{align}
(2.6) \quad Z &= \begin{pmatrix}
1 - \alpha_6 & \frac{\delta_3}{\alpha_1} \\
- \alpha_6 & 1
\end{pmatrix}
\end{align}

\begin{align}
(2.7) \quad Zy_{t-1} &= y^*_{t-1} = \begin{pmatrix} y^*_{t-1}(1) \\ y^*_{t-1}(2) \end{pmatrix} \\
(2.8) \quad Zc &= c^* = \begin{pmatrix}
\beta_1 \delta_1 & \frac{\delta_3}{\alpha_1} & 0 \\
0 & 0 & 1
\end{pmatrix}
\end{align}

\footnote{\((Z(2), Z(1)) \text{ in K-M-W, op. cit., p. 164 are different from our Z-transformation. They are equivalent to our } \theta^* \text{ defined in equation (2.12).} \)
or
\[ y_{t-1} = Z^{-1} y_{t-1} \]
(2.9)
\[ c = Z^{-1} c^* . \]

Thus, the transformed model under consideration is

(2.10)
\[ a_{t-1} = \rho a_{t-1-1} + u_{t-1} \]
(2.11)
\[ y_{t-1}^* = c^* a_{t-1-1} + c^* u_{t-1} \]
\[ i = 1, 2, \ldots, t-1 \]

with
\[ S^* = E \left( u_{t-1} u_{t-1}' \right) c^* \]
\[ J^* = c^* E(u_{t-1} u_{t-1}') c^* \]
\[ \phi^* = E(u_{t-1} u_{t-1}') = \phi . \]

Applying the general formulas given in Chapter I again, we obtain the following results:

The innovation process:

(2.12)
\[ \theta_{t-1}^* = y_{t-1}^* - c^* p_{t-1-1} . \]

The Kalman gain:

(2.13)
\[ K_{t-1}^* = (\rho \pi_{t-1-1} (c*p)' + S^*)(c*p)_{t-1-1} (c*p)' + J^*)^{-1} \]

and the discrete Riccati equation:

\[ \pi_{t-1}^* \rho_{t-1-1} p' + \phi_{t-1-1} \]
\[ - (\rho \pi_{t-1-1} (c*p)' + S^*)(c*p)_{t-1-1} (c*p)' \]
\[ + J^*)^{-1} (\rho \pi_{t-1-1} (c*p)' + S*') . \]

11Our results (2.13) and (2.14) are the same as the corresponding parts given in K-M-W, op. cit., p. 165.
Comparison of equations (2.12) through (2.14) with the corresponding solutions of Section 2 shows that the transformed results have the same forms as the original results. Substituting relationships in (2.9) into (2.3) through (2.5), we find that

\[ \pi_{t-i} = \pi^*_{t-i} \]

and

\[ K_{t-i} \theta_{t-i} = (K^*_{t-i} Z)(Z^{-1}_{t-i} \theta^*_{t-i}) = K^*_{t-i} \theta^*_{t-i}. \]

To provide a concrete counter-example to the first quotation cited at the beginning of this chapter, it is instructive to see that at \( t-1 = 1 \) in the third information period, both of our version and K-M-W's transformed version of the state estimation can be written in the form of equation (K-17) as follows:

\[ \dot{x}_1 = \rho \hat{x}_0 + K_1 \theta_1 \]

(2.15)

\[ = \rho \hat{x}_0 + K^*_1 \theta^*_1. \]

\[ \pi_{t-i} = \pi^*_{t-i} \] can be established as follows. (1) The assumption about the same initial state covariances gives \( \pi_0 = \pi^*_0 \). (2) Substituting \( Zc \) for \( c^* \) and \( Zy \) for \( y^* \) in equation (2.14) and using the result of (1) above, one finds that

\[ \pi_1 = \pi^*_1. \]

Applying this method recursively, one can establish the relationship \( \pi_{t-i} = \pi^*_{t-i} \) which, together with equation (2.9), can be used to establish the desired result

\[ K_{t-i} \theta_{t-i} = K^*_{t-i} \theta^*_{t-i}. \]
where

\[
K_1 = \frac{1}{\Gamma^* \Phi \Gamma} \begin{pmatrix} (1 - \alpha \frac{\delta_3}{\alpha_1}) \sigma^2(1) & \beta_1 \delta_3 \frac{\delta_3}{\alpha_1} \sigma^2(1) \\ \beta_1 \delta_3 \frac{\delta_3}{\alpha_1} \sigma^2(2) & \frac{\delta_3^2}{\alpha_1} \sigma^2(2) \end{pmatrix}
\]

\[
K_1^* = \frac{1}{\Gamma^* \Phi \Gamma} \begin{pmatrix} \beta_1 \delta_3 \sigma^2(1) & 0 \\ \frac{\delta_3}{\alpha_1} \sigma^2(2) & 0 \end{pmatrix}
\]

\[
0 \quad (\beta_1 \delta_3^2 + \frac{\delta_3^2}{\alpha_1} \sigma^2(2))
\]

with \( \Gamma^* \Phi \Gamma = (\beta_1 \delta_2)^2 \sigma^2(1) + \frac{\delta_3^2}{\alpha_1} \sigma^2(2) \).

\[
\theta_1 = y_1 - c \rho a_0
\]

and

\[
\theta_1^* = y_1^* - c \rho a_0
\]

Thus, given the same initial mean vector, the best estimation of the state vector derived from the transformed model is the same as that derived from the original model.

The Usefulness of Transformation

We have shown that K-M-W's justification of the need for a transformation in order to derive the Kalman gain and the associated
discrete Riccati equation is unfounded. This follows since our results for the best state estimation are identical to theirs. However, this does not mean that the linear transformation is a useless concept in the analysis of state estimation. The following discussion tries to explore the usefulness of the linear transformation of the model from different points of view.

So far we have assigned no information costs to the information production processes. In fact, if the cost of information is higher than the benefit from the use of information, the policymaker should stop the information-acquiring process. Application of information cost-benefit analysis to the forecast and decision-making process has been discussed in the economic and control theory literature. We do not intend to extend our analysis to this interesting field in this chapter, however, we shall justify the usefulness of linear transformation in light of the information cost-benefit analysis. There are many cases in which the cost of information can be reduced by the use of linear transformation. The following are three of them.

Case 1. Cost of computation and the original model. In general there are some linear transformations of the model coefficients

---

which can transform the original model to the ones that are easier to work with. For example, in view of the simplicity of $K^*$ given in equation (2.15), one finds that it is easier to work with K-M-W's transformed model than with the original model. Indeed, this reason can justify the need for transformation in K-M-W's paper.

Case 2. The Luenberger observer and the Kalman filter. The Kalman filter technique discussed above seems not difficult to work with. This is because the model under consideration is rather simple. In fact, the computation for the Kalman filter is almost formidable, if not impossible, for large systems. Thus, an easier but less optimal approach, the Luenberger observer is sometimes preferred to the Kalman filter for the state estimation. However, most observers given in control literature are discussed with the canonical measurement equations, i.e.,

$$y = (I, 0) X + v$$

where the coefficient matrix of the state $X$ consists of an identity matrix and an appropriate zero matrix. To utilize the technique, a

---


transformation is necessary in order to transform the original measurement equation to the desired canonical form.\textsuperscript{16}

Case 3. Measurement control and the Kalman filter. Since the Kalman filter technique is to filter the innovation process

$$\theta_{t-1} = y_{t-1} - E(y_{t-1}) = y_{t-1} - c\hat{p}_{t-1-1}.$$  

Thus, if the measurement vector gives almost perfect information about the state vector, i.e., if $y_{t-1}$ is almost equal to $c\hat{p}_{t-1-1}$ in the above equation (or we may say that $y_{t-1}$ is almost a noise free measurement vector), then the policymaker may get almost all the necessary information about the state from the measurement $y_{t-1}$ immediately without carrying out the costly Kalman filter computation.

The same argument is applicable to the components of the measurement vector. Under this situation, the policymaker will examine each single component of $y_{t-1}$ to see if there is any one-to-one relationship between a measurement variable and a state variable. Obviously, the policymaker cannot find out such relationships from the original measurement equation which allows information variables to be correlated. A transformation to make measurement variables uncorrelated is then necessary. Again, if the policymaker can find such a (almost perfect) one-to-one relationship between one measurement variable and one state variable, then there is no need to conduct the costly Kalman filtering procedure for the estimation of that particular variable.

\textsuperscript{16}Novak, \textit{op. cit.}, has given some clever equivalent state transformations which can transform the original state to the desired canonical measurement equation.
It suffices to illustrate the above arguments by the following examples. Let us consider K-M-W's transformed model and see the relationship between $y^*(2)$, i.e., money stock measurement, and $a(3)$, the disturbance term of the money supply function, as

\begin{align*}
\text{(2.16)} & \quad a_{t-i}(3) = \rho_3 a_{t-i-1}(3) + u_{t-i}(3) \\
\text{(2.17)} & \quad y^*_{t-i}(2) = \rho_3 a_{t-i-1}(3) + u_{t-i}(3).
\end{align*}

Applying the general formulas (2.13) and (2.14) gives $K^* = 1$ and $\pi^*(3) = 0$.

Although K-M-W give the above results, they do not provide enough interpretations. What system (2.16) and (2.17) implies is that the measurement variable contains all the information about the state variable\(^{17}\) therefore the innovation process is zero. There is no need to filter $y^*(2)$ for the state estimation. Further, let us consider a variant of K-M-W system\(^{18}\) where only estimates, rather than error-free observations, of the daily deposit stock are available to the policymaker in the third information. This variant shows

\(^{17}\)This argument becomes clearer if we utilize equation (2.1) and rewrite equations (2.16) and (2.17) respectively as

\begin{align*}
a_{t-i}(3) & = a_{t-i}(3) \\
y^*_{t-i}(2) & = a_{t-i}(3) .
\end{align*}

The measurement $y^*(2)$ gives the value of state, $a(3)$, immediately. Kalman filtering process should not be used.

