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The Ohio State University, Ph.D., 1977
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AN IMPACT OF A FLUID-FILLED SPHERICAL SHELL WITH A
SHALLOW SPHERICAL SHELL OF LARGER RADIUS:
A MODEL FOR HEAD INJURY

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By


* * * * *

The Ohio State University
1977

Reading Committee:
Prof. Ali E. Engin (Chairman)
Prof. Arthur W. Leissa
Prof. Karl K. Stevens

Approved By

Ali E. Engin
Adviser
Department of Engineering Mechanics
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VITA

April 13, 1945 ........ Born - Igbogbo Lagos, Nigeria
1964 ................. Meteorological Assistant IV, Nigerian Meteorological Services
1965-1968 ............ Technical Officer, Nigerian Ministry of Works, Lagos, Nigeria
1967: June ............ Dip. Civil Eng., College of Technology, Lagos, Nigeria
1971: July ............ B.S.C.E., University of Miami, Miami, Florida
1971-1976 ............ Fellowship, Nigerian Federal Government
1971-1972 ............ Teaching and Research Associate, Department of Civil Engineering, University of Miami, Miami, Florida
1972: August .......... M.Sc. (Civil Eng.), University of Miami, Miami, Florida
1973-1974 ............ Ph.D. student, Department of Civil Engineering, The Ohio State University, Columbus, Ohio
1973 .................. Research Associate, Department of Civil Engineering, The Ohio State University, Columbus, Ohio
1975-1977 ............ Ph.D. student, Department of Engineering Mechanics, The Ohio State University, Columbus, Ohio
1975-1977 ............ Graduate Administrative Associate, College of Engineering, The Ohio State University, Columbus, Ohio
1976: March .......... M.Sc. (Eng. Mech.), The Ohio State University, Columbus, Ohio
FIELDS OF STUDY

Major Field: Engineering Mechanics

Studies in Biomechanics. Professor Ali E. Engin


Studies in Concrete and Steel Design (plates, bins, shells, etc.). Professors Alfred Bishara and Morris Ojalvo

Studies in Applied Mathematics. Professors Stefan Drobot and Henry Colson

Studies in Dynamics. Professor Charles T. West

Studies in Plasticity. Professor Ting-Shu Wu

Studies in Elasticity. Professor Carl H. Popelar

Studies in Elastic Wave Propagation. Professor Karl F. Graff

Studies in Visoelasticity. Professor Karl K. Stevens
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NOMENCLATURE

A  Cross-sectional area of the viscoelastic disc

$B_i$  Bending moments along $\phi$-direction

$C_f$  Compressional wave in fluid, $[\mu/\rho_f]^{1/2}$

cosec$\phi$  cosec$\phi$

$C_s$  Compressional wave in shell, $[E/\rho_s(1 - \nu^2)]^{1/2}$

d$a_i$, $d_b_i$, $d_{c_i}$  Chord lengths of the deformed $i$th intervertebral disc

d$D_1$, $D_2$  Flectural stiffnesses $\frac{2E'n^3}{3}$, $\frac{2E'_{2n}^3}{3}$

d$i$  Length of $i$th disc

$E$, $E_1$, $E_2$  Young's Moduli of elasticity

$E'$, $E'_2$  $[E/(1 - \nu^2)]$, $[E_2/(1 - \nu^2)]$

e  Coefficient of restitution

$E_3$, $E_4$  Constants of mechanical model of viscoelastic solid

e$i$  Vertical centroidal eccentricity coordinate

$F(t)$  Acceleration pulse

$F_a$, $F_{a_i}$  Axial forces

$F_m$, $F_{m_i}$  Bending moments

$F_g$, $F_{s_i}$  Shear forces
NOMENCLATURE—Continued

\( h_i \)  
Horizontal components of centroidal eccentricity

\( H \)  
Heaviside function

\( I \)  
Area moment of inertia

\( i \)  
Subscript \( i = 1, 2, 3, \ldots, 25 \), unless otherwise stated

\( J \)  
Rotational moment of inertia

\( j_n(z) \)  
Spherical Bessel function

\( K_s \)  
Kinetic energy of shell particles

\( k, k_s \)  
Shape factors for shearing deformation of a beam and a shell respectively

\( L_i, \ell_i \)  
Length of \( i \)th vertebra

\( m_{f} \)  
Mass of fluid filling the spherical shell

\( m_0 \)  
Mass of both the spherical shell and its fluid

\( m_s \)  
Mass of empty spherical shell

\( n, n_i \)  
Normal unit vector components

\( P, P_i \)  
Axial force in \( z \)-direction

\( P_l \)  
Viscoelastic material constant

\( p \)  
Laplace time transform

\( P_n^m(\cos \theta) \)  
Legendre polynomial functions
NOMENCLATURE—Continued

\( Q, Q_i \) \hspace{1cm} \text{Shear force or shear forces in } x\text{-direction}

\( Q_s \) \hspace{1cm} \text{Potential energy}

\( q_0, q_1 \) \hspace{1cm} \text{Viscoelastic material parameters}

\( Q_n \) \hspace{1cm} \text{Shell normal reaction due to impact, } k_0 s^{3/2}

\( r_0, r_1 \) \hspace{1cm} \text{Viscoelastic material constants}

\( S \) \hspace{1cm} \text{Relative spherical shell displacement}

\( s \) \hspace{1cm} \text{Speed ratio of sound in fluid and in shell, } C_f/C_s

\( \ddot{S} \) \hspace{1cm} \text{Dimensionless relative displacement}

\( s_1, S_1 \) \hspace{1cm} \text{Viscoelastic parameter, shallow shell deflection}

\( S_c \) \hspace{1cm} \text{Progressive area of impact, } S_c = S_c(\phi, \theta)

\( S_{cm} \) \hspace{1cm} \text{Maximum area of impact}

\( t \) \hspace{1cm} \text{Time}

\( t_i \) \hspace{1cm} \text{Time at the beginning of impact}

\( t_r \) \hspace{1cm} \text{Relative time, } t - t_i

\( u, u_i \) \hspace{1cm} \text{Lagrangian displacement of the } i\text{th vertebral column along } x\text{-direction; meridional displacement of shell}

\( u_{li} \) \hspace{1cm} \text{First derivative of } u_i

\( \bar{v} \) \hspace{1cm} \text{Azimuthal (latitude) displacement of spherical shell}

\( v \) \hspace{1cm} \text{Non-dimensionalized azimuthal displacement}
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<td>$W, W_i$</td>
<td>Radial displacement of spherical shell or horizontal displacement of the vertebral column</td>
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<td>$W_s$</td>
<td>Shallow shell deflection at point of contact</td>
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<td>$W_{sp}$</td>
<td>Spherical shell deformation at point of contact</td>
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<td>$W_{11}$</td>
<td>First derivative of $W_i$</td>
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<td>$Y_n^m(\phi, \theta)$</td>
<td>Tesseral harmonics, $P_n^m(\cos \phi) \cos m\theta$</td>
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<td>$\sigma, \varepsilon$</td>
<td>Normal stress and strain</td>
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<td>$\tau, \gamma$</td>
<td>Shear stress and strain</td>
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<td>$\gamma_i$</td>
<td>Angle between vertebra adjoining the $i$th disc</td>
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<td>$\gamma$</td>
<td>Dynamic coefficient of friction at impact</td>
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<td>$\eta$</td>
<td>Viscoelastic parameter; angle, in sagittal plane, between acceleration impulse and caudocephalad direction, shallow shell coordinate</td>
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<tr>
<td>$\psi$</td>
<td>Pelvic angle or shallow shell coordinate</td>
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<td>$\phi_{11}$</td>
<td>First derivative of $\phi_1$</td>
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<td>$\phi, \phi_i$</td>
<td>Rigid body rotation of vertebral column or the spherical shell coordinate, $i = 1, 2, ..., 25$</td>
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<td>$\phi_{25}$</td>
<td>Rigid body rotation of the 25th vertebral column</td>
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<td>$\Delta_i$</td>
<td>$\frac{\partial^2}{\partial \phi^2} + \cot \frac{\partial}{\partial \phi} + \csc^2 \phi \frac{\partial^2}{\partial \theta^2}$</td>
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\( \nabla \) Gradient operator, \( \hat{e} \frac{\partial}{\partial r} + \hat{e}_\phi \frac{\partial}{\partial \phi} + \hat{e}_\theta \frac{\partial}{\partial \theta} \)

\( (, ) \) Comma or partial differentiation

\( | | \) Absolute values or determinant of a matrix

\( \mu \) Viscosity or viscoelastic parameter

\( \delta u, \delta w \) Virtual displacement

\( \delta q_i \) Virtual velocity

\( \delta \) Variation operator

\( \Pi_s \) Total energy in spherical shell

\( \Pi_f \) Total energy in fluid

\( \Pi \) Sum of the energy of spherical shell and the enclosed fluid

\( \zeta \) Shear-bending factor or ratio of \( r/a \)

\( a, a_2 \)

\[ \frac{1}{12} \left( \frac{h}{a} \right)^2, \frac{1}{12} \left( \frac{h_2}{a} \right)^2 \]

\( \alpha_i \) Angles between the axes of the vertebral column and the horizontal axis, \( z \) after deformation

\( \beta_i \) Angles between the axes of the vertebral column before deformation

\( \beta_\phi, \beta_\theta \) Rotational displacements in \( \phi \) and \( \theta \) directions

\( \delta s_i, \delta a_i \) Lateral, axial, and rotational deformations of the \( i \)th intervertebral disc
CHAPTER 1
INTRODUCTION

1.1. Preliminary and Purpose of Study

Much attention has been given to mathematical and experimental studies of head injuries and brain traumatic disturbances in the relevant literature. The common goal of all these studies is to acquire the ability to predict, assess for a proper treatment and reduce the amount of damages resulting from a head impact. Presently, we cannot frequently pick up brain damage electronically, but through inferences made from physiological and neurological malfunctions which are often shown at critical times. Histopathological observations on the primary effects of a fatal head injury involving contusions, lacerations, vascular lesions and nerve fiber degeneration including edema or distortion due to the creation of an intercranial hematoma, give alternative assessment methods. However, none of the above techniques gives a measure of the extent of the traumatic damage. A proper and prompt treatment of the victims of head injury with the right medication is the ideal medical practice. The skull is one of the most susceptible parts of the human body to sustain an injury. Consequently a direct quantitative measure of the damage inflicted to the head and brain is vital, but virtually impossible. Our last resort, then is to examine the two well-known mechanics of
production of this traumatic phenomenon:

1. Impulsive forces, which include a sudden motion without direct contact with any object; and

2. Impact or blow, involving collisions of the head with another solid object at an appreciable velocity.

The author has deliberately paid attention to collisions involving the head because it has not received as much treatment as it should. The author also examines the ensemble of vertebra, neck and head interactions rather than accept the traditional treatment of these portions of the human body in isolation.

According to Goldsmith (53), the mathematical description of the complete mechanical effects produced on any object as the result of an impact requires a knowledge of the collision configuration, the relative impact velocity and geometry and the composition and material properties of striker and target, both of these objects being considered as continuous structures. This mechanical effect initiates stress waves that eventually encompass the entire volume of each body, as well as additional "contact phenomena." By "contact phenomena," we mean temporary or permanent relative indentation of the two objects in the contact region. We must realize, then, that our treatment here is approximate because we cannot adequately obtain the in vivo biological composition and material properties of the human body.

Two primary viable head injury criteria evolve from as many schools of thought:

1. The rotational school believes that high rotational acceleration caused by impact induces high shear strains in the brain tissue as well as the rupturing of bridging veins;
2. The cavitation school feels that, due to impact, there exist some points in the brain where the pressure is sufficiently reduced to cause cavitation, and that the inevitable catastrophic collapse of these cavitation bubbles is the major source of brain damage.

Incidentally, the theme of this dissertation belongs to the cavitation school. As we shall see later, attention is given to the intracranial pressure, which has two components: (1) the induced pressure due to the deceleration of the system, and (2) the wave propagation due to local impact and resulting pressure distribution. However, it is obvious that in any realistic trauma, the head will experience both translational and rotational acceleration.

The unique aspect of this research is its ability to determine the force induced by collision rather than imposing arbitrary impulsive load as we shall later see when reviewing the previous works in this area.

1.2 Review of Previous Studies

The literature review can be divided into two primary groups: (1) Pure dynamic response of spherical shells, and (2) Analytical modeling of head injury.

(A) Pertinent Studies in Dynamic Response of Spherical Shells

The pertinent studies on the dynamic response of various shells began in 1882 when Lamb (125) formulated membrane theory in the study of closed spherical shells. About the same period, Rayleigh (204) and Love (145) presented two contradictory theories for the vibration of bells. Rayleigh's theory was inextensional while Love's was extensional
and flexural. However, with time, Love's theory is now accepted as the classical bending theory. (Since it is virtually impossible to mention every past work here, only the recent relevant work will be mentioned.)

In 1962 Medick (163) studies the response of a spherical shell to a concentrated load while Naghdi et al. (176) discussed the vibration of the elastic shells. Prasad (200) confined his study to spherical shells, using hemisphere and sphericaps. In 1966, axisymmetric response of a closed spherical shell to a nearly uniform radial impulse was studied by McIvor et al. (162). In the following year, Rand et al. (202) did some investigations involving fluid-filled spherical and spheroidal shells, the knowledge of which we specially carry forward to the study of head injury. Among other works of importance is the large impact deformation response of spherical shells, due to Haskell (79) in 1970.

(B) Studies Which Deal Directly with the Analytical Modeling of Head Injury

Of the two mechanisms for producing a head injury mentioned earlier, the one due to impulsive loading received the most extensive treatment in the literature. In the early sixties, there were many experimental studies on head injuries, among which are Gurdjian et al. (64) on photoelastic confirmation of the presence of shear strain at the craniospinal junction in a closed head injury, Lissner et al. (135) on the relation between acceleration and intracranial pressure changes in man, and Lindgren (133-135) on mechanical effects of head injury.
Anzelius (3) and Guttinger (74) are the forerunners of the vigorous mathematical modeling of brain damage. The dynamic response of an impulsive load on a mass of inviscid fluid contained in a rigid closed spherical shell was formulated in their studies. Because of the rigidity of the shell, only fluid which is analogous to the brain was affected. The impact point (pole) and the point opposite the impact point (counterpole) form sources of compression and tension (rarefraction) waves respectively; both travel in opposite directions towards the geometric center along which they either reinforce or annihilate each other, leading to a large pressure gradients, believed capable of causing brain damage. In an effort to simplify this model, Rayashi (80-81) proposed a one dimensional continuum model which appeared to be quite promising as an intermediate between the simplistic lumped-parameter and the complex two or three dimensional models simply because it has much fewer parameters, yet possesses identifiable and realistic injury mechanisms. He modeled the closed head impact problem as a fluid-filled, rigid but massless container with an attached spring striking a rigid wall. The container, fluid, and spring are analogous to the skull, the brain and the cerebrospinal fluid, and the composite properties of the helmet, skull, and the elasticity of impacted wall, respectively. Only approximate solutions were obtained for the limiting cases of soft and hard impacts. In 1975, Chandran and Liu (136) also studied the one dimensional problem but with a fluid-filled cylinder (container) attached to a spring-dash pot element striking a rigid wall. The exact closed-form wave propagation solution was obtained by exploiting the hyperbolic nature of the laplace transformed equations.
Engin (27-34), and Liu, 1971 (138), using a combination of membrane and bending theory for their axisymmetric spherical shell model, predicted a magnitude of head injury for a model subjected to a finite time duration pulse on the polar cap. These solutions utilized Legendre polynomial series, spherical bessel functions and the laplace transform technique for the time domain. Recently Chan et al. (14) has extended Engin's two-mode shell theory to a five-mode theory and obtained the non-axisymmetric response due to glancing blows. In 1976, Lankof et al., (126) included a continuum viscoelastic neck model to make the Chan's model more realistic. The author feels that this model can be made even more realistic and abandons the Lankof continuum neck model by accepting the head injury model as an ensemble of head, neck and vertebral column which are composed of rigid vertebra and viscoelastic vertebral discs.

Other important studies, which are essentially various extensions of two mode shell model of Engin with some slight modifications, are: (1) Dynamic analysis of fluid-filled spherical sandwich shell by Akkas (2) who used a finite difference method in obtaining his solution (2) Dynamic response of a fluid-filled spheroidal shell by Talhouni (244). Using spheroidal coordinates, he found high tensile stresses which are the usual criterion for fracture, in the skull, at the coup, counter-coup and a circle corresponding to a polar angle of about 35°. Brain damage was located at two points on the axis of symmetry, about half-way between the center of the poles, due to large negative pressure occurring at those points according to the cavitation theory. These results were very similar to those obtained by Engin in 1969. Other
**TABLE 1**

THE MATHEMATICAL METHODS AND MODELS OF HEAD IMPACT

<table>
<thead>
<tr>
<th>Indirect Impact</th>
<th>Direct Impact</th>
<th>Indirect Loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotation Only</td>
<td>Translation &amp; Rotation</td>
<td>Translation &amp; Rotation</td>
</tr>
<tr>
<td><strong>Rigid Spherical Shells</strong></td>
<td><strong>Rigid Skull</strong></td>
<td><strong>Discrete Parameter Models</strong> (Roberts et al., 1969, Allen, 1974, Roberts &amp; Thompson, 1975)</td>
</tr>
<tr>
<td><strong>One Dimensional Models</strong></td>
<td><strong>Three Dimensional Models</strong></td>
<td><strong>Viscoelastic Core &amp; Fluid Core</strong> (Chan, 1974)</td>
</tr>
<tr>
<td><strong>Elastic Tube</strong> (Kopecky &amp; Ripperger, 1969)</td>
<td><strong>Spherical Shell</strong></td>
<td><strong>Elastic-Shear &amp; Fluid Core</strong> (Merchant &amp; Crispins, 1974)</td>
</tr>
<tr>
<td><strong>Infinite Series</strong> (Engin &amp; Roberts, 1971)</td>
<td><strong>Exact Solution</strong> (Liu &amp; Chandran, 1974)</td>
<td><strong>Static Empty Layered Skull</strong> (Hardy &amp; Marcal, 1971)</td>
</tr>
<tr>
<td><strong>Three Dimensional Models</strong></td>
<td><strong>Rigid Tube Massless</strong> (Hayashi, 1969)</td>
<td><strong>Membrane Shell</strong> (Benedict et al., 1970)</td>
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<tr>
<td><strong>Exact Solution</strong> (Liu &amp; Chandran, 1974)</td>
<td><strong>Fluid Core</strong></td>
<td><strong>Bending Resistance</strong></td>
</tr>
<tr>
<td><strong>Viscoelastic Core</strong> (Liu &amp; Chandran, 1974)</td>
<td><strong>Elastic Core</strong> (Advani &amp; Owings, 1974)</td>
<td><strong>Multi-layer Shells</strong> (Gordon et al., 1973, Arras, 1973)</td>
</tr>
<tr>
<td><strong>Finite Element Method</strong></td>
<td><strong>Elastic Shell</strong></td>
<td><strong>Impulse Loading</strong> (Engin, 1969)</td>
</tr>
<tr>
<td><strong>Elastic Thin Sphere</strong></td>
<td><strong>Viscoelastic - Shell and Core</strong> (Hickling &amp; Wenner, 1973)</td>
<td><strong>Finite Duration Pulse</strong> (Liu et al., 1971, Kenner &amp; Goldsmith, 1972)</td>
</tr>
<tr>
<td><strong>Elastic Core</strong></td>
<td><strong>Membrane Shell</strong> (Benedict et al., 1970)</td>
<td><strong>Symmetric Pulse</strong> (Chan and Liu, 1974)</td>
</tr>
<tr>
<td><strong>Elastic Thin Ellipsoid</strong></td>
<td><strong>Bending Resistance</strong></td>
<td><strong>Head-Helmet</strong> (Knall et al., 1974)</td>
</tr>
<tr>
<td><strong>Elastic Core</strong></td>
<td><strong>Multi-layer Shells</strong> (Gordon et al., 1973, Arras, 1973)</td>
<td></td>
</tr>
</tbody>
</table>

*For the Ph.D. Dissertation, The Ohio State University, March 1977.
works of interest are: (1) Dynamic loading of a fluid-filled spherical shell by Kenner and Goldsmith (113-4); (2) Mathematical model of a head subjected to an axisymmetric impact by Hickling (85); (3) Intracranial pressures during head impact and theory of the mechanics of skull fracture by Engin (31-34); (4) Analytical investigation of the cavitation hypothesis of brain damage by Benedict et al. (9); and (5) Dynamic response of the human cadaver head compared to a simple mathematical model by Hodgson (92-95).

A summary of the contemporary studies in the field of head impact is shown in Table 1.

1.3 Organization

The subject matter of this research is primarily concerned with the theoretical analysis of a more sophisticated head injury model which consists of the vertebral column and the head. Chapter 2 deals essentially with the analysis of the role of the vertebral column as an impulse force transmitter to the head and as a restraint to prevent any random motion of the head. The obturator foramen together with the sacrum receives the impulsive input which may be a sudden acceleration or deceleration and/or impulsive force that has a tendency to impair or change the original state of the body. Chapters 3 and 4 deal with more elaborate extensions and variations of a fluid-filled spherical shell model, which, at the present, remains the most improved continuum mechanics model representing the human head when subjected to impulsive load or impact. Since an impact involves the collision of the spherical shell with another solid shell at an appreciable velocity, large linear accelerations and small angular accelerations generally characterize this
situation during the impact phase. In Chapter 2, the effect of the impulsive loading which is transmitted through the head-neck junction before impact results in a large angular motion, which, when severe, leads to a whip-lash or neck injury. Because the head injury trauma takes a relatively short time, a wave propagation approach has been adopted for the investigation of equations of motion of both the impacted and impacting shells. Moreover, the free vibration of the impacted shell is included for the simple fact that its dimension is finite. Spherical bessel functions and the legendre polynomial functions are the major mathematical functions which are utilized.

Chapter 5 plays a key role in initiating the numerical solution to displacements and a stress distribution in Chapters 3 and 4. It is this chapter that the force involved in the impact, the impact duration, and the coefficient of restitution are obtained. After obtaining the graphical solution of the force generated during impact, it is returned to Chapter 3 to numerically compute the stresses and displacements of the spherical shell and pressure distributions in the fluid which are needed to determine the actual possible skull and brain damage locations.
The head injury is treated as a trauma involving the ensemble of the vertebral column and the head to effectively represent the torso restraints on the activities of the head. In order to appreciate the previous efforts, the discussion or previous spine models naturally leads us to the subject matter of this chapter; discrete parameter models of the spine. The discussion begins with lumped parameter and continuum models.

**Lumped Parameter Models**

The drive for the spine dynamic modeling started as early as 1957 when Latham (128) subjected a double-mass spring-coupled system to a base excitation. The principal assumption was that the elastic characteristics of both the man and the seat cushion is represented by the spring. The primary interest here was to obtain the effect of cushion stiffness on the biodynamic response of the torso, while the basic idea was the dynamic overshooting, which is primarily the function of the rate of onset of the input pulse at a given acceleration level. Payne (197) continued this concept in 1962 and introduced a dynamic response index, DRI (defined as the ratio of the force to the body weight), which is a measure of dynamic overshoot.
In 1971, Hopkins (99), using a three-mass system, with a two-degrees of freedom, simulated the dynamic response of the torso to a low frequency vibration, which included the effects of the thoracic and abdominal viscera. But during the same period, Kaleps et al. (107) used four-mass model with a five-degree of freedom for simulating dynamic response of thorax, abdomen and axially loaded spine. This has an advantage of easy adaptation to simulate torso response in many dynamic conditions, over Hopkins three-mass model.

**Continuum Models**

Continuum models are classified as models with and without initial curvature, that is, straight uniform rod models with constant cross sections.

**(A) Straight Uniform Rod Models**

In 1958, Hess and Lombard (84) introduced an elastic continuum model of the spine with its one end free and the other loaded with acceleration impulse. However, the results gave a poor prediction. Liu and Murray (137), 1966, continued with the same model which was then assumed to include the head. The free end of the rod was now constrained by a rigid mass (head) while the other end was subjected to a unit step pulse. But two years later, Terry and Roberts (247), still continuing with the same model, used a Maxwellian viscoelastic model, rather than the previous elastic model. With optimization of the material constants, the formulation obtained from this model gave a good correlation with experimental acceleration data, using ramp acceleration input pulse.

In 1971 Shirazi (225) introduced a non-linear elastic model in which material constraints varied non-linearly with distance along the
rod while Rybicki et al. (219), modeled the spine by a porous elastic column containing a viscoelastic fluid supporting a rigid mass (head). The governing equations were solved by the finite difference technique. Soechting and Pasley (235), 1973, used $-G_x$ acceleration (horizontal deceleration) on a centrally loaded straight elastic rod, and from their results concluded that a lap-belted front seat passenger can have a maximum crash velocity of 6 mph with essentially an infinite deceleration to avoid a head impact with the windshield. In other words, the passenger with no anticipation must have a 15 in. motion with only a spinal muscular restraint.

(B) Curved Rod Models

Krause and Shrazi (124), 1971, modeled the lumbar spine as a curved rod and studied its bending response to periodic $+G_z$ excitation, while Li et al. (132) used a sinusoidal curved elastic column with one end attached to a mass (head) and the other loaded with a uniform acceleration. Material constants, Figure 8, and geometric properties determined by Moffatt et al. (171) were used. The head-torso mass eccentricity and restraints neglected by Li et al. was picked up by the Cramer's (19) curved beam-column model. This model includes a lordosis (concave anteriorly) and kyphosis (by means of eccentric linear inertia and rotary inertia) and shear effects. The formulation predicted axial load peak at about 35 msec. into the trapezoidal pulse of 80 msec. duration and the major portion of the spine went into tension at about 80 msec.
FIGURE 1  SIMPLIFIED HEAD-NECK STRUCTURE
FIGURE 2 VERTEBRAL COLUMN
(MODIFIED FROM GRANT'S ANATOMY)
FIGURE 3: AN INTERVERTEBRAL DISC
(SIDE VIEW MODIFIED FROM GRANT'S ANATOMY)
Discrete Parameter Models

The accuracy of the prediction of the impact effects on the brain is increased when the effects of the head-neck restraints are included in the head model for the head injury analysis. The efforts oriented towards this goal follow the approach used by Orne (187) in his discrete parameter model of the human spine in the Pilor Ejection Problem with some modifications for the present problem. The model consists of rigid vertebrae separated by deformable discs which are assumed to be viscoelastic in axial compression and elastic in shear and bending. Each vertebrae is assigned to support a portion of the eccentric torso. The vertebral column anatomy and the corresponding model are illustrated in Figures 2, 3, and 4.

The model is simple and realistic, yet gives an accurate prediction of vertebral column response to any impulse. The continuum presentation of the head-neck structure by Landkof, et al. (126) which has a static boundary condition (a stationary cervical and thoracic junction) is a complex formulation which gives an estimate of the head injury due to a relative neck-head response to an impulsive loading. The results, of course, correlate with the data of his similar experimental model, yet both are far from real because there is no head injury resulting from the motion of neck and head only without other parts of the body involved. That is, the cervical-thoracic junction will normally be set in motion.

In this dissertation, the following observations and assumptions are made in the mathematical formulation of vertebral column response
FIGURE 4 GEOMETRY OF INITIAL STATE

FIGURE 5 CANTILEVER DISC

FIGURE 6 DISC DEFORMATION $t > 0$
before the head impact ensures: (187)

1. Previous experimental studies have shown that vertebral bodies are much bigger than the intervertebral discs. Consequently, the mass of the intervertebral discs is neglected in comparison with mass attributed to the vertebrae.

2. We assume that all the inertia resistance to motion is attributed to rigid links whose geometry correspond to those of the individual vertebrae. In general, the center of mass does not coincide with the geometrical center of the links. For example, the mass of thoracic region is transmitted to the vertebrae by the ribs, acting as cantilever beams fixed to the vertebra.

3. Each rigid link has three-degrees-of-freedom in the sagittal plane, two translational and one rotational, hence a general rigid body motion can be executed, as shown in Figure 4.

4. All deformations in the spinal column occur in deformable links or "beam segments" whose geometry corresponds to those of individual discs. Each disc is capable of resisting axial, shear and bending deformations according to constitutive relations to be specified subsequently.

