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THE GRAPH RECONSTRUCTION PROBLEM

FOR (M,N) TREES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the The Ohio State University

by

John Howard LeFever A.B., M.S., M.S.

* * * * * *

The Ohio State University

1977

Reading Committee:

Dr. Dijen K. Ray-Chaudhuri
Dr. Richard Wilson
Dr. Thomas Dowling
Dr. Neil Robertson

Adviser
Department of Mathematics
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VITA

February 25, 1948 . . . . . . . . . . . . . . . . . . . . . . . . . . Born-Whittier, California

1970 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . A. B. Occidental College, Los Angeles, California

1970-1977 . . . . . . . . . . . . . . . . . . . . . . . . . . Teaching Associate, Department of Mathematics Ohio State University, Columbus, Ohio

1972 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . M. S., The Ohio State University, Columbus, Ohio

1977 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . M. S., The Ohio State University, Columbus, Ohio

FIELD OF STUDY

Major Field: Mathematics

Studies in Combinatorial Theory.
Professor D. K. Ray-Chaudhuri.
# Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acknowledgments</td>
<td>ii</td>
</tr>
<tr>
<td>Vita</td>
<td>iii</td>
</tr>
<tr>
<td>List of Figures</td>
<td>vi</td>
</tr>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Chapter</td>
<td></td>
</tr>
<tr>
<td><strong>I. Preliminaries</strong></td>
<td>4</td>
</tr>
<tr>
<td>Graphs</td>
<td></td>
</tr>
<tr>
<td>((m,n))-trees</td>
<td></td>
</tr>
<tr>
<td>Endvertices</td>
<td></td>
</tr>
<tr>
<td>Branches</td>
<td></td>
</tr>
<tr>
<td>Eccentricity, Radius and Centers of ((m,n))-trees</td>
<td></td>
</tr>
<tr>
<td>Degree of an (m)-Simplex and an (n)-Simplex</td>
<td></td>
</tr>
<tr>
<td>Automorphisms</td>
<td></td>
</tr>
<tr>
<td>Some Results on Graph Reconstruction</td>
<td></td>
</tr>
<tr>
<td><strong>II. Some Results on Reconstruction of ((M,N))-Trees</strong></td>
<td>26</td>
</tr>
<tr>
<td>Reconstruction of a (k)-tree with a ((k-1))-simplex</td>
<td></td>
</tr>
<tr>
<td>Neighborhood Contraction Reconstruction</td>
<td></td>
</tr>
<tr>
<td>Vertex Reconstrcutibility of ((m,n))-Trees</td>
<td></td>
</tr>
<tr>
<td><strong>III. Reconstruction of 2-Trees</strong></td>
<td>43</td>
</tr>
<tr>
<td><strong>IV. Reconstruction of 3-Trees</strong></td>
<td>73</td>
</tr>
<tr>
<td><strong>V. References</strong></td>
<td>124</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

Figure 1 ................................................. 72
INTRODUCTION AND SUMMARY

One of the more interesting problems in Graph Theory is known as Ulam's Conjecture, or the Reconstruction Conjecture. Suppose $G$ and $G'$ are graphs, and $f: V(G) \to V(G')$ is a bijection such that $G - v \cong G' - f(v)$ for every $v \in V(G)$. Then $G'$ is said to be a vertex reconstruction of $G$. Any graph is, of course, a reconstruction of itself, however it is interesting to ask if there can be a vertex reconstruction of a graph $G$ which is not isomorphic to $G$. Let $G = K_2$, and $G' = \overline{K}_2$. Then $G - v = K_1$ for each $v \in V(G)$ and $G' - v = K_1$, for each $v \in V(G')$. Therefore $G'$ is a vertex reconstruction of $G$, but $G \not\cong G'$. We may state Ulam's Conjecture (vertex version) as follows: Let $G$ be a graph, and $G'$ a vertex reconstruction of $G$. Then if $G \not\cong K_2$ or $\overline{K}_2$, $G \cong G'$.

Similarly, suppose $g: E(G) \to E(G')$ is a bijection, such that $G - e \cong G' - g(e)$ for every $e \in E(G)$. Then $G'$ is said to be an edge reconstruction of $G$. It is easy to see that if $G = K_{1,2} \cup K_1$ and $G' = K_2 \cup K_2$, or $G = K_{1,3}$ and $G' = K_3 \cup K_1$, then $G'$ is an edge reconstruction of $G$. We may state Ulam's Conjecture (edge version) as follows: If $G$ is a graph with at least four edges, and $G'$ is an edge reconstruction of $G$, then $G' \not\cong G$. 

1
If the vertex version of Ulam's Conjecture holds for $G$, $G$ is said to be vertex reconstructible. If the edge version holds for $G$, $G$ is said to be edge reconstructible.

There are two major approaches to Ulam's Conjecture. The first is to show that if $G'$ is a reconstruction of $G$, then any parameter (e.g., connectivity, chromatic number, etc.) assumes the same value for $G$ and $G'$. The second is to show that all members of a certain class of graphs are reconstructible.

The first important result was of the second type. In 1957, Kelly [69] showed that if $G$ is a tree, then $G$ is vertex reconstructible. Other classes of graphs, each member of which has been shown to be vertex reconstructible include disconnected graphs [69], regular graphs [69], unicyclic graphs [81], cacti [43], outerplanar graphs [45], and maximal planar graphs with at least 25 vertices, and minimum degree 5 [35].

Classes of graphs, each member of which have been shown to be edge reconstructible include the graphs $G$ for which $\epsilon > \frac{1}{2}v$, where $\epsilon = |E(G)|$ and $v = |V(G)|$ [79], the graphs $G$ for which $2^{e-1} > v$ [92], 4-connected planar graphs with minimum degree 5, and maximal planar graphs with minimum valency at least 4 [36].

We wish to consider a class of graphs introduced by Dewdney [28]; $(m,n)$-trees which can be considered as generalizations of trees. Our main theorems are:
THEOREM 1: If $G$ is an $(m,n)$-tree, with $n > m+1$, then $G$ is vertex reconstructible.

THEOREM 2: If $G$ is an $(m,n)$-tree, then $G$ is edge reconstructible.

THEOREM 3: If $G$ is a $(1,2)$-tree, then $G$ is vertex reconstructible.

THEOREM 4: If $G$ is a $(2,3)$-tree, then $G$ is vertex reconstructible.

The above discussion was concerned solely with finite graphs, i.e. graphs for which $|V(G)| < \infty$. We may well ask if the reconstruction conjecture is true for infinite graphs. That it is not is seen by the following example [38]: Let $G$ be a tree such that $V(G)$ is countable and the degree of every vertex of $G$ is countable, and let $G'$ be a countable number of copies of $G$. It is easy to see that $G'$ is a vertex reconstruction of $G$, and get $G \neq G'$. 
CHAPTER I

PRELIMINARIES

§1. Graphs. We define a graph $G$ to be an ordered pair $(V,E)$ of finite sets, where $E \subseteq P_2(V)$, the 2 element subsets of $V$. We call $V$ the set of vertices of $G$, and $E$ the set of edges. When more than one graph is being considered, we write $V(G)$ and $E(G)$ for $V$ and $E$. We let $v(G) = |V|$ and $e(G) = |E|$. If $e = \{v_1,v_2\} \in E$ then we say $e$ is incident with $v_1$ and $v_2$, $v_1$ and $v_2$ are incident with $e$, and $v_1$ and $v_2$ are adjacent. For $v \in V$, we define the degree of $v$ in $G$, written $\deg_G v$, to be the number of edges of $G$ incident with $v$. A vertex of $G$ is said to be isolated if $\deg_G v = 0$.

We denote by $K_n$ a graph $G = (V,E)$ where $|V| = n$, and $E = P_2(V)$, and by $K_{n,m}$ a graph $G = (V,E)$, where $V = V_1 \cup V_2$, $|V_1| = n$, $|V_2| = m$, $V_1 \cap V_2 = \emptyset$, and $E = \{e \in P_2(V): |e \cap V_1| = |e \cap V_2| = 1\}$.

An isomorphism from a graph $G$ to a graph $G'$ is a bijection $\phi: V(G) \rightarrow V(G')$ such that $\{v_1,v_2\} \in E(G)$ if and only if $\{\phi(v_1),\phi(v_2)\} \in E(G')$. $G$ and $G'$ are isomorphic, written $G \cong G'$, if there is an isomorphism from $G$ to $G'$. Note that $\cong$ is an equivalence relation on the class of all graphs.
A graph $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $v \in V(G)$, we define $G-v$ to be the subgraph of $G$ defined as follows: $V(G-v) = V(G) \setminus \{v\}$, $E(G-v) = \{e \in E(G) : v \notin e\}$. If $H$ is a subgraph of $G$, we define $G-H$ to be the subgraph defined by $V(G-H) = V(G) \setminus V(H)$, $E(G-H) = E(G) \setminus (E(H) \cup \{e \in E(G) : v \in e \text{ for some } v \in V(H)\})$. Similarly, if $e \in E(G)$, we define $G-e$ to be the subgraph of $G$ defined by $V(G-e) = V(G)$, $E(G-e) = E(G) \setminus \{e\}$. If $S \subseteq V \cup E$, we define $G \cdot S$ to be the smallest subgraph of $G$ containing $S$.

Suppose $G$ and $H$ are graphs. $S_G \subseteq V(G)$, $S_H \subseteq V(H)$. Suppose $f : S_G \rightarrow S_H$ is a bijection. We define the attachment of $G$ and $H$ via $f$, written $G \cup^f H$, to be the graph defined by $V(G \cup^f H) = (V(G) \setminus S_G) \cup (V(H) \setminus S_H) \cup f$, where we regard $f$ as a set of ordered pairs. $(a,b) \in E(G \cup^f H)$ if $a,b \in V(G) \setminus S_G$ and $(a,b) \in E(G)$, or $a,b \in V(H) \setminus S_H$ and $(a,b) \in E(H)$, or $a \in V(G) \setminus S_G$, $b = (x,y) \in f$, and $(a,x) \in E(G)$, or $a = (x,y) \in f$, $b \in V(H) \setminus S_H$, and $(y,b) \in E(H)$, or $a = (x_1,y_1) \in f$, $b = (x_2,y_2) \in f$, and either $(x_1,x_2) \in E(G)$ or $(y_1,y_2) \in E(H)$. As an example, let $V(G) = \{a,b,c\}$, $E(G) = \{[a,b],[b,c]\}$, $V(H) = \{x,y,z\}$, $E(H) = \{[x,y],[y,z]\}$, $S_G = \{b\}$, $S_H = \{y\}$ and $f = \{(b,y)\}$. Then $V(G \cup^f H) = \{a,c,y,z,(b,y)\}$ and $E(G \cup^f H) = \{[a,(b,y)],[c,(b,y)]\}$, $\{x,(b,y)\}$, $\{z,(b,y)\}$. The effect is to identify $x$ with $f(x)$ for each $x \in S_G$. 
§2. (m,n)-trees. We now proceed to define (m,n)-trees.

Definition. Let $G$ be a graph. If $H$ is a subgraph of $G$ and $H \sim K_n$, then $H$ is an $(n-1)$-simplex of $G$.

Definition. Let $G$ be a graph. An $(m,n)$ walk sequence of length $\ell$ of $G$ is a sequence $H_1, K_1, H_2, K_2, \ldots, H_\ell, K_\ell, H_{\ell+1}$ where $H_i$ is an $m$-simplex of $G$, for $i = 1, \ldots, \ell+1$, $K_j$ is an $n$-simplex of $G$ for $j = 1, \ldots, \ell$, and $H_i \subseteq K_j = H_{i+1}$ for $i = 1, \ldots, \ell$.

An $(m,n)$ walk sequence $H_1, K_1, \ldots, H_\ell, K_\ell, H_{\ell+1}$ is an $(m,n)$ cycle sequence if all the $n$-simplices are distinct, $H_1 = H_{\ell+1}$, but any two other $m$-simplices are distinct, and $\ell \geq 2$.

An $(m,n)$ walk sequence $H_1, K_1, \ldots, H_\ell, K_\ell, H_{\ell+1}$ of $G$ is an $(m,n)$ path sequence if all the terms are distinct.

If $H_1, K_1, \ldots, H_\ell, K_\ell, H_{\ell+1}$ is an $(m,n)$ path sequence (cycle sequence) of $G$, then the subgraph $\bigcup_{i=1}^{\ell} K_i$ is called an $(m,n)$ path (cycle) of $G$ of length $\ell$ from $H_1$ to $H_{\ell+1}$.

Definition. An $(m,n)$ path sequence $H_1, K_1, \ldots, K_\ell, H_{\ell+1}$ of $G$ is simple if whenever $v \in V(H_i) \cap V(H_j)$ for $i < j$ then $v \in V(H_i) \cap V(H_{i+1}) \cap \ldots \cap V(H_j)$.

A graph $G$ is $(m,n)$ simple if every $(m,n)$ path sequence of $G$ is simple.
Definition. A graph $G$ is $(m,n)$ connected if for any two $m$-simplices $H_1$ and $H_2$ of $G$, there is an $(m,n)$ walk sequence with first term $H_1$ and last term $H_2$. We say that such an $(m,n)$ walk sequence is an $(m,n)$ walk sequence from $H_1$ to $H_2$.

Definition. Let $n > m$ and $G$ be a graph. $G$ is an $(m,n)$-tree if $G \sim K_{m+1}$, or there is an $(n-m-1)$-simplex $K'$ of $G$, each vertex of which is adjacent to every vertex of an $m$-simplex $H'$ of $G - K'$, and to no other vertices of $G - K'$, and $G - K'$ is an $(m,n)$-tree. If $G$ is a $(k-1,k)$-tree, we say $G$ is a $k$-tree.

The following theorem is due to Dewdney [28].

2.1. Theorem. Let $G$ be an $(m,n)$ tree. Then:

1) If $J$ is an $\ell$-simplex of $G$, $\ell \leq m$, then $J$ is contained in an $m$-simplex of $G$.

2) $G$ is $(m,n)$ connected.

3) $G$ has no $(m,n)$ cycle sequences.

4) $G$ is $(m,n)$ simple.

Proof. If $G \sim K_{m+1}$, then certainly 1)-4) hold for $G$. Otherwise, suppose 1)-4) hold for all $(m,n)$ trees with fewer vertices than $G$. Let $K'$ be an $(n-m-1)$-simplex of $G$, each vertex of which is adjacent to every vertex of an $m$-simplex $H'$ of $G - K'$, and $G - K'$ is an $(m,n)$ tree. Then 1)-4) hold for $G - K'$. 

1) Suppose $J$ is an $\ell$-simplex of $G$, $\ell \leq m$. If $V(J) \cap V(K') = \emptyset$, then $J$ is an $\ell$-simplex of $G-K'$, and so is contained in an $m$-simplex of $G-K'$, hence of $G$. If $V(J) \cap V(K') \neq \emptyset$, then since the vertices of $K'$ are adjacent only to vertices in $H' \cup K'$, $J \subseteq H' \cup K'$. But $H' \cup K'$ is an $n$-simplex of $G$. Let $S$ be a set of $m-\ell$ vertices of $H' \cup K'$ not in $J$. Then $J \cup S$ is an $m$-simplex of $G$ containing $J$.

2) Suppose $H_1$ and $H_2$ are $m$-simplices of $G$. If $H_1$ and $H_2$ are $m$-simplices of $G-K'$, then there is an $(m,n)$ path sequence in $G-K'$ from $H_1$ to $H_2$. But this is an $(m,n)$ path sequence in $G$. If neither $H_1$ nor $H_2$ are $m$-simplices of $G-K'$, then $H_1 \cup H_2 \subseteq H' \cup K'$, an $n$-simplex of $G$. If $H_1 = H_2$, we are done. If $H_1 \neq H_2$, then $H_1$, $H' \cup K'$, $H_2$ is an $(m,n)$ path sequence from $H_1$ to $H_2$. Otherwise, we can assume $H_1 \subseteq H' \cup K'$ and $H_2 \subseteq G-K'$. There is an $(m,n)$ path sequence $H', K_1, \ldots, H_2$ in $G-K'$. But then $H_1$, $H' \cup K'$, $H', K_1, \ldots, H_2$ is an $(m,n)$ path sequence from $H_1$ to $H_2$.

3) Suppose $G$ has an $(m,n)$ cycle sequence $H_1, K_1, \ldots, K_\ell$, $H_{\ell+1} = H_1$. If $\bigcup_{i=1}^{\ell} V(K_i) \cap V(K') = \emptyset$, then it is an $(m,n)$ cycle sequence of $G-K'$. Therefore, there is a vertex $v \in V(K') \cap V(K_i)$ for some $i$. Such a vertex belongs to only one $n$-simplex of $G$, namely $H' \cup K'$. Therefore, $K_i = H' \cup K'$,
$H_i \subseteq H' \cup K'$, and $H_{i+1} \subseteq H' \cup K'$. The only $m$-simplex of $H' \cup K'$ contained in another $n$-simplex of $G$ is $H'$. Since $K_i \neq K_{i-1}$ (or $K_i$ if $i = 0$) and $K_i \neq K_{i+1}$ (or $K_i$ if $i = 0$), $H_i = H' = H_{i+1}$, contradicting either the fact that all $m$-simplices are distinct, or the fact that $i \geq 2$.

4) Let $H_1, K_1, \ldots, K_k, H_{k+1}$ be an $(m,n)$ path sequence of $G$. Suppose $H_k$ is an $m$-simplex in the sequence and $V(H_k) \cap V(K') \neq \emptyset$. Then $H_k \subseteq H' \cup K'$. But then $H' \cup K'$ is the only $n$-simplex containing $H_k$. So $k \neq 1$, $k+1$, then $K_k = K_{k+1} = H' \cup K'$, contradicting the fact that $H_1, K_1, \ldots, K_k, H_{k+1}$ is a path sequence. Hence $H_k = H_1$ or $H_k = H_{k+1}$. If $V(H_1) \cap V(K') \neq \emptyset$ and $V(H_{k+1}) \cap V(K') \neq \emptyset$, then since $K_1 = H' \cup K' = K_k$, $k = k$, and the path sequence is $H_1, H' \cup K', H_2$.

Now suppose that $v \in V(H_i) \cap V(H_j)$ and $i < j$. If $H_i \cup H_j \subseteq G-K'$, then $H_1, K_1, \ldots, K_{j-1}, H_j$ is an $(m,n)$ path sequence of $G-K'$. Since $G-K'$ is $(m,n)$ simple $v \in V(H_1) \cap \ldots \cap V(H_j)$. If $H_i, H_j \subseteq H' \cup K'$, then $i = 1$, $j = 1$, and there are no $m$-simplices in the $(m,n)$ path sequence between $H_i$ and $H_j$. Otherwise, we may assume $V(H_i) \cap V(K') \neq \emptyset$ and $H_j \subseteq G-K'$. Then $i = 1$, and the $(m,n)$ path sequence is $H_1, H' \cup K' ; H' = H_2, \ldots, H_{k+1}$. If $v \in V(H_1) \cap V(H_j)$ then $v \in V(H' \cup K') \cap V(G-K') = V(H')$. $H' = H_2, \ldots, H_{k+1}$ is an $(m,n)$ path sequence of $G-K'$, so $v \in V(H_1) \cap \ldots \cap V(H_j)$. But then $v \in V(H_2) \cap \ldots \cap V(H_j)$. Therefore $G$ is $(m,n)$ simple.
2.2. Lemma. Suppose $G$ is a graph, and $H_1$ and $H_2$ are $m$-simplices of $G$. If there is an $(m,n)$ walk sequence in $G$ with first term $H_1$, and last term $H_2$, then there is an $(m,n)$ path sequence in $G$ with first term $H_1$ and last term $H_2$.

Proof. Let $H_1, K_1, \ldots, K_p, H_{p+1} = H_2$ be an $(m,n)$ walk sequence from $H_1$ to $H_2$ of least length. If $H_i = H_j$ for $i \neq j$, then $H_1, K_1, \ldots, H_1 = H_j, \ldots, K_p, H_{p+1}$ is an $(m,n)$ path sequence of smaller length. Similarly, if $K_i \neq K_j$ for $i = j$, we can construct an $(m,n)$ path sequence of smaller length. Consequently, all terms in the sequence are distinct.

Definition. Let $G$ be an $(m,n)$ tree, and $H_1, H_2$ $m$-simplices of $G$. We define $d_{m,n}(H_1, H_2)$ to be the length of the shortest $(m,n)$ path from $H_1$ to $H_2$.

2.3. Lemma. Let $G$ be an $(m,n)$ tree, and $H, H'$ $m$-simplices of $G$. There is a unique $(m,n)$ path from $H$ to $H'$.

Proof. Since $G$ is $(m,n)$ connected, there is at least one such $(m,n)$ path. Suppose $d_{m,n}(H, H') = 1$ and there are two $(m,n)$ paths from $H$ to $H'$, $H, K, H'$, and $H = H_1, K_1, \ldots, K_p, H_{p+1} = H'$. If $K$ is not a term of $H_1, K_1, \ldots, K_p, H_{p+1}$, then $H_1, K_1, \ldots, K_p, H_{p+1}K, H$ is an $(m,n)$ cycle sequence of $G$. If $K$ is a term of the sequence, say $K = K_i$, then $H_1, K_1, \ldots, H_i, K_i, H_1$ is an $(m,n)$ cycle sequence of $G$. 


Suppose $d_{m,n}(H,H') = k$, and the theorem is true for any two $m$-simplices with $d_{m,n}(H,H') < k$. There is an $(m,n)$ path sequence $H = H_1, K_1, \ldots, K_k, H'_{k+1} = H'$ from $H$ to $H'$. Suppose there is another $(m,n)$ path sequence $H = H_1', K_1', \ldots, K'_k, H'_{k+1} = H'$ from $H$ to $H'$. If $H_2 = H_2'$, then $H_2, K_2, \ldots, K_k, H'_{k+1}$ and $H_2', \ldots, K'_k, H'_{k+1}$ are different $(m,n)$ path sequences from $H_2$ to $H'$. If $H_2 \neq H_2'$, then $H_2', \ldots, K'_k, H'_{k+1}$ and $H_2, K_2, H_1, K_1, \ldots, K_k, H'_{k+1}$ are different $(m,n)$ path sequences from $H_2'$ to $H'$. Since $d_{m,n}(H_2,H') < k-1$, this contradicts the induction hypothesis.

We may now define $d_{m,n}(H,H')$ to be the length of the $(m,n)$ path sequence from $H$, $H'$. We now wish to show that the four conditions in the previous theorem characterize $(m,n)$ trees.

2.4. Theorem. (Dewdney [28]) Suppose $G$ is a graph, and

1) If $J$ is a vertex or edge of $G$, then $J$ is contained
   in an $m$-simplex of $G$.

2) $G$ is $(m,n)$ connected.

3) $G$ has no $(m,n)$ cycle sequences.

4) $G$ is $(m,n)$ simple.

Then $G$ is an $(m,n)$ tree.

Proof. If $v(G) < m+1$, $G$ does not satisfy 1). So $v(G) \geq m+1$. If $v(G) = m+1$, and $G$ satisfies 1), then $G \simeq K_{m+1}$ and is an $(m,n)$ tree. Otherwise, suppose $G$ satisfies 1)-4) and
every graph with fewer vertices than \( G \) satisfying 1)–4) is an (m,n) tree.

Suppose \( H \) is an m-simplex of \( G \), \( K \) is an n-simplex of \( G \), and \( H \subseteq K \). If \( v \in V(K) \setminus V(H) \) and \( v \) is adjacent to \( w \in V(G) \setminus V(K) \), then \( (v,w) \) is contained in an m-simplex \( H' \) of \( G \). Let \( u \in V(H) \).

Since \( G \) is (m,n) connected, there is an (m,n) path sequence in \( G \) from \( H \setminus \{u\} \cup \{v\} \) to \( H' \), \( H \setminus \{u\} \cup \{v\} \), \( K_1, H_2, \ldots, H' \). Since \( G \) is (m,n) simple and \( v \in V(H') \), \( H \neq H_2 \). If \( K_1 \neq K \), then \( H \setminus \{u\} \cup \{v\} \) is an m-simplex contained in an n-simplex other than \( K \).

If \( K_1 = K \), then \( H_2 \) is an m-simplex contained in an n-simplex other than \( K \). In either case, \( K \) contains a m-simplex \( H' \neq H \), contained in an n-simplex other than \( K \).

Let \( v \in V(G) \). \( v \) is contained in an m-simplex \( H \). Since \( v \in V(G) \), there is a vertex \( w \in V(G) \setminus V(H) \). \( w \) is contained in an m-simplex \( H' \) of \( G \). Hence, \( G \) contains at least two m-simplices.

Let \( \ell \) be the length of a longest (m,n) path sequence in \( G \). Since \( G \) is finite, \( \ell \) exists, and \( \ell > 1 \). Let \( H_1, K_1, \ldots, K_\ell, H_{\ell+1} \) be an (m,n) path sequence in \( G \) of length \( \ell \). If \( K_\ell \) contains an m-simplex \( H' \neq H_\ell \), which is contained in an n-simplex other than \( K_\ell \), say \( K' \), then \( H_1, K_1, \ldots, K_\ell, H', K', H'' \), where \( H'' \) is any m-simplex in \( K' \) other than \( H' \), is an (m,n) path sequence of \( G \) of length \( \ell+1 \), since if \( H', K', H'' \) where already in the sequence, \( G \) would contain an (m,n) cycle sequence. Therefore, \( H_\ell \) is the only m-simplex
of $K$ contained in an $n$-simplex other than $K$. By an above argument, no vertex of $K\subset H$ is adjacent to a vertex not in $K$. Now consider $G-(K\subset H)$. Every $(m,n)$ walk sequence in $G-(K\subset H)$ is an $(m,n)$ walk sequence in $G$. Therefore $G-(K\subset H)$ is $(m,n)$ simple, and contains no $(m,n)$ cycle sequences. Suppose $J$ is an edge or vertex of $G-(K\subset H)$. $J$ is an edge or vertex of $G-(K\subset H)$. $J$ is contained in an $m$-simplex $H$ of $G$. If $H \notin G-(K\subset H)$, then $H \subseteq K$. But then $J \subseteq K$, so $J \subseteq H$. Suppose $H_a$ and $H_b$ are two $m$-simplices of $G-(K\subset H)$. There is an $(m,n)$ path sequence $H_a = H_{1}, H_{2}, \ldots, H_{i}, H_{j}, H_{j+1} = H_b$ in $G$. If this is not an $(m,n)$ path sequence in $G-(K\subset H)$, then $K$ is a term of the sequence. If $K_{i} = K_{i}$, then $H_{i} = H_{i+1} = H_{j}$, since $H_{j}$ is the only $m$-simplex in $K$, contained in another $n$-simplex of $G$. Therefore, $H_{1}, \ldots, H_{j+1}$ is an $(m,n)$ path sequence in $G-(K\subset H)$. We have shown that $G-(K\subset H)$ satisfies 1)-4), and since $\nu(G-(K\subset H)) < \nu(G)$, $G-(K\subset H)$ is an $(m,n)$ tree; and by definition, so is $G$.

§3. Endvertices.

Definition. Let $G$ be an $(m,n)$ tree. If $K'$ is an $(n-m-1)$-simplex of $G$, each vertex of which is adjacent to every vertex of an $m$-simplex $H'$, and to no others, and $G-K'$ is an $(m,n)$ tree, then $K'$ is said to be an end-simplex of $G$, and if $v$ is a vertex of an end-simplex of $G$, $v$ is said to be an endvertex of $G$.

If $G$ is an $(m,n)$-tree, we write $EV(G)$ for the set of endvertices of $G$. 


**Definition.** Suppose $G$ is an $(m,n)$ tree, and $K$ is an $n$-simplex of $G$. An $(m,n)$ path $H_1, K_1, H_2, K_2, \ldots, K_r, H_{r+1}$ is $K$-avoiding if $K_i \neq K$ for $i = 1, \ldots, r$. Suppose $H$ is an $m$-simplex of $G$. An $(m,n)$ path $H_1, K_1, H_2, K_2, \ldots, K_r, H_{r+1}$ is $H$-avoiding if $H_i \neq H$ for $i = 1, \ldots, r+1$. Suppose $G$ is an $(m,n)$ tree and $K$ is an $n$-simplex of $G$. Then we define the relation $\sim_K$ on the set of $m$-simplices as follows: $H_1 \sim_K H_2$ if there is a $K$ avoiding $(m,n)$ path from $H_1$ to $H_2$, or if $H_1 = H_2$.

**1.1. Proposition.** $\sim_K$ is an equivalence relation.

**Proof.** 1) $H \sim_K H$ by definition, for any $m$-simplex $H$.

2) Suppose $H_a \sim_K H_b$, then there is a $K$ avoiding $(m,n)$ path $H_a = H_1, K_1, \ldots, K_r, H_{r+1} = H_b$. But then $H_b = H_{r+1}, K_r, \ldots, K_1, H_1 = H_a$ is a $K$-avoiding path and $H_b \sim_K H_a$.

3) Suppose $H_a \sim_K H_b$ and $H_b \sim_K H_c$. Then there are $K$-avoiding paths $H_a = H_1, K_1, \ldots, K_r, H_{r+1} = H_b$ and $H_b = H_1, K_1, \ldots, K_s, H_{s+1} = H_c$. Then $H_a = H_1, K_1, \ldots, K_r$, $H_{r+1} = H_1, K_1, \ldots, K_s, H_{s+1} = H_c$ is an $(m,n)$ walk sequence which does not include $K$ as a term. But then there is an $(m,n)$ path sequence from $H_a$ to $H_c$ which avoids $K$. $H_a \sim_K H_c$.

**Definition.** Suppose $G$ is an $(m,n)$ tree, and $H$ is an $m$-simplex of $G$. Define $\sim_H$ on the $m$-simplices of $G$ as follows:
$H_1 \sim_H H_2$ if $H_1 = H_2$, or there is an $H$-avoiding $(m,n)$ path from $H_1$ to $H_2$.

**4.2. Proposition.** $\sim_H$ is an equivalence relation.

**Proof.** The proof is the same as above.

**Definition.** Let $K$ be an $n$-simplex of $G$. Then $B$ is a branch of $K$ in $G$ if $B = G \cdot S$, where $S$ is an equivalence class of $\sim_K$.

**Note:** If $K$ is an $n$-simplex of $G$, since each $(m+1)$ subset of $K$ is in a separate equivalence class, it follows that $K$ has exactly $\binom{n+1}{m+1}$ branches.

**Definition.** Let $H$ be an $m$-simplex of $G$. $B$ is a branch of $H$ in $G$ if $B = G \cdot (S \cup E(H))$ where $S$ is an equivalence class of $\sim_H$ and $S \neq H$.

**Note:** If $H$ is an $m$-simplex of $G$, $H$ may have any number of branches.

**4.3. Proposition.** The branches of an $m$-simplex $(n$-simplex) $H$ partition $V(G) \setminus V(H)$. i.e. If $v \in V(G) \setminus V(H)$, then $v$ is in exactly one branch of $H$.

**Proof.** $v$ is contained in an $m$-simplex $H'$ of $G$. Since $H'$ is contained in an equivalence class $\sim_m$ ($\sim_n$ if $H$ is an
n-simplex), \( v \) is contained in at least one branch of \( H \). Suppose \( v \) is contained in another branch of \( H \). Then \( v \) is contained in another \( m \)-simplex \( H'' \) of \( G \). There is an \((m,n)\) path \( H' = H_1, K_1, \ldots, K_{l'}, H_{l'+1} = H'' \) in \( G \). Since \( G \) is \((m,n)\) simple, \( v \) is a vertex of each term in the \((m,n)\) path. Hence, this is an \( H \)-avoiding \((m,n)\) path, and \( H' \sim_H H'' \). But then \( H' \) and \( H'' \) are contained in the same branch of \( H \).

If \( B \) is a branch of an \( n \)-simplex \( K \), and \( H \) is the \( m \)-simplex of \( K \) contained in \( B \), we say \( B \) is attached to \( K \) at \( H \).

**Definition.** Let \( H \) be a subgraph of \( G \), and \( H' \) be a subgraph of \( G' \). If \( f: V(H) \to V(H') \) is a bijection, we say \( G \) and \( G' \) are isomorphic with respect to \( f \), written \( G \sim_f G' \) if there is an isomorphism \( \phi: V(G) \to V(G') \) such that \( \phi|_{V(G)} = f \). If \( H \) is a subgraph of \( G \) and \( G' \), we say \( G \) and \( G' \) are isomorphic with respect to \( H \), written \( G \sim_H G' \), if \( G \sim_f G' \) where \( f: V(H) \to V(H) \) is the identity function on \( H \). If \( G \) is an \((m,n)\)-tree, \( H \) is an \( m \)-simplex or \( n \)-simplex of \( G \), and \( B \) and \( B' \) are branches of \( H \), we say \( B \) and \( B' \) are isomorphic, written \( B \sim B' \), if there is a bijection \( f: V(H) \to V(H') \), such that \( B \sim_f B' \).

**Definition.** If \( H \) is a subgraph of \( H' \), we define the group on \( H \) induced by \( H' \), written \( G_{H,H'}(H) \) to be the group of permutations \( f: V(H) \to V(H) \) such that \( H' \sim_f H' \). We define the orbits of \( H \) with respect to \( H' \) to be the orbits of \( G_{H,H'}(H) \).
Definition. Let B be a branch of an n-simplex K of an
(m,n) tree G. If v(B) = n+1, B is said to be a trivial branch.
Otherwise, B is a non-trivial branch.

§5. Eccentricity, Radius and Centers of (m,n)-Trees.

Definition. We define the (m,n) eccentricity of an m-simplex
H of G to be \( e_{m,n}(H) = \max_{H'} d_{m,n}(H,H') \), where the maximum is
taken over all m-simplices \( H' \) of G. We define the (m,n) radius
of G to be \( r_{m,n}(G) = \min_{H} e_{m,n}(H) \), where the minimum is taken over
all m-simplices \( H \) of G. We define the (m,n) center of G, \( C(G) \),
to be the union of all m-simplices \( H \) of G for which \( r_{m,n}(G) =
\epsilon_{m,n}(H) \).

5.1. Proposition. Let G be an (m,n) tree. C(G) is either
an m-simplex or an n-simplex of G.

Proof. If \( G \approx K_{m+1} \) or \( G \approx K_{n+1} \), then certainly the theorem
is true for G. Suppose \( v(G) > n+1 \), and the theorem is true for
every (m,n) tree \( G' \) with fewer vertices. If we delete the vertices
of a single end-simplex, the resulting graph is still on (m,n) tree.
Then the same is true if we delete the vertices of every end simplex
of G, as this is the result of removing them one at a time. Denote
the resulting graph by \( G' \). Let \( H \) be an m-simplex of \( G' \).
Suppose \( d_{m,n}(H,H') = \epsilon_{m,n}(H) \) in G. Let \( H = H_1, K_1, \ldots, K_{\ell}, H_{\ell+1} = H' \),
where \( \ell = \epsilon_{m,n}(H) \). \( H' \) is not contained in another n-simplex, or
we could find a longer \((m,n)\) path in \(G\). Therefore \((K_{\ell'} - (K_{\ell'}-H_{\ell'})\)

is an end-simplex of \(G\), and \(e_{m,n}(H)\) in \(G'\) is one less than

in \(G\). Therefore, the set of \(m\)-simplices with minimum eccentricity

are the same in \(G'\) as in \(G\). Therefore, the theorem holds for

\(G\).

The following corollary shows that \(C_{m,n}(G)\) is the middle term

of a longest \((m,n)\) path in \(G\).

**Corollary.** Let \(G\) be an \((m,n)\) tree, and \(H_1, K_1, H_2, \ldots, H_{\ell'}, K_{\ell'}, H_{\ell'+1}\)
a longest \((m,n)\) path in \(G\). If \(\ell\) is odd \(C(G) = \frac{K_{\ell'+1}}{2}\), and if

\(\ell\) is even \(C(G) = \frac{H_{\ell'+2}}{2}\).

**Proof.** If \(G \sim K_{m+1}\), then \(H_1 = G\) is a longest \((m,n)\) path

with \(\ell = 0\). \(C_{m,n}(G) = G = H_1 = \frac{H_{\ell'+2}}{2}\). If \(G \sim K_{n+1}\), then if \(H_1\)

and \(H_2\) are \(m\)-simplices of \(G\), \(H_1, K_1 = G\), \(H_2\) is a longest \((m,n)\)

path of \(G\) with \(\ell = 1\). \(C_{m,n}(G) = G = K_1 = \frac{K_{\ell'+1}}{2}\).