\(^{18}\)See the Appendix in K-M-W, op. cit., p. 171.
that an observation error, $v_{t-1}$, is added to equation (2.17) above such that the new measurement equation is

$$y_{t-1}^*(2) = y_{t-1}^*(2) + v_{t-1}$$

Then from the information cost-benefit analysis, the policymaker may conduct the Kalman filtering for $y^*(2)$ if the newly added observation error is rather noisy and may just take the estimate of $y^*(2)$, i.e., $\hat{y}^*(2)$, without using the filtering procedure if the added observation error is almost noise free.

In general, if there are two groups of measurement variables, one group contain almost perfect information about some of the state variables and the other group contain rather imperfect information about the rest of the state variables, then a cost-saving approach (not to apply the Kalman filtering) would require the policymaker to transform the original measurement equation to two uncorrelated groups and apply the Kalman filter to the rather noisy measurement group only.

Among the cases discussed above, case 1 and case 3 are directly taken from K-M-W's paper, but none of them has been given by K-M-W to justify the need for making transformation.

Section 4. Concluding Remarks

In the previous sections, we have shown the following points: First, although there are some reasons for the need for making transformation, but K-M-W's justification for the need for transformed
model is unfounded. The original model gives the same best state estimation as the transformed model.

Second, the Kalman filter approach introduced in this chapter is designed for analyses of multiple-equation systems, so, in principle, it is capable of dealing with analyses of optimal monetary strategies with multiple objective variables and unknown constants.

Third, the major contribution of this chapter is clear. We have shown that once the reduced form of the economic model is converted to the control theory format—the state-space representation of the model—then the desired Kalman gain and discrete Riccati equation can be found immediately from the general formulas given in Chapter I.

Finally, there are many linearly transformed models that can be derived from the original model, the K-M-W's transformed model for the third information period is only a special case of their theoretical IS-LM model.
Section 1: Introduction

In the previous chapter we have demonstrated that Kareken-Muench-Wallace's dynamic estimation procedure is essentially a special case of the Kalman filter technique. The next question that naturally arises is what is the role of dynamic estimation in the framework of stochastic monetary control process? K-M-W have provided the following control process. First, a reduced form equation specifies the relationship between the objective variable, $Y$, and the policy variable, $p$, as

$$(K-11) \quad Y_t = \delta_0 + \delta_1 p_t + \Delta_p a_{t-1} + \Delta u_t$$

where $Y_t$, $a_{t-1}$ and $u_t$ are unknown.

Second, conditional means of $Y_t$, $a_{t-1}$ and $u_t$, i.e., $\dot{Y}_t$, $\dot{a}_{t-1}$ and zero, respectively, are substituted for the unknowns in the above equation. The resulting equation is

$$(K-12) \quad \dot{Y}_t = \gamma_0 + \delta_1 p_t + \Delta_p \dot{a}_{t-1}.$$
from its conditional mean, $\hat{\gamma}_t$. The resulting optimal control strategy, $p_t$, is specified as

$$p_t = (\hat{\gamma}_t - \delta_0 - \Delta_p a_{t-1})/\delta_1.$$

As we have discussed in Chapter II, for example, in the second information period, the estimated autocorrelated disturbances, $\hat{a}_{t-1}$, are not available at day $t-1$; they are only available at day $t$. Thus, one question arises in K-M-W's control strategy: is the optimal monetary strategy, $\hat{\gamma}_t$, determined before or after the policy-maker obtains the estimation, $\hat{a}_{t-1}$? We consider three different control strategies according to the timing of information flow.

Case 1. If $\hat{a}_{t-1}$ is available before the determination of control strategy at day $t$, then an admissible strategy which depends on known magnitudes only can be derived from equations (K-12) and (K-15) as

$$\hat{\gamma}_t = \delta_0 + \delta_1 (p_t + \frac{1}{\delta_1} \Delta_p a_{t-1})$$

$$= \delta_0 + \delta_1 p^{**}$$

where $p$ is defined in equation (K-15), and

$$(3.1) \quad p^{**} = \hat{\gamma}_t + \frac{1}{\delta_1} \Delta_p a_{t-1}.$$

The optimal control $p^{**}$ depends on the stochastic variable, $\hat{a}_{t-1}$, so, $p^{**}$ is called the optimal stochastic control strategy.

If $\hat{a}_{t-1}$ is not available before the determination of control strategy at day $t$, then an admissible strategy should depend on some other known magnitudes. For this, we consider the following two cases:
Case 2. K-M-W’s control strategy. Since $\hat{a}_{t-1}$ is unknown at the beginning of day $t$, the control, $\hat{\nu}_t$, in equation (K-15) which depends on $\hat{a}_{t-1}$ is therefore not an admissible strategy. This shortcoming can be removed by the substitution of $\hat{\nu}_t$ from equation (K-12) into equation (K-15). The resulting control law is

$$\hat{\nu}_t = (Y_0 - \delta_0)/\delta_1$$

where $Y_0 = \delta_0 + \delta_1\hat{\nu}_t$.

This strategy is admissible. A close examination of this control strategy reveals that the control, $\hat{\nu}_t$, depends on a constant $\delta_0$ only, and does not depend on the estimated magnitudes, $\hat{a}_t$, which are known to the policymaker at day $t$. The drawback of K-M-W’s control strategy is that the role of estimation is completely disregarded in the policy decision process.

The purpose of the following strategy is designed to incorporate past information, $\hat{a}_{t-2}$, into the control process. Even $\hat{a}_{t-1}$ is not known at day $t$.

Case 3. A deterministic control strategy. By the substitution of $\hat{a}_{t-1}$ from equation (2.3) in Chapter II into equation (K-12) we obtain

$$\hat{\nu}_t = \delta_0 + \delta_1\hat{\nu}_t + \rho\Delta^1\hat{a}_{t-2} + \Delta^1K_{t-1}\theta_{t-1}$$

$$= \delta_0 + \delta_1\hat{\nu}_t + \Delta^1K_{t-1}\theta_{t-1}$$

where

$$\hat{\nu}_t = \frac{1}{\delta_1} \rho\Delta^1\hat{a}_{t-2}.$$ 

Since $p^*_t$ depends on known magnitudes $\hat{a}_{t-2}$ and $\delta_0$ and does not depend on
the unknown stochastic term \( K_{t-1}^\theta \). This control strategy will be
called deterministic.

Further, one can easily verify that if \( \hat{a}_{t-1} \) is known, then a
stochastic control strategy can be formulated as

\[
(3.2) \quad p_t^{**} = p_t^* + \frac{1}{b_1} \Delta^K_{t-1} \theta^t_{t-1}.
\]

The implication of above analysis is that if \( \hat{a}_{t-1} \) is available
after the determination of policy action, then a deterministic control
strategy is better than K-M-W's strategy since the former reacts to
estimated \( a_{t-2} \) while the latter does not. If \( \hat{a}_{t-1} \) is known, then an
optimal stochastic control strategy can be formulated either through
equation (3.1) or equation (3.2). If \( \hat{a}_{t-1} \) can be obtained through an
estimation stage (e.g., Kalman filter technique) then the optimal
stochastic control strategy is realized by cascading an estimator and
a deterministic controller as shown in equation (3.2) (or, equation
(3.1)). This is the so-called "separation theorem" in the control
theory literature.

K-M-W have shown the separation of a controller from an esti­
mator but do not give the concept of the optimal stochastic control

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2The separation theorem states that for linear systems with
 quadratic disutility function and subjected to additive white Gaussian
 noise disturbances, the optimal stochastic controller is realized by
cascading an optimal estimator with a deterministic optimum controller.
See, for example, Karl J. Astrom, Introduction to Stochastic Control
Witsenhausen, "Separation of Estimation and Control for Discrete Time
strategy, while Havrilesky\(^3\) has discussed the optimal stochastic control with information constraints but does not point out the possibility of separating an estimator from a controller. A purpose of this chapter is to synthesize K-M-W's approach and Havrilesky's in a Kalman filter and the separation theorem framework. Separation of a controller from an estimator for K-M-W's paper has been discussed above.\(^4\)

The rest of this chapter is devoted to the analysis of Havrilesky's model. Section 2 analyzes Havrilesky's model in a control theory framework and Section 3 concludes that Havrilesky's optimal stochastic monetary strategy can be regarded as cascading an optimal estimator with a deterministic controller.

Section 2: The Separation Theorem and the Optimal Monetary Strategy

Havrilesky's model can be represented in the state-space form as

\[
(3.3) \quad \min E \left( (X_1 - X^d)^T Q (X_1 - X^d) \right)
\]

subject to the state vector

\[
(3.4) \quad X_1 = FX_0 + GB
\]

and observation

\[
Y_0 = CX_0 + V
\]

where

\(^3\)Havrilesky, op. cit.

\(^4\)Although K-M-W consider a multiperiod quadratic loss function (K-M-W, op. cit., 159), but since they do not impose cost on policy instrument, the multiperiod optimization can be carried out each period separately. (See M. Aoki, Optimization of Stochastic Systems, (New York: Academic Press, 1967, 8, 66-67). Thus, the single period optimization discussed in Section 2 below for the Havrilesky's model can be applied to K-M-W's model, too.