5. The large mass of the pelvis plus the assumed presence of secured lap-type seat belts preclude any significant amount of rotation of the pelvis. This essentially implies a "clamped-beam" type of boundary condition at the base of the spine.

Figure 4 shows a curved, planar structural system consisting of alternating rigid and deformable links in which the loading and subsequent motions are confined to the plane. The ultimate goal is
determine the changes in the head-neck junction, due to the acceleration impulse on the seat. For example, we would like to know the axial force, shear force and the bending moment constraints at the time of impact of the head with the windshield.

In all automobiles the windshield is placed at a fixed distance from the front seat, and hence the time the head takes to get to the point of impact after the pelvis receives the acceleration impulse can be determined. Soechting and Pasley had predicted a minimum distance of 15 inches or maximum speed of 6 mph. If this is used as a criterion for an impact to take place, then it is obvious that on the contrary impact can be assumed to have occurred.

2.1. Force-Displacement Relation for Intervertebral Disc

Let us consider a uniform, viscoelastic cantilever beam for modelling of an intervertebral disc as shown in Figure 5. This beam is clamped at the left end and loaded by a lateral force, \( F_L \), an axial force, \( F_a \) and a bending moment \( F_m \). We assume that the material behavior of the beam is characterized by a three parameter, viscoelastic solid. From the usual assumptions of the Euler-Bernoulli beam theory, we obtain the strain due to bending and extension as:

\[
\epsilon(x,z,t) = U_x(x,t) - z W_{xx}(x,t)
\]  \( (2.1.1) \)

Let \( p_1 \), \( q_0 \), \( q_1 \) be the viscoelastic parameters, then the constitutive equation for a three-parameter viscoelastic solid is

\[
\sigma + p_1 \dot{\epsilon} = q_0 \epsilon + q_1 \dot{\epsilon}
\]  \( (2.1.2) \)

Throughout this dissertation the subscripts which are the elements of
the argument of any function denote the differentiation with respect to those subscripts, unless braces enclose the subscripts, in which case the subscripts denote the instantaneous value of the function; for example, \( M(x) \) denotes the value of the moment at length \( x \). Similarly, the dot placed on any function denotes a differentiation with respect to time. In equation (2.1.2) the stresses, \( \sigma \), and the strains, \( \varepsilon \), are functions of the two coordinates, \( x, z \), which define the sagittal plane and time, \( t \).

Multiplying equation (2.1.2) by \( z \) and integrating the result with respect to the area \( A \), gives:

\[
\int_A \sigma z \, dA + p_1 \int_A \sigma z \, dA = q_0 \int_A \sigma_x z \, dA - q_0 \int_A \sigma_{xx} z^2 \, dA + q_1 \int_A \sigma_x \, dA - q_1 \int_A \sigma_{xx} z^2 \, dA
\]  
(2.1.3)

If the following symbols are introduced in equation (2.1.3) and noting that \( I_{xz} = 0 \) because of symmetry,

\[
M(x) = \int_A \sigma z \, dA
\]
\[
I(x) = \int_A z^2 \, dA
\]
\[
F_a = \int_A \sigma \, dA
\]
\[
I(xz) = \int_A x \sigma \, dA
\]  
(2.1.4)

we obtain

\[
M(x) + p_1 M(x) = -[q_0 I(x) \sigma_{xx} + q_1 I(x) \sigma_{xx}]
\]  
(2.1.5)

Since the length of each intervertebral disc is small in comparison with the length of the vertebral column, \( I(x) \) can be replaced by a constant moment of inertia \( I \) for each disc without any ambiguity in notation.
From Figure 5, the internal moment, \( M(x) \) is given as

\[
M(x) = -[F_m + F_s(L - x)] \tag{2.1.6}
\]

If equation (2.1.6) is now introduced into equation (2.1.5), the following equation is obtained

\[
[1 + \rho_1(\frac{d}{dt})]F_m + [1 + \rho_1(\frac{d}{dt})]F_s(1 - x) =
I[q_0 + q_1(\frac{d}{dt})]W_{xx} \tag{2.1.7}
\]

The first and the second integration of equation (2.1.7) with respect to \( x \) give equations (2.1.8) and (2.1.9) respectively:

\[
[1 + \rho_1(\frac{d}{dt})]F_m + \frac{(L/20)[1 + \rho_1(\frac{d}{dt})]F_s}{I/L}[q_0 + q_1(\frac{d}{dt})]W_x = \tag{2.1.8}
\]

\[
[1 + \rho_1(\frac{d}{dt})]F_m + \frac{(2L/3)[1 + \rho_1(\frac{d}{dt})]F_s}{(2I/L^2)}[q_0 + q_1(\frac{d}{dt})]W = \tag{2.1.9}
\]

Subtracting equation (2.1.8) from equation (2.1.9) we obtain

\[
[1 + \rho_1(\frac{d}{dt})]F_s = -(\frac{6L}{I/L^2})[q_0 + q_1(\frac{d}{dt})]W_x + \tag{2.1.10}
\]

\[
(\frac{12I}{L^3})[q_0 + q_1(\frac{d}{dt})]W
\]

Next, equation (2.1.9) is multiplied by \( 3/4 \) and then it is subtracted from equation (2.1.8) and equation (2.1.11) results

\[
[1 + \rho_1(\frac{d}{dt})]F_m = \frac{(4I/L)}{[q_0 + q_1(\frac{d}{dt})]W_x} - \tag{2.1.11}
(\frac{6I}{L^2})[q_0 + q_1(\frac{d}{dt})]W
\]

If the rotation, \( W_x \) and the transverse displacement, \( W \), due to the bending moment, \( F_m \), are denoted by \( \delta \phi \) and \( \delta bs \) respectively, equations
(2.1.10) and (2.1.11) can be rewritten as follows:

\[ [1 + p_1(d/dt)]F_s = -(6I/L^2)[q_0 + q_1(d/dt)]\delta + (12I/L^3)[q_0 + q_1(d/dt)]\delta_{bs} \]  \hspace{1cm} (2.1.12)

\[ [1 + p_1(d/dt)]F_m = (4I/L)[q_0 + q_1(d/dt)]\delta - (6I/L^2)[q_0 + q_1(d/dt)]\delta_{bs} \]  \hspace{1cm} (2.1.13)

In order to obtain the conventional stiffness matrix from the last two equations, \( p_1 \) and \( q_1 \) are allowed to go to zero while \( q_0 \) attains the value \( E \), which is the modulus of elasticity.

\[
\begin{pmatrix}
F_s \\
F_m
\end{pmatrix} =
\begin{bmatrix}
12EI/L^3 & -6EI/L^2 \\
-6EI/L^2 & 4EI/L
\end{bmatrix}
\begin{bmatrix}
\delta_{bs} \\
\delta
\end{bmatrix}
\]  \hspace{1cm} (2.1.14)

Now let us go back to equation (2.1.2) and perform an integration with respect to area \( A \):

\[
\int_A \sigma dA + p_1 \int_A \dot{\sigma} dA = q_0 \int_A (U_x - \dot{z}W_{xx}) dA + q_1 \int_A (\dot{U}_x - \ddot{z}W_{xx}) dA
\]  \hspace{1cm} (2.1.15)

The symbols of equation (2.1.4) are introduced into equation (2.1.15) in order to get

\[
F_a + p_1 \dot{F}_a = q_0 A U_x + q_1 A \dot{U}_x
\]

or

\[
[1 + p_1(d/dt)]F_a = A[q_0 + q_1(d/dt)]U_x
\]  \hspace{1cm} (2.1.16)

Equation (2.1.16) integrated with respect to \( x \), gives

\[
[1 + p_1(d/dt)]F_a = (A/L)[q_0 + q_1(d/dt)]U
\]

If the axial extension, \( U \), due to the force, \( F_a \), is denoted by \( \delta a \), the above equation becomes
\[ [1 + p_1(d/dt)]F_a = (A/L)[q_0 + q_1(d/dt)]\delta a \]  

(2.1.17)

By considering a small element of the cantilever disc, the instantaneous bending moment, \( M(x) \), and the shear, \( V(x) \), can be related in the following manner:

\[ M(x)x + p_1 M(x)x = V(x) + p_1 V(x) \]  

(2.1.18)

If we recall equation (2.1.5), that is,

\[ M(x) + \frac{\partial}{\partial x}(\int_0^x IW xx) = \frac{\partial}{\partial x}(\int_0^x IW xx) \]

we can immediately realize that to obtain expression for equation (2.1.18) is a matter of differentiating equation (2.1.5) with respect to \( x \), which when carried out gives us

\[ V(x) + p_1 V(x) = q_0 IW xxx + q_1 IW xxx \]

but \( V(x) = F_s \). When \( F_s \) is now substituted for \( V(x) \), the following equation results

\[ [1 + p_1(d/dt)]F_s = I[q_0 + q_1(d/dt)]W xxx \]  

(2.1.19)

Equation (2.1.19) is then integrated three times and the transverse displacement, \( W \), obtained is due to the transverse force, \( F_s \). The displacement, \( W \), is denoted by \( \delta ss \)

\[ [1 + s_1(d/dt)]F_s = (A/Lk)[r_0 + r_1(d/dt)]\delta ss \]  

(2.1.20)

where \( s_1, r_0, r_1 \) are introduced in place of \( p_1, q_0 \) and \( q_1 \) to make the distinction between the axial and transverse directions. In equation (2.1.20)

\[ k = (6/L^2)(I/A) \]

Thus, the total deflection along \( Z \) direction is given as

\[ \delta s = \delta ss + \delta bs \]  

(2.1.21)
From equation (2.1.21)

\[ \delta ss = \delta s - \delta bs \]

If equations (2.1.20) and (2.1.13) are substituted into the above equation and \( \dot{\delta ss} \) solved for, the following equation for the time derivative of \( \delta ss \) ensues

\[ \dot{\delta ss} = [(kL/A) (p_1 - s_1) + (12I/L^2) s_1] [q_0 (\delta s - \delta ss) + q_1 \dot{\delta s}] - \\
(6I/L^2) s_1 [q_0 \delta \phi + q_1 \dot{\delta \phi}] - \tau_0 p_1 \delta ss) / [\tau_1 p_1 + \\
(12I/L^3)(kL/A) q_1 s_1 \]  \hspace{1cm} (2.1.22)

In a symbolic form the equation (2.1.22) is

\[ \dot{\delta ss} = f(\delta s, \delta ss, \delta \phi, \dot{\delta s}, \dot{\delta \phi}) \] \hspace{1cm} (2.1.23)

This is a first order differential equation. When the denominator goes to zero, equation (2.1.23) is no longer valid. The invalidity happens when any of the following conditions occurs:

(a) \( \tau_1 = s_1 = 0 \) pure elastic response in shear
(b) \( \tau_1 = q_1 = 0 \) Maxwellian in shear and tension
(c) \( p_1 = s_1 = 0 \) Kelvin-Voigt in shear and tension
(d) \( p_1 = q_1 = 0 \) elastic in tension

2.2. Geometry of Displacements

Figure 6a shows the geometry of two vertebral bodies connected by an intervertebral disc at the initial time, \( t = 0 \). The vertebral body DA of length \( L_1 \) makes an angle \( \phi_1^o \) with the horizontal line while the second vertebral body has an inclination of \( \phi_{i+1}^o \) with the horizontal
axis. The initial angular deviation of the vertebral axes is $\Delta \phi_i^0$, that is,

$$\Delta \phi_i^0 = \phi_i^0 - \phi_{i+1}^0$$  \hspace{1cm} (2.2.1)

At time, $t > 0$, the vertebral body DA gets rigid body displacements components $U_i$ and $W_i$ while the second vertebral body BE receives $U_{i+1}$ and $W_{i+1}$ at same time. The intervertebral disc deforms viscoelastically as shown in Figure 6b. The original coincident points C and B then separate as the intervertebral disc deforms. Let us define the lengths:

$$(AC)_i = d_{b1}$$

$$(BC)_i = d_{a1}$$  \hspace{1cm} (2.2.2)

$$(AB)_i = d_{c1}$$

In Figure 6c, the vertebral body is given a pure rotation. The instantaneous inclination of the two vertebral bodies are $\phi_i$ and $\phi_{i+1}$ while the relative deviation of their axes is $\Delta \phi_i$, which is defined algebraically as

$$\Delta \phi_i = \phi_i - \phi_{i+1}$$  \hspace{1cm} (2.2.3)

The combination of the planer displacements and rotation is shown in Figure 7 which shows the deformation of an intervertebral disc under the combined influence of the axial force, $F_a$, the transverse force, $F_s$ and the bending moment, $F_m$.

Refer to Figure 6a, b, c once more for the evaluation of the essential parameters. The length AB is given at any time by

$$AB = \left\{ (W_{i+1} - (W_i + l_i \sin \phi_i))^2 + (U_{i+1} - (U_i + l_i \cos \phi_i))^2 \right\}^{1/2}$$

Lengths DA and AB make angles $\phi_i$ and $\alpha_i$ with X-axis respectively while $\beta_i$ represents the relative angular deviations of AB and DA from the
horizontal axis after deformation of the intervertebral disc, such that

$$\beta_i = \alpha_i - \phi_i \tag{2.2.5}$$

The displacement components ($U_i, W_i$) represents the location of the end of intervertebral disc and the beginning of another vertebral body, $D_i$. If we include the angle $\phi_i$ to $U_i$ and $W_i$ we can locate the vertebral body $i$. Following this systematic numbering of the vertebral bodies and their correspondent variables, we can define $\alpha_i$ of Figure 6b as

$$\alpha_i = \arctan\left\{\frac{W_{i+1} - (W_i + l_i \sin \phi_i)}{U_{i+1} - (U_i + l_i \cos \phi_i)}\right\} \tag{2.2.6}$$

and its first derivative as

$$\dot{\alpha}_i = \frac{d}{dt}\left\{\frac{W_{i+1} - (W_i + l_i \sin \phi_i)}{U_{i+1} - (U_i + l_i \cos \phi_i)}\right\} \tag{2.2.7}$$

The components of deformation of the intervertebral disc in Figure 6b are

$$\begin{align*}
da_i &= d_i \sin \beta_i \tag{2.2.8} \\
db_i &= d_i \cos \beta_i
\end{align*}$$

The axial deformation of the disc is

$$\delta_{a_i} = db_i - dc_i \tag{2.2.9}$$

and its rate of change $\dot{\delta}_{a_i}$ is obtained by differentiating equations (2.2.8) and (2.2.9) and eliminating $db_i$ from the resulting two equations

$$\dot{\delta}_{a_i} = F_i(U_i, W_i, \phi_i, \dot{U}_i, \dot{W}_i, \dot{\phi}_i) \tag{2.2.11}$$
Figure 7. Intervertebral Disc Under Combined Forces: Axial, Lateral and Bending

Figure 8. Axial Compression of Two Vertebral Bodies Adjoined by an Intervertebral Disc
The lateral deformation of the disc is defined by

\[ \delta_{1} = \Delta a_{1} \]  

(2.2.12)

From the same Figure 6, the angular deformation \( \delta \phi \) is obtained by finding the difference of angular deviations of the axes of the two vertebral bodies joined together by the disc, at time \( t = 0 \) and at some other time \( t > 0 \). Thus,

\[ \delta \phi_{1} = (\phi_{i} - \phi_{i+1}) - (\phi_{i}^{0} - \phi_{i+1}^{0}) \]  

(2.2.13)

2.3. Axial, Shear and Bending Behavior

The material behavior of intervertebral discs under axial compression is that of viscoelastic solid. This idea is supported by Moffatt (171) in his relaxation experiment on the vertebral bodies and intervertebral discs under constant strain. The stress relaxes exponentially with time to some non-zero value, Figure 8 shows that the strain response to a suddenly applied and subsequently maintained axial stress consists of an initial and asymptotic elastic response (strain) with time.

As a result of the creep and relaxation tests conducted by Moffatt (171) and Hirsch (90) on intervertebral discs, a three-parameter solid model is used here to simulate the axial behavior of the discs. This is essentially equivalent to (Figure 9a and b), (a) a spring in series with a Kelvin-Voigt model, or (b) a spring in parallel with a Maxwell model, see Stevens (240) and Flugge (43).

The differential equation for the intervertebral disc model is

\[ \sigma + p_{2} \dot{\sigma} = q_{0} \varepsilon + q_{1} \dot{\varepsilon} \]  

(2.3.1)
Characteristics of a Three-Parameter Viscoelastic Solid (187,43)
In equation (2.3.1), the viscoelastic parameters are expressed in terms of the physical constants as

\[
q_0 = \frac{E_3 E_4}{(E_3 + E_4)}
\]

\[
p_2 = \frac{1}{\mu(E_3 + E_4)}
\]

\[
q_1 = \frac{E_4}{(E_3 + E_4)\mu}
\]

where \(E_3, E_4, \mu\) are mechanical model constants. Equation (2.1.17) is applied to a uniform rod of length \(L\), with the above mechanical constants replacing the model parameters \(q_0, q_1, p_2\), where

\[
\sigma = \frac{F_a}{A}
\]

\[
\varepsilon = \frac{\delta_a}{L}
\]

For shear behavior of the model, the differential equation is

\[
\tau + S_1 \dot{\tau} = r_0 \gamma + r_1 \gamma
\]

which is the same as (2.1.21), where

\[
\tau = \frac{F_s}{A}, \text{ shear stress}
\]

\[
\gamma = \frac{\delta A}{l k}, \text{ shear strain.}
\]

From (2.1.21), (2.1.17) and (2.1.12)

\[
\dot{F}_s = [F_s + (A/k)(r_0 \delta_{ss} + r_1 \dot{\delta}_{ss})]/s_1
\]

\[
\dot{F}_a = (1/p_1)[-F_a + (A/l)(q_0 \delta_a)]
\]

\[
\dot{F}_m = (1/p_1)[-F_m - (6I/k^2)q_0(\delta_s - \dot{\delta}_{ss}) + q_1 + q_1(\delta_s - \dot{\delta}_{ss})] + (4I/l)(q_0 \delta_\phi + q_1 \dot{\delta}_\phi)
\]

The equations (2.1.23) and (2.3.5a) are systems of first order ordinary differential equations of the form
\[
\frac{df_k}{dt} = f_k(t, \delta_s^k, \delta_{ss}^k, \delta_a^k, \phi) \frac{d\delta_s^k}{dt} \frac{d\delta_{ss}^k}{dt} \frac{d\delta_a^k}{dt} \frac{d\delta\phi}{dt} 
\]

\(k = 1, 2, 3, \ldots, 25\)  

Equation (2.3.5b) is one hundred and seventy-five ordinary linear differential equations which is already in the standard form usually required for numerical solutions by Runge-Kutta type schemes.

To simplify the system, the intervertebral discs are assumed to possess only axial viscoelastic properties. Incidentally, the literature contains experimental data only on axial properties of the intervertebral discs. Equations (2.1.14) and (2.3.5a) reduce to

\[
F_s = A\delta_q^0 / kL \delta_{ss}^k 
\]

(2.3.6)

\[
F_s = (12I\delta_q^0 / L^3)(\delta_s - \delta_{ss}) - (6I\delta_q^0 / L^2)\delta\phi 
\]

(2.3.7)

\[
F_m = -(6I\delta_q^0 / L^2)(\delta_s - \delta_{ss}) + (4I\delta_q^0 / L)\delta\phi 
\]

(2.3.8)

\[
F_a + p_1 \dot{\phi} = (A/L)(\delta_q^0 \delta_a + \delta_1 \delta_a) 
\]

(2.3.9)

From (2.3.6) through (2.3.9) after some algebraic manipulations, we obtain

\[
F_s = \frac{12I\delta_q^0}{L^3(4\zeta - 3)} \delta_s - \frac{6I\delta_q^0}{L^2(4\zeta - 3)} \delta\phi 
\]

(2.3.10)

\[
\dot{\phi} = [-F_a + \frac{A}{L}(\delta_q^0 \delta_a + \delta_1 \delta_a)] / p_1 
\]

(2.3.11)

\[
F_m = -\frac{6I\delta_q^0}{L^2(4\zeta - 3)} \delta_s + \frac{4\zeta I\delta_q^0}{L(4\zeta - 3)} \delta\phi 
\]

(2.3.12)

where

\[
\zeta = 1 + \frac{3I\delta_q^0}{L^2} (\frac{kL}{k0A}) 
\]

(2.3.13)

and \(\zeta - 1\) is the bending-shear stiffness ratio.
2.4. Equations of Motion of Vertebral Column

Before we introduce the equations of motion of the vertebral column let us briefly examine the movements of the vertebral bodies. All three regions of the articular processes of the vertebral bodies permit flexion, extension, and mediolateral movements. In addition to these movements the cervical vertebra also allow sideways look in upward direction with comfort, while the thoracic vertebra permit mediolateral rotations (see Figure 10).

These movements can be described by acceleration components \((\omega_i, u_i, \phi_i)\) of the Lagrangian centroidal coordinate system \((x_i, x_i, \phi_i)\). The instantaneous position of a vertebral body is described as shown in Figure 11 by

\[
Z_i = W_i + h_i \sin \phi_i + e_i \cos \phi_i
\]

and

\[
x_i = u_i + h_i \cos \phi_i - e_i \sin \phi_i
\]

which are the horizontal and the vertical components of the center of mass of the vertebral body at time, \(t > 0\). The dimensions \(h_i\) and \(e_i\) define the centroidal eccentricity of the vertebral body.

Reactions of the intervertebral disc on the vertebral body can be generalized in the following manner with the aid of Figures 7 and 11 to include all the twenty-five vertebral bodies and twenty-five intervertebral discs

\[
F_s = F_{s1}
\]

\[
F_a = F_{a1}
\]

\[
F_m = F_{m1}
\]
FIGURE 10
MOVEMENTS OF FREE VERTEBRAE

CERVICAL  THORACIC  LUMBAR
FIGURE 11  FREE BODY DIAGRAM OF A VERTEBRAL BODY AND ASSOCIATED COORDINATES (187)
\[ \delta_s = \delta_{si} \]  
\[ \delta_a = \delta_{ai} \]  
\[ \delta_\phi = \delta_{\phi i} \]  
\[ \delta_{bs} = \delta_{bsi} \]

where the subscript \( i \) will now stand for any intervertebral disc or vertebral body along the vertebral column. The bending moment acting at the bottom of an \( i \)th disc is expressed as:

\[ B_i = F_{mi} - F_{si} b_i \]  
(2.4.3)

and the components of the shear, \( Q_i \), and axial forces, \( P_i \), along the X- and Z- axis are:

\[ Q_i = F_{si} \cos \phi_i + F_{ai} \sin \phi_i \]  
(2.4.4)

and

\[ P_i = -F_{si} \sin \phi_i + F_{ai} \cos \phi_i \]  
(2.4.5)

where \( \phi_i \) represent the instantaneous orientation of the vertebral body.

The equation of motion of the vertebral body can now be written

\[ M_{1x} = Q_1 - Q_{1-1} \]  
(2.4.6)

\[ M_{1y} = P_1 - P_{1-1} \]  
(2.4.7)

\[ J_{1z} = (F_{mi-1} - B_i) - C_{2i} Q_{1-1} + C_{3i} Q_i + C_{1i} P_{1-1} + C_{4i} P_i \]  
(2.4.8)

By carrying out a pertinent differentiations on equations (2.4.1), we obtain two equations, which are acceleration components of the mass center

\[ \ddot{z}_1 = \ddot{w}_1 + C_{1i} \dot{\phi}_i^2 - C_{2i} \ddot{\phi}_i \]  
(2.4.9)

and
\[ \ddot{x}_1 = \ddot{u}_1 + c_{11} \dot{\phi}_1^2 + c_{11} \phi_1 \]  
(2.4.10)

If we now make use of equations (2.4.9) and (2.4.10) in equations (2.4.6) and (2.4.7), we obtain another pair of acceleration equations in terms of the applied forces

\[ \ddot{u}_1 = \frac{1}{m_1}(Q_i - Q_{i-1}) + \frac{c_{2i}}{J_1}\{(F_{mi} - B_i) - c_{2i}Q_{i-1} + c_{3i}Q_i + c_{1i}P_{i-1} + c_{4i}P_i\} - c_{1i} \phi_i^2 \]  
(2.4.11)

and

\[ \ddot{u}_1 = \frac{1}{m_1}(P_i - P_{i-1}) + \frac{c_{1i}}{J_1}\{(F_{mi} - B_i) - c_{2i}Q_{i-1} + c_{3i}Q_i + c_{1i}P_{i-1} + c_{4i}P_i\} - c_{2i} \phi_i^2 \]  
(2.4.12)

Also from equation (2.4.8) we have

\[ \phi_i = \frac{1}{J_1}\{(F_{mi} - B_i) - c_{2i}Q_{i-1} + c_{3i}Q_i + c_{1i}P_{i-1} + c_{4i}P_i\} \]  
(2.4.13)

where \( J_1 \) is the centroidal moment of inertia about an axis perpendicular to the sagittal plane and \( c_{ji} \) \((j = 1, 2, 3, 4)\) that appear in equations (2.4.6)-(2.4.13) are

\[ c_{1i} = -(h_i \sin \phi_i + e_i \cos \phi_i) \]
\[ c_{2i} = -(h_i \cos \phi_i - e_i \sin \phi_i) \]  
(2.4.14)
\[ c_{3i} = h_i \cos \phi_i + e_i \sin \phi_i \]
\[ c_{4i} = h_i \sin \phi_i - e_i \cos \phi_i \]

Equations (2.4.11) through (2.4.13) can now be written as a system of ordinary linear differential equations of first order in
anticipation for computer algorithm by utilizing the following:

\[ \ddot{W}_i = \dot{W}_{1i}, W_{1i} = \dot{W}_1 \]

\[ \ddot{U}_i = \dot{U}_{1i}, U_{1i} = \dot{U}_1 \quad (i = 1, 2, \ldots, 25) \quad (2.4.15) \]

\[ \ddot{\Phi}_i = \dot{\Phi}_{1i}, \Phi_{1i} = \dot{\Phi}_1 \]

To the system of first order differential equation of the vertebra, we must also include the equation (2.3.11)

\[ \dot{F}_{ai} = [-F_{ai} + \frac{A}{L}(q_0 \delta_{ai} + q_1 \dot{a}_i)]/p_1 \quad (2.4.16) \]

Since there are twenty-five vertebra, the above equations result in one hundred and seventy-five ordinary linear differential equations of first order.
CHAPTER 3

THE DYNAMIC RESPONSE OF THE FLUID FILLED
SPHERICAL SHELL

Figure 12 shows the collision of the closed spherical shell and the shallow shell while Figure 13 gives the free body diagram of the spherical shell, the analysis of which is the subject of this chapter.

3.1 Formulation of Deformation

Equation for the Spherical Shell

Linear shell theory which includes membrane (extensional), bending (inestensional), rotatory inertia and transverse shear effects is used. The three-dimensional system coordinates \((r, \phi, \theta)\) are embedded in the two-dimensional system (surface coordinates) \((\phi, \theta)\) for the mid-surface of the spherical shell. In order to evaluate the variations along the thickness of the spherical shell, a normal coordinate is also introduced so that the coordinates \((\phi, \theta, \hat{n})\) correspond to the displacement components \((U^*, V^*, W^*)\).

\(\phi\)-axis is meridional, while \(\theta\)-axis is aximuthal. The rigid body components of displacement and velocity just before impact are \((U_{25}, \phi_{25}, W_{25})\) and \((\dot{U}_{25}, \phi_{25}, \dot{W}_{25})\), respectively. However, these are not useful information for deformation analysis. The radius of mid-surface of the spherical shell is \(a\) and any other point which is not on the midsurface is described as
FIGURE 12 IMPACT MODEL

Fluid-filled spherical shell is impacting a shallow shell of larger radius.
Figure 13. Stress Distribution at Impact

As an aid to the understanding of the coordinate system, the $\phi$-axis should be imagined to pass through the midpoints of both the forehead and the area of the neck-head junction which is the only plane of symmetry. The $\phi$-axis, orthogonal to the $\phi$-axis passes through the middle of the forehead and almost through the ear drums.
\[ r^* = a \pm \zeta. \]

or

\[ r^* = a(1 \pm \zeta) \]

where

\[ \zeta = \frac{\zeta}{a} \]

The rotational components along \( \phi \) and \( \theta \) are \( \beta_\phi \) and \( \beta_\theta \), respectively.

