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MIXED LAGRANGE AND HERMITE-FEJÉR INTERPOLATION

DISSERTATION

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By
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* * * * *

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INTRODUCTION

In 1914, G. Faber [6] proved his famous result which may be stated as follows: For any system of nodes \( \{x_{i_n}, \ldots, x_{n n}\} \), \( n=1,2,\ldots \), on a finite interval \([a,b]\), there is a function \( f^* \in C[a,b] \) such that the Lagrange interpolation polynomials \( L_n(f^*;x) \) fail to converge uniformly to \( f^*(x) \) on \([a,b]\).

In 1916, L. Fejér [7],[8] introduced the Hermite-Fejér interpolation polynomials, called by him "step parabolae", where the derivatives are prescribed to be zeros at the nodes. A general case is obtained if the prescribed derivatives are bounded but not necessarily zeros. In [7], L. Fejér treated the case with zeros of the Legendre polynomials as nodes. He proved that, for any \( f \in C[-1,1] \), \( H_n(f;x) \) tends to \( f(x) \) uniformly in every closed subinterval of \((-1,1)\), but \( H_n(f;1) \) and \( H_n(f;-1) \) tend to \( \frac{1}{2} \int_{-1}^{1} f(x)dx \) which is, in general, different from \( f(1) \) and \( f(-1) \), respectively.

However, in his later paper, L. Fejér [9],[10],[11] proved that \( H_n(f;x) \) tends uniformly to \( f(x) \) on \([-1,1]\) if the nodes are Jacobi nodes with parameters \( 0 > \alpha, \beta > -1 \). So, one would ask why the polynomials \( H_n(f;x) \) behave better, with respect to convergence, than \( L_n(f;x) \)? One can ask further what the
convergence property is if derivatives at some nodes are omitted.

We call "a mixed type interpolation polynomial" (MLHP in short) a polynomial with minimal degree and with the properties as follows: It interpolates a given function on two different kinds of nodes. On the first kind of nodes, the Lagrange points or L-points in short, no additional condition is required. While on the second kind of nodes, the Hermite-Fejér points or HF-points in short, we assume the polynomial has a vanishing derivatives. In the general case where we assume the polynomial has a derivative not necessarily zero, the node is called Hermite point or H-point in short. MLHP was studied first by E. Feldheim [12] in 1942. He proved that, based on the Chebyshev nodes, the MLHP, where all the nodes but one were H-points, converged uniformly to the given continuous function on \([-1,1]\). Following the ideas of E. Feldheim, it may be conjectured that the prescription of the derivatives regulated the behaviour of \(H_n\) process. Perhhaps, P. Turán suggested, omitting derivatives at a "few" exceptional points would not disturb the convergence property of the modified H-F polynomials of lower degree. The result obtained, however, does not prove our expectations. Turán [32] essentially showed that the MLHP over Chebyshev nodes with only one L-point \(y\), which is confined in a proper subinterval of \((-1,1)\), converges uniformly to \(f\) in \([-1,1]\).
if and only if $f$ belongs to some well-defined linear subspace of C. P. Vértesi [35] also showed that the mixed interpolation process over Chebyshev nodes with two properly chosen L-points becomes an unbounded operator on $C[-1,1]$. These results were extended to a fixed number $k$ of L-points by A. Meir, A. Sharma and J. Tzimbalario [21]. Then A. Meir raised the question: How does the MLHP behave if there are unbounded number of L-points?

Our studies of MLHP also are linked to a problem of S. N. Bernstein [2], who constructed a sequence $\{A_n(f;x)\}$ of polynomial operators depending on a fixed constant $c>0$ and satisfying the conditions

(i) $A_n(f;x)$ is a linear operator mapping $C[-1,1]$ into polynomials of degree $\leq n(1+c)$.

(ii) $A_n(f;x_{kn})=f(x_{kn})$, $k=1,2,\ldots,n$, and

(iii) for every $f \in C[-1,1]$, $A_n(f;x)$ tends uniformly to $f(x)$ on $[-1,1]$.

In Bernstein's construction, the $x_{kn}$'s were taken to be Chebyshev nodes. Later, P. Erdős [5] gave necessary and sufficient conditions which a triangular matrix of nodes must satisfy in order that there exists for every fixed $c>0$ a sequence of Bernstein type interpolation polynomials. Subsequently, G. Freud [14] proved that, under Erdős's condition, there exists for every $c>0$ a Bernstein type interpolation procedure which gives approximation in optimal order in $C$-norm. Namely, the property (iii) can be replaced
by

\[ |A_n(f; x) - f(x)| \leq K_1(c)E_{n-1}(f). \]

A stronger result was later proved by G. Freud and A. Sharma [17] over Jacobi nodes and ±1. There a Timan type estimate

\[ |J_n^{(\alpha, \beta)}(f; x) - f(x)| \leq C(n^{1-\alpha\beta} + n^{-\alpha\beta}) \]

holds. But the polynomials \( J_n^{(\alpha, \beta)}(f; x) \) interpolate \( f(x) \) only at Jacobi nodes which do not include ±1. Thus the interpolatory property is not quite fulfilled. G. Freud and A. Sharma [18] then modified \( J_n^{(\alpha, \beta)} \) into interpolation polynomials on the Jacobi nodes and ±1. And they proved that for \( -\frac{1}{2} \leq \alpha, \beta \leq \frac{3}{2} \) the modified interpolation polynomials \( A_n^{(\alpha, \beta)} \) satisfy the more precise Telyakovski-Gopengauz type estimate which is

\[ |A_n^{(\alpha, \beta)}(f; x) - f(x)| \leq Cn^{1/2}. \]

Since a polynomial with degree \( \leq n(1+c) \) can be achieved by properly distributing the HF-points and L-points, one would ask if some mixed interpolation processes can serve as the linear operators considered by S. N. Bernstein and be used to solve related problems in this area.

This thesis deals with convergence and order of convergence for various cases of MLHP. A detailed introduction is given in the beginning of each chapter. In chapter I, some recent and earlier results of E. Egerváry and P. Turán [4] and P. Szász [27],[29] are extended. A
A special case of our Theorem 1.3 shows that the MLHP of any \( f \in C \left[ -1,1 \right] \) with Chebyshev nodes as HF-points and \( \pm 1 \) L-points converges uniformly to \( f(x) \) on \( [-1,1] \). On the contrary, as observed by D. L. Berman [1], the Hermite-Fejér interpolation process over \( \pm 1 \) and Chebyshev nodes does not converge at \( x = 0 \) even for the simple function \( f(x) = |x| \).

This produces a strong negative result on the conjecture that the prescription of derivatives may regulate the behavior of the interpolation process. In chapter II, a result of P. Vértesi [34] is extended to Jacobi nodes and to the zeros of orthogonal polynomials with respect to a class of weight functions introduced by G. Freud [15]. We proved that the mixed interpolation processes with unbounded number of L-points converge uniformly, as precise as the Hermite-Fejér one, inside the intervals which are free from limiting points of L-points. Chapter III is a joint work of Professor G. Freud and the author. There we investigated a particular type of MLHP. Besides the independent interest of the result, the argument applied could be useful in related problems of interpolation. In chapter IV, the result of chapter III is extended both in the system of nodes and in the classes of functions considered. We hope that the results of this thesis have given a more satisfactory answer to the question of A. Meir and to some earlier problems in MLHP as well.
1. Introduction and Results. In this paper, we shall first extend a convergence theorem recently proved by P. Szász [29]. Let \( w_n(x) = P_n^{(\alpha, \beta)}(x) \), the Jacobi polynomial of degree \( n \) with parameters \( \alpha, \beta > -1 \). Let \( x_{vn}, v=1,2,\ldots,n \), be the \( n \) zeros of \( w_n(x) \) in \((-1,1)\) and we define \( x_{on} = 1 \) and \( x_{n+1, n} = -1 \). Let \( f \) be a continuous function on \([-1,1]\). The generalized almost-step parabola \( S_n(x) \) of the function \( f \) discussed in [29] is the unique polynomial of degree \( \leq 2n \) satisfying

\[
S_n(x_{vn}) = f(x_{vn}) \quad (v=0,1,\ldots,n)
\]

\[
S'_n(x_{vn}) = y'_{vn} \quad (v=1,2,\ldots,n)
\]

with

\[
|y'_{vn}| \leq \Delta \quad (v=1,2,\ldots,n)
\]

where \( \Delta \) is a fixed non-negative number. The close form of \( S_n(x) \) can be obtained. It is
Theorem. Assume that \(0<\alpha<1\) and \(-1<\beta<0\). Then \(S_n(x)\) converges to \(f(x)\) on \([-1,1]\). Moreover, the convergence is uniform in each subinterval \([-1,1-\delta]\) with \(0<\delta<2\).

In the special case with \(\alpha=\frac{1}{2}\) and \(\beta=-\frac{1}{2}\), it was proved in an earlier paper of P. Szász [28] that the convergence is uniform on the whole interval \([-1,1]\).

Our extended results are as follows:

Theorem 1.1. Assume that \(-\frac{1}{2}<\alpha<0\), \(-1<\beta<0\) and \(\alpha\geq \beta\). Then \(S_n(x)\) converges to \(f(x)\) uniformly on \([-1,1]\).

Theorem 1.2. Assume that \(-\frac{1}{2}<\alpha\), \(-1<\beta<0\) and \(\alpha\geq \beta\). Then \(S_n(x)\) converges to \(f(x)\) on \([-1,1]\). Moreover, the convergence is uniform in each subinterval \([-1,1-\delta]\) with \(0<\delta<2\).

Let \(H_n(\omega_n;f;x)\) (\(H_n(x)\) in short) be the Hermite-Fejér interpolation polynomial of the function \(f\) over the zeros of \(\omega_n(x)\). Then we have

\[
(1.3) \quad S_n(x) = \sum_{v=1}^{n} f(x_{vn}) \frac{1-x}{1-x_{vn}} (1+c_{vn}(x-x_{vn})) l_{vn}^2(x)
\]

where

\[
(1.4) \quad c_{vn} \frac{1}{1-x_{vn}} - \frac{\omega_n(x_{vn})}{\omega_n(x_{vn})^2} \quad (v=1,2,\ldots,n)
\]

(s.[29],(5)).
Corollary 1.3. In the special case of mixed Lagrange and Hermite-Fejér interpolation, namely $y_{vn} = 0$ for all $v = 1, 2, \ldots, n$, if $\alpha \geq -\frac{1}{2}$ and $\alpha \geq \beta > -1$, then we have

$$\|s_n(f) - f\| \leq \|H_n(f) - f\|$$

where $\|\cdot\|$ is the usual supremum norm on $[-1,1]$.