\(^5\)Our equations (3.3) and (3.4) correspond to Havrilesky's equation (20) and equations (1) through (6), respectively.
\[
Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} Y_1 \\ A \\ m \\ M \end{pmatrix}, \quad X_0 = \begin{pmatrix} Y_0 \\ A \\ m \\ M \end{pmatrix}, \quad Y_0 = \begin{pmatrix} A^* \\ m^* \\ M^* \end{pmatrix}.
\]

\[
C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & a & -b & e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}, \quad V = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.
\]

**Y**: income (with \(Y_1\) unknown magnitude).

**X^d**: desired state vector.

\[A = A_0 + w_1; \text{ intercept of IS curve; } A \text{ is normally distributed with mean } A_0 \text{ and variance } \sigma^2 w_1; \text{ i.e. } A \sim N(A_0, \sigma^2 w_1).\]

\[m = m_0 + w_2; \text{ intercept of the money demand function; } m \sim N(m_0, \sigma^2 w_2).\]

\[M = M_0 + w_3; \text{ intercept of the money supply function; } M \sim N(M_0, \sigma^2 w_3).\]

**B**: base money; control variable.

\(z_i, \ i = 1, 2, 3\): error of observation: \(z_i \sim N(0, \sigma^2 z_i), \ i = 1, 2, 3.\)

\[\text{covariance } (z_i, z_j) = \text{covariance } (w_i, w_j) = 0, \text{ for } i \neq j.\]

\[\text{covariance } (w_i, z_j) = 0, \text{ for } i, j = 1, 2, 3.\]

\[a = \frac{u}{su+kh}, \ b = \frac{he}{su+kh}; \ a, b, e, s, u, k, \text{ and } h \text{ are known constants.}\]

\(\star\): denotes a measured (estimated) magnitude.
and subscripts attached to the variables other than disturbances \( w_i \) and \( z_j \): \( i = 0 \) stand for the decision period, or the initial period; \( i = 1 \) stands for the time after the decision period and will be called the future period. Finally the initial mean vector and the covariance matrix of the state variable are assumed to be known as 

\[
E(X_0) = \overline{X}_0 = \hat{X}_0 = \begin{pmatrix} A_0 \\ m_0 \\ M_0 \end{pmatrix}
\]

and 

\[
E((X_0 - E(X_0))(X_0 - E(X_0))') = P_0 = \Sigma_0 = \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & \sigma_{w_1}^2 & 0 & 0 \\ 0 & 0 & \sigma_{w_2}^2 & 0 \\ 0 & 0 & 0 & \sigma_{w_3}^2 \end{pmatrix}
\]

with 

\[
\sigma_y^2 = a^2 \sigma_{w_1}^2 + \left(- \frac{b}{e} \right)^2 \sigma_{w_2}^2 + \left( \frac{b}{e} \right)^2 \sigma_{w_3}^2
\]

where 

\[
E(X_i) = \overline{X}_i \quad \text{and} \quad E(X_i - \overline{X}_i)(X_i - \overline{X}_i)' = P_i, \quad i = 0, 1,
\]

denote the mean and the covariance, respectively, when information is not used. \( E(X_i) = \hat{X}_i \) and \( E(X_i - \hat{X}_i)(X_i - \hat{X}_i)' = P_i, \quad i = 0, 1, \) denote the mean and the variance respectively when information is used.

Since in the real world, the policymaker may predict the future state variable before he decides the magnitude of policy action, the policy-making process therefore can be decomposed into two stages. The
first is a prediction (estimation) stage and the second is a deterministic control stage. The results of these two stages crucially depend on how the current information \( y_0 \) is used.

Case A: When the Current Information is not Used.

1. Prediction stage.

If the policymaker does not use the current information, then the predicted value of future state can be obtained by a simple interpolation from the current state and the control action,

\[
\overline{X}_1 = F \overline{X}_0 + GB
\]

with the covariance of prediction\(^6\)

\[
\Sigma_1 = FEQF'.
\]

2. Control stage--The certainty-equivalence principle.

The disutility can be decomposed into two parts,

\[\Sigma_1 = E(X_1 - \overline{X}_1)(X_1 - \overline{X}_1)' =
\begin{pmatrix}
a^2\sigma_{w_1}^2 + \left(\frac{b}{e}\right)^2 \left(\sigma_{w_2}^2 + \sigma_{w_3}^2\right) & \frac{-b}{e} \sigma_{w_1}^2 & \frac{b}{e} \sigma_{w_1}^2 \\
\frac{a}{e} \sigma_{w_2}^2 & \sigma_{w_2}^2 & 0 & 0 \\
\frac{-b}{e} \sigma_{w_3}^2 & 0 & \sigma_{w_3}^2 & 0 \\
\frac{b}{e} \sigma_{w_3}^2 & 0 & 0 & \sigma_{w_3}^2
\end{pmatrix}\]
\[ W = \min E((X_1 - \overline{X}_1 + \overline{X}_1 - X^d)')Q(X_1 - \overline{X}_1 + \overline{X}_1 - X^d)) \]
\[ = \min E(\overline{X}_1 - X^d)'Q(\overline{X}_1 - X^d) + \min E x_1'Qx_1 \]
\[ = W_1 + W_2 \]

where \( x_1 = X_1 - \overline{X}_1 \).

The first part of the disutility function, \( W_1 \), is obtained by a deterministic control strategy because both the predicted value and the desired value are known. The second part, \( W_2 \), is not controllable because \( X_1 \) is unknown, which in turn implies that \( x_1 \) is unknown. Since the expected value of \( x_1 \) is zero, no control action should be taken against \( W_2 \).

An elementary way to solve the optimal deterministic control problem\(^7\) gives the optimal strategy, \( B^* \) (base money), as

\[ (3.5) \quad B^*_0 = -L \overline{X}_1 + B^d \]

where

\[ L = (G'QG)^{-1}G'Q = \left( \begin{array}{cccc} \frac{1}{b} & 0 & 0 & 0 \end{array} \right) \]

\[ \overline{X}_1 = FX_0 \]

\[ B^d = LX^d = \frac{a}{b} A^d - \frac{1}{c} m^d + \frac{1}{e} M^d \]

\( L \) stands for the deterministic reaction coefficient.

\(^7\)Differentiating \( E(\overline{X}_1 - X^d)'Q(\overline{X}_1 - X^d) \) with respect to control variable \( B \) and setting the result equal to zero, we then obtain the optimal deterministic controller \( B^*_0 \) shown in equation (3.5).
\( \tilde{X}_1 \) stands for the pure prediction of future state from the initial conditions, independent of a deterministic control action.

\( B^d \) is a constant which steers the current state toward the desired state, independent of the prediction stage.\(^8\)

or

\[
(3.6) \quad B^* = -\frac{a}{b} A_0 + \frac{1}{e} M_0 - \frac{1}{e} M_0 + B^d.
\]

By substituting the optimal control strategy into the disutility function, equation (5), the disutility due to the uncontrollable part is\(^9\)

\[
W_2 = \min \text{Ex}^1 Qx_1 = \text{trace } Q \tilde{X}_1
\]

\[
= a^2 \sigma^2_w + \frac{b^2}{e^2} \cdot \sigma^2_w + \frac{b^2}{e^2} \cdot \sigma^2_w .
\]

Case B: When the Current Information is Used.

1. Prediction stage.

When information is used, the best prediction of the future state \( \hat{X}_1 \) is given as

\(\text{We assume that the desired value, } y^d, \text{ is autonomously determined, so } B^d \text{ can be separated from the prediction component } \tilde{X} \text{ in equation (3.5).}\)

\(\text{Since income variance (Havrilesky, eq. (19)) can be expressed as } E(y - \hat{y})^2, \text{ Havrilesky seems not interested in the cost associated with steering the current state toward the desired state, i.e., the cost attached to } B_0, \text{ } E(y^d - \hat{y})^2, \text{ we omit the discussion of this type of cost, too. Without loss of generality, we will compare } W_2 \text{ with } I_2 \text{ only, where } W_2 \text{ is generated without using information and } I_2, \text{ generated with the use of information. We do not compare } W_1 \text{ with } I_1 \text{ here for simplicity.}\)
(3.7) \( \hat{X}_1 = F \bar{X}_0 + GB_0 + K \theta \)

where

\[
(3.8) \quad \theta = \begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix} = (y_0 - C \bar{X}_0), \text{ a known vector; } \theta_1 = W_i + Z_i.
\]

\[
K = \Sigma_0 C' (C \Sigma_0 C' + J)^{-1}
\]

\[
J = E(VV') = \begin{pmatrix}
\sigma^2 & 0 & 0 \\
0 & \sigma^2 & 0 \\
0 & 0 & \sigma^2
\end{pmatrix}
\]

\( K \) is the stochastic partial regression coefficient vector, or, the Kalman gain, and \( R \) is the covariance matrix of the residuals of the observation vector.

\( y_0 - C \bar{X}_0 \) is a known magnitude because both \( y_0 \) and \( C \bar{X}_0 \) are known. The Kalman gain is explicitly given as:

\[
K = \begin{pmatrix}
\frac{a_1 \sigma^2_w}{\Delta_1} & \frac{-b/e \sigma^2}{\Delta_2} & \frac{b/e \sigma^3}{\Delta_3} \\
\frac{a_2 \sigma^2_w}{\Delta_2} & \frac{a_2 \sigma^2_w}{\Delta_2} & 0 \\
\frac{a_3 \sigma^2_w}{\Delta_3} & \frac{a_3 \sigma^2_w}{\Delta_3} & 0
\end{pmatrix}
\]

where \( \Delta_i = \sigma^2_w + \sigma^2_{z_i} \) for \( i = 1, 2, 3 \).
Notice that equation (3.8) reflects the information vector in the decision period and it is not previously available.

The covariance of $X_t$, or the discrete Riccati equation, with the use of information is

$$P_t = \mathbb{E}(X_t - \hat{X}_t)(X_t - \hat{X})'$$

$$= \mathbb{E}C_0F' - (\mathbb{E}C_0C' + J)^{-1}(\mathbb{E}C_0C')' .$$

\[\text{The discrete Riccati equation at } t=1 \text{ is explicitly given as:}\]

$$P_1 = \begin{pmatrix} a^2 \sigma^2_w 1 - \frac{\sigma^2_{w_1}}{\Delta_1} & \sigma^2_w 1 - \frac{\sigma^2_{w_1}}{\Delta_1} & 0 & 0 \\ \sigma^2_w 1 - \frac{\sigma^2_{w_1}}{\Delta_1} & a^2 \sigma^2_w 1 - \frac{\sigma^2_{w_2}}{\Delta_2} & \sigma^2_w 1 - \frac{\sigma^2_{w_1}}{\Delta_2} & 0 \\ 0 & \sigma^2_w 1 - \frac{\sigma^2_{w_2}}{\Delta_2} & a^2 \sigma^2_w 1 - \frac{\sigma^2_{w_2}}{\Delta_2} & 0 \\ 0 & 0 & \sigma^2_w 1 - \frac{\sigma^2_{w_3}}{\Delta_3} & a^2 \sigma^2_w 1 - \frac{\sigma^2_{w_3}}{\Delta_3} \end{pmatrix}$$
The right hand side of equation (3.9) bears an interesting implication for the value of information. The first term, \( \mathbf{P}_0 \mathbf{F}' \), is the same as the covariance matrix of the state when no information is used; subtracting the second term \( (\mathbf{CE}_0 \mathbf{C}' + J)^{-1}(\mathbf{CE}_0 \mathbf{C}') \), a semi-positive-definitive matrix, from the first term shows the reduction of covariance due to the use of information. This fact leads the result of \( \mathbf{P}_1 \preceq \Sigma_1 \).