The components of displacement are described by Taylor's expansion about \( \zeta = 0 \) with the appropriate higher order terms neglected for linearity.

\[ U^*(\zeta, \phi, \theta, t) = u^*(\phi, \theta, t) + \zeta \beta_\phi(\phi, \theta, t) \]

\[ V^*(\zeta, \phi, \theta, t) = v^*(\phi, \theta, t) + \zeta \beta_\theta(\phi, \theta, t) \]

\[ W^*(\zeta, \phi, \theta, t) = w^*(\phi, \theta, t) \]  

The \( \phi \)-axis passes through the midpoint of both the contact area and the neck area at the neck-head junction, the boundary of the only plane of symmetry; and the \( \theta \)-axis passes through the middle of contact area and almost through the ears.

Following the general method outlined in Appendix A-3

\[ \alpha = \phi \]

\[ \beta = \theta \]

we obtain

\[ A^*_\phi = R^*_\phi = R^*_\theta = a \]

\[ A^*_\theta = a \sin \phi \]

The non-dimensionalized displacement components are

\[ U = u + \zeta \beta_\phi \]

\[ V = v + \zeta \beta_\theta \]

\[ W = w \]
where

\[
\begin{align*}
U &= \frac{U^*}{a} \\
u &= \frac{u^*}{a} \\
V &= \frac{V^*}{a} \\
V &= \frac{v^*}{a} \\
W &= \frac{W^*}{a} \\
v &= \frac{w^*}{a}
\end{align*}
\]

(3.1.5)

**Strain-Displacement Equations**

The strain-displacement equations at any point along the thickness of the shell are given as (123,130, appendix A-2):

\[
\begin{align*}
\varepsilon_\phi^{(\xi)} &= \frac{1}{1 + \zeta} (\partial u / \partial \phi + w) + \zeta \partial \beta / \partial \phi \\
\varepsilon_\theta^{(\xi)} &= \frac{1}{1 + \zeta} (\csc \phi \partial v / \partial \theta + \cot \phi \partial u / \partial \theta + \partial w / \partial \theta) + \zeta (\csc \phi \partial \beta / \partial \theta + \cot \phi \beta) \\
\gamma_{\phi\theta}^{(\xi)} &= \frac{1}{1 + \zeta} (\partial v / \partial \phi + \csc \phi \partial u / \partial \theta + \csc \phi \partial \beta / \partial \theta - \beta \cot \phi) + \zeta (\partial \beta / \partial \theta + \csc \phi \partial \beta / \partial \theta - \beta \cot \phi) \\
\gamma_{\phi n}^{(\xi)} &= \frac{1}{1 + \zeta} (\partial w / \partial \phi + \partial \beta / \partial \phi + \beta) \\
\gamma_{\theta n}^{(\xi)} &= \frac{1}{1 + \zeta} (\csc \phi \partial w / \partial \theta - \partial v / \partial \theta + \partial \beta / \partial \theta + \beta)
\end{align*}
\]

(3.1.6)

**Stress-Strain Equations**

\[
\begin{align*}
\sigma_\phi &= \sigma_\phi^{(\xi)} = \frac{E}{1 - \nu^2} (\varepsilon_\phi^{(\xi)} + \nu \varepsilon_\theta^{(\xi)}) \\
\sigma_\theta &= \sigma_\theta^{(\xi)} = \frac{E}{1 - \nu^2} (\nu \varepsilon_\phi^{(\xi)} + \varepsilon_\theta^{(\xi)})
\end{align*}
\]

(3.1.7)
where $E$, $G$, $\nu$, and $k_s$ are Young's Modulus, shear modulus, Poisson's ratio and the correction factor of transverse shear, respectively.

Integrating (3.1.7) over the thickness of the shell gives the following stress resultants as functions of displacements:

\[
N_\phi(u,v,w) = K\langle u,\phi + w \rangle + v(u\cot\phi + v,\theta \csc\phi + w) \tag{3.1.8}
\]

\[
N_\theta(u,v,w) = K\langle v(u,\phi + w) + (u\cot\phi + v,\theta \csc\phi + w) \rangle \tag{3.1.9}
\]

\[
N_{\phi\theta}(u,v,w) = \frac{1}{2}K(1 - \nu)\langle u,\theta \csc\phi + v,\phi - v\cot\phi \rangle \tag{3.1.10}
\]

\[
M_\phi = \frac{D}{a}\langle \beta_{\phi,\phi} + v(\beta_{\theta,\theta} \csc\phi + \beta_{\phi} \cot\phi) \rangle \tag{3.1.11}
\]

\[
M_\theta = \frac{D}{a}\langle v\beta_{\phi,\phi} + \beta_{\theta,\theta} \csc\phi + \beta_{\phi} \cot\phi \rangle \tag{3.1.12}
\]

\[
M_{\phi\theta} = \frac{1}{2}D(1 - \nu)/a\langle \beta_{\phi,\phi} + \beta_{\theta,\theta} \csc\phi - \beta_{\theta} \cot\phi \rangle \tag{3.1.13}
\]

\[
Q_\phi = \frac{K(1 - \nu)}{2K_s}\langle \beta_{\phi} + w,\phi \rangle \tag{3.1.14}
\]

\[
Q_\theta = \frac{K(1 - \nu)}{2K_s}\langle \beta_{\theta} + w,\theta \rangle \tag{3.1.15}
\]

where

\[
D = Eh^3/12(1 - \nu^2), \quad K = Eh/(1 - \nu^2), \quad \text{and}
\]

\[
K_s = 2k_s/(1 - \nu) \tag{3.1.16}
\]

$K_s$ is an averaging coefficient for the shear. $N_\theta$, $N_\phi$, $N_{\phi\theta}$ are membrane-stress resultants; $M_\phi$, $M_\theta$, $M_{\phi\theta}$ are the moment resultants; $Q_\phi$, $Q_\theta$ are the transverse-shear resultants.
3.2. Equations of Motion of a Fluid-Filled Shell

The equations of motion for the fluid-filled spherical shell are obtained by an energy method. The total energy of the system is defined as

\[ \Pi(u, v, w, \beta_\phi, \beta_\theta, q_i; F_i) = \Pi_s(u, v, w, \beta_\phi, \beta_\theta; F_i) + \Pi_f(q_i; F_i) \quad (3.2.1) \]

The contribution of energy from both the fluid and the shell are treated differently as \( \Pi_s \) and \( \Pi_f \), but they are interrelated by the common boundary conditions at the solid-fluid interface known as continuity conditions. For the case of a spherical shell and inviscid fluid, the pressure, \( P_a \), relates the equations of motion of the shell particles obtained from \( \Pi_s \) to the equations of motion of the fluid derived directly from the fluid energy rate, that is, the fluid power. The spherical shell is assumed to be homogeneous and non-porous solid.

(a) Derivation of Equations of Motion for the Spherical Shell

The energy equation of the spherical shell is defined as

\[ \Pi_s = K_s + Q_s \quad (3.2.2) \]

where the above notations are defined in the following manner:

Kinetic energy,

\[ K_s = \frac{1}{2} \rho_s \int \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right] dv_s \]

Potential energy,

\[ Q_s = U - V \quad (3.2.3) \]

Internal energy,

\[ U = \frac{1}{2} \int \left[ \sigma_{\phi \phi} + \sigma_{\theta \phi} + 2\tau_{\phi \phi} \gamma_{\phi \phi} + 2\tau_{\theta \phi} \gamma_{\phi \theta} + 2\tau_{n \phi} \gamma_{\phi n} + 2\tau_{n \theta} \gamma_{n n} \right] dv_s \]
Work done by external forces,

\[ V = \int \left( F_r w + F_\phi u + F_\theta v \right) dA_s \]

The external forces in Figure 13 are local forces defined with the aid of step-functions, H:

- \( F_\theta = 0 \)
- \( F_r = -P_a + R^*_l \)
- \( F_\phi = F^*_l - \gamma R^*_n = F^*_L \)
- \( R^*_l = R^*_n + R^*_n \)

If \( \theta^* = \pi/2 + \theta \), the generalized forces, \( F^*_L, F^*_n, \) and \( R^*_t \) can be defined as follows:

\[
F^*_L = F^*_L(t)[H(\phi + \phi_n) - H(\phi - \phi_n)] [H(\theta^* + \theta_n) - H(\theta^* - \theta_n)]
\]

\[
F^*_n = F^*_n(t)[H(\phi + \phi_n) - H(\phi - \phi_n)] [H(\theta^* + \theta_n) - H(\theta^* - \theta_n)]
\]

\[
R^*_t = \frac{\gamma R^*_n}{R^*_n}
\]

\[
R^*_n = R^*_n(t)[H(\phi + \phi_c) - H(\phi - \phi_c)] [H(\theta + \theta_c) - H(\theta - \theta_c)]
\]

\[ \gamma = \tan \gamma = \frac{R^*_t}{R^*_n} \]

where \((\phi_n, \theta_n)\) and \((\phi_c, \theta_c)\) are the angular coordinates bounding the spherical surface area of the neck-head junction and the contact area respectively.

From the free body of the spherical shell at impact, the following equations are obtained:

1. For no rotation at impact, the moment about the centroid of the sphere with contact area, \( S_c \) and neck area, \( S_n \), is:

\[
J_{25} \phi_25 + F_L a_s n - R^*_n a_s c = 0
\]  

(3.2.4a)
(2) The dynamic equilibrium along the x-axis is

\[ S_c R^* + F_a - m_0 U_{25} = 0. \]  \hspace{1cm} (3.2.4b)

where the equivalent force \( F_L \) is

\[ F_L = \frac{[F_s - F_m/a]}{S_n} \]

\( F_a, F_s, F_m \) are the axial force, transverse force and the bending moment at the neck-head junction which are discussed in Chapter Two. \( R^* \) is the shear stress due to frictional force and \( U_{25} \) is the rigidbody vertical velocity of the spherical shell just before impact. The dynamic frictional coefficient, \( \gamma \), must be chosen so that equations (3.2.4a) and (3.2.4b) are satisfied. The mass of the shell and the fluid is \( m_o \).

The Action Integral is obtained by taking the time integration of equation (3.2.2), thus

\[ \Pi_A = \frac{1}{t} \int_0^t \int_{V_s} \left\{ \left[ \rho S \left( \frac{\partial u_i}{\partial t} \right) \right]^2 + \sigma_{ij} \varepsilon_{ij} \right\} dV_s - \frac{1}{2} \int_{A_s} [F_i u_i] dA dt \] \hspace{1cm} (3.2.5)

where \( u_i \) is the orthogonal displacement components,

\[ u_i = (u, v, w, \beta_\phi, \beta_\theta), \quad i = 1, 2, 3, 4, 5 \] \hspace{1cm} (3.2.6)

and \( (\varepsilon_{ij}, \sigma_{ij}) \) are strain and stress tensors defined by equations (3.1.6) and (3.1.7). Repeated subscripts denote summation.

The variation operator \( \delta \) is applied on the Action Integral to obtain the admissible variations of \( (u, v, w, \beta_\phi, \beta_\theta) \) or virtual displacements \( \delta u, \delta v, \delta w, \delta \beta_\phi, \) and \( \delta \beta_\theta \). Thus

\[ \delta \Pi_A = \delta \int_0^t \Pi_s (u, v, w, \beta_\phi, \beta_\theta) dt \] \hspace{1cm} (3.2.7)

such that \( \delta \Pi_A = 0 \)

When this operation is carried out and simplified, the resulting equation from equation (3.2.7) is set to zero to obtain a 5x5 matrix.
equation of motion along with its boundary conditions.

\[ A_{kj}(u,v,w,\beta,\phi,\theta) = g_{kl} \]  
\[ B_{kj}(u,v,w,\beta,\phi,\theta) = 0 \]  

(3.2.8)  
(3.2.9)

where \( A_{kj} \) and \( B_{kj} \) are linear partial differential operators which are respectively defined by the 5x5 matrices, \( [a_{kj}] \) and \( [b_{kj}] \). The matrix, \( [g_{kl}] \) is generalized force components. Thus

\[ A_{kj} = [a_{kj}] \]
\[ B_{kj} = [b_{kj}] \]  
\[ g_{kl} = a^{2}[F_{\phi},F_{\theta},F_{r},0,0]^T \]  

All the matrix elements are defined in the supplementary definition section at the end of the chapter.

The above technique is the Hamiltonian variation principle, which has the beauty of converting a mechanic problem into a geometric problem that can be more readily solved.

(b) Derivation of Equation of Motion for the Fluid

(i) A hypothetic rigid body motion for a chunk of fluid with a uniform external pressure \( P \)

Let \( G(\dot{q}) \) be Gibb's function which is an analogy of Königs <188> that behaves like kinetic energy when used with rigid body motion, and \( \delta W \) be a virtual work, and \( n_i \) be unit normal components \( (n_r,n_\phi,n_\theta) \).

\[ G(\dot{q}) = \rho_f \dot{q}_i \dot{q}_i \]  

(3.2.11a) 

Virtual rate of work done by pressure, \( P \), is given as

\[ \delta W = -Pn_i \delta q_i \]  

(3.2.11b)
where $\delta q_1$ is virtual velocity. Differentiating (3.2.11a) with respect to $\dot{q}_1$ and dividing (3.2.11b) by $\delta q_1$ we obtain

$$\frac{\partial G}{\partial q_1} = \rho_f \dot{q}_1$$

$$\frac{\partial W}{\partial q_1} = -Pn_1$$

From equation of motion

$$\int \frac{\partial G}{\partial q_1} dv + \int \frac{\partial W}{\partial q_1} dA = 0$$

we obtain

$$\int \rho_f \dot{q}_1 ^i dv - \int P_1 n_1 dA = 0$$

or

$$\rho_f \dot{q}_1 ^i - P_1 = 0$$

(3.2.12)

For a spherical shell, the normal to the surface area is along the radial direction which the pressure acts since an inviscid fluid is chosen for the model. The normal unit vector components, $n_1$, can be replaced by $n$

$$\rho_f \dot{q}_n = P_n$$

For an irrotational fluid

$$\epsilon_{kk} = q_{k,k} = 0$$

(3.2.13)

Therefore $q_1$ can be represented by a potential function $\phi(r,\phi,\theta,t)$ such that

$$q_1 = -\phi, i$$

(3.2.14)

Now substitute (3.2.13) and (3.2.14) into (2.3.26) to obtain

$$-\rho_f \dot{\phi}, i = 0$$

or

$$\nabla_p = -\rho_f \nabla (d\phi/dt)$$

(3.2.15)
Since in this case \( \frac{d}{dt} = \frac{d}{\partial t} \) for the constant fluid density, \( \rho_f \),
equation (3.2.15) can be written after integrating and putting the
canstant of integration to zero

\[
P_a = -\rho_f \frac{\partial \phi}{\partial t} (a, \phi, \theta, t) \tag{3.2.16}
\]

(ii) Wave equation for an irrotational fluid

Differentiating \( \Pi_f \)

\[
\dot{\Pi}_f = G(\dot{q}) - \dot{U} + \dot{W}(r=a) \tag{3.2.17}
\]

where \( G(\dot{q}) = \frac{1}{2} \rho_f \dot{q}_i \dot{q}_i \)

\[
\dot{U} = \frac{1}{2} \sigma_{ij} \dot{e}_{ij}
\]

\[
\dot{W}(r=a) = \left[ \rho \dot{q}_i - \rho_j \delta_{ij} \right] \big|_{r=a}
\]

When \( q_i \) is orthogonal velocity vector components, the strain and the
stress rates are defined as follows:

\[
\dot{e}_{ij} = \frac{1}{2} (q_{i,j} + q_{j,i}) \tag{3.2.18}
\]

\[
\sigma_{ij} = (-P + \lambda \dot{e}_{kk}) \delta_{ij} + 2 \mu \dot{e}_{ij} \tag{3.2.19}
\]

where \( \mu \) and \( \lambda \) are first and second coefficient of viscosity.

Use equation (3.2.13) in equation (3.2.19) to obtain

\[
\sigma_{ij} = -P \delta_{ij} + 2 \mu \dot{e}_{ij} \tag{3.2.20}
\]

The equations (3.2.18) through (3.2.20) are now used in equation

(3.2.17) to obtain

\[
\dot{\Pi}_f = \frac{1}{2} \int_{V} \left[ \rho_f \dot{q}_i \dot{q}_i + P_a \delta_{ij} \dot{e}_{ij} - 2 \mu \dot{e}_{ij} \dot{e}_{ij} \right] dV \tag{3.2.21}
\]

Let \( \delta \), the variation operator, be applied to equation (3.2.21) to obtain

\[
\delta \dot{\Pi}_f = \int_{V} \left( \rho_f \dot{q}_i - \mu q_{i,j} \right) \delta q_i + \int_{S} \left( \rho_f \dot{q}_i - P \delta_{ij} n_j \right) \delta q_i ds \tag{3.2.22}
\]

since

\[
\delta \dot{\Pi}_f = 0
\]
for any admissible motion, then, equation (3.2.22) becomes
\[(\rho \frac{\partial q_i}{\partial x} - \mu q_{i,j} + \rho_f q - p_{ij} \delta_{ij}) \delta q_i = 0 \]  \hspace{5cm} (3.2.23)

Now using equation (3.2.12) in (3.2.23), obtain
\[\rho \frac{\partial q_i}{\partial x} - \mu q_{i,j} = 0 \]  \hspace{5cm} (3.2.24)

or
\[\frac{\partial^2 q_i}{\partial t^2} = \frac{C^2}{\rho} q_i \]  \hspace{5cm} (3.2.25)

where
\[C^2 = \frac{\mu}{\rho_f}\]
\[v^2 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r}) + \frac{1}{r} \csc \phi \frac{\partial}{\partial \phi} (\sin \phi \frac{\partial}{\partial \phi}) + \]
\[\frac{1}{r^2} \csc^2 \phi \frac{\partial^2}{\partial \phi^2} \]  \hspace{5cm} (3.2.27)

Using equation (3.2.14) in (3.2.25), we obtain
\[\frac{\partial^2 \phi}{\partial t^2} = C^2 v^2 \phi \]  \hspace{5cm} (3.2.28)
3.3. Transformation of Variables (14,176)

To uncouple the displacements variable of equation (3.2.8) and obtain a simplified system of five partial differential equations, we make the following changes:

\[ U^* = H\phi - \psi \sin \phi \]
\[ V^* = H,0 \csc \phi \]
\[ \beta_\phi = G,\phi - A\sin \phi \]
\[ \beta_\theta = G,\theta \csc \phi \]
\[ w^* = w^* \]

Now substitute (3.3.1) in (3.2.8) to obtain

\[ C_{k\ell}(H^*, G, W^*, \psi^*, \Lambda) = f_{k\ell} \]

the expansion of which is (3.3.3) through (3.3.7)

\[
C_{11}(H^*) + C_{12}(G) + C_{13}(W^*) + C_{14}(\psi^*) + C_{15}(\Lambda) + \\
\left(\frac{1 + \nu}{E_h}\right) a_F^\phi = 0
\]

(3.3.3)

\[
C_{21}(H^*) + C_{22}(G) + C_{23}(W^*) + C_{24}(\psi^*) + C_{25}(\Lambda) + \\
\left(\frac{1 - \nu}{E_h}\right) a_F^\theta = 0
\]

(3.3.4)

\[
C_{31}(H^*) + C_{32}(G) + C_{33}(W^*) + C_{34}(\psi^*) + C_{35}(\Lambda) + \\
\frac{2a(1 + \nu)}{E_h} F = 0
\]

(3.3.5)

\[
C_{41}(H^*) + C_{42}(G) + C_{43}(W^*) + C_{44}(\psi^*) + C_{45}(\Lambda) = 0
\]

(3.3.6)

\[
C_{51}(H^*) + C_{52}(G) + C_{53}(W^*) + C_{54}(\psi^*) + C_{55}(\Lambda) = 0
\]

(3.3.7)

where \( C_{k\ell} \) is a linear partial differential operator whose matrix elements \( [C_{k\ell}] \) are defined in Supplementary Definition Section.
After some algebraic manipulations and using

\[
V_1 = \frac{\partial^2}{\partial \phi^2} + \cot \phi \frac{\partial}{\partial \phi} + \csc \phi \frac{\partial^2}{\partial \theta^2}
\]

these five governing equations of spherical shell finally emerge as:

\[
\begin{align*}
&\{V_1^2 + 2(1 - \frac{1}{2k_s}) - \frac{2\rho_s a^2(1 - \nu^2)k_1}{E} \frac{\partial^2}{\partial t^2}\psi^* + \\
&\frac{1}{k_s} - \frac{2\rho_s a^2(1 - \nu^2)k_2}{E} \frac{\partial^2}{\partial t^2}\Lambda = \frac{2(1 + \nu)a}{Eh}(F_\phi - F_\theta)\\
&\frac{12a^2}{h^2k_s} - \frac{4\rho_s a^2(1 - \nu^2)}{E} \frac{\partial^2}{\partial t^2}\psi^* + \\
&\{V_1^2 + 2(1 - \frac{6a^2}{h^2k_s}) - \frac{2a^2(1 - \nu^2)}{E} k_r \frac{\partial^2}{\partial t^2}\Lambda = 0\\
&\{V_1^2 + (1 - \nu)(1 - \frac{1}{k_s}) - \frac{\rho_s a^2(1 - \nu^2)k_1}{E} \frac{\partial^2}{\partial t^2}\psi^* + \\
&\{a(1 - \frac{1 - \nu}{2k_s}) - \frac{\rho_s a^2(1 - \nu^2)k_2}{E} \frac{\partial^2}{\partial t^2}G + \quad (3.3.10)\\
&\{(1 + \nu + \frac{1 - \nu}{2k_s}) W^* + \frac{a(1 + \nu)}{Eh} F_\theta = \\
&\quad (\frac{1 + \nu}{2})\psi^*, \phi \sin \phi + 2\psi^* \cos \phi \alpha
\end{align*}
\]
\[
\left\{ \frac{6a(1 - \nu)}{h^2k_s} - \frac{2\rho_s a(1 - \nu^2)}{E} \frac{\partial^2}{\partial t^2}\right\}H^* + \]

\[
\left\{ \frac{\nu^2}{h^2k_s} + (1 - \nu)(1 - \frac{6a^2}{h^2k_s}) - \frac{\rho_s a^2(1 - \nu^2)}{E} \frac{kr}{\partial t^2}\right\}G - \]

\[
\left\{ \frac{6a(1 - \nu)}{hk_s} \right\}W^* = \left( \frac{1 + \nu}{2} \right) \lambda, \phi \sin \phi + 2\lambda \cos \phi \tag{3.3.11} \]

\[
\left\{ (1 + 2k_s)(\frac{1 + \nu}{1 - \nu})v^2 \right\}H^* - \left\{ a\nu^2 \right\}G - \]

\[
\left\{ \frac{\nu^2}{4k_s(1 - \nu)} - \frac{2\rho_s a^2(1 - \nu^2)k_s}{E} \frac{\partial^2}{\partial t^2}\right\}W^* - \]

\[
\frac{2a^2(1 + \nu)k_s}{Eh} F_r = (1 + 2k_s)(\frac{1 + \nu}{1 - \nu}) (\psi^*, \phi \sin \phi + 2\psi \cos \phi) a - (\lambda, \phi \sin \phi + 2\lambda \cos \phi) a \tag{3.3.12} \]

Boundary conditions are obtained from equation (3.2.9) in addition to
\[
\frac{\partial}{\partial t} W^*(\phi, \theta, t) = \frac{\partial}{\partial r^*} \phi^*(r^*, \Theta, \phi, t) \tag{3.3.13} \]

Non-Dimensionalized Equations

In addition to equation (3.1.5), we have
\[
\tau = \frac{C_s t}{a} \]

\[
C_s = \left[ \frac{E}{\rho_s (1 - \nu^2)} \right]^{0.5} \tag{3.3.14} \]

\[
\phi(r, \phi, \Theta, \tau) = \phi^*(r^*, \phi, \Theta, t)(a/c_s)^2 \]

\[
r = r^*/a \]

Following the method of Vander Naut (1932), which is widely used and improved upon by Prasad (1964) and Shah, et al. (1969), which has
already been adopted in equation (3.3.1), we have the non-dimensionalized form of (3.3.1):

\[
\begin{align*}
&u = \frac{3H}{\phi} - \psi \sin \phi \\
&v = \frac{3H}{\theta} \csc \phi \\
&\beta_\phi = \frac{3G}{\phi} - \Lambda \sin \phi \\
&\beta_\theta = \frac{3G}{\theta} \csc \phi \\
w &= w
\end{align*}
\]

(3.3.15)

Laplace transform is used in equation (3.3.15) for a transition from time domain to the \(p\)-domain, that is

\[(\phi, \theta, \tau) \rightarrow (\phi, \theta, p)\]

Thus, we have

\[
L\{u, v, w, \beta_\phi, \beta_\theta\} = \int_0^\infty e^{-pt} \{u, v, w, \beta_\phi, \beta_\theta\} dt
\]

\[
= \{\bar{u}, \bar{v}, \bar{w}, \bar{\beta}_\phi, \bar{\beta}_\theta\}
\]

and similarly

\[
L\{H, \psi, G, \Lambda\} = \{\bar{H}, \bar{\psi}, \bar{G}, \bar{\Lambda}\}
\]

The transformed governing equation of shell (3.3.8) through (3.3.13) are

\[
D_{k\ell} (\bar{H}, \bar{G}, \bar{W}, \bar{\psi}, \bar{\Lambda}) = f_{k\ell}
\]

(3.3.17)

In the Supplementary Definition Section [\(D_{k\ell}\)] is the 5x5 matrix of the linear partial differential operator \(D_{k\ell}\) and the expanded form gives

\[
\begin{align*}
&d_{14} \bar{\psi} + d_{15} \bar{\Lambda} = \frac{2(1 + \nu)}{Eh} (\bar{\beta}_\phi - \bar{\beta}_\theta) \csc \phi \\
&d_{24} \bar{\psi} + d_{25} \bar{\Lambda} = 0 \\
&d_{31} \bar{H} + d_{32} \bar{G} + d_{33} \bar{W} + d_{34} \bar{\psi} = 0 \\
&d_{41} \bar{H} + d_{42} \bar{G} + d_{43} \bar{W} + d_{45} \bar{\Lambda} = 0
\end{align*}
\]

(3.3.18) - (3.3.21)
\[ d_{51} \bar{H} + f_{52} \bar{G} + d_{53} \bar{W} + d_{54} \bar{V} + d_{55} \bar{I} = \]
\[ - \frac{2}{(1 - \nu)} f_k \bar{p} \Phi(1, \phi, 0, \rho) + \frac{2(1 + \nu) a}{E_h} \bar{R} \]
\[ p \bar{W} = \frac{\partial}{\partial r} \Phi(1, \phi, \theta, \rho) \quad (3.3.22b) \]

**Initial Conditions**

1. \( W(\phi, \theta, 0) = 0 \)
2. \( \frac{\partial W}{\partial \tau}(\phi, \theta, 0) = 0 \)
3. \( u(\phi, \theta, 0) = 0 \)
4. \( \frac{\partial u}{\partial \tau} = 0 \)
5. \( \phi(r, \phi, \theta, 0) = f(r, \phi, \theta) \)
6. \( \frac{\partial \Phi}{\partial \tau}(r, \phi, \theta) = P_r |_{r=a} \) \quad (3.3.22c)
7. \( V(\phi, \theta, 0) = 0 \)
8. \( \frac{\partial V}{\partial \tau}(\phi, \theta, 0) = 0 \)
9. \( \beta_\phi(\phi, \theta, 0) = 0 \)
10. \( \beta_\theta(\phi, \theta, 0) = 0 \)
11. \( \frac{\partial \beta_\phi}{\partial \tau}(\phi, \theta, 0) = 0 \)
12. \( \frac{\partial \beta_\theta}{\partial \tau}(\phi, \theta, 0) = 0 \)

And from (3) and (7) we can infer that \( \frac{\partial H}{\partial \phi}(\phi, \theta, 0) - \psi(\phi, \theta, 0) \csc \phi = 0 \) and \( \frac{\partial}{\partial \theta} H(\phi, \theta, 0) = 0 \), imply that for any \( \phi \) and \( \theta \)

\[ H(\phi, \theta, 0) = \psi(\phi, \theta, 0) = 0 \quad (3.3.22d) \]