In 1958, E. Egerváry and P. Turán [4] studied generalized quasi-step parabolae. The generalized quasi-step parabola $Q_n(x)$ of the function $f$ is the unique polynomial of degree $\leq 2n+1$ satisfying

$$Q_n(x_{vn}) = f(x_{vn}) \quad (v = 0, 1, \ldots, n+1)$$

(1.5)

$$Q_n'(x_{vn}) = y'_{vn} \quad (v = 1, 2, \ldots, n)$$

and (1.2). They proved that $Q_n(x)$ converges uniformly to $f(x)$ on $[-1,1]$ if $\alpha = \beta = 0$. Later in 1959, P. Szász [27] proved the same conclusion for the case $\alpha = \beta = \frac{1}{2}$. Here, we shall give extension to the following:

Theorem 1.4. Assume that $-\frac{1}{2} \leq \alpha < 0$. Then $Q_n(x)$ converges uniformly to $f(x)$ on $[-1,1]$.

Corollary 1.5. Assume $y'_{vn} = 0$ for all $v = 1, 2, \ldots, n$. If $\alpha = \beta \geq -\frac{1}{2}$, then

$$\|Q_n(f) - f\| \leq \|H_n(f) - f\|.$$  

In the case that $\alpha = \beta = \frac{1}{2}$ and $y'_{vn} = 0$ for all $v = 1, 2, \ldots, n$, R. B. Saxena [28] gave the following estimate

$$\|Q_n(f) - f\| \leq 5\omega(f; \frac{\log n}{n}),$$

which is the order of convergence of Hermite-Fejér interpolation process over Chebyshev nodes given by
E. Moldovan [22]. It is worthwhile to mention here the surprising phenomenon observed by D. L. Berman [1]. Berman proved that the Hermite-Fejér interpolation process over \( \pm 1 \) and Chebyshev nodes diverges at the point \( x=0 \) for the simple function \( |x| \). On the other hand, by corollary 1.5, if the prescription of zero derivative is omitted at \( x=\pm 1 \), then the mixed Lagrange and Hermite-Fejér interpolation process converges uniformly on the whole interval and

\[
\| Q_n(f) - f \| \leq C_n \sum_{k=1}^{n} \omega(f; \frac{1}{k})
\]

which is the precise order of convergence of Hermite-Fejér interpolation process over Chebyshev nodes obtained by R. Bojanic [3].

2. Auxiliary Estimates. It is well-known that

\[
H_n(x) = \sum_{v=1}^{n} \frac{f(x_{vn})[1+(\beta-\alpha)-(\alpha+\beta+2)x_{vn}(x-x_{vn})]1^2_{vn}(x)}{1-x_{vn}}
\]

where

\[
l_{vn}(x) = \frac{w_n(x)}{(x-x_{vn})w_n'(x_{vn})}
\]

for \( v=1,2,\ldots,n \).

Since \( H_n(x) \) is of degree \( \leq 2n-1 \) and satisfies \( H_n(x_{vn}) = f(x_{vn}) \) and \( H'_n(x_{vn}) = 0 \) for \( v=1,2,\ldots,n \), it is clear, by (1.3), that we have

\[
H_n(x) = \sum_{v=1}^{n} \frac{f(x_{vn})}{1-x_{vn}}(1+C_{vn}(x-x_{vn}))1^2_{vn}(x) + H_n(1)(\frac{w_n(x)}{w_n(1)})^2.
\]
Consequently,

\( (2.3) \ |S_n(x) - f(x)| \leq |S_n(x) - H_n(x)| + |H_n(x) - f(x)| \)

\( \leq |f(1) - H_n(1)| \left( \frac{\omega_n(x)}{\omega_n(1)} \right)^2 + |H_n(x) - f(x)| \)

\[ + \left| \sum_{v=1}^{n} y_v \cdot \frac{1-x}{1-x_{vn}} (x-x_{vn})^2 \right|_{vn}(x). \]

Throughout this paper, we will assume that

\( (2.4) \ x_{j+1}, n \leq x \leq x_j \quad (j=0, 1, \ldots, n) . \)

Then, by lemma 2 in [23], we have, for \( x > 0 , \)

\( (2.5) \ |x - x_{vn}| \sim |j^2 - v^2| n^{-2} \quad (v=1, 2, \ldots, n; v \neq j, j+1) . \)

we shall apply the formula (S. (8.9.2) in [30])

\( (2.6) \ |p_n^{(\alpha, \beta)}(x_{vn})| \sim v^{-\alpha-3/2} n^{\alpha+2} \quad (0 \leq x_{vn} < 1; v=1, 2, \ldots, n) \)

And by using the formula (S. (4.1.3) in [30])

\( (2.7) \ p_n^{(\alpha, \beta)}(x) = (-1)^{n} p_n^{(\beta, \alpha)}(-x) , \)

we have the asymptotic estimate

\( (2.8) \ |p_n^{(\alpha, \beta)}(x_{vn})| \sim (n+1-v)^{-\beta-3/2} n^\beta+2 \)

\[ (-1 < x_{vn} < 0; v=1, 2, \ldots, n) . \]

By (2.1), (2.5), (2.6), and (2.8), we have

\( (2.9) \ |f(1) - H_n(1)| \)

\[ = \left| \sum_{x_{vn} > 0} f(1) - f(x_{vn}) \right| \frac{1 - \alpha + \beta - (\alpha + \beta + 1)}{\omega_n(x_{vn})^2} \frac{1}{(1+x_{vn})(1-x_{vn})^2} \]

\[ \leq \left| \sum_{1 \leq x_{vn} > 0} \right| + \left| \sum_{-1 < x_{vn} < 0} \right| \]
Since \( \beta > 1 \), we have

\[
2.10 \quad \sum_{v=1}^{n} \frac{v^{2\beta+1}}{n^{2\beta+2}} \leq C_4
\]

A simple calculation yields

\[
2.11 \quad \sum_{v=1}^{n} \frac{v^{2\alpha-1}}{n^{2\alpha}} \leq C_5 \quad (\alpha > 0)
\]

\[
\sum_{v=1}^{n} \frac{v^{2\alpha-1}}{n^{2\alpha}} \leq C_6 \log n \quad (\alpha = 0)
\]

From (2.9), (2.10) and (2.11), we have

\[
2.12 \quad \|f(1) - H_n(1)\| \leq C_7 \|f\| \frac{w^2_n(x)}{w_n(1)} \quad (\alpha > 0)
\]

\[
\|f\| \|w^2_n(x)\| \log n + \frac{1}{n} \quad (\alpha = 0)
\]

Lemma 2.1. For \( \alpha \geq \frac{1}{2} \) and \( \beta > 1 \), we have

\[
2.13 \quad \left| \sum_{v=1}^{n} \frac{y_v}{1-x_v n} \frac{1-x}{1-x_v n} \right|^{\frac{1}{2}} (x, v, n)
\]

\[
\leq C_9 \Delta (w_n^2(x) \log n + \frac{1}{n}) \quad (x \in [-1, 1]).
\]

Proof. Let \( s = j, j+1 \) but \( s \neq 0, n+1 \). It is clear that

\[
2.14 \quad \left| \frac{1-x}{1-x_v n} \right| \leq C_{10}.
\]

We have following estimate

\[
2.15 \quad 1^2_{sn}(x) = \lambda_n(x) \sum_{v=1}^{n} \frac{1^2_v(x)}{\lambda_n(x)} \leq \lambda_n(x) \sum_{v=1}^{n} \frac{1^2_v(x)}{\lambda_n(x)}
\]

where \( \lambda_n(x) \) is the Christoffel function. (S. [13], § I.3).
It is known that

\[(2.16) \quad \lambda_n^{-1}(x) = \sum_{v=1}^{n} \frac{i^2_{vn}(x)}{\lambda_n(x_{vn})}\]

\[(S. [13], \S I.4. (4.7))\]

and

\[(2.17) \quad \frac{\lambda_n(x_{sn})}{\lambda_n(x)} = o(1)\]

\[(S. [24], (2.1)).\]

Therefore, we have

\[(2.18) \quad |l_{sn}(x)| \leq C_{11}.\]

Thus, by (1.2), (2.5), (2.14) and (2.18), we have

\[(2.19) \quad |y_{sn} \frac{1-x}{1-x_{sn}} l_{sn}(x) | \leq C_{12} A_n^{-1}.\]

Now, by (1.2), (2.5), (2.6), and (2.8), we have, for \(x \geq 0\),

\[(2.20) \quad \sum_{v=1}^{n} \sum_{v \neq j, j+1}^{n} \left| y_{vn} \frac{1-x}{1-x_{vn}} (x-x_{vn}) \right| l_{vn}(x) \leq \Delta \sum_{v=1}^{n} \frac{1-x}{1-x_{vn}} \frac{1}{|x-x_{vn}|} \frac{w_n(x)}{w'_n(x_{vn})}^2\]

\[\leq \Delta \left( \sum_{1 \geq v_{vn} > 0} + \sum_{-1 \leq x_{vn} < 0} \right)\]

\[\leq \Delta \left( \sum_{v \neq j, j+1}^{n} + \sum_{v \neq j, j+1}^{n} \right)\]

\[\leq C_{13} \Delta \left( \sum_{v=1}^{n} \frac{n^2}{v^2} \frac{(i+1)^2}{n^2} \frac{n^2}{|j^2-v^2|} \frac{v^{2a+3}}{n^{2a+4}} \right)\]
It is not hard to show (cf. Lemma 3 and 4 in [23]) that for $\tau > 0$,

$$
(2.21) \quad \sum_{v=1}^{n} \frac{v^{\tau}}{\Sigma |j^2 - v^2|} \leq C_{14} ((j+1)^{\tau-1} + n^{\tau-1}) \log n \quad (j=0,1,\ldots,n).
$$

The Lemma follows from (2.19), (2.20), (2.21) and a similar argument for $x<0$.

3. Proofs.

3.1 Proof of Theorem 1.1. Since both $-1<\alpha, \beta <0$, it is well-known that

$$
(3.1) \quad |f(x)-H_n(x)| \to 0 \text{ uniformly on } [-1,1].
$$

Since $\alpha \geq \frac{1}{2}$ and $\alpha \geq \beta$ it is easily inferred (e.g. (7.32.2) in [30]) that

$$
(3.2) \quad |\omega_n(x)| \leq |\omega_n(1)| n^{\alpha} \quad (x \in [-1,1]).
$$

From (3.1) and (3.2), we know

$$
(3.3) \quad |f(1)-H_n(1)| \left( \frac{\omega_n(x)}{\omega_n(1)} \right)^2 \to 0 \text{ uniformly on } [-1,1].
$$

The theorem follows from (3.1), (3.3) and Lemma 2.1.

Q.E.D.
3.2. Proof of Theorem 1.2. It is sufficient to prove it for the case \( \alpha \geq 0 \). Since \( \beta < 0 \), again by (7.32.2.) in [30], we have

\[
(3.4) \quad |w_n(x)| = o(n^v) \quad (x \in [-1,1-\delta]),
\]

where \( v = \text{MAX} \{ \beta, -\frac{1}{2} \} \).