The separation theorem is used again here. However, the conditional mean, \( \hat{X}_1 \), rather than the unconditional mean, \( \bar{X}_1 \), is used to decompose the disutility function

\[
I = \min E((X_1 - \hat{X}_1 + \hat{X}_1 - X^d)'Q(X_1 - \hat{X}_1 + \hat{X}_1 - X^d))
\]

\[
= \min E(\hat{X}_1 - X^d)'Q(\hat{X}_1 - X^d) + \min E(X_1 - \hat{X})'Q(X_1 - \hat{X})
\]

\[\tag{3.10}
= I_1 + I_2.
\]

Following the same procedure as that for case A, we find the optimal stochastic control strategy, \( B^*_{**} \), as

\[
B^*_{**} = -L\hat{X}_1 + B^d
\]

\[\tag{3.11}
= B^d - \frac{a}{b}A_0 + \frac{1}{\epsilon}M_0 - \frac{1}{\epsilon}M_0 + c_1\theta_1 + c_2\theta_2 + c_3\theta_3
\]

\[= B^*_{**} + c_1\theta_1 + c_2\theta_2 + c_3\theta_3
\]

\[\text{Notice that the control coefficient } L \text{ is the same as in the case A. This is so because } L \text{ only depends on the form of disutility function, it is independent of the pure predicted value of the future state.}\]
where $\hat{X}_1 = \hat{Y}_0 + K\theta$ for a pure prediction of $X_1$, and, $L$ and $B^d$ are the same as that specified in equation (3.6);

$$c_1 = \frac{a}{b} \frac{\sigma^2_{w_1}}{\Delta_1}$$

$$c_2 = \frac{1}{e} \frac{\sigma^2_{w_2}}{\Delta_2}$$

$$c_3 = -\frac{1}{e} \frac{\sigma^2_{w_3}}{\Delta_3}$$

$$\Delta_i = \sigma^2_{w_i} + \sigma^2_{z_i}$$

By substituting the optimal control function, equation (3.11) into equation (3.10), the disutility expressed in terms of $I_2$ is

$$I_2 = E(X_1 - \hat{X}_1)'Q(X_1 - \hat{X}_1) = \text{trace } Q \hat{P}_1$$

$$= a^2 \sigma^2_{w_1} (1 - \frac{1}{\Delta_1}) + \frac{b}{e} \sigma^2_{w_2} (1 - \frac{1}{\Delta_2}) + \frac{b}{e} \sigma^2_{w_3} (1 - \frac{1}{\Delta_3}) .$$

Again the value of information is shown by a reduction of disutility from $W_2$ to $I_2$.

Section 3: Concluding Remarks

Given the initial means $A_0$, $m_0$, $M_0$, the previous development of this chapter shows that the stochastic monetary strategy can be separated into two stages: a prediction (estimation) and a deterministic control. The results are summarized as follows.
1. The separation theorem.

Havrilesky's optimal stochastic monetary strategy is shown to be the result of the following steps:

i) A deterministic control strategy, $B^*_0$, is given in equation (3.6).

ii) A prediction stage result is shown in equation (3.7). The income component, $\gamma$, of equation (3.7) is given as

$$\gamma = (0,a,-b \frac{b}{e},e)(X^0_0 + bB^*_0 + \frac{a_1 \sigma^2}{\Delta_1}, \frac{a_2 \sigma^2}{\Delta_2}, \frac{b \sigma^2}{\Delta_3} \theta)$$

iii) To obtain the optimal stochastic control, we re-write the result in ii) above as

$$\gamma = (0,a,-b \frac{b}{e},e)(X^0_0 + b(B^*_0 + \frac{1}{b}(\frac{a_1 \sigma^2}{\Delta_1}, \frac{-b \sigma^2}{\Delta_2}, \frac{b \sigma^2}{\Delta_3} \theta)))$$

$$= (0,a,-b \frac{b}{e},e)(X^0_0 + b(B^*_0 + c_1 \theta_1 + c_2 \theta_2 + c_3 \theta_3))$$

$$= (0,a,-b \frac{b}{e},e)(X^0_0 + bB^{**}_0)$$

where

$$B^{**}_0 = B^*_0 + c_1 \theta_1 + c_2 \theta_2 + c_3 \theta_3$$

Equation (3.14) shows that the optimal stochastic control $B^{**}_0$ is obtained by cascading a deterministic control, $B^*_0$, and a stochastic reaction function $c_1 \theta_1 + c_2 \theta_2 + c_3 \theta_3$.

The stochastic reaction coefficients $c_i$, $i = 1,2,3$ can be interpreted as the product of the deterministic reaction coefficient, $\frac{1}{b}$,
and the income component of Kalman gain matrix (see the first row of Kalman gain matrix given in n. 10). When estimates are very poor (i.e., $\sigma_i^2$ is relatively large), the absolute values of Kalman gain decreases, which, in turn, implies that the absolute values of stochastic reaction coefficients decrease. Thus, Kalman gain, or, the estimation stage, plays a very important role in the analysis of optimal stochastic monetary strategy. The relationships among estimation variances, $\sigma_i^2$, reaction coefficients, $c_i$, and various monetary strategies are illustrated by the example given in point 2 below.

2. The specification of Havrilesky's optimal monetary strategy.

Havrilesky has specified the optimal policy reaction function as

\[ (3.15) \quad B = B_0 + c_1 A^* + c_2 M^* + c_3 M^* \]

where \( B = B^{**} \).

Strictly speaking, since Havrilesky is unable to specify the term \( B_0 \) in the above equation, his control law is not admissible. Different interpretation of the term \( B_0 \) may give different results on the control magnitude. If \( B_0 \) is interpreted as a "constant," e.g., defining \( B_0 \) as the same as our deterministic control part,

\[ B_0 = B^*_0 \]

Then given the initial means of the intercepts, this definition of \( B_0 \) is consistent with Havrilesky's fixed monetary strategy for if $\sigma_i^2 \to \infty$, $i = 1,2,3$, equations (3.6), (3.11) and (3.15) would give the same result.
Unfortunately, \( B_0 \) so defined does not satisfy Havrilesky's other strategies. For example, under a money supply strategy \( z_1^2 + \infty \), \( z_2^2 + \infty \) and \( z_3^2 \) or \( c_1 \rightarrow 0 \), \( c_2 \rightarrow 0 \) and \( c_3 = -\frac{1}{e} \), the optimal monetary strategy as specified in equation (3.11) is different from equation (3.15) by an absolute magnitude

\[
\left| -\frac{1}{e}M^* - \left(-\frac{1}{e}M_0\right) \right| = \left| -\frac{1}{e}M_0 \right| = \text{bias}.
\]

In other words, equation (3.15) implies that the policy-maker overreacts to the survey data \( M^* \) by a bias \( \left| -\frac{1}{e}M_0 \right| \). In general, the rest of Havrilesky's strategies also contain biases. The actual magnitude of bias for each monetary strategy depends on the relative magnitudes of \( A_0 \), \( m_0 \) and \( M_0 \).

The bias associated with equation (3.15) can be removed only if \( B_0 \) is correctly defined. This can be done by the substitution of \( \theta_1 = A^* - A_0 \), \( \theta_2 = m^* - m_0 \) and \( \theta_3 = M^* - M_0 \) into equation (3.11), then the resulting equation is

\[
B^{**} = B_0 + c_1A^* + c_2m^* + c_3M^* ,
\]

where \( B_0 = B^* - c_1A_0 - c_2m_0 - c_3M_0 \).

3. Income variance.

One of the implications of the above formulation is that our income variance is the same as that specified by Havrilesky. To show this, we start with our income variance given in equation (3.13) as

\[
I_2 = \text{trace} \ Q P_1
\]

Further, we rewrite \( P_1 \) defined in equation (3.9) as
\[ P_1 = F \left( \Sigma_0 - (\Sigma_0 C') (\Sigma_0 C' + J)^{-1} (\Sigma_0 C')' \right) F' \]
\[ = F \left( \Sigma_0 - N (\Sigma_0 C' + J) N' \right) F' \]

where \[ N = (\Sigma_0 C') (\Sigma_0 C' + J)^{-1}. \]

Now, by making use of the following matrix identity,\(^{13}\)
\[ \Sigma_0 - N (\Sigma_0 C' + J) N' = (I - NC) \Sigma_0 (I - NC)' + NJN' \]

our income variance, \( I_2 \), can be rewritten as
\[ I_2 = \text{trace} \left[ Q \left( F \left( (I - NC) \Sigma_0 (I - NC)' + NJN' \right) F' \right) \right] \]

or
\[ = \sigma^2 \left( a - a \frac{\sigma^2_{w_1}}{\Delta_1} \right)^2 + \sigma^2 \left( a \frac{\sigma^2_{w_1}}{\Delta_1} \right)^2 \]
\[ + \sigma^2 \left( \frac{\sigma^2_{w_2}}{\Delta_2} \right)^2 + \sigma^2 \left( \frac{\sigma^2_{w_2}}{\Delta_2} \right)^2 \]
\[ + \sigma^2 \left( \frac{\sigma^2_{w_3}}{\Delta_3} \right)^2 \]

Finally, utilizing the definitions of \( c_1, c_2 \) and \( c_3 \) given in equation (3.12), we get
\[ I_2 = \sigma^2 \left( a + bc_1 \right)^2 + \sigma^2 \left( \frac{b}{e} + bc_2 \right)^2 + \sigma^2 \left( \frac{b}{e} + bc_3 \right)^2 \]
\[ + \sigma^2 \left( bc_1 \right)^2 + \sigma^2 \left( bc_2 \right)^2 + \sigma^2 \left( bc_3 \right)^2. \]

The last expression is exactly the same as the income variance obtained by Havrilesky. However, the income variance only reflects the loss associated with the uncontrollable part of equation (3.10). Theoretically speaking, a misspecification of the reaction function in Havrilesky's case would result in a different policy magnitude from the optimal policy solution, and therefore would result in a different square deviation between the "optimal" value and the desired value of the objective variable. In other words, the controllable part of the loss function of equation (3.10) would be sensitive to different policy reaction magnitude. Havrilesky neither did specify a complete reaction function, nor did he analyze the controllable part of the loss function. His analysis of optimal monetary strategy is therefore incomplete.