Now suppose \(G\) is an \((m,n)\) tree, and the theorem is true for
every \((m,n)\) tree with fewer vertices. Let \(H_1, K_1, H_2, \ldots, H_{\ell'}, K_{\ell'}, H_{\ell'+1}\)
be a longest \((m,n)\) path in \(G\). Construct \(G'\) as in the lemma.

Then \(C_{m,n}(G) = C_{m,n}(G') \cdot K_1 - (K_1 - H_2)\), and \(K_{\ell'} - (H_{\ell'} - K_{\ell'})\) are end

simplices of \(G\). Therefore \(H_1, K_1, \ldots, H_{\ell'}, K_{\ell'}, H_{\ell'+1}\) where

\(H'_1 = H_{i+1}\), and \(K'_i = K_{i+1}\) for \(i = 1, \ldots, \ell-2\) and \(\ell' = \ell-2\) is an
(m,n) path in $G'$ of greatest length. $\ell$ and $\ell'$ are of the same parity. If $\ell$ is odd, $C_{m,n}(G) = C_{m,n}(G') = \frac{\ell' + 1}{2} = \frac{\ell - 1}{2} = \frac{\ell + 1}{2}$.

If $\ell$ is even, $C_{m,n}(G) = C_{m,n}(G') = \frac{\ell' + 2}{2} = \frac{\ell'}{2} = \frac{\ell + 2}{2}$.

**Definition.** Let $v \in EV(G)$ if $v$ is an end-vertex of a longest path in $G$, $v$ is called a peripheral vertex of $G$, otherwise $v$ is called a non-peripheral end-vertex of $G$. We denote the set of peripheral vertices of $G$ by $EV_p(G)$, and the set of non-peripheral end-vertices by $EV_n(G)$. If $B$ is a branch of $C(G)$, $B$ is called a peripheral branch of $C(G)$ if $B$ contains a peripheral vertex. $B$ is called a non-peripheral branch otherwise. We denote the union of the peripheral branches by $B_p(G)$, and the union of the non-peripheral branches by $B_n(G)$.

§6. **Degree of an m-Simplex and an n-Simplex.**

**Definition.** If $H$ is an m-simplex of an $(m,n)$ tree $G$, the degree of $H$ in $G$, written $deg_G H$ is the number of n-simplices of $G$ containing $H$. If $K$ is an n-simplex of $G$, the degree of $K$ in $G$, written $deg_G K$ is the number of m-simplices contained in $K$ which are contained in an n-simplex other than $K$.

The following proposition allows us to count the number of end vertices in a $k$-tree by knowing the degrees of the $k$-1, and $k$ simplices of $G$. 

6.1. Proposition. Let $G$ be a $k$-tree. $G \not\cong K_{k+1}$. Then

$$|EV(G)| = \Sigma (\deg_H - 2) + \Sigma (\deg_K - 2) + 2$$

$H$ a $k$-simplex $K$ a $k$-simplex

$\deg_H \geq 2$ $\deg_K \geq 2$

Proof. Let

$$f(G) = \Sigma (\deg_H - 2) + \Sigma (\deg_K - 2) + 2$$

$H$ a $k$-simplex $K$ a $k$-simplex

$\deg_H \geq 2$ $\deg_K \geq 2$

Suppose $G = H \cup \{u\} \cup \{v\}$, where $H \sim K_k$, and $u, v$ are adjacent to every vertex of $H$. Then $H$ is the only $(k-1)$-simplex of degree $\geq 2$, and $\deg_H = 2$. There are no $k$-simplices of degree $\geq 2$. So $f(G) = 2$, and $|EV(G)| = 2$. Suppose $G$ is a $k$-tree, and for every $k$-tree $G'$ with fewer vertices, the proposition is true. Let $u$ be an end-vertex of $G$, adjacent to every vertex of a $(k-1)$-simplex $H$. Let $G' = G - u$.

Case 1: $\deg_{G'}H = 1$.

Let $K$ be the unique $k$-simplex of $G'$ containing $H$.

Case a: $\deg_{G'}K = 1$.

In this case, one vertex of $K$ is an end-vertex of $G'$, but not of $G$. Then $|EV(G)| = |EV(G')|$. $\deg_H = 2$, $\deg_K = 2$, $k > 2$. 


\[ \deg_G H' = \deg_G H' \text{ for } H' \neq H, \text{ and } \deg_G K' = \deg_G K' \text{ for } K' \neq K. \]

Then \( f(G) = f(G') \).

**Case b:** \( \deg_G K \geq 2 \).

In this case, no vertex of \( K \) is an end-vertex of \( G' \). Then

\[ |EV(G)| = |EV(G')| + 1. \]

\[ \deg_G H = 2, \deg_G K = \deg_G K+1, \text{ and } \deg_G H' = \deg_G H \text{ for } H' \neq H, \text{ and } \deg_G K' = \deg_G K' \text{ for } K' \neq K. \]

Then \( f(G) = f(G') + 1 \).

**Case 2:** \( \deg_G H \geq 2 \).

Then no vertex of \( H \) is an end-vertex of \( G' \), so

\[ |EV(G)| = |EV(G')| + 1. \]

\[ \deg_G H = \deg_G H+1, \text{ and } \deg_G H' = \deg_G H \text{ for } H' \neq H, \]

and \( \deg_G K = \deg_G K \) for every \( k \)-simplex \( K \). Therefore \( f(G) = f(G') + 1 \).

**§7. Automorphisms.**

**7.1. Theorem.** Let \( G \) be a \( k \)-tree. Suppose \( \alpha \) is an automorphism which fixes every end-vertex of \( G \). Then if \( u \neq w \) but \( \alpha(u) = \alpha(w) \), then \( u \) and \( w \) are adjacent to every vertex of \( G \).

**Proof.** The proof will be by induction on \( v(G) \). If \( G = K_k \) or \( K_{k+1} \) or \( K_1 \cup f K_2 \) where \( K_1 \sim K_2 \sim K_{k+1} \), and \( f: H_1 \rightarrow H_2 \) where \( H_1 \) and \( H_2 \) are \((k-1)\)-simplices of \( K_1 \) and \( K_2 \), then the theorem is true for \( G \). Note that \( K_1 \cup f K_2 \) consists of a \( k \)-simplex \( K' \) with two vertices adjacent to each vertex of \( K' \).
Suppose \( v(G) = n > k+2 \), and for every \( k \) tree \( G' \) with \( v(G) < n \), the theorem is true. Let \( v \) be an end-vertex of \( G \). \( v \) is adjacent to every vertex of a \((k-1)\)-simplex \( H \).

Case 1: No vertex of \( H \) is an end-vertex of \( G-v \). Then \( \alpha|_{G-v} \) fixes every end-vertex of \( G-v \). Therefore if \( \alpha(u) = w \) and \( \alpha(w) = u \), \( u \) and \( w \) are adjacent to every vertex of \( G-v \).

Therefore, \( u \) and \( w \) are adjacent to each other and to every vertex of \( H \). If \( (u,w) \cap V(H) = \emptyset \), then \( H \cup \{u,w\} \) is a \((k+1)\)-simplex of \( G \). This is impossible. Therefore, we may assume \( u \in V(H) \).

Since \( \alpha \) is an automorphism, and \( (u,v) \in E(G) \), then \( (\alpha(u),v) \in E(G) \).

Therefore, \( (u,w) \subseteq V(H) \), and so \( u \) and \( w \) are adjacent to \( v \) and so to every vertex of \( G \).

Case 2: There is a vertex \( x \) in \( H \) which is an end-vertex of \( G-v \). Since \( G \neq K_{k+1} \) or \( K_1 \cup K_2 \), there is only one such \( x \).

Hence \( x \) is the only vertex adjacent to \( v \) with degree \( k+1 \). So \( \alpha(x) = x \). But then \( \alpha|_{V(G)\setminus\{v\}} \) fixes every end-vertex of \( G-v \).

So \( u, w \) must be vertices of \( H \), and are adjacent to every vertex of \( G \).

§8. Some Results on Graph Reconstruction.

We now proceed to prove some results dealing with Graph Reconstruction. A very useful result is due to Kelly [69]. Let \( S(F,G) \) be the number of subgraphs of \( G \) isomorphic to \( F \).
8.1. Lemma. Let $G'$ be a vertex reconstruction of $G$. If $\nu(F) < \nu(G)$, then $S(F, G) = S(F, G')$.

Proof. Let $f$ be a bijection $f: V(G) \to V(G')$ such that $G-v \sim G'-f(v)$ for every $v \in V(G)$. Since $\nu(F) < \nu(G)$ a given subgraph isomorphic to $F$ occurs in $G-v$ for $\nu(G) - \nu(F)$ vertices of $G$. Therefore

$$S(F, G) = \sum_{v \in V(G)} S(F, G-v)/(\nu(G) - \nu(F))$$

$$S(F, G') = \sum_{v \in V(G)} S(F, G'-f(v))/(\nu(G') - \nu(F))$$

But $S(F, G-v) = S(F, G'-f(v))$, and $\nu(G) - \nu(F) = \nu(G') - \nu(F)$. Therefore $S(F, G) = S(F, G')$.

Corollary. If $G$, $G'$, and $f$ are as in the lemma, then the number of subgraphs of $G$ isomorphic to $F$ containing a vertex $v$ is equal to the number of subgraphs of $G'$ isomorphic to $F$ containing $f(v)$.

Proof. This number is just $S(F, G) - S(F, G-v) = S(F, G') - S(F, G'-f(v))$.

Corollary. If $G$, $G'$, and $f$ are as in the lemma, then $\deg_G v = \deg_{G'} f(v)$.

Proof. This is just the number of subgraphs $F \sim K_2$ containing $v$ and $f(v)$. 
Analogously, we have the proposition:

8.2. **Proposition.** If $G'$ is an edge reconstruction of $G$, and $\varepsilon(F) < \varepsilon(G)$, $S(F, G) = S(F, G')$.

Proof. As above, a given subgraph isomorphic to $F$ occurs in $G-e$ for $\varepsilon(G)-\varepsilon(F)$ edges of $G$. So

$$S(F, G) = \sum_{e \in E(G)} S(F, G-e)/\varepsilon(G)-\varepsilon(F)$$

$$S(F, G') = \sum_{e \in E(G)} S(F, G-g(e)/\varepsilon(G')-\varepsilon(F)$$

Since $S(F, G-e) = S(F, G'-g(e))$ for every $e$, and $\varepsilon(G) - \varepsilon(F) = \varepsilon(G') - \varepsilon(F)$, $S(F, G) = S(F, G')$.

8.3. **Theorem.** Let $G$ and $G'$ be graphs. $G'$ is a vertex reconstruction of $G$ if and only if for any graph $F$ with $\nu(F) < \nu(G)$, $S(F, G) = S(F, G')$.

Proof. Necessity follows from the previous proposition. The subgraphs of $G$ obtained by deleting one vertex are precisely the maximal subgraphs with $\nu(G)-1$ vertices. It will be sufficient to show that $G$ and $G'$ have the same collection of edge maximal subgraphs with $\nu(G)-1$ vertices. Let $\mathcal{F}_1$ be the collection of all graphs with $\nu(G)-1$ vertices, and $i$ edges, for $i = 0, \ldots, \binom{\nu(G)-1}{2}$. If $F \in \mathcal{F}_{\binom{\nu(G)-1}{2}}$, and $S(F, G) \geq 1$, then
F is a maximal subgraph of G and G' with v(G)-1 vertices. Suppose for j = i,...,(v(G)-1) , G and G' have the same collection of maximal subgraphs contained in \( \mathcal{F}_j \). If F is a graph in \( \mathcal{F}_j \), since every subgraph of G in \( \mathcal{F}_j \) is contained in a unique maximal subgraph of G, F occurs as a maximal subgraph of G exactly \( S(F,G) - \sum_{j=1}^{(v(G)-1)} S(F,F') \) times, where the sum is taken over all F' which are maximal subgraphs of G in
\[
(\binom{v(G)-1}{2}) \cup \bigcup_{j=1}^{v(G)-1} \mathcal{F}_j.
\]

Since \( S(F,G) = S(F,G') \), then G and G' have the same maximal subgraphs in \( \mathcal{F}_j \). Therefore G and G' have the same collection of subgraphs obtained by deleting one vertex, and G' is a vertex reconstruction of G.

The following Corollary is due to Greenwell [50].

**Corollary.** If G is vertex reconstructible and has no isolated vertices, then G is edge reconstructible.

**Proof.** Suppose G' is an edge reconstruction of G. By Kelly's Lemma, \( S(F,G) = S(F,G') \) for every F with \( e(F) < e(G) \). Now since G has no isolated vertices, if \( v(F) < v(G) \), \( e(F) < e(G) \). Therefore, \( S(F,G) = S(F,G') \) for every graph F with \( v(F) < v(G) \), and G' is a vertex reconstruction of G. But then G \( \sim G' \).
CHAPTER II

SOME RESULTS ON RECONSTRUCTION OF (M, N)-TREES

§9. Reconstruction of a k-Tree with a Labeled (k-1)-Simplex.

The following lemma is extremely important in what follows.

9.1. Lemma. Let G and G' be k-trees. Suppose K is a (k-1)-simplex of G, and K' is a (k-1) simplex of G'. Let $E(V(G)) = L \cup U \cup (E(V(G)) \cap K)$ be a partition of $E(V(G))$. We call $L$ the set of labeled end vertices of G, and $U$ the set of unlabeled end vertices of G. Let $g$ be an injection $g: K \cup L \to K' \cup E(V(G))$ such that $g(K) = K'$. If there is a bijection $f: U \to E(V(G')) \setminus g(K \cup L)$, having the property that for every $v \in U$, there is an isomorphism $\phi_v: G-v \to G'-f(v)$ such that $\phi_v|_{K \cup L} = g$, and if $|u| \geq 2$, there is an isomorphism $\phi: G \to G'$ such that $\phi|_{K \cup L} = g$.

Proof. The proof is by induction on $v(G)$. For simplicity, we will denote $f(v)$ by $v'$. If the conditions of the theorem are satisfied by $G$, then $v(G) \geq k+2$. Suppose $v(G) = k+2$. Then $L = \emptyset$, and $|U| = 2$. Then if $U = \{u_1, u_2\}$, $u_1$ and $u_2$ are adjacent to every vertex of $K$. Since $G-u_i \cong G'-u_i'$ with an isomorphism $\phi_{u_i}$ such that $\phi_{u_i}|_{K} = g$, then $u_i'$ is adjacent to every vertex of $K'$. It follows that $f \cup g$ is an isomorphism from $G$ to $G'$.  

26
Suppose \( v(G) = n \), and for every pair of \( k \)-trees \( H \) and \( H' \) with fewer vertices, the theorem is true. Let \( b \) be the number of branches of \( K \) in \( G \).

Case 1: \( b \geq 2 \).

Let \( B_1, B_2, \ldots, B_b \) be the branches of \( K \) in \( G \). Suppose that for some branch \( B_i \) of \( K \), \( V(B_i) \cap U = \emptyset \). Let \( G_i = G \setminus (B_i \setminus K) \).

Then \( v(G_i) < v(G) \). Suppose \( v \in U \). Then \( \phi_i = \phi_v |_{G \setminus (B_i \setminus K)} \) is an isomorphism from \( G \setminus (B_i \setminus K) \) to \( G' \setminus (\phi_v(B_i) \setminus K') \) such that \( \phi_i |_{K \cup L \setminus (L \cap B_i)} = \phi |_{K \cup L \setminus (L \cap B_i)} \cdot \)

Then if we define the labeled vertices of \( G_i \) to be \( L \setminus V(B_i) \) and the unlabeled vertices of \( G_i \) to be \( U \), then \( G_i \) satisfies the conditions of the theorem. By the induction hypothesis, there is an isomorphism \( \phi_i \) from \( G_i \) to \( G' \setminus (\phi(B_i) \setminus K') \) such that \( \phi_i |_{K \cup (L \setminus B_i)} = \phi |_{K \cup (L \setminus B_i)} \cdot \)

Now let \( \phi = \phi_i \cup \phi_v |_{B_i} \). Then \( \phi \) is an isomorphism from \( G \) to \( G' \), such that \( \phi |_{K \cup L} = \emptyset \).

Suppose that for every branch \( B_j \) of \( K \), \( V(B_j) \cap U \neq \emptyset \), and for some branch \( B_i \) \( V(B_i) \cap L \neq \emptyset \). Let \( u \in U \cap V(B_i) \) and \( v \in U \cap V(B_j) \) where \( i \neq j \). Then \( \phi = \phi_u |_{G \setminus (B_i \setminus K)} \cup \phi_v |_{B_i} \) is an isomorphism from \( G \) to \( G' \) such that \( \phi |_{L \cup K} = \emptyset \).

Suppose now that \( L = \emptyset \). We wish to show that \( G' \) has the same branches as \( G \). i.e. \( K' \) has \( b \) branches in \( G' \), \( B_1', \ldots, B_b' \) and there is a permutation \( \mu: \{1, 2, \ldots, b\} \rightarrow \{1, 2, \ldots, b\} \) such that
In that case $G \sim G'$. We may assume that $\nu(B_1) \leq \nu(B_2) \leq \ldots \leq \nu(B_b)$. Suppose $\nu(B_i) = k+1$, for $i = 1, \ldots, b$. If $\nu(B'_i) > k+1$, let $v'$ be an end vertex of $G'$ in $B'_i$, and let $u' \neq v'$ be an end vertex of $G'$, not in $B'_i$. Then $G'-u'$ has a branch with $> k+1$ vertices, but $G-u$ does not. Therefore $\nu(B'_i) = k+1$ for every $i$, and since $\nu(G) = \nu(G')$, $G$ and $G'$ have the same branches.

Otherwise, $\nu(B_b) > k+1$. Let $v \in EV(B_b)$. $\phi_v(B_b)$ is a branch of $K'$ with $> k+1$ vertices. Let $u \in EV(B_b)$. $G-u$ has $b$ branches of $K$. Consequently, $G'-u'$ has $b$ branches of $K'$, and $G'$ has no fewer branches of $K'$, than $G$ has of $K$. Since $\nu(\phi(B_b)) > k+1$, by symmetry, $G$ has no fewer branches of $K$ than $G'$ has of $K'$. Therefore $K'$ has $b$ branches in $G'$.

Suppose $\nu(B_1) = k+1$, and $v \in EV(B_1)$. Then $K$ has $b-1$ branches in $G-v$, as does $K'$ in $G'-v'$. Therefore $v'$ is adjacent to every vertex of $K'$. Consequently, $\phi = \phi_v \cup (v,v')$ is an isomorphism from $G$ to $G'$, such that $\phi_{|K \cap L} = \varphi$.

Suppose $\nu(B_i) > k+1$, $i = 1, \ldots, b$. If $\nu(B'_i) = k+1$, let $v' \in EV(G') \cap V(B'_i)$. $K'$ has $b-1$ branches in $G'$, while $K$ has $b$ branches in $G-v$. Therefore, $\nu(B'_i) > k+1$, for $i = 1, \ldots, b$.

Suppose $\nu(B_i) = k_i$, $i = 1, \ldots, b$, and let $v \in EV(B_i)$. $K$ has a branch in $G-v$ with $k_i$ vertices. So $K'$ has a branch $B'$ in $G-v'$ with $k'_i$ vertices. $B'$ is not a branch of $K'$ in $G'$, or else for $u' \in EV(G') \cap V(B') \setminus K'$, $K'$ has a branch in $G'-u'$.
with \( k_1 - 2 \) vertices, as would \( K \) in \( G - u' \), which is impossible. Therefore, \( \phi_v(B_2), \ldots, \phi_v(B_b) \) are branches of \( K' \) in \( G' \), and \( \phi_v|_{B_1}: B_1 \to \phi_v(B_1) \) is an isomorphism such that \( \phi_v|_K = g \). We can relabel the branches of \( K' \) in \( G' \) so that \( B'_1 = \phi_v(B_1) \) for \( i = 2, \ldots, b \).

Let \( m_1 \) be the number of branches of \( K \) in \( G \) with \( k_1 \) vertices, \( m_2 \) the number with \( k_1 + 1 \) vertices, and \( m_3 \) the number with \( \geq k_1 + 2 \) vertices. Since \( v(B_1) = v(B'_1) \), we have \( v(B_i) = v(B'_i) \) \( i = 1, \ldots, b \).

Case a: \( m_3 > 0 \).

Let \( u \in EV(B_1) \). \( K \) has \( m_1 \) branches in \( G - u \) with \( k_1 \) vertices, and \( m_2 + m_3 \) branches with \( \geq k_1 + 1 \) vertices, as does \( K' \) in \( G' - v' \). Consequently, \( B'_1, \ldots, B'_b \) are branches of \( K' \) in \( G' \), and \( \phi_u|_{B_1 \cup \ldots \cup B_{m_1}} \cup \phi_v|_{B_{m_1+1} \cup \ldots \cup B_b} \) is an isomorphism of the desired type.

Case b: \( m_3 = 0 \), \( m_2 > 0 \).

If \( |EV(B_1 \cup \ldots \cup B_{m_1})| \geq 2 \), then let \( H = B_1 \cup \ldots \cup B_{m_1} \) and \( H' = B'_1 \cup \ldots \cup B'_b \). For every \( u \in EV(H) \), \( K' \) has a branch in \( G' - u' \) with \( k_1 - 1 \) vertices, and so \( u' \in EV(H') \). Therefore, \( H \) and \( H' \) satisfy the hypothesis of the theorem. Since \( v(H) < v(G) \), \( H \sim g H' \). For \( u \in EV(H) \), \( \phi_u(B_{m_1+1} \cup \ldots \cup B_b) = B'_{m_1+1} \cup \ldots \cup B'_b \)
and so \( G \sim G' \). Suppose \( |EV(B_1 \cup \ldots \cup B_m)| = 1 \). Then \( m_1 = 1 \).

Let \( u_1 \in EV(B_1) \). \( \phi_{u_1}(B_2 \cup \ldots \cup B_b) = B_2' \cup \ldots \cup B_b' \), and so 
\[
B_2 \cup \ldots \cup B_b \sim B_2' \cup \ldots \cup B_b'.
\]
If we can show that \( B_1 \sim B_1' \), we are done. For every \( u \in EV(B_2 \cup \ldots \cup B_b) \), \( K \) and \( K' \) have at least one branch in \( G-u \) and \( G'-u' \) with \( k_1 \) vertices. If \( B_1 \not\sim B_1' \) then \( B_1-u \sim B_1' \) and \( B_1 \sim B_1'-u' \) for every \( u \in B_1 \), for each \( i = 2, \ldots, b \). But then if \( B_2 \sim B_2' \), \( B_2-u \sim B_2'-u' \) and so 
\[
B_1 \sim B_1'.
\]

Case c: \( m_2 = m_3 = 0 \).

Suppose \( B_i \sim K B_j \) for every \( i, j \). If \( b \geq 3 \) and \( B_i \not\sim K, B_j' \), then let \( h \neq i, j \), and \( u' \in EV(B_h) \). Then \( \phi_{u_1}^{-1}(B_i') \not\sim \phi_{u_1}^{-1}(B_j) \) which contradicts our assumption. Therefore \( B_1 \sim B_1' \sim K, B_i \) for \( i = 2, \ldots, b \). If \( b = 2 \), then let \( u_1 \in EV(B_1) \) and \( u_2 \in EV(B_2) \). Then \( B_1 \sim \phi_{u_1}(B_1) \), and \( B_2 \sim \phi_{u_2}(B_2) \), so \( B_1 \sim B_1' \sim K, B_2' \), and so \( G \sim G' \).

Otherwise, for a branch \( B_i \), the number of branches \( B \) such that \( B \sim K B_i \) is the maximum number of such branches of \( K \) in \( G-u \) for all \( u \in EV(G) \). But this is the number of branches \( B' \) of \( K' \) in \( G' \) such that \( B_i \sim B' \). We conclude that \( G \sim G' \).
§10. Neighborhood Contraction Reconstruction.

The following theorem was originally proved by Harary and Palmer [60].

10.1. Theorem. Let $G$ and $G'$ be trees. Suppose there is a bijection $f: \text{EV}(G) \to \text{EV}(G')$ such that for $v \in \text{EV}(G)$, $G-v \cong G'-f(v)$, then $G \cong G'$.

Proof. For $v \in \text{EV}(G)$, we denote $f(v)$ by $v'$. If $v \in \text{EV}(G)$, $\nu(G) = \nu(G-v) + 1 = \nu(G-v') + 1 = \nu(G')$. Therefore $\epsilon(G) = \epsilon(G')$ as well.

Suppose $|\text{EV}(G)| = 2$. Then $G$ and $G'$ are paths of length $\epsilon(G) = \epsilon(G')$, and so $G \cong G'$.

Suppose $|\text{EV}(G)| = 3$, $\text{EV}(G) = \{v_1, v_2, v_3\}$. Then $G$ and $G'$ each have a vertex of degree 3, say $v_0$ and $v_0'$. If $\deg_{G-v_0} v_0 = 3$ for each $v \in \text{EV}(G)$, then $\phi_v(v_0) = v_0'$ for each $v \in \text{EV}(G)$. Define $g(v_0) = v_0'$, and we may apply Lemma 9.1 and conclude that $G \cong G'$. If $v_1$ is adjacent to $v_0$, but $v_2$ and $v_3$ are not, then for $i = 2, 3$ there is an isomorphism $\phi_{v_1}' = G-v_1 \to G'-v_1'$ such that $\phi_{v_1}'(v_1) = v_1'$. Define $g(v_1) = v_1'$, and we may apply Lemma 9.1 and conclude that $G \cong G'$. If $v_1$ and $v_2$ are adjacent to $v_0$, but $v_3$ is not, then $G-v_1$ is a path for $i = 1, 2$ with $v_3$ as an end-vertex. Then there is an isomorphism $\phi_{v_1}' = G-v_1 \to G'-v_1'$, such
Case 2: $b = 1$

By symmetry, if $K'$ has $\geq 2$ branches in $G'$, then $K$ has $\geq 2$ branches in $G$. Hence $K'$ has 1 branch in $G'$. Let the vertices of $K$ be $v_1, \ldots, v_k$, $v$ the vertex of $G$ adjacent to each vertex of $K$, and $w$ the vertex of $G'$ adjacent to each vertex of $K'$. Define $B_i$ to be the branch of $K \cup \{v\}$ at $K \cup \{v\} - \{v_i\}$, and $B'_i$ the branch of $K' \cup \{w\}$ at $K \cup \{w\} - \{v_i\}$ for $i = 1, \ldots, k$. If $u \in EV(G)$, $\phi_u(v) = w$, and so $\phi_u(B_i - u) = B'_i - w$ for $i = 1, \ldots, k$.

Suppose 2 branches of $K \cup \{u\}$, $B_i$ and $B_j$, are non-trivial, and $v(B_i) = k_i$, $v(B_j) = k_j$. Let $u_i \in EV(B_i)$, and $u_j \in EV(B_j)$. Then $v(\phi_{u_i}(B_i)) = k_i - 1$, $v(\phi_{u_j}(B_i)) = k_i$, and $v(\phi_{u_j}(B_j)) = k_j - 1$. Consequently, $u_i' \in V(B_i')$ and $u_j' \in V(B_j')$. So $\phi = \phi_{u_i}|(G-B_i) \cup K \cup \phi_{u_j}|B_i$ is an isomorphism such that $\phi|K = g$.

Suppose only one branch $B_i$ is non-trivial. Consider the $k$-tree $B_i$. We can extend $g'|_{B_i}$ to $g'$, so that $g'(v) = w$, and $g'|_{K-\{u_i\}} = g$. $B_i$ satisfies the hypothesis of the theorem, and since $v(B_i) = v(G) - 1$, by the inductive hypothesis, there is an isomorphism $\phi': B_i \to B'_i$ such that $\phi'|_{B_i \cup \{v\} - \{u_i\}} = g'$. Then $\phi = \phi' \cup \{(u_i, u_i')\}$ is an isomorphism from $G$ to $G'$ such that $\phi|K = g$. 
that \( \phi_i^*(v) = v' \) for \( i = 1, 2 \). Define \( g(v) = v' \), and we may apply Lemma 9.1 and conclude that \( G \sim G' \). If all three are adjacent to \( v \), then \( G \sim K_{1,3} \sim G' \).

Suppose \( |EV(G)| \geq 4 \). The length of a longest path in \( G \) is equal to the length of a longest path in any subgraph \( G-v \). But the same is true for \( G' \). Therefore, the paths of greatest length in \( G \) and \( G' \) are of the same length. Consequently \( G \) and \( G' \) both have a 1-center or both have a 2-center.

Case 1: \( G \) has a 1-center, say \( v \). Let \( v' \) be the center of \( G' \).

If \( v \) has \( \geq 3 \) peripheral branches in \( G \), then \( G-v \) has a path of length \( e_{0,1}(G) \), for every \( v \). Consequently, \( \phi_v(v) = v' \). Define \( g(v) = v' \), and we may apply Lemma 9.1 and conclude that \( G \sim G' \).

If \( v \) has 2 peripheral branches in \( G \), and each has \( \geq 2 \) peripheral vertices, then \( \phi_v(v) = v' \) for each \( v \in EV(G) \), and as above we may conclude that \( G \sim G' \).

Suppose \( v \) has 2 peripheral branches in \( G \), and \( |EV(B_n(G))| \geq 2 \). Recall that \( B_n(G) \) is the union of the non-peripheral branches of \( G \), and \( B_p(G) \) is the union of the peripheral branches of \( G \).

If \( v \in EV(B_n(G)) \), then \( \nu(B_p(G)) = \nu(B_p(G-v)) = \nu(B_p(G'-v')) \leq \nu(B_p(G')) \). By symmetry \( \nu(B_p(G')) \leq \nu(B_p(G)) \). Therefore
\[ \nu(B_p(G)) = \nu(B_p(G')) , \text{ and } \nu' \in EV(B_n(G')) . \] Since \( \phi_{\nu}(v_0) = v'_0 \), we may define \( g(v_0) = v'_0 \), and use Lemma 9.1 to conclude that \( B_n(G) \simeq B_n(G') \). Since \( \phi_{\nu}(B_p(G)) = B_p(G') \), \( B_p(G) \simeq B_p(G') \), and \( G \simeq G' \).

Suppose \( v_0 \) has 2 peripheral branches in \( G \), and \( |EV(B_n(G))| = 1 \). Let \( v_n \in EV(B_n(G)) \). \( \nu(B_p(G)) = \nu(B_p(G-v_n)) = \nu(B_p(G'-v_n)) \).

There is a vertex \( v_p \in EV(B_p(G)) \) such that \( G-v_p \) has a longest path. \( \nu(B_p(G'-v'_p)) = \nu(B_p(G-v_p)) = \nu(B_p(G)) - 1 \). Therefore \( \nu(B_p(G)) = \nu(B_p(G')) \), and \( v_n \in EV(B_n(G')) \). Since \( \phi_{\nu}(B_p(G)) = B_p(G') \), \( B_p(G) \simeq B_p(G') \). Since \( \nu(G) = \nu(G') \) and \( |EV(G)| = |EV(G')| \), \( B_n(G) \simeq B_n(G') \), and \( G \simeq G' \).

Suppose \( v_0 \) has 2 peripheral branches in \( G \), and no non-peripheral branches. If \( v'_0 \) had non-peripheral branches in \( G \), by symmetry, and the above case, so would \( v_0 \).

Let the branches of \( v_0 \) be \( B_1 \) and \( B_2 \), with \( \nu(B_1) \leq \nu(B_2) \).

Suppose \( |EV(B_1)| = 1 \). Since \( |EV(G)| \geq \frac{1}{4} \), \( \nu(B_2) \geq \nu(B_1) + 2 \).

Let \( v_1 \in EV(B_1) \). \( G'-v' \) has a 2-center \( (x',y') \) with branches \( B'_x, B'_y \) where \( x' \in V(B'_x), y' \in V(B'_y) \), and \( B'_x \) has one end-vertex. Since \( G'-v'_1 \) does not contain a longest path, \( v'_1 \) is a peripheral vertex contained in a branch \( B' \) of \( v'_0 \). If \( B'-v' = B'_x \), then \( v'_0 = x \), and \( v'_0 \) has a branch in \( G' \) with one end-vertex. If \( B'-v' = B'_y \), then \( v'_0 = y \), and \( v'_0 \) has a branch in \( G' \) with one end-vertex.
Let $B'_1$ and $B'_2$ be the branches of $v'_0$ in $G'$ where $|EV(B'_1)| = 1$, and $u'_1 \in EV(B'_1)$. If $|EV_p(B'_2)| \geq 2$, then for every $v \in EV(B'_2)$, $\phi_v(v'_0) = v'_0$, $\phi_v(v'_1) = u'_1$. Then by letting $L = \{v'_1\}$, and $g = \{(v'_0, v'_0), (v'_1, u'_1)\}$, we may conclude by Lemma 9.1 that $G \sim G'$. If $|EV_p(B'_2)| = 1$, then since $|EV_p(G)| = |EV_p(G')|$, $|EV_p(B'_2)| = 1$. Let $v_2 \in EV_p(B'_2)$, and $u'_2 \in EV_p(B'_2)$. By letting $L = \{v'_1, v'_2\}$ and defining $g = \{(v'_1, u'_1), (v'_2, u'_2), (v'_0, v'_0)\}$, we may again conclude by Lemma 9.1 that $G \sim G'$.

Suppose $B_1$ and $B_2$ have at least two end-vertices. By symmetry, the branches of $v'_0$ have at least two end-vertices. Let $v_1 \in EV(B_1)$, such that $G-v_1$ contains a longest path, and $v_2 \in EV(B_2)$ such that $G-v_2$ contains a longest path. $v'_0$ has branches with $v(B_1) - 1$, $v(B_2)$ vertices in $G'-v'_1$. Therefore, if $B'_1$ and $B'_2$ are the branches of $v'_0$ in $G'$, with $v(B'_1) \leq v(B'_2)$, $v(B_2) \leq v(B'_2)$. By symmetry, we may conclude that $v(B'_1) \leq v(B_2)$, $v(B_2) = v(B'_2)$, $v(B_1) = v(B'_1)$.

Suppose $v(B_1) < v(B'_2) - 1$. Then $\phi_{v_1}(B_2) = B'_2$ and $\phi_{v_2}(B_1) = B'_1$. So $B'_1 \sim B_1$, $B'_2 \sim B_2$, and $G \sim G'$.

Suppose $v(B_1) = v(B'_2) - 1$. $\phi_{v_1}(B_2) = B'_2$, so $B'_2 \sim B_2$. If $v \in EV(B_2)$ such that $G-v$ contains a longest path, then $v'_0$ has branches with $v(B'_1)$, $v(B_2) - 1$ vertices in $G'-v'$ and so $v' \in EV(B'_1)$. If for some $v \in EV(B_2)$, $\phi_v(B_1) = B'_1$, then $B_1 \sim B'_1$, and $G \sim G'$. If not then for every $v$, $B'_1 \sim B_1$, $v \sim B'_1$-v \sim B_1$, and again $G \sim G'$. 
Suppose \( v(B_1) = v(B_2) \). If \( B_1 \not\sim B_2 \), then since \( \phi_{v_1}(B_2) \) and \( \phi_{v_2}(B_1) \) are branches of \( v'_0 \) in \( G' \), \( G \sim G' \). Suppose \( B_1 \sim B_2 \). If \( B_1 \not\sim B_2 \), by symmetry \( B_1 \not\sim B_2 \). Therefore \( \phi_{v_1}(B_2) \sim B_2 \sim B_1 \sim \phi_{v_2}(B_1) \), and \( G \sim G' \).

Case 2: \( G \) and \( G' \) have 2-centers.

Let \( B_1 \) and \( B_2 \) be the branches of the center of \( G \), \( B_1' \) and \( B_2' \) the branches of the center of \( G' \), and \( v(B_1) \leq v(B_2) \) and \( v(B_1') \leq v(B_2') \).

Suppose \( |EV(B_1)| \geq 2 \) and \( |EV(B_2)| \geq 2 \). If \( v_1 \in EV(B_1) \) such that \( G-v_1 \) contains a longest path, and \( v_2 \in EV(B_2) \) such that \( G-v_2 \) contains a longest path, then the branches of the center in \( G'-v_1 \) contains \( v(B_1)-1 \) and \( v(B_2) \) vertices. Therefore \( v(B_2) \leq v(B_2') \). By symmetry, \( v(B_2') \leq v(B_2) \). And so \( v(B_2) = v(B_2') \), and \( v(B_1) = v(B_1') \).

If \( v(B_1) < v(B_2)-1, \phi_{v_1}(B_2) = B_2', \) and \( \phi_{v_2}(B_1) = B_1' \), and so \( B_1 \sim B_1', B_2 \sim B_2', \) and \( G \sim G' \).