Also, by Theorem 14.6 in [30], we have

\[
(3.5) \quad |f(x) - H_n(x)| \to 0 \quad \text{uniformly on} \quad [-1,1-\delta].
\]

From (2.3), (3.4), (3.5), (2.12), and (2.13), we have

\[
|S_n(x) - f(x)| \to 0 \quad \text{uniformly on} \quad [-1,1-\delta].
\]

Since \( S_n(1) = f(1) \), we have the convergence on \([-1,1]\). Q.E.D.

3.3. Proof of Theorem 1.4. We have (see [27])

\[
Q_n(x) = f(-1)(1-x)\left(\frac{w_n(x)}{w_n(1)}\right)^2 + f(1)(1+x)\left(\frac{w_n(x)}{w_n(1)}\right)^2
\]

\[
+ \sum_{v=1}^{n} f(x_{vn})\frac{1-x^2}{(x-x_{vn})^2}((x-x_{vn})\overline{c}_{vn} + 1)\overline{c}_{vn}^2(x)
\]

\[
+ \sum_{v=1}^{n} y_{vn}\frac{1-x^2}{(x-x_{vn})^2}\overline{c}_{vn}^2(x)
\]

where \( \overline{c}_{vn} = \frac{2x_{vn}w_n'(x_{vn})}{1-x^2_{vn}w_n'(x_{vn})} \) for \( v=1,2,-\ldots,n \).

Based on the idea that we use to prove Theorem 1.1 and 1.2, we have

\[
(3.6) \quad |Q_n(x) - f(x)|
\]

\[
\leq |f(1) - H_n(1)|\left(\frac{w_n(x)}{w_n(1)}\right)^2 + |f(-1) - H_n(-1)|\left(\frac{w_n(x)}{w_n(-1)}\right)^2
\]
We find that

\[ \left| \sum_{v=1}^{n} y_v \cdot \frac{1-x^2}{1-x_{vn}^2} (x-x_{vn}^2)^2 \right| \leq 2 \Delta \left[ \sum_{1>\nu_{vn}>0} \frac{1-x_{vn}^{-1}}{1-x_{vn}} \cdot |x-x_{vn}^2| \right] \]  

\[ + \sum_{-1<\nu_{vn}<0} \frac{1+x_{vn}^{-1}}{1+x_{vn}} \cdot |x-x_{vn}^2| \right]. \]

Since \(-\frac{1}{2} < \alpha = \beta < 0\), by (3.1), (3.2), and (2.13) along with their similar argument with respect to \(\beta\), we have

(3.7) \( |Q_n(x) - f(x)| \leq 2C_\alpha n^{2\alpha} \log n + O(\mathcal{w}(f; n^{\alpha})) \)  \( (x \in [-1, 1]) \).

Q.E.D.

3.4. Proofs of corollaries.

Corollary 1.3 is a consequence of (2.3) and (3.2).

Corollary 1.5 is a consequence of (3.6) and (3.2).

Q.E.D.
CHAPTER II
A REMARK
ON HERMITE-FEJÉR INTERPOLATION OMITTING SOME DERIVATIVES

1. Introduction. The aim of this paper is to give a more satisfactory answer to a question of A. Meir: How does the Hermite-Fejér interpolating process behave if we omit the derivatives at possibly infinitely many nodes? Let's call a node Hermite-Fejér point (HF-point in short), if we assume a zero derivative at this point. And, we call a node Lagrange point (L-point in short), if no derivative is prescribed. In his paper [34], P.O. Vértesi gave an upper estimate for the precision of that mixed type of interpolating process over the Chebyshev nodes. His estimate shows that, if the number of L-points is \( m_n \) with \( m_n = \varrho_n \log n \) where \( \varrho_n \to 0 \) as \( n \to \infty \), then, inside the intervals which are free from limiting points of L-points, the mixed type of interpolating process is at least as good as Hermite-Fejér interpolating process. We shall extend this result to all Jacobi nodes with both parameters \( \alpha, \beta > -1 \) and to the zeros of the orthogonal polynomials with respect to a class of weight functions introduced by G. Freud [15]. Furthermore, we shall show that the mixed type of
interpolating process is as precise as the Hermite-Fejér one for certain classes of functions inside the intervals considered.

2. Results. Let \( W(x) \) be a weight function on \([-1,1]\). We assume that either \( W(x) = (1-x)^\alpha \cdot (1+x)^\beta \) with \( \alpha, \beta > -1 \) or \( (1-x)^\delta W(x) \) is nondecreasing and \( (1+x)^\tau W(x) \) nonincreasing for some real \( \delta \) and \( \tau \). Let \( \omega_n(x) = p_n(W; x) \) be the orthogonal polynomial of degree \( n \) with respect to the weight function \( W \). Here we assume that \( p_n(W; x) \) is normalized in the sense that

\[
(2.1) \quad \int_{-1}^{1} p_n^2(W; x) W(x) dx = 1
\]

and \( p_n(W; x) \) has leading coefficient \( \gamma_n(W) > 0 \). It is known that \( p_n(W; x) \) has \( n \) simple zeros \( x_{kn}, k=1,2,\ldots,n, \) in \((-1,1)\).

Let \( \omega_1(x) = \prod_{i=1}^{r_n} (x-x_{in}) \) where \( x_{in}, i=1,2,\ldots,r_n \),

are chosen to be HF-points and \( \omega_2(x) = \prod_{j=r_n+1}^{n} (x-x_{jn}) \)

where \( x_{jn}, j=r_n+1, r_n+2,\ldots,n \) are L-points. For any \( f \in C([-1,1]) \), the interpolating polynomial of degree \( \leq 2n-1-m_n \) with \( m_n + r_n = n^1 \) can be expressed as follows:

\[
(2.2) \quad H_{n,m}(\omega_1; \omega_2; f; x) = \sum_{k=1}^{n} f(x_{kn}) h_{kn}(x)
\]

\(^1\)In what follows, we shall write \( m,r \) instead of \( m_n, r_n \).
where

\[(2.3)\quad h_{in}(x) = \left[1 - (x - x_{in}) \left( \frac{\omega_{n}(x_{in})}{2\omega_{n}(x_{in})} + \frac{\omega_{n}(x_{in})}{2\omega_{in}(x_{in})} \right) \right] \]

\[(2.4)\quad h_{jn}(x) = l_{jn}(x) \cdot \frac{n}{\prod_{k=r+1}^{j} \frac{x_{jn} - x_{kn}}{x_{jn} - x_{kn}}} \quad (j = r+1, r+2, \ldots, n),\]

and

\[(2.5)\quad l_{kn}(x) = \frac{\omega_{n}(x)}{(x - x_{kn}) \omega_{n}(x_{kn})} \quad (k = 1, 2, \ldots, n).\]

(S. [25], section 3.8).

Let \( \Omega \) be a non-decreasing function which satisfies

\[(2.6)\quad \Omega(2h) \leq 2\alpha(h) \quad (h \geq 0),\]

and

\[(2.7)\quad C(\Omega) = \{ g \in C([-1, 1]) | \omega(g; h) \leq \alpha(h); h \geq 0 \}.\]

Then we have

Theorem 2.1. Let \((A, B) \subseteq (-1, 1)\) be free from limiting points of \(L\)-points. If \(f \in C(\Omega)\) and \(m_n = [\rho_n \log n]\)

where \(\rho_n \to 0\) as \(n \to \infty\), then in any \([a, b] \subseteq (A, B)\)

we have

\[|f(x) - H_{n, m}(f; x)| \leq C_1 \left( \frac{\Omega(2)}{1 - C_2 \cdot n} \right) \sum_{k=m+1}^{n} \frac{\Omega(1)}{k} \quad (x \in [a, b]).\]

Let \(\text{Lip}_{M, \gamma}\) be the set of functions belonging to \(\text{Lip}_\gamma\) and with

a coefficient in the \(\text{Lip}\)-condition \(\leq M\).

Then we have

Theorem 2.2. Let \((A, B) \subseteq (-1, 1)\) be free from limiting points
of \( L \)-points. If \( f \in \operatorname{Lip}_\gamma \) with \( 0 < \gamma \leq 1 \) and \( M > 0 \) and 
\[ m_n = \left\lfloor \rho_n \log n \right\rfloor \] 
where \( \rho_n \to 0 \) as \( n \to \infty \), then in any \( [a, b] \subset (A, B) \)
we have \(^2\)

\[
\sup_{f \in \operatorname{Lip}_\gamma} \|f - H_n^m(f)\| \sim \begin{cases} 
M_n^{-\gamma} & \text{if } 0 < \gamma < 1 \\
\log n & \text{if } \gamma = 1.
\end{cases}
\]

3. Proofs of theorems. Let \( \Delta = \min(a - A, B - b) \). Since

the interval \((A, B)\) is free from limiting points of \( L \)-points,
there exists a positive integer \( n_0 \) such that

\[
|x - x_{jn}| \geq \Delta \quad (x \in [a, b], j = r+1, r+2, \ldots, n, n \geq n_0).
\]

After a little calculation, we have

\[
\frac{\omega_n'(x_{in})}{2\omega_n'(x_{in})} = \frac{\omega_n''(x_{in})}{2\omega_n'(x_{in})} = \frac{n}{\sum_{j=r+1}^{m} \frac{1}{x_{in} - x_{jn}}} \quad (i=1, 2, \ldots, r).
\]

Let \( x \in [a, b] \). Then

\[
\sum_{j=1+1}^{n} \frac{|x_{kn} - x_{jn}|}{x - x_{jn}} \leq \left( \frac{2}{\Delta} \right)^m \leq \frac{C_6 \rho_n}{5^n}.
\]

However, we also have

\[
\sum_{j=r+1}^{n} \frac{|x_{kn} - x_{jn}|}{x - x_{jn}} \leq \frac{\sum_{j=r+1}^{n} |x_{kn} - x| + |x - x_{jn}|}{|x - x_{jn}|} \leq \left( \frac{1 + \frac{1}{m+1}}{m+1} \right)^m \leq C_7
\]

\[
(x - x_{kn} \leq \frac{\Delta}{m+1}; k = 1, 2, \ldots, r).
\]

We follow the usual notation

\[
v_{kn}(x) = 1 - (x - x_{kn}) \frac{\omega_n'(x_{kn})}{\omega_n'(x_{kn})} \quad (k=1, 2, \ldots, n).
\]

\(^2\)Here \( \| \cdot \| \) is the usual supremum norm on \([a, b]\) and \( u \lesssim v \) means \( u \leq C_3 v \) and \( v \leq C_4 u \). We also assume that all \( C_1, C_2, \ldots \) are constants \( > 0 \) depending on \( a, b, A, B, \) and \( \omega(x) \) only.
It was proved that

\[ |v_{kn}(x)| \leq C_8 (1-x_{kn}^2)^{-1} \quad (x \in [a, b]; k=1,2,\ldots,n). \]

(S. [30], (14.5.2) and S. [13], (16)).