Similarly,

\[ G(\phi, \theta, 0) = \Lambda(\phi, \theta, 0) = 0 \]
Recurrence Relation (91)

The following recurrence relations are useful tools in solving the equations. Let $f^m_n(\cos\phi)$ be spherical associated Legendre polynomial function

$$
cos\phi f^m_n(\cos\phi) = \frac{(n - m + 1)}{2n + 1} f^m_{n+1}(\cos\phi) + \frac{(m + n)}{2n + 1} f^m_{n-1}(\cos\phi)
$$

$$
sin\phi \frac{d}{d\phi} f^m_n(\cos\phi) = n \frac{(n - m + 1)}{2n + 1} p^m_{n+1}(\cos\phi) - \frac{(n + 1)(n + m)}{2n + 1} p^m_{n-1}(\cos\phi)
$$

If we let $n \rightarrow n - 1$ in $f^m_{n+1}$ and $n \rightarrow n + 1$ in $f^m_{n-1}$, we obtain $f^m_n$ so that

$$
A_{mn} \cos\phi f^m_n(\cos\phi) = A_{mn+1} \frac{n - m}{2n - 1} f^m_n(\cos\phi) + \frac{(n + m + 1)}{2n + 3} A_{n-1} f^m_n(\cos\phi)
$$

$$
A_{mn} \sin\phi \frac{d}{d\phi} f^m_n(\cos\phi) = A_{mn+1} \frac{-n - m}{2n - 1} (n - 1) f^m_n(\cos\phi) - \frac{(n + 2)(n + m + 1)}{2n + 3} f^m_n(\cos\phi)
$$

$$
2A_{mn} \cos\phi f^m_n(\cos\phi) + A_{mn} \sin\phi \frac{d}{d\phi} f^m_n(\cos\phi) =
$$

$$
-A_{mn-1} \frac{(n + m + 1)n}{2n + 3} f^m_n(\cos\phi) + A_{mn+1} \frac{(n + 1)(n - m)}{2n - 1} f^m_n(\cos\phi)
$$

$$
\gamma^2 \cos m\phi f^m_n(\cos\phi) = -\gamma^2 \cos m\phi p^m_n(\cos\phi)
$$

$$
\gamma^4 \cos m\phi f^m_n(\cos\phi) = \lambda^2 \cos m\phi p^m_n(\cos\phi)
$$
We now assume a solution to the transformed equation of the shell in (3.3.17):

\[
H(\phi, \theta, p) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} H_{nm}(p) Y_{n}^{m}(\phi, \theta)
\]

\[
\bar{G}(\phi, \theta, p) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \bar{G}_{nm}(p) Y_{n}^{m}(\phi, \theta)
\]

\[
\bar{W}(\phi, \theta, p) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \bar{W}_{nm}(p) Y_{n}^{m}(\phi, \theta)
\]

\[
\bar{U}(\phi, \theta, p) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \bar{U}_{nm}(p) Y_{n}^{m}(\phi, \theta)
\]

(3.3.27)

where \( Y_{n}^{m}(\phi, \theta) \) is the tesseral harmonic and it obeys the same recurrence relations as the associated Legendre polynomial function, \( P_{n}^{m}(\cos \phi) \), in equations (3.3.23) through (3.3.25).

\[
Y_{n}^{m}(\phi, \theta) = P_{n}^{m}(\cos \phi) \cos m \theta
\]

(3.3.28)

It is further assumed that all the acting forces can be expanded in the form of spherical Legendre polynomial functions

\[
\bar{R}(p) = \bar{R}_{n1}^*(p) + \bar{F}_{n1}^*(p)
\]

\[
= \sum_{n=1}^{\infty} \sum_{m=0}^{n} \bar{R}_{nm}(p) Y_{n}^{m}(\phi, \theta)
\]

and

\[
\bar{F}_{\phi}(p) = \bar{F}_{L1}^*(p) - \gamma \bar{R}_{n1}^*(p)
\]

\[
= \sum_{n=1}^{\infty} \sum_{m=0}^{n} \bar{F}_{nm}(p) Y_{n}^{m}(\phi, \theta)
\]

(3.3.29)
where

\[ \mathcal{R}_{mn}(p) = r \mathcal{R}^*(p) \]

\[ \mathcal{F}_{mn}(p) = r \mathcal{F}^*(p) \]

\[ \mathcal{R}^*(p) = \{ \mathcal{F}_N(p), \mathcal{F}_n(p) \} \]

\[ \mathcal{F}^*(p) = \{ \mathcal{F}_L(p), \mathcal{F}_n(p) \} \]

\[ \tilde{\phi} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \tilde{\omega}_{mn}(p) \frac{j_n \left( \frac{i pr}{s} \right)}{j_n'(\frac{i pr}{s})} Y_n^m(\phi, \theta) \]

where \( j_n \left( \frac{i pr}{s} \right) \) is a spherical bessel function and is related to the cylindrical bessel function as follows:

\[ j_n(z) = \sqrt{\frac{2}{\pi z}} J_{n+\frac{1}{2}}(z) \]

\[ j_n'(z) = \frac{d}{dz} j_n(z) \]  

(3.3.30)

Substitute (3.3.27) and (3.3.29) in (3.3.18) through (3.3.22a), multiply the results by \( \mathcal{P}_L^m(\cos \phi) \sin \theta \) and integrate from 0 to \( \pi \) and note the equation (3.3.28) to get:

\[ [bk_1 p^2 + \lambda_n - \alpha_s] \bar{\omega}_{mn} + [bk_2 p^2 - e_s] \bar{\Lambda}_{mn} = \beta \bar{F}_{mn} \]  

(3.3.31a)

\[ [2b p^2 - e_s] \bar{\omega}_{mn} + [bk_r p^2 + \alpha \lambda_n - \gamma_s] \bar{\Lambda}_{mn} = 0 \]  

(3.3.31b)

\[ [k_1 p^2 + \lambda_n + a_s] \bar{H}_{mn} + [k_2 p^2 - g_s] \bar{C}_{mn} - \]

\[ [b_s] \bar{\omega}_{mn} - [\gamma_{mn}] \bar{\omega}_{mn+1} + [\lambda_{mn}] \bar{\omega}_{mn-1} = 0 \]  

(3.3.31c)

\[ [2 p^2 - g_s] \bar{H}_{mn} + [ak_r p^2 + \lambda_n + \theta_s] \bar{C}_{mn} + \]

\[ [a s] \bar{\omega}_{mn} - [\alpha_{mn}] \bar{\omega}_{mn+1} + [\lambda_{mn}] \bar{\omega}_{mn-1} = 0 \]  

(3.3.31d)
When we solve the first two equations of (3.3.31), we obtain group I equations:

\[ \bar{v}_{mn} = -A(p)\bar{\lambda}_{mn} \]  

(3.3.32)

\[ \bar{\lambda}_{mn} = -B(p)\bar{v}_{mn} \]  

(3.3.33)

Now using (3.3.32) in the remaining three equations of (3.3.31) and solving them simultaneously, we obtain group II equations:

\[
\begin{bmatrix}
    h_{n1} & g_{n1} & w_{n1} \\
    h_{n2} & g_{n2} & w_{n2} \\
    h_{n3} & g_{n3} & w_{n3}
\end{bmatrix}
\left(\bar{H}_{mn}, \bar{C}_{mn}, \bar{W}_{mn}\right)^T = \left(P_{n1}, P_{n2}, P_{n3}\right)^T
\]

(3.3.34)

where the above 3x3 matrix is defined in Supplementary Definition section.

Solve equation (3.3.33) by Cramer's rule and obtain:

\[ \bar{H}_{mn} = \Delta_h(p)/\Delta_{n2}(p) \]  

\[ \bar{C}_{mn} = \Delta_g(p)/\Delta_{n2}(p) \]  

\[ \bar{W}_{mn} = \Delta_w(p)/\Delta_{n2}(p) \]

(3.3.35)

where

\[ \Delta_{n2}(p) = h_{n1}(g_{n2}w_{n3} - w_{n2}g_{n3}) - g_{n1}(h_{n2}w_{n2} - w_{n2}h_{n3}) + w_{n1}(h_{n2}g_{n3} - g_{n2}h_{n3}) \]

\[ \Delta_h(p) = P_{n1}(g_{n2}w_{n3} - w_{n2}g_{n3}) - g_{n1}(P_{n2}w_{n3} - w_{n2}P_{n3}) + w_{n1}(P_{n2}g_{n3} - g_{n2}P_{n3}) \]
\[ \Delta_w(p) = h_{n1}(g_{n2}p_{n3} - g_{n3}p_{n2}) - g_{n1}(h_{n2}p_{n3} - h_{n3}p_{n2}) + \]
\[ p_{n1}(h_{n2}g_{n2} - h_{n3}g_{n2}) \]  
\[ (3.3.35) \]
\[ \Delta_g(p) = h_{n1}(p_{n2}w_{n3} - w_{n2}p_{n3}) - p_{n1}(h_{n2}w_{n3} - w_{n2}h_{n3}) + \]
\[ w_{n1}(h_{n2}p_{n3} - p_{n2}h_{n3}) \]

Following the development of (3.3.27), \( \overline{F}_{mn} \) and \( \overline{R}_{mn} \) can be written in anticipation of final results as
\[ \overline{F}_{mn}(p) = r_{mn}\overline{F}^*(p) \]
\[ \overline{R}_{mn}(p) = r_{mn}\overline{R}^*(p) \]  
\[ (3.3.36) \]

We are now ready to take the Laplace inverse of the solution to get us back to time domain. Due to the physical nature of the problem, \( \bar{H}, \bar{G}, \bar{W}, \bar{\psi} \) and \( \bar{\Lambda} \) satisfy the following conditions naturally [17]:

(a) All the variables are defined for \( t \geq 0 \) and they are all of \( O(e^t) \) and their time derivatives are sectionally continuous.

(b) The integral

\[ \int_{c-iR}^{c+iR} \frac{f(p)}{h(p)} e^{pt} dp \]

has all its poles along the imaginary axis.

Then

\[ \int_{c-iR}^{c+iR} e^{pt} \frac{f(p)}{h(p)} dp = \int_{\Gamma_1} e^{pt} \, dp + \int_{\Gamma_2} e^{pt} \, dp + \int_{\Gamma_3} e^{pt} \, dp \]

\[ (3.3.37) \]

But

\[ \lim_{R \to \infty} \int_{\Gamma_1} (...) \, dp + \int_{\Gamma_2} (...) \, dp + \int_{\Gamma_3} (...) \, dp = 0 \]
Figure 14. Path of Inverse Laplace Integration.

and hence we have

\[ \int_{c-iR}^{c+iR} \exp(pt) \frac{f(p)}{h(p)} \, dp = \lim_{R \to \infty} \int_L \exp(pt) \frac{f(p)}{h(p)} \, dp \]

The inverse

\[ L^{-1} \frac{\exp(pt)f(p)}{h(p)} = \sum_{p_n \in \mathbb{R}} \text{Res} \left[ \frac{\exp(pt)f(p)}{h(p)} \right] \]

where

\[ h'(p) = \frac{dh(p)}{dp} \]

and \( p_n \) is the roots of the equation \( h(p) = 0 \). The above is now used to evaluate the convolution integrals in the following manner:

Let \( \text{Con} \{f(p)/h(p), g(p)\} \) be convolution of \( f(p)/h(p) \) and \( g(p) \), such that

\[ \text{Con} \{f(p)/h(p), g(p)\} = \frac{f(p)}{h(p)} * g(p) \]

\[ \text{Con} \{f(p)/h(p), g(p)\} = \text{Con} \cdot \text{Con} \{f(p)/h(p)\} * \text{Con} \{g(p)\} \]

\[ L^{-1}\{\text{Con} \{f(p)/h(p), g(p)\}\} = \text{Con} \{L^{-1}\{f(p)/h(p)\}, L^{-1}\{g(p)\}\} \]
We can write this result in a more useful form in preparation for the
rest of the chapters. The Laplace inverse of the final result is

\[
L^{-1}\{\text{Con } v[f(p)/h(p), g(p)]\} = \sum_{\ell=1}^{\infty} \text{Res } \left[ f(p)/h(p) \right] \int_{0}^{\tau} g(\tau - \xi) \sin \omega_{n\ell} \xi \, d\xi
\]

Equation (3.3.39) or its equivalent equation immediately above is used
here throughout whenever a transient motion is needed.

**Frequency for Group I** \( n > 1 \)

The frequency spectra obtained from this group are torsional
modes. The spectral diagram is shown in Figure 15.

\[
\Lambda_{n1} = \frac{2a}{(1 - \nu)^2} (k_1 k_r - 2k_2)p^4 - \left( \frac{2}{1 - \nu} \right) \{\alpha(k_1 + k_r)(\lambda_n - 2) + \frac{1}{k_s}(k_1 + k_2) + \frac{2a}{k_s}(k_r + 1)\}p^2 + (\lambda_n + 2)[\alpha(\lambda_n - 2) + \frac{1}{k_s}(1 + 2a)] + \frac{1}{k_r^2}
\]

The derivative of frequency equation (3.3.40) is
\[ \Delta'_{n1} = \frac{8a}{(1 - \nu)^2 (k_1 k_r - 2k_2)p^3 - \left( \frac{4}{1 - \nu} \right) (\alpha (k_1 + k_r) (\lambda_n - 2) + \frac{1}{k_s} (k_1 + k_2) + \frac{2a}{k_s} (k_r + 1)) p \] \\

(3.3.41)

**Frequency for Group II \( n \geq 1 \)**

\[ \Delta_{n2}(p) = \left[ k_1 p^2 + \lambda_n \alpha + (1 - \nu)(1 - 2k_s \alpha)/2k_s \right] \left[ k_s/(1 - \nu) \right] k_s + \]

\[ f_j_n(ip/s)/(ip/s)j''(p)] p^2 + \lambda_n + 4k_s (1 + \nu)/(1 - \nu) - \]

\[ (1 - \nu) \lambda_n/2k_s \right] - \left[ k_2 p^2 - (1 - \nu)/2k_s \right] \left( 2a p^2 - \right. \]

\[ (1 - \nu)/2k_s \left] \left[ k_s + f_j_n(ip/s)/(ip/s)j''(p) \right]/(1 - \nu) + \lambda_n + 4k_s (1 + \nu)/(1 - \nu) \right] + \lambda_n \left[ \left( 1 + \nu \right) + (1 - \nu)/2k_s \right] \lambda_n + \]

\[ \left[ \left( 1 + \nu \right) + (1 - \nu)/2k_s \right] \left[ 2a p^2 - (1 - \nu)/2k_s \right] \lambda_n + \]

\[ \left[ \left( 1 + \nu \right) + (1 - \nu)/2k_s \right] \lambda_n \left[ k_s + 2k_r (1 + \nu)/(1 - \nu) \right] \]

(3.3.42)

The derivative of the frequency equation (3.3.42) is \( \Delta'_{n2} \).

\[ \Delta'_{n2} = 2k_1 p \left[ \left[ a k_r p^2 + \lambda_n \alpha + (1 - \nu)(1 - 2k_s \alpha)/2k_s \right] \left[ 2k_s p^2 + k_s + \right. \right. \]

\[ f_j_n(ip/s)/(ip/s)j''(p)] + \lambda_n + 4k_s (1 + \nu)/(1 - \nu) - \]

\[ (1 - \nu) \lambda_n/2k_s \right] - 2k_2 p \left( 2a p^2 - (1 - \nu)/2k_s \right) \left[ 2k_s p^2 + k_s + \right. \]

\[ f_j_n(ip/s)/(ip/s)j''(p)]/(1 - \nu) + \lambda_n + 4k_s (1 + \nu)/(1 - \nu) - \]

\[ \lambda_n \left[ \left( 1 + \nu \right) + (1 - \nu)/2k_s \right] + \left( 1 + \nu \right) + (1 - \nu)/2k_s \right] \left[ 4 \lambda_n \alpha P + \right. \]

\[ 2a k_r p \lambda_n \left[ 1 + 2k_s (1 + \nu)/(1 - \nu) \right] + \left[ k_1 p^2 + \right. \]

\[ \lambda_n \right] \left( 1 - \nu \right) (1 - 2k_s)/2k_s \left[ 2a k_r p^2 + k_s + f_j_n(ip/s)/(ip/s) \cdot \right. \]

\[ j''(p)]/(1 - \nu) + \lambda_n + 4k_s (1 + \nu)/(1 - \nu) \right] + \left[ a k_r p^2 + \lambda_n \alpha + \right. \]

\[ \right) \left( 1 - \nu \right) \left[ 2a k_r p^2 \right. \]

(3.3.43)
The frequency spectra plots in Figures 15, 16 and 17 are the roots of the frequency equations (3.3.40) and (3.3.42). The graph in Figure 16 is particularly obtained by neglecting the fluid contribution to equation (3.3.42), that is, the spherical Bessel function terms. Figure 17 shows some of the infinite frequency spectra for a fluid filled spherical shell with a distinct rigid body mode, \( \Omega = 0 \).
Figure 17. Non-torsional frequency spectra for a fluid-filled spherical shell

The nondimensional frequency, $\Omega$ is the ratio of the linear speed, $\omega$ and the wave speed in the spherical shell, $C_g$. 
\[(1 - \nu)(1 - 2k_s^2)k_s^2 \frac{p^2 d}{dp} B(ip/s) + 2pB(ip/s) - \]
\[\{k_2^2 p^2 - (1 - \nu)/2k_s\} \{(2\alpha p^2 - (1 - \nu)/2k_s\} \frac{p^2 d}{dp} B(ip/s) + \]
\[2pB(ip/s)\} + \frac{4\alpha p^2 B(ip/s) + \lambda_n + 4k_s(1 + \nu)/(1 - \nu)]\}

where \( j_n'(p) = j_n(ip/s) \) in (3.3.42-3) (3.3.43)

In anticipation of our final results, we define the following expressions:

\[D_n(p) = \Delta_{n2}(p)\Delta'_{n1}(p) + \Delta_{n1}(p)\Delta'_{n2}(p)\]
\[D_n(s_{n\xi}) = \Delta_{n2}(s_{n\xi})\Delta'_{n1}(s_{n\xi}) + \Delta_{n1}(s_{n\xi})\Delta'_{n2}(s_{n\xi})\]
\[d_{n\xi}(s_{n\xi}) = \Delta'_{n1}(s_{n\xi})\]

\[= 8\alpha(k_1^2 - 2k_2)(s_{n\xi})^3/(1 - \nu)^2 - \]
\[4\alpha(k_1 + k_2)(\lambda_n - 2) + \frac{1}{k_s}(k_1 + k_2) + \]
\[\frac{2\alpha}{k_s}(k_1(r + 1))(s_{n\xi})/(1 - \nu)\] (3.3.44)

\[F_L(\tau) = \{F_s - F_m/a\}/S_n\] (3.3.45)

Equation (3.3.35) can be written in a more useful form:

\[\Delta_h(p) = \{[\Delta_{11}(p) - \gamma\Delta_{12}(p)], \Delta_{11}(p), \Delta_{12}(p)\}\]
\[\{\bar{F}_n(p), \bar{F}_L(p)\}\] (3.3.46)

\[\Delta_g(p) = \{[\Delta_{21}(p) - \gamma\Delta_{22}(p)], \Delta_{21}(p), \Delta_{22}(p)\}\]
\[\{\bar{F}_n(p), \bar{F}_L(p)\}\] (3.3.47)

\[\Delta_w(p) = \{[\Delta_{31}(p) - \gamma\Delta_{32}(p)], \Delta_{31}(p), \Delta_{32}(p)\}\]
\[\{\bar{F}_n(p), \bar{F}_L(p)\}\] (3.3.48)
\[ N_h(s \Omega \ell) = \{ [\Delta_{11}(p) - \gamma \Delta_{12}(p)], \Delta_{11}(p), \Delta_{12}(p) \}_{p = -i s \Omega \ell} \] (3.3.49)

\[ N_g(s \Omega \ell) = \{ [\Delta_{21}(p) - \gamma \Delta_{22}(p)], \Delta_{21}(p), \Delta_{22}(p) \}_{p = -i s \Omega \ell} \] (3.3.50)

\[ N_w(s \Omega \ell) = \{ [\Delta_{31}(p) - \gamma \Delta_{32}(p)], \Delta_{31}(p), \Delta_{32}(p) \}_{p = -i s \Omega \ell} \] (3.3.51)

\[ N_A(s \Omega \ell) = \beta_v (2b s^2 n^2 - e_s) / d_{n_j}(s \Omega n_j) \{-\gamma, 1\} \] (3.3.52)

\[ N_Y(s \Omega \ell) = \beta_v (b a_k s^2 n^2 + \alpha \lambda_n + x_s) / d_{n_j}(s \Omega n_j) \{-\gamma, 1\} \] (3.3.53)

where

\[ \Delta_{11}(p) = -\{ b_s (a k_r p^2 + \alpha \lambda_n + \theta_s) + g_s (k_2 p^2 - g_s) \} \{ m_n \Delta_{11}(p) \} \] (3.3.54)

\[ \Delta_{12}(p) = b_{m n} \beta_v (b a_k s p^2 + \alpha \lambda_n + x_s) \{ (a k_r p^2 + \alpha \lambda_n + \theta_s) (B(i p / s) p^2 + \lambda_n^2 + y_s) - g_s \lambda_n \} - (k_2 p^2 - g_s) \{ a \beta_v (2 b p^2 - e_s) [B(i p / s) p^2 + \alpha \lambda_n + \theta_s] - g_s a_{m n} \beta_v \} \] (3.3.55)

\[ \Delta_{21}(p) = b_s (2 a p^2 - g_s) - g_s (k_1 p^2 + \lambda_n + a_s) \{ m_n \Delta_{11}(p) \} \] (3.3.56)

\[ \Delta_{31}(p) = \{(k_1 p^2 + \lambda_n + a_s)(a k_r p^2 + \lambda_n + \theta_s) - (k_2 p^2 - g_s)(2 a p^2 - g_s) \} \{ m_n \Delta_{11}(p) \} \] (3.3.57)
\[ \Delta_{22}(p) = \left[ b_{mn} \alpha \beta \nu (2bp^2 - e_s) \left[ B(ip/s)p^2 + \lambda_n + y_s \right] - \right. \]
\[ g_s a_{mn} \nu \left[ f_s (b_s p^2 + \alpha \lambda_n + X_s) - 2bp^2 + e_s \right] (k_1 p^2 + \lambda_n + a_s) - \]
\[ b_{mn} \nu (b_s p^2 + \alpha \lambda_n + X_s) [B(ip/s)p^2 + \lambda_n + y_s] (2ap^2 - g_s) + \]
\[ g_\nu \left[ f_s \lambda_n + b_{mn} \nu \left( 2ap^2 - g_s \right) \left[ f_s (b_s p^2 + \alpha \lambda_n + X_s) - \right. \]
\[ \left. (2bp^2 - e_s) \right] + \alpha \beta b_{mn} f_\nu \lambda_n (2bp^2 - e_s) \right] \quad (3.3.58) \]

\[ \Delta_{32}(p) = \left[ a_{mn} f_s (b_s p^2 + \alpha \lambda_n X_s) \beta_\nu - \beta_\nu (2bp^2 - e_s) \right] \left[ (k_1 p^2 + \right. \]
\[ \lambda_n + a_s) \left( a_r p^2 + \alpha \lambda_n + \theta_s \right) - \left( k_2 p^2 - g_s \right) (2ap^2 - g_s) \right] - \]
\[ (k_2 p^2 - g_s) f_s \lambda_n \beta_\nu \left( ab_p^2 - e_s \right) + \]
\[ b_{mn} \nu (2ap^2 - g_s) (b_s p^2 + \alpha \lambda_n + X_s) (a_r p^2 + \alpha \lambda_n + \theta_s) \]
\[ \right] \quad (3.3.59) \]

For \( n = 0 \),

\[ \Delta_{02} \text{ is the frequency transcendental equation} \]
\[ \Delta_{02}(p) = k_1 + f \left( \frac{j_0(ip/s)}{(ip/s)j_0(ip/s)} p^2 + 2(1 + v) = 0 \quad (3.3.60) \right. \]

and the derivative of \( \Delta_{02}(p) \) at \( p = -s_0 \Omega_\ell \) is

\[ d_{02}(s_0 \Omega_\ell) = b k_s \left( \left( s f - 2k_s \right) \Omega_0 \ell - 2f \frac{j_0(\Omega_0 \ell)}{j_0'(\Omega_0 \ell)} + \Omega_0 \ell \frac{j_0'(\Omega_0 \ell)}{j_0'(\Omega_0 \ell)} \right)^2 \]
\[ \left. \right) \quad (3.3.61) \]

From (3.3.27) for \( n = m = 0 \)

\[ W(\phi, \theta, \tau) = -\frac{2k_s (1 - v^2) a}{E h_s} \sum_{\ell = 1}^{\infty} \frac{j_0^2(\Omega_0 \ell)}{j_0'(\Omega_0 \ell)} \int_{0}^{\tau} \int_{0}^{r_0} \int_{0}^{1} B^*(\xi) \sin s_0 \Omega_\ell (\tau - \xi) \sin \xi \left( \tau - \xi \right) d\xi \]
\[ \frac{d_{02}(s_0 \Omega_\ell)}{d_{02}(s_0 \Omega_\ell)} \quad (3.3.62) \]
Since

\[ \hat{\phi}_{mn} = \hat{W}_{mn} \frac{j_n(ipr)}{j'_{n}(\frac{ip}{s})} \]

\[ \phi_{mn}(s\Omega_{n\ell}) = \hat{W}_{mn}(s\Omega_{n\ell}) \frac{j_n(\Omega_{n\ell}^r)}{j'_{n}(\Omega_{n\ell}^r)} \]  

(3.3.63)

It should be noted that in finding the roots of the frequency equation of the second kind, \( j_n'(\Omega) \) or \( j_n'(ip/s) \) cannot attain the value of zero. Consequently, we must look at:

\[ j_n\left(\frac{ipr}{s}\right) / [j_n\left(\frac{ip}{s}\right) ip] \]

as a multiplicative factor that relates the fluid potential equation \( \Phi(r,\phi,\theta,\tau) \) to the spherical shell normal response, \( w(\phi,\theta,\tau) \), that is

\[ \Phi(r,\phi,\theta,\tau) = \sum_{\ell=1}^{\infty} \frac{j_n(\Omega_{n\ell}^r)}{j'_n(\Omega_{n\ell}^r)\Omega_{n\ell}^r} w(\phi,\theta,\tau) \]  

(3.3.64)

Therefore, \( \phi \) is completely defined, thus

\[ \Phi(r,\phi,\theta,\tau) = \sum_{\ell=1}^{\infty} \frac{2k_s(1+\nu)}{Eh} ar_{00} \frac{j_0(\Omega_{0\ell}^r)j'_0(\Omega_{0\ell}^r)}{d_0(\Omega_{0\ell}^r)} \int_0^\tau \frac{3}{\delta \tau} R_1(\xi) \cdot \]

\[ \sin s\Omega_{0}^r(\tau - \xi) d\xi + \sum_{n=1}^{\infty} \sum_{m=0}^{n} Y_n^m(\phi,\theta) \frac{j_n(\Omega_{n\ell}^r)}{\Omega_{n\ell}^r j'_n(\Omega_{n\ell}^r)} \frac{N_w(s\Omega_{n\ell}^r)}{D_n(s\Omega_{n\ell}^r)} \]

\[ \int_0^\tau \frac{3}{\delta \tau} \left\{ \begin{array}{c} R_n(\xi) \\ F_N(\xi) \end{array} \right\} \sin s\Omega_{n\ell}^r(\tau - \xi) d\xi \]  

(3.3.65)

From equation (3.3.22c)
\[ P_a = \sum_{\ell=1}^{\infty} \frac{2k_s(1 + v)a}{\text{Eh}} J_{mn} \left( \frac{J_0(\Omega_0 \ell r)}{\ell_0 \ell_0 d_0(\Omega_0 \ell r)} \right) J_0(\Omega_0 \ell r) \int_0^\tau \frac{\partial^2}{\partial \tau^2} R(\xi). \]

\[ \sin s_{\Omega_n \ell}(\tau - \xi) d\xi + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} Y_m^m(\phi, \theta) \frac{J_n(\Omega_{n \ell r})}{\ell_n \ell_n' \ell_n} \frac{N_w(\Omega_{n \ell})}{D_n(\Omega_{n \ell})} R_n(\xi). \]

\[ \int_0^\tau \frac{\partial^2}{\partial \tau^2} \left[ \begin{array}{c}
R_n(\xi) \\
F_N(\xi) \\
F_L(\xi)
\end{array} \right] \sin s_{\Omega_{n \ell}}(\tau - \xi) d\xi \]