In short, we have shown that the modern control theory can be effectively used for the analysis of stochastic monetary strategy and that the estimation stage has played a dominant role in the formulation of the stochastic control strategy.

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14 Havrilesky, op. cit., equation (19), 1048.

15 See Section 4 of Chapter IV below for evidence of the effect of reaction coefficients on the controllable part of the loss function.
APPLICATION OF THE SEPARATION THEOREM TO THE
ST. LOUIS EQUATION AND THE MINNIE EQUATION

Section 1: Introduction

In recent years there have been some application of modern control theory to the conduct of open market operation, such as that, Pindyck, R.S. and Roberts, S.M. consider the trade-offs between short-run targets; Tinsley, P.A. considers the effectiveness of partial monthly information in revising quarterly forecasts; Kalchbrenner, J.H. and Tinsley, P.A. consider a feedback strategy for monetary policy. It is worth noting that the filtering approaches used in Tinsley, and Kalchbrenner and Tinsley are essentially special cases of the techniques of the Kalman filtering processes.

The purpose of this chapter is to provide control experiments to illustrate the effectiveness of the Kalman filtering process in the design of optimal monetary policy. The total spending equation of the St. Louis model is used to represent the state equation and the demand

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for money equation of the MINNIE model; a small version of the MIT-PENN-SSRC econometric model is used to represent the measurement equation.\(^2\) Then a Kalman filtering process is applied to the demand for money equation of the MINNIE model for the purpose of extracting money market information to improve the estimation (prediction) of the total spending equation of the St. Louis model. The result is surprisingly good. The Kalman filtering process is able to reduce the standard error of the total spending equation of 71 billion dollars\(^3\) to roughly .0001 billion dollars, which in turn results in a great improvement in design of optimal monetary strategy. We found that during the period under consideration, the total value of the quadratic loss function when Kalman filter is used is roughly only half of the loss when Kalman filter is not used.

Furthermore, in order to compare our approach with the "add factor" filtering approach suggested by Tinsley, an "add factor" filtering process is used to substitute the Kalman filtering process for the estimation and control experiments. We found that the results show virtually no improvement for the case using "add factor" filter as compared with the case without using filter.


\(^3\) See n. 6 below.
Section 2: The Estimation Stage

In this section we will present an application of Kalman filtering process to improve the estimation of the state equation. A revised version of the total spending equation of the St. Louis model is used to represent the state equation, i.e.,

\[(4.1) \quad X_t = FX_{t-1} + GM_t + b_t + v_{1,t}\]

where

\[
X_t = \begin{pmatrix}
Y_t \\
M_t \\
M_{t-1} \\
M_{t-2} \\
M_{t-3} \\
M_{t-4}
\end{pmatrix}
\quad F = \begin{pmatrix}
\begin{bmatrix}
1 & m_{12} & m_{13} & m_{14} & m_{15} & m_{16}
\end{bmatrix} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

\[
G_t = \begin{pmatrix}
m_0 \\
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\quad b_t = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\quad v_{1,t} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\(Y_t\): nominal GNP

\(M_t\): narrowly defined money supply

\(m_{12}, m_{13}, m_{14}, m_{15}, m_{16}\): coefficients, given

\(^4\)See Appendix A, equation (A.3). Datum for \(Y\) and \(M\) are given in Table 3 and Table 4 in the text of this chapter. Datum for \(m_0, m_{1j}\), \(j=2,\ldots,6\), and \(b_t\) are given in Appendix A.
b_t : exogenous factors, given
v_{1t} : residual.

In the real world, current GNP is unobservable and the lagged GNP is observable, therefore in equation (4.1), \( Y_t \) of \( X_t \) is assumed to be unknown, and is supposed to be estimated from the known quantities \( X_{t-1}, M_t, b_t \) and coefficients \( F, G \).

One of the basic purposes of monetary indicator analysis developed in the previous chapters is to extract information from the observable money market conditions to improve the estimation of the real sector of the economy, \( Y_t \). We therefore introduce a revised version of the demand for money function of the MINNIE model as the measurement equation to represent the money market conditions for state estimation purposes.\(^5\)

\[
(4.2) \quad r_t = H_t X_t + f_t + v_{2,t}
\]

where \( H_t = (h_{1t}, h_{2t}, h_{3t}, 0, 0, 0) \), given.

\( r_t \) : three months treasury bill rates, given
\( f_t \) : exogenous factors, given
\( v_{2,t} \) : residual.

Notice that \( Y_t \) element of \( X_t \) is unknown in equation (4.2). However its information is contained in the behavior of \( r_t \).

Simulations were conducted for III/1970 through IV/1974 to generate residuals \( v_{1,t} \) and \( v_{2,t} \), which in turn, are used to estimate

\(^5\) See Appendix A, equation (A.5) for derivation of equation (4.2), \( r_t, H_t \) and \( f_t \).
the variance-covariance matrix of \( v_{1,t} \) and \( v_{2,t} \), i.e., \( V \), as follows.\(^6\)

\[
V = \begin{pmatrix}
V_{11} & V_{12} \\
V_{12} & V_{22}
\end{pmatrix}
\]

where

\[
V_{11} = \begin{pmatrix} 71.38147 & 0 \\
0 & 0 \end{pmatrix}, \quad V_{12} = \begin{pmatrix} 0.58068 \\
0 & 0 \end{pmatrix}
\]

\[ V_{22} = 0.25162 \]

\( 0 \): (5 x 1) zero vector

\( 0_k \): (5 x 5) zero matrix.

Let values in \( t-1 \) stand for the initial conditions, i.e.,

\(^6\)Simulations for equations (A.1) and (A.2) of Appendix A were conducted for periods of III/1970 to IV/1974. Our estimate of "standard error" of the St. Louis equation (A.1), i.e., the first element of \( V_{11} \) is 71.38147 (billions of dollars) which is far exceeding the estimate of 5.61 as given in the original St. Louis equation (equation (A.1)). However, our estimate of "standard error" of the MINNIE equation based on equation (A.2), say, \( U_{22} = E(u_{2t} - E(u_{2t}))' \) was 0.00002, which is far less than the estimate of 0.0068 as given in the original MINNIE equation (equation (A.2)). The estimate of \( U_{22} \) was then used to calculate \( V_{22} \) by utilizing the relationship

\[
\bar{v}_{2,t} = \frac{1}{(.0615)} \text{ in RTB}_t / \text{RTB}_t \cdot u_{2,t} \text{ given in equation (A.5). Furthermore, } V_{11}, V_{12}, V_{22} \text{ are defined as follows:
}\]

\[
V_{11} = E(v_{1t} - E(v_{1t})) (v_{1t} - E(v_{1t}))'
\]

\[
V_{12} = E(v_{1t} - E(v_{1t})) (v_{2t} - E(v_{2t}))'
\]

\[
V_{22} = E(v_{2t} - E(v_{2t})) (v_{2t} - E(v_{2t}))'
\]

For simplicity \( n=18 \) quarters is used to calculate the expected values, not adjusted for degree of freedom.
Let \( \bar{X}_t \) and \( \Sigma_t \) stand respectively for the expected value and variance-covariance matrix of \( X_t \) given the measurement at quarter \( t-1 \), but the measurement at quarter \( t \) is not used. Similarly, \( \hat{X}_t \) and \( \Pi_t \) stand respectively for the expected value and variance-covariance matrix of \( X_t \) given the measurements from quarter \( t-1 \) to \( t \).

Then

Case 1. When the measurement equation (4.2) at quarter \( t \) is not used, the state estimation is \( \bar{X}_t \),

\[
(4.4) \quad \bar{X}_t = FX_{t-1} + GM_t + b_t
\]

where the initial condition \( X_{t-1} \) and \( b_t \) are known.

The associated variance-covariance matrix is

\[
(4.5) \quad E_{11,t} = v_{11}
\]

Case 2. When the measurement equation (4.2) at quarter \( t \) is used, the state estimation is \( \hat{X}_t \).

\[
(4.6) \quad \hat{X}_t = FX_{t-1} + GM_t + b_t + k_t(r_t - H_t \bar{X}_t - f_t)
\]

where \( X_{t-1} \) and \( b_t \) are known; the estimates \( \bar{X}_t \) is given in equation (4.4).

\[\text{Equations (4.6), (4.7) and (4.8) are derived from the system of equations (4.1) and (4.2). For details, see Kwakernaak, H. and Sivan, R., op. cit., 550-551, equations (6.528) through (6.532).}\]
The Kalman gain is

\[ k_t = (V_{11} H'H_t + V_{12}) \cdot (H_t V_{11} H_t' + 2H_t V_{12} + V_{22})^{-1}. \]

The updated variance-covariance matrix is

\[ \Pi_{11}, t = (I - k_t H_t) V_{11} - k_t V_{12}. \]

Case 3. When "add factor \( v_{2t} \)" is used as information at quarter \( t \), as suggested by Tinsley, the state estimation is as follows,

\[ \hat{x}_t = FX_{t-1} + GM_t + b_t + k_t v_{2,t}, \]

where initial condition \( x_{t-1}, b_t \) and \( v_{2,t} \) are assumed to be known.

The gain vector is

\[ k_t = V_{12} V_{22}^{-1}. \]

The variance-covariance matrix is

\[ \Pi_t = V_{11} - V_{12} k_t V_{22}. \]

Notice that we have assumed that GNP is unobservable at quarter \( t \) but its lagged value, GNP_{t-1}, is observable at quarter \( t \), which is equivalent to say that given the initial condition, known GNP_{t-1}, there needs only "one period" prediction techniques to predict GNP. Equation (4.4) through equation (4.11) are designed for one period prediction only.