And

\[ \sum_{x-x_{kn} \geq \Delta \text{ m+1}} |v_{kn}(x)| \leq C_9 n^{-1} (m+1)^2 \quad (x \in [a, b]). \]

(S. [16], (3.3) and Lemma 4.1).

To obtain (3.7) above, actually we had applied a well-known estimate for Jacobi polynomials. In virtue of Lemma 3 in [13], the very same estimate

\[ |p_n(W;x)| \leq C_{10} \quad (x \in [a, b]) \]

also holds for Freud's weight functions.

In the case \( W(x) = (1-x)^\alpha (1+x)^\beta \), it is easily inferred (e.g. from [30], (4.3.4) and (8.9.2)) that

\[ |p_n^{(k)}(W;x_{kn})|^{-1} \leq C_{11} n^{-1} \quad (k=1,2,\ldots,n). \]

We get from (2.5), (3.8), and (3.9)

\[ \sum_{x-x_{kn} \geq \Delta \text{ m+1}} |x-x_{kn}| h_{kn}^2(x) \leq C_{12} (m+1)^{-1} n^{-1} \quad (x \in [a, b]). \]

We claim that (3.10) also holds for Freud's weight functions.

We need the following proved results from [13].

\[ l_{kn}(x) = \frac{\gamma_{n-1}(W)}{\gamma_n(W)} \lambda_n(W;x_{kn}) \frac{p_{n-1}(W;x_{kn})}{p_n(W;x)} \]

\[ (k=1,2,\ldots,n), \]

\[ 0 \leq \frac{\gamma_{n-1}(W)}{\gamma_n(W)} \leq 1, \]
\[ n \sum_{k=1}^{n} \lambda_n(W; x_{kn}) p_{n-1}^2(W; x_{kn}) = \int_{-1}^{1} p_n^2(W; x) W(x) dx = 1, \]

where

\[ \lambda_n(W; x) = \left[ \sum_{k=0}^{n-1} p_k^2(W; x) \right]^{-1} \]

is the Christoffel function.


Freud's condition implies that \( \sqrt{1-x^2} W(x) \) is bounded.

Therefore, we have

\[ \lambda_n(W; x_{kn}) \leq C_1 n^{-1} \quad (k=1, 2, \ldots, n). \]

(S. [13], §III. Lemma 3.2)

Thus (3.10) follows from (3.8), (3.11), (3.12), (3.13), and (3.14).

It follows from (3.10) that

\[ \leq C_{14} (m+1)^2 n^{-1} \quad (x \in [a, b]). \]

Note that \( W(x) \), either Jacobi or Freud's, satisfies

\[ 0 < C_{15} \leq W(x) \leq C_{16} \quad (x \in [A, B]). \]

Consequently, if \( \xi'_n \) and \( \xi''_n \) are two consecutive zeros of \( p_n(W; x) \), then

\[ C_{17} n^{-1} \leq |\xi'_n - \xi''_n| \leq C_{18} n^{-1} \quad (\xi'_n, \xi''_n \in [A, B]), \]

\[ C_{19} n^{-1} \leq \lambda_n(W; x_{kn}) \leq C_{20} n^{-1} \quad (x_{kn} \in [A, B]), \]

and

\[ |l_{kn}(x)| \leq C_{21} \quad (x, x_{kn} \in [A, B]). \]
Applying the method of (4.10) in [16], we get from (3.8), (3.11), (3.12), (3.17), (3.18), and (3.19)

$$\sum_{|x-x_{kn}|<\frac{\Delta}{m+1}} |f(x)-f(x_{kn})| \cdot |x-x_{kn}| l_{kn}^2(x)$$

$$\leq C_{22} \cdot \frac{1}{n} \sum_{v=m+1}^{n} \Omega\left(\frac{1}{v}\right) \cdot \frac{1}{v} \quad (x \in [a,b]).$$

Also, from (4.9) and (4.10) of [16], we obtain

$$\sum_{|x-x_{kn}|<\frac{\Delta}{m+1}} |f(x)-f(x_{kn})| \cdot v_{kn}(x) \cdot l_{kn}^2(x)$$

$$\leq C_{23} \cdot \frac{1}{n} \sum_{v=m+1}^{n} \Omega\left(\frac{1}{v}\right) \quad (x \in [a,b]).$$

Proof of Theorem 2.1. From (2.2), we have

$$f(x)-H_{n,m}(f;x)= \sum_{|x-x_{in}|<\frac{\Delta}{m+1}} (f(x)-f(x_{in})) h_{in}(x)$$

$$+ \sum_{|x-x_{jn}|\geq\frac{\Delta}{m+1}} (f(x)-f(x_{jn})) h_{jn}(x)$$

By (2.3), (2.4), (3.2), (3.3), (3.7), (3.10), and (3.15), we have

$$\sum_{|x-x_{jn}|\geq\frac{\Delta}{m+1}} \frac{1+C_2\rho_n}{x-x_{jn}} \cdot h_{jn}(x) \leq C_{24} \cdot \sum_{|x-x_{jn}|\geq\frac{\Delta}{m+1}} \Omega(2n) \cdot \frac{1+C_2\rho_n}{x-x_{jn}} \cdot h_{jn}(x) \leq C_{24} \cdot \sum_{|x-x_{jn}|\geq\frac{\Delta}{m+1}} \Omega(2n) \cdot \frac{1+C_2\rho_n}{x-x_{jn}} \cdot h_{jn}(x) \leq C_{24} \cdot \sum_{|x-x_{jn}|\geq\frac{\Delta}{m+1}} \Omega(2n) \cdot \frac{1+C_2\rho_n}{x-x_{jn}} \cdot h_{jn}(x)$$

By (2.3), (3.2), (3.4), (3.20), and (3.21), we have

$$\sum_{|x-x_{in}|<\frac{\Delta}{m+1}} (f(x)-f(x_{in})) h_{in}(x)$$

$$\leq C_{27} \cdot \sum_{|x-x_{in}|<\frac{\Delta}{m+1}} \frac{|f(x)-f(x_{in})| \cdot v_{in}(x) \cdot l_{kn}^2(x)}{x-x_{in}}$$

$$+ C_{27} \cdot \sum_{|x-x_{kn}|<\frac{\Delta}{m+1}} \frac{|f(x)-f(x_{kn})| \cdot |x-x_{kn}| \cdot l_{kn}^2(x)}{x-x_{kn}}$$
The theorem follows from (3.22), (3.23), and (3.24). Q.E.D.

Proof of Theorem 2.2. In virtue of Lemma 5.2. in [16], we may assume that, for any \( x \in [a, b] \),

\[
\forall k \in \{1, 2, \ldots, n\}, \quad |x - x_k| < \frac{\Delta}{m+1};
\]

It is easy to show that, for \( x \in [a, b] \), we have

\[
\frac{n}{j=r+1} \frac{x_{in} - x_{jn}}{x - x_{jn}} \geq 0 \quad (|x - x_{in}| < \frac{\Delta}{m+1}, i=1, 2, \ldots, r).
\]

In case \( m=0 \), then the left hand side of (3.26) is 1.

Therefore we have

\[
\frac{n}{j=r+1} \frac{x_{in} - x_{jn}}{x - x_{jn}} \geq 2^n > 0 \quad (|x - x_{in}| < \frac{\Delta}{m+1}, i=1, 2, \ldots, r).
\]

By (3.22), (2.3), (3.2), (3.4), (3.20), and (3.23), we obtain

\[
|\mathcal{B}(x) - H_{n,m}(f;x)| \geq \frac{\sum |f(x) - f(x_{in})| v_{in}(x) l_{in}^2(x) \cdot \frac{n}{j=r+1} \frac{x_{in} - x_{jn}}{x - x_{jn}}}{|x - x_{in}| < \frac{\Delta}{m+1}} - C_{27}^m - 1 \frac{n}{v=m+1} \frac{\Omega(\frac{1}{v})}{v} - C_{24}^m \frac{\Omega(2n)}{v}^2 + C_2^m n \quad (x \in [a, b]).
\]

In virtue of (3.17), if \( n_0 \) is large enough, then, for each \( n \geq n_0 \), there exists a \( \xi \in [a, b] \) which is a zero of \( p_{n+1}(W;x) \).

Then

\[
|p_{n}(W; \xi)| \geq C_{28} > 0.
\]

(S. [16], Lemma 5.1.).

Following the constructive proof of Theorem 5.1. in [16],
there exists a function $g_\xi$ such that

\begin{equation}
(3.30) \quad g_\xi \in C(\Omega),
\end{equation}

\begin{equation}
(3.31) \quad g_\xi(x_{in}) - g_\xi(x_{in}) = g_\xi(x_{in}) \geq 0 \quad (|\xi - x_{in}| < \frac{\Delta}{m+1}; i=1,2,\ldots, r),
\end{equation}

and

\begin{equation}
(3.32) \quad \sum_{|\xi - x_{in}| \leq \frac{\Delta}{m+1}} (g_\xi(x_{in}) - g_\xi(x_{in})) v_{in}(\xi) l_{in}(\xi) \geq C_{29} p_n^2(W; \xi) n^{-1} \sum_{v=m+1}^{n} \Omega(\frac{1}{v}).
\end{equation}

In view of (3.25), (3.27), (3.29), (3.31), and (3.32), we have

\begin{equation}
(3.33) \quad \sum_{|\xi - x_{in}| \leq \frac{\Delta}{m+1}} (g_\xi(x_{in}) - g_\xi(x_{in})) v_{in}(\xi) l_{in}(\xi) \cdot \frac{n}{\prod_{j=r+1}^{\infty} (\xi - x_{in})} \geq C_{30} n^{-1} \sum_{v=m+1}^{n} \Omega(\frac{1}{v}).
\end{equation}

In the case that $C(\Omega) = \text{Lip}_M \gamma, \quad 0 < \gamma \leq 1,$

we get from (3.28) and (3.33)

\begin{equation}
(3.34) \quad \sup_{f \in \text{Lip}_M \gamma} \|f - H_{n,m}(f)\| \geq \begin{cases} 
M^{-\gamma} & (0 < \gamma < 1) \\
M^{-1} \log n & (\gamma = 1)
\end{cases}
\end{equation}

By Theorem 2.1. and (3.34), we have the desired result.

Q.E.D.
CHAPTER III
ON MIXED LAGRANGE AND HERMITE-FEJÉR INTERPOLATION

1. Introduction. We call "a mixed type interpolation polynomial" (MLHP in short) a polynomial with minimal degree and with the properties as follows: It interpolates a given function on two different kinds of nodes. On the first kind of nodes, denoted by $y_j$ (the Lagrange points or L-points in short), there is no additional condition required. While on the second kind of nodes, denoted by $x_k$ (the Hermite-Fejér points or HF-points in short), we assume the polynomial has a vanishing derivative. MLHP was studied first by E. Feldheim [12] in 1942. He proved\(^1\) that, based on the Chebyshev nodes, the MLHP, where all the nodes but one were HF-points, converged uniformly to the given continuous function on the interval $[-1,1]$. In 1958, E. Egerváry and P. Turán [4] investigated the case where both end points of the interval $[-1,1]$ were L-points.