If \( v(B_1) = v(B_2)-1 \), then \( \phi_{v_1}(B_2) = B_2' \). If \( v \in EV(B_2) \) such that \( G-v \) contains a longest path, then \( v_0' \) has branches with \( v(B_1) \), and \( v(B_2)-1 \) vertices in \( G'-v' \), so \( v' \in EV(B_2') \). If for some such \( v \), \( \phi_{v}(B_1) = B_1' \), then \( B_1 \sim B_1' \) and \( G \sim G' \). If not then for all such \( v \), \( B_1' \sim B_2-v \sim B_2'-v' \sim B_1 \), and \( G \sim G' \).
Suppose \( \nu(B_1) = \nu(B_2) \). If \( B_1 \not\sim B_2 \), then \( \phi_{\nu_1}(B_2) \) and \( \phi_{\nu_2}(B_1) \) are the branches of \( \nu_0 \), and \( G \simeq G' \).

Suppose \( B_1 \sim B_2 \). Then if \( B_1 \not\sim B_2' \), by symmetry \( B_1 \not\sim B_2 \).

So \( B_1 \sim B_2' \) and if \( \phi_{\nu_2}(B_2) = B_2' \), \( B_1 \sim B_2' \sim B_2 \sim B_1 \) and \( G \simeq G' \).

If \( |EV(B_1)| = 1 \), then by symmetry, and the above cases, \( |EV(B_1')| = 1 \). Let the end vertex of \( B_1 \) be \( w \), that of \( B_1' \) be \( w' \).

If \( |EV_p(B_2)| \geq 2 \), then for any \( v \in EV(B_2) \), \( \phi_{\nu}(w) = w' \), and by defining \( g = \{(w,w')\} \) we may conclude by Lemma 9.1 that \( G \simeq G' \). If \( |EV_p(B_2)| = 1 \), then since \( |EV_p(G)| = |EV_p(G')| \), \( |EV_p(B_2')| = 1 \). Let \( w_2 \in EV(B_2) \), and \( w'_2 \in EV(B_2') \). By defining \( L = \{w_2\} \) and \( g = \{(w,w'),(w_2,w'_2)\} \) we again conclude by Lemma 9.1 that \( G \simeq G' \). This concludes the proof of the theorem.

Let \( G \) be a graph, and \( v \in V(G) \). The neighborhood of \( v \) in \( G \) is defined to be \( N_G(v) = \{v' \in V(G) : \{v,v'\} \in E(G)\} \). We define the graph \( G*v \), called the neighborhood contraction of \( G \) at \( v \), as follows:

\[
V(G*v) = V(G) \cup (N_G(v) \setminus \{v\}) \cup N_G(v).
\]

\[
E(G*v) = \{e : e = \{v_1,v_2\} \text{ where } \{v_1,v_2\} \subseteq V(G) \setminus \{v\} \cup N_G(v) \text{ or } e = \{N_G(v),v_1\} \text{ where there is a } v_2 \in N_G(v) \text{ and } \{v_1,v_2\} \in E(G)\}.
\]
We may ask if the following is true: Let $G$ and $G'$ be graphs. If there is a bijection $f: V(G) \to V(G')$ such that $G* v \cong G'* f(v)$ for every $v \in V(G)$, then $G \cong G'$.

Although this appears to be difficult, we can draw certain conclusions about $G$ and $G'$.

10.2. Proposition. $v(G) = v(G')$.

Proof. $f$ is a bijection.

10.3. Proposition. $\deg_G v = \deg_{G'} f(v)$.

Proof. $\deg_G v = v(G) - v(G*v) = v(G') - v(G'* f(v)) = \deg_{G'} f(v)$.

10.4. Proposition. $\epsilon(G) = \epsilon(G')$.

Proof. $\epsilon(G) = \frac{1}{2} \sum_{v \in V(G)} \deg_G v = \frac{1}{2} \sum_{v' \in V(G')} \deg_{G'} v' = \epsilon(G')$.

10.5. Proposition. If $G$ is connected, then so is $G'$.

Proof. For any $v \in V(G)$, the number of components of $G*v$ is the same as of $G$.

10.6. Theorem. Suppose $G$ is a tree, and $G'$ is a graph. If there is a bijection $f: V(G) \to V(G')$, such that $G* v \cong G'* f(v)$, then $G \cong G'$.

Proof. Since $G'$ is connected, $v(G') = v(G)$, and $\epsilon(G') = \epsilon(G)$, $G'$ is a tree. Suppose $v \in EV(G)$. Since $|N_G(v)| = 1$, $G* v \cong G-v$. 
Therefore $f|_{EV(G)}$ is a bijection from $EV(G)$ to $EV(G')$ such that $G - v \cong G' - f(v)$. By the previous theorem, $G \cong G'$. 

§11. Vertex Reconstructibility if $n-m \geq 2$.

11.1. Theorem. Suppose $G$ is an $(m,n)$-tree, with $n-m \geq 2$. If $G'$ is a graph and $f: v(G) \to V(G')$ is a bijection such that for each $v \in V(G)$, there is an isomorphism $\phi_v: G-v \to G'-f(v)$ then $G \cong G'$.

Proof. For simplicity we write $v'$ for $f(v)$. If $G \cong K_{m+1}$, then $G' \cong K_{m+1}$, and $G \cong G'$.

Suppose $\nu(G) > m+1$. Let $v \in EV(G)$. $v \in V(K)$ where $K$ is an end-simplex of $G$, each vertex of which is adjacent to every vertex of an $m$-simplex $H$ and no others. $K \cup H\setminus\{v\}$ is the only $(n-1)$-simplex of $G-v$ which is not contained in an $n$-simplex of $G$. $\phi_v(K \cup H\setminus\{v\})$ is the only $(n-1)$-simplex of $G-v'$ which is not contained in an $n$-simplex of $G-v'$. By Kelly's Lemma, $\deg_G v = \deg_G v' = n$, $v$ and $v'$ are each contained in exactly one $n$-simplex, and for any vertex $w \in V(G)$, $\deg_G w = \deg_G w' \geq n$. $v'$ is adjacent to each vertex of $\phi(K \cup H\setminus\{v\})$. But then $\phi_v \cup \{(v,v')\}$ is an isomorphism from $G$ to $G'$. 
§12. Edge Reconstructibility of $(m,n)$-Trees.

We now proceed to show that if $G$ is an $(m,n)$-tree, $G$ is edge reconstructible.

12.1. Lemma. Let $G$ be an $(m,n)$-tree. Let $C = (v_1, v_2, \ldots, v_k, v_1)$ be a cycle of $G$, where $k \geq 4$. Then $C$ has a diagonal, i.e. there are two adjacent vertices $v_i, v_j$ in $C$ with $j \neq i-1, i+1$, and $(v_i, v_j) \notin \{v_k, v_1\}$.

Proof. Proof is by induction on $|V(G)|$. If $G \cong K_{m+1}$, then since any two vertices are adjacent, the theorem is true for $G$.

Suppose $|V(G)| = k$, and for any $(m,n)$-tree $G'$ with fewer vertices the theorem is true for $G'$. Let $K'$ be an $(n-m-1)$ simplex of $G$, each vertex of which is adjacent to every vertex of an $m$-simplex $H'$ of $G-K'$, and no others, and $G-K'$ is an $(m,n)$-tree. Suppose $C = (v_1, v_2, \ldots, v_k, v_1)$ is a cycle of $G$. If $C$ is a cycle of $G-K'$, then by the induction hypothesis, $C$ has a diagonal. If $C$ is a cycle of $H' \cup K'$, then since $H' \cup K' \cong K_{n+1}$, $C$ has a diagonal. Otherwise, by a relabeling of the vertices if necessary, we may assume $v_1 \in V(K')$. Let $i$ be the largest index such that $(v_1, v_2, \ldots, v_i) \subseteq V(H' \cup K')$, and $j$ the smallest index such that $(v_j, v_{j+1}, \ldots, v_k, v_1) \subseteq V(H' \cup K')$. Then $v_{i+1} \in V(G-(H' \cup K'))$ and $v_{j-1} \in V(G-(H' \cup K'))$, so $i+1 \neq j$. But $v_i$ and $v_j$ are adjacent, and therefore $(v_i, v_j)$ is a diagonal of $C$. 


12.2. Theorem. Let $G$ be an $(m,n)$ tree and $G'$ a graph. If there is a bijection $g: E(G) \rightarrow E(G')$ such that for every $e \in E(G)$ there is an isomorphism $\phi_e: G-e \rightarrow G'-g(e)$, then $G \sim G'$.

Proof. Since $(m,n)$ trees have no isolated vertices, by Greenwell's Theorem, $G \sim G'$ if $n-m \geq 2$, or $m = 0$, $n = 1$. The case $m = 1$, $n = 2$ will follow from our proof of the vertex reconstructibility if $G$ is a 2-tree. Suppose $G$ is a $k$-tree, and $k \geq 3$. If $G \sim K_k$, $K_{k+1}$, or $K \cup_f K'$ where $K \sim K' \sim K_{k+1}$ and $f$ is a bijection from a $k$-simplex of $K$ to a $k$-simplex of $K'$, $G$ is vertex reconstructible since in each case $G$ contains a vertex adjacent to every other vertex. (If $v$ is such a vertex, then $\deg_G v = \deg_{G'} v = v(G) - 1$, so $\phi_v \cup [(v,v')]$ is an isomorphism.) Otherwise, by Greenwell's Theorem, $G'$ is a vertex reconstruction of $G$. Let $f: V(G) \rightarrow V(G')$ be a bijection such that $G-v \sim G'-f(v)$ for every $v \in V(G)$. Let $v_0$ be an end vertex of $G$. $G'-f(v_0)$ is a $k$-tree. $\deg_G v_0 = \deg_g f(v_0) = k$, and $v_0$ and $f(v_0)$ are each contained in a $k$-simplex. Therefore $f(v_0)$ is adjacent to every vertex of a $(k-1)$-simplex of $G'-f(v_0)$, and $G'$ is a $k$-tree. Suppose $v$ is an end vertex of $G$. $v$ is adjacent to every vertex of a $(k-1)$-simplex of $H$. There is a vertex $w$ adjacent to every vertex of $H$, and a vertex $u$ adjacent to a $(k-1)$-simplex of $H \cup \{w\}$ since $k \geq 3$. There are two vertices $x, y$ of $H$ adjacent to $u, v$, and $w$. Let $e = \{x, y\}$. Then $(\phi_e(v), \phi_e(x), \phi_e(w), \phi_e(y))$ and $(\phi_e(v), \phi_e(x), \phi_e(u), \phi_e(y))$ are
cycles of $G - g(e)$ which have no diagonal. The only possibility is that $g(e) = \{\phi_e(x), \phi_e(y)\}$ and then $\phi_e$ is an isomorphism from $G$ to $G'$. 
CHAPTER III

§13. Reconstruction of 2-Trees

13.1 Theorem. Let $G$ be a 2-tree, and $G'$ a graph. Suppose there is a bijection $f: V(G) \rightarrow V(G')$ such that for every $v \in V(G)$, there is an isomorphism $\phi_v: G-v \rightarrow G'-f(v)'$. Then $G \sim G'$.

Proof. For simplicity, we will denote $f(v)$ by $v'$. If $G \sim K_2$ or $G \sim K_3$, let $v \in V(G)$. $v$ is adjacent to every other vertex of $G$. By Kelly's Lemma, $v'$ is adjacent to every vertex of $G-v'$, and so $\phi_v \cup \{(v, v')\}$ is an isomorphism from $G$ to $G'$. Otherwise, let $v$ be an end-vertex of $G$. $\deg_g(v) = \deg_{g'}(v') = 2$. $G'-v'$ is a 2-tree, and by Kelly's Lemma, $v'$ is contained in a 2-simplex of $G'$. Therefore, by definition, $G'$ is a 2-tree.

Case 1: $|EV(G)| = 2$. Let $v \in EV(G)$, adjacent to $u$ and $w$, where $u$ is an end-vertex of $G-v$. Now, $\deg_{g'-u'}\phi_u'(v) = 1$, and so $u'$ is adjacent to $\phi_u(v)$ and $\phi_u'(w)$, since $\phi_u(v)$ and $\phi_u'(w)$ are adjacent. The third vertex of $G'-u'$ adjacent to $u'$ is adjacent to $\phi_u'(w)$. $u'$ is also adjacent to an end-vertex of $G'-u'-\phi_u(v)$. Suppose $\phi_u'(w)$ is not an end-vertex of $G'-u'-\phi_u(v)$. Then there is an end-vertex of $G'-u'-\phi_u(v)$ adjacent to $\phi_u'(w)$. If there is only one such vertex, it must be $\phi_u(X)$, where $X$ is the unique end-vertex of $G-u-v$. But then $\phi_u \cup \{(u, u')\}$ is an
isomorphism from $G$ to $G'$. If there are two such vertices, then $w$ is adjacent to every vertex of $G$, and so $\phi_w \cup \{(w,w')\}$ is an isomorphism. Suppose $\phi_u(w)$ is an end-vertex of $G'-u'-\phi_u(v)$. Then $\phi_u(w)$ is adjacent to two vertices of $G'-u'-\phi_u(v)$, say $\phi_u(x)$ and $\phi_u(y)$. Suppose $u$ is adjacent to $x$. Let $\phi_u(z)$ be the vertex of $G'-u'-\phi_u(v)-\phi_u(w)$ adjacent to $\phi_u(x)$ and $\phi_u(y)$. If there are no other vertices in $G'$, then $G' \sim G'-u' \cup \{u\}$ where $u$ is adjacent to $\phi_u(v)$, $\phi_u(w)$ and $\phi_u(x)$. If there are other vertices in $G'$, then either $\phi_u(x)$ or $\phi_u(y)$ is an end-vertex of $G'-u'-\phi_u(v)-\phi_u(w)$, and so $\deg_{G'-u'} \phi_u(x) \neq \deg_{G'-u'} \phi_u(y)$. Therefore, if $u'$ is adjacent to $\phi_u(y)$, $G$ and $G'$ do not have the same degree sequence. Consequently, $u'$ is adjacent to $\phi_u(x)$, and $\phi_u \cup \{(u,u')\}$ is an isomorphism from $G$ to $G'$. We may now assume $|EV(G)| \geq 3$.

Case 2: $r_{1,2}(G) = 1$. If $v$ is a vertex in the center of $G$, then $v$ is adjacent to every vertex of $G$. But then $\phi_v \cup \{(v,v')\}$ is an isomorphism from $G$ to $G'$. We may now assume $r_{1,2}(G) > 1$.

Case 3: $|EV(G)| = 3$. By Lemma 6.1, $G$ contains either a 1-simplex $H$ of degree 3, or a 2-simplex $K$ of degree 3. Note that by Lemma 6.1, such a 2-simplex or 3-simplex must be unique. Suppose $G$ contains a 1-simplex $H$ of degree 3. Let $F$ be the graph consisting of two adjacent vertices, and 3 vertices adjacent to them. Since $r_{1,2}(G) > 1$, $v(F) < v(G)$, and so $S(F,G) = S(F,G')$,
and $G'$ contains a 1-simplex $H'$ of degree 3. Let the branches of $H$ be $B_1$, $B_2$, $B_3$, and $v_i$ the end-vertex of $B_i$, for $i = 1, 2, 3$.

Suppose $v(B_i) > 3$, for $i = 1, 2, 3$. Then $G \setminus v_i$ and $G' \setminus v_i'$ contain a 1-simplex of degree 3. Therefore each branch of $H'$ contains $> 3$ vertices. Suppose $v(B_i) = k_i$ for $i = 1, 2, 3$, and $k_1 \leq k_2 \leq k_3$. The branches of $H'$ in $G' \setminus v_i'$ contain $k_1 - 1$, $k_2$, and $k_3$ vertices. Suppose a branch $B'$ of $H'$ contains $k_1 - 1$ vertices. Let $v_i'$ be the end-vertex of $B'$. Then $H$ has a branch in $G \setminus v_i$ with $k_1 - 2$ vertices. This is clearly not possible. Therefore $H'$ has branches in $G'$ with $k_1$, $k_2$ and $k_3$ vertices. Note that $\phi_{v_i}(H) = H'$ for $i = 1, 2, 3$, as $H$ and $H'$ are the unique 1-simplices in $G \setminus v_i$ and $G' \setminus v_i'$ respectively, of degree 3. Since $B_i$ has one end-vertex and $v(B_i) > 3$, then $G_{B_i}(H) = E$, for each $i$, where $E$ is the group consisting of one element.

Suppose $k_1 < k_2 - 1$. Let $B_1$ be the branch of $H'$ in $G'$ which contains $k_1$ vertices. $\phi_{v_2}|B_1 = \phi_{v_3}|B_1$, so if we define $g: V(H) \cup \{v_1\} \rightarrow V(H') \cup \{v_1'\}$ by $g = \phi_{v_2}|V(H) \cup \{v_1\}$, we may conclude by Lemma 9.1, that $G \simeq G'$.

Suppose $k_1 = k_2 - 1 < k_3 - 1$. Let $B_3$ be the branch of $H'$ in $G'$ with $k_3$ vertices. $\phi_{v_1}|B_3 = \phi_{v_2}|B_3$, so if we define $g: V(H) \cup \{v_3\} \rightarrow V(H') \cup \{v_1'\}$ by $g = \phi_{v_1}|V(H) \cup \{v_3\}$, we may conclude by Lemma 9.1 that $G \simeq G'$. 
Suppose \( k_1 = k_2 - 1 = k_3 - 1 \). \( \phi_{v_1}(B_2) \) and \( \phi_{v_1}(B_3) \) are branches of \( H' \) in \( G' \), denote \( \phi_{v_1}(B_2) \) by \( B'_2 \) and \( \phi_{v_1}(B_3) \) by \( B'_3 \).

If \( B_2 \not\sim B_3 \), then \( \phi_{v_2}|B_3 = \phi_{v_2}|B_3 \), and by defining
\[
g = \phi_{v_1}|_{V(H) \cup \{v_3\}}
\]
we may conclude by Lemma 9.1 that \( G \sim G' \).

Suppose \( B_2 \sim B_3 \). If \( B_1 \not\sim B_2-V_2 \), then \( \phi_{v_2}|B_1 = \phi_{v_2}|B_1 \), and we may similarly conclude by Lemma 9.1, that \( G \sim G' \). Suppose \( B_1 \sim B_2-V_2 \). If \( B_1 \not\sim H B_2-V_2 \), then \( \phi_{v_2}(B_1) \sim H \phi_{v_2}(B_2-V_2) \), and \( G' \sim G'-V_2 \cup \{V_2\} \) where \( V_2 \) is adjoined to \( \phi_{v_2}(B_2-V_2) \) in the unique way so that \( \phi_{v_2}(B_2-V_2) \cup \{V_2\} \sim B_2 \). But then \( G \sim G' \). If \( B_1 \not\sim H B_1-V_i \) for \( i = 1,2 \), then \( \phi_{v_2}|B_1 = \phi_{v_2}|B_1 \), and by defining
\[
g = \phi_{v_2}|_{V(H) \cup \{V_1\}}
\]
we may conclude by Lemma 9.1 that \( G \sim G' \).

Suppose \( k_1 = k_2 < k_3 \). Then \( \phi_{v_1}|B_3 = \phi_{v_1}|B_3 \), and by defining
\[
g = \phi_{v_1}|_{H \cup \{V_3\}}
\]
we may conclude by Lemma 9.1 that \( G \sim G' \).

Suppose \( k_1 = k_2 = k_3 \). If one branch (say \( B_1 \)) is not isomorphic to the others, then \( \phi_{v_2}|B_1 = \phi_{v_2}|B_1 \) and by defining \( g = \phi_{v_2}|_{V(H) \cup \{V_1\}} \), we may conclude by Lemma 9.1 that \( G \sim G' \). Suppose \( B_1 \sim B_2 \sim B_3 \).

If \( B_1 \sim H B_2 \sim H B_3 \), then \( B_1 \sim \phi_{v_3}(B_1) \sim H \phi_{v_3}(B_2) \). Similarly, if the branches of \( H' \) are \( B'_1, B'_2, B'_3 \), \( B_1 \sim B'_1 \sim H' B'_2 \sim H' B'_3 \). If \( g: V(H) \to V(H') \) is the function for which \( B_1 \sim g B'_1 \), then \( B_2 \sim g B'_2 \) and \( B_3 \sim g B'_3 \), and \( G \sim G' \). Suppose \( B_1 \sim H B_3 \) and \( B_1 \not\sim B_2 \).
Then \( \phi_{v_1}(B_2) \sim_{H', \phi_{v_1}(B_3)} \), but \( \phi_{v_2}(B_1) \not\sim_{H', \phi_{v_2}(B_3)} \) and \( \phi_{v_3}(B_1) \not\sim_{H', \phi_{v_3}(B_2)} \). Then \( H' \) has 3 branches, two of which are isomorphic with respect to \( H' \) but the third is not. But then \( G \approx G' \).

Now, suppose \( \nu(B_1) = 3 \) and \( \nu(B_i) \geq 4 \), for \( i = 2, 3 \). Let \( \nu(B_i) = k_i \) for \( i = 1, 2, 3 \), where \( k_1 < k_2 \leq k_3 \). \( G' - v'_1 \) has no 1-simplex of degree 3. Consequently, \( v'_1 \) is the end-vertex of a branch of \( H' \) with 3 vertices. \( H' \) has branches in \( G' - v'_2 \) with 3 = \( k_1 \), \( k_2 - 1 \), and \( k_3 \) vertices. If \( H' \) has a branch \( B' \) in \( G' \) with \( k_2 - 1 \) vertices, then \( v'_3 \in EV(B') \) and \( H \) has a branch in \( G - v_3 \) with \( k_2 - 2 \) vertices. This is clearly impossible. Therefore, \( H' \) has branches in \( G' \) with \( k_1, k_2 \) and \( k_3 \) vertices.

Suppose \( k_2 = k_3 \). Then \( H' \) is the center of \( G' - v'_1 \) and so \( \phi_{v_1}(H) = H' \). But then \( \phi_{v_1} \cup \{(v_1, v'_1)\} \) is an isomorphism from \( G \) to \( G' \).

Suppose \( k_2 < k_3 \). Then \( \phi_{v_2}(B_3) \) is a branch of \( H' \) in \( G' \). If \( B_2 \not\sim_{B_3 - v_3} \), then \( \phi_{v_2}|H = \phi_{v_3}|H \). Then if we define \( g = \phi_{v_2}|V(H) \cup \{(v_1, v'_1)\} \), we may conclude that \( G \approx G' \). If \( B_2 \sim_{B_3 - v_3} \), then there is an isomorphism \( \phi'_3 : G - v_3 \to G' - v'_3 \) such that \( \phi'_3(B_2) = \phi_{v_2}(B_2 - v_2 \{v'_2\}) \). But then \( \phi'_3|H = \phi_{v_2}|H \). Again we may conclude by Lemma 9.1, that \( G \approx G' \).

Now suppose \( \nu(B_1) = \nu(B_2) = 3 \) and \( \nu(B_3) \geq 4 \). Let \( v \) and \( u \) be the vertices of \( H \), and \( v \) be the vertex of \( B_3 \) adjacent to \( v \).
and $u$. We can assume there is a vertex $x$ of $B_3$ adjacent to $w$ and $u$. $\deg_G v = 4$, $\deg_{G'-v} \phi_v(v_1) = \deg_{G'-v} \phi_v(v_2) = 1$. Consequently, $v'$ is adjacent to $\phi_v(v_1)$, $\phi_v(v_2)$, and $\phi_v(u)$, and to one other vertex of $G'$ adjacent to $\phi_v(u)$. As in the case that $|EV(G)| = 2$, we may deduce that $v'$ is adjacent to $\phi_v(w)$ or an end-vertex of $G'-v'-\phi_v(v_1)-\phi_v(v_2)$, in which case $u$ is adjacent to every vertex of $G$. In either case, $G \cong G'$.

Suppose $G$ has a 2-simplex $K$ of degree 3. By symmetry, $G'$ has a 2-simplex $K'$ of degree 3. Let the branches of $K$ in $G$ be $B_1$, $B_2$, $B_3$, and $v(B_i) = k_i$ for $i = 1, 2, 3$. Suppose $v_i \in EV(B_i)$ for $i = 1, 2, 3$. Assume that $k_1 \leq k_2 \leq k_3$.

Suppose $k_1 \geq 4$. Then for every $v \in EV(G)$, $K'$ has 3 branches in $G'-v'$. But then every branch of $K'$ in $G'$ contains $\geq 4$ vertices. $K'$ has branches in $G'-v_1$ with $k_1-1$, $k_2$, and $k_3$ vertices. If $K'$ had a branch $B'$ in $G'$ with $k_1-1$ vertices, then if $v_1' \in EV(B')$, $K$ would have a branch in $G-v_1$ with $k_1-2$ vertices. This is clearly impossible. Therefore $K'$ has branches in $G$ with $k_1$, $k_2$, and $k_3$ vertices.

Suppose $k_2 < k_3$. $\phi_{v_1}(B_3) = \phi_{v_2}(B_3)$ and $\phi_{v_1}(B_3)$ is a branch of $K'$ in $G'$. Since $G_{B_3}(K) = E$, $\phi_{v_1}|B_3 = \phi_{v_2}|B_3$. If we define $g = \phi_{v_1}|K \cap B_3 \cup \{(v_3, v'_3)\}$ we may conclude by Lemma 9.1 that $G \cong G'$.

Suppose $k_1 < k_2 - 1 = k_3 - 1$. Then $\phi_{v_2}(B_1) = \phi_{v_3}(B_1)$ and $\phi_{v_2}(B_1)$ is a branch of $K'$ in $G'$. If we define $g = \phi_{v_2}|K \cap B_1 \cup \{(v_1, v'_1)\}$ we may again conclude that $G \cong G'$. 

Suppose \( k_1 = k_2 - 1 = k_3 - 1 \). Then \( \phi_{v_1}(B_2) \) and \( \phi_{v_1}(B_3) \) are branches of \( K' \) in \( G \). Denote \( \phi_{v_1}(B_2) \) by \( B_2' \) and \( \phi_{v_1}(B_3) \) by \( B_3' \). If \( B_2 \not\sim B_3 \), then \( \phi_{v_2}(B_3) = B_3' \) and \( \phi_{v_2}|K = \phi_{v_1}|K \). If we define \( g = \phi_{v_2}|V(K) \cup \{v_3\} \), we may conclude by Lemma 9.1 that \( G \sim G' \).

Suppose \( B_2 \sim B_3 \). If \( B_2 \not\subseteq B_1 \), then \( \phi_{v_2}(B_1) = \phi_{v_3}(B_1) \) and \( \phi_{v_2}(B_1) \) is a branch of \( K' \) in \( G' \). Now, \( \phi_{v_2}|K = \phi_{v_3}|K \), and if we define \( g = \phi_{v_2}|V(K) \cup \{v_1\} \) we may again conclude by Lemma 9.1 that \( G \sim G' \).

Suppose \( B_1 \sim B_3 \). If \( B_2 \not\subseteq B_1 \), then \( \phi_{v_2}(B_1) = \phi_{v_3}(B_1) \) and \( \phi_{v_2}(B_1) \) is a branch of \( K' \) in \( G' \). Now, \( \phi_{v_2}|K = \phi_{v_3}|K \), and if we define \( g = \phi_{v_2}|V(K) \cup \{v_1\} \), we may conclude by Lemma 9.1 that \( G \sim G' \).

Suppose \( B_2 \sim B_3 \). If \( B_2 \not\subseteq B_1 \) \( \not\subseteq B_1 \), then \( G' \sim G \) \( \phi_{v_2}|V(K) \cup \{v_2\} \) where \( v_2 \) is adjoined to \( \phi_{v_2}(B_2 \not\subseteq v_2) \) so that \( \phi_{v_2}(B_2 \not\subseteq v_2) \cup \{v_2\} \cup \phi_{v_2}(B_3) \sim B_2 \cup B_3 \). If \( B_2 \not\subseteq B_1 \), then \( \phi_{v_2}|B_1 = \phi_{v_3}|B_1 \), and again by Lemma 9.1, we may conclude that \( G \sim G' \).

Suppose \( k_1 = k_2 = k_3 \). If some branch, (say \( B_1 \)), is not isomorphic to either of the others, then \( \phi_{v_2}|B_2 = \phi_{v_2}|B_1 \), and we may similarly conclude that \( G \sim G' \). Suppose \( B_1 \sim B_2 \sim B_3 \). Let \( h \) and \( h' \) be functions as defined above. If \( B_1 \sim h B_2 \sim h B_3 \), then \( \phi_{v_1}(B_2) \sim h', \phi_{v_1}(B_3) \), and \( \phi_{v_2}(B_1) \sim h', \phi_{v_2}(B_3) \). Consequently, if
are the branches of $K'$, then $B_1 \sim h$, $B_2 \sim h$, $B_3$, and so $G \sim G'$. If $B_1 \nsubseteq h$ $B_2 \nsubseteq h$ $B_3$, then $\phi_{v_1}(B_2) \sim h$, $\phi_{v_1}(B_3)$, and $\phi_{v_2}(B_1) \nsubseteq h$ $\phi_{v_2}(B_3)$. Consequently, $K'$ has 3 branches $B_1'$, $B_2'$, $B_3'$ each isomorphic to $B_1$, two of which are isomorphic with respect to $h'$, and the third not. Again, $G \sim G'$.

Now, suppose $k_1 = 3$, and $k_2 \geq 4$, $\deg_{G'-v_1}K' = 2$, and $\deg_{G'-v_2}K' = \deg_{G'-v_3}K' = 3$. Therefore $K'$ has one branch in $G'$ with 3 vertices, and two with $\geq 4$ vertices. $K'$ has branches in $G'-v_2$ with 3 = $k_1$, $k_2-1$, $k_3$ vertices. If $K'$ has a branch $B'$ in $G'$ with $k_2-1$ vertices, let $v' \in EV(B')$. Then $K$ has branches in $G-v$ with $k_1$, $k_2-2$, $k_3$ vertices. This is not possible. Therefore $K'$ has branches in $G'$ with $k_1$, $k_2$, $k_3$ vertices. Let $u \in V(K)$ be the vertex of $V(B_2) \cap V(B_3)$, and $v,w$ the other vertices of $K$.

Let $h: V(K) \rightarrow V(K)$ be defined by $h(u) = u$, $h(v) = w$, and $h(w) = v$. If $\phi_{v_2}|_K = \phi_{v_3}|_K$, then we may define $g: V(B_1) \rightarrow V(B_1)$ by $g = \phi_{v_2}|_{B_1}$, and conclude from Lemma 9.1 that $G \sim G'$. If not, then $B_2 \nsubseteq h$ $B_3 \nsubseteq v_3$, and there is an isomorphism $\phi_{v_3}': G-v_3 \rightarrow G'-v_3'$ such that $\phi_{v_3}'|_K = \phi_{v_2}|_K$, and again we may conclude that $G \sim G'$.

Suppose $k_1 = k_2 = 3$. Let $v$ be the vertex of $V(B_1) \cap V(B_2)$ and $u,w$ the other vertices of $K$. $\deg_{G}v = 4$, and $\deg_{G-v}v_1 = \deg_{G-v}v_2 = 1$. Consequently, $v'$ is adjacent to $\phi_{v}(v_1)$ and $\phi_{v}(v_2)$. Since any edge of $G'$ is contained in a triangle, $v'$ is also adjacent to $\phi_{v}(u)$ and $\phi_{v}(w)$. But then $\phi_{v} \cup \{(v,v')\}$ is an isomorphism from $G$ to $G'$. 


We may now assume that \(|E(V(G))| \geq 4\). If the center \(C\) of \(G\) is a 1-simplex, we say that \(C\) is a 2-center. If \(C\) is a 2-simplex, we say that \(C\) is a 3-center.

Case 4: \(G\) has a 2-center with \(\geq 3\) peripheral branches. Let \(F\) be a longest \((1,2)\)-path in \(G\). By Kelly's Lemma, \(S(F,G') > 1\). Again by Kelly's Lemma, \(G'\) has no \((1,2)\)-path of greater length than \(F\). Therefore \(G'\) has a 2-center. Let \(C\) be the center of \(G\), and \(C'\) the center of \(G'\). If \(v \in V(G)\setminus V(C)\), then \(G-v\) and \(G'-v'\) contain a \((1,2)\)-path of greatest length. But this means that \(C'\) has \(\geq 3\) peripheral branches.

Suppose now that \(C\) has non-peripheral branches. Let \(v \in V(B_p(G))\), adjacent to the vertices of \(C\). \(\deg_G C = \deg_{G-v} C + 1 = \deg_{G-v} C' + 1 \geq \deg_{G'} C'\). By symmetry, \(\deg_G C \leq \deg_{G'} C'\). Therefore, \(\deg_G C = \deg_{G'} C'\), and \(v\) is adjacent to the vertices of \(C'\). By Kelly's Lemma, the number of longest \((1,2)\)-paths in \(G'-v'\) is less than the number in \(G'\). Therefore, \(v' \in V(B_p(G'))\), and \(\phi_v(B_p(G)) = B_n(G')\), and \(C\) has non-peripheral branches. Now, let \(u \in EV(B_n(G))\). \(\phi_u(B_p(G)) \subseteq B_p(G')\). Since \(\nu(G) = \nu(G')\), and \(\nu(B_n(G)) = \nu(B_n(G'))\), it follows that \(\nu(B_p(G)) = \nu(B_p(G'))\) and \(\phi_u(B_p(G)) = B_p(G')\). If \(G_p(C) = C_2\), then if \(g = \phi_u|V(C)\), \(B_n(G) \sim G_n(G')\), and so \(G \sim G'\).

Similarly if \(G_p(G) = C_2\), we have \(G \sim G'\). We may assume that \(G_p(G) = G_n(C) = E\). Let \(g = V(C) \to V(C')\) be such that \(B_n(G) \sim g B_n(G')\). Suppose there are \(m\) peripheral branches, and let \(v_1, v_2, \ldots, v_m\) be the vertices of \(B_p(G)\) adjacent to the vertices of
C. If B is a branch of C, we denote by \( n(B, G) \) and \( n(B, G-v) \), the number of branches \( B_i \) such that \( B \sim C B_i \) in \( G \) and \( G-v \) respectively. We denote by \( n(B, G') \) and \( n(B, G'-v') \), the number of branches \( B'_i \) such that \( B \sim G B'_i \) in \( G' \) and \( G'-v' \) respectively.

If \( B_i \sim C B_j \) for each \( i,j = 1, \ldots, m \), then let \( k \neq i,j \).

If \( v_k(B_i) \sim C, v_k(B_j) \), and we may conclude that \( B'_i \sim C, B'_j \) for each \( i,j = 1, \ldots, m \). Since \( B_i \sim G v_2(B_i) \), we see that \( B_i \sim G i \) for \( i = 1, \ldots, m \). Therefore \( B_i(G) \sim G B_i(G') \), and \( G \sim G' \). Otherwise, let \( B_i \) be a branch of \( C \), and suppose \( B_i \not\sim C B_j \). Then \( n(B_i, G) = n(B_i, G-v_j) = n(B_i, G'-v'_j) \leq n(B_i, G') \). By symmetry, \( n(B_i, G') \leq n(B_i, G) \) and so \( n(B_i, G) = n(B_i, G') \). We may again conclude that \( B_i(G) \sim G B_i(G') \), and \( G \sim G' \).

Now suppose that \( C \) has no non-peripheral branches in \( G \). By symmetry, and the above argument, \( C' \) has no non-peripheral branches in \( G' \). Let \( u \in EV(G) \). \( \deg_G C = \deg_{G-u} C = \deg_G G-u, C' \leq \deg_{G'} C' \). By symmetry, \( \deg_G C' \leq \deg_G C \), and so \( \deg_G C' = \deg_G C \). Let \( v_1, v_2, \ldots, v_m \) be the vertices of \( G \) adjacent to the vertices of \( C \). If \( B \) is a branch of \( C \), the number of branches \( B_i \) such that \( B_i \sim B \) is the sum of the numbers in \( G-v_i \) divided by \( m-1 \). But this is the number of branches \( B'_i \) of \( C' \) such that \( B \sim B'_i \). Suppose some branch, say \( B_1 \), is not isomorphic to any other branch. Let \( B'_1 \) be the branch of \( C' \) such that \( B_i \sim B'_1 \), and \( B'_2, \ldots, B'_m \) the other branches of \( C' \). If \( v_1 \in V(B'_1) \), adjacent to the vertices
of $C$, then $\phi_{V_1}(B_2 \cup \ldots \cup B_m) = B_2' \cup \ldots \cup B_m'$.
If $G_{B_1}(C) = C_2$, let $g = \phi_{V_1}|V(C) : B_1 \sim B_1'$, and so $G \sim G'$. Suppose $G_{B_1}(C) = E$.