(3.3.66)

Equation of the steady state motion is obtained as follows:

For \( p = \pm is_{\Omega_{n \ell}} \)

Using equation (3.3.39), we have

\[ \int_0^\tau \frac{\partial^2}{\partial \tau^2} \left[ \begin{array}{c}
R_n(\xi) \\
F_N(\xi) \\
F_L(\xi)
\end{array} \right] \sin s_{\Omega_{n \ell}}(\tau - \xi) d\xi \]

(3.3.67)

Using equations (3.3.67) we obtain the inverse of equation (3.3.33) and the results of this process are as follow:

\[ H(\phi, \theta, \tau) = \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} Y_m^m(\phi, \theta) \frac{N_h(\Omega_{n \ell})}{D_n(\Omega_{n \ell})} \int_0^\tau \frac{R_n(\xi)}{F_N(\xi)} \sin s_{\Omega_{n \ell}}(\tau - \xi) d\xi \]

\[ G(\phi, \theta, \tau) = \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} Y_m^m(\phi, \theta) \frac{N_g(\Omega_{n \ell})}{D_n(\Omega_{n \ell})} \int_0^\tau \frac{R_n(\xi)}{F_N(\xi)} \sin s_{\Omega_{n \ell}}(\tau - \xi) d\xi \]
\[ W(\phi, \theta, \tau) = \sum_{\xi=1}^{\infty} - \frac{k_s(1 + \nu) \alpha_m}{\pi l \Omega_0 (s \Omega_0 \xi)} \int_{0}^{1} \left[ \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{y_n^m(\phi, \theta)}{D_n(s \Omega_0 \xi)} \int_{0}^{\tau} \left\{ \frac{R_n(\xi)}{F_L(\xi)} \right\} \sin s \Omega_0 \xi (\tau - \xi) \, d\xi \right] \sin s \xi \nu \xi (\tau - \xi) \, d\xi \]

\[ A(\phi, \theta, \tau) = \sum_{\xi=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} y_n^m(\phi, \theta) N_A(s \Omega_0 \xi) \int_{0}^{\tau} \left\{ \frac{R_n(\xi)}{F_L(\xi)} \right\} \sin s \Omega_0 \xi (\tau - \xi) \, d\xi \]

\[ \Psi(\phi, \theta, \tau) = \sum_{\xi=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} y_n^m(\phi, \theta) N_\Psi(s \Omega_0 \xi) \int_{0}^{\tau} \left\{ \frac{R_n(\xi)}{F_L(\xi)} \right\} \sin s \Omega_0 \xi (\tau - \xi) \, d\xi \]

(3.3.68)

3.4 Stress Components

From equation (3.1.7), assuming \( \zeta \ll a \), so that \( 1 + \zeta = 1 \),

\[ \sigma_\phi = \left( \frac{E}{1 - \nu^2} \right) [H, \phi + \psi, \phi \sin \phi - \psi \cos \phi + \psi] \]

\[ \nu(-m^2 \csc^2 \phi H + \cot \phi H, \phi - \cos \psi + \psi) \]

\[ \sigma_\theta = \left( \frac{E}{1 - \nu^2} \right) [\csc^2 \phi (-m^2) H + \cot \phi H, \phi - \cos \psi + \psi] \]

\[ \nu(H, \phi + \psi, \phi \sin \phi - \psi \cos \phi + \psi) \]

\[ \tau_{\phi \theta} = G[2 \csc \phi H, \phi \theta - 2 \cot \phi \csc \phi H, \theta - \psi, \theta] \]

\[ \tau_{\phi n} = \frac{G}{k_s} [W, \phi - H, \phi + \nu, \phi + (\psi - \lambda) \sin \phi] \]

\[ \tau_{\theta n} = \frac{G}{k_s} [\csc \phi (W, \phi - H, \theta + V, \theta)] \]

(3.3.69)
3.5 Supplementary Definitions A-3

\[ K_1 = 1 + h^2/12a^2 \]
\[ K_2 = h^2/6a^2 \]
\[ K_r = 1 + 3h^2/20a^2 \]
\[ S = c/c_s \]
\[ c_s = E/\rho_s(1 - \nu^2)^{0.5} \]
\[ \alpha = h^2/12a^2 \]
\[ K_s = 1 \]

\[(A-3.2.1)\]

\[ a_{11} = \frac{-Eh}{(1 - \nu^2)} \frac{\partial^2}{\partial \phi^2} + \cot \phi \frac{\partial}{\partial \phi} \frac{\partial^2}{\partial \theta^2} + \frac{1 - \nu}{2} \csc^2 \phi \frac{\partial^2}{\partial \phi \partial \theta} - \]
\[ \left( \frac{1 - \nu}{2K_s} + \nu + \cot^2 \phi \right) + \left( \rho a^2 h K \right) \frac{\partial^2}{\partial t^2} \]

\[ a_{12} = \frac{Eh}{(1 - \nu^2)} \csc \phi \left( \frac{3 - \nu}{2} - \frac{\partial}{\partial \theta} - \frac{1 + \nu}{2} \frac{\partial^2}{\partial \phi \partial \theta} \right) \]

\[ a_{13} = -\frac{E h_1}{(1 + \nu)} \left( 1 + \frac{1}{2K_s} \right) \frac{\partial}{\partial \phi} \]

\[ a_{14} = -\frac{E h a}{2(1 + \nu)K_s} - \frac{\rho a h^3}{6} \frac{\partial^2}{\partial t^2} \]

\[ a_{15} = 0 \]

\[ a_{21} = \frac{-Eh}{(1 - \nu^2)} \left( \frac{1 + \nu}{2} \csc \phi \frac{\partial^2}{\partial \phi \partial \theta} + \frac{3 - \nu}{2} \csc \phi \cot \phi \frac{\partial}{\partial \theta} \right) \]
\[
a_{22} = -\frac{Eh}{(1 - \nu^2)} \left( \frac{1 - \nu}{2} \frac{\partial^2}{\partial \phi^2} + \cot \phi \frac{\partial}{\partial \phi} + 1 - \cot^2 \phi - \frac{1}{K_s} \right) + \\
csc^2 \phi \frac{\partial^2}{\partial \theta^2} + \rho a^2 h K_1 \frac{\partial^2}{\partial t^2}
\]

\[
a_{23} = -\frac{Eh}{(1 + \nu)} \left( 1 + \frac{1}{2K_s} \right) \csc \phi \frac{\partial}{\partial \theta}
\]

\[
a_{24} = [0]
\]

\[
a_{25} = -\frac{Eh}{2(1 + \nu)K_s} + \rho a^2 h K_2 \frac{\partial^2}{\partial t^2}
\]

\[
a_{31} = -\frac{Eh}{(1 + \nu)} \left[ (1 + \frac{1}{2K_s}) (\frac{\partial}{\partial \phi} + \cot \phi) \right]
\]

\[
a_{32} = \frac{Eh}{(1 + \nu)} \left( 1 + \frac{1}{2K_s} \right) \csc \phi \frac{\partial}{\partial \theta}
\]

\[
a_{33} = -\frac{Eh}{2(1 + \nu)K_s} \frac{\partial^2}{\partial \phi^2} + \cot \phi \frac{\partial}{\partial \phi} + \csc^2 \phi \frac{\partial^2}{\partial \phi \partial \theta} + \\
\frac{2Eh}{(1 + \nu)} + \rho a^2 h K_1 \frac{\partial^2}{\partial t^2}
\]

\[
a_{34} = -\frac{Eh}{2(1 + \nu)K_s} \frac{\partial}{\partial \phi} + \cot \phi
\]

\[
a_{35} = -\frac{Eh}{2(1 + \nu)K_s} \csc \phi \frac{\partial}{\partial \theta}
\]

\[
a_{41} = -\frac{Eh}{2(1 + \nu)K_s} + \frac{\rho a h}{6} \frac{\partial^2}{\partial t^2}
\]

\[
a_{42} = [0]
\]

\[
a_{43} = \frac{Eh}{2(1 + \nu)K_s} \frac{\partial}{\partial \phi}
\]

\[
a_{44} = -\frac{Eh^3}{12(1 - \nu^2)} \frac{\partial^2}{\partial \phi^2} + \cot \phi \frac{\partial}{\partial \phi} + \frac{(1 - \nu)}{2} \frac{\partial^2}{\partial \theta^2} - (\cot^2 \phi + \nu) + \\
\frac{Eh^2}{2(1 + \nu)K_s} + \frac{\rho a^2 h K r}{12} \frac{\partial^2}{\partial t^2}
\]
\[ a_{45} = \frac{Eh^3}{12(1 - \nu^2)} - \left(\frac{1 + \nu}{2}\right) \csc\phi \frac{\partial^2}{\partial \phi \partial \theta} + \left(\frac{3 - \nu}{2}\right) \csc\phi \cot\phi \frac{\partial}{\partial \theta} \]

\[ a_{51} = [0] \]

\[ a_{52} = -\frac{Eh a^3}{2(1 + \nu)K_s} + \frac{\rho a\hbar^3}{6} \frac{\partial^2}{\partial t^2} \]

\[ a_{53} = \frac{Eh a}{2(1 + \nu)K_s} \csc\phi \frac{\partial}{\partial \theta} \]

\[ a_{54} = -\frac{Eh^3}{12(1 - \nu^2)} \left(\frac{1 + \nu}{2}\right) \csc\phi \frac{\partial^2}{\partial \phi \partial \theta} + \left(\frac{3 - \nu}{2}\right) \csc\phi \cot\phi \frac{\partial}{\partial \theta} \]

\[ a_{55} = -\frac{Eh^3}{12(1 - \nu^2)} \left(\frac{1 - \nu}{2}\right) \left(\frac{\partial^2}{\partial \phi^2} + \cot\phi \frac{\partial}{\partial \phi} + 1 - \cot^2 \phi\right) + \csc^2 \phi \frac{\partial^2}{\partial \theta^2} + \frac{Eh a^2}{2(1 + \nu)K_s} + \frac{\rho a^2 h^3 K_r}{12} \frac{\partial^2}{\partial t^2} \]

(A-2.3.2)

\[ b_{11} = \frac{Eh}{2(1 + \nu)} \csc\phi \frac{\partial}{\partial \phi} \]

\[ b_{12} = \frac{Eh}{2(1 + \nu)} \frac{\partial}{\partial \phi} - \cot\phi \]

\[ b_{21} = \frac{Eh}{(1 - \nu^2)} \frac{\partial}{\partial \phi} + \cot\phi \]

\[ b_{22} = \frac{Eh}{(1 - \nu^2)} \csc\phi \frac{\partial}{\partial \theta} \]

\[ b_{23} = \frac{Eh}{(1 - \nu)} \]

\[ b_{31} = -\frac{Eh}{2(1 + \nu)K_s} \]

\[ b_{32} = \frac{Eh}{2(1 + \nu)K_s} \csc\phi \frac{\partial}{\partial \theta} \]

\[ b_{33} = \frac{Eh R}{2(1 + \nu)K_s} \]
\[ b_{44} = \frac{Eh^3}{24(1 + \nu)} \csc \phi \frac{\partial}{\partial \theta} \]

\[ b_{45} = \frac{Eh^3}{24(1 + \nu)} \left( \frac{\partial}{\partial \phi} - \cot \phi \right) \]

\[ b_{54} = \frac{Eh^3}{12(1 - \nu^2)} \left( \nu \frac{\partial}{\partial \phi} + \cot \phi \right) \]

\[ b_{55} = \frac{Eh^3}{12(1 - \nu^2)} \csc \phi \frac{\partial}{\partial \theta} \]

\[ b_{13} = b_{14} = b_{15} \]

\[ b_{24} = b_{25} = 0 \]

\[ b_{34} = b_{35} = 0 \]

\[ b_{41} = b_{42} = b_{43} = 0 \]

\[ b_{51} = b_{52} = b_{53} = 0 \]

\[ (A-3.3.1) \]

\[ C_{11} = \frac{3}{\partial \phi^3} + \cot \phi \frac{\partial^2}{\partial \phi^2} - (\nu + \cot^2 \phi + \frac{1 - \nu}{2K_s}) \frac{\partial}{\partial \phi} + \]

\[ \csc^2 \phi \frac{\partial^3}{\partial \phi \partial \theta^2} - 2\csc^2 \phi \frac{\partial^2}{\partial \phi^2} - \frac{\rho a^2(1 - \nu^2)K_s}{E} \frac{\partial^3}{\partial t^2 \partial \phi} \]

\[ C_{12} = a \left( \frac{1 - \nu}{2K_s} \right) \frac{\partial}{\partial \phi} - \frac{\rho a^2(1 - \nu^2)K_s}{E} \frac{\partial^3}{\partial t^2 \partial \phi} \]

\[ C_{13} = (1 + \nu) \frac{\partial}{\partial \phi} + \left( \frac{1 - \nu}{2K_s} \right) \]

\[ C_{14} = -a \frac{\partial^2}{\partial \phi^2} + 3\cot \phi \frac{\partial}{\partial \phi} + \left( \frac{1 - \nu}{2} \right) \csc^2 \phi \frac{\partial^2}{\partial \theta^2} - \]

\[ \frac{a^2(1 - \nu^2)K_1}{E} \frac{\partial^2}{\partial t^2} - (1 + \nu + \frac{1 - \nu}{2K_s}) \]
\[ C_{15} = a \sin \phi \frac{\rho a^2(1 - \nu^2) K_1}{E} \frac{\partial^2}{\partial t^2} - \frac{(1 - \nu)}{2K_s} \]

\[ C_{21} = \csc \phi \frac{\partial^3}{\partial \phi \partial \theta} + \cot \phi \frac{\partial^2}{\partial \theta \partial \phi} + (1 - \nu)(1 - \frac{1}{2K_s}) \frac{\partial}{\partial \theta} + \csc \phi \frac{\partial^3}{\partial \theta^3} - \frac{\rho a^2(1 - \nu^2) K_1}{E} \frac{\partial^3}{\partial t^2 \partial \theta} \]

\[ C_{22} = a \csc \phi \frac{(1 - \nu)}{2K_s} \frac{\partial}{\partial \theta} - \frac{\rho(1 - \nu^2) a^2 K_2}{E} \frac{\partial^3}{\partial t^2 \partial \theta} \]

\[ C_{23} = \csc \phi (1 + \nu + \frac{1 - \nu}{2K_s}) \frac{\partial}{\partial \theta} \]

\[ C_{24} = -a(\frac{1 + \nu}{2}) \frac{\partial^2}{\partial \phi \partial \theta} + 2\cot \phi \frac{\partial}{\partial \theta} \]

\[ C_{25} = 0 \]

\[ C_{31} = -(1 + 2K_s) \frac{1 + \nu}{1 - \nu} \frac{\partial^2}{\partial \phi^2} + \cot \phi \frac{\partial}{\partial \phi} + \csc \phi \frac{\partial^2}{\partial \phi \partial \theta} \]

\[ C_{32} = a \frac{\partial^2}{\partial \phi^2} + \cot \phi \frac{\partial}{\partial \phi} + \csc \phi \frac{\partial^2}{\partial \theta^2} \]

\[ C_{33} = \frac{\partial^2}{\partial \phi^2} + \cot \phi \frac{\partial}{\partial \phi} + \csc \phi \frac{\partial^2}{\partial \theta^2} - \frac{2\rho a^2(1 - \nu^2) K_1 K_2}{E} \frac{\partial^2}{\partial t^2} - \frac{4K_s(1 + \nu)}{1 - \nu} \]

\[ C_{34} = a(1 + 2K_s) \frac{1 + \nu}{1 - \nu} (\sin \phi \frac{\partial}{\partial \phi} + 2\cos \phi) \]

\[ C_{35} = -a[\sin \phi \frac{\partial}{\partial \phi} + 2\cos \phi] \]

\[ C_{41} = \frac{6a(1 - \nu)}{h^2 K_s} \frac{\partial}{\partial \phi} - \frac{2\rho a(1 - \nu^2)}{E} \frac{\partial^3}{\partial t^2 \partial \phi} \]
\[
C_{42} = \frac{\partial^3}{\partial \phi^3} + \cot \phi \frac{\partial^2}{\partial \phi^2} - (\nu + \cot \phi + \frac{6a^2(1-\nu)}{h^2K_s} \frac{\partial}{\partial \phi} + \\
\csc^2 \phi \frac{\partial^3}{\partial \phi \partial \theta^2} - 2\csc^2 \phi \cot \phi \frac{\partial^2}{\partial \theta^2} - \frac{\rho a^2(1-\nu^2)K_r}{E} \frac{\partial^3}{\partial t^2 \partial \phi}
\]
\[
C_{43} = -\frac{6a(1-\nu)}{h^2K_s} \frac{\partial}{\partial \phi}
\]
\[
C_{44} = -a \sin \phi \frac{6(1-\nu)}{h^2K_s} - \frac{2\rho a(1-\nu^2)}{E} \frac{\partial^2}{\partial t^2}
\]
\[
C_{45} = -\sin \phi \frac{\partial^2}{\partial \phi^2} + 3\cot \phi \frac{\partial}{\partial \phi} - (1 + \nu + \frac{6a^2(1-\nu)}{h^2K_s}) + \\
\left(\frac{1}{2}\nu\right) \csc^2 \phi \frac{\partial^2}{\partial \theta^2} - \frac{\rho a^2(1-\nu^2)}{E} K_r \frac{\partial^2}{\partial t^2}
\]
\[
C_{51} = \csc \phi \frac{6a(1-\nu)}{h^2K_s} \frac{\partial}{\partial \theta} - \frac{2\rho a(1-\nu^2)}{E} \frac{\partial^3}{\partial t^2 \partial \theta}
\]
\[
C_{52} = \csc \phi \frac{\partial^3}{\partial \theta \partial \phi^2} + \cot \phi \frac{\partial^2}{\partial \theta \partial \phi} + (1 - \frac{6a^2}{h^2K_s})(1 - \nu) \frac{\partial}{\partial \theta} + \\
\csc^2 \phi \frac{\partial^2}{\partial \theta^2} - \frac{\rho a^2(1-\nu^2)}{E} K_r \frac{\partial^3}{\partial t^2 \partial \theta}
\]
\[
C_{53} = -\csc \phi \frac{6a(1-\nu)}{h^2K_s} \frac{\partial}{\partial \theta}
\]
\[
C_{54} = 0
\]
\[
C_{55} = -\left(\frac{1 + \nu}{2}\right) \frac{\partial^2}{\partial \phi \partial \theta} + 2\cot \phi \frac{\partial}{\partial \theta}
\]

(A-3.3.2)

\[
d_{11} = d_{12} = d_{13} = 0
\]
\[
d_{21} = d_{22} = d_{23} = 0
\]
\[
d_{35} = d_{44} = 0
\]
\[ d_{14} = \left[ \nu^2 + 2\left(1 - \frac{1}{K_s}\right) - \frac{2K_1}{1 - \nu} p^2 \right] \]

\[ d_{15} = \left(\frac{2}{1 - \nu}\right) K_2 p^2 - \frac{1}{K_s} \]

\[ d_{24} = \left(\frac{4\alpha}{1 - \nu}\right) p^2 - \frac{1}{K_s} \]

\[ d_{25} = \alpha\left(\frac{2}{1 - \nu} K_r p^2 - \nu^2 - 2\right) + \frac{1}{K_s} \]

\[ d_{31} = K_1 p^2 - \nu^2 - 1 + \nu + \frac{1 - \nu}{2K_s} \]

\[ d_{32} = K_2 p^2 - \frac{1 - \nu}{2K_s} \]

\[ d_{33} = -1 + \nu \frac{1 - \nu}{2K_s} \]

\[ d_{34} = \left(\frac{1 + \nu}{2}\right) \sin\phi \frac{\partial}{\partial \phi} + 2\cos\phi \]

\[ d_{41} = 2\alpha p^2 - \frac{1 - \nu}{2K_s} \]

\[ d_{42} = \alpha(K_r p^2 - \nu^2 - 1 + \nu) + \frac{1 - \nu}{2K_s} \]

\[ d_{43} = [(1 - \nu)/2K_s] \]

\[ d_{45} = -\left[\alpha\left(\frac{1 + \nu}{2}\right) \sin\phi \frac{\partial}{\partial \phi} + 2\cos\phi\right] \]

\[ d_{51} = [(1 + 2K_s \frac{1 + \nu}{1 - \nu}) \nu^2] \]

\[ d_{52} = [\nu^2] \]

\[ d_{53} = \left(\frac{2}{1 - \nu}\right) K_1 K_2 p^2 - \nu^2 + 4K_s \frac{1 + \nu}{1 - \nu} \]
\[ d_{54} = -[(1 + 2K_s \frac{1 + \nu}{10\nu})(\sin \phi \frac{3}{\partial \phi} + 2\cos \phi)] \]

\[ d_{55} = [\sin \phi \frac{3}{\partial \phi} + 2\cos \phi] \]

\[ v_1^2 = \frac{3^2}{\partial \phi^2} + \cot \phi \frac{3}{\partial \phi} + \csc \phi \frac{3^2}{\partial \theta^2} \]

\[ F = \frac{d}{d\theta} (p_\theta) \]  
(A-2.3.4)

\[ v^2 = \frac{1}{r^2} \frac{3}{\partial r^2} (r^2 \frac{3}{\partial r}) + \frac{1}{r} \csc \phi \frac{3}{\partial \phi} (\sin \phi \frac{3}{\partial \theta}) + \frac{1}{r} \csc^2 \phi \frac{3^2}{\partial \theta^2} \]  
(A-2.3.5)

\[ PW(\phi, \theta, p) = \frac{3}{\partial r} \phi(1, \phi, \theta, p) \]

(A-3.3.3)

\[ a_s = (1 - \nu)(\frac{1}{2K_s} - 1) \]

\[ b = \frac{2}{(1 - \nu)} \]

\[ d = (1 + \nu)/2 \]

\[ \beta_v = 2(1 + \nu)a/Eh \]

\[ \theta = -a(1 - \nu) + (1 - \nu)/2K_s \]

\[ g_s = (1 - \nu)/2K_s \]

\[ d_s = 2(1 - 1/K_s) \]

\[ e_s = 1/K_s \]

\[ x_s = 1/K_s - 2a \]

\[ y_s = 4K_s (1 + \nu)/(1 - \nu) \]

\[ b_s = [(1 + \nu) + (1 - \nu)/2K_s] \]

\[ f_s = 1 + 2K_s (1 + \nu)/(1 - \nu) \]

\[ \lambda_n = n(n + 1) \]

\[ B(\frac{1p}{s}) = bK_s[K_s + f \int_n (\frac{1p}{s})/(\frac{1p}{s}) j_n(\frac{1p}{s})] \]

\[ f = \rho_f/\rho_s \]
\begin{align*}
\frac{h_{n1}}{h_{n2}} &= K_1 p^2 + \lambda_n + a_s \\
\frac{h_{n2}}{h_{n3}} &= 2\alpha p^2 - g_s \\
g_{n1} &= K_2 p^2 - g_s \\
g_{n2} &= \alpha K_1 p^2 + \lambda_n + \Theta_s \\
g_{n3} &= \lambda_n \\
W_{n1} &= b_s \\
W_{n2} &= g_s \\
W_{n3} &= B (i \rho / s) p^2 + \lambda_n + y_s \\
S_c (\phi, \theta) &= \delta \sin \phi \\
C_{mn} &= \beta \nu K_1 \frac{2n + 1}{2\pi} \frac{(n - m)!}{(m + n)!} \left[ \int_{\pi/2-\Theta}^{\pi/2+\Theta} \int_{\phi_N}^{\phi} Y_n^m (\phi, \theta) \, d\theta \, d\phi \right] S_c (\phi, \theta) \\
A_{mn} &= \lambda_{mn} \frac{2n + 1}{2\pi} \frac{(n - m)!}{(m + n)!} \left[ \int_{\pi/2-\Theta}^{\pi/2+\Theta} \int_{\phi_N}^{\phi} Y_n^m (\phi, \theta) \, d\theta \, d\phi \right] S_c (\phi, \theta) \\
\lambda_{mn} &= \frac{2\pi (m + n)!}{s} \frac{2n^3 + n^2 (1 + 2m) - 4n + 3nm - 2m + 1}{(2n - 1)(2n + 3)(2n + 1)} \\
Y_{mn} &= - \frac{2\pi (n + m)!}{(2n + 1)(2n - 1)(n - m - 1)!} [(1 + n)d - 2]
\end{align*}
\[ \Delta_{mn} = \frac{2\pi(n + m + 1)!}{(2n + 1)(2n + 3)(n - m)!} \begin{bmatrix} \text{nd} + 2 \end{bmatrix} \]

\[ r_{mn} = \frac{2n + 1}{2\pi} \frac{(n - m)!}{(n + m)!} \left[ \int_{\phi = 0}^{\phi = N} \int_{\theta = 0}^{\theta = \theta} Y_{n}^{m}(\phi, \theta) d\phi d\theta \right] s_{c}(\phi, \theta) \]

\[ b_{mn} = (\gamma_{mn} + \Delta_{mn}) \frac{(2n + 1)(n - m)!}{2\pi(n + m)!} \left[ \int_{\phi = 0}^{\phi = N} \int_{\theta = 0}^{\theta = \theta} Y_{n}^{m}(\phi, \theta) d\phi d\theta \right] s_{c}(\phi, \theta) \]

\[ A(p) = [b\alpha K_{s} p^{2} + \alpha_{n} + x_{s}] / [2bp^{2} - e_{s}] \]

\[ B(p) = \beta_{\psi} [2bp^{2} - e_{s}] / [2b^{2} K_{1} p^{4} + (2\lambda_{n} + K_{2} - 2d_{s} - e_{s} K_{1}) p^{2} + e_{s} (d_{s} - \lambda_{n} - 1)] \]

\[ P_{n1} = A(p)B(p)b_{mn} F_{\Phi}(p) \]

\[ P_{n2} = aB(p)b_{mn} F_{\Phi}(p) \]

\[ P_{n3} = C_{mn} R(p) + B(p) [A(p)f_{s} - 1] a_{mn} F_{\Phi}(p) \]
CHAPTER 4

DYNAMIC RESPONSE OF THE SHALLOW SHELL

Shell Response to Impact

For the equation of the shallow spherical shell, we recall equations (3.3.18) through (3.3.22) with the following modifications:

1. \( \overline{F}_\phi - \overline{P}_\theta = \gamma \overline{R}_n \)
2. \( \phi(1, \phi, \theta, \rho) = 0 \)
3. \( P = +i \omega \)
4. \( f = 0 \)
5. \( s = 1 \)
6. \( a + a_2, a_2 > a \), radius of closed spherical shell
7. \( \phi \rightarrow \psi \)
8. \( \theta \rightarrow \Xi \)
9. \( \overline{R}_{mn}(\rho) = d_{mn} \overline{R}(\rho) \)

The resulting equations are

\[
(bk_1 \rho^2 + \lambda_n - d_s) \overline{\psi}_{mn} + (bk_2 \rho^2 - e_s) \overline{\Lambda}_{mn} = \beta \gamma \overline{R}_{mn}(\rho) \quad (4.2)
\]

\[
(2bap^2 - e_s) \overline{\psi}_{mn} + (bak \rho^2 + \alpha \lambda_n + \chi_s) \overline{\Lambda}_{mn} = 0 \quad (4.3)
\]

\[
(k_1 \rho^2 + \lambda_n + a_s) \overline{H}_{mn} + (k_2 \rho^2 - g_s) \overline{G}_{mn} - (b_s) \overline{W}_{mn} + (\gamma_{mn} + \Delta_{mn}) \overline{\psi}_{mn} = 0 \quad (4.4)
\]
The time of wave propagation from the center of the finite shallow shell to its boundary is in the same order of magnitude as the impact duration, the transient solution of the shallow shell due to a prescribed local velocity input is also considered.