\(^1\)Actually, he proved the result for the mixed Lagrange and Hermite (not just Hermite-Fejér) interpolation.
and the zeros of a Legendre polynomial were HF-points. They proved that this MLHP converged uniformly to the given function on [-1,1]. The same statement for the zeros of a Chebyshev polynomial of second kind was proved by P. Szász [27]. For the kind of MLHP studied by E. Feldheim described above, if the only L-point y is confined to a closed subinterval of [-1,1], namely \( |y| \leq 1-\delta, \delta>0 \), P. Turán [32] showed that this interpolation process converged uniformly only for functions in a well-defined linear subspace of C. This result was extended to a fixed number k of L-points by A. Meir, A. Sharma, and J. Tzimbalario [21]. Answering a problem formulated by A. Meir, P. Vértesi [34] investigated the behaviour of the MLHP in the case when the number of L-points is unbounded. He proved that, based on the Chebyshev nodes, the mixed interpolation was, roughly speaking, as good as Hermite-Fejér interpolation on the intervals which were free from limiting points of L-points. This conclusion was extended by C. D. Liu [20] to the zeros of orthogonal polynomials with respect to Jacobi and Freud's weight functions.

In our present paper, we discuss the degree of convergence for the case that L-points and HF-points alternate in order. The HF-points in our case are zeros of a Jacobi polynomial \( P_n^{(\alpha,\beta)}(x) \) with \( 0>\alpha,\beta>\frac{1}{2} \) and the L-points are the local extrema of \( P_n^{(\alpha,\beta)}(x) \). We shall prove that these MLHP have similar approximation capability as Lagrange
interpolation polynomials.

Let \( w_1(x) \) and \( w_2(x) \) be polynomials which have simple zeros only, the zeros of \( w_1(x) \) are the HF-points, while those of \( w_2(x) \) are the L-points. The essential tool we use in our argument is the following formula, representing MLHP:

\[
F_n(w_1, w_2; f; x) = H_n(w_1; f; x) + w_1^2(x) L(w_2; \frac{f-H}{w_1}, x)
\]

where \( H_n(w_1; f; x) \) is the Hermite-Fejér interpolation polynomial based on the zeros of \( w_1(x) \) and \( L(w_2; e; x) \) is the Lagrange interpolation polynomial based on the zeros of \( w_2(x) \). It is clear that \( F_n(w_1, w_2; f; x) = F_n(c_1 w_1, c_2 w_2; f; x) \) for arbitrary constants \( c_1, c_2 \neq 0 \). Applying formula (1.1), we believe, some earlier results could have been proven in a simple fashion.

2. Results. Let \( x_{kn} (k=1, 2, \ldots, n) \) be the zeros of the nth Jacobi polynomial \( p_n(x) = p_n^{(\alpha, \beta)}(x)^2 \) and \( y_{jn} (j=1, 2, \ldots, n-1) \) the zeros of \( \frac{d}{dx} p_n^{(\alpha, \beta)}(x) \). It is easily inferred (e.g. from [30], (4.21.7)) that

\[
\frac{d}{dx} p_n^{(\alpha, \beta)}(x) = n(n+\alpha+\beta+1) p_{n-1}^{(\alpha+1, \beta+1)}(x).
\]

For \( f \in C [-1, 1] \), let \( F_n(f; x) = F_n(p_n; p_n'; f; x) \) be the uniquely defined polynomial which has degree at most \( 3n-2 \) satisfying

\[
p_n^{(\alpha, \beta)}(x) \text{ is normalized in the sense that the integral } \int_{-1}^{1} p_n^{(\alpha, \beta)}(x)^2 (1-x)^{\alpha}(1+x)^{\beta} \, dx = 1 \text{ and the leading coefficient } y_n > 0.
\]
(2.2) \( F_n(f;x_{kn})=f(x_{kn}), \quad F_n'(f;x_{kn})=0 \quad (k=1,2,\ldots,n) \)

and

(2.3) \( F_n(f;y_{jn})=f(y_{jn}) \quad (j=1,2,\ldots,n-1). \)

We investigate the uniform convergence of \( F_n(f;x) \) to \( f(x) \)
on \([-1+\delta, 1-\delta]\) where \( \delta \) is an arbitrary but fixed positive number \(<1\).

Let \( \text{Lip}_M \) be the set of functions belonging to \( \text{Lip}_Y \) and

with a coefficient in the Lip-condition \( \leq M. \)

Our results are as follows:

Theorem 2.1. For any \( f \in \text{Lip}_M \) with \( 0<\gamma<1 \) and \( \gamma >0 \), we have \(^3\)

\[
\|F_n(f)-f\|=O\left(\frac{\log n}{n}\right)
\]

Let \( \Omega \) be a non-decreasing function which satisfies

(2.4) \[ \Omega(2h)\leq 2\Omega(h) \quad (h>0). \]

And we define

(2.5) \[ C(\Omega) = \{g \in C[-1,1] \mid \omega(g;h)\leq \Omega(h) \quad \text{for all } h>0 \}. \]

Then we have the following

Theorem 2.2. For \( \alpha, \beta \geq -\frac{1}{2} \) and any \( \Omega \) which satisfies (2.4),

we have

\[
\sup_{f \in C(\Omega)} \|F_n(f)-f\| \geq A \Omega\left(\frac{1}{n}\right) \log n,
\]

where \( A \) depends on \( \alpha, \beta, \) and \( \delta \) only.

\(^3\)In this paper all \( O \)-estimates are such that only constant factors depending on \( \alpha, \beta, \gamma \) and \( \delta \) are involved. And \( \| \cdot \| \)
is the usual supremum norm on \([-1+\delta, 1-\delta] \).
From the two theorems above, we conclude that

\[(2.6) \quad \sup_{f \in \text{Lip}_M^\gamma} \| f_n - f \|_{M^\gamma} \leq n^{-\gamma} \log n \quad (0 < \gamma < 1)^4.\]

3. **Auxiliary Estimates.** Let the zeros of \( p_n(x) \) and \( p_n'(x) \) be ordered as follows:

\[(3.1) \quad x_{n1} < x_{n-1} < x_{n-1} < \cdots < x_2 < x_2 < x_1 < x_1.\]

The Hermite-Fejér interpolation polynomial of a function \( g \) with nodes \( x_{kn} \), \( k = 1, 2, \ldots, n \), is

\[(3.2) \quad H_n(p_n; g; x) = \sum_{x_{kn}}^{n} g(x_{kn}) v_{kn}(x) l_k(p_n(\alpha, \beta); x),\]

where

\[(3.3) \quad v_{kn}(x) = \frac{1-x[\alpha-\beta+(\alpha+\beta+2)x_{kn}]+(\alpha-\beta)x_{kn}+(\alpha+\beta+1)x^2_{kn}}{1-x^2_{kn}}\]

and

\[(3.4) \quad l_k(p_n(\alpha, \beta); x) = \frac{p_n(x)}{(x-x_{kn})/p_n(x_{kn})(x-x_{kn})}.\]

Let \([a, b] \subset (-1, 1)\) be fixed. In what follows, \( C_3, C_4, \ldots \) will be constants depending on \( \alpha, \beta, \gamma, a, \) and \( b \) only. It was proved in [33], (2.1) and (3.2), that, for \( \alpha, \beta > 1,\)

\[(3.5) \quad |g(x) - H_n(g;x)| \leq C_3 \sum_{k=1}^{n} \omega(g; k/n) \frac{1}{n^{1/2}} \quad (x \in [a, b]).\]

In particular, if \( g \in \text{Lip}_M^\gamma \) with \( 0 < \gamma < 1 \) and \( M > 0 \), then

\[(3.6) \quad |g(x) - H_n(g;x)| \leq C_4 n^{-\gamma} \quad (x \in [a, b]).\]

---

4. Here \( u \sim v \) means \( u = O(v) \) and \( v = O(u) \).
We know
\begin{equation}
L(p_n^{\frac{f-H}{p_n}};x) = L(P^{(\alpha+1, \beta+1)}_{n-1};\frac{f-H}{p_n};x) = \sum_{j=1}^{n-1} \frac{(f-H)(y_{jn})}{p_n^2(y_{jn})} L(P^{(\alpha+1, \beta+1)}_{n-1};x),
\end{equation}
where
\begin{equation}
l_j(p^{(\alpha+1, \beta+1)}_{n-1};x) = \frac{p'_n(x)}{(x-y_{jn})p_n''(y_{jn})} (j=1,2,\ldots,n-1)
\end{equation}

Since
\begin{equation}
|p_n(x)| \sim \sqrt{p_n^{(\alpha, \beta)}(x)},
\end{equation}
we have
\begin{equation}
|p_n(x)| \leq C_5 \quad (x \in [a, b]),
\end{equation}
\begin{equation}
|p'_n(x_{kn})| \sim k^{-\alpha+3/2} n^{\alpha+5/2} \quad (0 \leq x_{kn} < 1, k=1,2,\ldots).
\end{equation}
(S. [30], (4.3.4), (7.32.5) and (8.9.2)).

Let $x_{0n} = 1$ and $x_{n+1} = -1$. Then for $x \geq 0$ and $x \in [x_{m+1}, x_m, x_n, m=0,1,\ldots$, we have
\begin{equation}
|x-x_{kn}| \sim n^{-2} |m^2-k^2| \quad (k=0,1,\ldots,k+m,m+1)
\end{equation}
(S. [23], Lemma 2).

In virtue of
\begin{equation}
p_n^{(\alpha, \beta)}(x) = (-1)^n p_n^{(\beta, \alpha)}(-x),
\end{equation}
similar asymptotic estimates hold for $-1 \leq x < 0$ and $-1 \leq x_{kn} < 0, k=n+1, n, \ldots$.

By Lemma 3, 4, and 5 of [23] and by (3.13), we have
\begin{equation}
\sum_{k=1}^{n} |l_k(p^{(\alpha, \beta)}_n;x)| \leq 1 + C_6 |p_n(x)| \log n \quad (x \in [-1,1]),
\end{equation}
which is true for all $\alpha, \beta \geq -\frac{1}{2}$. Thus, in particular, we have
\[ (3.15) \quad \sum_{j=1}^{n-1} |l_j(P_{n-1}^{(\alpha+1, \beta+1)}; x)| \leq C_7 \log n \quad (x \in [a, b]). \]

Following the proofs of Lemma 3 and 4 of [23], it is not difficult to show

Lemma 3.1. For $\tau \geq 0$, we have
\[ \sum_{k=1}^{n} \frac{k^2}{(k^2 - m^2)} \leq B(m \tau^{-1} + n \tau^{-1}) \log n \quad (m=1, 2, \ldots, n), \]

where $B$ is a constant depending on $\tau$ only.