If $B \not\sim B_1$ is a branch of $C$, the number of branches $B_i$ such that $B_i \sim C B$ is the sum of the numbers in $G_{v_i}$, $i = 2, \ldots, m$, divided by $m-1$, and if $g: V(C) \to V(C)$ is such that $B_1 \sim g B_1'$, then this number is the number of branches $B_1'$ such that $B_1 \sim g B_1'$, and so $G \sim G'$. Suppose there is a branch not isomorphic to $B_1$. We may assume by the above argument that there are at least two such branches.

Let $B_1^*$ be the union of all branches $B_i$ such that $B_i \sim B_1$, and $B_1^*$ the union of all branches $B_1'$ of $C'$ such that $B_1' \sim B_1$. Let $B_2^*, B_2'^*$ be the union of all branches not isomorphic to $B_1$ in $G$ and $G'$ respectively. $\phi_{V_1}(B_2^*) = B_2'^*$, and so $B_2^* \sim B_2'^*$. If $G_{B_1^*}(C) = C_2$, then let $g = \phi_{V_1}|V(C) : B_1 \sim g B_1'$, and so $G \sim G'$.

Suppose $G_{B_1^*}(C) = E$. Let $g: V(C) \to V(C')$ be such that $B_1^* \sim g B_1'^*$. We can, by reasoning analogous to above, conclude that if $B$ is a branch of $C$ and $B \not\sim B_1$, then the number of branches $B_i$ such that $B_i \sim C B_1$ is the number of branches $B_i'$ such that $B_i \sim g B_1'$, and then that $G \sim G'$. Now, suppose that $B_1 \sim B_i$ for all $i$. If $G_{B_1}(C) = C_2$, then $B_1 \sim B_i'$ for each $i$, where $g: V(C) \to V(C')$ is any function, and so $G \sim G'$. If $G_{B_1}(C) = E$, let $k_1$ be the number of branches $B_i$ such that $B_1 \sim C B_i$, and $k_2 = m-k_1$. By relabeling the branches, we may assume that $k_1 \geq k_2$. Similarly,
let \( k_1' \) be the number of branches \( B_1' \) of \( C' \) such that \( B_1' \sim C, B_1' \), and \( k_2' = m-k_1' \). Similarly, we may assume that \( k_1' \geq k_2' \). If we show that \( k_1 = k_1' \), it will prove that \( G \sim G' \). If \( k_2 = 0 \), then for all \( v_k, B_i \sim C B_j \) for any branches \( B_i, B_j \) in \( G-v_k \). But then \( B_i \sim C, B_j' \) for any \( i,j \), and \( k_2 = 0 \). If \( k_2 > 0 \), let \( v \) be a vertex of a branch \( B_k \), such that \( B_k \not\sim C B_1 \), adjacent to the vertices of \( C \). Then \( G'-v' \) has collections of \( k_1 \) and \( k_2-1 \) branches such that if \( B_i' \), \( B_j' \) are in the same collection \( B_i' \sim C, B_j' \), and if \( B_i', B_j' \) are not in the same collection \( B_i \not\sim C, B_j \). Therefore \( k_1' = k_1 \) or \( k_1+1 \). By symmetry \( k_1' \leq k_1 \) and so \( k_1 = k_1' \) and \( G \sim G' \).

Case 5: \( G \) has a 2-center \( C \) with exactly two peripheral branches. We may again conclude that \( G' \) has a 2-center \( C' \). If \( C' \) had \( \geq 3 \) peripheral branches, then by symmetry, and Case 4, so would \( C \). Therefore \( C' \) has exactly 2 peripheral branches.

Suppose each peripheral branch of \( C \) has \( \geq 2 \) peripheral vertices. If \( v' \in EV_p(G') \), then by Kelly's Lemma \( v \in EV_p(G) \). \( G-v \) contains a longest \((1,2)\)-path for each such \( v \), and then \( G'-v' \) does also. Consequently, each peripheral branch of \( C' \) in \( G' \) contains \( \geq 2 \) peripheral vertices.

Suppose \( C \) has no non-peripheral branches in \( G \). Then for \( v \in EV_p(G) \), \( C \) has no non-peripheral branches in \( G-v \), and \( C' \) has no non-peripheral branches in \( G'-v' \). Therefore \( C' \) has no
non-peripheral branches in $G'$. Let the branches of $C$ in $G$ be $B_1$, $B_2$, where $v(B_1) \leq v(B_2)$. If $v_1 \in EV(B_1)$, then $C'$ has branches in $G'-v_1'$ with $v(B_1)-1$, and $v(B_2)$ vertices. If $C'$ has a branch $B'$ in $G'$ with $v(B_1)-1$ vertices, then let $u' \in EV(B')$. $C$ has branches in $G-u$ with $v(B_1)-2$, $v(B_2)$ vertices. This is impossible, and so $C'$ has branches in $G'$ with $v(B_1)$ and $v(B_2)$ vertices, and $B'_1 = \phi_{v_1}(B_2)$ is a branch of $C'$ in $G'$. Let $k_1 = v(B_1)$ and $k_2 = v(B_2)$. Suppose $k_1 < k_2-1$. Let $v_2 \in EV(B_2)$. $\phi_{v_2}(B_2-v_2) = B'_2-v'_2$, and $B'_1 = \phi_{v_2}(B_1)$ is a branch of $C'$ in $G$.

If $G_{B_2}(C) = C_2$, then let $g: V(C) \to V(C')$ be a function such that $B_1 \sim g B_1'$. Then $B_2 \sim g B_2'$, and $G \sim G'$. If $G_{B_2}(C) = E$, then if $g: V(C) \to V(C')$ is the function such that $B_2 \sim g B_2'$, for $u_1 \in EV(B_1)$, $\phi_{u_1}(C) = g$. But then $B_1 \sim g B_1'$ by Lemma 9.1, and so $G \sim G'$. Suppose $k_1 = k_2-1$. If $G_{B_2}(C) = E$, then again, we may conclude by Lemma 9.1 that $G \sim G'$. If $G_{B_2}(C) = C_2$, let $v_2 \in EV(B_2)$. Since $G'$ has branches in $G'-v_2'$ with $v(B_1)$, $v(B_2)-1$ vertices, $v_2' \in EV(B_2')$. If $B_1 \sim B_2-v_2$, then there is an isomorphism $\phi'_{v_2}: G-v_2 \to G'-v_2'$ such that $\phi'_{v_2}(B_2-v_2) = B_2'-v_2'$, and so $B_1 \sim B_1'$. If $B_1 \not\sim B_2-v_2$, then $\phi_{v_2}(B_2-v_2) = B'_2-v'_2$, and so $B_1 \sim B_1'$. In either case, if $g: V(C) \to V(C')$ is a function such that $B_1 \sim g B_1'$, then $B_2 \sim g B_2'$, and so $G \sim G'$. Suppose
\[ k_1 = k_2. \] If \( B_1 \sim B_2 \), then for every \( v \in EV(G) \), \( C \) has a branch \( B \sim B_1 \) in \( G-v \). But then if \( B_1' \) and \( B_2' \) are branches of \( C' \) in \( G' \), \( B_1' \sim B_2' \sim B_1 \). If \( G_{B_1'}(C) = C_2 \), let \( g: V(C) \rightarrow V(C') \) be such that \( B_1 \sim g B_1 \). Then \( B_2 \sim g B_2 \), and \( G \sim G' \). If \( G_{B_1'}(C) = E \), then there is an automorphism \( \phi: G \rightarrow G \) such that \( \phi(B_1') = B_2 \), and \( \phi(B_2) = B_1 \). Therefore, without any loss of generality we may suppose that if \( v \in EV(B_1) \), then \( v' \in EV(B_1') \). Let \( g: V(C) \rightarrow V(C') \) be the function such that \( B_2 \sim g B_2' \). Then for all \( v \in EV(B_1) \), \( \phi_v|V(C) = g \).

By Lemma 9.1, we may conclude that \( B_1 \sim g B_1' \), and \( G \sim G' \). If \( B_1 \not\sim B_2 \) let \( v_1 \in EV(B_1) \) and \( v_2 \in EV(B_2) \). \( C' \) has a branch \( B_1' \sim B_1 \) in \( G'-v_1' \) and hence in \( G \), and a branch \( B_2' \sim B_2 \) in \( G'-v_1 \) and hence in \( G \). If \( G_{B_2'}(C) = C_2 \), let \( g: V(C) \rightarrow V(C') \) be a function such that \( B_1 \sim g B_1' \). Then \( B_2 \sim g B_2' \), and \( G \sim G' \). If \( G_{B_2'}(C) = E \), then if \( g: V(C) \rightarrow V(C') \) is the function such that \( B_2 \sim g B_2' \), for each \( v \in EV(B_1) \), \( \phi_v|V(C) = g \), and we may conclude by Lemma 9.1 that \( B_1 \sim g B_1' \), and \( G \sim G' \).

Now, suppose \( C \) has non-peripheral branches in \( G \). By symmetry, \( C' \) has non-peripheral branches in \( G' \). Let \( v \in EV(B_p(G)) \) and \( u \in EV(B_n(G)) \). \( v(B_p(G)) = v(B_p(G-u)) = v(B_p(G'-u')) \leq v(B_p(G')) \).

By symmetry, \( v(B_p(G')) \leq v(B_p(G)) \). Therefore \( v(B_p(G)) = v(B_p(G')) \), \( v(B_n(G)) = v(B_n(G')) \), \( \phi_v(B_n(G)) = B_n(G') \), and \( \phi_u(B_p(G)) = B_p(G') \).

Consequently, \( B_n(G) \sim B_n(G') \), and \( B_p(G) \sim B_p(G') \). If \( G_{B_n}(G)(C) = C_2 \),
then if $g: V(C) \to V(C')$ is a function such that $B_p(G) \simeq g B_p(G')$, then $B_n(G) \simeq g B_n(G')$, and $G \simeq G'$. If $G_n(C) = E$, and $g: V(C) \to V(C')$ is the function such that $B_n(G) \simeq g B_n(G')$, then for all $v \in EV(B_p(G))$, $\phi_v|V(C) = g$, and so by Lemma 9.1, $B_p(G) \simeq g B_p(G')$, and $G \simeq G'$.

Now, suppose one peripheral branch of $C$, $B_1$, has exactly one peripheral vertex, and the other, $B_2$, has $\geq 2$ peripheral vertices. Let $v \in EV(B_1)$. $v$ is the only end-vertex of $G$ such that $G-v$ does not contain a longest $(1,2)$-path of $G$. But then $v'$ is the only end-vertex of $G'$ such that $G'-v'$ does not contain a longest $(1,2)$-path of $G'$. This means that one peripheral branch of $C'$, $B_1'$ has one peripheral vertex, and the other branch, $B_2'$ has $\geq 2$ peripheral vertices. Suppose $C$ has non-peripheral branches. Let $v \in EV_p(G)$ such that $G-v$ contains a longest $(1,2)$-path. Then since $v' \in EV_p(G')$, $v(B_p(G)) = v(B_p(G-v)) + 1 = v(B_p(G'-v')) + 1 = v(B_p(G'))$, and therefore $C'$ has non-peripheral branches. Let $u \in EV_n(G)$. Then $\phi_u(B_p(G)) = B_p(G')$ and so $B_p(G) \simeq B_p(G')$. Let $B'_1 = \phi_v(B_1)$ and $B'_2 = \phi_u(B_2)$. $\phi_v(B_n(G)) = B_n(G')$, and so $B_n(G) \simeq B_n(G')$. If $G_n(C) = C_2$, then if $g: V(C) \to V(C')$ is a function such that $B_p(G) \simeq g B_p(G')$, then $B_n(G) \simeq g B_n(G')$, and so $G \simeq G'$. If $G_n(C) = E$, then let $g: V(C) \to V(C')$ be the function such that $B_n(G) \simeq g B_n(G')$. For every $v \in EV(B_2)$, $\phi_v|V(C) = g$. 


If for \( v \in EV(B_2) \), \( B_1 \not\cong B_2-v \), then \( \phi_v(B_1) = B_1' \), and \( \phi_v(B_2-v) = B_2'-v' \). If for \( v \in EV(B_2) \), \( B_1 \cong B_2-v \), then there is an isomorphism \( \phi_v': G-v \rightarrow G'-v' \) such that \( \phi_v'(B_1) = B_1' \) and \( \phi_v'(B_2-v) = B_2'-v' \). In any case, we may conclude that \( B_1 \cong B_1' \), and by Lemma 9.1, that \( B_2 \cong B_2' \), and so \( G \cong G' \).

Suppose now that \( C \) has no non-peripheral branches. By symmetry and the above case \( C' \) has no non-peripheral branches. Let \( B_1 \) be the branch of \( C \) with one peripheral vertex, and \( B_2 \) the other, \( B_1' \) the branch of \( C' \) with one peripheral vertex, and \( B_2' \) the other. If \( B_2 \) has \( \geq 3 \) peripheral vertices, then let \( v \in EV_p(B_2) \) \( \phi_v(B_1) = B_1' \) since \( B_2'-v_2' \) contains \( \geq 2 \) peripheral vertices, and \( v' \in EV_p(G') \). But then \( B_1 \cong B_1' \), and \( \nu(B_2) = \nu(B_2') \). Therefore, for all \( v \in EV(B_2) \) \( \phi_v(B_1) = B_1' \). Since \( B_1 \) has exactly one peripheral vertex \( G_{B_1}(C) = E \) and if \( g: V(C) \rightarrow V(C') \) is the function such that \( B_1 \cong g B_1' \), then we may conclude by Lemma 9.1 that \( B_2 \cong g B_2' \), and \( G \cong G' \). If \( B_2 \) has exactly two peripheral vertices, let \( u \in EV_n(G) \). If \( u \in EV(B_1) \) then the longest \((1,2)\)-path of \( G \) containing \( u \) contains a peripheral vertex of \( B_2 \), and there are two such paths. Consequently, there are two \((1,2)\)-paths of \( G' \) containing \( u' \) of greatest length. But then \( u' \in EV_n(B_1') \). By symmetry, if \( v' \in EV_n(B_1') \), then \( v \in EV_n(B_1) \). We may conclude that if \( u \in EV_n(B_1) \), then \( u' \in EV_n(B_1') \) for \( i = 1,2 \). For all \( v \in EV(B_2) \), if \( \phi_v(B_2-v) = B_1' \), then \( B_1 \cong B_2-v \) and so there is an isomorphism \( \phi_v': G-v \rightarrow G'-v' \) such that \( \phi_v'(B_2-v) = B_2'-v' \). If \( g: V(C) \rightarrow V(C') \) is the function such that \( B_1 \cong g B_1' \),
then we may conclude by Lemma 9.1 that $B_2 \sim_{g} B_2'$, and so $G \cong G'$.

Now, suppose that each peripheral branch of $C$ contains one peripheral vertex. Then $G$ and $G'$ each contain exactly one $(1,2)$-path of greatest length. Let $P$ be the longest $(1,2)$-path of $G$, and $P'$ the longest $(1,2)$-path of $G'$. By Kelly's Lemma there is an isomorphism $\theta: P \to P'$. Let $v_1,v_2$ be the end-vertices of $P$.

Suppose there is no automorphism $\alpha: P \to P$ such that $\alpha(v_1) = v_2$, $\alpha(v_2) = v_1$, then for all $u \in EV_n(G)$, $\phi_u|V(C) \cup \{v_1,v_2\} = \theta|V(C) \cup \{v_1,v_2\}$ and we may conclude by Lemma 9.1 that $G \cong G'$.

Now, suppose there is such an automorphism $\alpha: P \to P$. If $C$ has no non-peripheral branches in $G$, then for every $u \in EV_n(G)$, $C$ has no non-peripheral branches in $G-u$, and so $C'$ has no non-peripheral branches in $G'-u'$. Therefore $C'$ has no non-peripheral branches in $G'$. Let $B_1, B_2$ be the peripheral branches of $C$ in $G$, $v_1 \in EV_p(B_1)$ and $v_2 \in EV_p(B_2)$. We may suppose that $v(B_1) \leq v(B_2)$. Suppose that $B_1$ has only one end-vertex. Then by Kelly's Lemma $v_1$ and $v_1'$ are each contained in exactly one $(1,2)$-path of length $l$ for $1 \leq l \leq r_{1,2}(G)$.

But this means that $v_1'$ is contained in a branch, say $B'_1$, which contains exactly one end-vertex. Let $B'_2$ be the other branch of $C'$. If $u \in EV_n(G)$, then $u \in V(B_2)$ and $u' \in V(B'_2)$. Since $P \cong P'$, $B_1 \sim B'_1$. Let $g: V(C) \to V(C')$ be the function such that $B_1 \sim_{g} B'_1$. Then for each $u \in EV_n(G)$ $\phi_u|V(C) \cup \{v_2\} = g|V(C) \cup \{v_2\}$, and we may conclude that $B_2 \sim_{g} B_2'$, and so $G \cong G'$. Now suppose $B_1$ and $B_2$
each contain \( \geq 2 \) end-vertices. By symmetry the branches of \( C' \) each have \( \geq 2 \) end-vertices. Let \( u_1 \in EV_n(B_1) \). Then \( C' \) has branches in \( G' - u' \), with \( v(B_1) - 1 \), and \( v(B_2) \) vertices. If \( C' \) has a branch \( B' \) in \( G' \) with \( v(B_1) - 1 \) vertices, then let \( u' \in EV_n(B') \). \( C \) has branches in \( G - u \) with \( v(B_1) - 2 \) and \( v(B_2) \) vertices. This is clearly not possible. Therefore \( C' \) has branches in \( G' \) with \( v(B_1) \) and \( v(B_2) \) vertices. Suppose \( v(B_1) < v(B_2) - 1 \). Let \( B'_1 \) be the branch of \( C' \) with \( v(B_1) \) vertices and \( B'_2 \) the other. If \( u_1 \in EV_n(B_1) \) and \( u_2 \in EV_n(B_2) \), then \( \phi_{u_1}(B_2) = B'_2 \), and \( \phi_{u_2}(B_1) = B'_1 \). Therefore, \( B_1 \sim B'_1 \) and \( B_2 \sim B'_2 \). \( G' \) is isomorphic to \( B_1 \cup h(B_2) \) where \( h: V(C) \to V(C') \) is some function. Let \( V(C) = \{x, y\} \).

If \( h(x) = x \) and \( h(y) = y \), then \( G \sim G' \). If \( h(x) = y \) and \( h(y) = x \), then since \( G_{B_1} \cap P(C) = G_{B_2} \cap P(C) = E \), \( (P \cap B_1) \cup (P \cap B_2) \neq (P \cap B_1) \cup h(P \cap B_2) \). But \( P' \sim (P \cap B_1) \cup h(P \cap B_2) \). Since \( P \sim P' \), this is impossible. Therefore \( h(x) = x \), \( h(y) = y \), and so \( G \sim G' \).

Suppose \( v(B_1) = v(B_2) - 1 \). Let \( u_1 \in EV_n(B_1) \), and \( u_2 \in EV_n(B_2) \).

Then \( \phi_{u_1}(B_2) \sim B'_2 \). If \( B_1 \sim B_2 \cdot u_2 \), then \( B'_1 \sim B'_2 \cdot u_2 \) and \( B'_1 \sim B_1 \).

If \( B_1 \not\sim B_2 \cdot u_2 \), then \( \phi_{u_2}(B_1) = B'_1 \). In either case \( B_1 \sim B'_1 \) and \( B_2 \sim B'_2 \). By the above reasoning, we may conclude that \( G \sim G' \).

Suppose \( v(B_1) = v(B_2) \). Let \( u_1 \in EV_n(B_1) \) and \( u_2 \in EV_n(B_2) \). Then \( \phi_{u_1}(B_2) \) and \( \phi_{u_2}(B_1) \) are branches of \( C' \) in \( G' \). By symmetry, if \( B'_1 \) and \( B'_2 \) are branches of \( C' \) in \( G' \), then \( C \) has a branch
B\_1 \sim B\_1' \text{ and a branch } B\_2 \sim B\_2'. \text{ We may finally conclude that } C'
has branches B\_1' \text{ and } B\_2' \text{ such that } B\_1 \sim B\_1' \text{ and } B\_2 \sim B\_2'. \text{ Again by the above reasoning, this implies that } G \sim G'.

Now suppose \( C \) has non-peripheral branches. By symmetry and the above case \( C' \) has non-peripheral branches. Let \( u \in \text{EV}(B\_p(G)) \).
Then \( \nu(B\_p(G)) = \nu(B\_p(G-u)) = \nu(B\_p(G'-u')) \). By symmetry \( \nu(B\_p(G')) \leq \nu(B\_p(G)) \), and so \( u' \in \text{EV}(B\_n(G')) \) and \( \phi\_u(B\_p(G)) = B\_p(G') \). If \( |\text{EV}(B\_p(G))| \geq \frac{4}{5} \), let \( v \in \text{EV}(B\_p(G)) \). Since \( \nu(B\_p(G'-v')) = \nu(B\_p(G'))-1 \), \( v' \in \text{EV}(B\_p(G')) \), and \( \phi\_v(B\_n(G)) = B\_n(G') \). We conclude that \( B\_p(G) \sim B\_p(G') \) and \( B\_n(G) \sim B\_n(G') \). If \( G\_B\_n(C) = C_2 \), let \( g: V(C) \rightarrow V(C') \) be a function such that \( B\_p(G) \sim B\_p(G') \). \( B\_n(G) \sim B\_n(G') \), and so \( G \sim G' \). Suppose \( G\_B\_n(C) = E \). Let \( g: V(C) \rightarrow V(C') \) be the function such that \( B\_n(G) \sim B\_n(G') \). Let the peripheral branches of \( C \) be \( B\_1 \) and \( B\_2 \), and the peripheral branches of \( C' \) be \( B\_1' \) and \( B\_2' \) where \( B\_1 \sim B\_1' \) and \( B\_2 \sim B\_2' \). We may assume \( \nu(B\_1) \leq \nu(B\_2) \). Suppose \( B\_1 \) has exactly one end-vertex. Then for any \( u \in \text{EV}_n(B\_p(G)) \), \( u \in \text{EV}(B\_2) \), and \( u' \in \text{EV}(B\_1') \). Then \( \phi\_u(B\_1) = B\_1' \) and \( \phi\_u|V(C) = g \). Then by Lemma 9.1 we may conclude that \( B\_2 \sim g B\_2' \), and so \( G \sim G' \). If \( B\_1 \) and \( B\_2 \) each have \( \geq 2 \) end-vertices, then let \( u\_1 \in \text{EV}(B\_1) \), and \( u\_2 \in \text{EV}(B\_2) \). If \( \nu(B\_1) < \nu(B\_2)-1 \), then \( \phi\_u\_1(B\_2) = B\_2' \) and \( \phi\_u\_1|V(C) = g \). \( \phi\_u\_2(B\_1) = B\_1' \) and \( \phi\_u\_2|V(C) = g \). Therefore \( B\_1 \sim g B\_1' \), \( B\_2 \sim g B\_2' \), and \( G \sim G' \). If \( \nu(B\_1) = \nu(B\_2)-1 \), then \( \phi\_u\_1(B\_2) = B\_2' \), \( \phi\_u\_1|V(C) = g \) and so \( B\_2 \sim g B\_2' \).
If \( B_1 \cong_{C} B_2^{-u_2} \), then there is an isomorphism \( \phi'_{u_2} : G^{-u_2} \rightarrow G'^{-u'_2} \) such that \( \phi'_{u_2}(B_1) = B_1' \). If \( B_1 \not\cong_{C} B_2^{-u_2} \), then \( \phi'_{u_2}(B_1) = B_1' \).

In either case, since \( \phi'_{u_2}|_{V(C)} = g \) we have that \( B_1 \cong_{g} B_1' \) and \( G \cong G' \). If \( v(B_1) = v(B_2) \), \( \phi'_{u_1}(B_2) \) and \( \phi'_{u_2}(B_1) \) are branches of \( C' \). Therefore \( C' \) has a branch \( B_1' \) such that \( B_1 \cong_{g} B_1' \) and a branch \( B_2' \) such that \( B_2 \cong_{g} B_2' \). By symmetry, if \( B_1' \) and \( B_2' \) are the peripheral branches of \( C' \) in \( G' \), then \( C \) has a branch \( B_1 \) such that \( B_1 \cong_{g} B_1' \) and \( B_2 \cong_{g} B_2' \). Consequently, we see that \( B_p(G) \cong g B_p(G') \) and so \( G \cong G' \).

Now suppose \( |EV(B_p(G))| = 3 \). Let \( u \in EV(B_n(G)) \cdot v(B_p(G)) = v(B_p(G-u)) = v(B_p(G'-u')) \leq v(B_p(G')) \). By symmetry, \( v(B_p(G')) \leq v(B_p(G)) \). Therefore \( v(B_p(G)) = v(B_p(G')) \), and \( \phi_u(B_p(G)) = B_p(G') \).

Suppose \( B_1 \) is the peripheral branch of \( C \) with one end-vertex, and \( B_2 \) the peripheral branch with two. Since \( G_{B_1} = E \), and there is no isomorphism \( \theta : B_2 \rightarrow B_1 \), then \( G_{B_p} = E \). Let \( g : V(C) \rightarrow V(C') \) be the unique function such that \( B_p(G) \cong g B_p(G') \). If \( |EV(B_n(G))| \geq 2 \), then for all \( u \in EV(B_n(G)) \), \( \phi_u|_{V(C)} = g \), and we may conclude by Lemma 9.1 that \( B_n(G) \cong g B_n(G') \), and that \( G \cong G' \). Suppose \( |EV(B_n(G))| = 1 \). If \( u \in EV_n(B_p(G)) \), \( \phi_u(B_n(G)) = B_n(G') \), and so \( B_n(G) \cong B_n(G') \). If \( v(B_n(G)) \geq 5 \), let \( v \in EV(B_n(G)) \). \( G_{B_n(G)-v}(C) = E \), and there is a unique way to adjoin \( v \) to \( B_n(G)-v \), and \( v' \) to \( B_n(G')-v' \) so that \( B_n(G)-v \cup \{v\} \cong B_n(G')-v' \cup \{v'\} \cong B_n(G) \).
Consequently, \( \phi_v \cup \{(v,v')\} \) is an isomorphism from \( G \) to \( G' \). If \( \nu(B_n(G)) = 4 \), let \( v \in V(B_n(G)) \) adjacent to the vertices of \( C \). \( v \) is adjacent to the end-vertex of \( B_n(G) \) as well. Since \( \deg_{G'}(v)(C') = 2 \), \( v' \) is adjacent to the vertices of \( C' \) and to the end-vertex of \( B_n(G') \). Consequently, \( \phi_v \cup \{(v,v')\} \) is an isomorphism from \( G \) to \( G' \). If \( \nu(B_n(G)) = 3 \), then of course \( G \simeq G' \). Again we conclude that \( G \simeq G' \).

Now, suppose \( |EV(B_p(G))| = 2 \). By symmetry and the above cases \( |EV(B_p(G'))| = 2 \). Then \( B_p(G) = P \) and \( B_p(G') = P' \). Suppose \( G_p(C) = E \). Then if \( g: V(C) \to V(C') \) is the function such that \( B_p(G) \simeq g B_p(G') \), for all \( u \in EV(B_n(G)) \), \( \phi_u|V(C) = g \), and we may conclude by Lemma 9.1, that \( B_n(G) \simeq g B_n(G') \) and that \( G \simeq G' \). Suppose \( G_p(C) = C_2 \). Let \( v \in EV(B_p(G)) \). \( \phi_v(C) = C' \) since \( C' \) is the 1-simplex of the center of \( G'-v' \) with degree \( \geq 3 \). Let \( g = \phi_v|V(C) \). Let \( \phi_v(B_2) = B_2' \). Suppose \( C' \) has a unique branch \( B' \) such that \( B_1 \cup B_2 \simeq g B_1' \cup B_2' \). Then \( B_1' \cup B_2' = B_p(G')-v' \), and \( \phi_v(B_n(G)) = B_n(G') \). If there are \( \geq 2 \) such branches \( B' \), then there is an isomorphism \( \phi' \) such that \( \phi'(B_p(G)-v) = B_p(G')-v' \), and then \( \phi'(B_n(G)) = B_n(G') \). In either case \( B_n(G) \simeq g B_n(G') \). Let \( g: V(C) \to V(C') \) be a function such that \( B_n(G) \simeq g B_n(G') \). Then since \( G_p(C) = C_2 \), \( B_p(G) \simeq g B_p(G') \), and so \( G \simeq G' \).
Case 6: $G$ has a 3-center $C$, and all three branches of $C$ are peripheral branches. By Kelly's Lemma, the (1,2)-paths of $G$ and $G'$ are of the same length. Therefore $G'$ has a 3-center $C'$. Since $|EV(G)| \geq 4$, there is a vertex $u \in EV(G)$ such that $C$ has 3 peripheral branches in $G-u$. But then $C'$ has 3 peripheral branches in $G'-u'$, and so $C'$ has 3 peripheral branches in $G$.

Let $C = \{x,y,z\}$, and $h: V(C) \to V(C')$ a function defined by $h(x) = y$, $h(y) = z$, and $h(z) = x$. If $B_1$ and $B_2$ are branches of $C$, we say that $B_1 \simeq_r B_2$ if $B_1 \simeq_h B_2$ or $B_2 \simeq_h B_1$. Let the branches of $C$ be $B_1$, $B_2$, and $B_3$, where $\nu(B_1) \leq \nu(B_2) \leq \nu(B_3)$. Let $k_i = \nu(B_i)$ for $i = 1,2,3$. Let $u \in EV(B_1)$. Then $C'$ has branches with $k_1-1$, $k_2$, and $k_3$ vertices in $G'-u'$. If $C'$ has a branch $B'$ with $k_1-1$ vertices, then let $v' \in EV(B')$. $C$ has a branch in $G-v$ with $k_1-2$ vertices. This is not possible.

Therefore $C'$ has branches in $G$ with $k_1$, $k_2$, and $k_3$ vertices, and $\phi_u(B_2 \cup B_3) = B'_2 \cup B'_3$ where $B'_2$ and $B'_3$ are branches of $C'$ in $G'$ with $\nu(B'_2) = k_2$, and $\nu(B'_3) = k_3$.

Suppose $k_1 < k_2-1$. Let $u_2 \in EV(B_2)$ $B'_1 = \phi_{u_2}(B_1)$ is a branch of $C'$ in $G'$. If $G_{B'_1}(C) = C_2$, then if $g: V(C) \to V(C')$ is a function such that $B_2 \cup B_3 \simeq_g B'_2 \cup B'_3$, and $B_1 \simeq_g B'_1$, and so $G \simeq G'$. If $G_{B'_1}(C) = E$, then let $g: V(C) \to V(C')$ be the function such that $B_1 \simeq_g B'_1$. For every $v \in EV(B_2 \cup B_3)$,
\( \phi_v|V(C) = g \), and we may conclude by Lemma 9.1 that \( B_2 \cup B_3 \sim B_2' \cup B_3' \) and that \( G \sim G' \).

Suppose \( k_1 = k_2 - 1 < k_3 - 1 \). Let \( B_1' \) be the branch of \( C' \) with \( k_1 \) vertices. If \( G_{B_3}(C) = E \), let \( g: V(C) \rightarrow V(C') \) be the function such that \( B_3 \sim g B_3' \). For all \( u \in EV(B_1 \cup B_2) \), \( \phi_u(B_3) = B_3' \) and \( \phi_u|V(C) = g \). Therefore we may conclude by Lemma 9.1 that \( B_1 \cup B_2 \sim B_1' \cup B_2' \), and that \( G \sim G' \). Suppose \( G_{B_3}(C) = C_2 \). Let \( x \in V(B_1) \cap V(B_2) \), and \( e: V(C) \rightarrow V(C) \) a function such that \( e(x) = x, e(y) = z \), and \( e(z) = y \). Let \( u \in EV(B_2) \). Since \( C \) has branches in \( G' - u' \), with \( k_1, k_2 - 1 \) and \( k_2 \) vertices, \( u' \in EV(B_2) \). If \( B_2 - u \not\sim e B_1 \), then \( B_2' - u' \not\sim B_1 \), and so \( \phi_u(B_1) = B_1' \), and if \( g: V(C) \rightarrow V(C') \) is the function such that \( B_2 \cup B_3 \sim g B_2' \cup B_3' \), then \( \phi_u|V(C) = g \), and so \( B_1 \not\sim g B_1' \), and \( G \not\sim G' \). If \( B_2 - u \sim e B_1 \), then there is an isomorphism \( \phi_u': G - u \rightarrow G' - u' \) such that \( \phi_u'(B_2 - u) = B_2' - u' \). Then \( \phi_u'(B_1) = B_1' \) and \( g: V(C) \rightarrow V(C) \) is the function such that \( B_2 \cup B_3 \sim g B_2' \cup B_3' \), then \( \phi_u|V(C) = g \), and \( B_1 \sim g B_1' \) and so \( G \sim G' \).

Suppose \( k_1 = k_2 - 1 = k_3 - 1 \). Let \( B_1 \) be the branch of \( C' \) with \( k_1 \) vertices. Let \( u \in EV(B_2) \). If \( B_2 - u \not\sim B_1 \), then \( B_1 \sim B_2 - u \sim B_2' - u' \sim B_1' \). If \( B_2 - u \not\sim e B_1 \), then since \( B_2 - u \sim B' - u' \), \( \phi_u(B_1) = B_1' \). In either case \( B_1 \sim B_1' \). Let \( x \in V(B_2) \cap V(B_3) \), and \( e: V(C) \rightarrow V(C) \) be defined by \( e(x) = x, e(y) = z \), and \( e(z) = y \). If \( B_2 \sim e B_3 \), then let \( g: V(C) \rightarrow V(C') \) be a function such that \( B_1 \sim g B_1' \). Then
$B_2 \cup B_3 \sim g B'_2 \cup B'_3$, and $G \sim G'$. If $B_2 \not\sim_e B_3$, then $G_{B_2 \cup B_3}(C) = E$. If $|\text{EV}(B_2)| \geq 2$, let $g: V(C) \to V(C')$ be the function such that $B_2 \cup B_3 \sim g B'_2 \cup B'_3$. Then we may conclude by Lemma 9.1 that $B_1 \sim g B'_1$. Suppose $|\text{EV}(B_1)| = 1$. If $v(B_1) \geq 5$, let $v \in \text{EV}(B_1)$. $G_{B_1 \cup V}(C) = E$, and there is a unique way to adjoin $v$ to $B_1-v$ and $v'$ to $B'_1-v'$ so that $B_1-v \cup \{v\} \sim B'_1-v' \cup \{v'\} \sim B_1$. Consequently, $\phi_v \cup \{(v,v')\}$ is an isomorphism from $G$ to $G'$.

If $v(B_1) = 4$, let $v \in V(B_1)$ adjacent to the vertices of $C$. $v$ is adjacent to the end-vertex of $B_1$ as well. Since $\deg_{G'} v, (C') = 2$, $v'$ is adjacent to the vertices of $C'$ and to the end-vertex of $B'_1$. Consequently, $\phi_v \cup \{(v,v')\}$ is an isomorphism from $G$ to $G'$. If $v(B_1) = 3$, then of course $G_{B_1}(C) = C_2$, and $B_1 \sim g B'_1$, and so $G \sim G'$.

Suppose $k_1 = k_2 < k_3 - 1$. Let $u \in \text{EV}(B_3)$. Then $B_1 = \phi_u(B_1)$ and $B'_2 = \phi_u(B_2)$ are branches of $C'$ in $G'$, and $\phi_u(B_1 \cup B_2) = B'_1 \cup B'_2$. If $G_{B_2}(C) = C_2$, then let $g = \phi_u|V(C)$. $B_3 \sim g B'_3$, and so $G \sim G'$. If $G_{B_3}(C) = E$, let $g: V(C) \to V(C')$ be the function such that $B_3 \sim g B'_3$. Then for all $v \in \text{EV}(B_1 \cup B_2)$, $\phi_v|V(C) = g$, and we may conclude by Lemma 9.1 that $B_1 \cup B_2 \sim g B'_1 \cup B'_2$.