4.1 Vibration Due to Local Velocity Input

To obtain the vibration of the shallow shell, equations (4.2) through (4.6) are now used. The forcing function $R_n(p)$ is set zero. The initial conditions are:

$$ \{u, v, w\} = \{0, 0, \frac{w_{25}}{c_s}\} $$

(4.1.1)

where $w_{25}(t)$ is the rigid body velocity of the head or closed spherical shell. Equations of motion of the shallow spherical shell are:

$$ [\epsilon_{ij}] \{\bar{\psi}_{mn}, \bar{\Lambda}_{mn}, \bar{H}_{mn}, \bar{G}_{mn}, \bar{W}_{mn}\}^T = \{0, 0, 0, \frac{w_{25}}{c_s}\}^T $$

(4.1.2)

where $[\epsilon_{ij}]$ is a 5x5 matrix, the elements of which are the coefficients of $\bar{\psi}_{mn}, \bar{\Lambda}_{mn}, \bar{H}_{mn}, \bar{G}_{mn}, \bar{W}_{mn}$. Specifically:

for $p = +1\omega$

$$ \epsilon_{11} = -bk_1\omega^2 + \lambda_n - d_s $$

$$ \epsilon_{12} = -bk_2\omega^2 - e_s $$

$$ \epsilon_{13} = \epsilon_{14} = \epsilon_{15} = 0 $$
To solve equation (4.1.2) we must assume a solution of functions 

\[ \Psi(\phi, \theta, \tau), \Lambda(\phi, \theta, \tau), H(\phi, \theta, \tau), G(\phi, \theta, \tau) \text{ and } W_{mn}(\phi, \theta, \tau) \] 

which must satisfy the boundary conditions. The following boundary conditions are
specifically chosen for the solution of the equations:

1. \( \Delta T \Xi = \pm \Xi_0 \)
   - (a) \( U(\psi, \pm \Xi_0, \tau) \neq 0 \)
   - (b) \( V(\psi, \pm \Xi_0, \tau) = 0 \)
   - (c) \( W(\psi, \pm \Xi_0, \tau) = 0 \)
   - (d) \( \beta_\psi(\psi, \pm \Xi_0, \tau) \neq 0 \)
   - (e) \( \beta_\Xi(\psi, \pm \Xi_0, \tau) \neq 0 \)

2. \( \Delta T \psi = \pm \psi_0 \)
   - (f) \( U(\pm \psi_0, \Xi, \tau) = 0 \)
   - (g) \( V(\pm \psi_0, \Xi, \tau) = 0 \)
   - (h) \( W(\pm \psi_0, \Xi, \tau) = 0 \)
   - (i) \( \beta_\psi(\pm \psi_0, \Xi, \tau) = 0 \)
   - (j) \( \beta_\Xi(\pm \psi_0, \Xi, \tau) = 0 \)

We must realize that there are many ways to satisfy equations (4.1.4) and (4.1.5). However, the following functions are chosen to satisfy the equations:

\[
H(\psi, \eta, \tau) = \sum_{n=1}^{\infty} \sum_{m=0}^{5} \sum_{k=1}^{n} H_{mn}(\omega_{nk}) p_n^m(x) \cos mn \sin \omega_{nk} \tau \quad (4.1.6)
\]

\[
\psi(\psi, \eta, \tau) = 0 \quad (4.1.7)
\]

\[
W(\psi, \eta, \tau) = \sum_{n=1}^{\infty} \sum_{m=0}^{5} \sum_{k=1}^{n} W_{mn}(\omega_{nk}) p_n^m(x) \cos mn \sin \omega_{nk} \tau \quad (4.1.8)
\]

\[
\Lambda(\psi, \eta, \tau) = 0 \quad (4.1.9)
\]
\[ G(\psi, \eta, \tau) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{k=1}^{5} G_{mn}(\omega_{nk}) p_n^m(x) \cos 2m\eta \sin \omega_{nk} \tau \quad (4.1.10) \]

where

\[ \beta = \cos \frac{\pi \psi}{\psi_0} \]

\[ X = \cos \frac{\pi \psi}{2\psi_0} \quad (4.1.11) \]

\[ \eta = \frac{\pi \Omega}{2\xi_0} \]

and

\[ H_{mn}(\omega_{nk}) = \frac{a}{c_s} \omega_{25} \{ e_{11}e_{22}(e_{34}e_{45} - e_{35}e_{44}) - 
\]

\[ e_{12}e_{21}(e_{14}e_{45} - e_{15}e_{44}) \}/\Delta_0(\omega_{nk}) \]

\[ G_{mn}(\omega_{nk}) = \frac{a}{c_s} \omega_{25} \{ e_{22}e_{11}(e_{32}e_{45} - e_{35}e_{43}) - 
\]

\[ e_{12}e_{21}(e_{33}e_{45} - e_{35}e_{43}) \}/\Delta_0(\omega_{nk}) \]

\[ W_{mn}(\omega_{nk}) = \frac{a}{c_s} \omega_{25} \{ e_{11}e_{22} - e_{21}e_{22} \}\{ e_{33}e_{44} - e_{43}e_{34} \}/\Delta(\omega_{nk}) \]

\[ \Delta_0(\omega_{nk}) = d_0(\omega_{nk})B_{00}(\omega_{nk}) + d_{00}(\omega_{nk})B_0(\omega_{nk}) \quad (4.1.14) \]

\[ d_0(\omega) = \frac{4a}{(1 - \nu)^2} (k_1k_r - 2k_2) \omega^4 - \left( \frac{2}{1 - \nu} \right) \alpha(k_r + k_1)(\lambda_n - 2) + 
\]

\[ \frac{1}{k_s(k_1 + k_2)} + \frac{2a}{k_s}(k_r + 1) \omega^2 + 
\]

\[ (\lambda_n - 2)(\alpha(\lambda_n - 2) + \frac{1}{k_s}(1 + 2\alpha)) + \frac{1}{k_s^2} \quad (4.1.16) \]
Equations (4.1.16) and (4.1.17) are the frequency equations for the torsional and non-torsional groups. Also, the following quantities:

\[ \Delta_{mn}^* = \frac{\Delta_{mn}}{2\psi_0} \]
\[ \gamma_{mn}^* = \frac{\gamma_{mn}}{2\psi_0} \]
\[ \lambda_{mn}^* = \frac{\lambda_{mn}}{2\psi_0} \]

are used in (4.1.2b), where

\[ \Delta_{mn}, \gamma_{mn} \text{ and } \lambda_{mn} \] are defined in the supplementary definitions of Chapter Three.

4.2. **Vibration of the Spherical Shallow Shell Due to Impact Force**

Equations (4.2) through (4.6) are also used here in the determination of the response due to the impact forcing function \( R_n \). When the first two equations, (4.2) and (4.3), are solved simultaneously we obtain:

\[ \bar{A}_{mn}(p) = E(p)\bar{R}_{mn}(p) \quad (4.2.1) \]
\[ \bar{V}_{mn}(p) = D(p)\bar{R}_{mn}(p) \quad (4.2.2) \]

where
\[ E(p) = (2b_0 p^2 - e_s) \beta \gamma/d_0(p) \]

\[ D(p) = (b_0 k_0 p^3 + \alpha \lambda + X_0) \gamma \beta /d_0(p) \]

and

\[ d_0(\omega) = [d_0(p)]_{p=\omega} \text{ in (4.6)} \]

We have already seen that the torsional frequency spectral \( \omega_{sk} \) is obtained from \( d_0(\omega) = 0 \). We solve the last three equations for \( H_{mn} \), \( \bar{G}_{mn} \) and \( \bar{W}_{mn} \) in terms of \( R_{mn} \) while the values of \( \bar{V}_{mn} \) and \( \bar{A}_{mn} \) from (4.2.3) are substituted. If we let \([q_{ij}]\) be a 3x3 coefficient matrix of \( \bar{H}_{mn} \), \( \bar{G}_{mn} \) and \( \bar{W}_{mn} \) and \( r_j \) be the corresponding vector of 3x1 matrix coefficient of \( \bar{R}_{mn} \), then we have

\[ [q_{ij}][\bar{H}_{mn}, \bar{G}_{mn}, \bar{W}_{mn}]^T = [r_j]^T \bar{R}_{mn}, \quad i = j = 1, 2, 3 \quad (4.2.4) \]

We must recognize at once that the determinant,

\[ \text{Det}(q_{ij}) = d_{00}(p) \quad (4.2.5) \]

is defined in (4.1.17) and by Cramer's rule

\[ \bar{H}_{mn}(p) = \begin{vmatrix} r_1 & q_{12} & q_{13} \\ r_2 & q_{22} & q_{23} \\ r_3 & q_{32} & q_{33} \end{vmatrix} \quad \frac{\bar{R}_{mn}(p)/d_{00}(p)}{d_{00}(p)} \quad (4.2.6) \]

\[ \bar{G}_{mn}(p) = \bar{R}_{mn}(p) \begin{vmatrix} q_{11} & r_1 & q_{13} \\ q_{21} & r_2 & q_{23} \\ q_{31} & r_3 & q_{33} \end{vmatrix} \quad \frac{1}{d_{00}(p)} \quad (4.2.7) \]
\[
\tilde{w}_{mn}(p) = \frac{\tilde{R}_{mn}(p)}{d_{00}(p)}
\]

where \( | | \) denotes determinant of the matrix and its elements are defined in supplementary definitions (A-4).

Alternatively, we can take the inverse directly and obtain equations (4.2.9) through (4.2.11) with the denominator \( (d_0^2 d_{00}^2) \) replaced by \( \Delta_{00}(\omega) \) in anticipation of the inverse Laplace transform since the product of \( d_0(\omega) \) and \( d_{00}(\omega) \) goes to zero for the frequency spectra.

Thus,
\[
\tilde{h}_{mn}(p) = \frac{\{g_{11}s_1 + g_{12}s_2 + g_{13}s_3\} \tilde{R}_{mn}(p)/\Delta_{00}(p)}{\Delta_{00}(p)}
\]

\[
\tilde{g}_{mn}(p) = \frac{\{g_{21}s_1 + g_{22}s_2 + g_{23}s_3\} \tilde{R}_{mn}(p)/\Delta_{00}(p)}{\Delta_{00}(p)}
\]

\[
\tilde{w}_{mn}(p) = \frac{\{g_{31}s_1 + g_{32}s_2 + g_{33}s_3\} \tilde{R}_{mn}(p)/\Delta_{00}(p)}{\Delta_{00}(p)}
\]

The Laplace inverse of equations (4.2.9-11) are

\[
H_{mn}(\tau) = \int_0^\tau e^{\xi p} \frac{g_{11}s_1 + g_{12}s_2 + g_{13}s_3}{\Delta_{00}(p)} d_{mn}R_n(\tau - \xi) d\xi
\]

\[
G_{mn}(\tau) = \int_0^\tau e^{\xi p} \frac{g_{21}s_1 + g_{22}s_2 + g_{23}s_3}{\Delta_{00}(p)} d_{mn}R_n(\tau - \xi) d\xi
\]

\[
W_{mn}(\tau) = \int_0^\tau e^{\xi p} \frac{g_{31}s_1 + g_{32}s_2 + g_{33}s_3}{\Delta_{00}(p)} d_{mn}R_n(\tau - \xi) d\xi
\]

\[
\psi_{mn}(\tau) = \int_0^\tau e^{\xi p} \frac{g_{11}s_1 + g_{12}s_2 + g_{13}s_3}{\Delta_{00}(p)} d_{mn}R_n(\tau - \xi) d\xi
\]
\[ \Lambda_{mn}(\tau) = \int_{0}^{\tau} e^{\xi p} \frac{\beta \gamma(bak_r - p^2 + \alpha \lambda_n + x_s)}{B_0(p)} d_{mn} R_n(\tau - \xi) \, d\xi \]

Let

\[ H_{mn}^s(\omega_{nj}) = \frac{|g_{11}^s| + g_{12}^s + g_{13}^s|}{\Delta_{00}(p)} |d_{mn}| \]

\[ G_{mn}^s(\omega_{nj}) = \frac{|g_{21}^s| + g_{22}^s + g_{23}^s|}{\Delta_{00}(p)} |d_{mn}| \]

\[ W_{mn}^s(\omega_{nj}) = \frac{|g_{31}^s| + g_{32}^s + g_{33}^s|}{\Delta_{00}(p)} |d_{mn}| \]

\[ \Lambda_{mn}^s(\omega_{nj}) = \frac{\beta \gamma(bak_r p^2 + \alpha \lambda_n + x_s)}{B_0(p)} |d_{mn}| \]

\[ \gamma_{mn}^s(\omega_{nj}) = \frac{\beta \gamma(2bp^2 - e_s)}{B_0(p)} |d_{mn}| \]

where \(| |\) denotes absolute values so that the solution can be written in a shorter form. The five modes of the forcing motion are subscripted as \(H_s, G_s, W_s, \gamma_s\) and \(\Lambda_s\) and are defined, using equations (4.2.13) as follow:

\[ H_s(\psi, n, \tau) = \sum_{j=1}^{5} \sum_{n=1}^{\infty} \sum_{m=0}^{n} H_{mn}^s(\omega_{nj}) P_n^m(x) \cos m \phi \int_{0}^{\tau} R_n(\tau) \sin \omega_{nj}(\tau - \xi) \, d\xi \]

\[ G_s(\psi, n, \tau) = \sum_{j=1}^{5} \sum_{n=1}^{\infty} \sum_{m=0}^{n} G_{mn}^s(\omega_{nj}) P_n^m(x) \cos 2m \phi \int_{0}^{\tau} R_n(\tau) \sin \omega_{nj}(\tau - \xi) \, d\xi \]
4.3. **Total Response** of the Shallow Shell

The shallow shell motion is completely described by the combination of the equations (4.1.6-10) and (4.2.14)

\[
W_\psi (\psi, \eta, \tau) = \sum_{j=0}^{5} \sum_{n=1}^{\infty} \sum_{m=0}^{n} W_{jmn}^{s} (\omega) p_{n}^{m}(X) \cos \omega_{mnj} \int_{0}^{\tau} R_{n}(\tau) \sin \omega_{nj}(\tau - \xi) \, d\xi
\]

\[
A_\psi (\psi, \eta, \tau) = \frac{2\psi_{0}}{\pi} \sum_{j=0}^{5} \sum_{n=1}^{\infty} \sum_{m=0}^{n} A_{jmn}^{s} (\omega) \frac{d p_{n}^{m}(X)}{d\psi} \int_{0}^{\tau} \cos 2\omega_{mnj} R_{n}(\tau) \sin \omega_{nj}(\tau - \xi) \, d\xi
\]

\[
\psi_{s} (\psi, \eta, \tau) = \frac{2\psi_{0}}{\pi} \sum_{j=0}^{5} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \psi_{jmn}^{s} (\omega) \frac{d p_{n}^{m}(X)}{d\psi} \int_{0}^{\tau} \cos 2\omega_{mnj} R_{n}(\tau) \sin \omega_{nj}(\tau - \xi) \, d\xi
\]
\[(5.3.4)\]

\[
\mathfrak{P} \left\{ (\mathcal{E} \ast \mathcal{F}) \right\} \leq \mathcal{N} \mathfrak{P} \left\{ \mathfrak{T} \mathcal{F} \mathcal{M} \right\} \sum_{1}^{(\mathcal{C} \mathcal{M})} \left( \mathcal{U} \mathcal{M} \mathcal{N} \mathcal{P} \mathcal{Q} \mathcal{S} \mathcal{T} \mathcal{U} \mathcal{V} \mathcal{W} \mathcal{X} \mathcal{Y} \mathcal{Z} \right) \frac{\phi_{p}}{(\mathcal{X})_{u \mathcal{m} \mathcal{p}}} \frac{(I_{= \mathcal{L} \mathcal{O} \mathcal{M} \mathcal{N}} \mathcal{P} \mathcal{Q} \mathcal{S} \mathcal{T} \mathcal{U} \mathcal{V} \mathcal{W} \mathcal{X} \mathcal{Y} \mathcal{Z})}{(I_{= \mathcal{L} \mathcal{O} \mathcal{M} \mathcal{N}} \mathcal{P} \mathcal{Q} \mathcal{S} \mathcal{T} \mathcal{U} \mathcal{V} \mathcal{W} \mathcal{X} \mathcal{Y} \mathcal{Z})} = (1', \mathcal{U} \mathcal{F} \mathcal{V})
\]

\[(5.3.4)\]

\[
\mathfrak{P} \left\{ (\mathcal{E} \ast \mathcal{F}) \right\} \leq \mathcal{N} \mathfrak{P} \left\{ \mathfrak{T} \mathcal{F} \mathcal{M} \right\} \sum_{1}^{(\mathcal{C} \mathcal{M})} \left( \mathcal{U} \mathcal{M} \mathcal{N} \mathcal{P} \mathcal{Q} \mathcal{S} \mathcal{T} \mathcal{U} \mathcal{V} \mathcal{W} \mathcal{X} \mathcal{Y} \mathcal{Z} \right) \frac{\phi_{p}}{(\mathcal{X})_{u \mathcal{m} \mathcal{p}}} \frac{(I_{= \mathcal{L} \mathcal{O} \mathcal{M} \mathcal{N}} \mathcal{P} \mathcal{Q} \mathcal{S} \mathcal{T} \mathcal{U} \mathcal{V} \mathcal{W} \mathcal{X} \mathcal{Y} \mathcal{Z})}{(I_{= \mathcal{L} \mathcal{O} \mathcal{M} \mathcal{N}} \mathcal{P} \mathcal{Q} \mathcal{S} \mathcal{T} \mathcal{U} \mathcal{V} \mathcal{W} \mathcal{X} \mathcal{Y} \mathcal{Z})} = (1', \mathcal{U} \mathcal{F} \mathcal{V})
\]
4.4 Supplementary Definitions A-4

(Also See A-3)

\[ L(\omega) = -\left(\frac{1}{k_s} + \frac{2}{1 - \nu} k_2 \omega^2\right)\{\lambda_n - 2(1 - \frac{1}{k_s}) - \frac{2}{1 - \nu} k_1 \omega^2\} \]

\[ g_{11} = -\lambda_n [\alpha \lambda_n - (1 - \nu)(\alpha - \frac{1}{2k_s}) + \alpha k_r \omega^2][L(\omega)f_s + 1] - \alpha [\Delta_{mn}^* + \gamma_{mn}^*] \lambda_n \]

\[ g_{12} = \lambda_n L(\omega)[\gamma_{mn}^* + \Delta_{mn}^*] - \frac{1}{2k_s} k_2 \omega^2 [L(\omega)f_s + 1] \lambda_{mn}^* \]

\[ g_{13} = -\alpha [\Delta_{mn}^* + \gamma_{mn}^*] \left[\frac{1 + \nu}{2k_s} - k_2 \omega^2\right] + L(\omega)(\gamma_{mn}^* + \Delta_{mn}^*) [\alpha \lambda_n - (1 - \nu)(\alpha - \frac{1}{2k_s}) + \alpha k_r \omega^2] \]

\[ g_{21} = \alpha [\Delta_{mn}^* + \gamma_{mn}^*] f_s \lambda_n - \lambda_n \left[\frac{1 - \nu}{2k_s} - 2\omega^2\right] [L(\omega)f_s + 1] \]

\[ g_{22} = -\lambda_n (1 - \nu)(1 - \frac{1}{k_s}) + k_1 \omega^2 [L(\omega)f_s + 1] \lambda_n + \]

\[ f_s \lambda_n L(\omega) [\gamma_{mn}^* + \Delta_{mn}^*] \]

\[ g_{23} = L(\omega)(2\omega^2 - \frac{1 - \nu}{2k_s} \gamma_{mn}^* + \Delta_{mn}^*) - \alpha [\Delta_{mn}^* + \gamma_{mn}^*] \lambda_n - (1 - \nu)(1 - \frac{1}{k_s}) + k_1 \omega^2 \]

\[ g_{31} = \left[\frac{1}{2k_s} - 2\omega^2\right] \lambda_n - f_s \lambda_n [\alpha \lambda_n - (1 - \nu)(\alpha - \frac{1}{2k_s}) + \alpha k_r \omega^2] \]

\[ g_{32} = f_s \lambda_n (k_2 \omega^2 - \frac{1 - \nu}{2k_s}) + \lambda_n (\lambda_n - (1 - \nu)(1 - \frac{1}{k_s}) + k_2 \omega^2) \]

\[ g_{33} = [\alpha \lambda_n - (1 - \nu)(\alpha - \frac{1}{2k_s}) + \alpha k_r \omega^2][\lambda_n - (1 - \nu)(1 - \frac{1}{k_s}) +]

\[ k_1 \omega^2] + [k_2 \omega^2 - \frac{1 - \nu}{2k_s} \left[\frac{1 - \nu}{2k_s} - 2\omega^2\right] \]
The first derivatives of \( d_0(\omega) \) and \( d_{00}(\omega) \) are:

\[
B_0(\omega) = \frac{16a}{(1-v)^2} (k_1^2 - 2k_2)\omega^3 - \frac{4}{1-v} \{ \alpha(k_1 + k_r)(\lambda_n - 2) + \\
\frac{1}{k_s}(k_1 + k_2) + \frac{\alpha}{k_s}(k_r + 2) \} \omega 
\]

\[
B_{00}(\omega) = \frac{6k_{1b}k_s}{1-v} (k_1^2 - 2k_2)\omega^5 - 4(2k_2k_1b_k s g_s + \alpha(\lambda_n + y_s)) \cdot \\
(k_1^2 - 2k_2) + k_{1ab}k_r k_s(\lambda_n + a_s) + b_k k_1^2(\lambda_n + \theta_s) \} \omega^3 + \\
2(2k_2g_s(\lambda_n + y_s) - b_k k_s g_s - k_2 g_s f_s \lambda_n + \\
b_k k_s(\lambda_n + a_s)(\lambda_n + y_s) - b_s \lambda_n(k_2 + ak_r) \} \omega 
\]

\[
d_{mn} = \frac{2n + 1}{4\pi} \frac{(n-m)!}{(n+m)!} \int_{-\psi_1}^{\psi_1} Y_m(\psi, \eta) d\psi d\eta 
\]  \hspace{1cm} (A-4.6)

The following are read in terms of \( \omega \) when we substitute \( p = \pm i\omega \).

\[
q_{11} = [k_1 p^2 + \lambda_n + a_s] 
\]

\[
q_{12} = [k_2 p^2 - g_s] 
\]

\[
q_{13} = [b_s] 
\]

\[
q_{21} = [2a p^2 - g_s] 
\]

\[
q_{22} = [\alpha \lambda_n + \theta_s + a_k p^2] 
\]  \hspace{1cm} (A-4.7)

\[
q_{23} = [g_s] 
\]

\[
q_{31} = [-f_s \lambda_n] 
\]

\[
q_{32} = [\lambda_n] 
\]

\[
q_{33} = [bk_s p^2 + \lambda_n + y_s] 
\]
\[ r_1 = D(\omega)\gamma(\gamma_{mn} + \Delta_{mn})\overline{R}_{mn} \]

\[ r_2 = \gamma\alpha(\gamma_{mn} + \Delta_{mn})\overline{R}_{mn} E(\omega) \]

\[ r_3 = \left( \beta_{\nu a} + (\frac{f_s D(\omega) - E(\omega)}{\lambda_{mn}})\overline{R}_{mn} \right)^{s_n} \]

\[ s_1 = \beta_{\nu a} \gamma [2b_p^2 - e_s] b_{mn} \]

\[ s_2 = \beta_{\nu a} \gamma [b_{akp}^2 + \alpha\lambda + x_s] \]

\[ s_3 = \beta_{\nu a} c_{mn} + a_{mn} \left[ f_s [2b_p^2 - e_s] - [b_{akp}^2 + \alpha\lambda + x_s] \right] \]
The subject of this chapter is exclusively an elastic impact with no attendant energy loss. A quasi-static analysis is made in which the gradual increase of the contact area is taken into account at all stages of impact from beginning to the end. The quasi-static analysis assumptions have been fully confirmed experimentally by Goodier et al. [55] in the case of impact between a ball and a half-space and Conway et al. [18] for the impact of a steel ball on nylon layers.

5.1. Impact and Restitution

During the collision, the shells possess transient reaction force components, $Q_n$, the induced normal force, and $Q_f$, the induced frictional force. To be able to determine completely both the shallow shell deflection and the spherical shell deformation which in turn affects the behavior of the fluid, the transient reaction forces must be determined. These forces become maximum at a relatively small time after collision and decrease with time until separation occurs. The approach here, then, is the same as Phillips and Calvit [199], and Filippov [41]. The basic assumption here is that the relative deflection
D. of the impacted shallow shell is directly proportional to the force, \( \hat{Q}_n \) caused by the spherical shell.

\[
D = \lambda_0 \hat{Q}_n \tag{5.1.1}
\]

where

\[
\hat{Q}_n = \int_0^T Q_n \, dt \tag{5.1.2}
\]

\[
\lambda_0 = \frac{1}{16h_2} \sqrt{\frac{3}{\rho_2 E_2}}
\]

The spherical shell response, \( W_{sp}(\phi, \theta, \tau) \) and the shallow shell response, \( W_s(\psi, \Xi, \tau) \) are given by equations (3.3.68) and (4.3.1). The two equations are restated here:

\[
W_{sp}(\phi, \theta, \tau) = -\frac{2k_s(1 - \nu)a}{Ehs} \sum_{\ell=1}^\infty \frac{[j_0'(\Omega_{\ell})]^2}{d_{02}(\Omega_{\ell})} \int_0^\tau R(\xi) \sin[s\Omega_{\ell}(\tau - \xi)] \, d\xi + \\
\sum_{\ell=1}^\infty \sum_{n=1}^\infty \sum_{m=0}^\infty P^m_n(X) \frac{N_w(s\Omega_{n\ell})}{D_n(s\Omega_{n\ell})} \cos m\theta \int_0^\tau R(\xi) F_{N}(\xi), F_{L}(\xi) \right] \sin[s\Omega_{n\ell}(\tau - \xi)] \, d\xi
\tag{5.1.3}
\]

\[
W_s(\psi, \Xi, \tau) = \sum_{n=1}^\infty \sum_{m=0}^\infty P^m_n(X) \cos mn \{ \sum_{j=1}^5 W^{s}(\omega_{nj}) \int_0^\tau R(\xi) \sin[\omega_{nj}(\tau - \xi)] \, d\xi + \\
\sum_{j=1}^5 W^{s}(\omega_{nj}) \sin[\omega_{nj}\tau] \}
\tag{5.1.4}
\]

Let \( S_1 \) be the relative displacement between the shallow shell and the spherical shell radial deformations such that

\[
S_1 = W_s - W_{sp} \tag{5.1.5}
\]

and let \( S \) be the additional displacement of the center of the fluid shell system due to penetrations in both shells:

\[
S = Z - S_1 \tag{5.1.6}
\]
where $Z$ is the rigid body absolute displacement of the center of mass of the rigid fluid shell system.