Lemma 3.2. For $\alpha, \beta \geq -\frac{1}{2}$, we have
\[ |P_n(x)| \geq C_8 > 0 \quad (j=1, 2, \ldots, n-1). \]

Proof. The $p_n$'s satisfy the relation
\[ (1-x^2)p_n'(x) = -nx_p_n(x) + u_n p_n(x) + v_n p_{n-1}(x), \]
where $v_n \sim n$. (S. [30], (4.5.7).)

By differentiating both side of the equation above and inserting $x = y_n$, we have
\[ (3.16) \quad (1-y_n^2)p_n''(y_n) = -np_n(y_n) + v_n p_{n-1}(y_n) \quad (j=1, 2, \ldots, n-1). \]

In virtue of the differential equation, (S. [30], (4.2.1)),
\[ (1-x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x] y' + n(n+\alpha+\beta+1)y = 0, \]
we have
\[ (3.17) \quad |p_n(y_n)| \sim |p_{n-1}(y_n)| \quad (j=1, 2, \ldots, n-1). \]

Then
\[ (3.18) \quad \frac{1}{p_n(y_n)} \left| \frac{p_n(y_n)}{p_{n-1}(y_n)} \right| = \frac{\gamma_n}{\gamma_{n-1}} \lambda_n(y_n) \quad (j=1, 2, \ldots, n-1), \]

where $\lambda_n(x)$ is the Christoffel function. (S. [13], §I.4).
The Lemma follows from the facts that
\[ 0 < \frac{y_{n-1}}{y_n} \leq 1 \] and \[ \lambda_n(x) = 0 \left( \frac{1}{n} \right) \] for \( x \in [-1, 1] \).

(S. [13], § I.7. and § III.3. Lemma 3.2). Q.E.D.

Lemma 3.3. Let \( g \in C [-1, 1] \) and \( g(x) = 0 \) for \( x \in [-1, 1] \setminus (a, b) \). Then

\[
(3.19) \quad \left| \frac{g(y_{jn}) - H_n(g; y_{jn})}{p_n^2(y_{jn})} \right| \leq C \frac{\sum_{k=1}^{n} w(g,k) \frac{1}{n^2}}{\sum_{k=1}^{n} \frac{1}{n^2}} (j = 1, 2, \ldots, n-1).
\]

Proof. By (3.5) and Lemma 3.2, it is clear that (3.19) holds for \( y_{jn} \in \left[ \frac{-1+a}{2}, \frac{1+b}{2} \right] \). Since \( g(x) = 0 \) for \( x \notin (a, b) \), in view of (3.2), we have

\[
(3.20) \quad g(x) - H_n(g;x)
\]

\[
= \sum_{k=0}^{n} (g(1) - g(x_{kn})) v_{kn}(x) \frac{1}{p_n(x)}
\]

\[
= \sum_{k=0}^{n} (g(-1) - g(x_{kn})) v_{kn}(x) \frac{1}{p_n(x)}
\]

\[
+ \sum_{k=0}^{n} (g(-1) - g(x_{kn})) v_{kn}(x) \frac{1}{p_n(x)}
\]

\[
(3.3), (3.4), (3.11), (3.12), \text{ and } (3.20)
\]

It follows from (3.3), (3.4), (3.11), (3.12), and (3.20) that

\[
(3.21) \quad |g(x) - H_n(g;x)|
\]

\[
\leq C_0 n^{-2} \sum_{k=1}^{n} \frac{w(g,k^2)}{n^2} p_n^2(x) \quad (x \in \left[ \frac{-1+a}{2}, \frac{1+b}{2} \right]).
\]

In particular, (3.21) holds for \( y_{jn} \in \left[ \frac{-1+a}{2}, \frac{1+b}{2} \right] \).
Consequently, (3.19) holds for all \( y_{jn}, j=1,2,\ldots,n-1 \).

Q.E.D.

Lemma 3.4. Let \( g \in L^p_{\gamma, M_1} \) with \( 0 < \gamma \leq 1 \) and \( M_1 > 0 \). Then there exists a sequence of polynomials \( \pi_m(x), m=1,2,\ldots \), such that \( \text{deg} \pi_m \leq m \) and

\[
(3.22) \quad |g(x) - \pi_m(x)| \leq C_{11} M_1 \left( \frac{\sqrt{1-x^2} + \frac{1}{m}}{m} \right)^\gamma \quad (x \in [-1,1]),
\]

and

\[
(3.23) \quad |\pi'_m(x)| \leq C_{12} M_1 \left( \frac{\sqrt{1-x^2} + \frac{1}{m}}{m} \right)^{-1} \quad (x \in [-1,1]).
\]

Proof. The existence of \( \pi_m(x), m=1,2,\ldots \), which satisfy (3.22) is the strong form of Jackson's theorem. (S. [31], 5.2.1).

Then for integer \( t \geq 0 \), we have

\[
(3.24) \quad |\pi_{2t+1}(x) - \pi_2t(x)|
\]

\[
\leq |\pi_{2t+1}(x) - g(x)| + |g(x) - \pi_2t(x)|
\]

\[
\leq C_{11} M_1 \left( \frac{\sqrt{1-x^2} + \frac{1}{2t+1}}{2t+1} \right)^\gamma \left( \frac{\sqrt{1-x^2} + \frac{1}{2t+2}}{2t+2} \right)
\]

\[
\leq 5C_{11} M_1 \left( \frac{\sqrt{1-x^2} + \frac{1}{2t+1}}{2t+1} \right)^\gamma.
\]

Consequently,

\[
(3.25) \quad |\pi'_{2t+1}(x) - \pi'_2t(x)| \leq C_{13} M_1 \left( \frac{\sqrt{1-x^2} + \frac{1}{2t+1}}{2t+1} \right)^\gamma - 1.
\]

(S. [31], 4.8.72. (35) and (36)).

Therefore, for integer \( s \geq 0 \), we obtain
Using (3.26) above, we obtain (3.23). Q.E.D.

Lemma 3.5. Let $g \in \text{Lip}_\gamma$ with $0 < \gamma < 1$ and $M_1 > 0$.

Assume that $g(x) = 0$ for $x \in [a, b]$. Let $[a', b'] \subset (a, b)$ where $a'$ and $b'$ depend on $a$ and $b$. Then

$$\frac{|g(y_{in}) - H_n(g; y_{in})|}{p_n^2(y_{in})} \leq C 161 \frac{\log n}{n^\gamma} \quad (j=1, 2, \ldots, n-1).$$

Moreover,

$$\frac{|g(y_{in}) - H_n(g; y_{in})|}{p_n^2(y_{in})} \leq C 17 M_1 n^{-1} \quad (y_{jn} \in [a', b']; j=1, 2, \ldots, n-1).$$

Proof. Let $\pi_m(x)$, $m=1, 2, \ldots$, be a sequence of polynomials satisfying (3.22) and (3.23).

It is clear that Hermite interpolating polynomials reproduce polynomials with degree less than $2n$.

Therefore, we have

$$(3.27) \quad \pi_n(x) = H_n(\pi_m; x) + \sum_{k=1}^n (x-x_{kn})^2 (p_n; x) \pi_n'(x_{kn}).$$

(S. [30], (14.1.9) and (14.1.10)).

Using (3.27), we have
(3.28) \[ |g(x) - H_n(g;x)| \leq |(g - \Pi_n)(x) - H_n(g - \Pi_n;x)| \]

\[
= \sum_{k=1}^{n} \frac{(x-x_{kn})^2}{k!} \frac{\pi_n(x_{kn})}{(p_n;x)}.
\]

Since \(-1/2 < \alpha, \beta < 0\), based on (3.25), we have

(3.29) \[ |(g - \Pi_n)(x) - H_n(g - \Pi_n;x)| = o(1)w(g;\frac{1}{n}) \quad (x \in [-1,1]). \]

(S. [30], Theorem 14.5. and (14.1.11)).

From (3.4), (3.11), (3.12), (3.23), and Lemma 3.1, we have the following estimate for \( y_{jn} > 0 \):

(3.30) \[ \left| \sum_{k=1}^{n} (y_{jn} - x_{kn}) \frac{1^2}{k!} (p_n; y_{jn}) \pi_n(x_{kn}) \right| \]

\[ \leq \sum_{k=1}^{n} \left| y_{jn} - x_{kn} \right|^{-1} \left| \frac{p_n(y_{jn})}{p'_n(x_{kn})} \right|^2 \left| \pi_n(x_{kn}) \right| \]

\[ \leq C_1 g_{M1} \left\{ \sum_{k=1}^{n} \frac{n^2}{k^{2+2\gamma}} \frac{k^{2\alpha+3}}{n^{2\alpha+5}} \frac{n^{2-2\gamma}}{k^{1-\gamma}} \right. \]

\[ + \sum_{m=1}^{n} \left| (n-j+1)^2 - m^2 \right| \frac{m^{2\alpha+3}}{n^{2\alpha+5}} \frac{n^{2-2\gamma}}{m^{1-\gamma}} \}

\[ \left. \frac{p_n(y_{jn})}{p_n(y_{jn})} \right| \]

\[ \leq C_1 g_{M1} \frac{\log n}{n^{\gamma}} \cdot p_n(y_{jn}) \]

In virtue of (3.13), the same estimate holds for \( y_{jn} < 0 \). Therefore (3.30) is true for all \( j = 1, 2, \ldots, n-1 \).

For \( x_{kn} > 0 \), by (3.4), (3.11), (3.12), (3.14), (3.23), and
Lemma 3.2, we have

\[(3.31) \quad \left| (y_{jn} - x_{kn}) \right|^2 \left( p_n(y_{jn}) \right) \ |
\]

\[\leq \frac{p_n(y_{jn})}{p_n(x_{kn})} \cdot \left( p_n(y_{jn}) \right) \ |
\]

\[\leq C_{20} \frac{\frac{2\alpha + 3}{2} \cdot \frac{2 - 2\nu}{\alpha + 5/2} \cdot k^{1 - \nu}}{n} \cdot (1 + C_6 |p_n(y_{jn})| \cdot \log n) \cdot |p_n(y_{jn})|\]

\[\leq C_{21} \frac{\log n}{n^{1/2}} \cdot p_n^2(y_{jn}) \quad (j=1,2,\ldots,n-1).
\]

By symmetry, this is also valid for $x_{kn} < 0$. Thus (3.31) is true for all $k=1,2,\ldots,n$ and $j=1,2,\ldots,n-1$. In particular, it is true for $k=j$, $j+1$ for each $j=1,2,\ldots,n-1$. Thus the first estimate of this lemma follows from (3.28), (3.29), (3.30), (3.31), and Lemma 3.2.