Suppose $k_1 = k_2 = k_3 - 1$. Let $B'_1$ and $B'_2$ be the branches of $C'$ with $k_1$ vertices. If $G_{B_3}(C) = E$, then let $g: V(C) \to V(C')$ be the function such that $B_3 \sim g B'_3$. Then for all $u \in \text{EV}(B_1 \cup B_2)$,
\( \phi_u|_C = g \), and we may conclude by Lemma 9.1, that \( B_1 \cup B_2 \sim B_1' \cup B_2' \), and so \( G \sim G' \). If \( G_{B_3'}(C) = C_2 \), then let \( u_1 \in EV(B_1) \) and \( u_2 \in EV(B_2) \). Then \( \phi_{u_1}(B_1 \cup B_3') = B_1' \cup B_3' \) for some \( i = 1,2 \), and \( \phi_{u_2}(B_1 \cup B_3) = B_1' \cup B_3' \) for some \( i = 1,2 \). We may conclude that \( C' \) has a branch \( B' \) such that \( B_1 \cup B_3 \sim B' \cup B_3' \) and a branch \( B'' \) such that \( B_2 \cup B_3 \sim B'' \cup B_3' \). By symmetry \( C \) has a branch \( B \) such that \( B \cup B_3 \sim B_1 \cup B_3' \) and a branch \( B* \) such that \( B* \cup B_3 \sim B_1' \cup B_3' \).

Suppose \( B_1 \cup B_3 \sim B_1' \cup B_3' \). Then the above implies that \( B_2 \cup B_3 \sim B_1 \cup B_3' \). Suppose \( k_1 = k_2 = k_3 \). Let \( B \) be a branch of \( C \) in \( G \), and \( u \in EV(G) \setminus EV(B) \). Then \( \phi_u(B) \) is a branch of \( C' \) in \( G' \) isomorphic to \( B \). Suppose \( B_1 \sim B_2 \) are branches of \( C \) in \( G \), and \( u \in EV(G) \setminus EV(B_1 \cup B_2) \). Then \( \phi_u(B_1) \) and \( \phi_u(B_2) \) are branches of \( C' \) in \( G' \) isomorphic to \( B_1 \) and \( B_2 \). By symmetry, we may conclude that if \( B' \) is a branch of \( C' \) in \( G' \), there is a branch of \( C \) in \( G \) isomorphic to \( B' \). The above implies that if \( B \) is a branch of \( C \), the number of branches \( B_1 \) of \( C \) so that \( B_1 \sim B \) is the same as the number of branches \( B_1' \) such that \( B_1' \sim B \).

Suppose a branch of \( C \), say \( B_1 \), is not isomorphic to either of the others. Let \( B_1' \sim B_1 \) be a branch of \( C' \). If \( G_{B_1}(C) = E \), then let \( g: V(C) \rightarrow V(C') \) be the function such that \( B_1 \sim gB_1 \). Then for every \( u \in EV(B_2 \cup B_3) \), \( \phi_u|_C = g \), and we may conclude by Lemma 9.1 that \( B_2 \cup B_3 \sim gB_2 \cup B_3' \), and so \( G \sim G' \). If \( G_{B_1}(C) = C_2 \),
then let $u \in EV(B_1)$. Let $\phi_u(B_2 \cup B_3) = B_2' \cup B_3'$, and let $g = \phi_u|V(C)$. Then since $G_{B_1}(C) = C_2$, $B_1 \simeq g B_1'$, and so $G \simeq G'$.

Suppose $B_1 \simeq B_2 \simeq B_3$. Let $B_1'$, $B_2'$ and $B_3'$ be the branches of $C'$. If $B_1 \simeq B_2 \simeq B_3'$, then for any $u \in EV(G)$, if $B_1'$, $B_2'$, and $B_3'$ are the branches of $C'$ in $G-u'$ with $k_1$ vertices, then $B_1 \simeq B_2 \simeq B_3$. This implies that $B_1' \simeq B_2' \simeq B_3'$, and $G \simeq G'$. Otherwise we may assume $B_1 \not\simeq B_2 \simeq B_3$. By symmetry, it is not the case that $B_1' \simeq B_2' \simeq B_3'$. Therefore, by relabeling the branches of $C'$ if necessary, $B_1 \not\simeq B_2 \simeq B_3'$, and so $G \simeq G'$.

Case 7: $G$ has a 3-center and exactly two peripheral branches. By symmetry, and the above cases, $G'$ has a 3-center with exactly two peripheral branches.

Suppose there are exactly two end-vertices in $B_1(G)$. Then for every $u \in EV(B_n(G))$, $C'$ has non-peripheral branches and exactly two end-vertices in the peripheral branches in $G'-u'$. This implies that the peripheral branches of $C'$ in $G'$ have exactly two end-vertices. If $u \in EV(B_n(G))$, then $\phi_u(B_1(G)) = B_1(G')$. If $G_{B_1}(C) = E$, then let $g: V(C) \rightarrow V(C')$ be the function such that $B_1(G) \simeq g B_1(G')$. Then for every $u \in EV(B_n(G))$, $\phi_u|V(C) = g$, and we may conclude by Lemma 9.1 that $B_n(G) \simeq B_n(G')$, and so $G \simeq G'$. If $G_{B_1}(C) = C_2$ let $B_1$ and $B_2$ be the peripheral branches of $C$ and let $v_1 \in EV(B_1)$. The center $H$ of $G'-v_1'$ is the $1$-simplex of $C$ contained in $\phi_{v_1}(B_2)$. $\phi_{v_1}(B_2)$ is the branch of $H$.
which contains exactly one end-vertex. This implies that \( \phi_{v_1}(C) = C' \).

\( B_n(G') = B_n(G'-v') \). Hence \( \phi_{v_1}(B_n(G)) = B_n(G') \). Let \( g = \phi_{v_1}|V(C) \).

Since \( G_p(B_n(G)) = C_2 \), \( B_p(G) \cong B_p(G') \), and \( G \cong G' \).

Suppose \( |EV(B_p(G))| \geq 3 \), and \( C \) has a non-trivial non-peripheral branch. If \( v \in EV(B_p(G)) \), \( \deg_{G'-v'}(C') = 3 \), and so \( C' \) has a non-trivial non-peripheral branch in \( G' \). If \( u \in EV(B_n(G)) \), \( v(B_p(G)) = v(B_p(G-v)) = v(B_p(G'-v')) \leq v(B_p(G')) \). By symmetry, \( v(B_p(G')) \leq v(B_p(G)) \). Therefore \( v(B_p(G)) = v(B_p(G')) \), \( \phi_u(B_p(G)) = B_p(G') \), and \( \phi_u(B_n(G)) = B_n(G') \), and so \( B_p(G) \cong B_p(G') \), and \( B_n(G) \cong B_n(G') \). Suppose \( G_{B_p(G)}(C) = E \). If \( |EV(B_n(G))| \geq 2 \), let \( g: V(C) \to V(C') \) be the function such that \( B_p(G) \cong B_p(G') \). For each \( u \in EV(B_n(G)) \), \( \phi_u|V(C) = g \), and we may conclude by Lemma 9.1 that \( B_n(G) \cong B_n(G') \), and so \( G \cong G' \). Suppose \( |EV(B_n(G))| = 1 \). If \( v(B_n(G)) \geq 5 \), and \( u \in EV(B_n(G)) \), \( G_{B_n(G)}-u(C) = E \), and so there is a unique way to adjoin \( u \) to \( B_n(G)-u \), and \( u' \) to \( B_n(G')-u' \), so that \( B_n(G)-u \cup \{u\} \cong B_n(G')-u' \cup \{u'\} \cong B_n(G) \). Therefore \( \phi_u \cup \{(u,u')\} \) is an isomorphism from \( G \) to \( G' \). If \( v(B_n(G)) = 4 \), and \( u \in V(B_n(G)) \), adjacent to the vertices of \( C \cap (B_n(G)) \), \( u \) is adjacent to the end-vertex of \( B_n(G) \). \( u' \) is adjacent to the vertices of \( C' \cap B_n(G') \), and to the end-vertex of \( B_n(G') \). Therefore \( \phi_u \cup \{(u,u')\} \) is an isomorphism from \( G \) to \( G' \). If \( v(B_n(G)) = 3 \), then of course \( G_{B_n(G)}(C) = C_2 \), and so \( B_n(G) \cong B_n(G') \), and \( G \cong G' \).
If $G_p(C) = C_2$, then if $\phi: V(C) \rightarrow V(C')$ is the function such that $B_n(G) \simeq B_n(G')$ then $B_p(G) \simeq B_p(G')$ and $G \simeq G'$. 

Suppose now that $C$ has no non-trivial non-peripheral branch.

By symmetry, and the above cases, $C'$ has no non-trivial non-peripheral branch. Let $B_1$ and $B_2$ be the peripheral branches of $C_1$ and suppose $\nu(B_1) \leq \nu(B_2)$. If $B_1$ has exactly one end-vertex $v_1$, then for $l = 1,2,\ldots,r_{1,2}(G)$, $v_1$ is contained in exactly one $(1,2)$-path of length $l$. By Kelly's Lemma, the same is true for $v'_1$. Therefore $v'_1$ is the end-vertex of a peripheral branch of $C'$ with exactly one end-vertex. If $B_2$ has exactly one peripheral vertex $v_2$, then let $u \in EV_n(G)$. $B'_1 = \phi_u(B_1)$ is a branch of $C'$. Let $B'_2$ be the other. If $u_1, u_2 \in EV_n(G)$, then $\phi_{u_1}|V(C) \cup \{v_1\} \cup \{v_2\} = \phi_{u_2}|V(C) \cup \{v_1\} \cup \{v_2\}$, and by defining $g = \phi_{u_1}|V(C) \cup \{v_1\} \cup \{v_2\}$, we may conclude by Lemma 9.1 that $G \simeq G'$. If $B_2$ has $\geq 2$ peripheral vertices, then $\nu(B_1) < \nu(B_2)-1$. If $u \in EV(B_2)$, $B'_1 = \phi_u(B_1)$ is a branch of $C'$ in $G'$. Let $B'_2$ be the other branch of $C'$ and $g = \phi_u|V(C)$. For each $v \in EV(B_2)$, $\phi_{v}|V(C) = g$, and we may conclude by Lemma 9.1 that $B_2 \simeq B'_2$, and so $G \simeq G'$.

Suppose $B_1$ has $\geq 2$ end-vertices. If $B'_1$ and $B'_2$ are the peripheral branches of $C'$, then by symmetry, and the above cases, $B'_1$ and $B'_2$ have $\geq 2$ end-vertices. Let $x \in V(B'_1) \cap V(B_2)$ and $x' \in V(B'_1) \cap V(B_2')$. We say that $B \simeq_x B'$ if there is an isomorphism $\phi: B \rightarrow B'$ such that $\phi(x) = x'$ and $B \simeq_x B_2$ if there is an
isomorphism $\phi: B_1 \rightarrow B_2$ such that $\phi(x) = x$. Let $u_1 \in EV(B_1)$ such that $G-u_1$ contains a longest $(1,2)$-path. $C'$ has branches in $G'-u_1'$ with $v(B_1)-1$ and $v(B_2)$ vertices. If $C'$ has a branch $B'$ in $C'$ with $v(B_1)-1$ vertices, then let $v' \in EV(B')$ such that $G'-v'$ contains a longest $(1,2)$-path. Then $C$ has branches in $G-v$ with $v(B_1)-2$ and $v(B_2)$ vertices. This is impossible. Therefore $C'$ has branches in $G'$ with $v(B_1)$ and $v(B_2)$ vertices, and $B'_2 = \phi_u(B_2)$ is a branch of $C'$ in $G'$. Note that $B_2 \sim B'_2$. Let $u_2 \in EV(B_2)$ such that $G-u_2$ contains a longest $(1,2)$-path. If $B_1 \not\sim x B_2-u_2$, then $B_1 \not\sim x B'_2-u_2'$, and so $B'_1 = \phi_{u_2}(B_1)$ is a branch of $C'$ in $G'$. Note that $B_1 \sim B'_1$. If $B_1 \sim x B_2-u_2$, then there is an isomorphism $\phi_{u_2}' : G-u_2 \rightarrow G'-u_2'$ such that $\phi_{u_2}'(B_2-u_2) = B'_2-u_2'$ and so $B'_1 = \phi_{u_2}'(B_1)$ is a branch of $C'$ in $G'$, and $B_1 \sim x B'_1$.

By the above reasoning and symmetry, if $B$ is a branch of $C$ in $G$, the number of branches $B_i$ of $C$ such that $B \sim x B_i$ is the same as the number of branches $B'_i$ of $C'$ such that $B \sim x B'_i$. But this implies that $G \sim G'$. 

§14. End-vertex Reconstruction of 2-Trees

We may ask if the analogue of Theorem 10.1 holds for 2-trees. That is, if $G$ and $G'$ are 2-trees, and $f: EV(G) \rightarrow EV(G')$ is a bijection such that $G-v \sim G'-f(v)$ for each $v \in EV(G)$, then does it necessarily follow that $G \sim G'$? We see that for the graphs $G$ and $G'$, shown in Figure 1, this is not the case.

The end-vertices of $G$ and $G'$ are $v_1$, $v_2$, and $v'_1$, $v'_2$, respectively. For $i = 1, 2$, $G-v_i \sim G'-v'_i \sim H$, and yet $G \not\sim G'$. 

![Figure 1](image-url)
15.1. Lemma. Let $G$ and $G'$ be 3-trees, $H$ and $H'$ 2-simplices of $G$ and $G'$ respectively. Suppose for some $f: V(H) \to V(H')$, $G \simeq f(G')$. Suppose also that $v \in EV(G)$, $v' \in EV(G')$, and there is a function $g: V(H) \to V(H')$ such that there is an isomorphism $\phi_v: G - v \to G' - v'$, such that $\phi_v|_{V(C)} = g$. If $G_g(H) = C_3$, then $G \simeq g G'$.

Proof. Let $v_1, \ldots, v_b$ be the vertices of $G$ adjacent to all vertices of $H$, $V(H) = \{x_1, x_2, x_3\}$ and the branches of $H$ be $B_1, \ldots, B_b$. We define $B_i^j$ to be the branch of $V(H) \cup \{v_i\}$ at $V(H) \cup \{v_i\}\{x_j\}$, and $B_i^* = \bigcup_j B_i^j$. Suppose $v$ is adjacent to all vertices of $H$. Then $v'$ is adjacent to all vertices of $H'$, and so clearly $G \simeq G'$. Suppose $v$ is not adjacent to all vertices of $H$, and $v \in EV(B_i^*)$. Then $v' \notin EV(\phi_v(B_2^*) \cup \phi_v(B_3^*))$. If $x_1' = \phi_v(x_1)$, and $G \not\simeq g G'$, then $G \simeq g G'$ where $g'(x_1) = x_1'$, $g'(x_2) = x_3'$, and $g(x_3) = x_2'$. But this implies that if $h: V(H) \to V(H)$ is the function such that $h(x_1) = x_1$, $h(x_2) = x_3$, $h(x_3) = x_2'$,
We wish to show that 3-trees are reconstructible. We will assume the following: \( G \) is a 3-tree. \( G' \) is a graph. \( f: V(G) \rightarrow V(G') \) is a bijection, having the property that there is an isomorphism \( \phi_v: G-v \rightarrow G'-f(v) \) for every \( v \in V(G) \). For simplicity, we will write \( v' \) for \( f(v) \). We will show that, necessarily, \( G' \cong G \).

**15.2. Theorem.** If \( G \) is a 3-tree, \( G \) is reconstructible.

**Proof.** Case 1: \( r_{2,3}(G) = 1 \).

Since \( r_{2,3}(G) = 1 \), there is a 2-simplex \( H \) of \( G \), the vertices of which are adjacent to every other vertex of \( G \). Let \( v \in V(H) \). Since \( \deg_G v = \deg_{G'} v' \), \( v' \) is adjacent to every vertex of \( G' \). Therefore \( \phi_v \cup [(v,v')] \) is an isomorphism from \( G \) to \( G' \).

We may now assume that \( r_{2,3}(G) \geq 2 \).

Case 2: \( G \) has 2-end-vertices.

\( G \) is a \((2,3)\)-path. Let \( u \) be an end vertex of \( G \). \( u \) is adjacent to a 2-simplex of \( G \), say \( \{v,w,x\} \). There is a vertex \( y \) adjacent to \( v \), \( w \), and \( x \), and a vertex \( z \) adjacent to \( w \), \( x \),
and \( y \). (We may relabel the vertices if necessary.) Consider \( G-v \), \( G'-v' \). \( \deg_{G'-v'} \phi_v(u) = 2 \), so \( v' \) is adjacent to \( \phi_v(u) \), and since \( G' \) is a 3-tree, \( \phi_v(w) \) and \( \phi_v(x) \) as well. The fourth vertex of \( G' \) adjacent to \( v' \) is a vertex adjacent to \( \phi_v(w) \) and \( \phi_u(x) \). Since \( v' \) is an end-vertex of \( G'-\phi_v(u) \), \( G'-\phi_v(u)-v' \) is a 3-tree with 2 end-vertices. \( v' \) is adjacent to an end-vertex of \( G'-\phi_v(u)-v' \). If \( \phi_u(w) \) and \( \phi_u(x) \) are not end-vertices of \( G'-\phi_v(u)-v' \), then \( \phi_v(y) \) is an end-vertex of \( G'-\phi_v(u)-v' \). If \( v' \) is adjacent to \( \phi_v(y) \), we are done, since then \( \phi_v \cup \{(v,v')\} \) is an isomorphism. If not, then \( v' \) is adjacent to the other end-vertex of \( G'-\phi_v(u)-v' \), and \( G'-\phi_v(u)-v' \) is a (2,3)-path, every vertex of which is adjacent to \( \phi_v(w) \) and \( \phi_v(x) \). Then \( G' \) is isomorphic to the graph obtained by joining \( v' \) to \( \phi_v(u) \), \( \phi_v(x) \), \( \phi_v(w) \), and \( \phi_v(y) \) in \( G'-v' \). Therefore \( G' \sim G \).

Otherwise, suppose \( \phi_v(x) \) is an end-vertex of \( G'-\phi_v(u)-v' \). In this case, only \( \phi_v(y) \) and \( \phi_v(z) \) are adjacent to \( \phi_v(w) \) and \( \phi_v(x) \). If \( \deg_{G'-\phi_v(u)-v'} \phi_v(w) = \deg_{G'-\phi_v(u)-v'} \phi_v(z) \), then \( \phi_v(w) \) and \( \phi_v(z) \) are adjacent to every vertex of \( G'-\phi_v(u)-v' \), and \( G' \) is isomorphic to the graph obtained by joining \( v' \) to \( \phi_v(u) \), \( \phi_v(x) \), and \( \phi_v(y) \), and so \( G' \sim G \). If not, then since \( G' \) and \( G \) have the same degree sequence, \( v' \) is adjacent to \( \phi_v(y) \), and \( \phi_v \cup \{(v,v')\} \) is an isomorphism.
Case 3: $|\text{EV}(G)| = 3$. By Proposition 6.1, $G$ has a 2-simplex $H$ such that $\deg_{G}H = 3$, or a 3-simplex $H$ such that $\deg_{G}H = 3$. If $G$ has a 2-simplex $H$ such that $\deg_{G}H = 3$, let $F = K_3 \cup \{u_1, u_2, u_3\}$ where $u_1$, $u_2$, and $u_3$ are adjacent to all vertices of $K_3$. By Kelly's Lemma, $S(F, G) = S(F, G') = 1$, and $G'$ has a 2-simplex $H'$ such that $\deg_{G'}H' = 3$. Similarly, if $G$ has a 3-simplex $H$ such that $\deg_{G}H = 3$, $G'$ has a 3-simplex $H'$ such that $\deg_{G'}H' = 3$.

Let the branches of $H$ be $B_1$, $B_2$, and $B_3$, and $k_1 = \nu(B_1)$, for $i = 1, 2, 3$. We may assume that $k_1 \leq k_2 \leq k_3$. Let $v_1 \in \text{EV}(B_1)$, and $w_1$ the vertex of $B_i$ adjacent to 3 vertices of $H$. Suppose $k_1 = k_2 = k_3$, and $H$ is a 2-simplex. There is a vertex $v \in V(H)$ such that $\deg_{B_3} v = 1$. $v$ is adjacent to $v_1$, $v_2$, the vertices adjacent to $v_1$, and $v_2$, and an end-vertex adjacent to two vertices of $H$. The same is true for $v'$. Therefore $\phi_v \cup \{(v, v')\}$ is an isomorphism from $G$ to $G'$.

Otherwise, by Kelly's Lemma, if $r$ is the length of the $(2,3)$-path $B_3$, $v_3'$ and $v_3$ are each contained in exactly one $(2,3)$-path of length $\ell$ for each $\ell = 1, 2, ..., r$. This implies $B_3 \sim B_3'$ where $B_3'$ is the branch of $H'$ containing $v_3'$. By Kelly's Lemma, $v_3$ and $v_3'$ are contained in one subgraph isomorphic to $B_1 \cup B_3 \cup w_2$ and one isomorphic to $B_2 \cup B_3 \cup w_1$. This implies that $H'$ has branches $B_1'$ and $B_2'$ such that $B_1 \cup B_3 \sim B_1' \cup B_3'$, and $B_2 \cup B_3 \sim B_2' \cup B_3'$. This implies that $G \sim G'$. We may now assume that $|\text{EV}(G)| \geq 4$. 
Case 4: G and G' have 3-centers C and C', and \( \geq 3 \) peripheral branches. Suppose C has at least one non-peripheral branch. Let \( v \in EV(B_n(G)) \). 

Therefore, \( v(B_p(G)) \leq v(B_p(G')) \). Let \( w \in V(B_p(G)) \), adjacent to the vertices of C. Since \( r_{2,3}(G) > 1 \), \( w' \notin EV(G') \). If \( G'' \) is the largest 3-tree contained in \( G' \), containing \( C' \), \( \deg_{G''}C = \deg_{G'}C' - 1 \). Therefore, \( w' \) is adjacent to the vertices of \( C' \), and \( w' \in V(B_p(G)) \). But then \( \phi(B_n(G)) = B_n(G') \), and \( B_n(G) \sim B_n(G') \). Therefore, \( v(B_p(G)) = v(B_p(G')) \), \( \phi(B_p(G)) = B_p(G') \), and \( B_p(G) \sim B_p(G') \).

Suppose \( G(B_n(G)) = S_3 \). If \( g: V(C) \rightarrow V(C') \) is a function such that \( B_p(G) \sim g B_p(G') \), \( B_n(G) \sim g B_n(G') \), and so \( G \sim G' \).

Suppose \( G(B_n(G)) = C_3 \). Then by Lemma 15.1, for \( u \in EV(B_n(G)) \), if \( g = \phi_u|V(C) \), \( B_n(G) \sim g B_n(G') \), and so \( G \sim G' \).

Suppose \( G(B_n(G)) = C_2 \) with orbits \( \{x,y\} \) and \( \{z\} \). Let \( B_n(G) = B_1 \cup B_2 \cup \ldots \cup B_k \), where \( v(B_1) \leq v(B_2) \leq \ldots \leq v(B_k) \). Suppose \( k \geq 2 \), \( B_1 \sim C B_2 \sim C \ldots \sim C B_k \), and \( B_1 \not\sim C B_i \) for any other \( i \).

Suppose \( k_1 \geq 2 \). If \( G(B_1(G)) = S_3 \), let \( u \in EV(B_1) \). If \( u' \in EV(B_1) \), \( B_1 \sim B_1' \). If \( g: V(C) \rightarrow V(C') \) is a function such that \( G-(B_1-C) \sim g \) \( G-(B_1'-C') \), \( B_1 \sim g B_1' \), and so \( G \sim G' \).

Suppose \( G(B_1(G)) = C_2 \) where the orbits of \( B_1 \) in \( C \) are \( \{x,y\} \) and \( \{z\} \). If \( G(B_2 \cup \ldots \cup B_k(G)) = C_2 \), then the orbits of \( \phi_u(B_2 \cup \ldots \cup B_k(G)) \)
in \( C' \) are the same as the orbits of \( B_n(G') \), and \( \phi_u|_{G-(B_n-B)} \) can be extended to an isomorphism from \( G \) to \( G' \). If \( G_{B_2U...UB_k}(C) = S_3 \), let \( z \in EV(B_n)\setminus EV(B_1U...UB_k) \). Then \( \phi_z(B_1U...UB_k) = \phi_v(B_1U...UB_k) \) and the orbits of \( B_n(G') \) in \( C \) are the orbits of \( \phi_z(B_1U...UB_k) \). Then we can extend \( \phi_z|_{G-B_n(G)} \) to an isomorphism from \( G \) to \( G' \). Otherwise, there are branches \( B_{k_1+1} \sim C \sim \cdots \sim C B_{2k_1} \) such that \( B_1 \sim r B_{k_1+1} \) where \( r(x) = y \), \( r(y) = x \), and \( r(z) = z \). Then the orbits of \( B_n(G') \) in \( C' \) are \( \phi_u([x,y]) \), and we can extend \( \phi_u|_{B_n(G)} \) to an isomorphism from \( G \) to \( G' \). Suppose \( k_1 = 1 \). If \( G_{B_1}(C) = S_3 \), then if \( u \in EV(B_2) \), \( B_1 \sim g \phi_u(B_1) \) where \( g \) is any bijection \( g: V(C) \rightarrow V(C') \). Then if \( z \in EV(B_1) \), \( \phi_z|_{B_1}(G) \) can be extended to an isomorphism from \( G \) to \( G' \). If \( G_{B_1}(C) \neq C_2 \), with orbits \( \{x,y\} \) and \( \{z\} \), then we may conclude that \( G \sim G' \) by the same argument as the Case \( k_1 \geq 2 \). Suppose \( G_{B_1}(C) = C_2 \) with orbits \( \{x,y\} \) and \( \{z\} \). If \( G_{B_2U...UB_k}(C) = C_2 \) with orbits \( \{x,y\} \) and \( \{z\} \) then if \( z \in EV(B_1) \), \( \phi_w(B_1UUB_2U...UB_k) \) has the same orbits in \( C' \) as \( \phi_z(B_2U...UB_k) \) and we can extend \( \phi_z|_{B_1}(G) \) to an isomorphism from \( G \) to \( G' \). Otherwise, \( G_{B_2U...UB_k}(C) = S_3 \), and the orbits of \( B_n(G') \) in \( C' \) are the same as \( \phi_v(B_1) \) where \( v \in V(B_2) \) and \( v \) is adjacent to the vertices of \( C \). Then we can extend \( \phi_u|_{B_1}(G) \) to an isomorphism from \( G \) to \( G' \).
Now, suppose \( k = 1 \). Let \( v_1 \) be the vertex of \( B_1 \) adjacent to each vertex of \( C \). Suppose the branch of \( \{x,y,z,v_1\} \) at \( \{x,y,v_1\} \) has \( k_1 \) vertices, and the branches at \( \{x,z,v_1\} \) and \( \{y,z,v_1\} \) have \( k_2 \) vertices. If \( k_1 = 3 \) or \( k_2 = 3 \), the orbits of \( B_n(G') \) in \( C' \) are the same as the orbits of \( B_n(G) \). Therefore, we may extend \( \phi_{v_1|B_p(G)} \) to an isomorphism from \( G \) to \( G' \). Otherwise, if \( k_1 \leq k_2 \), and \( u \in \text{EV}(B_1) \) in the branch at \( \{x,y,v_1\} \), the orbits of \( B_n(G') \) in \( C' \) are \( \phi_u([x,y]) \) and \( \phi_u([z]) \), and so we can extend \( \phi_u|B_p(G) \) to an isomorphism from \( G \) to \( G' \). If \( k_2 < k_1 \) and \( u \in \text{EV}(B_n(G)) \) in the branch at \( \{y,z,v_1\} \), the orbits of \( B_n(G') \) in \( C' \) are \( \phi_u([x,y]) \) and \( \phi_u([z]) \). Then we may extend \( \phi_u|B_p(G) \) to an isomorphism from \( G \) to \( G' \).

Suppose \( G_{B_n(G)}(C) = E \). Then let \( g \) be the unique bijection \( g: V(C) \to V(C') \) such that \( B_n(G) \cong B_n(G') \). Let \( w_1', \ldots, w_k' \) be the vertices of \( B_p(G) \) adjacent to the vertices of \( C \). Then since \( \text{deg}_{g}|w_1', C' - 1 \), \( w_1', \ldots, w_k' \) are the vertices of \( B_p(G') \) adjacent to the vertices in \( C' \). If \( B \) is a peripheral branch, the number of branches \( B_1 \) such that \( B \cong B_1 \) is the sum of the numbers in \( G-w_i \), for \( i = 1, \ldots, k \), divided by \( k \). But this is the same as the number of branches \( B' \) of \( C' \) such that \( B_1 \cong B' \). Hence \( G \cong G' \).

Suppose \( C \) has no non-peripheral branches. Each vertex of \( G \) is contained in a longest \((2,3)\)-path of \( G \). By Kelly's Lemma, the
same is true of $G'$. Therefore $C'$ has no non-peripheral branches in $G'$. If $w \notin V(C)$, then $G-w$ contains a longest $(2,3)$-path. The same is true of $G'$, and so $C'$ has $\geq 3$ peripheral branches in $G'$. If $v \in V_p(G)$, then $G-w$ contains a longest $(2,3)$-path. Since, if not $v'$ would be adjacent to the vertices of $C'$, and then $r_{2,3}(G') = 1$ or $v'$ is contained in a non-peripheral branch of $C'$.

If $w$ is adjacent to the vertices of $C$, then $\deg_{G-w}C = \deg_{G'-w}C = \deg_{G'}C' - 1$, and so $w'$ is adjacent to the vertices of $C'$. Suppose $B$ is a branch of $C$ in $G$. The number of branches $B_i$ such that $B_i \sim B$ is the sum of the number occurring in each of the subgraphs $G-w$, for each such $w$, divided by $\deg_{G-w}C$. But the number of branches $B_i'$ such that $B_i' \sim B$ is computed by the same calculation. Therefore $C$ and $C'$ have the same number of branches $B_i$ such that $B_i \sim B_i'$.

Let the branches of $C$ in $G$ be $B_1, B_2, \ldots, B_{i_1}, B_{i_2}, \ldots, B_{i_n}$, where $B_{i_j} \sim B_{i_j}$ for each $j, i_1, i_2$, and $B_{i_j} \sim B_{i_j}$ for each $j$. Let $B^*_{i_j} = B^*_{i_1} \cup \ldots \cup B^*_{i_j}$.

Suppose $k \geq 2$. For a given $j$, there are branches $B^*_{i_j}, \ldots, B^*_{i_j}$ each isomorphic to $B^*_j$. Let $w \in V(G) \setminus V(B^*_j)$ adjacent to the vertices of $C$. Then $\phi_w(B^*_j) = B^*_{i_j} \cup \ldots \cup B^*_{i_j}$, and so $B^*_j \sim B^*_{i_j} \cup \ldots \cup B^*_{i_j}$. Let $B_{i_j} = B^*_j \cup \ldots \cup B^*_{i_j}$.
Suppose for some \( j \), \( G_{B_j^*}(C) = E \). If there are \( \geq 2 \) branches \( B_i \neq B_j^* \), let \( B \) be a branch of \( C \), such that \( B \notin B_j^* \). The number of branches \( B_i \) such that \( B_i \sim C B \) is the sum of the number in \( G-w \), for all \( w \in V(G) \setminus V(B_j^*) \) adjacent to the vertices of \( C \). Let \( g \) be the unique bijection \( g: V(C) \rightarrow V(C') \) such that \( B_j^* \sim g B_j^* \). Then the number of branches \( B_i' \) of \( C' \) such that \( B_i' \sim B_j^* \) is the same number. But then \( G \sim G' \). Suppose there is one branch \( B \notin B_j^* \).

If \( |EV(B)| = 2 \), and \( v(B) \neq v(B_j^*)+1 \), then for \( v \in EV(B) \), \( \phi_v|C = g \).

We may conclude by Lemma 9.1 that \( G \sim G' \). Suppose \( |EV(B)| \geq 2 \) and \( v(B) = v(B_j^*)+1 \). If \( \phi_v(B-v) \leq B_j^* \) for some \( v \), then \( \phi_v(B-v) \sim C' B_j^* \), and so there is an isomorphism \( \phi_v' \) such that \( \phi_v'(B_j^*) = B_j^* \). If we define \( g: V(C) \rightarrow V(C') \) by \( g = \phi_v'|C \), we may conclude by Lemma 9.1 that \( G \sim G' \). Suppose there is only one end-vertex in \( B_j^* \). If \( G_B(C) = C_2 \), let \( w \) be the vertex of \( B \) adjacent to the vertices of \( C \). Then \( G \sim G-w \cup \{w\} \) where \( w \) is adjacent to either of the end vertices of \( B-C-T \{w\} \). But \( G' \sim G'-w \cup \{w'\} \), where \( w' \) is adjacent to either end-vertex of \( \phi_w(B-C-T \{w\}) \) and then \( G \sim G' \). If \( G_B(C) = E \), let \( B' \) be the branch of \( C' \) such that \( B' \sim B \), and let \( g \) be the unique bijection \( g: V(C) \rightarrow V(C') \) such that \( B \sim g B' \). If we consider all vertices \( w \in V(G) \setminus V(B) \) adjacent to the vertices of \( C \), then if \( B_1 \) is a branch of \( C \), \( B_1 \neq B \), the number of branches \( B_i \), such that \( B_i \sim C B_1 \), is equal to the sum over all such \( w \), of the number in \( G-w \), divided by \( \text{deg}_G C-2 \).
But then the number of branches $B'_i$ such that $B'_{j-} \sim g B'_i$ is the same. Consequently, $G \sim G'$.

Suppose for some $j$, $G_{B_j^*}(C) = S_3$. Then let $w \in V(B'_j)$ adjacent to the vertices of $C$. Then $G \sim G-w-B_j^* \cup g B_j^*$, where $g: V(C) \to V(C)$ is any bijection. $G' \sim G'-w'-B_j^* \cup g B_j^*$ where $g: V(C') \to V(C')$ is any bijection. But then $G \sim G'$.

Suppose that for no $j$ is $G_{B_j^*}(C) = E$ or $S_3$. If for some $j$, $G_{B_j^*}(C) = C_3$, let $v \in V(B'_j)$. $\phi_v(B_j^* - v) = B_j^*-v'$. By Lemma 15.1, $G \sim G'$.

The only case left is that $G_{B_j^*}(C) = C_2$ for every $j$. Suppose $k \geq 3$, and for each $j_1, j_2$, the orbits of $B_{j_1}^*$ and $B_{j_2}^*$ in $C$ are the same, say $\{x, y\}$ and $\{z\}$. Then if $v \in V(G) \setminus V(B_{j_1}^* \cup B_{j_2}^*)$

$\phi_v(B_{j_1}^* \cup B_{j_2}^*) = B_{j_1}^* \cup B_{j_2}^*$. If $u \in V(B_{j_1}^*)$, $G \sim G-u-B_{j_1}^* \cup g B_{j_2}^*$

where $g: V(C) \to V(C)$ is a bijection, and $g(\{x, y\}) = \{x, y\}$

$G' \sim G'-u'-\phi_u(B_{j_1}^* - u) \cup h B_{j_1}^*$ where $h: V(C') \to V(C')$ is a bijection

where $h(\text{an orbit of } B_{j_1}^*) = \text{an orbit of } B_{j_2}^*$. But then $G \sim G'$.

If not and $k = 3$, then suppose $B_1^*$ has orbits $\{x, y\}$ and $\{z\}$ in $C$, $B_2^*$ has orbits $\{x, z\}$ and $\{y\}$ in $C$ and $B_3^*$ has orbits $\{y, z\}$ and $\{x\}$ in $C$. Then $G \sim B_1^* \cup B_2^* \cup h B_3^*$ where $h(\{y, z\}) = \{y, z\}$.

$G' \sim B_1^* \cup B_2^* \cup h B_3^*$, where $h(\text{an orbit of } B_3^*) \neq \text{an orbit of } B_1^* \text{ or } B_2^*$. Then $G \sim G'$.
Suppose \( k = 2 \). Let \( w_1, \ldots, w_n \) be the vertices of \( B_1^* \) adjacent to the vertices of \( C \). Suppose the orbits of \( B_1^* \) in \( C \) are \( \{x, y\} \) and \( \{z\} \). Let \( k_1 \) be the total number of vertices of \( B_1^* \) in all branches of \( \{x, y, w\} \) for any \( w \), and \( k_2 \) the number in all branches of \( \{x, z, w\} \) for any \( w \). If \( k_1 = 3 \) or \( k_2 = 3 \), then the orbits of \( B_1^* \) in \( C \) are the same as for \( B_1^* \). Then we may extend \( \phi_w \) to an isomorphism from \( G \) to \( G' \). Otherwise, if \( k_1 < k_2 \), let \( u \in EV(B_1^*) \), for \( u \) in a branch of \( \{x, y, w\} \) for some \( w \). Then \( \phi_w(\{x, y\}) \) is an orbit of \( B_1^* \) in \( C' \). Therefore we may extend \( \phi_w|_{B_2^*} \) to an isomorphism from \( G \) to \( G' \). If \( k_1 < k_2 \), let \( u \in EV(B_1^*) \), for \( u \) in a branch of \( \{x, z, w\} \) for some \( w \). Then \( \phi_w(\{x, y\}) \) is an orbit of \( B_1^* \) in \( C' \), and we may extend \( \phi_u|_{B_2^*} \) to an isomorphism from \( G \) to \( G' \).