If we define $S_c(\phi_c, \theta_c, \tau) = S_c(\phi_c, \theta_c)$ as the instantaneous area of contact in this context, we immediately assume the temporal behavior of $S_c(\phi_c, \theta_c)$. Hereafter, the subscript $c$ will designate contact between two shells. We shall also use $S_c$ for $S_c(\phi_c, \theta_c)$ and $S_{cm}$ for $S_c(\phi_{c_{max}}, \theta_{c_{max}})$. The mathematical value of $S_c$ is

$$S_c = \int_{\theta_c}^{\theta_c} \int_{\phi_c}^{\phi_c} (R \phi)(R \theta)$$

$$= a^2 \int_{\theta_c}^{\theta_c} \int_{\phi_c}^{\phi_c} \sin \phi \phi \theta$$

$$S_c = 2a^2 \theta \cos \phi_c$$

$R_n(\tau)$ is the normal stress at the contact, such that $Q(\tau)$ can be defined as

$$Q(\tau) = S_c R_n(\tau)$$

Hence, the first derivative of $S_1$ can also be defined as

$$\frac{dS_1}{d\tau} = S_c R_n(\tau)$$

from equation (5.1.6) and the second derivative is

$$\frac{d^2S_1}{d\tau^2} = S_c R_n$$

$S_c$ is a variable which depends on $\theta_c$ and $\phi_c$ which in turn are defined as

$$\theta_c = \theta_c(\theta)$$

$$\phi_c = \phi_c(\phi)$$
The rigid body acceleration of the spherical shell is \( \frac{d^2 Z}{d\tau^2} \) where

\[
\frac{d^2 Z}{d\tau^2} = - \frac{S \cdot R_n(\tau)}{M_0}
\]  

(5.1.12)

where \( M_0 \) is the total mass of fluid-shell system. If we invoke the Hertzian approach, the relations between the impacting force and the penetrations in the shallow shell and spherical shell are

\[
Q_n(S) = a_0 S^{3/2}
\]

or

\[
R_n(S) = a_0 S^{3/2} \quad \text{or} \quad \frac{S^{3/2}}{S_c}
\]

(5.1.13)

S has a temporal behavior also, so that

\[
Q_n(\tau) \equiv Q(S)
\]

\[
R_n(\tau) \equiv R_n(S)
\]

(5.1.14)

and

\[
a_0 = \frac{4a^2}{3} \left\{ E_1 E_2 / (E_1' + E_2') \right\}
\]

(5.1.15)

where \( E_1' = E_1/(1 - \nu_1^2) \) and \( E_2' = E_2/(1 - \nu_2^2) \). The rigid body acceleration after impact is

\[
\frac{d^2 Z}{d\tau^2} = - \frac{a_0}{M_0} S^{3/2}
\]

(5.1.16)

From equation (5.1.5)

\[
\frac{d^2 S}{d\tau^2} = \frac{d^2 Z}{d\tau^2} - \frac{d^2 S_1}{d\tau^2}
\]

(5.1.17)

Equation (5.1.17) is simplified a little after using equations (5.1.2) and (5.1.3) to eliminate \( Z, S_1 \), and we have the following results
\[
\frac{d^2 S}{dt^2} + \lambda_1 s^{3/2} = \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \int_0^\tau \left[ \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\xi} \sum_{m=0}^{\infty} \lambda_2 \int_0^{\xi} S^{3/2}(\xi) \sin[s_\Omega(\tau - \xi)] d\xi + \lambda_3 \int_0^{\tau} S^{3/2}(\xi) \sin[s_\Omega(\tau - \xi)] d\xi + \lambda_4 \int_0^{\tau} S^{3/2}(\xi) \sin[\omega_k(\tau - \xi)] d\xi + a_0^2 \frac{dS^{3/2}}{dt}
\]

\[
\sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \lambda_5 \int_0^{s_\Omega_n} \sin[s_\Omega(\tau - \xi)] d\xi - \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \lambda_6 \int_0^{s_\Omega_n} \sin[\omega_k(\tau - \xi)] d\xi - \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_0^{s_\Omega_n} \left( \sum_{\xi=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k) - \sum_{\xi=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}}.}}\right.\right]
\]

where

\[
\lambda_1 = a_0^2 \lambda_1 - \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k) - \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}}.}}\right.\right]
\]

\[
\lambda_2 = - \frac{2\alpha_0^2 \lambda_2}{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}}.}}\right.\right]
\]

\[
\lambda_3 = - \frac{\lambda_3}{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}}.}}\right.\right]
\]

\[
\lambda_4 = a_0^2 p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k) / \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{p_n^m(\xi) \cos[\theta(s_\Omega_n)] w_{nm}(\omega_k)}}.}}\right.\right]
\]

\[
\{0, F_N, F_L \}^T = (5.1.18)
\]

\[
\{0, F_N, F_L \}^T = (5.1.19)
\]

\[
\{0, F_N, F_L \}^T = (5.1.20)
\]

\[
\{0, F_N, F_L \}^T = (5.1.21)
\]

\[
\{0, F_N, F_L \}^T = (5.1.22)
\]
\[ \lambda_5 = a_0^m \cos \theta (s \Omega_n) N_n(s \Omega_n) / D_n(s \Omega_n) \]  
\[ \lambda_6 = a_0^m (X \cos \eta \Omega_{nn})(\omega_{nn})^2 \]  
and
\[ \mu = \cos \phi \]
\[ X = \cos \psi \]
\[ \eta = \pi \pi/2 \Xi_0 \]

Equation (5.1.18) is an integro-differential equation that must be solved to obtain the value of S. Since this is an exceedingly complex equation, a numerical technique for solving such equation must be used. Finite difference approach is used here to obtain the solutions. The important observation is that separation occurs as soon as S is negative and Q(\tau) or R(\tau) at this time, \( \tau_c \), becomes zero.

Relations Between the Angle used for the Spherical Shell and the Shallow Shell

From equation (5.1.7) area of contact on spherical shell, \( S_c \), must be the same area on the shallow shell. Hence
\[ a_c^2 \theta \cos \phi_c = a_2^2 \Xi \cos \psi \]
And from the above equation, \( \Xi \) is defined as
\[ \Xi = a_c^2 \cos \phi_c / a_2^2 \cos \psi \theta \]  
\[ \Xi = a_c^2 \cos \phi_c / a_2^2 \cos \psi \theta \]  
Elemental length along the circumference of the contact area on both shallow shell and the closed spherical shell are equal and hence
\[ a_\phi c = a_2 \psi c \]
In a functional form with $\phi_c$ as independent variable, the above equation is

$$\psi_c = \frac{a}{a_2} \phi_c$$

(5.1.27)

Of course, the same result is achieved when the role of $\xi_c$ and $\theta_c$ are interchanged with a corresponding interchange of the role of $\psi_c$ and $\phi_c$ in equations (5.1.26) and (5.1.27).

The area $S_c$ can also be expanded in terms of $P_n^m(\cos \phi) \cos m \theta$.

$$S_c = \sum_{n=1}^{\infty} \sum_{m=0}^{n} S_{cmn} P_n^m(\cos \phi) \cos m \theta$$

(5.1.28)

where

$$S_{cmn} = \frac{2a^2 \phi_c}{m} \sin m \theta - \frac{1}{m} (1 - \cos m \theta) \frac{2n + 1}{2} \frac{(n - m)!}{(n + m)!} \int_{-\phi_c}^{\phi_c} \cos p_n^m(\cos \phi) \, d\phi$$

(5.1.29)

Algorithm for Numerical Solution

The continuous function, $S(\xi)$ of equation (5.1.18) is discretized so that

$$\tau = \sum_{j=1}^{N} (\Delta \xi)_j$$

(5.1.30)

and its temporal argument

$$\tau_j = \sum_{k=1}^{j} (\Delta \xi)_k$$
We can have the Taylor's expansion of $S_j$ and $dS_j/d^2$ about $\xi = \xi_{j-1}$ and retain only first four terms or cubic part of the expansion of $S_j$ and the quadratic part of $dS_j/d^2$, thus

$$
S_j = S_{j-1} + \Delta \xi \frac{dS_{j-1}}{d\xi} + \frac{(\Delta \xi)^2}{2} \frac{d^2S_{j-1}}{d\xi^2} + \frac{(\Delta \xi)^3}{6} \frac{d^3S_{j-1}}{d\xi^3} \quad (5.1.31)
$$

and

$$
\frac{dS_j}{d\xi} = \frac{dS_{j-1}}{d\xi} + \Delta \xi \frac{d^2S_{j-1}}{d\xi^2} + \frac{(\Delta \xi)^2}{2} \frac{d^3S_{j-1}}{d\xi^3} \quad (5.1.32)
$$

Equation (5.1.18) is changed from a continuous form to a discrete form in such a manner that a closed solution of the stieltjes (or hereditary) integral is obtained

$$
\frac{d^2S_j}{d\xi^2} = -\lambda_1 S^{3/2} - \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} \lambda_2 \sum_{k=1}^{1} s_k^{3/2} \Delta \xi \sin(s\Omega_{nk}(\tau - \tau_k)) \Delta \xi - \\
\sum_{l=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \lambda_3 \sum_{k=1}^{1} s_k^{3/2} \Delta \xi \sin(s\Omega_{nk}(\tau - \tau_k)) - \\
5 \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \lambda_4 \sum_{t=1}^{1} s_t^{3/2} \Delta \xi \sin(\omega_k(\tau - \tau_k)) - \\
\sum_{l=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \lambda_5 \sum_{k=1}^{1} (s\Omega_{nk}) \{0, F_{Nk}, F_{Lk}\} \sin(s\Omega_{nk}(\tau - \tau_k)) \Delta \xi + \\
\sum_{l=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \lambda_6 \sum_{k=1}^{1} F_{nk} \sin(s\Omega_{nk}(\tau - \tau_k)) - \\
5 \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \lambda_6 \sin(\omega_{nk}) \frac{dS_j}{d\xi} - \lambda_0 \frac{dS_j}{d\xi} \quad (5.1.33)
$$

where $\lambda_1, ..., \lambda_6$ are defined in equations (5.1.19) through (5.1.24).
In order to evaluate the fourth term of equation (5.1.31), we must define the following

\[
\frac{d^3 s_j}{d\xi^3} = \left\{ \frac{d^2 s_j}{d\xi^2} - \frac{d^2 s_j}{d\xi^2} \right\} (\Delta \xi)_j \quad (5.1.34)
\]

Since the analysis is not self-starting, we need initial conditions at time \( t = 0 \).

Initial Displacement

\[ s_j(0) = s_0 = 0 \quad (5.1.35) \]

Initial Velocity

\[
\frac{ds_j(0)}{d\xi} = \hat{s}_j(0) = \frac{\dot{w}_{25}}{c_s} \quad (5.1.36)
\]

where \( \dot{w}_{25}/c_s \) is the non-dimensionalized rigid body velocity of the head obtained in Chapter Two.

Other Initial Conditions

\[
\frac{d^2 s_j(0)}{d\xi^2} = \frac{\ddot{w}_{25}a}{c_s^2} \quad (5.1.37)
\]

\[
\frac{d^3 s_j}{d\xi^3} = 0
\]

For compact computation, it is wise to normalize \( s_j \) by multiplying by a factor \( c_s/\dot{w}_{25} \) so that

\[
\hat{s}_j = \frac{c_s}{\dot{w}_{25}} s_j \quad (5.1.38)
\]
and equation (5.1.36) becomes
\[ \frac{d\hat{S}}{d\xi} = \hat{S}_j(0) = 1.0; \quad \frac{d^2\hat{S}}{d\xi^2} = \hat{S}(0) = \frac{\hat{\omega}_{25} a}{c_s} \]

Also, $S$ is replaced by $\hat{S}$ in the previous equations. All the $\lambda$'s, especially in equation (5.1.33) must be modified by the same factor.

When using digital computer, we must follow equations (5.1.30) through (5.1.38) in a reverse order to obtain the value of $S$. The computer quits computation as soon as $S$ is negative and we know that the end of impact or collision is reached and separation begins. The time at which separation occurs is the impact duration

\[ \tau_c = \sum_{j=1}^{N} (\Delta\xi)_j \] (5.1.39)

and the coefficient of restitution

\[ e = -\left[ \frac{d\hat{S}}{d\xi} + \frac{d\hat{S}_1}{d\xi} \right] \xi = \xi_c \] (5.1.40)

where of course $\tau_c$ and $\xi_c$ are identically equal and

\[ \frac{d\hat{S}_1}{d\xi} = \lambda_0 \hat{S}^{3/2} \]

\[ \lambda_0 = \frac{c_s}{\hat{\omega}_{25}} \]
CHAPTER 6

NUMERICAL RESULTS

6.1. Discussions and Numerical Results of the Mathematical Model of Vertebral Column Response to Collisions

The mathematical expressions obtained in the second chapter provide a realistic input data for the subsequent head impact analysis. Attention has been particularly focussed on a lap-belted occupant, of a moving vehicle, whose vertebral column structure is idealized by an ensemble of 25 rigid bones adjoined by twenty four viscoelastically deformable discs. The sacrum is assumed rigid enough that its boundary with the fifth lumbar vertebrae can be regarded as a "fixed boundary."

The extended anthropometric data and the quiescent conditions, Table 3, are used as the input data in solving the differential equations that predict the instantaneous body positions and the internal body forces after collisions. Displacement fields for 10 mph and 50 mph vehicle collisions are shown in Figures 18 and 19 respectively. The internal force distributions due to static body weight of the 10 mph and 50 mph vehicle occupants are separately shown in Figures 20 and 21. Figures 22 and 23 give the same internal force distribution at about 20 msecs after vehicle deceleration.
The bending moment which is mainly acting on the first few lumbar vertebra is probably due to the fixed end assumption. Large compressive axial force in Figure 23 can cause anterior lip type of fracture of the vertebra body. The distinct differences in the values of the internal forces of the vertebral column of the 10 mph and 50 mph vehicle occupants show that the body is subjected to excessive strains with increasing collision speed.

Since no head impact occurs in 10 mph vehicle collision, it is perhaps reasonable to assume that occupant in the vehicle is not hurt under normal circumstances.

6.2. Head Impact Phase

(a) Preliminary Remarks

The following ideal conditions are utilized in arriving at the numerical solutions: water has been chosen as material for brain. The skull cap is idealized by a thin, homogeneous, isotropic spherical shell. The shell data in Table 5 are also chosen to reflect the limitations of the applied theories: the thickness/radius ratio of the shell is at most 0.02 which is within the limit of thin shell theory and the calculated value of wave speed, $C_s$ (103,333 in/sec), is in a close agreement with the wave speed of 106,000 in/sec through the skull mentioned by Goldsmith's paper (52)

Muller's subroutine is used to obtain the roots of the frequency equations of group I and group II, in case of an empty spherical shell. The frequency spectra for these two groups are shown in Figures 15 and 16. In case of a fluid filled spherical shell, the frequency equation
becomes much more difficult because of the presence of spherical bessel functions. A special computer program is written to obtain the frequencies shown in Figure 17.

In order to obtain a convergent results, we have taken the triple summation

$$\sum_{\ell=0}^{\infty} \sum_{n=1}^{(10,15)} \sum_{m=0}^{(10,15)} \sum_{n=1}^{(10,15)} \sum_{m=0}^{(10,15)}$$

This essentially means that a total number of $(10)(10)(10!)$ and $(15)(15)(15!)$ single summations are undertaken for each values of the stress and fluid pressure respectively.

(b) Results

The mathematical expressions in Chapter 5 is programed to obtain the dynamic contact stress distributions. The penetration and the stress distribution plots are shown in Figure 24. Typical computer results are at the end of the computer program in Appendix C and also tabulated in Table 4. The part of the computer program which is designed to find the contact force distribution is capable of accepting or rejecting the cone angles defining the contact area: the computer accepts or rejects on the basis of the separation criterion during the iteration process.

This method of obtaining the dynamic contact stress distribution is regarded as "direct contact" approach in the summary of survey of previous work in the field of head injury modelling in Chapter One. Once the proper distribution of the dynamic contact stresses are found then the second phase of the analysis, which involves the determination
of stresses in the impacting shell and pressures in the fluid, is performed.

Beyond Figure 24, the temporal definitions of all the encircled numbers are given in Table 6. During the course of the analysis of the impact process in the previous chapter, the coordinate system is carefully chosen to utilize the advantage of $\theta = 0$ symmetry plane. The resultants of the shearing stresses induced at both contact areas and the head-neck junction, and the equivalent bending moment at the same junction are all assumed to lie on this plane. Consequently, stresses induced on $\phi$-direction are larger than those stresses in $\theta$-direction.

Representative results are summarized in Figures 25 through 30 and their explanations are:

1. In Figure 25, each curve shows all possibly existing numerically maximum pressure in the fluid enclosed by the spherical shell at a specified time. The maximum negative pressure occurs at $\phi = 90^\circ$ and time, $t = 17.4$ \(\mu\)secs which corresponds to the occurrence of the peak of dynamic contact stress.

2. Figure 26 shows pressure wave distribution for $\theta = 10^\circ$, and time $t = 2.9$ \(\mu\)secs. Each curve gives the picture of the pressure distribution as seen by an observer riding from $\phi = 0^\circ$ to $\phi = 180^\circ$ along a constant circle prescribed by the specified radius given in terms of the fraction of the spherical shell radius. In other words, each curve in Figure 26 represents a time frozen picture of the pressure distribution along a circle defined by a constant radius.

3. The pressure wave propagation along the diameter defined by the coordinates (90,60) and (270,240) is shown in Figure 27. This
particular plot is presented because the maximum negative pressure seen in Figure 25 occurs at a location on this diameter specified by $r = -0.4$. The pressure distribution corresponding time $t$ is omitted for the sake of clarity. The Figures 25 through 27 enable us to locate the position of maximum negative pressure (rarefaction pressure) which promotes cavitation activities that can lead to a serious brain damage.

4. A typical stress field in the dynamic contact region is shown in Figure 28 for $\phi = 1^\circ$. In this region the shearing stresses are small, but a close study of the overall results reflects that the shearing stresses increase in magnitude to the same significant order as the normal stresses which themselves have a little growth towards the head-neck junction. The large growth and concentration of the shearing stresses is partially due to the inability of the fluid to transmit the shear waves.

5. The plots of midsurface normal stresses, $\sigma_{\phi\phi}(\phi,30)$ and $\sigma_{\theta\theta}(\phi,30)$ against $\phi$ are shown in Figures 29 and 30. In these two figures the circle defined by $r$ represents zero stress level; tensile stresses are plotted inside and compressive stresses outside of the circle. The maximum tensile and compressive stresses are located along the plane, $\phi = 0^\circ$. Also from a close study of all the results, a conclusion can be drawn that the damages occurring in the skull lie on the skull segment described by the coordinates $(\pm 90^\circ, \pm 90^\circ)$. But when the buffer action of the skull sutures is considered, that is, allowing a reasonable reduction in the obtained values of the normal stresses for effect of sutures, the location of the skull damage can in effect be reduced to the skull segment described by the coordinates $(\pm 30^\circ, \pm 90^\circ)$. 
automobile speed before collision is 10 mph

Instantaneous poises in an automobile accident simulated by -20g acceleration impulse of 25msecs. duration

Figure 18. Instantaneous poises in an automobile accident simulated by -20g acceleration impulse of 25 msecs. duration.
Figure 19

Instantaneous Body Poses in an Automobile Accident Which is Simulated by a - 20g Acceleration Impulse of 25m secs Duration
Figure 20. Internal forces of the vertebral column at time, t = 0 msecs.
$v_0 = 50\text{mph}$

-20g acceleration

Impulse of 25msecs duration

Figure 21 Internal forces of the vertebra at t=0msec

**Diagram Details:***

- **Axial Force ($F_a$)** and **Shear Force ($F_s$)** are plotted against time.
- **Bending Moment ($F_m$)** is also plotted.
- The graph shows the forces acting on the vertebra, with annotations indicating the forces and moments at various points.
- The head-neck junction and the height of the vertebral column above lumbar (L5) are marked on the graph.
Figure 22. Internal forces of the vertebral column at time, t = 20 μsecs.
Figure 23. Internal forces of the vertebral column at time, $t = 20.1$ msecs.
FIGURE 24 CONTACT FORCE DISTRIBUTION AND PENETRATION

TIME (µsecs.)

1. DYNAMIC CONTACT FORCE (lbs x 10^2)
2. DYNAMIC CONTACT STRESS (lbs/in^2 x 10^2)
3. PENETRATION (in x 10^-3)
FIGURE 25  MAXIMUM PRESSURE DISTRIBUTION
FIGURE 26 PRESSURE WAVES ALONG THE MERIDIAN

FOR
TIME = 2.9\mu\text{secs}
ANGLE \Theta = 10^\circ
Figure 2.7 Maximum pressure variation with time at $(\theta, \phi) = [(90, 60), (270, 240)]$.
Figure 28: Stress field at $\phi = 1^\circ$ in contact region.
FIGURE 29 NORMAL STRESS, $\sigma_{\theta \theta}$ AT VARIOUS TIME
FIGURE 30  NORMAL STRESS $\sigma_{00}$ ($\Phi, 30$) AT VARIOUS TIME
6.3 Conclusion and Recommendation for Future Research

In view of the overall results of this analysis, the following conclusions can be drawn:

1. When we apply the cavitation criterion, the brain damage as shown by the pressure plots in Figures 25, 26, and 27, occurs at a distance of $0.4 \times$ (model radius) from the geometric center of the model towards the counter coup, where rarefaction or maximum negative pressure occurs. This result agrees with that obtained by Engin (27).

2. There is a strong possibility of skull fracture in the area bounded by $(\pm 30^\circ, \pm 90^\circ)$ coordinates. The tensile stresses obtained in this region are well above the tensile strength of the skull bone.

3. The maximum value of the dynamic contact stress between the fluid-filled spherical shell and the target shell is about 2400 psi for an impact velocity of 500 in/sec.

4. For an impact velocity of 500 in/sec., duration of impact between the fluid-filled spherical shell and the value of the maximum penetration is 0.00626 in. occurring at time, $t=15.96\mu$ secs.

In any future research to improve on this present work, the following ideas may be incorporated:

1. Timoshenko Beam theory can be used for the intervertebral disc model instead of the Euler-Bernoulli beam theory used in Chapter two. The order of the differential equations for each intervertebral disc will naturally increase, thus adding more complexity to the numerical analysis. This improvement is very necessary since the thickness of the intervertebral disc is much smaller than its cross-section. This property qualifies the disc as a 'thick beam'.
3. In order to study the significance of the shear stresses between shell-fluid interface a viscous fluid should be considered instead of the inviscid fluid used here for the brain. Alternatively, the brain can be regarded as a jelly substance that can be modeled by a viscoelastic material.

4. The skull model can be made of composite layers to improve on the "homogeneity" assumption previously made.

5. Inclusion of plastic wave theory to the elastic wave approach for the dynamic shell impact will, no doubt, increase the accuracy of the obtained results.
BIBLIOGRAPHY


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(33) Engin, A.E., "Intracranial Pressures During Head Impact", Digest of the 10th International Conference on Medical and Biological Engineering, Paper No. 4-16, p. 73, 1973.


(38) Evans, F.G., Stress and Strain in Bones, Thomas Springfield, Ill. 1957.


(45) "Foreign Object Impact Damage to Composites", American Society For Testing and Materials, STP 568.


(100) Huang, H., and Wang, Y.F., "Early-Time Interaction of Spherical Acoustic Waves and a Cylindrical Elastic Shell".


APPENDIX A

A-1. Viscoelastic Stress-Strain Relation

For elastic material, the Hooke's Law relating stress to strain can be expressed in the notation of Timoshenko et al. [250] as:

\[
\begin{align*}
\sigma_x &= \lambda \epsilon_x + 2\mu \epsilon_x \\
\sigma_y &= \lambda \epsilon_y + 2\mu \epsilon_y \\
\sigma_z &= \lambda \epsilon_z + 2\mu \epsilon_z \\
\tau_{xy} &= \nu \sigma_{xy} \\
\tau_{yz} &= \nu \sigma_{yz} \\
\tau_{zx} &= \nu \sigma_{zx}
\end{align*}
\]

where

\[
\epsilon = \epsilon_x + \epsilon_y + \epsilon_z
\]

\(\lambda, \mu\) are the Lame elastic constants

\[
\lambda = E\nu/[(1 + \nu)(1 - 2\nu)]
\]

\[
\mu = E/[2(1 + \nu)]
\]

The viscoelastic constitutive equations are obtained by applying the Biot's form of correspondence principle, replacing the elastic constant \(\lambda\) and \(\mu\) by the viscoelastic operators \(R\) and \(Q\) respectively. These operators are defined as follows.
where summation may be finite or infinite and the coefficient $p_i$, $q_i$, $r_i$ and $s_i$ are determined only by material tests. Equations (A-1.1) are given in a most general form

$$f_i(\sigma_i, \dot{\sigma}_i, \gamma_i, \ldots) = A_{ij}g_j(\varepsilon_i, \dot{\varepsilon}_i, \varepsilon_i, \ldots)$$

(A-1.3)

For a direct use here, the following properties are imposed on the viscoelastic constitutive law:

(i) Whenever a subscript $i$ of (A-1.2) is zero,

$$R = q_0/p_0 = \lambda$$

$$Q = s_0/r_0 = \mu$$

(A-1.4)

equation (A-1.1) must be recovered for the elastic material constitutive relation.

(ii) Stress must be a function of strain and strain rate only, that is, the Kelvin-type of materials. The immediate consequence of the assertion is that the thermoelastic effect that could lead to a very bad non-linear relation is eliminated. Thus

$$\sigma = g_1(\varepsilon, \dot{\varepsilon})$$

$$\tau = g_2(\gamma, \dot{\gamma})$$

(A-1.5)

These two equations are satisfied only if equation (A-1.2) is of the form
\[ R = \left[ \lambda + \eta_1 \frac{\partial}{\partial t} \right] \]
\[ Q = \left[ \mu + \eta_2 \frac{\partial}{\partial t} \right] \]

The constitutive equations for this kind of viscoelastic materials are

\[ \sigma_x = \lambda \varepsilon + 2\mu \varepsilon_x + \eta_1 \dot{\varepsilon} + 2\eta_2 \dot{\varepsilon}_x \]
\[ \tau_{xy} = \mu \gamma_{xy} + \eta_2 \dot{\gamma}_{xy} \]  \hspace{1cm} (A-1.7)

\[ etc. \]

Assumption Made

1. Material properties are isotropic, otherwise \( \lambda, \mu, \eta_1, \eta_2 \) will have different values along any direction, such as \( x, y, z \).

2. **Incompressibility.** For small strains, the incompressibility of a material is equivalent to assuming an infinite bulk modulus, or a poisson's ratio of one-half.

\[ R = \text{indefinite} \]
\[ Q = E + \eta_1 \frac{\partial}{\partial t} \]

Consequently, the normal stresses \( \sigma_x, \sigma_y, \sigma_z \) are indeterminate, and all deformations are the result of shear deformations, and

\[ \tau = G \gamma + \eta_2 \dot{\gamma} \]

3. **Elastic behavior under hydrostatic pressure.** This permits a finite bulk modulus that is not time dependent; even in practice, if the bulk modulus is slightly time dependent some sort of "averaging value" would be used as a constant. The bulk modulus \( k \) is defined as

\[ K = R + \frac{2}{3} Q \]
\[ = (K)_{\text{elastic}} + (\eta_1 + \frac{2}{3} \eta_2) \frac{\partial}{\partial t} \]
A time dependent bulk modulus requires that

\[ n_2 = - \frac{3}{2} n_1 = n \]

Consequently

\[ Q = \mu + n \frac{\partial}{\partial t} \]

\[ R = (K)_{\text{elastic}} - \frac{2}{3} Q \]  

(A-1.8)

\[ \Gamma_{ij} = \frac{1}{g_{ii}} \frac{\partial}{\partial \alpha^i} \sqrt[3]{g_{ii}} \]

(A-2.3)

A-2. Small Displacement Theory in Orthogonal Curvilinear Coordinates

Some of the applicable results of small displacement theory [262] are only stated here. Length of an arc in the curvilinear coordinate \( \alpha^1, \alpha^2 \) is

\[ (ds)^2 = g_{11}(d\alpha^1)^2 + g_{22}(d\alpha^2)^2 + g_{33}(d\alpha^3)^2 \]

(A-2.1)

Let \( g = \det(g_{ij}) \), and since \( g_{ij} = 0 \) for \( i \neq j \)

\[ g = g_{11}g_{22}g_{33} \]

(A-2.2)

The Christoffel symbols is reduced to

\[ \Gamma_{ij}^{\alpha} = \frac{1}{g_{ii}} \frac{\partial}{\partial \alpha^i} \sqrt[3]{g_{ii}} \]

\[ \Gamma_{ij} = \Gamma_{ji} = \frac{1}{g_{ii}} \frac{\partial}{\partial \alpha^j} \sqrt[3]{g_{ii}} \]

\[ \Gamma_{ij} = - \frac{\sqrt[3]{g_{ii}}}{g_{ii}} \frac{\partial}{\partial \alpha^i} \frac{\partial}{\partial \alpha^j} \]

\[ \Gamma_{ijk} = 0 \quad \text{for } i, j, k \text{ all different} \]

\[ \Gamma_{ij} = 0 \quad \text{for } i, j, k \text{ all different} \]
Linear Strain-Displacement Relations

\[ 2f_{i1} = 2g_{i1} \frac{3v_i}{\partial \alpha_i} + \sum_{j=1}^{3} v_j \frac{3g_{ij}}{\partial \alpha_j} \quad \text{for } i \neq j \quad (A-2.4) \]

\[ 2f_{ij} = g_{i1} \frac{3v_i}{\partial \alpha_j} + g_{jj} \frac{3v_j}{\partial \alpha_i} \]

Components of the displacements and body forces may be alternatively defined with respect to the local Cartesian coordinates, \( \hat{e} \):

\[ g_{i} = \sqrt{g_{ii}} \hat{e}_i \]

\[ u = \sum_{i=1}^{3} u^i \hat{e}_i \]

\[ p = \sum_{i=1}^{3} u^i \hat{e}_i \quad \text{(body force)} \quad (A-2.5) \]

\[ u^i = \sqrt{g_{ii}} v^i \]

\[ \varepsilon_{ij} = \frac{f_{ij}}{\sqrt{g_{ii}g_{jj}}} \]

\[ \sigma_{ij} = \sqrt{g_{ii}g_{jj}} \tau_{ij} \]

where \((u^i, \varepsilon_{ij}, \sigma_{ij})\) are displacements, strains, and stresses along the local coordinates and \((v^i, f_{ij}, \tau_{ij})\) are curvilinear displacement, strain, and stresses.