Since $g(x) = 0$ for $x \in [a,b]$, from (3.2), we have

\[(3.32) \quad g(x) - H_n(g;x) = \sum (g(x) - g(x_{kn}))v_{kn}(x)\frac{1}{p_n(x_{kn})} \sum_{[a',b']} \]

\[(x \in [-1,1]).
\]

Now, by (3.4), (3.11), (3.12), and (3.32), in the cases $y_{jn} \in [a',b']$, we have

\[(3.33) \quad |g(y_{jn}) - H_n(g;y_{jn})| \leq C_{22} \sum_{k=1}^{n} \left| g(x_{kn}) \right| \frac{1}{1 - x_{kn}^2} \cdot \left( \frac{p_n(y_{jn})}{p_n(x_{kn})} \right)^2
\]

\[\leq C_{23} \left\{ \sum_{k=1}^{n} \frac{n^2 \cdot k^{2\alpha + 3}}{n^{2\alpha + 5}} + \sum_{m=1}^{n} \frac{n^2 \cdot m^{2\beta + 3}}{n^{2\beta + 3}} \right\} \cdot p_n^2(y_{jn})
\]

\[\leq \frac{C_{24} \cdot M_1 \cdot p_n^2(y_{jn})}{n}.\]
This proves the second part of the Lemma. Q.E.D.

Modifying (3.30) and (3.31), we can prove

Lemma 3.6. Let \( R(x) = Ax + B \). Then

\[
\sum_{k=1}^{n} (y_{jn} - x_{kn})^2 (p_n(x_{kn}) R'(x_{kn})) \leq C_{24} A \frac{\log n}{n} p_n^2(y_{jn})
\]

\((j=1,2,\ldots,n-1)\).

Lemma 3.7. Let \( x', y' \in [a, b] \) be two consecutive nodes from (3.1). Then

\[
\frac{C_{25}}{n} \geq |x' - y'| \geq \frac{C_{26}}{n}.
\]

Proof. We assume that \( x' \) is a HF-point and \( y' \) a L-point.

Since both \( p_n(x) \) and \( p_n'(x) \) are Jacobi polynomials, it can be easily inferred (e.g. [13], §111.5, Theorem 5.1) that

\[
|x' - y'| \leq \frac{C_{25}}{n}.
\]

By mean value theorem, there is an \( u \) which lies between \( x' \) and \( y' \), such that

\[
(3.34) \quad |p_n(y')| = |p_n(x') - p_n(y')| = |x' - y'| \cdot |p_n'(u)|.
\]

Applying (3.10) and Bernstein's inequality on \([\frac{-1+a}{2}, \frac{1+b}{2}]\), we have

\[
(3.35) \quad |p_n'(x)| \leq C_5 n \quad (x \in [a, b]).
\]

From (3.34), (3.35), and Lemma 3.2, we have

\[
(3.36) \quad |x' - y'| \geq \frac{C_{26}}{n}.
\]

Q.E.D.

4. Proof of the main result.

Proof of Theorem 2.1. Let \( R(x) \) be the linear equation whose graph intersects that of \( f(x) \) at \( x = -1 + \frac{\delta}{4}, 1 + \frac{\delta}{4} \), namely,
(4.1) \[ R(x) = \frac{f(-l+\frac{\delta}{4})-f(-\frac{\delta}{4})}{2-\frac{\delta}{2}}(x-1+\frac{\delta}{4})+f(1-\frac{\delta}{4}). \]

Next, we define two continuous functions \( f_1 \) and \( f_2 \) on \([-1,1]\) as follows:

\[
(4.2) \quad f_1(x) = \begin{cases} 
  f(x)-R(x) & \text{if } x \in [-1+\frac{\delta}{4}, 1-\frac{\delta}{4}], \\
  0 & \text{otherwise}
\end{cases}
\]

\[
(4.3) \quad f_2(x) = \begin{cases} 
  0 & \text{if } x \in [-1+\frac{\delta}{4}, 1-\frac{\delta}{4}], \\
  f(x)-R(x) & \text{otherwise}
\end{cases}
\]

It is clear that \( f(x) = f_1(x) + f_2(x) + R(x) \) and

\[
(4.4) \quad \omega(f_i;h) = O(1) \omega(f;h) \quad (h \geq 0 \text{ and } i=1,2).
\]

It follows that

\[
(4.5) \quad f(x)-H_n(f;x) = \sum_{i=1}^{2} (f_i(x)-H_n(f_i;x)) + R(x)-H_n(R;x)
\]

Now, from (3.7), (3.10), (3.15), (4.2), (4.4) and Lemma 3.3, we have

\[
(4.6) \quad p_n^2(x)L(p_n;\frac{f_1-\frac{\delta}{2}}{p_n};x) = O(1) \frac{\log n}{n} \quad (x \in [-1+\delta, 1-\delta]).
\]

And, from (3.7), (3.10), (3.15), (4.3), (4.4), and Lemma 3.5, we have, for \( x \in [-1+\delta, 1-\delta] \),

\[
(4.7) \quad p_n^2(x)L(p_n;\frac{f_2-\frac{\delta}{2}}{p_n};x) = O(1) \left| \sum_{j=1}^{n-1} (f_2(y_{jn})-H_n(f_2;y_{jn})) \right|\left| p_n^2(y_{jn}) \right|
\]

\[
= O(1) \left| \sum_{y_{jn} \in [-1+\delta/2, 1-\delta/2]} \sum_{y_{jn} \in [-1+\delta/2, 1-\delta/2]} \right|
\]
\[ = 0(1)\frac{\log n}{n} + 0(1)\frac{\log n}{n^\gamma} \sum_{j=1}^{n-1} \frac{1}{y_j} \cdot \begin{cases} \left(\frac{1}{2}, 1 - \frac{\delta}{2}\right) \\
\end{cases} \]

since \[ \sum_{j=1}^{n-1} \frac{1}{y_j} = 0(1) \quad \text{for } x \in [-1 + \delta, 1 - \delta]. \]

(S. [13, §III.6.6.4]).

By (3.7), (3.15), (4.1), (3.27), and Lemma 3.6, we have

\[ p_n^2(x) L(p_n; \frac{R-H(R)}{p_n^2}; x) = 0(1) \left(\frac{\log n}{n}\right)^2 \quad (x \in [-1 + \delta, 1 - \delta]). \]

By virtue of (3.6), (4.5), (4.6), (4.7), (4.8), and (1.1), we have

\[ \| F_n(f) - f \| = 0(1) M_n^{-\gamma} + O(\frac{\log n}{n^\gamma} + O(1) \left(\frac{\log n}{n}\right)^2 \]

\[ = 0(1) M_n^{-\gamma} \quad \text{Q.E.D.} \]

Proof of Theorem 2.2. We shall construct, for each \( n \geq n_0 \) with \( n_0 \) large enough, a function \( g_n \in C(\Omega) \) such that

\[ |F_n(g_n; x^*) - g_n(x^*)| \geq A_1 \Omega \left(\frac{1}{n}\right) \log n \]

for some \( x^* \in (-1 + \delta, 1 - \delta) \). In what follows, \( A_2, A_3, \ldots \) will be constants depending on \( \alpha, \beta \), and \( \delta \) only. In virtue of Lemma 3.7, we can assume that \( n_0 \) is so large that, for each \( n \geq n_0 \), at least two consecutive zeros of \( p_n(x), x_{t_n} \) and \( x_{t+1,n} \), say, are in the interval \( (-1 + \delta, 1 - \delta) \).

Let \( x_{t_1,n} (x_{t_1,n}, \text{resp.}) \) be the smallest (largest resp.) zero of \( p_n(x) \) in the interval \( (-1 + \delta, 1 - \delta) \). We define a
piecewise linear function $f_n$ which has vertices at
\[ x_{ln} \leq y_{l-1}, n \leq \cdots \leq y_{tn} \leq x_{tn} \leq y_{t-1}, n \leq \cdots \leq y_{rn} \leq y_{rn} \]
such that
\[
(4.10) \quad f_n(-1) = f_n(1) = f_n(x_{kn}) = 0 \quad (k = r, r+1, \ldots, l)
\]
\[
f_n(y_{in}) = (-1)^i \quad (i = r, r+1, \ldots, t-1).
\]
\[
f_n(y_{jn}) = (-1)^{j+1} \quad (j = t, t+1, \ldots, l-1).
\]
Now let
\[
(4.11) \quad g_n(x) = \frac{1}{2} \Omega \left( \frac{C_{26}}{n} \right) f_n(x) \quad (x \in [-1,1]),
\]
where $C_{26}$ is the constant given in (3.36) with $[a, b] = [-1+5, 1-5]$ assumed. We claim that
\[
(4.12) \quad g_n \in C(\Omega).
\]
It is clear that
\[
(4.13) \quad \omega(g_n; h) = \frac{1}{2} \Omega \left( \frac{C_{26}}{n} \right) \omega(f_n; h) \quad (h \geq 0).
\]
Since $f_n(x)$ is piecewise linear, by (3.36) and (4.10), we have
\[
(4.14) \quad \omega(f_n; h) \leq \begin{cases} 
\frac{h \cdot \frac{n}{C_{26}}}{2} & (0 \leq h \leq \frac{C_{26}}{n}) \\
\frac{C_{26}}{n} & (\frac{C_{26}}{n} \leq h).
\end{cases}
\]
In addition, it follows from (2.4) that
\[
(4.15) \quad \frac{\Omega(h_2)}{h_2} \leq \frac{2 \Omega(h_1)}{h_1} \quad \text{if } 0 \leq h_1 \leq h_2.
\]
Therefore, by (4.13), (4.14), and (4.15), we have
\[
\omega(g_n; h) \leq \Omega \left( \frac{C_{26}}{n} \right) \leq \Omega(h) \quad \text{if } h \geq \frac{C_{26}}{n},
\]
\[\omega(g_n; h) \leq \frac{3}{2} h \cdot \frac{n}{C_{26}} \alpha\left(\frac{C_{26}}{n}\right) (0 \leq \frac{C_{26}}{n}).\]

This proves (4.12).

Since \(g_n(x_{kn}) = 0\) for all \(k=1,2,\ldots,n\), it follows that \(H_n(p_n; g_n; x) = 0\). Thus, by (4.10) and (4.11), we have

\[
(4.16) \quad (g_n - H_n)(y_{jn}) = \begin{cases} 0, & j = r, r+1, \ldots, t-1 \cr (-1)^{j+1} \frac{n}{n} \alpha\left(\frac{C_{26}}{n}\right), & j = t, t+1, \ldots, 1-1. \end{cases}
\]

For \(x \in (y_{tn}, x_{tn})\), in view of the sign changes of \(l_j(p_n; x)\) and \(g_n(y_{jn})\) for \(j = r, r+1, \ldots, 1-1\), we find that all \(g_n(y_{jn}) \cdot l_j(p_n; x)\), for \(j = r, r+1, \ldots, 1-1\), are of same sign.