Now, suppose \( k = 1 \). Let the branches of \( C \) in \( G \) be \( B_1, \ldots, B_n \) and those of \( C' \) in \( G' \) be \( B_1', \ldots, B_n' \). If \( G_{B_1}(C) = S_3 \), then \( G \cong B_2 \cup \ldots \cup B_n \cup B_1 \) where \( h: V(C) \to V(C) \) is any bijection. Let \( v \in EV(B_1), G' \cong \phi_v(B_2 \cup \ldots \cup B_n) \cup B_1^* \), where \( g: V(C') \to V(C') \) is any bijection. Then \( G \cong G' \). If \( G_{B_1}(C) = C_3 \), let \( v \in EV(B_1) \). Then by Lemma 15.1 if \( g: V(C) \to V(C') \) is a function such that \( B_2 \cup \ldots \cup B_n \cong B_2' \cup \ldots \cup B_n' \), \( B_1 \cong B_1' \), and so \( G \cong G' \). Suppose \( G_{B_1}(C) = C_2 \). If for each \( i \), the orbits of \( B_i \) in \( C \) are the same, say \( \{x, y\} \) and \( \{z\} \), then for every \( u \in EV(G) \), any two branches of \( C' \) in \( G'-u' \) have the same orbits. But then any two branches
of $C'$ in $G'$ have the same orbits in $C'$, and so $G \cong G'$. Suppose $n = 3$, and any two branches of $C$ have different orbits in $C$. Then for any $v \in EV(G)$, the two branches of $C'$ in $G'-v'$ have different orbits in $C'$. This means the branches of $C'$ in $G'$ have different orbits, and $G \cong G'$. If $n = 3$ and $B_1$ and $B_2$ have the same orbits in $C$ but $B_3$ does not, then if $u \in EV(B_1), v \in EV(B_3)$, the two branches of $C'$ in $G'-v'$ have the same orbits in $C'$. Then two branches of $C'$ in $G'$ have the same orbits in $C'$, and the third does not. But then $G \cong G'$.

Now suppose $n \geq 4$, and $k_1$ branches of $C$ in $G$ have orbits $\{x,y\}$ and $\{z\}$ in $C$, $k_2$ have orbits $\{x,z\}$ and $\{y\}$ in $C$, and $k_3$ with orbits $\{y,z\}$ and $\{x\}$ in $C$, where we may assume $k_1 \geq k_2 \geq k_3$, (by relabeling the vertices of $C$ if necessary). Suppose $k_3 > 0$. Since $n \geq 4$, $k_1 \geq 2$. Let $u \in EV(G)$, where $u$ is in a branch with orbits $\{x,y\}$ and $\{z\}$ in $C$. Then $C'$ has at least one branch in $G'-u'$ with any possible orbit in $C'$. Let $k'$ be the largest number of branches of $C'$ in $G'$ which have the same orbits in $C'$. If $v \in EV(G)$ is in a branch with orbits $\{y,z\}$ and $\{x\}$ in $C$, then $k' \geq$ the largest number of branches of $C'$ in $G'-v'$ with the same orbits in $C' = k_1$. By symmetry, $k_1 \geq k'$, and so $k_1 = k'$. Then if $k''$ is the second largest number of branches of $C'$ in $G'$ with the same orbits in $C'$, $k''$ is the second largest number of such branches in $G'-v'$. But then, $G'$ and $G$ have the same numbers of branches which have orbits in $C$ and $C'$ in common, and
$G \cong G'$. Suppose $k_3 = 0$. Then $k_2 > 0$. If $v \in EV(G)$ and $v$ is a vertex in a branch with orbit \{x, z\} and \{y\} in $C$, then if $k'$ is as above, $k' \geq$ the maximum number of branches of $C'$ with the same orbits in $C'$ in $G'-v = k_1$. By symmetry, $k' \leq k_1$, and so $k_1 = k'$, $C$ and $C'$ have the same number of branches with orbits in $C$ and $C'$ in common, and $G \cong G'$.

The only case left is that $G_{B_1}(C) = E$. Let $w$ be the vertex of $B_1$ adjacent to the vertices of $C$. Let $D_1$ be the branch of \{x, y, z, w\} at \{x, y, w\}, $D_2$ the branch at \{x, z, w\} and $D_3$ the branch at \{y, z, w\}, and suppose $v(D_1) \leq v(D_2) \leq v(D_3)$. (We may relabel the vertices of $C$ if necessary.) We will say $D_i \sim D_j$ if $D_i \sim_h D_j$ where $h(w) = w$. Suppose $v(D_1) \geq 4$, and $D_i \not\sim D_j$ for $i \neq j$. Let $u \in EV(D_1)$. $\phi_u(B_1-u) \cup \{u\}$ is a branch of $C'$ and $B_1 \sim \phi_u(B_1-u) \cup \{u\}$, since $v(D_1) \leq v(D_2) \leq v(D_3)$. If $B_1 \sim \phi_u(B_1-u) \cup \{u\}$, $B_1-u \sim \phi_u(B_1-u)$, and therefore $\phi_u$ may be extended to an isomorphism from $G$ to $G'$. Suppose $D_i \sim D_j$ for all $i,j$. Let $r: V(C) \cup \{w\} \to V(C) \cup \{w\}$ be such that $r(x) = y$, $r(y) = z$ and $r(z) = x$. It is not the case that $D_i \sim r D_j$ for all $i,j$, or else $G_{B_1}(C) \leq C_3$. Suppose $D_1 \sim r D_2$. Then $D_1 \sim \theta D_2$ where $\theta(x) = x$, $\theta(y) = z$ and $\theta(z) = y$. We may assume that $D_2 \sim r D_3$.

Let $u \in EV(B_1)$. Then there is a unique way to adjoin $D_1$ to $B_1-D_1$ so that $(B_1-D_1) \cup D_1 \cong B_1$, and so a unique way to adjoin $D_1$ to $\phi_u(B_1-D_1)$ so that $\phi_u(B_1-D_1) \cup D_1 \cong B_1$. But $G'$ is isomorphic to such a graph, and $G' \cong G$. Suppose $D_1 \cong D_2$ but $D_1 \not\cong D_3$. Let
r be defined as above. If $D_1 \simeq r D_2$, and $u \in EV(D_1)$, then
$G' \simeq G' - u \Phi_u(D_1) \cup D_1$ where $\Theta$ is the only bijection, for which
$B_1 \simeq \Phi_u(D_1 - D_1) \cup D_1$, and then $G \simeq G'$. If $D_1 \not\simeq r D_2$, there is
a unique way to adjoin $D_1$ to $\Phi_u(D_1 - D_1)$ so that $B_1 \simeq \Phi_u(D_1 - D_1) \cup D_1$
and again $G \simeq G'$. The cases $D_1 \simeq D_3$ and $D_2 \simeq D_3$ are handled
similarly. Suppose $v(D_1) = 3$, and $v(D_2) \geq 4$. If $v(D_2) < v(D_3)$,
then if $w$ is the vertex of $B_1$ adjacent to the vertices of $C$, there
is a unique way to adjoin $B_1'$ to $G' - \Phi_w(B_1 - w)$ so that $\Phi_w(B_1 - w') \simeq$
$\Phi_w(B_1 - w)$. If $v(D_2) = v(D_3)$, then $v(D_2) \geq 2$ or $G_{B_1}(C) = C_2$.
Then if $u \in EV(D_2)$, there is a unique way to adjoin $B_1'$ to $G' - \Phi_u(B_1 - u)$
so that $\Phi_u(B_1) - u' \simeq \Phi_u(B_1 - u)$ and so $G \simeq G'$. Suppose $v(D_1) = v(D_2) =$
$4$, and $v(D_3) \geq 6$. Then there is a unique way to adjoin $B_1'$ to
$G - \Phi_w(B_1 - w)$ so that $\Phi_w(B_1 - w') \simeq \Phi_w(B_1 - w)$ and so $G \simeq G'$.

Case 5: $C$ has exactly two peripheral branches. If $C'$ has
$\geq 3$ peripheral branches, then by symmetry, and the previous cases,
so does $C$. Suppose $C$ has non-peripheral branches in $G$. Let $F$
be the graph $P \cup h K_4$ where $P$ is a longest path of $G$, and
$h: K \to C$ where $K_4 \cong K \cong K_3$ and $C$ is the center of $P$. $S(F,G) =$
$S(F,G')$, and so $G'$ has non-peripheral branches.

Suppose $|EV(B_{p}(G))| \geq 3$. Let $v \in EV(B_{p}(G))$, such that $G-v$
contains a longest $(2,3)$-path of $G$. Then since $v' \in V(B_{p}(G'))$,
$\Phi_v(B_{p}(G)) = B_{p}(G')$. Let $u \in EV(B_{p}(G))$. Then $\Phi_u(B_{p}(G)) = B_{p}(G')$.
Therefore $B_{p}(G) \simeq B_{p}(G')$, and $B_{n}(G) \simeq B_{n}(G')$. 
If \( G_B(C) = S \), then \( G \sim B_p(G) \cup B_p(G) \) where \( h : V(C) \to V(C) \) is any bijection. But \( G' \sim B_p(G') \cup h', B_p(G') \), where \( h' : V(C') \to V(C') \) is any bijection. But then \( G \sim G' \).

If \( G_B(C) = C_3 \), then by Lemma 15.1 \( G \sim G' \).

If \( G_B(C) = S_3 \) or \( C_3 \), \( G \sim G' \) analogously.

Suppose \( G_B'(C) = E \), and \( \lvert EV(B_n(G)) \rvert \geq 2 \). Then for \( \phi_1, \phi_2 \in EV(B_n(G)) \), \( \phi_1 \mid C = \phi_2 \mid C \). If we define \( g : V(C) \to V(C') \) by \( g = \phi_1 \mid C \), we may conclude from Lemma 9.1, that \( B_n(G) \sim B_n(G') \), and so \( G \sim G' \).

Suppose \( \lvert EV(B_n(G)) \rvert = 1 \). If \( G_B(G)-u(C) = E \), there is a unique way to adjoin \( u \) to \( B_n(G)-u \) so that \( B_n(G)-u \cup \{u\} \sim B_n(G) \). The same is true for \( B_n(G') \) and \( u' \), so \( \phi_u \cup \{(u,u')\} \) is an isomorphism from \( G \) to \( G' \). If \( G_B(G)(C) = C_2 \) or \( S_3 \), then let \( w \) be the vertex of \( B_n(G) \) adjacent to the vertices of \( C \). \( G \sim G-w \cup \{w\} \) where \( w \) is adjacent to an end-vertex of \( B_n-w-C \), which is adjacent to two vertices of \( C \). But \( w' \in V(B_n(G)) \), and is adjacent to the vertices of \( C' \), and \( G' \) is isomorphic to the graph formed in that way, and so \( G \sim G' \).

Now, suppose \( \lvert EV(B_n(G)) \rvert \geq 2 \), and \( G_B(G)(C) = C_2 \). Let \( B_1 \) and \( B_2 \) be the peripheral branches of \( C \), and \( B_1' \) and \( B_2' \) the peripheral branches of \( C' \). Suppose \( B_1 \sim B_1' \) and \( B_2 \sim B_2' \). Suppose \( \lvert EV(B_1) \rvert \geq 2 \), and \( B_1 \not\sim B_2 \). If \( G_B(G)(C) = G_B(G)(C) = C_2 \), then if
\( v(B_1) \leq v(B_2) \), let \( v \in \text{EV}(B_1) \). \( \phi_v(B_2) = B_2' \), and there is a unique way to adjoin \( B_1' \) to \( B_2' \cup B_n(G') \) so that \( G_{B_1}(C') = C_2 \). But then \( \phi_v|_{B_2' \cup B_n(G)} \) can be extended to an isomorphism from \( G \) to \( G' \).

Suppose \( G_{B_1}(C) = C_2 \), and \( G_{B_2}(C) = S_3 \). If for some \( v \in \text{EV}(B_2) \),

\( B_2 - v \not\sim B_1 \), then \( \phi_v(B_1 \cup B_n(G)) = B_1' \cup B_n(G') \). \( G' \sim B_1' \cup B_n(G') \cup h', B_2' \)

where \( h': V(C') \rightarrow V(C') \) is any bijection. Then \( G \sim G' \). If for each \( v \in \text{EV}(B_2) \), \( B_2 - v \not\sim B_1 \), there is a vertex \( v \in \text{EV}(B_2) \) such that the orbits of \( B_1 \) and of \( B_2 - v \) in \( C \) are the same. For such a vertex \( v \), \( B_1' \sim C', B_2' - v \), and there is an isomorphism \( \phi'_v: G - v \rightarrow G' - v' \)

such that \( \phi'_v(B_1) = B_1' \). But then \( B_1' \cup B_n(G) \sim B_1' \cup B_n(G') \) and by the above reasoning \( G \sim G' \).

Suppose \( |\text{EV}(B_2(G))| \geq 3 \), and \( B_1 \not\sim B_2 \). If \( G_{B_1}(C) = C_2 \), and \( B_1 \) and \( B_2 \) have the same orbits in \( C \), then let \( u \in \text{EV}(B_1) \) such that \( G - u \) contains a longest \((2,3)\)-path of \( G \). Then \( \phi_u(B_2 \cup B_n) = B_2' \cup B_n(G') \). But then if we adjoin \( B_1' \) to \( B_2' \cup B_n(G') \) so that \( B_1' \) and \( B_2' \) have the same orbits, we get a graph isomorphic to \( G \).

Suppose \( B_1 \) and \( B_2 \) have different orbits in \( C \). Let the orbits of \( B_1 \) be \([x,y]\) and \([z]\), and the orbits of \( B_2 \), \([x,z]\) and \([y]\).

If \( G_{B_1}(C) = E \), let \( u \in \text{EV}(B_2) \) such that \( G - u \) contains a longest \((2,3)\)-path of \( G \), and \( g: V(C) \rightarrow V(C') \) the unique bijection such that \( B_n(G) \sim g B_n(G') \). \( \phi_u(B_1) = B_1' \), and so \( B_1 \sim g B_1' \) and we may similarly deduce that \( B_2 \sim g B_2' \), and \( G \sim G' \). If \( G_{B_1}(G)(C) = C_2 \),
and the orbits of $B_n(G)$ are $\{y,z\}$ and $\{x\}$, and if $u \in EV(B_2)$, such that $G-u$ contains a longest path, $\phi_u(B_1 \cup B_n(G)) = B'_1 \cup B_n(G')$, and $B'_1$ and $B_n(G')$ do not have the same orbits in $C'$. Similarly, $B'_2$ and $B_n(G')$ do not have the same orbits in $C'$, if $B_n(G)$ and $B_1$ have the same orbits in $C$, and again we may deduce that $B'_1$ and $B_n(G')$ have the same orbits in $C'$. Consequently, $G \simeq G'$. Suppose $G_{B_1}(C) = E$ then there is an isomorphism $\theta: B_1 \cup B_2 \to B_1 \cup B_2$ such that $\theta(B_1) = B_2$ and $\theta(B_2) = B_1$. We may assume $\theta(z) = z$, $\theta(x) = y$ and $\theta(y) = x$. If $G_{B_n}(G) = E$, let $u \in EV(B_2)$ such that $G-u$ contains a longest $(2,3)$-path of $G$. Then if $g$ is defined as above, $\phi_u(B_1 \cup B_n(G)) = B'_1 \cup (B_n(G'))$ and so $B'_1 \simeq B_1$. Similarly, $B_2 \simeq B'_2$, and $G \simeq G'$. If $G_{B_n}(C) = C_2$, with orbits $\{x,y\}$ and $\{z\}$, then $\phi_u(B_1 \cup B_n(G)) = B'_1 \cup B_n(G')$, and then the orbits of $B_n(G')$ in $C'$ are the same as $B'_1 \cup B'_2$ and $G \simeq G'$.

Now, suppose $|EV(B_p(G))| \geq 3$, and $B_1 \not\simeq B_2$. Then the orbits of $B_1$ in $C$ are the same as $B_2$, and as $B_1 \cup B_2$. If $|EV(B_1)| \geq 2$ and $|EV(B_2)| \geq 2$, and $\nu(B_2) \leq \nu(B_1)$, let $u \in EV(B_2)$ such that $G-u$ has a longest $(2,3)$-path of $G$. $\phi_u(B_1 \cup B_n(G)) = B'_1 \cup B_n(G')$. Therefore $B_1 \cup B_n(G) \simeq B'_1 \cup B_n(G')$. $G' \simeq B'_1 \cup (G')_h \cup B'_2$, where $h': V(C') \to V(C')$ and $h'$(an orbit of $B'_2$) is an orbit of $B'_1$. But then $G \simeq G'$. If $|EV(B_1)| = 1$, let $u \in EV(B_2)$ such that $G-u$ contains a longest $(2,3)$-path of $G$. If $B_2-u \not\simeq B_1$, then $\phi_u(B_1 \cup B_n(G)) = B'_1 \cup B_n(G')$, and so $B_1 \cup B_n(G) \simeq B'_1 \cup B_n(G')$. But
then there is a unique way to adjoin $B_2$ to each so that $B_2$ and $B_1$, $B_2'$ and $B_1'$ have the same orbits in $C$ and $C'$, and $G \cong G'$. If $B_2-u \cong B_1$, then there is an isomorphism $\varphi'_u: B_1 \cup B_n(G) \rightarrow B_1' \cup B_n(G')$. Again, $G \cong G'$.

Now, suppose $|EV(B_n(G))| = 2$. Let $u$ be the end-vertex of $B_1$, $v$ the end-vertex of $B_2$. Then $u'$ is the end-vertex of $B_1'$, and $v'$ is the end-vertex of $B_2'$. The center of $G'-u'$ is a 3-simplex $K$ which has two branches in $G'-u'$. $C$ is the 2-simplex in $K$ for which the branch at $K$ has one end-vertex. The same is true for $v'$. Consequently, $\varphi'_u(C) = C'$, $\varphi'_v(C) = C'$. Since $G_{B_1}(C) = E$, $G'$ is isomorphic to the graph we get by adjoining $v'$ to $\varphi'_u(B_2-v)$ in $G'-v'$, so that $B_1' \cup \varphi'_u(B_2-v) \cup \{v\} \cong B_1 \cup B_2$. But then $G \cong G'$.

Suppose $C$ has no non-peripheral branches. Let the branches of $C$ be $B_1$ and $B_2$, and those of $C'$ be $B_1'$ and $B_2'$. Let $L = \{v \in EV(G): G-v$ contains no longest $(2,3)$-path of $G\}$ and $U = EV(G) \setminus L$.

Suppose $L = \emptyset$. Then for each $u \in EV(G)$ $G-u$ contains a longest $(2,3)$-path of $G$. But then $G'-u'$ contains a longest $(2,3)$-path of $G'$, and so each branch of $C'$ has $>2$ peripheral branches. Suppose $\nu(B_1) \leq \nu(B_2)$. Let $u \in EV(B_1)$. $C'$ has branches with $\nu(B_1)-1$, and $\nu(B_2)$ vertices in $G'-u'$. If $C'$ has branches with $\nu(B_1)-1$, and $\nu(B_2)+1$ vertices in $G'$, then suppose $\nu(B_1') = \nu(B_2')-1$ and let $v' \in EV(B_1')$. $C$ has a branch with $\nu(B_2)+1$ vertices in
This is impossible, and so \( C' \) has branches with \( v(B_1) \), \( v(B_2) \) vertices, and \( B_2 \sim \phi_u(B_2) \). Let \( B'_1 = \phi_u(B_1) \).

Suppose \( v(B_1) < v(B_2) - 1 \). Then if \( v \in \text{EV}(B_2) \) \( \phi_v(B_1) = B'_1 \) and so \( B_1 \sim B'_1 \), \( B_2 \sim B'_2 \).

If \( G_{B_1}(C) = S_3 \), then \( G' \sim B'_1 \cup g(B_2') \) where \( g': V(C') \to V(C') \) is any bijection. But then \( G \sim G' \).

If \( G_{B_1}(C) = C_3 \), let \( u \in \text{EV}(B_1) \) \( \phi_u(B_2) = B'_2 \). Then by Lemma 15.1, if \( g = \phi_u|V(C) \), \( B_1 \sim g B'_1 \), and so \( G \sim G' \).

If \( G_{B_2}(C) = S_3 \) or \( G_{B_2}(C) = C_3 \), then analogously we may conclude that \( G \sim G' \).

Suppose \( G_{B_1}(C) = E \). Then for all \( v_1, v_2 \in \text{EV}(B_2) \) \( \phi_{v_1}|C = \phi_{v_2}|C \), and if we define \( g: V(C) \to V(C') \) by \( g = \phi_{v_1}|C \), we can conclude from Lemma 9.1, that \( B_1 \sim g B'_2 \) and so \( G \sim G' \). If \( G_{B_2}(C) = E \), then we may analogously conclude that \( G \sim G' \).

Suppose \( G_{B_1}(C) = G_{B_2}(C) = C_2 \). Let \( w \) be the vertex of \( B_2 \) adjacent to the vertices of \( C \). Suppose the orbits of \( B_2 \) and \( B_1 \) in \( C \) are \( \{x,y\} \) and \( \{z\} \). Then let \( D_1, D_2, D_3 \) be the branches of \( \{x,y,z,w\} \) at \( \{x,w,z\} \), \( \{y,z,w\} \) and \( \{x,y,z\} \) respectively. \( D_1 \sim \{w,z\} D_2 \) and \( G_{D_3}(\{x,y,z\}) = C_2 \) with orbits \( \{x,y\} \) and \( \{z\} \).

Suppose \( v(D_3) = 3 \), \( v(B_2) \geq 8 \), and so \( v(D_1) \geq 5 \). If \( u \in \text{EV}(D_1) \), then the orbits of \( B'_2 \) in \( C' \) are \( \phi_u(\{x,y\}) \) and
φ_u(\{z\}) , since φ_u(\{x,y,w\}) is the only branch of φ_u(\{w,x,y,z\}) which has only 3 vertices. Then B'_1 and B'_2 have the same orbits in C' and G \simeq G'. Suppose v(D_1) = 3. Then again, since v(B_2) \geq 5 , if u \in EV(B_2), the orbits of B'_2 in C' are φ_u(\{x,y\}) and φ_u(\{z\}) since φ_u(\{x,y,w\}) is the only branch of φ_u(\{w,x,y,z\}) which has more than 3 vertices. Then B'_1 and B'_2 have the same orbits in C'. Suppose v(D_1) > 3 and v(D_3) > 3. If v(D_3) = v(D_1)+1, let u \in EV(D_1). We see that the orbits of B'_2 in C' are φ_u(\{x,y\}) and φ_u(\{z\}) . Then B'_1 and B'_2 have the same orbits in C' and G \simeq G'. Otherwise, let u \in EV(D_3) , again we see that the orbits of B'_2 in C' are φ_u(\{x,y\}) and φ_u(\{z\}) and G \simeq G'.

If B'_1 and B'_2 do not have the same orbits in C, then by symmetry, neither do B'_1 and B'_2 , and again G \simeq G'.

Now, suppose v(B'_1) = v(B'_2)-1. For every v \in EV(B'_2), C' has a branch in G'-v' which is isomorphic to B'_1 . If in each case φ_v(B'_2-v) = B'_1 , then φ_v(B'_1) = B'_2-v'. But then B'_2-v' \simeq B'_1-v' for every v, and consequently B'_1 \simeq B'_1 .

If G_{B'_1}(C) = S_3 , then G' \simeq B'_2 \cup g', B'_1 where g': V(C') \rightarrow V(C') is any bijection. But then G \simeq G'. If G_{B'_2}(C) = S_3 , then we analogously conclude that G \simeq G'.

Suppose G_{B'_2}(C) = E . Then for v_1,v_2 \in EV(B'_1) , φ_{v_1}|_C = φ_{v_2}|_C . If we define g: V(C) \rightarrow V(C') by g = φ_{v_1}|_C , we may conclude that B'_1 \simeq g B'_1 and then G \simeq G'.
Suppose $G_{B_1}(C) = C_3$. Let $u \in EV(B_1)$, $\phi_u(B_2) = B_2'$, and by Lemma 15.1, if $g = \phi_u|V(C)$, then $B_1 \simeq B_1'$ and so $G \simeq G'$.

Suppose $G_{B_2}(C) = C_3$. If for some $v \in EV(B_2)$, $B_2-v \not\simeq B_1'$, then $\phi_v(B_1) = B_1'$, and $\phi_v(B_2-v) \simeq B_2'-v'$. As above, we see that $G \simeq G'$. If for every $v \in V(B_2)$, $B_1 \simeq B_2-v$, then let $w_1$ be the vertex of $B_1$ adjacent to the vertices of $C$ and $w_2$ the vertex of $B_2$ adjacent to the vertices of $C$. Suppose the branch of $\{w_1,x,y,z\}$ at $\{x,y,w_1\}$ contains the fewest vertices. Then if $v$ is an end-vertex in the branch of $\{w_2,x,y,z\}$ at $\{x,y,w_2\}$, there is an automorphism $\theta: G-v \rightarrow G-v$ so that $\theta(B_1) = B_2-v$ and $\theta(B_2-v) = B_1$. Then there is an isomorphism $\phi_v = G-v \rightarrow G'-v'$ such that $\phi_v(B_1) = B_1'$.

If we define $g = \phi_v|V(C)$, we may conclude by Lemma 15.1, that $B_1 \simeq B_1'$, and $G \simeq G'$.

Now, suppose $G_{B_2}(C) = C_2$. If $G_{B_1}(C) = C_2$, then we may use the above techniques to determine whether or not $B_1'$ and $B_2'$ have the same orbits in $C'$. We can see that $B_1$ and $B_2$ have the same orbits in $C$ if and only if $B_1'$ and $B_2'$ have the same orbits in $C'$.

Now, suppose $G_{B_1}(C) = E$, and $G_{B_2}(C) = C_2$. If $B_2-v \not\simeq B_1$ for all $v \in EV(B_2)$, then for all such $v$, $\phi_v(B_1) = B_1'$ and for $v_1,v_2 \in EV(B_2)$, $\phi_{v_1}|C = \phi_{v_2}|C$. By defining $g: V(C) \rightarrow V(C')$ by $g = \phi_{v_1}|C$, we can conclude from Lemma 9.1 that $G \simeq G'$. Otherwise, let the orbits of $B_2$ be $\{x,y\}$ and $\{z\}$, let $g$ be the unique function such that $B_1 \simeq g B_1'$, and $x'$, $y'$, and $z'$ be $g(x)$, $g(y)$.
and \( g(z) \) respectively, \( w' \) the vertex of \( B_1 \) adjacent to \( C' \), and \( w \) be the vertex of \( B_1 \) adjacent to \( x, y \) and \( z \). Let \( D_1 \) be the branch of \( \{x,y,z,w\} \) at \( \{x,z,w\} \), \( D_2 \) the branch at \( \{y,z,w\} \) and \( D_3 \) the branch at \( \{x,y,w\} \). Suppose \( v(D_1) = 3 \) and \( v(D_2) = 3 \).

Then for every \( v_1, v_2 \in EV(B_1) \), \( \phi_{v_1}(\{x,y\}) = \{x',y'\} \) is an orbit of \( B_2' \) in \( C' \). Therefore the orbits of \( B_2' \) in \( C' \) are \( \{x',y'\} \) and \( \{z'\} \). Therefore \( G \cong G' \). The same is true if \( v(D_3) = 3 \). Suppose \( v(D_3) \geq \frac{5}{4} \) and \( v(D_1) \) or \( v(D_2) \geq \frac{5}{4} \). If \( u \in EV(D_1 \cup D_2) \), \( \phi_u(D_3) \) is the branch of \( \{x',y',z',w'\} \) at \( \{x',y',w'\} \). If \( v \in EV(D_3) \), \( \phi_u(D_1 \cup D_2) \) is the union of the branches of \( \{x',y',z',w'\} \) at \( \{x',z',w'\} \) and \( \{y',z',w'\} \). If \( \{x,y\} \) is an orbit of \( D_3 \) in \( C \), then \( G' \cong G' - u' - \phi_u(D_3 - u) \cup D_3 \cong G \), where \( D_3 \) is adjoined in such a way that \( \{x',y'\} \) is an orbit of \( D_3 \) in \( C' \). If \( \{x,y\} \) is not an orbit of \( D_3 \) in \( C \), then for any \( u_1, u_2 \in EV(B_1) \), \( \phi_{u_1 \cup u_2}(C) = \phi_{u_2}(C) \), and if we define \( g = \phi_{u_1 \cup u_2}(C) \) we may conclude by Lemma 9.1, that \( B_1 \cong B_1' \), and so \( G \cong G' \).

Now, suppose \( k_1 = k_2 \), if \( u \in EV(B_1) \), \( \phi_u(B_2) \) is a branch of \( C' \) in \( G' \). If \( v \in EV(B_2) \), \( \phi_v(B_1) \) is a branch of \( C' \) as well. Therefore \( C' \) has branches \( B_1', B_2' \) such that \( B_1 \cong B_1' \). Suppose \( B_1 \not\cong B_2 \). If \( G_{B_1}(C) = S_3, C_3 \) or \( E \), then we may employ the same techniques as in previous cases, and conclude that \( G \cong G' \). The same is true for the cases \( G_{B_2}(C) = S_3, C_3 \) or \( E \).
In the case that \( G_{B_1}(C) = G_{B_2}(C) = C_2 \), we may employ previous techniques, and conclude that \( B_1 \) and \( B_2 \) have the same orbits in \( C \) if and only if \( B_1' \) and \( B_2' \) have the same orbits in \( C' \).

Now, suppose that \( B_1 \sim B_2 \). If \( G_{B_1}(C) = S_3 \) or \( C_3 \), we may use a previous argument and conclude that \( G \sim G' \). If \( G_{B_1}(C) = C_2 \), then we may similarly conclude that \( B_1 \) and \( B_2 \) have the same orbits in \( C \) if and only if \( B_1' \) and \( B_2' \) have the same orbits in \( C' \).

Suppose \( G_{B_1}(C) = E \). We may assume that for \( u \in EV(B_2) \), \( u' \in EV(B_2') \). If \( g: V(C) \rightarrow V(C') \) is the function such that \( B_1 \sim g B_1' \), then by Lemma 9.1, \( B_2 \sim g B_2' \), and so \( G \sim G' \).

Suppose \( |L| = 1 \). Let \( v_1 \in L \) be the peripheral vertex of \( B_1 \). \( v_1' \) is the unique peripheral vertex of a branch \( B_1' \) of \( C' \). If \( |EV(B_1)| = 1 \), then \( v_1 \) and \( v_1' \) are contained in a unique \((2,3)\)-path of length \( \ell \) for each \( \ell = 1, 2, \ldots, r_{2,3}(G) \). This implies that \( |EV(B_1')| = 1 \). For all \( u \in EV(B_2) \), \( \phi_u(B_1) = B_1' \), and so \( B_1 \sim g B_1' \).

If \( G_{B_1}(C) = E \), let \( g: V(C) \rightarrow V(C') \) be the function such that \( B_1 \sim g B_1' \). For \( u \in EV(B_2) \), \( \phi_u|V(C) = g \), and so \( B_2 \sim g B_2' \) and \( G \sim G' \).

Suppose \( G_{B_1}(C) = C_2 \). Let the orbits of \( B_1 \) in \( C \) be \( \{x_1, x_2\} \) and \( \{x_3\} \), those of \( B_1' \) be \( \{x_1', x_2'\} \) and \( \{x_3'\} \), and \( x_4, x_4' \) the vertices of \( B_2 \) and \( B_2' \) adjacent to the vertices of \( C \) and \( C' \). Let \( D_1 \) and \( D_1' \) be the branches of \( H = \{x_1, x_2, x_3, x_4\} \) and
$H' = \{x'_1, x'_2, x'_3, x'_4\}$ at $H\backslash\{x'_1\}$ and $H'\backslash\{x'_1\}$ respectively. Suppose $D_1$ or $D_2$, and $D_3$ is non-trivial. If $u_1 \in EV(D_1 \cup D_2)$ and
$u_3 \in EV(D_3)$, $\phi_{u_1}(D_3) \subseteq D'_3$, and $\phi_{u_2}(D_1 \cup D_2) \subseteq D'_1 \cup D'_2$. Therefore $v(D_3) \leq v(D'_3)$ and $v(D_1 \cup D_2) \leq v(D'_1 \cup D'_2)$. But this implies that $v(D_3) = D'_3$, $v(D_1 \cup D_2) = v(D'_1 \cup D'_2)$, $\phi_{u_1}(D_3) = D'_3$, and $\phi_{u_2}(D_1 \cup D_2) = D'_1 \cup D'_2$. Therefore, $B_1 \cup D_3 \sim B'_1 \cup D'_3$ and $B_1 \cup D_2 \cup D_3 \sim B'_1 \cup D'_2 \cup D'_3$.

If $(x_1x_2) \in G_{D_1}$, where $(x_1x_2)$ is the transposition $t: (x_1, x_2) \rightarrow (x'_1, x'_2)$ where $t(x_1) = x_2$ and $t(x_2) = x_1$, then if $g: V(C) \rightarrow V(C')$ is a function such that $B_1 \cup D_2 \cup D_3 \sim g(B_1 \cup D'_2 \cup D'_3)$, $D_1 \sim g(D'_1)$ and so $G \sim G'$. If $(x'_1x'_2) \in G_{D_2 \cup D_3}$, we similarly have that $G \sim G'$. Otherwise, at least one of $D_1$ and $D_2 \cup D_3$ has $\geq 2$ end-vertices. Suppose $|EV(D_1)| \geq 2$. Then if $g: V(C) \rightarrow V(C')$ is the function such that $B_1 \cup D_2 \cup D_3 \sim g(B_1 \cup D'_2 \cup D'_3)$, then by Lemma 9.1, $D_1 \sim g(D'_1)$, and so $G \sim G'$. The case when $|EV(D_2 \cup D_3)| \geq 2$ is similar.

Suppose $D_1$ is trivial. Let $v(D_2) \leq v(D_3)$. If $u \in EV(D_2)$, then $\phi_u(B_1 \cup D_3) = B'_1 \cup D'_3$ for a branch $D'_3$. If $v \in EV(D_3)$ and $D_3 \sim v \sim D_2$ then there is a branch $D'_3$ such that $B_1 \cup D_2 \sim B'_1 \cup D'_2$. This implies that $G \sim G'$. Suppose $D_2$ and $D_3$ are trivial. Then if $v_1 \in EV(B_1)$, and $H'$ is the center of $G'-v'_1$, $C'$ is the 2-simplex of $H'$ which has a branch $B'$ with only one end-vertex. But $B' = B'_1 - v'_1$. There is a unique way to adjoin $v_1$ to $B'_1 - v'_1$ and $v'_1$ to $B'$ so that $B_1 - v_1 \cup \{v_1\} \sim B' \cup \{v'_1\} \sim B_1$. Therefore $G \sim G'$. 


Suppose $|\text{EV}(B_1)| \geq 2$, and $u_1 \in \text{EV}(B_1) \setminus \text{EV}_p(B_1)$, $\phi_{u_1}(B_2) = B'_2$, and so $B_2 \sim B'_2$. If $u_2 \in \text{EV}(B_2)$, and if $\phi_{u_2}(B_2-u_2) = B'_1$, $B_2-u_2 \sim B'_2-u_2 \sim B_1$, and so $B_1 \sim B'_1$. If $\phi_{u_2}(B_2-u_2) = B'_2-u_2$, then $B_1 \sim B'_1$.

If $G_{B_1}(C) = E$, then for all $v' \in \text{EV}(B_2)$ there is an isomorphism $\phi'_v \colon G-v \rightarrow G'-v'$ such that $\phi'_v(B_1) = B'_1$. If $g \colon V(C) \rightarrow V(C)$ is the function such that $B_1 \sim g B'_1$, then by Lemma 9.1, we may conclude that $B_1 \sim g B'_2$, and so $G \sim G'$. If $G_{B_1}(C) \neq E$, then $G_{B_1}(C) = C_2$. Let the orbits of $B_1$ in $C$ be $\{x_1, x_2\}$ and $\{x_3\}$. If $P_1$ is the longest $(2,3)$-path contained in $B_1$, and $P'_1$ is the longest $(2,3)$-path in $B'_1$, the orbits of $P_1$ and of $B_1$ in $C$ are the same. If $u_1 \in \text{EV}(B_1) \setminus \text{EV}_p(B_1)$, $\phi_{u_1}(P_1) = P'_1$. If $g = \phi_{u_1}|V(C)$, then $B_1 \sim g B'_1$ and so $G \sim G'$.