---

8 P.A. Tinsley, op. cit., especially 33-34, equation (14) and 41, n. 29.
The estimated Kalman gain for case 2 is listed in Table 2. The decline from .0221 of I/1972 to .0158 of III/1974, mainly reflects the decrease in $\ln RTB_t/RTB_e$ associated with linearization of the demand for money function described in equation (A.5) of Appendix A. In case 3, the first element of gain vector is $-2.30775$ which shows an opposite direction of adjustment from Kalman gain given in case 2. The difference reflects the difference in definitions of gain matrices.\(^9\)

The estimated one-period prediction state variance-covariance matrices for three cases are, respectively, as follows,

\begin{equation}
\Sigma_{11} = \begin{pmatrix}
71.3815 & 0' \\
0 & 0
\end{pmatrix}
\end{equation}

\begin{equation}
\Pi_{11,t} = \begin{pmatrix}
.00012 & 0' \\
0 & 0
\end{pmatrix}, \quad t = I/1972, \text{ and for } \Pi_{11,t+i}, \quad i=1,...,ii, \text{ given in Table 2.}
\end{equation}

\begin{equation}
\Pi_{11} = \begin{pmatrix}
70.0414 & 0' \\
0 & 0
\end{pmatrix}
\end{equation}

It is clear from our estimation that the variances of case 2 given in Table 2 are far less than that of case 1 and case 3. This reflects the power of the Kalman filtering process to reduce the one period prediction variance.

\(^9\)For further explanations, see section 4 of this chapter.
### TABLE 2

Estimated Kalman Gain and Updated State Variance for Case 2^a,^b

<table>
<thead>
<tr>
<th>Quarter, t+i, t=I/1972, i=1,...,11</th>
<th>Kalman Gain: k_{t+i}</th>
<th>Updated State Variance: ( \pi_{t+i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I/1972</td>
<td>0.0221</td>
<td>0.00012</td>
</tr>
<tr>
<td>II/</td>
<td>0.0217</td>
<td>0.00014</td>
</tr>
<tr>
<td>III/</td>
<td>0.0210</td>
<td>0.00009</td>
</tr>
<tr>
<td>IV/</td>
<td>0.0200</td>
<td>0.00013</td>
</tr>
<tr>
<td>I/1973</td>
<td>0.0188</td>
<td>0.00010</td>
</tr>
<tr>
<td>II/</td>
<td>0.0176</td>
<td>0.00009</td>
</tr>
<tr>
<td>III/</td>
<td>0.0157</td>
<td>0.00012</td>
</tr>
<tr>
<td>IV/</td>
<td>0.0165</td>
<td>0.00015</td>
</tr>
<tr>
<td>I/1974</td>
<td>0.0164</td>
<td>0.00014</td>
</tr>
<tr>
<td>II/</td>
<td>0.0158</td>
<td>0.00009</td>
</tr>
<tr>
<td>III/</td>
<td>0.0158</td>
<td>0.00013</td>
</tr>
<tr>
<td>IV/</td>
<td>0.0167</td>
<td>0.00010</td>
</tr>
</tbody>
</table>

^a^ Only the first element of vector k is listed.

^b^ Only the first diagonal element of variance-covariance matrix, \( \pi_{t+i} \), is listed.
Section 3: The Separation Theorem and the Optimal Monetary Strategy

In the previous chapters we have shown that the estimation stage and the control stage can be solved separately and then be combined to find the optimal monetary strategy. Also, we have stressed the point that different monetary strategy mainly reflects the response of the policymaker to different state variable estimation. The control stage itself is quite straightforward and does not affect much on the policy decision process.

The purpose of this section is to combine each one of the three different results of state estimation given in the previous section with the same control stage result to access the effectiveness of estimation stage on the design of optimal monetary policy strategy.

The policymaker is assumed to minimize the following expected quadratic loss function for the period of II/1972 to III/1974, i.e.,

\[(4.15) \quad \text{Min } W = \frac{1}{2} E \sum_{i=1}^{10} (X_{t-1}^e - X_{t+1}^d)'Q(X_{t+1} - X_{t+1}^d) + I_1 + I_2\]

where

\[I_1 = \frac{1}{2} E \sum_{i=1}^{10} (X_{t+1}^e - X_{t+1}^d)'Q(X_{t+1} - X_{t+1}^d);\]

\[I_2 = \frac{1}{2} E \sum_{i=1}^{10} (X_{t+1}^e - X_{t+1}^d)'Q(X_{t+1} - X_{t+1}^d);\]

\[I_2 \text{ is not controllable.}\]

\[X_{t+1}^e: \text{ estimated conditional mean of state variable at quarter } t+1,\]

\[i = 1, \ldots, 10, \quad t = I/1972\]

\[X_{t+1}^d: \text{ desired value of state variable } X_{t+1}, \text{ defined as}\]
Penalty cost; the diagonal elements of \( Q \) are arbitrarily given as \((50, 100, 0, 0, 0, 0)\) and all the off-diagonal elements are zeros.\(^{10}\)

The control stage solution always takes the following form:\(^{11}\)

\[
M^*_{t+1} + L_{t+1} \cdot \frac{X^e_{t+i-1}}{e_{t+i}}, \; i = 1, \ldots, 10, \; t = I/1972
\]

where \( L_{t+i}, \; i = 1, \ldots, 10 \), stand for feedback control coefficients which are not affected by different forms of \( X^e \).

Notice that the above equation implies that the policymaker minimizes equation (4.15) subject to the constraint of state variable behavior, \( X_{t+i}, \; i = 1, \ldots, 10, \; t = I/1972 \). For purposes of illustration let us consider the policy decision process at the beginning of quarter \( t+1 \), where the policymaker is given with known \( X_{t-1}, M_t, b_{t+1}, \) and \( X^e_t \). He has to predict \( X_{t+1} \) and select \( M^*_{t+1} \), subject to the constraint of the behavior of \( X_{t+1} \) as follows,

\[
(4.16) \quad X_{t+1} = FX_t + GM_{t+1} + b_{t+1} + v_{t-1,t+1}
\]

where the initial condition \( X_{t-1} \), and \( b_{t+1} \) are known, \( t = I/1972 \).

The one-period estimation of \( X_t \), i.e., \( X^e_t \) in equation (4.16) can take any one of the three forms, \( \dot{X}_t, \ddot{X}_t \), and \( \dddot{X}_t \) as described in previous

\(^{10}\) The diagonal elements of \( Q \) are arbitrarily chosen. The higher penalty cost imposed on money stock than on GNP is designed to reflect the desire of the Federal Reserve System to maintain an orderly money market condition for the periods under consideration. This can be justified from Table 4 in which, for the period of II/1972 to III/1974, \( M^d \) is almost equal to \( M \); whereas in Table 3, \( Y^d \) is considerably different from \( Y \).

sections. By substituting $X_t^e$ into equation (4.16), we have a two-period prediction value $X_{t+1}^e$, which may take either one of the following forms,

(4.17) $\overline{X}_{t+1} = F\overline{X}_t + GM_{t+1} + b_{t+1}$

where $\overline{X}_t$ is given in equation (4.4).

(4.18) $\hat{X}_{t+1} = FX_t + GM_{t+1} + b_{t+1}$

where $\hat{X}_t$ is given in equation (4.6).

(4.19) $\hat{X}_{t+1} = F\hat{X}_t + GM_{t+1} + b_{t+1}$

where $\hat{X}_t$ is given in equation (4.9).

The estimated two-period prediction variance-covariance matrices for $\overline{X}_{t+1}$, $\hat{X}_{t+1}$ and $\hat{X}_{t+1}$ are given respectively as follows,

(4.20) $F_{11} F' + V_{11}$

(4.21) $F_{11,t} F' + V_{11}$

(4.22) $\Sigma_{11,t} F' + V_{11}$

In general, at the beginning of quarter $t+1$, $i = 1,\ldots,10$, the policymaker is always given with known $X_{t+i-2}^e$ and $b_{t+i}$, $i = 1,\ldots,10$, as initial conditions, and use one-period prediction $X_{t+i-1}^e$ to obtain two-period prediction $X_{t+i}^e$. The related two-period prediction covariance matrices are given from equations (4.20) through (4.22) except that in equation (4.21) "t" is replaced by "$t+i-1$," $i = 1,\ldots,10$.

The detailed description of the optimal solution derived by periodic control revision is given in Appendix B. The result of the optimal
GNP, $\bar{Y}^*$, $\hat{Y}^*$ and the optimal money stock $\bar{M}^*$, $\hat{M}^*$, $\hat{M}^*$ are given in Table 3 and Table 4, and in Figure 1 and Figure 2.

The resulting expected losses of the quadratic loss function for II/1972 to III/1974 are derived by substituting the results given in Tables 3 and 4 into equation (4.15). The results for the following three cases are given in Table 5.

Case 1: When $\bar{X}_{t+i}^*$ and $(\bar{F}^*_t + V^*_t)$, $i = 1, \ldots, 10$, $t = I/1972$, are used to find the loss values $I_1$ and $I_2$.

Case 2: When $\hat{X}_{t+i}^*$ and $(\hat{F}^*_{t+i-1} + V_{11})$, $i = 1, \ldots, 10$, are used to find the loss values $I_1$ and $I_2$.

Case 3: When $\hat{X}_{t+i}^*$ and $(\hat{F}^*_{11} + V_{11})$, $i = 1, \ldots, 10$ are used to find the loss values $I_1$ and $I_2$.

The loss values of $I_1$ and $I_2$ for case 1 are 18,460 and 35,691 respectively, and for case 3 are 19,692 and 35,356 respectively. There are no significant differences existing between loss values of these two cases. However, the loss values of $I_1 = 8,416$ and $I_2 = 17,845$ of case 2 are far less than that of case 1 and case 3. The results strongly support the use of Kalman filtering process in the design of optimal monetary strategy.

Section 4: Concluding Remarks

In the previous sections we have decomposed the optimal stochastic control strategy into two stages, i.e., the estimation stage and the control stage, and we also have decomposed the expected loss function into two parts, i.e., the controllable part $I_1$, and the uncontrollable part $I_2$. The loss associated with the controllable part can be lowered
### Table 3

GNP Paths Produced by Alternative Strategies<sup>a</sup>

(Billions of Dollars)

<table>
<thead>
<tr>
<th>Quarter t+i</th>
<th>Actual Desired</th>
<th>Optimal Solutions&lt;sup&gt;b&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Y$</td>
<td>$Y^d$</td>
</tr>
<tr>
<td>II/1972</td>
<td>1,143</td>
<td>1,143</td>
</tr>
<tr>
<td>III/</td>
<td>1,169</td>
<td>1,172</td>
</tr>
<tr>
<td>IV/</td>
<td>1,205</td>
<td>1,207</td>
</tr>
<tr>
<td>I/1973</td>
<td>1,249</td>
<td>1,245</td>
</tr>
<tr>
<td>II/</td>
<td>1,278</td>
<td>1,278</td>
</tr>
<tr>
<td>III/</td>
<td>1,309</td>
<td>1,340</td>
</tr>
<tr>
<td>IV/</td>
<td>1,344</td>
<td>1,339</td>
</tr>
<tr>
<td>I/1974</td>
<td>1,359</td>
<td>1,361</td>
</tr>
<tr>
<td>II/</td>
<td>1,384</td>
<td>1,386</td>
</tr>
<tr>
<td>III/</td>
<td>1,416</td>
<td>1,412</td>
</tr>
</tbody>
</table>

<sup>a</sup>Y's stand for GNP's. Only GNP elements of $X$ matrices are concerned here.