Thus by (3.7), (4.15), (4.16), and Lemma 3.2, we have

\[
(4.17) \quad p_n^2(x) \cdot \frac{|L(p_n; g_n - H_n; x)|}{p_n^2(x)} = p_n(x) \sum_{j=r}^{1-1} \frac{g_n(y_{jn})}{p_n^2(y_{jn})} \cdot l_j(p_n; x) \geq A_2 \alpha\left(\frac{1}{n}\right) p_n^2(x) \sum_{j=r}^{1-1} |l_j(p_n; x)|.
\]

\[
= A_2 \alpha\left(\frac{1}{n}\right) p_n^2(x) \left\{ \sum_{j=1}^{n-1} |l_j(p_n; x)| - \sum_{j=1}^{n-1} |l_j(p_n; x)| \right\}.
\]

In virtue of a known estimate (S. [23], Theorem 1), we have

\[
(4.18) \quad \sum_{j=1}^{n-1} |l_j(p_n; x)| \geq A_3 \frac{|p'(x)|}{n} \log n \quad (x \in [-1+\epsilon, 1-\epsilon]).
\]
Using Lemma 3.2, we get

\[ \left| \int_{y_{tn}}^{x_{tn}} p_n^2(x)p_n'(x)dx \right| = \frac{|p_n^3(y_{tn})|}{3} \geq A_4. \]

By virtue of Lemma 3.7, this infers that

(4.19) \[ |p_n^2(x^*)p_n'(x^*)| \geq A_5 \cdot n \text{ for some } x^* \in (y_{tn}, x_{tn}). \]

By (4.18) and (4.19), we have

(4.20) \[ p_n^2(x^*) \sum_{j=1}^{n-1} |l_j(p_n'; x^*)| \geq A_6 \log n. \]

On the other hand,

(4.21) \[ \sum_{j=1}^{n} |l_j(p_n'; x)| = 0(1) \quad (x \in [-\frac{1+S}{2}, \frac{1-S}{2}]). \]

(S. [13], §III. 6. (6.4)).

Combining (3.10) with (4.12), we have

(4.22) \[ p_n^2(x^*) \sum_{j \in \mathcal{N}} |l_j(p_n'; x^*)| = 0(1). \]

From (4.17), (4.20), and (4.22), we have

(4.23) \[ p_n^2(x^*) |L(p_n'; \frac{g_n-H_n(g_n)}{p_n^2}; x^*)| \geq A_7 \alpha_n(\frac{1}{n}) \log n. \]

By our construction,

(4.24) \[ |g_n(x) - H_n(g_n; x)| = |g_n(x)| = 0(1) \quad \alpha_n(\frac{1}{n}) \quad (x \in [-1, 1]). \]

From (1.1), (4.23), and (4.24), we conclude that

\[ |F_n(g_n; x^*) - g_n(x^*)| \]

\[ \geq p_n^2(x^*) |L(p_n'; \frac{g_n-H_n(g_n)}{p_n^2}; x)| - |g_n(x^*) - H_n(g_n; x)| \]

\[ \geq A_1 \alpha_n(\frac{1}{n}) \log n. \quad \text{Q.E.D.} \]
1. Introduction and result. In their joint paper [19], G. Freud and the author investigated the convergence of a mixed type interpolation, in which the zeros of Jacobi polynomial $p_n^{(\alpha, \beta)}(x)$ with $0 > \alpha, \beta > -\frac{1}{2}$ are Hermite-Fejér points and the local extrema of $p_n^{(\alpha, \beta)}(x)$ are Lagrange points. They proved that, for $0 < \gamma < 1$,

$$\sup_{f \in \text{Lip}_M} \|F_n(f) - f\| \sim M n^{-\gamma} \log n.$$  

In this paper, we assume $\alpha, \beta > -\frac{1}{2}$, and drop the requirement $\alpha, \beta < 0$. We can prove the following

Theorem

$$\sup_{f \in \mathcal{C}^{(\Omega)}} \|F_n(f) - f\| \sim \Omega(n^{\frac{1}{n}}) \log n,$$

where $\Omega$ is any non-decreasing function satisfying (2.4) in [19].

2. Proof. In Theorem 2.2 of [19], we had proved that

$$(2.1) \quad \sup_{f \in \mathcal{C}^{(\Omega)}} \|F_n(f) - f\| \geq \Omega(n^{\frac{1}{n}}) \log n. \quad (\alpha, \beta > -\frac{1}{2}).$$

In the present paper, we shall follow all the notations defined in [19], unless otherwise specified.
Therefore, it is sufficient to prove that

\[(2.2) \sup_{f \in \mathbb{C}(\mathbb{R})} \|F_n(f) - f\| = O(1) \frac{1}{\log n}.\]

Let \(f \in \mathbb{C}(\mathbb{R})\). By Jackson's theorem, there exists a sequence of polynomials \(\Pi_m\), \(m = 1, 2, \ldots\), with \(\deg \Pi_m \leq m\) such that

\[(2.3) |f(x) - \Pi_m(x)| = O(1) \mathcal{O}(\frac{1}{m}) \quad (x \in [-1, 1]).\]

Now, for \(h > \frac{1}{n}\), we have

\[(2.4) \omega(\Pi_m; h) \leq \omega(f; h) + \omega(f - \Pi_m; h) = O(1) \mathcal{O}(h).\]

Consequently,

\[(2.5) \Pi_n'(x) = O(1) \frac{n}{\sqrt{1-x^2}} \mathcal{O}(\frac{1}{n}).\]

(S. [31], §4.12, (22)).

It is easily inferred (e.g. from [25], section 3.8) that

\[(2.6) F_n(f; x) = \sum_{i=1}^{n} f(x_{in}) h_{in}(x) + \sum_{j=1}^{n-1} f(y_{jn}) t_{jn}(x)\]

where

\[(2.7) h_{in}(x) = (1 - 2(x-x_{in})^2 \frac{p_n'(x_{in})}{p_n(x_{in})})^{1/2} \frac{p_n(x)}{p_n'(x)} (i=1, 2, \ldots, n),\]

and

\[(2.8) t_{jn}(x) = 1_j(p_n'; x)(\frac{p_n(x)}{p_n'(y_{jn})})^2 (j=1, 2, \ldots, n-1).\]

Also, for any polynomial \(\Pi(x)\) of degree \(\leq 3n-2\), we have

\[(2.9) \Pi(x) = \sum_{i=1}^{n} \Pi(x_{in}) h_{in}(x) + \sum_{j=1}^{n-1} \Pi(y_{jn}) t_{jn}(x) + \sum_{i=1}^{n} \Pi'(x_{in})(x-x_{in})^2 \frac{p_n(x)}{p_n'(x_{in})} \frac{p_n'(x)}{p_n'(x_{in})}\]

...
Let $q_n(x) = f(x) - \pi_n(x)$. 

By (2.6) and (2.9), we have 

\begin{equation}
|f(x) - p_n(f,x)| \leq |q_n(x) - p_n(q_n;x)| 
\end{equation}

\begin{align*}
&+ \left| \sum_{i=1}^{n} \Pi'_n(x_{in})(x-x_{in}) \right| \frac{1}{i!} \frac{p'_n(x)}{p'_n(x_{in})} \\
& \leq \sum_{i=1}^{n} |q_n(x) - q_n(x_{in})| \cdot |h_{in}(x)| \\
&+ \sum_{j=1}^{n-1} |q_n(x) - q_n(y_{jn})| \cdot |t_{jn}(x)| \\
&+ \left| \sum_{i=1}^{n} \Pi'_n(x_{in})(x-x_{in}) \right| \frac{1}{i!} \frac{p'_n(x)}{p'_n(x_{in})} \\
&\text{Since } x \in [-1+\delta,1-\delta], \text{ by (3.12) and (3.35) in [19], we have} \\
&\begin{equation}
|\frac{p'_n(x)}{p'_n(x_{in})}| = O(1) \quad (x \in [-1+\delta,1-\delta]). 
\end{equation}

A simple calculation yields 

\begin{equation}
|1 - 2(x-x_{in}) \frac{p'_n(x_{in})}{p'_n(x_{in})}| = O(1)(1-x_{in}^2)^{-1} \quad (i=1,2,\ldots,n). 
\end{equation}

Following the proof of Theorem 2.1 in [33], we get from 
(2.7), (2.3), (2.11), and (2.12) the estimate below 

\begin{equation}
\sum_{i=1}^{n} |q_n(x) - q_n(x_{in})| \cdot |h_{in}(x)| = O(1) \Omega\left(\frac{1}{n}\right) \log n \quad (x \in [-1+\delta,1-\delta]). 
\end{equation}

In virtue of (3.10), (3.15), and Lemma 3.2 in [19], we get 
from (2.8) and (2.3) the following
(2.14) \[ \sum_{j=1}^{n-1} |q_n(x) - q_n(y_j)| \cdot |t_j(x)| = O(1) \Omega(\frac{1}{n}) \log n \quad (x \in [-1+\delta, 1-\delta]). \]

Let \( x_{m+1}, r < x \leq x_m, n \) and \( s = m, m+1. \)

Since \( x \in [-1+\delta, 1-\delta] \), we may assume that \( x \in [-1+\frac{\delta}{2}, 1-\frac{\delta}{2}] \) for \( n \geq n_0 \) with \( n_0 \) a positive integer large enough.

Consequently,

(2.15) \[ |l_{sn}(x)| = O(1) \]

(2.16) \[ |x - x_{sn}| = O(1) n^{-1} \]

and

(2.17) \[ \frac{1}{1 - x_{sn}^2} = O(1) \]

(S. [13], Theorem 5.1).

Let's assume that \( x \geq 0. \)

now

(2.18) \[ \sum_{i=1}^{n} \frac{1}{n} |u_n'(x_{in})| \cdot |x - x_{in}| \cdot \frac{1}{i^2(p_n; x)} \cdot \frac{p_n'(x)}{p_n(x_{in})} \]

\[ \leq \sum_{s=m}^{m+1} \left| + \sum_{x_{in} > 0} \frac{1}{x_{in}^{1/2}} \frac{1}{p_n(x_{in})} \right| \]

\[ = \sum_{s=m}^{m+1} \left| + \sum_{x_{kn} < 0} \frac{1}{x_{kn}^{1/2}} \frac{1}{p_n(x_{kn})} \right| \]

By (2.5), (2.11), (2.15), (2.16), and (2.17), we have

(2.19) \[ \sum_{s=m}^{m+1} = O(1) \Omega(\frac{1}{n}) \]

and, by (3.4), (3.11), (3.12), (3.35), and Lemma 3.1 of [19], we have
Symmetrically, we can show that

\[(2.21) \quad \sum_{x_{kn} < 0} \left| \frac{n}{\log n} \right|^{1/2} = O\left(\frac{\log n}{n} \right).\]

Therefore, we obtain

\[(2.22) \quad \sum_{i=1}^{n} \frac{|p'(x_{in})|}{|x-x_{in}|^{1/2} (p_n(x)) \cdot \left| \frac{p'(x)}{p_n'(x_{in})} \right|} = O\left(\frac{\log n}{n} \right).\]

Again, by a similar argument, we can show that (2.22) also holds for \(x < 0\).

We get (2.2) from (2.10), (2.13), (2.14), and (2.22).
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