Suppose $|L| = 2$. Let $v_1$ and $v_2$ be the peripheral vertices of $B_1$ and $B_2$ respectively, and $v'_1$ and $v'_2$ be the peripheral vertices of $B'_1$ and $B'_2$ respectively. Let $P$ and $P'$ be the longest $(2,3)$-paths of $G$ and $G'$. By Kelly's Lemma, $P \sim P'$. Suppose $G_{P}(C) = E$. Then if $g \colon V(C) \cup \{v_1, v_2\} \rightarrow V(C') \cup \{v'_1, v'_2\}$ is the function such that $P \sim g P'$, we may conclude by Lemma 9.1, that $G \sim G'$.

Suppose $G_{P}(C) \neq E$. Then $G_{P}(C) = C_2$. Let $B_1$ and $B_2$ be the branches of $C$, $B'_1$ and $B'_2$ the branches of $C'$ with $v(B_1) \leq v(B_2)$,
and \( v(B_1') \leq v(B_2') \). As above, we may conclude, after relabeling \( B_1' \) and \( B_2' \), if necessary, that \( B_1 \sim B_1' \) and \( B_2 \sim B_2' \). Suppose \( G_{B_1}(C) = C_2 \). If \( u \in EV_1(B_1) \), then \( \phi_u(P_1) = P_1' \). If \( g = \phi_u|V(C) \), \( B_1 \sim g \cdot B_1' \). If \( G_{B_2}(C) = C_2 \), we similarly have that \( G \sim G' \). Suppose \( G_{B_1}(C) = G_{B_2}(C) = E \). If \( |EV(B_1)| \geq 3 \), and \( u \in EV_2(B_2) \), there is an isomorphism \( \phi'_u : G-u \to G'-u' \) such that \( \phi'_u(B_1) = B_1' \). If \( g = \phi_u|V(C) \cup L \), we may conclude by Lemma 9.1 that \( B_2 \sim B_2' \), and so \( G \sim G' \). If \( |EV(B_1)| \geq 3 \), we similarly have that \( G \sim G' \). Suppose \( |EV(B_1)| = |EV(B_2)| = 2 \). There is an isomorphism \( \phi_{v_1}' : G-v_1 \to G'-v_1' \) such that \( \phi_{v_1}'(B_2) = B_2' \). There is a unique longest \((2,3)\)-path \( Q \) of \( G-v_1 \) and \( Q' \) of \( G'-v_1' \) such that \( G_q(C) = C_2 \). There is a unique way to adjoin \( v_1 \) to \( B_1-v_1 \) and \( v_1' \) to \( B_1'-v_1' \) so that \( Q \cup v_1 \sim Q' \cup v_1' \). This implies that \( G \sim G' \).

Case 6: \( G \) contains a \( 4 \)-center \( C \), which has \( 4 \) peripheral branches. By Kelly's Lemma, the longest \((2,3)\)-paths of \( G \) and \( G' \) are of the same length, and so \( G' \) has a \( 4 \)-center \( C' \). If \( |EV(G)| = 4 \), and \( r_{2,3}(G) = 2 \), then \( r_{2,3}(G') = 2 \), and the end-vertices of \( G \) and \( G' \) are adjacent to 3 vertices in the center. If a branch of \( C' \) contains 2 end-vertices \( v_1 \) and \( v_2 \), let \( v_3 \in EV(G') \setminus \{v_1, v_2\} \). Then \( C \) contains a branch in \( G-v_3 \) with 2 end-vertices. This is impossible. Therefore, each branch of \( C' \) in \( G' \) contains one end-vertex and \( G \sim G' \). If \( |EV(G)| = 4 \) and \( r_{2,3}(G) \geq 3 \), then by symmetry
and the above argument, \( r_{2,3}(G) \geq 3 \). If \( v \in EV(G) \), then
\[ \deg_{G - v} C = 4 \], and so \( \deg_{G' - v'} C' = 4 \). Since \( G' \) has no non-peripheral end-vertices, each branch of \( C \) in \( G \) is peripheral. Otherwise, if \( |EV(G)| \geq 5 \), let \( v \in EV(G) \), such that each branch of \( C \) in \( G - v \) is peripheral. Then each branch of \( C' \) in \( G' - v' \) is peripheral. In any case, \( C' \) has 4 peripheral branches in \( G' \).

Let the branches of \( C \) in \( G \) be \( B_1, B_2, B_3 \) and \( B_4 \), \( k_i = v(B_i) \), \( i = 1,2,3,4 \), and \( k_1 \leq k_2 \leq k_3 \leq k_4 \). Let \( v \in EV(B_1) \). \( C' \) has branches in \( G' - v' \) with \( k_1 - 1, k_2, k_3 \) and \( k_4 \) vertices. If \( C' \) has a branch \( B' \) in \( G' \) with \( k_1 - 1 \) vertices, then let \( u' \in EV(B') \). \( C \) has a branch in \( G - u \) with \( k_1 - 2 \) vertices. This is impossible. Therefore \( C' \) has branches in \( G' \) with \( k_1, k_2, k_3, \) and \( k_4 \) vertices.

Case i: \( k_3 < k_4 \). Let \( B'_4 \) be the branch of \( C' \) in \( G' \) with \( k_4 \) vertices. If \( u \in EV(B'_1) \), then \( \phi_u(B'_4) = B'_4 \), and \( B'_4 \sim B'_4 \).

Suppose \( G_{B_4}(C) = E \). Let \( g: V(C) \to V(C) \) be the unique function such that \( B'_4 \sim g B'_4 \). Then for all \( u \in EV(G) \setminus EV(B_4) \), \( \phi_u|V(C) = g \). By Lemma 9.1, we may conclude that \( B_1 \cup B_2 \cup B_3 \sim g G'(V(B'_4) \setminus V(C)) \), and so \( G \sim G' \).

Suppose \( G_{B_4}(C) = C_2 \). Let \( V(C) = \{x,y,z,w\} \), and the orbits of \( B_4 \) in \( C \) be \( \{x,y\}, \{z\}, \) and \( \{w\} \), where \( B_4 \) is a branch of \( C \) at \( \{x,y,z\} \). Let \( B_x \) be the branch of \( C \) at \( \{x,y,w\} \), \( B_y \) the branch of \( C \) at \( \{x,w,z\} \) and \( B_z \) the branch of \( C \) at \( \{y,w,z\} \). If \( u \in EV(G) \setminus EV(B_4) \), then \( \phi_u(B_4) = B'_4 \). Let \( x' = \phi_u(x), y' = \phi_u(y), \)
$z' = \phi_u(z)$ and $w' = \phi_u(w)$. If $u \not\in \mathrm{EV}(G) \setminus \mathrm{EV}(B_4)$, then

$\phi_v(z) = z'$, $\phi_v(w) = w'$, and $\phi_v([x',y']) = [x',y']$. Let $B'_z$ be the branch of $C'$ at $[x',y',w']$, $B'_y$ the branch of $C'$ at $[x',w',z']$ and $B'_x$ the branch of $C'$ at $[y',w',z']$. If $u \in \mathrm{EV}(B'_x) \cup \mathrm{EV}(B'_y)$, then $v(B'_z - u') = v(B'_z)$, and if $v \in \mathrm{EV}(B'_z)$, then $v(B'_z - v') = v(B'_z) - 1$. Therefore, $u' \in \mathrm{EV}(B'_x) \cup \mathrm{EV}(B'_y)$, and $v' \in \mathrm{EV}(B'_z)$. Now, $\phi_u(B'_4 \cup B'_z) = B'_4 \cup B'_z$. If $(xy) \in G_{B'_z}(C)$, then

$G_{B'_z \cup B'_4}(C) = G_{B'_z}(C)$. $\phi_v(B'_4 \cup B'_x \cup B'_y) = B'_4 \cup B'_x \cup B'_y$ and

$G' \simeq B'_4 \cup B'_x \cup B'_y \cup g_{B'_z}$ where $g : V(C') \to V(C)$ is any function such that $g([x',y']) = [x',y']$, $g(y') = z'$ and $g(w') = w'$. But then $G \simeq G'$. If $(xy) \not\in G_{B'_z}(C)$, then $G_{B'_z \cup B'_4}(C) = E$. Let $g : V(C) \to V(C')$ be the function such that $B'_4 \simeq g_{B'_4}$ and $B'_z \simeq g_{B'_z}$. For each $u \in \mathrm{EV}(B'_x) \cup \mathrm{EV}(B'_y)$, $\phi_u|V(C) = g$. By Lemma 9.1, we may conclude that $B'_x \cup B'_y \simeq g_{B'_x} \cup B'_y$ and then $G \simeq G'$.

Suppose $G_{B'_z}(C) = C_3$. If $k_2 < k_3$, then $G_{B'_3 \cup B'_4}(C) = E$. Let $B'_3$ be the branch of $C'$ in $G'$ with $k_3$ vertices, and $u \in \mathrm{EV}(B'_1 \cup B'_2)$.

$\phi_u(B'_3 \cup B'_4) = B'_3 \cup B'_4$. Let $g = \phi_u|V(C)$. For every $v \in \mathrm{EV}(B'_1 \cup B'_2)$, $\phi_v|V(C) = g$, and we may conclude by Lemma 9.1, that $B'_1 \cup B'_2 \simeq G' - (B'_3 \cup B'_4) - C$, and then $G \simeq G'$. If $k_1 < k_2 = k_3$, then $G_{B'_2 \cup B'_3 \cup B'_4}(C) = E$. Let $u \in \mathrm{EV}(B'_1)$. $B'_2 = \phi_u(B_2)$, $B'_3 = \phi_u(B_3)$, and $B'_4 = \phi_u(B_4)$ are branches of $C'$ in $G'$, and $B'_2 \cup B'_3 \cup B'_4 \simeq B'_2 \cup B'_3 \cup B'_4$. Let $g = \phi_u|V(C)$. If $|\mathrm{EV}(B'_1)| \geq 2$, then for each
v \in E(V(B_1))$, $\phi_v|V(C) = g$, and we may conclude by Lemma 9.1 that if $B_1'$ is the branch of $C'$ with $k_1$ vertices, then $B_1 \sim B_1'$ and so $G \sim G'$. If $|E(V(B_1))| = 1$, let $B_1'$ be the branch of $C'$ with $k_1$ vertices, and $v \in V(B_2)$ adjacent to 3 vertices of $C$.

Then $\phi_v(B_1 \cup B_4) = B_1' \cup B_4'$. $G_{B_1 \cup B_4}(C) = E$, and so there is a unique way to adjoin $B_1'$ to $B_2 \cup B_3 \cup B_4'$ such that $B_1 \cup B_4 \sim B_1' \cup B_4'$. But then $G \sim G'$. If $k_1 = k_2 = k_3$, let $V(C) = \{x,y,z,w\}$ where $B_4$ is the branch at $\{x,y,z\}$, and $V(C') = \{x',y',z',w'\}$ where $B_4'$ is the branch at $\{x',y',z'\}$. Let $h: V(C) \to V(C)$ be defined by $h(x) = y$, $h(y) = z$, $h(z) = x$, and $h(w) = w$, and $h': V(C') \to V(C')$ be defined by $h'(x') = y'$, $h'(y') = z'$, $h'(z') = x'$, and $h'(w') = w'$. We say that $B_i \overset{r}{\sim} B_j$ if $B_i \overset{h}{\sim} B_j$, or $B_j \overset{h}{\sim} B_i$, and $B_i \overset{r}{\sim} B_j$ if $B_i \overset{h}{\sim} B_j$, or $B_j \overset{h}{\sim} B_i$. If $B_1 \overset{r}{\sim} B_2 \overset{r}{\sim} B_3$, then for $u \in E(V(B_1))$, $\phi_u(B_2) \overset{r}{\sim} \phi_u(B_3)$. Similarly, we may conclude that if $B_1'$, $B_2'$, and $B_3'$ are the branches of $C'$ with $k_1$ vertices, then $B_1' \overset{r}{\sim} B_2' \overset{r}{\sim} B_3'$. Since $\phi_u(B_2 \cup B_4) \overset{r}{\sim} B_2 \cup B_4$, this implies that $G \sim G'$. Suppose one branch of $C$, say $B_1$, is not isomorphic to either of the others with respect to $r$. Let $u_2 \in E(V(B_2))$ and $u_3 \in E(V(B_3))$. Then $\phi_{u_2}(B_1) \not\overset{r}{\sim} \phi_{u_2}(B_3)$, and $\phi_{u_3}(B_1) \not\overset{r}{\sim} \phi_{u_3}(B_2)$. Therefore if $B_1' = \phi_{u_2}(B_1)$, there is no branch $B' \neq B_1'$ such that $B' \overset{r}{\sim} B_1'$. Now, $G_{B_1 \cup B_4}(C) = E$, and if $g: V(C) \to V(C')$ is the function such that $B_1 \cup B_4 \overset{g}{\sim} B_1' \cup B_4'$, then for each $u \in E(V(B_2 \cup B_3))$, 

...
\( \phi_u|V(C) = g \), and by Lemma 9.1, if \( B_2^1 \) and \( B_3^1 \) are the two other branches of \( C' \), then \( B_2 \cup B_3 \cong B_2^1 \cup B_3^1 \), and \( G \cong G' \).

Suppose \( G_{B_4^1}(C) = S_3 \). Let \( V(C) = \{x,y,z,w\} \) with \( B_4 \) the branch of \( C \) at \( \{x,y,z\} \), and \( V(C') = \{x',y',z',w'\} \) with \( B_4' \) the branch of \( C' \) at \( \{x',y',z'\} \). If \( k_2 < k_3 \), and \( G_{B_3^1 \cup B_4^1}(C) = E \), then let \( B_3^1 \) be the branch of \( C' \) with \( k_3 \) vertices, and \( u \in EV(B_1) \).

If we define \( g = \phi_u|V(C) \), then for all \( v \in EV(B_1 \cup B_2) \), \( \phi_v(B_3 \cup B_4) = B_3^1 \cup B_4^1 \), and \( \phi_v|V(C) = g \). We may conclude by Lemma 9.1 that if \( B_1^1 \) and \( B_2^1 \) are the two other branches of \( C' \), then \( B_1 \cup B_2 \cong B_1^1 \cup B_2^1 \), and so \( G \cong G' \). If \( k_2 < k_3 \) and \( G_{B_3^1}(C) \neq E \), then if \( B_3 \) is the branch of \( C \) at \( \{x,y,w\} \) then \( G_{B_3 \cup B_4^1}(C) = \{(x,y),e\} \). Again, let \( B_3^1 \) be the branch of \( C' \) with \( k_3 \) vertices, and we may suppose that \( B_3^1 \) is the branch of \( C' \) at \( \{x',y',w'\} \). If we show that \( C' \) has branches \( B_1^1 \) and \( B_2^1 \) such that \( B_1 \cup B_3 \cup B_4 \cong B_1^1 \cup B_3^1 \cup B_4^1 \) and \( B_2 \cup B_3 \cup B_4 \cong B_2^1 \cup B_3^1 \cup B_4^1 \), this will imply that \( G \cong G' \). Let \( v \in EV(B_1^1) \). \( B_2^1 = \phi_v(B_2) \) is a branch of \( C' \) such that \( B_2 \cup B_3 \cup B_4 \cong B_2^1 \cup B_3^1 \cup B_4^1 \). If \( k_1 < k_2-1 \), then for \( v \in EV(B_2^1) \), \( B_2^1 = \phi_v(B_1^1) \) is a branch of \( C' \), and \( B_1 \cup B_3 \cup B_4 \cong B_1 \cup B_3 \cup B_4^1 \). If \( k_1 = k_2-1 \), and \( v \in EV(B_2) \), then \( v' \in EV(B_2^1) \). Let \( h : V(C) \rightarrow V(C) \) be defined by \( h(x) = y \), \( h(y) = x \), \( h(z) = z \), and \( h(w) = w \). For all such \( v \), \( C' \) has a branch \( B \) in \( G'-v' \) isomorphic to \( B_1 \). If for any \( v_1^1 \), \( v_2 \in EV(B_2) \), \( B_2^1-v_2 \cong B_2-v_2 \), and \( B_1 \not\sim B_2-v_2 \), then \( \phi_v(B_1) \neq B_2-v_2 \), and so
$B_1 \cup B_3 \cup B_4 \cong B_1' \cup B_3' \cup B_4'$. If $B_1 \cong B_2 - v_1$ for some $v_1 \in EV(B_2)$, then $B_1 \cup B_3 \cup B_4 \cong B_1' \cup B_3' \cup B_4'$. Otherwise, $B_1'$ is the only branch of $C'$ with $k_1$ vertices which appears in all $G - v$ for $v \in EV(B_2)$. If $v \in EV(B_2)$ such that $B_1 \not\cong B_2 - v$ then $B_1' = \phi_v(B_1)$ is a branch $B_1'$ such that $B_1 \cup B_3 \cup B_4 \cong B_1' \cup B_3' \cup B_4'$. If $k_1 = k_2$, then let $u_1 \in EV(B_1)$ and $u_2 \in EV(B_2)$. $B_1 = \phi_{u_2}(B_1)$ is a branch such that $B_1 \cup B_3 \cup B_4 \cong B_1' \cup B_3' \cup B_4'$. It may happen that $B_1' = B_2'$. By symmetry, $C$ has a branch $B$ such that $B \cup B_3 \cup B_4 \cong B_1' \cup B_3' \cup B_4'$ and a branch $B'$ such that $B' \cup B_3 \cup B_4 \cong B_2' \cup B_3' \cup B_4'$. We may therefore assume that $B_1' = B_1$. Therefore $G \cong G'$.

Now, suppose $k_2 = k_3$. If $k_1 < k_2 - 1$ and $G_{B_1 \cup B_4}(C) = E$, then if $B_1'$ is the branch of $C'$ with $k_1$ vertices, and $u \in EV(B_2 \cup B_3)$, $\phi_u(B_1 \cup B_4) = B_1' \cup B_4'$. Let $g = \phi_u|V(C)$. For every $v \in EV(B_2 \cup B_3)$, $\phi_v|V(C) = g$. If $B_2'$ and $B_3'$ are the branches of $C'$ with $k_2$ and $k_3$ vertices, we may conclude by Lemma 9.1 that $B_2 \cup B_3 \cong g B_2 \cup B_3'$, and so $G \cong G'$. If $k_1 < k_2 - 1$, and $G_{B_1 \cup B_4}(C) \neq E$, then if $\{x,y\} = V(B_1) \cap V(B_4)$, and $G_{B_1 \cup B_4}(C) = \{e,(xy)\}$. Again, if $u \in V(B_2 \cup B_3)$, $\phi_u(B_1 \cup B_4) = B_1' \cup B_4'$, where $B_1'$ is the branch of $C'$ with $k_1$ vertices. If $v \in EV(B_1)$, $v' \in EV(B_1')$, and $B_2'$ and $B_3'$ are the branches of $C'$ with $k_2$ vertices, then $\phi_v(B_2 \cup B_3 \cup B_4) = B_2' \cup B_3' \cup B_4'$. If $\{x',y'\} = V(B_1') \cap V(B_1)$, then $\phi_v|V(B_2 \cup B_3 \cup B_4) \cup \phi_u|V(B_1) \setminus V(C)$ is an isomorphism from $G$ to $G'$. Suppose $k_1 = k_2 - 1$. 

Since \( r_{2,3}(G) \geq 3 \), there are vertices \( v_2 \in V(B_2) \) and \( v_3 \in V(B_3) \) such that \( v_2, v_3 \notin EV(G) \), and \( v_2, v_3 \) are adjacent to 3 vertices of \( C \). \( \phi_{v_2}(B_1 \cup B_4) = B'_1 \cup B'_4 \) where \( B'_1 \) is the branch of \( C' \) with \( k_1 \) vertices. Let \( B'_2 \) and \( B'_3 \) be the branches of \( C \) with \( k_2 \) vertices. There is a branch \( B' = \phi_{v_3}(B_2) \) of \( C' \) such that

\[
B_1 \cup B_2 \cup B_4 \sim B'_1 \cup B'_1 \cup B'_4 ,
\]
and a branch \( B'' = \phi_{v_2}(B_3) \) such that

\[
B_1 \cup B_3 \cup B_4 \sim B'_1 \cup B'' \cup B'_4 .
\]
By symmetry, there are branches \( B^* \) and \( B^{**} \) of \( C \) such that \( B_1 \cup B^* \cup B_4 \sim B'_1 \cup B'_2 \cup B'_4 \) and

\[
B_1 \cup B^{**} \cup B_4 \sim B'_1 \cup B'_3 \cup B'_4 .
\]
But this implies that we may relabel \( B_2' \), \( B_3' \) if necessary, so that \( B_1 \cup B_2 \cup B_4 \sim B'_1 \cup B'_2 \cup B'_4 \) and

\[
B_1 \cup B_3 \cup B_4 \sim B'_1 \cup B'_3 \cup B'_4 .
\]

Now suppose \( k_1 = k_2 = k_3 \). Let \( B'_1 \), \( B'_2 \) and \( B'_3 \) be the branches of \( C' \) with \( k_1 \) vertices. Let \( \{x, y, z\} = V(C) \cap V(B'_4) \), and \( V(C) = \{x, y, z, w\} \). We will say \( B'_1 \sim_w B'_j \) if there is an isomorphism \( \phi: B'_1 \rightarrow B'_j \) such that \( \phi(w) = w \). Let \( h: V(C) \rightarrow V(C) \) be defined by \( h(x) = y \), \( h(y) = z \), \( h(z) = x \), and \( h(w) = w \). Let \( V(C') = \{x', y', z', w'\} \) and \( V(C') \cap V(B'_4') = \{x', y', z', w'\} \). Similarly, if \( B'_i \) and \( B'_j \) are branches of \( C' \), we say \( B'_i \sim_{w'} B'_j \) if there is an isomorphism \( \phi': B'_i \rightarrow B'_j \) such that \( \phi'(w') = w' \). We also define \( h' = V(C') \rightarrow V(C') \) by \( h'(x') = y' \), \( h'(y') = z' \), \( h'(z') = x' \), and \( h'(w') = w' \). We say \( B'_i \sim_r B'_j \) if \( B'_i \sim_h B'_j \) or \( B'_j \sim_h B'_i \), and \( B'_i \sim_r B'_j \) if \( B'_i \sim h' B'_j \), or \( B'_j \sim h' B'_i \). Suppose

\[
B_1 \sim_r B_2 \sim_r B_3 .
\]
Then if \( u \in EV(B_1) \), \( B_2 \sim \phi_u(B_2) \sim_r \phi_u(B_3) \).
Similarly, for any two branches of $B_i', B_j'$ of $C'$, $B_i \sim_{r'} B_j'$.

But then $B_1 \sim_{w} B_1, B_2 \sim_{r} B_2'$, and so $G \sim G'$. If $B_1 \sim_{r} B_2 \sim_{r} B_3$, but $B_1 \not\sim_{w} B_2$, then let $u_i \in EV(B_i)$ for $i = 1, 2, 3$. Let $\phi_{u_1}(B_2) \sim_{w} \phi_{u_2}(B_3)$, and $\phi_{u_3}(B_1) \not\sim_{w} \phi_{u_3}(B_2)$, but $\phi_{u_3}(B_1) \not\sim_{w} \phi_{u_3}(B_2)$.

Therefore, after relabeling the branches of $C'$ if necessary, we conclude that $B_1 \sim_{w} B_1, B_2 \sim_{r} B_2'$ but $B_1 \not\sim_{w} B_2$. But then $G \sim G'$.

Suppose $B_1 \not\sim_{w} B_2$ and $B_1 \not\sim_{w} B_3$. If $B_1 \sim_{w} B_2 \sim_{w} B_3'$, then by symmetry and the above case, $B_1 \sim_{w} B_2 \sim_{w} B_3$. Let $u_i \in EV(B_i)$, for $i = 1, 2, 3$. Let $\phi_{u_1}(B_1) \sim_{w} \phi_{u_2}(B_1)$, $\phi_{u_2}(B_2) \not\sim_{w} \phi_{u_3}(B_2)$, and $\phi_{u_3}(B_3)$. By relabeling, if necessary, $B_1' = \phi_{u_2}(B_1)$. Now

$$\phi_{u_2}(B_1 \cup B_4) = B_1' \cup B_4'. \text{ If } G_{B_1 \cup B_4}(C) = E, \text{ then let } g = \phi_{u_2}|V(C).$$

For every $u \in EV(B_2 \cup B_3)$, $\phi_u|V(C) = g$, and we may conclude by Lemma 9.1 that $B_2 \cup B_3 \sim g B_2 \cup B_3'$ and $G \sim G'$. If $G_{B_1 \cup B_4}(C) \neq E$, then if $(x, y) = V(B_1) \cap V(B_4)$, $G_{B_1 \cup B_4}(C) = \{e, (xy)\}$.

$\phi_{u_1}(B_2 \cup B_3 \cup B_4) = B_1' \cup B_3' \cup B_4'$. But $G' \sim G' \backslash (B_1 \backslash C) \cup B_1$, where $B_1 \cup B_4' \sim B_1 \cup B_4$, and then $G \sim G'$.

Case ii: $k_2 < k_3 = k_4$. Let $u \in EV(B_1)$. Let $B_3' = \phi_u(B_3)$ and $B_4' = \phi_u(B_4)$ are the branches of $C'$ with $k_3$ vertices. Let $B_1'$ and $B_2'$ be the branches of $C'$ with $k_1$ and $k_2$ vertices. If $G_{B_3 \cup B_4}(C) = E$, let $g = \phi_u|V(C)$. For every $v \in EV(B_1 \cup B_2)$, $\phi_v|V(C) = g$, and we may conclude by Lemma 9.1 that $B_1 \cup B_2 \sim g B_1 \cup B_2$. \hspace{1cm}
and so $G \cong G'$. Otherwise, let $\{x, y, z\} = V(C) \cap V(B_3)$ and $\{x, y, w\} = V(C) \cap V(B_4)$. By relabeling $B'_1$ and $B'_2$ if necessary, we may assume that $\phi_u(B_2) = B'_1$, and therefore that $B_2 \cup B_3 \cup B_4 \cong B'_2 \cup B'_3 \cup B'_4$.

Suppose $G_{\overline{B_3 \cup B_4}}(C) = \{e, (xy)\}$. If $k_1 < k_2 - 1$, then if $u_1 \in EV(B_1)$, $u'_1 \in EV(B'_1)$ and if $u_2 \in EV(B_2)$, then $u'_2 \in EV(B'_2)$, and so if $g = \phi_{u_1}|V(C)$, and $v \in EV(B_1 \cup B_2)$, then $\phi_v|V(C) = g$, and we may conclude by Lemma 9.1 that $B_1 \cup B_2 \cong B'_1 \cup B'_2$, and therefore that $G \cong G'$. If $k_1 = k_2$, let $u_2 \in EV(B_2)$, $B' = \phi_{u_2}(B_2)$ is a branch of $C'$ such that $B_1 \cup B_2 \cup B_4 \cong B'_1 \cup B'_3 \cup B'_4$. By symmetry $C$ has branches $B^*$ and $B^{**}$ such that $B^* \cup B_3 \cup B_4 \cong B'_1 \cup B'_3 \cup B'_4$, and $B^{**} \cup B_3 \cup B_4 \cong B'_2 \cup B'_3 \cup B'_4$. Therefore, we may assume, by relabeling $B'_1$ and $B'_2$ if necessary, that $B_1 \cup B_3 \cup B_4 \cong B'_1 \cup B'_3 \cup B'_4$ and $B_1 \cup B_3 \cup B_4 \cong B'_2 \cup B'_3 \cup B'_4$, and so that $G \cong G'$. If $k_1 = k_2 - 1$, let $u_1 \in EV(B_1)$ and $g = \phi_{u_1}|V(C)$.

Let $v_2 \in V(B_2)$, adjacent to 3 vertices of $C$, and such that $v_2$ is not an end-vertex. $\phi_{v_2}(B_1) = B'_1$, and so $B_1 \cong B'_1$. If $u_2 \in EV(B_2)$ and $\phi_{u_2}(B_2 - u_2) = B'_1$, then $B_1 \cong B_2 - u_2$, and there is an isomorphism $\phi'_{u_2} : G - u_2 \to G' - u_2$ such that $\phi'_{u_2}(B_1) = B'_1$. Therefore for all $v \in EV(B_1 \cup B_2)$, there is an isomorphism $\phi' : G - v \to G' - v'$ such that $\phi'|V(C) = g$, and therefore we may conclude by Lemma 9.1, that $B_1 \cup B_2 \cong B'_1 \cup B'_2$, and so $G \cong G'$. 


Suppose there is an isomorphism \( \alpha: B_3 \to B_4 \) such that 
\( \alpha((x,y)) = (x,y) \). If \( k_1 < k_2 - 1 \), then let \( u_1 \in EV(B_1) \) and 
\( u_2 \in EV(B_2) \). \( \phi_{u_1}(B_2) = B'_2 \) and \( \phi_{u_2}(B_1) = B'_1 \). If \( (zw) \in G_{B_1} \), 
then \( G' \simeq G' - u_1 \setminus (B'_1 \setminus C') \cup h B_1 \) where \( h \) is a function such that 
\( h(V(B_1) \cap V(B_3) \cap V(B_4)) = V(B'_1) \cap V(B'_3) \cap V(B'_4) \). If \( (zw) \notin G_{B_1} \), 
then let \( g: V(C) \to V(C') \) be the function such that \( B_1 \cup B_3 \cup B_4 \simeq_{g} B'_1 \cup B'_3 \cup B'_4 \). For every \( v \in EV(B_2) \), \( \phi_v|V(C) = g \), and since 
k_2 > k_1, \( |EV(B_2)| \geq 2 \). We may conclude by Lemma 9.1 that \( B_2 \simeq_{g} B'_2 \) 
and so \( G \simeq G' \).

If \( k_1 = k_2 - 1 \), let \( u_1 \in EV(B_1) \). \( \phi_{u_1}(B_2) = B'_2 \) and so \( B_2 \simeq_{g} B'_2 \) 
let \( v_2 \in V(B_2) \), such that \( v_2 \notin EV(B_2) \), and \( v_2 \) is adjacent to 
the vertices of \( C \cap B_2 \). Then \( \phi_{v_2}(B_1) = B'_1 \), and so \( B_1 \simeq_{g} B'_1 \). If 
\( (zw) \in G_{B_1} \), then by the same reasoning as above \( G \simeq G' \). If 
\( (zw) \notin G_{B_1} \), and \( u \in EV(B_2) \), and if \( \phi_u(B_1) = B'_2 - u' \), there is 
an isomorphism \( \theta: G-u \to G-u \) such that \( \theta(B_1) = B_2 - u \), \( \theta(B_2 - u) = B_1 \), 
and \( \theta([z,w]) = [z,w] \). We conclude that there is an isomorphism 
\( \phi'_u = G-u \to G'-u' \) such that \( \phi'_u(B_1) = B'_1 \). If \( g: V(C) \to V(C') \) is 
the function such that \( B_1 \cup B_3 \cup B_4 \simeq_{g} B'_1 \cup B'_3 \cup B'_4 \), then we may 
conclude that \( B_2 \simeq_{g} B'_2 \), and so that \( G \simeq G' \).

If \( k_1 = k_2 \), let \( u_1 \in EV(B_1) \), and \( u_2 \in EV(B_2) \). \( \phi_{u_1}(B_2) \) and 
\( \phi_{u_2}(B_1) \) are branches of \( C' \) in \( G' \). By symmetry \( C \) has branches 
\( B_1^* \) and \( B_2^* \) such that \( B_1^* \simeq_{g} B'_1 \) and \( B_2^* \simeq_{g} B'_2 \). We may assume that
If \((zw) \in G_{B_1}(C)\), then similarly, \(G \sim G'\). Suppose \((zw) \notin G_{B_1}(C)\), and \((zw) \notin G_{B_2}(C)\). If \(|EV(B_1)| = |EV(B_2)| = 1\), then we may assume \(u_1' \in EV(B_1')\) and \(u_2' \in EV(B_2')\). So \(\phi_{u_1'}|V(C) = \phi_{u_2'}|V(C)\). If \(g = \phi_{u_1'}|V(C)\), then we conclude by Lemma 9.1 that \(B_1 \cup B_2 \sim B_1' \cup B_2'\) and so \(G \sim G'\). If \(|EV(B_1)| \geq 2\), then since \(k_1 = k_2\), \(|EV(B_2)| \geq 2\). If \(B_1 \not\sim_h B_2\) where \(h: V(C) \to V(C)\) is such that \(h(z,w) = (z,w)\), then for every \(u \in EV(B_2)\), \(\phi_u(B_1) = B_1'\), and if \(g: V(C) \to V(C')\) is such that \(B_1 \cup B_3 U B_4 \sim g B_1' \cup B_3' U B_4'\), we conclude from Lemma 9.1 that \(B_2 \sim g B_2'\). If \(B_1 \sim h B_2\), then let \(\theta: B_1 \to B_2\) be an isomorphism such that \(\theta|V(C) = h\). For \(u \in EV(B_1)\), \(G-u \sim G-\theta(u)\). Then we may assume that for \(u \in EV(B_1)\), \(u' \in EV(B_1')\). If \(g: V(C) \to V(C')\) is the function such that \(B_2 \cup B_3 U B_4 \sim g B_1' \cup B_3' U B_4'\), we may conclude by Lemma 9.1, that \(B_1 \sim g B_1'\), and so \(G \sim G'\).

Case iii: \(k_1 < k_2 = k_3 = k_4\). Again we let \(B_1'\) be the branch of \(C'\) with \(k_1\) vertices. Let \(u_1 \in EV(B_1)\), then \(\phi_{u_1}(B_2 U B_3 U B_4) = B_2' U B_3' U B_4'\). If \(k_1 < k_2 - 1\) then let \(v \in EV(B_2 U B_3 U B_4)\). If \(k_1 = k_2 - 1\), let \(v \in V(B_2 U B_3 U B_4) \setminus EV(B_2 U B_3 U B_4)\). In either case \(\phi_v(B_1) = B_1'\), and so \(B_1 \sim B_1'\). Suppose \(G_{B_1}(C) = E\). If \(k_1 < k_2 - 1\), let \(u_i \in EV(B_i)\) for \(i = 2,3,4\). If \(k_1 = k_2 - 1\), let
u_1 \in V(B_1) \setminus EV(B_1)$. In either case, let $g: V(C) \to V(C')$ be the function such that $B_1 \sim g B_1$. Then $\phi_{u_3} (B_1 \cup B_2) = B_1 \cup B''$ for some $B''$.

Therefore there is a branch $B''$ of $C'$ such that $B_1 \cup B_2 \sim g B_1 \cup B''$. Similarly there are branches $B''$ and $B'''$ such that $B_1 \cup B_3 \sim g B_1 \cup B''$ and $B_1 \cup B_4 \sim g B_1 \cup B''$. This implies that $G \sim G'$.

Suppose $G_{B_1}(C) = S_3$. If $g: V(C) \to V(C')$ is any function such that $B_1 \cup B_2 \cup B_3 \cup B_4 \sim g B_1 \cup B_3 \cup B_4$, then since $G_{B_1}(C) = S_3$, $B_1 \sim B_1$ and so $G \sim G'$. Suppose $G_{B_1}(C) = C_3$. If $u_1 \in EV(B_1)$ and $g = \phi_{u_1}|V(C)$, then by Lemma 15.1, $B_1 \sim g B_1$, and so $G \sim G'$. Suppose $G_{B_1}(C) = C_2$.

Let the orbits of $B_1$ in $C$ be $\{x_3, x_4\}, \{x_2\}, \{x_1\}$ where $B_1$ is the branch of $C$ at $\{x_1, x_2, x_3, x_4\}$. Suppose $B_2$ is the branch of $C$ at $\{x_1, x_3, x_4\}$. If $k_1 < k_2 - 1$, let $u_3 \in EV(B_3)$. If $k_1 = k_2 - 1$, let $u_3 \in V(B_3) \setminus EV(B_3)$. In either case, $\phi_{u_3} (B_1 \cup B_2) = B_1 \cup B''$ where $B''$ is a branch of $C'$. If $(x_3, x_4) \notin G_{B_2}(C)$, then let $g: V(C) \to V(C')$ be the function such that $B_1 \cup B_2 = g B_1 \cup B''$. Then there is a branch $B''''$ of $C'$ such that $B_1 \cup B_2 \cup B_3 \sim g B_1 \cup B'' \cup B''''$. Similarly there is a branch $B''''$ of $C'$ such that $B_1 \cup B_2 \cup B_4 \sim g B_1 \cup B'' \cup B''''$. This implies that $G \sim G'$. Suppose $(x_3, x_4) \in G_{B_2}(C)$.