<sup>b</sup>For derivation of optimal solutions, see Appendix B.
### TABLE 4
Money Stocks Produced by Alternative Strategies\(^a\)
(Billions of Dollars)

<table>
<thead>
<tr>
<th>Quarter</th>
<th>Actural</th>
<th>Desired</th>
<th>Optimal Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(M)</td>
<td>(M^d)</td>
<td>(\tilde{M^*})</td>
</tr>
<tr>
<td>II/1972</td>
<td>242.9</td>
<td>242.9</td>
<td>249.6</td>
</tr>
<tr>
<td>III/</td>
<td>247.7</td>
<td>247.9</td>
<td>253.2</td>
</tr>
<tr>
<td>IV/</td>
<td>253.3</td>
<td>253.0</td>
<td>258.6</td>
</tr>
<tr>
<td>I/1973</td>
<td>257.6</td>
<td>257.7</td>
<td>263.6</td>
</tr>
<tr>
<td>II/</td>
<td>262.3</td>
<td>262.0</td>
<td>269.6</td>
</tr>
<tr>
<td>III/</td>
<td>265.9</td>
<td>265.8</td>
<td>269.6</td>
</tr>
<tr>
<td>IV/</td>
<td>269.2</td>
<td>269.3</td>
<td>271.7</td>
</tr>
<tr>
<td>I/1974</td>
<td>273.1</td>
<td>273.3</td>
<td>271.8</td>
</tr>
<tr>
<td>II/</td>
<td>278.1</td>
<td>277.5</td>
<td>272.8</td>
</tr>
<tr>
<td>III/</td>
<td>280.7</td>
<td>280.7</td>
<td>280.4</td>
</tr>
</tbody>
</table>

\(^a\)For derivation of optimal solutions, see Appendix B.

\(^b\)\(\tilde{M^*}\) path is similar to \(\tilde{M^*}\) and is omitted in Figure 2.
Billions of Dollars

Figure 1

GNP Paths Produced by the Alternative Strategies
Figure 2

Money Stocks Produced by the Alternative Strategies
TABLE 5

Expected Losses of Alternative Strategies\(^a\)

<table>
<thead>
<tr>
<th></th>
<th>Controllable Losses (I_1)</th>
<th>Uncontrollable Losses (I_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1: (\bar{Y}^<em>, \bar{M}^</em>)</td>
<td>18,410</td>
<td>35,691</td>
</tr>
<tr>
<td>Case 2: (\hat{Y}^<em>, \hat{M}^</em>)</td>
<td>8,415</td>
<td>17,845</td>
</tr>
<tr>
<td>Case 3: (\hat{Y}^<em>, \hat{M}^</em>)</td>
<td>19,692</td>
<td>35,356</td>
</tr>
</tbody>
</table>

\(^a\)Results are derived from equation (4.15), Table 3 and Table 4.
by applying a feedback control to a more efficient state estimation. Whereas the loss associated with uncontrollable part can only be reduced by the improvement of the state estimation and cannot be affected by the control action.

Furthermore, the definition of Kalman filter implies that the effectiveness of using monetary indicators to improve the estimation of the ultimate objective variables relies upon high correlations between the monetary indicators and the ultimate objective variables. And we have shown that the Kalman filtering process has generated high correlation between the innovation vector of the measurement equation, \(v_{2t} + H_{t-1} v_{1t}\), and the residual of the state equation, \(v_{1t}\). As a result a weighted sum of the updated state covariances of the Kalman filter estimation, measured by the uncontrollable part of the loss function, is much lower than the corresponding loss generated by the case without using filter (case 1).

The "add factor" filtering process also relies upon the correlation between the residual of the measurement equation \(v_{2t}\) and the residual of the state equation \(v_{1t}\). However, since the correlation is low, the "add factor" approach can only make a negligible reduction in the uncontrollable part of the loss function.

Furthermore, by comparing the results of the different stochastic control strategies, we find that the application of a feedback control to the state estimation generated by Kalman filtering process gives the lowest value of the controllable part of the loss function. The application of the same feedback control to the estimation without using filtering process gives the second lowest corresponding loss.
Whereas the case with the use of the "add factor" filtering process gives the highest value of the controllable part of the loss function. The reason is that in our examples, the sign of the "add factor" gain is different from that of the Kalman gain. And as a result the use of "add factor" filtering process even worsens the loss associated with the controllable part.

The result of the "add factor" filtering process implies that Havrilesky's analysis, which we criticized in Chapter III above, is indeed incomplete since neither did it provide a complete policy reaction function, nor did it provide an analysis for the effect of the policy reaction coefficients on the controllable part of the loss function. The effectiveness of optimal stochastic control solutions should be evaluated on the basis of the losses associated with the controllable part and the uncontrollable part.
CHAPTER V

SUMMARY

This dissertation has dealt with a topic related to one current research stream in monetary economics, that is the finding of optimal monetary strategy with the use of information variables. We have shown that some analytical techniques employed by economists are quite restrictive. For example, Havrilesky considers static model with a single policy objective variable; Kareken, Muench and Wallace (K-M-W) although consider dynamic models but they are unable to solve the multiple-objective variables case. This is evidenced by their use of transformations in dealing with multiple sources of information. To improve the state of arts, we have introduced a Kalman filter approach to monetary strategy and provided control experiments to show the effectiveness of the Kalman filtering process.

By the use of a Kalman filter approach, we have shown:

(1) Monetary indicators and target problems can be viewed as the estimation problem and the deterministic control problem, respectively.

(2) With respect to Havrilesky's work, we have explicitly shown the importance of monetary indicators (estimation stage) in the formulation of stochastic monetary strategies; we also give a complete specification for his policy reaction function. Also, we have indicated
that his analysis is incomplete because no analysis on the effect of the reaction coefficients on the controllable part of the expected loss function has been made.

(3) With regard to K-M-W's work, we point out that multiple information as well as multiple objective variables can be analyzed in a Kalman filter framework; we also provide a more reasonable deterministic reaction function than that proposed by K-M-W.

(4) The Kalman filter approach has been shown to be applicable to K-M-W's dynamic model as well as to Havrilesky's static model.

(5) In our control experiments for the period of II/1972 to III/1974, the total spending equation of the St. Louis model is used to represent the state equation and the demand for money equation of the MINNIE model is used to represent the measurement equation. The application of the Kalman filtering process to the measured information enables us to reduce the standard error of the St. Louis equation from the estimated value of 71 billion dollars to 0.0001 billion dollars. This evidence strongly supports the use of the Kalman filtering process in the design of optimal stochastic monetary strategy.

Lastly, a part of our contribution, i.e., our Chapter II on the Kareken, Muench and Wallace's work of the optimal open market strategy analysis is a pioneering work of the application of the Kalman filtering process to the monetary indicator analysis. Our result was published
in November 1973\textsuperscript{1} which is almost two years earlier than other developments of the Kalman filter applications to the monetary analyses.\textsuperscript{2}


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I. The original versions of the total spending equation of the St. Louis model and the demand money for equation of the MINNIE model.

1. Alphabetic listing of variables:

   E  High-employment Federal budget expenditures, in billions of dollars.
   M  Nominal narrowly defined money stock, in billions of dollars, Federal Reserve Bank of St. Louis data.
   MD Nominal demand deposits adjusted at all commercial banks.
   N  Population, in billions.
   RTB Treasury bill rates, quarterly average.
   XGNF Real gross national product, in billions of 1958 dollars.
   Y  Nominal GNP, in billions of dollars, Federal Reserve Bank of St. Louis data.
   ZCT Ceiling rate on passbook saving deposits, quarterly average.
   ZDRA Federal Reserve discount rate, quarterly average.

2. Estimated equations ("t" statistics appear with each regression coefficients, enclosed by parentheses):

   The total spending equation of the St. Louis model:

The demand for money equation of the MINNIE model:

Sample period: I/1947 - IV/1973

\[
\begin{align*}
\ln \left( \frac{MD_t}{Y_t} \right) &= 0.2799 \ln \left( \frac{MD_{t-1}}{Y_{t-1}} \right) - 0.0615 \ln RTB_t \\
&\quad - 0.123 \ln ZCT_t + Q1 + Q2 + 0.0779 \ln \left( \frac{ZDRA_t}{ZDRA_{t-1}} \right) \\
&\quad - 0.3393 \ln \left( \frac{X_{GDP,t}}{N_t} \right) - 0.519 - 0.9 \hat{U}_{t-1}
\end{align*}
\]

standard error = .0068

where: \( Q1 = \begin{cases} 
-0.1271 & \text{if time < II/1960} \\
0 & \text{otherwise}
\end{cases} \)

\[ Q2 = \begin{cases} 
-0.051 & \text{if time > II/1960} \\
0 & \text{otherwise}
\end{cases} \]

II. The versions of the equations used in the text.

1. The state equation: the total spending equation of the St. Louis model is rearranged as follows:

\[
\begin{align*}
Y_t &= Y_{t-1} + m_0 M_t + m_1 M_{t-1} + m_2 M_{t-2} + m_3 M_{t-3} \\
&\quad + m_4 M_{t-4} + m_5 M_{t-5} + b_t + v_{1,t}
\end{align*}
\]
2. The measurement equation: the demand for money equation of the MINNIE model is renormalized on RTB for Kalman filtering purposes; MD in equation (A.2) is replaced by $M$ as follows:

\[
\begin{align*}
(\text{A.4}) & \quad \text{RTB}_t = \frac{1}{(0.0615) \ln \text{RTB}_t} \cdot \left[ (0.7201 \frac{\ln Y_t}{Y_t}) Y_t \
& \quad - \frac{\ln M_t}{M_t} \cdot M_t + (0.2799 \frac{\ln M_{t-1}}{M_{t-1}}) \cdot M_{t-1} \
& \quad + d_t + u_{2,t} \right] \\
\end{align*}
\]

where:
\[
d_t = -0.123 \ln \text{ZCT}_t + 0.0779 \ln \left( \frac{\text{ZDRA}_t}{\text{ZDRA}_{t-1}} \right) \\
-0.3393 \ln \left( \frac{\text{XGNP}_t}{N_t} \right) - 0.468 + 0.9u_{t-1}.
\]

Notice that in equation (A.4), the new intercept given in $d_t$, -0.468 is derived by adding $Q_2 = 0.051$ to the original intercept of equation (A.2), -0.519. Furthermore the identity
\[
\ln X_t = \frac{\ln X_t}{X_t} \cdot X_t
\]
is used in converting equation (A.2) to (A.4).
For simplicity, equation (A.4) can be redefined as follows:

\( r_t = h_{1,t} Y_t + h_{2,t} M_t + h_{3,t} M_{t-1} + f_t + v_{2,t} \)

where: \( r_t = RTB_t \)

\( h_{1t}, h_{2t}, h_{3t} \) are equal to the coefficients shown in the corresponding parts of equation (A.4): they are variable coefficients changing from time to time, and are assumed to be known to the policymaker.\(^1\)

\[
\begin{align*}
\log RTB_t &= \log RTB_{t-1} + 0.615 \cdot d_t \\
\log RTB_t &= \log RTB_{t-1} + 0.615 \cdot u_{2,t}
\end{align*}
\]

\( \log Y_t = 0.7201 - \frac{0.615}{\log RTB_t} \cdot u_{2,t} \)

\( Y_t \) is unknown in quarter \( t \).

Theoretically speaking, \( Y_t \) should be replaced by an updated estimate of \( Y_t \) for the computation of \( h_{1t} \). However, since the added computation involved is cumbersome if not impossible, we assume without loss of generality that \( h_{1t} \) is known to the policymaker as defined above.

Similarly, \( h_{1,t+i}, i=1,...,10 \) are computed by assuming that \( Y_{t+i}, i=1,...,10 \) are known in quarter \( t+i, i=1,...,10 \) respectively.

A similar simplification can be found in Peter Walsh and J.B. Cruz, Jr., "Neighboring Stochastic Control of an Econometric Model," Annals of Economic and Social Measurements, May 2, 1976.
<table>
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<th>Quarter</th>
<th>RTB</th>
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<tr>
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<tr>
<td>II/</td>
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<td>4.5</td>
<td>4.75</td>
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<td>4.5</td>
<td>4.96</td>
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<td>IV/</td>
<td>4.23</td>
<td>4.5</td>
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<td>4.50</td>
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<tr>
<td>II/</td>
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<td>4.5</td>
<td>4.50</td>
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<td>III/</td>
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<td>4.5</td>
<td>4.50</td>
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<td>IV/</td>
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The Raw Data of Population, Real GNP and Federal Budget Expenditures (billions)

<table>
<thead>
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<th>Quarter</th>
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<th>E</th>
</tr>
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<tbody>
<tr>
<td>I/1971</td>
<td>0.20621</td>
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<tr>
<td>II/</td>
<td>0.20672</td>
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<tr>
<td>III/</td>
<td>0.20720</td>
<td>747.1</td>
<td>218.3</td>
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<tr>
<td>IV/</td>
<td>0.20777</td>
<td>759.1</td>
<td>223.3</td>
</tr>
<tr>
<td>I/1972</td>
<td>0.20814</td>
<td>770.9</td>
<td>232.2</td>
</tr>
<tr>
<td>II/</td>
<td>0.20857</td>
<td>786.6</td>
<td>239.9</td>
</tr>
<tr>
<td>III/</td>
<td>0.20898</td>
<td>798.1</td>
<td>235.5</td>
</tr>
<tr>
<td>IV/</td>
<td>0.20944</td>
<td>814.1</td>
<td>259.3</td>
</tr>
<tr>
<td>I/1973</td>
<td>0.20982</td>
<td>832.8</td>
<td>258.9</td>
</tr>
<tr>
<td>II/</td>
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<td>IV/</td>
<td>0.21097</td>
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<td>269.5</td>
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<td>279.0</td>
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<td>315.1</td>
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APPENDIX B

DERIVATION OF THE OPTIMAL STOCHASTIC CONTROL SOLUTIONS
This appendix will present periodic stochastic control revision strategies for the period from II/1972 to III/1974 in which stochastic control solutions are derived based on the initial information constraints and are revised according to the updated information.

Let three cases (Case 1 to 3) of the state estimation given in Chapter IV serve as the relevant information constraints, then the control revisions in response to the updated information can be shown as follows.

Consider Case 2 of Chapter IV in which the expected value of the state variable at quarter \( t+1 \), \( \hat{X}_{t+1} \), as shown in equation (4.18), is derived from known initial condition \( X_{t+1-2} \) and known estimate \( \hat{X}_{t+1-1} \), \( t+1/1972, i=1\ldots10. \)

(i) At the beginning of the first control quarter \( t+1 \), the state estimation (prediction of equation (4.18) can be rewritten as equation (B.1)

\[
\hat{X}_{t+1} = F\bar{X}_t + GM^*_{t+1} + b_{t+1} + Fk(v_{2,t} + H_{t,1,v_{t_1,t}}) 
\]

where: \( \bar{X}_t \) is given in equation (4.4).\(^2\)

\(^1\)We use the solution versions given in G.C. Chow, Analysis and Control of Dynamic System, 1975, Chapters 7 and 8, to find the optimal solutions for state and control variables.

\(^2\)(\( x_t - H_t \xi - f_t \)) in equation (4.18) can be shown equal to \( (v_{2t} + H_{t,1,v_{t_1,t}}) \) by substituting \( \bar{X}_t \) in equation (4.4) for \( X_t \) in equation (4.2) and rearranging terms in the resulting equation (4.2).
Notice that at the beginning of the first control quarter \( t+1, t = I/1972 \), the only measurement available is for quarter \( t \), no measurements available for future quarters \( t+i, i=1,\ldots,10 \). Therefore the case \( \underline{X}_{t-1} \) of equation (4.4) is used to predict the future states, i.e., equation (B.1) can be generalized as equation (B.2) in which all predicted states are conditioned on the same information set \( (v_{2,t} + H_{t-1,t}) \) at quarter \( t \),

\[
\begin{align*}
\hat{X}_{t+1} &= \underline{F} \underline{X}_{t+1-1} + \underline{G} \underline{X}_{t+1} + b_{t+1} + F_{t}(v_{2,t} + H_{t-1,t}) \\
\text{where: } i=1,\ldots,10, \ t=I/1972; \ \underline{X}_{t-1} = X_{t-1} \text{ is given; } \\
\underline{X}_{t} \text{ is given in (4.4).}
\end{align*}
\]

The policymaker at the beginning of \( t+1 \) would find optimal solutions \( \hat{M}_{t+1}^{*} \) and \( \hat{X}_{t+1}^{*} \) for \( i=1,\ldots,10 \) which minimize the quadratic loss function of equation (4.15) for \( t+1, i=1,\ldots,10 \), subject to constraints of \( \hat{X}_{t+1}^{*}, i=1,\ldots,10 \) of equation (B.2). The results, \( \hat{X}_{t+1}^{*} \) and \( \hat{M}_{t+1}^{*} \) i.e., for \( i=1 \) only, are listed on the II/1972 rows of Table 3 and Table 4 respectively.

(ii) Similarly at the beginning of \( t+2, t=I/1972 \), \( \text{GNP}_{t} \) is known i.e., \( X_{t} \) is known. Using \( X_{t} \) to replace \( X_{t-1} \) as known initial condition and using new measurements \( M_{t+1}, r_{t+1} \) etc. to generate new information \( (v_{2,t+1} + H_{t+1} v_{1,t+1}) \), the policymaker could update state predictions, \( \hat{X}_{t+2} \) for \( i=2,\ldots,10 \) conditioned on information \( (v_{2,t+1} + H_{t+1} v_{1,t+1}) \) and revise
the feedback control strategies accordingly. The policymaker can therefore find the revised optimal solutions \( \hat{X}_{t+i}^* \) and \( \hat{M}_{t+i}^* \), \( i=2, \ldots ,10 \). Again \( \check{X}_{t+2}^* \) and \( \check{M}_{t+2}^* \) are given on the III/1972 rows of Table 3 and Table 4 respectively.

(iii) The stochastic control experiments for the beginnings of quarters \( t+3, \ldots ,t+10 \) of Case 2 can be conducted in a similar way as described in (i), (ii) above with the substitutions of the appropriate time subscripts. And again, only the optimal solutions \( \hat{X}_{t+i}^* \) and \( \hat{M}_{t+i}^* \) of the \( (t+i) \)th quarter experiment are given on the corresponding rows of Table 3 and Table 4.

In general, for Case 1 solutions, by assuming the information set \( (v_{t+i-1} + H_{t+i-1} V_{t+i-1}) \), \( i=1, \ldots ,10 \) of Case 2 above equal to zeros and following the procedures described in (i) to (iii) above, the policymaker can find the optimal solutions \( \check{X}_{t+i}^* \) and \( \check{M}_{t+i}^* \), \( t=I/1972, i=1, \ldots ,10 \) as given in Table 3 and Table 4.

By the same token, for Case 3 solutions, by using fixed value, \( \tilde{k}_t \) of equation (4.10), substituting for \( k_{t+i-1} \), \( i=1, \ldots ,10 \) and using \( v_{2,t+i-1} \) substituting for \( (v_{2,t+i-1} + H_{t+i-1} V_{t+i-1}) \) and following the procedures (i) to (iii) of Case 2 above, the policymaker can find the optimal solutions, \( \check{X}_{t+i}^* \) and \( \check{M}_{t+i}^* \), \( t=I/1972, i=1, \ldots ,10 \) as given in Table 3 and Table 4.