If $k_1 < k_2 - 1$, let $u_2 \in EV(B_2)$. If $k_1 = k_2 - 1$, let $u_2 \in V(B_2) \setminus EV(B_2)$. In either case, $\phi_{u_2} (B_1 \cup B_3 \cup B_4) = B_1 \cup B'' \cup B''''$ for some branches of $C'$, $B''$ and $B''''$. If $g: V(C) \to V(C')$ is any function such that
\[ B_1 \cup B_3 \cup B_4 \cong g \circ B_1 \cup B'' \cup B^iv, \] then since \((x_3x_4) \in G_{B_2}(C),\]
\[ B_2 \cong g \circ B'', \] and so \( G \cong G' \).

Case iv: \( k_1 = k_2 = k_3 = k_4 \). If \( B_1 \cong B_2 \cong B_3 \cong B_4 \), then for every \( v \in EV(G) \), \( C \)' has 3 branches in \( C'-v \)' isomorphic to each other, and to \( B_1 \). But then \( B_1 \cong B_1 \cong B_2 \cong B_3 \cong B_4 \). Otherwise, if \( B_1 \) is a branch of \( C \) in \( G \), the number of branches \( B \) such that \( B \cong B_1 \) is the maximum number in any \( G-u \) for \( u \in EV(G) \).

But the same is true for a branch \( B_1 \) of \( C \)' . Therefore \( C \) and \( C \)' have the same number of branches isomorphic to a given branch.

Suppose \( G_{B_1}(C) = S_3 \) for some \( i \), (say \( i = 1 \)). Let \( u_1 \in EV(B_1) \), \( \phi_{u_1}(B_2 \cup B_3 \cup B_4) = B_2 \cup B_3 \cup B_4 \) for some branches \( B_2 \cup B_3 \cup B_4 \). Since a branch occurs with the same frequency in \( G \) as in \( G' \), \( B_1 \cong B_1 \). If \( g: V(C) \to V(G') \) is a function such that \( B_2 \cup B_3 \cup B_4 \cong g \circ B_2 \cup B_3 \cup B_4 \). Since \( G_{B_1}(C) = S_3 \), \( B_1 \cong g \circ B_1 \), and \( G \cong G' \).

Suppose \( G_{B_1}(C) = C_3 \) for some \( i \), (say \( i = 1 \)). Let \( u_1 \in EV(B_1) \), and \( u_1' \in EV(B_1') \). If \( g = \phi_{u_1}|V(C) \), then by Lemma 15.1
\[ B_1 \cong g \circ B_1 \), and so \( G \cong G' \).

We may assume that for each \( i \), \( G_{B_1}(C) = C_2 \) or \( E \).

Suppose one branch (say \( B_1 \)) is not isomorphic to any other.
Let \( B_1 \) be the branch of \( C \) such that \( B_1 \cong B_1 \), and \( B_2 \), \( B_3 \),
and $B_4'$ the other branches of $C'$. If $u_1 \in \text{EV}(B_1)$, then

$\phi_{u_1}(B_2 \cup B_3 \cup B_4) = B_2' \cup B_3' \cup B_4'$, and so $B_2 \cup B_3 \cup B_4 \sim B_2' \cup B_3' \cup B_4'$.

If $G_{B_1}(C) = E$, then let $g: V(C) \to V(C')$ be the function such that $B_1 \sim g B_1'$. For every $v \in \text{EV}(B_2 \cup B_3 \cup B_4)$, $\phi_v(B_1) = B_1'$ and $\phi_v|V(C) = g$. We may conclude by Lemma 9.1 that $B_2 \cup B_3 \cup B_4 \sim g B_2' \cup B_3' \cup B_4'$, and so $G \sim G'$. If $G_{B_1}(C) = C_2$, let the orbits of $B_1$ in $C$ be $\{x_1\}, \{x_2\}, \{x_3, x_4\}$ where $B_1$ is the branch of $C$ at $\{x_2, x_3, x_4\}$. Suppose $B_2$ is the branch of $C$ at $\{x_1, x_3, x_4\}$.

If $(x_3, x_4) \in G_{B_2}(C)$, then let $u_2 \in \text{EV}(B_2)$. $\phi_{u_2}(B_1 \cup B_3 \cup B_4) \sim B_1' \cup B_3' \cup B_4'$. If $g = \phi_{u_2}|V(C)$, then $B_1 \cup B_2 \sim g B_1' \cup B_2'$, and so $G \sim G'$. If $(x_3, x_4) \notin G_{B_2}(C)$, then $G_{B_1 \cup B_2}(C) = E$.

For every $v \in \text{EV}(B_3 \cup B_4), \phi_v|V(C) = g$, and we may conclude by Lemma 9.1 that $B_3 \cup B_4 \sim g B_3' \cup B_4'$, and so $G \sim G'$.

Suppose $B_1 \sim B_2 \neq B_3 \sim B_4$. Let $B_1'$ and $B_2'$ be the branches of $C'$ isomorphic to $B_1$ and $B_2$. If $u_3 \in \text{EV}(B_3)$, then $\phi_{u_3}(B_1 \cup B_2) = B_1' \cup B_2'$, so $B_1 \cup B_2 \sim B_1' \cup B_2'$. If $G_{B_1 \cup B_2}(C) = E$, let $g = V(G) \to V(G')$ be the function such that $B_1 \cup B_2 \sim g B_1' \cup B_2'$. For each $v \in \text{EV}(B_3 \cup B_4), \phi_v|V(C) = g$, and by Lemma 9.1, $B_3 \cup B_4 \sim g B_3' \cup B_4'$, and so $G \sim G'$. If $G_{B_1 \cup B_2}(C) = C_2$, let $u_2 \in \text{EV}(B_2), \phi_{u_2}(B_1 \cup B_3 \cup B_4) = B_1' \cup B_3' \cup B_4'$. If $g = \phi_{u_2}|V(C)$, then since $G_{B_1 \cup B_2}(C) = C_2$, $B_1 \cup B_2 \sim g B_1' \cup B_2'$ and so $G \sim G'$. 
Suppose \( B_1 \simeq B_2 \simeq B_3 \simeq B_4 \), and \( G_{B_1}(C) = C_2 \). Suppose two branches of \( C \), say \( B_1 \) and \( B_2 \), have orbits in \( C \) in common. If \( B_3 \) and \( B_4 \) have orbits in \( C \) in common, then \( G \) has 2 subgraphs isomorphic to \( B_1 \cup B_2 \). By Kelly's Lemma, \( G' \) has 2 subgraphs isomorphic to \( B_1 \cup B_2 \). This means that \( G \) has branches \( B'_1 \) and \( B'_2 \) with orbits in \( C' \) in common and \( B'_3 \) and \( B'_4 \) with orbits in \( C \) in common. Since \( B_1 \not\simeq B'_1 \), this implies that \( G \not\sim G' \). If \( B_3 \) and \( B_4 \) do not have orbits in \( C' \) in common, then \( C' \) has branches \( B'_1 \) and \( B'_2 \) with orbits in \( C' \) in common, and \( B'_3 \) and \( B'_4 \) without orbits in \( C' \) in common. Suppose \( B'_3 \) and \( B'_4 \) each have their 2-element orbits in one of \( V(B'_1) \) or \( V(B'_2) \), say \( V(B'_1) \). Then by Kelly's Lemma, \( G' \) has a subgraph isomorphic to \( B_1 \cup B_3 \cup B_4 \). This implies that \( G \not\sim G' \). If not, then by symmetry, the same is true of \( G' \), and so \( G \not\sim G' \).

Suppose no two branches of \( C \) have orbits in common. Consider the function \( p: \{1,2,3,4\} \rightarrow \{1,2,3,4\} \) defined as follows: If the 2-element orbit of the branch \( B_i \) is in \( V(B_j) \), then \( p(i) = j \). Similarly, we define the function \( p' \) for the branches in \( C' \). By the above case, then do not exist integers \( i,j \in \{1,2,3,4\} \) such that \( p(i) = j \) and \( p(j) = i \). There are two possibilities for \( p \). By relabeling the branches if necessary, we may assume that either \( p(1) = 2 \), \( p(2) = 3 \), \( p(3) = 4 \), \( p(4) = 2 \) or \( p(1) = 2 \), \( p(2) = 3 \), \( p(3) = 4 \), \( p(4) = 1 \). In the first case, by Kelly's Lemma, \( G' \) has a subgraph isomorphic to \( B_2 \cup B_3 \cup B_4 \), and \( p' \) is of the same form.
as \( P \). In the second case, \( G' \) has no such subgraph, and again 
\( P' \) is of the same form as \( P \). In the first case, let \( B'_2, B'_3, B'_4 \) be the branches of \( C' \) such that \( B_2 \cup B_3 \cup B_4 \cong B'_2 \cup B'_3 \cup B'_4 \), 
\( G' \) is obtained by adjoining \( B'_1 \) to \( C' \) in \( B'_2 \cup B'_3 \cup B'_4 \). But adjoining \( B'_1 \) in any manner would result in a graph isomorphic to 
\( G \). In the second case, let \( u_1 \in EV(B'_1) \). \( \phi_{u_1}(B_2) \) is the only 
branch of \( C' \) in \( \phi_{u_1}(B_2 \cup B_3 \cup B_4) \) such that there is no branch 
whose 2-element orbit is in \( V(B_2) \). This implies that \( C' \) has 
branches \( B'_1 \) whose 2-element orbits is in \( V(B_2) \), and that 
\( \phi_{u_1}|V(C) \) may be extended to an isomorphism from \( B_1 \) to \( B'_1 \). This 
means that \( G \cong G' \). Suppose \( G_{B_1}(C) = E \). Assume \( V(B_4) \cap V(C) = \{x_1, x_2, x_3\} \). We define the vertices \( x_1^i, x_2^i, x_3^i \) for \( i = 1, 2, 3, 4 \) 
as follows: if \( \theta_i = V(C) \rightarrow V(C) \) is the unique function such that 
\( B_1 \cong \theta_i B_1 \), then \( x_j^i = \theta_i(x_j) \) for \( j = 1, 2, 3 \). If \( x \in V(C) \) and 
\( x \in V(B_{i_1}) \cap V(B_{i_2}) \cap V(B_{i_3}) \), we say \( x \) is of type \( j_1, j_2, j_3 \) if 
\( x = x_{j_1}^{i_1} = x_{j_2}^{i_2} = x_{j_3}^{i_3} \). We make no distinction between different orderings 
of \( j_1, j_2, \) and \( j_3 \). We similarly define the type of a vertex \( x' \) 
of \( C' \). Note that for 3 branches \( B_i, B_j, B_k \), there is a subgraph 
of \( G' \), \( B'_i \cup B'_j \cup B'_k \cong B_i \cup B_j \cup B_k \). This implies that the type 
of \( v \in V(B_i) \cap V(B_j) \cap V(B_k) \) is the same as the type of 
\( v' \in V(B'_i) \cap V(B'_j) \cap V(B'_k) \). We may conclude that the list of types 
of vertices of \( C \) is the same as the list of types of vertices of \( C' \).
Suppose for a vertex $x$ of type $j_1, j_2, j_3$, say $j_1 = j_2 = j_3 = 1$. We may assume $x = x_4 = V(B_1) \cap V(B_2) \cap V(B_3)$. For no other vertex of $C$ does the type contain more than one 1. Let $u_4 \in EV(B_4)$. 

$\phi_{u_4}(B_1 \cup B_2 \cup B_3) = B'_1 \cup B'_2 \cup B'_3$ for some branches $B'_1, B'_2, B'_3$. The type of the vertex $\phi_{u_4}(x_4)$ is $1, 1, 1$. Similarly, for no other vertex of $C'$ does the type contain more than one 1. This implies that for all $u \in EV(B_1 \cup B_2 \cup B_3)$, $\phi_u(B_4) = B'_4$, and if $g: V(C) \to V(C')$ is the function such that $B_4 \sim g B'_4$, then $\phi_u|V(C) = g$. We may conclude by Lemma 9.1, that $B_1 \cup B_2 \cup B_3 \sim g B'_1 \cup B'_2 \cup B'_3$, and so $G \sim G'$. Suppose we do not have the above case, but for two vertices $x$ and $y$ of $C$, the types of $x$ and $y$ each contain two 1's. Let the type of $x$ be $1, 1, j_1$, and the type of $y$ be $1, 1, j_2$, and suppose $(x, y) \subseteq V(B_1)$. For $u \in EV(B_1)$, let $B'_1 \cup B'_2 \cup B'_3 = \phi_u(B_2 \cup B_3 \cup B_4)$. $G$ is isomorphic to a graph obtained by adjoining $B_1$ to $B'_2 \cup B'_3 \cup B'_4$, so that two vertices of $C'$ have types containing two 1's, and those two vertices have types $1, 1, j_1$, and $1, 1, j_2$. But there is only one way to adjoin $B_1$ and satisfy the above conditions, and the resulting graph is isomorphic to $G$. Now suppose the type of each vertex is $1, 2, 3$. Let $B_2 \cup B_3 \cup B_4 \sim B'_2 \cup B'_3 \cup B'_4$, for some branches $B'_2, B'_3, B'_4$ of $C'$. $G'$ is isomorphic to a graph obtained by adjoining $B_1$ to $B'_2 \cup B'_3 \cup B'_4$, so that each vertex of $C'$ has type $1, 2, 3$. But there is only one way to adjoin $B_1$ and satisfy the above conditions, and the resulting graph is isomorphic to $G$. 
Case 7: G contains a 4-venter C with exactly 3 peripheral branches. By a previous argument G' contains a 4-center C'. If C' had 4 peripheral branches, by symmetry and Case 6, C would also.

Let $B_1$, $B_2$, and $B_3$ be the peripheral branches of $C_1$ and $v_1 \in EV_p(B_1)$, $v_2 \in EV_p(B_2)$, and $v_3 \in EV_p(B_3)$. Since $|EV(G)| \geq 4$, there is a vertex $v \in EV(G) \setminus \{v_1, v_2, v_3\}$. C' has 3 peripheral branches in G'-v', and so C' has 3 peripheral branches.

We first consider the case that C has a non-peripheral branch.

Let $v \in EV_p(G)$. $deg_{G'-v} C' = 4$, and so C' has a non-peripheral branch. Let the non-peripheral branch of C be $B_4$, and the non-peripheral branch of C' be $B_4'$. If $u \in EV(B_4)$, $\nu(B_p(G)) = \nu(B_p(G-\{u\})) = \nu(B_p(G'))$. By symmetry, $\nu(B_p(G')) \leq \nu(B_p(G))$. This shows that $\nu(B_p(G)) = \nu(B_p(G'))$, and $\phi_u(B_p(G)) = B_p(G')$. Let $B_p(G') = B_1' \cup B_2' \cup B_3'$. Suppose for some i, $i = 1, 2, 3$, and $v \in EV(B_i)$. $\phi_v(B_4) = B_4'$, then $B_4 \sim B_4'$. Otherwise, for every i, and $v \in EV(B_i)$, $B_i-v \sim B_i'$. But then $B_i'-v' \sim B_i'$. Since $B_i \sim B_i'$ for some branch $B_i$ of C' this implies that $B_4 \sim B_4'$.

Let $V(C) = \{x_1, x_2, x_3, x_4\}$, and $B_1$ the branch of C at $V(C) - \{x_4\}$.

Suppose $G_{B_4}(C) = S_3$. Then if $g: V(C) \to V(C')$ is any function such that $B_p(G) \sim g B_p(G')$, then $B_4 \sim g B_4'$, and so $G \sim G'$. 


Suppose $G_{B_4}(C) = C_3$. Let $u \in EV(B_4)$, then $u' \in EV(B_4')$.

Let $g = \phi_u|V(C)$. Then, by Lemma 15.1, $B_4 \sim_{g} B_4'$, and $G \sim G'$.

Suppose $G_{B_4}(C) = E$. Let $g: V(C) \rightarrow V(C')$ be the function such that $B_4 \sim_{g} B_4'$, and $x'_i = g(x_i)$. We wish to show that if $B_4'$ is the branch of $C'$ at $V(C') \setminus \{x'_i\}$, then $B_1 \sim_{g} B_1'$. If $v(B_2) \neq v(B_4) + 1$, let $u \in EV(B_2)$. If $v(B_2) = v(B_4) + 1$, let $u \in V(B_2) \setminus EV(B_2)$. In either case, $\phi_u(B_4) = B_4'$, $\phi_u(B_1) \subseteq B_1'$, and so $v(B_1) \leq v(B_1')$. By symmetry, $v(B_1') \leq v(B_1)$. Therefore $v(B_1') = v(B_1')$, and $\phi_u(B_1) = B_1'$, and so $B_1 \sim_{g} B_1'$. Similarly, $B_2 \sim_{g} B_2'$, and $B_3 \sim_{g} B_3'$. This shows that $G \sim G'$.

Suppose $G_{B_4}(C) = C_2$. Without loss of generality, the orbits of $B_4$ in $C$ are $\{x_1, x_2\}$, $\{x_3\}$, and $\{x_4\}$. Let $g: V(C) \rightarrow V(C')$ be a function such that $B_4 \sim_{g} B_4'$, and $x'_i = g(x_i)$. Let $B_3'$ be the branch of $C'$ at $x'_1$, $x'_2$, $x'_3$. If $v(B_4) \neq v(B_2) - 1$, let $v \in EV(B_2)$. If $v(B_4) = v(B_2) - 1$, let $v \in V(B_2) \setminus EV(B_2)$. In either case, $\phi_v(B_3) \subseteq B_3'$, and $v(B_3) \leq v(B_3')$. By symmetry, $v(B_3') \leq v(B_3)$, and $\phi_v(B_3) = B_3'$. Therefore $B_3 \cup B_4 \sim B_3' \cup B_4'$. If $(x_1, x_2) \not\in G_{B_3}(C)$, then $B_1' = \phi_v(B_1)$ is a branch of $C'$ such that $B_1 \cup B_3 \cup B_4 \sim B_1' \cup B_3' \cup B_4'$. We may similarly show that there is a branch $B_2'$ of $C'$ such that $B_2 \cup B_3 \cup B_4 \sim B_2' \cup B_3' \cup B_4'$. This implies that $G \sim G'$. Suppose $(x_1, x_2) \in G_{B_3}(C)$. If $v(B_4) \neq v(B_3) - 1,$
let $u \in EV(B_3)$. If $v(B_4) = v(B_3) - 1$, let $u \in V(B_3) \setminus EV(B_3)$.

In either case, $\phi_u(B_4) = B_4'$, and $u' \in V(B_3')$. $\phi_u(B_1 \cup B_2 \cup B_4) = B_1' \cup B_2' \cup B_4'$, where $B_1'$ and $B_2'$ are branches of $C$. Let $g = \phi_u \mid V(C)$. Since $(x_1, x_2) \in G_{B_3}(C), B_3 \sim g B_3'$, and so $G \sim G'$.

Now, suppose $C$ has no non-peripheral branch. By symmetry, and the above case, $C'$ has no non-peripheral branch. As above, we let $V(C) = \{x_1, x_2, x_3, x_4\}$, and $B_1$ be the branch of $C$ at $V(C) \setminus \{x_1\}$. Let the peripheral branches of $C$ be $B_1, B_2$ and $B_3$, where $v(B_1) \leq v(B_2) \leq v(B_3)$. Let $B_1', B_2'$, and $B_3'$ be the peripheral branches of $C'$, and $x_4' \in V(B_1') \cap V(B_2') \cap V(B_3')$. If $r_{2,3}(G) = 2$, then by Kelly's Lemma, $r_{2,3}(G') = 2$, and $x$ and $x'$ are adjacent to every other vertex of $G$ and $G'$. This implies that $\phi_x \cup \{(x, x')\}$ is an isomorphism from $G$ to $G'$. Otherwise, for each $u \in EV(G)$, $C$ and $C'$ have 3 non-trivial branches in $G-u$ and $G'-u'$, respectively. Let $u_1 \in EV(B_1)$. $C'$ has branches in $G'-u_1$ with $v(B_1) - 1$, $v(B_2)$, and $v(B_3)$ vertices. If $C'$ has a branch $B'$ with $v(B_1) - 1$ vertices, let $u' \in EV(B')$. Then $C$ has a branch in $G-u$ with $v(B_1) - 2$ vertices. This is impossible.

Therefore the branches of $C'$ in $G'$ have $v(B_1)$, $v(B_2)$, and $v(B_3)$ vertices and $\phi_{u_1}(B_2)$ and $\phi_{u_1}(B_3)$ are branches of $C'$ in $G'$. Let $k_1 = v(B_1)$, for $i = 1, 2, 3$, and $g' = \{(x_4, x_4')\}$.

Suppose $k_2 < k_3$. Let $B_3'$ be the branch of $C'$ with $k_3$ vertices $B_2' = \phi_{u_1}(B_2)$ is a branch of $C'$ such that $B_2 \cup B_3 \sim g B_2 \cup B_3'$. 


If \( k_1 \neq k_2 - 1 \), let \( u_2 \in \text{EV}(B_2) \). If \( k_1 = k_2 - 1 \), let \( u_2 \in V(B_2) \setminus \text{EV}(B_2) \). In either case, \( B_1' = \phi_{u_2}(B_1) \) is a branch of \( C' \) such that \( B_1 \cup B_3 \sim_g B_1' \cup B_3' \). By symmetry, if \( B' \), \( B'' \), and \( B_3' \) are the branches of \( C' \), there are branches \( B^* \) and \( B^{**} \) of \( C \), such that \( B^* \cup B_3' \sim_g B' \cup B_3' \) and \( B^{**} \cup B_3' \sim_g B'' \cup B_3' \). This implies that \( G \sim G' \).

Suppose \( k_1 < k_2 = k_3 \). If \( B_1' \) is the branch of \( C' \) with \( k_1 \) vertices, we can, by the same reasoning as above, show that \( C' \) has branches \( B_2' \) and \( B_3' \) such that \( B_1 \cup B_2 \sim_g B_1' \cup B_2' \) and \( B_1 \cup B_3 \sim_g B_1' \cup B_3' \). This implies that \( G \sim G' \).

Suppose \( k_1 = k_2 = k_3 \). If \( B_i \) and \( B_j \) are branches of \( C \) we say \( B_i \sim x_i B_j \) if there is an isomorphism \( \theta \) from \( B_i \) to \( B_j \) such that \( \theta(x_i) = x_i \). Similarly, if \( B_i' \) and \( B_j' \) are branches of \( C' \) we say \( B_i' \sim x_i B_j' \), if there is an isomorphism \( \theta' \) from \( B_i' \) to \( B_j' \) such that \( \theta(x_i') = x_i' \). Suppose \( B_i \sim x_i B_2 \sim x_i B_3 \). Then if \( u_1 \in \text{EV}(B_1) \), \( B_2 \sim_g \), \( \phi_{u_1}(B_2) \sim x_i \), \( \phi_{u_1}(B_3) \). We can similarly show that if \( B' \) is a branch of \( C' \), then \( B_1 \sim B' \). Let \( h : V(C) \rightarrow V(C) \) be defined by \( h(x_1) = x_2 \), \( h(x_2) = x_3 \), \( h(x_3) = x_1 \), and \( h(x_4) = x_4 \). We say \( B_i \sim h B_j \), if \( B_i \sim h B_j \) or \( B_j \sim h B_i \). Similarly, we define \( h' : V(C') \rightarrow V(C') \), and \( B_i' \sim B_j' \) if \( B_i' \sim h B_j' \) or \( B_j' \sim h B_i' \). If \( B_1 \sim B_2 \sim B_3 \), then \( \phi_{u_1}(B_2) \sim \phi_{u_1}(B_3) \). We can similarly show that \( B_1 \sim B_2 \sim B_3 \), and so \( G \sim G' \). Otherwise, there is one
branch, say $B_1$, such that $B_1 \not\sim B_2$, and $B_1 \not\sim B_3$. This implies that it is not the case that $B_1 \sim B_2 \sim B_3$. If $u_1 \in EV(B_1)$, then $\phi_{u_1}(B_2) \sim \phi_{u_2}(B_3)$. But then if $B'_3$ is the third branch of $C'$, $B'_1 \not\sim \phi_{u_1}(B_2)$. This implies that if $g = \phi_{u_1}|V(C)$, then $B_1 \sim g B'_1$, and so $G \sim G'$. 

Suppose it is not the case that $B_1 \sim x_4 B_2 \sim x_4 B_3$. We may assume that $B_1 \not\sim x_4 B_2$, and $B_1 \not\sim x_4 B_3$. If $u_1 \in EV(B_1)$, and $u_2 \in EV(B_2)$ then $B_1 \not\sim g, \phi_{u_1}(B_2), B_1 \not\sim g, \phi_{u_1}(B_3)$, and $B_1 \sim g, \phi_{u_2}(B_1)$. Therefore there is a unique branch $B'_1$ such that $B_1 \sim B'_1$. Let $B'_2$ and $B'_3$ be the other branches of $C'$. If $(x_2 x_3) \notin G(B_1, C)$, then if $g = \phi_{u_2}|V(G)$, for each $u \in EV(B_2 \cup B_3)$, $\phi_u|V(C) = g$.

We may conclude by Lemma 9.1 that $B_2 \cup B_3 \sim g B'_2 \cup B'_3$. If $(x_2 x_3) \in G(B_1, C)$, then $\phi_{u_1}(B_2 \cup B_3) = B_2 \cup B'_3$. If $g = \phi_{u_1}|V(C)$, then since $(x_2 x_3) \in G(B_1, C)$, $B_1 \sim g B'_1$, and so $G \sim G'$.

Case 8: $G$ contains a 4-center $C$ with exactly 2 peripheral branches. By symmetry, and the previous cases, $G'$ contains a 4-center $C'$ with exactly 2 peripheral branches.

Suppose $C$ has non-peripheral branches. Since $|EV(G)| \geq 4$, there is a vertex $u \in EV(G)$ such that $G-u$ contains a longest $(2,3)$-path, and $\deg_{G-u} C \geq 3$. But then $\deg_{G'-u} C' \geq 3$, and so $C'$
has non-peripheral branches. If \( v \in EV(B_n(G)) \), then \( \nu(B_p(G)) = \nu(B_p(G-v)) = \nu(B_p(G'-v')) \leq \nu(B_p(G')) \). By symmetry, \( \nu(B_p(G')) \leq \nu(B_p(G)) \). Therefore \( \nu(B_p(G)) = \nu(B_p(G')) \) and so \( \phi_v(B_p(G)) = B_p(G') \).

Let \( V(C) = \{x_1, x_2, x_3, x_4\} \), \( V(C') = \{x'_1, x'_2, x'_3, x'_4\} \), \( B_1 \) the branch of \( C \) at \( V(C) \backslash \{x_1\} \) and \( B'_1 \) the branch of \( C' \) at \( V(C') \backslash \{x'_1\} \).

Suppose \( B_1 \), and \( B_2 \) are the peripheral branches of \( C \), and \( B'_1 \), and \( B'_2 \) are the peripheral branches of \( C' \).

Suppose \( |EV(B_1)| = 1 \). If \( G_{B_1 \cup B_2}(C) = E \), then if \( g: V(C) \to V(C') \) is the function such that \( B_1 \cup B_2 \cong g B_1 \cup B_2 \), for each \( u \in EV(B_n(G)) \), \( B_n(G) \cong g B_n(G') \), and so \( G \cong G' \). Otherwise, let \( u_2 \in EV_{p}(B_2) \). If \( H' \neq C' \) is the center of \( G'-u_2' \), \( C' \) is the 3-simplex containing \( H' \) with degree \( \geq 3 \). Therefore \( \phi_{u_2}(C) = C' \), and \( \phi_{u_2}(B_1) \) is a peripheral branch of \( C' \). Note that if \( \alpha \in G_{B_1 \cup B_2}(C) \), \( \alpha([x_3, x_4]) = [x_3, x_4] \), and \( \alpha([x_1, x_2]) = [x_1, x_2] \).

This implies that \( \phi_{u_2}([x_1, x_2]) = [x'_1, x'_2] \) and \( \phi_{u_1}([x_3, x_4]) = [x'_3, x'_4] \), and that \( \phi_{u_2}(B_1 \cup B_n(G)) = \phi_{u_2}(B_1) \cup B_n(G') \). If \( g = \phi_{u_2}|V(C) \), then \( B_p(G) \cong g B_p(G') \), and so \( G \cong G' \).

Suppose \( |EV(B_1)| \geq 2 \), and \( |EV(B_2)| \geq 2 \). Let \( k_1 = \nu(B_1) \), \( k_2 = \nu(B_2) \), and suppose \( k_1 \leq k_2 \). If \( u_1 \in EV(B_1) \) such that \( G-u_1 \) contains a longest \((2, 3)\)-path of \( G \), then \( \phi_{u_1}(B_n(G) \cup B_2) = B_n(G') \cup B_2 \) where \( B_2' \) is a peripheral branch of \( C' \).
If there is no automorphism \( \theta : B_p(G) \rightarrow B_p(G) \) such that
\[
\theta(B_1) = B_2, \quad \text{and} \quad \theta(B_2) = B_1 \quad \text{and} \quad (x_3, x_4) \notin G_{B_2}(C),
\]
then there is a unique way to adjoin \( B_1 \) to \( B_n(G) \cup B_2 \), and \( B_n(G') \cup B_2' \), so that \( B_1 \cup B_2 \cong B_1 \cup B_2' \cong B_p(G) \). This implies that \( G \cong G' \).

Suppose \((x_3, x_4) \in G_{B_2}(C)\), let \( v \in EV(B_2) \) such that \( G \) contains a longest \((2,3)\)-path. If there is an automorphism \( \alpha : G - v \rightarrow G - v \) such that \( \alpha(B_1 \cup B_n(G)) = (B_2 - v) \cup B_n(G) \), then there is an isomorphism \( \phi'_v : G - v \rightarrow G' - v' \) such that \( \phi'_v(B_1 \cup B_n(G)) = B_1 \cup B_n(G') \). If there is no such automorphism, then of course, \( \phi'_v(B_1 \cup B_n(G)) = B_1 \cup B_n(G') \).

Since \((x_3, x_4) \in G_{B_2}(C)\), if \( g = \phi'_v|V(C) \), \( B_2 \cong g B_2 \), and so \( G \cong G' \).

Suppose there is an automorphism \( \theta : B_p(G) \rightarrow B_p(G) \), such that
\[
\theta(B_1) = B_2, \quad \text{and} \quad \theta(B_2) = B_1 . \quad \text{Let} \quad g : V(C) \rightarrow V(C) \quad \text{be a function such that} \quad B_2 \cup B_n(G) \cong g B_2 \cup B_n(G') . \quad \text{Then} \quad B_1 \cong g B_1 , \quad \text{and so} \quad G \cong G' .
\]

Now suppose \( G \) has no non-peripheral branches. By symmetry and the previous case, neither does \( G' \). Let the peripheral branches of \( C \) be \( B_1 \) and \( B_2 \), with \( B_1 \) the branch of \( C \) at \( \{x_2, x_3, x_4\} \), and \( B_2 \) the branch of \( C \) at \( \{x_1, x_3, x_4\} \). Let \( k_1 = \nu(B_1) \), \( k_2 = \nu(B_2) \), and we may assume that \( k_1 \leq k_2 \). Let \( \{x_3', x_4'\} = B_1' \cap B_2' \).

We will say \( B_1 \cong B_1' \) if there is an isomorphism \( \alpha \) from \( B_1 \) to \( B_1' \), such that \( \alpha([x_3, x_4]) = [x_3', x_4'] \).
Suppose \(|\text{EV}(B_i)| \geq 2\), for \(i = 1, 2\). Let \(u_1 \in \text{EV}(B_1)\) such that \(G-u_1\) contains a longest \((2,3)\)-path of \(G\). Then \(C'\) has branches in \(G'-u_1\) with \(k_1-1\), and \(k_2\) vertices. If \(C'\) has a branch \(B'\) in \(G'\) with \(k_1-1\) vertices, let \(v' \in \text{EV}(B')\) such that \(G'-v'\) contains a longest \((2,3)\)-path. Then \(C\) has a branch in \(G-v\) with \(k_1-2\) vertices. This is not impossible. Therefore \(C'\) has branches in \(G'\) with \(k_1\) and \(k_2\) vertices, and \(B'_2 = \phi_{u_1}(B_2)\) is a branch \(C'\) in \(G'\).

Suppose \(k_1 < k_2-1\). If \(u_2 \in \text{EV}(B_2)\) such that \(G-u_2\) contains a longest \((2,3)\)-path of \(G\), then \(B'_1 = \phi_{u_2}(B_1)\) is the branch of \(C'\) with \(k_1\) vertices.

Suppose \(k_1 = k_2-1\). If \(B_1 \neq B'_1\), then for each \(v \in \text{EV}(B_2)\), such that \(G-v\) contains a longest \((2,3)\)-path of \(G\), \(B'_1 \sim B_2 - v'\), and \(B_1 \sim B_2 - v\). But if \(v \in \text{EV}(B_2)\), there is a vertex \(v' \in \text{EV}(B_2')\) such that \(B_2-v \sim B_2-v'\). Therefore \(B_1 \sim B'_1\).

Suppose \(k_1 = k_2\). Let the other branch of \(C'\) be \(B'_1\). If \(u_2 \in \text{EV}(G)\) such that \(G-u_2\) contains a longest \((2,3)\)-path of \(G\), then \(\phi_{u_2}(B_1)\) is a branch of \(C'\). Therefore \(C'\) has branches \(B' \sim B_1\), and \(B'' \sim B_2\) in \(G'\). By symmetry, \(C\) has branches \(B^* \sim B'_1\) and \(B^{**} \sim B'_2\) in \(G'\). Therefore \(B_1 \sim B'_1\).
If \((x_3, x_4) \in G_{B_1}(C)\), let \(g: V(C) \rightarrow V(C')\) be a function such that \(B_2 \sim g B_1\). Then \(B_1 \sim g B_1\), and so \(G \sim G'\). If \((x_3, x_4) \in G_{B_2}(C)\), we similarly conclude that \(G \sim G'\). Suppose \((x_3, x_4) \in G_{B_i}(C)\) for \(i = 1, 2\). We define \(L = \{v \in EV(G): G-v \text{ contains a longest (2,3)-path of } G\}\). If \(u \in EV(G) \setminus L\), then if \(u \in EV(B_2)\), and \(B_1 \sim B_2 - u\), there is an isomorphism \(\phi_u^v: G-v \rightarrow G'-v'\) such that \(\phi_u^v(B_1) = B_1'\). In any case, for \(u \in EV(G) \setminus L\), there is an isomorphism \(\phi_v^v: G-v \rightarrow G'-v'\) such that \(\phi_u^v(B_1) = B_1'\). For \(u_1, u_2 \in EV(G) \setminus L\), \(\phi_{u_1}^v|V(C) \cup L = \phi_{u_2}^v|V(C) \cup L\), and so by Lemma 9.1, \(G \sim G'\).

Now, suppose \(B_1\) has exactly one end-vertex \(v_1\). By Kelly's Lemma, \(v_1\) and \(v_1'\) are each contained in exactly one path of length \(t\), for \(t = 1, 2, ..., r_{2,3}(G)\). Therefore, \(v_1'\) is the end-vertex of a branch \(B_1'\) with exactly one end-vertex. If \(u \in EV(B_2)\) such that \(G-u\) contains a longest (2,3)-path of \(G\), then \(\phi_u(B_1) = B_1'\).

Suppose \((x_3, x_4) \notin G_{B_1}(C)\). We define \(L = \{v \in EV(G): G-v \text{ contains a longest (2,3)-path of } G\}\). For all \(u_1, u_2 \in EV(G) \setminus L\), \(\phi_{u_1}^v|V(C) \cup L = \phi_{u_2}^v|V(C) \cup L\), and so by Lemma 9.1, \(G \sim G'\). Suppose \((x_3, x_4) \in G_{B_1}(C)\). The center \(H'\) of \(G'-v_1'\) has 2 branches, one with exactly one end-vertex. That branch is \(B_1'-v_1'\). There is a unique way to adjoin \(v_1\) to \(B_1-v_1\) and \(v_1'\) to \(B_1'-v_1'\) so that \((B_1-v_1) \cup \{v_1\} \sim (B_1'-v_1') \cup \{v_1'\} \sim B_1\). Therefore \(\phi_{v_1} \cup ((v_1, v_1'))\) is an isomorphism from \(G\) to \(G'\).

This completes the proof.
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