The strains can be written in terms of local displacement

\[ \varepsilon_{ii} = \frac{1}{2g_{ii}} \left[ 2g_{ii} \frac{3}{\partial \alpha_i} \left( \frac{u^i}{\sqrt{g_{ii}}} \right) + \sum_{j=1}^{3} \frac{3g_{ij}}{\partial \alpha_j} \frac{u^j}{\sqrt{g_{jj}}} \right] \quad i = j \quad (A-2.6) \]

and
\[ \varepsilon_{ij} = \frac{1}{2\sqrt{g_{ii}g_{jj}}} \left[ g_{ii} \frac{\partial^2}{\partial x^i \partial x^j} (u^1) + g_{jj} \frac{\partial^2}{\partial x^j \partial x^i} (u^1) \right] \quad i \neq j \quad (A-2.6) \]

A-3. Geometry of Shell

Position vectors defining the points \( P_0, Q_0, P, Q \) are functions of curvilinear space variables \( \alpha \) and \( \beta \)

\[
\overrightarrow{OP_0} = \vec{p}_0(\alpha, \beta)
\]
\[
\overrightarrow{OQ_0} = \vec{q}_0(\alpha, \beta)
\]
\[
\overrightarrow{OP} = \vec{p}(\alpha, \beta)
\]
\[
\overrightarrow{OQ} = \vec{q}(\alpha, \beta)
\]

(A-3.1)

Coordinates \( \alpha \) and \( \beta \) coincide with the lines of curvature of the mid-surface and the unit vectors along these directions are respectively \( \hat{u}_\alpha \) and \( \hat{u}_\beta \) which are defined thus

\[
\hat{u}_\alpha = \frac{1}{A} \frac{\partial \vec{p}_0}{\partial \alpha}
\]
\[
\hat{u}_\beta = \frac{1}{B} \frac{\partial \vec{p}_0}{\partial \beta}
\]

where

\[
A^2 = \frac{\partial \vec{p}_0}{\partial \alpha} \frac{\partial \vec{p}_0}{\partial \alpha}
\]

and

\[
B^2 = \frac{\partial \vec{p}_0}{\partial \beta} \frac{\partial \vec{p}_0}{\partial \beta}
\]

(A-3.2)

The length, \( ds_0 \), of a line element between two neighboring points whose coordinates are \((\alpha, \beta)\) and \((\alpha + d\alpha, \beta + d\beta)\) is given by

\[
(ds_0)^2 = d\vec{p}_0 \cdot d\vec{p}_0 = A^2(d\alpha)^2 + B(d\beta)^2
\]

(A-3.3)

Let \( \hat{n}_0 \) be the unit vector perpendicular to the mid-surface of the shell, such that \( \hat{u}_\alpha, \hat{u}_\beta \) and \( \hat{n}_0 \) are mutually orthogonal
Figure 31. Geometry of shell before and after deformation.
\[ \hat{n}_0 = \hat{u}_\alpha \times \hat{u}_\beta \]

(A-3.4)

Let also \( R_\alpha \) and \( R_\beta \) be the radii of curvature along \( \alpha \) and \( \beta \) respectively.

The geometry of the midsurface gives rise to the following matrix [108]

\[
\frac{\partial}{\partial \alpha} \begin{bmatrix} \hat{u}_\alpha \\ \hat{u}_\beta \\ \hat{n}_0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{B} \frac{\partial A}{\partial \beta} & \frac{A}{R_\alpha} \\ \frac{1}{B} \frac{\partial A}{\partial \beta} & 0 & 0 \\ -\frac{A}{R_\alpha} & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{u}_\alpha \\ \hat{u}_\beta \\ \hat{n}_0 \end{bmatrix}
\]

\[
\frac{\partial}{\partial \beta} \begin{bmatrix} \hat{u}_\alpha \\ \hat{u}_\beta \\ \hat{n}_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \frac{\partial B}{\partial \alpha} & 0 \\ -1 \frac{\partial B}{\partial \alpha} & 0 & \frac{B}{R_\beta} \\ 0 & -\frac{B}{R_\beta} & 0 \end{bmatrix} \begin{bmatrix} \hat{u}_\alpha \\ \hat{u}_\beta \\ \hat{n}_0 \end{bmatrix}
\]

By using the principle of partial differentiation in Euclidean surfaces, the following identities are obtained

\[
\frac{\partial^2 \hat{u}_\alpha}{\partial \alpha \partial \beta} = \frac{\partial^2 \hat{u}_\alpha}{\partial \beta \partial \alpha}
\]

\[
\frac{\partial^2 \hat{u}_\beta}{\partial \alpha \partial \beta} = \frac{\partial^2 \hat{u}_\beta}{\partial \beta \partial \alpha}
\]

(A-3.6)

\[
\frac{\partial^2 \hat{n}_0}{\partial \alpha \partial \beta} = \frac{\partial^2 \hat{n}_0}{\partial \beta \partial \alpha}
\]

When equations (A-3.5) are used in equations (A-3.6), these three identities lead to the following relations:
The first two of equation (A-3.7) is called Codazzi conditions and the last equation, the Gauss condition. Let \(Q_0\) be any point which is not on the midsurface whose position is defined by

\[
\bar{Q}_0 = \bar{q}_0 = \bar{p}_0 + \xi \bar{n}_0 \quad (A-3.8)
\]

where \(\xi\) is the distance of the point from the midsurface of the shell. From (A-3.8) it is obvious that any point in the shell before deformation can be specified by the set of orthogonal curvilinear coordinates \((\alpha, \beta, \xi)\).

Let \((\bar{e}_1, \bar{e}_2, \bar{e}_3)\) be the local base vectors such that

\[
\bar{e}_1 = \frac{\partial \bar{p}}{\partial \alpha} = A(1 + \frac{\xi}{R_\alpha}) \hat{u}_\alpha
\]

\[
\bar{e}_2 = \frac{\partial \bar{p}}{\partial \beta} = B(1 + \frac{\xi}{R_\beta}) \hat{u}_\beta
\]

\[
\bar{e}_3 = \frac{\partial \bar{p}}{\partial \xi} = \hat{n}_0
\]

where \(\hat{u}(\xi), \hat{u}(\xi), \hat{n}(\xi)\) are unit vectors at any point in the shell, \(\xi\), away from the midsurface

\[
P_0 = P_0(\alpha, \beta), \quad Q_0 = Q_0(\alpha + d\alpha, \beta + d\beta)
\]

\[
P = P(\alpha + u, \beta + u), \quad Q = Q(\alpha + u + d(u + \alpha), \beta + u + d(u + \beta))
\]

\[
\bar{PQ} = \bar{dP}
\]
\[ d\vec{p} = \vec{p}_{0,\alpha} d\alpha + \vec{p}_{0,\beta} d\beta + \vec{p}_{0,\xi} d\xi. \]

\[ = A(1 + \frac{\xi}{R\alpha}) \hat{u}(\xi) d\alpha + B(1 + \frac{\xi}{R\beta}) \hat{u}(\xi) d\beta + \hat{n}(\xi) d\xi \]

\[(ds)^2 = d\vec{p}_0d\vec{p} \]

\[ = \sum_{k,j=1}^{3} g_{jk} d\alpha^j d\alpha^k \quad \text{(A-3.10)} \]

where

\[ \alpha^1 = \alpha \]

\[ \alpha^2 = \beta \]

\[ \alpha^3 = \xi \]

\[ g_{11} = A^2(1 + \frac{\xi}{R\alpha})^2 \]

\[ g_{22} = B^2(1 + \frac{\xi}{R\beta})^2 \quad \text{(A-3.11)} \]

\[ g_{33} = 1 \]

\[ g_{jk} = 0 \quad j \neq k \]

Equation (A-3.10) can be used in equation (A-3.6) to obtain the strain field.
APPENDIX B

TABLE 2

BREAKING STRENGTH OF VERTEBRAE (RUFF),* Kgs

<table>
<thead>
<tr>
<th>Vertebra</th>
<th>19</th>
<th>21</th>
<th>21</th>
<th>23</th>
<th>33</th>
<th>36</th>
<th>38</th>
<th>43</th>
<th>44</th>
<th>46</th>
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</thead>
<tbody>
<tr>
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<td>640</td>
<td>540</td>
<td>600</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Th9</td>
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<td>720</td>
<td>700</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Th10</td>
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<td>770</td>
<td>730</td>
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<td></td>
</tr>
<tr>
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<td>755</td>
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<td></td>
<td></td>
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<tr>
<td>Th12</td>
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<td>800</td>
<td></td>
<td></td>
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<td></td>
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</tr>
<tr>
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<td>900</td>
<td>800</td>
<td>800</td>
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<td></td>
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</tr>
<tr>
<td>L2</td>
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<tr>
<td>L4</td>
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<td>950</td>
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<td></td>
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</tr>
<tr>
<td>L5</td>
<td>1020</td>
<td></td>
<td>1000</td>
<td>1200</td>
<td></td>
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<table>
<thead>
<tr>
<th>Vertebral Level</th>
<th>Disc Area</th>
<th>Disc Height</th>
<th>Area Moment of Inertia</th>
<th>Vertebra Height</th>
<th>Mass Ecc. $e_i^*$</th>
<th>Init. Config. $u_i^*$</th>
<th>$w_i^*$</th>
<th>Mass</th>
<th>Rotatory Inertia</th>
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<td>.306</td>
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<td>1.50</td>
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<td>3.00**</td>
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<td>.117</td>
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<td>1.04</td>
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<td>0.20</td>
<td>9.15</td>
<td>.0117</td>
<td>.117</td>
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<tr>
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<td>.119</td>
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<td>-0.20</td>
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TABLE 3--Continued

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<th>Disc Area</th>
<th>Disc Height</th>
<th>Area Moment of Inertia</th>
<th>Vertebra Height</th>
<th>Mass Ecc.</th>
<th>Init. Config.</th>
<th>Rotatory Inertia</th>
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</table>

*For "uniaxial" response: set $u_i^0$, $e_i$ and pelvic angle $\psi$, equal to zero.

†Disc data are associated with vertebra immediately above, in this case, L5.

**Arbitrary datum.

\[ W(1) = 2.7686 \quad \text{U}(1) = -0.2758 \]

<table>
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<td>z Coord.</td>
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<td>Slope</td>
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<td>Initial</td>
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<td>Length</td>
<td>Slope</td>
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### TABLE 3--Continued

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</thead>
<tbody>
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<td><strong>x Coord.</strong></td>
<td><strong>z Coord.</strong></td>
<td><strong>Length</strong></td>
<td><strong>Slope</strong></td>
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</tr>
<tr>
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Initial Conditions at Time = 0.0, with Initial Stepsize = 0.00100

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<td><strong>z Coord.</strong></td>
<td><strong>Length</strong></td>
<td><strong>Slope</strong></td>
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<tr>
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### TABLE 5

**DATA FOR THE SHELLS' NUMERICAL COMPUTATION**

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<th>Spherical Shell</th>
<th>Shallow Shell</th>
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<td>0.888 lbm/in³</td>
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<td>Modulus of elasticity</td>
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<td>4.7 x 10⁶ lbf/in²</td>
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<td>Lamé constant, ν</td>
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<td>Radius a</td>
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<td>Thickness of shell</td>
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<td>0.30 in</td>
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<td>Fluid density</td>
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<td>Wave speed in fluid</td>
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<td>Value</td>
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APPENDIX C

COMPUTER PROGRAM FOR THE IMPACT ANALYSIS

The summary and explanation of the program are as follows:

Main Program

1. Representation of Hereditary integral for the stress distribution.

2. Search for the contact force distribution (stress distribution) by solving the differential equation involving the penetration of the closed spherical shell and the shallow shell into each other.

   The stress distribution search is done in two ways: (1) a program with only time variation for different contact area. This assumes a constant contact area and (2) a program in which time and the two coordinates ($\phi, \theta$) of the contact area are simultaneously varied.

3. (a) Computation of the five modes variables $W$, $H$, $G$, $\Lambda$ and $\Psi$: $W$, $H$ and $\Lambda$ are for linear displacements whereas $G$ and $\Psi$ are for angular displacements.

   (b) Computation of the fluid pressure; skull stresses ($\sigma_\phi, \sigma_\theta, \tau_{\phi\theta}, \tau_{\phi z}, \tau_{\theta z}$) and strain field ($\epsilon_0, \epsilon_\phi, \gamma_{\phi\theta}, \gamma_{\phi z}, \gamma_{\theta z}$) for every 30° increments in $\phi$ direction and 10° increments in $\theta$ directions.
Subroutine Root 1

In this subroutine, the frequency equations are solved to obtain the frequency spectra needed for the computation of the amplitude of the vibration of the spherical shell.

The subroutine also makes use of the following subroutines designed for the generation of the spherical Bessel functions: subroutines FUNC, BSLRA, BESSEL, SERIES, BRECUR and FRECUR.

Subroutine Root 2

Subroutine Root 2 is the only place where the frequency spectral for the shallow shell is obtained. The amplitudes of the vibration are also computed for use in the search for stress distribution in the main program.

Subroutine B4

This subroutine merely simplifies the individual calling of the following subroutines in the main program:

Subroutine B2—compute derivative of group I frequency equation
Subroutine B3—compute derivative of group II frequency equation
Subroutine BFUN—compute $j_n(r\Omega)$,

$$k_1 + f_s \frac{j_n(\Omega)}{\Omega j_n'(\Omega)}$$

and derivatives, for preparation of pressure computation in the main program

Subroutine Group 1—amplitude of spherical shell vibration is computed

Subroutine Group 1 calls subroutines CMN, ASLEG1, ASLEGD
Subroutine CMN calls subroutine ASLEGI

Subroutine ASLEGI calls subroutine ASLEGI

Subroutine ASLEGD calls subroutine ASLEGI

Subroutine ASLEGI calls subroutine FCTRL

Subroutine ASLEGI—generate the spherical Legendre polynomials

\[ P_n^m(\cos \phi), \quad P_n^m(\cos \phi) \cos \theta, \quad P_n^m(\cos \phi) \sin \theta \]

Subroutine ASLEGI—generate the integral of spherical Legendre polynomials

\[ \int_{-\alpha}^{\alpha} P_n^m(\cos \phi) d\phi \]

Subroutine CMN—\[ \int_{-\theta}^{\theta} \int_{-\phi}^{\phi} P_n^m(\cos \phi) \cos \theta d\phi d\theta, \]

\[ \int_{\pi/2-\theta}^{\pi/2} \int_{-\phi}^{\phi} P_n^m(\cos \phi) \cos \theta d\phi d\theta \]

Subroutine ASLEGD—\[ \frac{d P_n^m}{d \phi}(\cos \phi), \quad \frac{d^2 P_n^m}{d \phi^2}(\cos \phi), \quad \frac{d P_n^m}{d \phi}(\cos \phi) \cos \theta, \]

\[ \frac{d^2 P_n^m}{d \phi^2}(\cos \phi) \cos \theta \]

Subroutine FCTRL—generates the factorial of any supplied number

Subroutine DDET3 and DDET5 used in the main program are available in the Fortran Library. DDET3 and DDET5 are double precision subroutines used to obtain first and second derivatives of tabulated values using three points and five points derivative, a combination of forward and backward finite difference.

Subroutine Muller is used in both subroutine Root 1 and Root 2 to obtain the complex frequencies. This subroutine is also available in the Fortran Library.
Main Program

Sub. Root 1

①

Sub. Root 1

Sub. FUNC  Sub. BSLRA  BESSEL  Sub. SERIES  Sub. FRECUR

Sub. BRECUR

②

Sub. Root 2

Sub. FCTRL  Sub. ASLEG1  Sub. ASLEG1

④

Sub. Group 1

Sub. CMN

Sub. ASLEG1  Sub. ASLEG1  Sub. FCTRL

③

Sub. B4

Since this program requires a lot of computer storage more than what is available, an overlay structure is used. The overlay grouping are

1. B4, CMN, Group 1, ASLGD, BFUNC, B2, B3
2. Root 1, FUNC, BSLRA, BESSEL, SERIES, BRECUR, FRECUR
3. Root 2
4. PRPLOT, Y03PCK

This means that only main program, subroutines ASLEG1, ASLEG2 and FCTRL can communicate with any of the groups. No one group can communicate with the other group since they are temporarily out of the computer's memory.
DIMENSION P1(20), P2(20), P3(20), P4(20), P5(20), P6(20)
EQUIVALENCE (Y1, CCM1), (Y2, CFE1), (Y3, CFF1), (Y4, CFFTA1)
EQUIVALENCE (Y5, CCMF11), (Y6, CMTA1), (Y7, CFF1), (Y8, CFF1)
EQUIVALENCE (Y10, HFI1), (Y11, HFI1), (Y12, HFI1), (Y13, HFI1)
EQUIVALENCE (Y14, WTA11), (Y15, GF11), (Y16, GTA11), (Y17, HFI1)
EQUIVALENCE (Y18, HTAS11), (Y19, HFI1), (Y20, HFI11), (Y21, GF1A1)
EQUIVALENCE (JK, KJ)
PRINT 6000
WRITE(10,3000)
WRITE(10,7000)
WRITE(10,16)
WRITE(11,12000)
WRITE(11,13000)
WRITE(11,17)
WRITE(12,14004)
WRITE(12,15000)
WRITE(12,711)
WRITE(12,1150)
CAGA=0.1
FE=0.000
XNT=0
KOUNT=0
DEL1=5.0-06
TAUL=0.0
TAU=0.00
UT=1.0-04
D1W25=4.9999991
D2W25=9.1599457
FNE=1.7478600
FL=0.1216000
DF4=FN
DFL=FL
DO 1007 J=1,3
DNW11=1.0 J=0.00
DNW11=1.0 J=0.00
1007 CONTINUE
READ 2,A2,H2,V3,K52,RHD02,E2
PRINT 1,A2,H2,V3,K52,RHD02,E2
READ15=SI1,V1,AL1,HI1,CF1,HRDF,RHD01
PRINT 5,SI1,V1,AL1,HI1,CF1,HRDF,RHD01
READ15=101,EP,EPZ
CALL UNDORF
CALL ROOT1
E11=EI1-(1-E11*42)
E21=EI2-(1-E21*42)
E32=EI1+EI2*3,
EE=EI2*E21
AA=AI1
G=386.4
LAMD0=9.99EE/E32
E1=EI1+0.5/(1-E11)
P1=3.14159265400
DFE=E1/180
DTA=P1/180
P1=P1/180
FL=FL+5.0/17.0
GRAVITY=G
WEIGHT=H30S1+HOF1+4.*PI2+10*3.3
MASS=HEIGHT/GRAVITY
M0=M110D2W29
A31+23=FN
A31=31=FL
A5(1,2)=0.00
A5(1,3)=0.00
ALP=3.0*R0S2*21
ALPP=1.6*R0S2*21
ALPH0=DSQR(A5P)/ALPP
FEN=PI/4.*
ATN=PI/4.*
A1=DSQR(A1)
K0=DEEG1E01/E32
DK0=A11+6.*DO/E32
BAK0=K0*DK0
CTAS0=PI/12.*
CFEE0=PI/8.*
FEE=6.0DOP1/180.00
TAS=6.0DOP1/180.00
FEE=6.00
TAS=6.00
FEE=PI/20.*
TAS=PI/20.*
FEE=PI/36.*DO
TAS=PI/36.*DO
DEL=DELI*CS/A1
200 CONTINUE
TAS=PI/6.*
TAU=PI/6.*
TAU0=0.00
TAU=PI/4.*
TAU=0.00
XX(I)=TAU
TIME=TAU
C GO TO 40
FEE=FE0+DFEE
SIC=PI/2.*
IF(FEE=EQ=SIC) GO TO 200
TAS=CTASKS
TA=CTASKS/PI
FE=FEE/180.00
C IF(TA=LT.1.00)OR(FE LT.0.00) STOP
Z=DCOU1(FEE)
IF(FEE EQ.0.00) GO TO 200
CFEE=A1+FE0/A2
CFEE1=CFEE+PI/0.5/CFEE0
X=DCOS(CFEE1)
FB=CFEE1/2.*
BFE=DCOS(FB)
C T=1/AI/AI.02*I/Z/X/10TAS
CMU=0.5P/CTAS/CSTAS
SC=2.00*TAS*Z
UTAI=UTAI+TAU
PRINT 201,UTAI,SC,TA,FE
C CALL ROOT2
KNT*KNT=1
CALL ROOTZ(KNT)
F(1)=0.00
ATA3(1) = 0.00
D23(1) = 0.00
D11 = 0.00
D21(1) = 0.00
D22(1) = 0.00
D4S(1) = 0.00
D2S(1) = 0.00
D0 202 JK = 1, 20
YW1(KJ) = 0.00
D33(KJ) = 0.00
YW(KJ) = 0.00
D24(KJ) = 0.00
Y1(KJ) = 0.00
Y2(KJ) = 0.00
Y3(KJ) = 0.00
Y4(KJ) = 0.00
Y5(KJ) = 0.00
Y6(KJ) = 0.00
Y8(KJ) = 0.00
Y9(KJ) = 0.00
Y13(KJ) = 0.00
Y14(KJ) = 0.00
Y15(KJ) = 0.00
RNS(KJ) = 0.00
YW(KJ) = 0.00

202 CONTINUE
D15(1) = D1W25/CS
D25(1) = D2*25*AI/CS/CS
RNS(1) = 0.0
TIM(1) = 0.00
SS(1) = 0.*
ASS5 = SS(11) * A1
ARNS5 = RNS(11) * MO
PRINT 5000 * CS, D1S(11) + D2S(11)
PRINT 5000 * CS, D1S(11) + D2S(11)

205 CONTINUE
PRINT 7000 * D1S(11) + D2S(11) + TAUK + S(11) + RNS(1)
DQ 2000 KJ = 2, 20
J = KJ - 1
DEL = 0.05
JE = KJ
YW2 = 0.0
YWZ = 0.0
YW1 = 0.0
YW2 = 0.0
YWZ = 0.0
D25(KJ) = 0.00
Y1(KJ) = 0.00
ATA3(KJ) = 0.00
ARN52(KJ) = 0.00
YW(KJ) = 0.00
D23(KJ) = 0.00

177
D12(KJ)=0.00
FDR=0.0
SY0=0.0
S11=0.0
SY10=0.0
SY0=0.0
DELD2=DELD2
DELD2=DELD2
IF(KJ, EQ, 20) WRITE(7, 204) (RNS11), I=1,19
TAU=TAU+DEL
TAU1=TAU/C5*PAI1.0*D6
TIMKJ=TAU
SS1(KJ)=SS1(JJ)+DEL*DIS1(JJ)+DELD2*DIS2(JJ)/2.00*
9
DELD3+DIS3(JJ)/6.00+DELD4*DIS4(JJ)/24.00
IF(SS1(KJ), LE, 0.00) DIS1(KJ)=1
IF(SS1(KJ), LE, 0.00) JC=KJ
IF(IS1(KJ), LE, 0.0) FEEC=FEE
IF(IS1(KJ), LE, 0.0) IAU=TAU
IF(IS1(KJ), LE, 0.0) ITASC=TAU
IF(IS1(KJ), LE, 0.0) SCC=SC
C
IF(IS1(KJ), GT, 1.00) GO TO 200
IF(IS1(KJ), GT, 0.00) AHI+KJ=GE+20
GO TO 200
IF(IS1(KJ), LE, 0.00) RNS1(KJ)=0.00
IF(IS1(KJ), LE, 0.00) F(KJ)=0.00
IF(IS1(KJ), LE, 0.00) WRITE(7, 204) (RNS11), F(I), I=1, KJ
204 FORMAT(5F16.6)
AS11=SS1(KJ)
AS2=AS1+AS1
AS3=DSQRT(AS2)
C
IMPACT FORCE IN DIMENSIONAL FORM=RNS(E1E2/E1+E2)
F(KJ)=BAK0*AS3*AI1+AA/DX0
RNS1(KJ)=F(KJ)/AA/SC
S11=BAK0*AS3/MASS
WRITE(8,70001) FE,TA,F(KJ),SC,TAU1+KJ
WRITE(7,70001) FE,TA,F(KJ),SC,TAU1+KJ
70001 FORMAT(2F5.0,F16.5)
A51=KNS(KJ)
A311+=-KNS(KJ)
203 CONTINUE
IF(IS1(KJ), LT, 0.00) F(KJ)=0.00
IF(IS1(KJ), LT, 0.00) RNS1(KJ)=0.00
IF(IS1(KJ), LT, 0.00) RNS1(KJ)=0.00
DIS1(KJ)=DIS1(JJ)+DELD2*DIS2(JJ)/2.00+DELD3*DIS3(JJ)/6.00+DELD4*DIS4(JJ)/24.00
TEST=SS1(KJ)-SS1(JJ)
ASS=SS1(KJ)+A91
IF(ISTEST, EQ, 0.00) GO TO 2000
ASS2=SS1(KJ)
A53(KJ)=TAU1
IF(IS2(KJ), GT, 0.00) ANS2(KJ)=0.00
IF(IS2(KJ), GT, 0.00) ARNS2(KJ)=F(KJ)
IF(IS2(KJ), LT, 0.00) SSC=SS1(JJ)
IF(IS2(KJ), LT, 0.00) RNSC=0.00
IF(IS2(KJ), LT, 0.00) WRITE(3, 15) TAUC+SCC+RNSC+SSC+JC
15 FORMAT(5E15.5)
IFIDEL=E0,00) DIRNS1(KJ)=0.0
IFIDEL=E0,00) D2RNS1(KJ)=0.0
RNS1=NS1(KJ)
IFIDEL+GT,0.00) DIR=KNS+RNS1(JJ)/DEL
IFIDEL+GT,0.00) D2RNS1(KJ)=DIR-DIRNS1(JJ)/DEL
DIRNS1(KJ)=DIR
310
ARNS*RVS(KJ)
Y(KJ)=0.00
YS=0.00
Y(KJ)=0.00
Y=0.00
DO 1004 NUM=1,6
DO 1003 I=1,6
IF(NUM.GT.20) GO TO 1002
IF(NUM.EQ.1.AND.I.EQ.1) GO TO 1006
N=NUM-1
IF(KJ.LT.3) CALL B4(NUM,1)
ATAU=SCMEGA(NUM+I1)*TAU
DATAU=ATAU*DEL/TAU
DO 5110 KJ=I+KJ
TV=SCMEGA(NUM+I1)*TAU-TIM(K1)*S
RI=KNS(K1)*DEL
5110 Y=YS*AT=OGEN(TH)
DM=SDPGAIN(NUM+I1)
FNN=FMN(1.00-910S)(TAU))/OM
FLL=FMN(1.00-OOS)(TAU))/OM
DDW=0.00
IF(KJ.GT.3) GO TO 1002
DO 1002 J=1,3
IF(KJ.LT.3) DNM1(NUM+I1,J)=DNM1(I)
1002 CONTINUE
DDW=-DNW1(NUM+I1,J)*Y*M111(NUM+I1)+FNN+DNW11(NUM+I1,J)*FLL
IF(I.LT.5) GO TO 1001
IF(I.LT.5) K=I
DATAS=OMEG(S(NUM+I1)*DEL
ATAS=OMEGS(NUM+I1)*TAU
DDMN=CMN(NUM+I1)*OSIN(ATA)
DO 5111 KJ=K+KJ
TS=OMEGS(NUM+I1)*TAU-TIM(K1)*S
RI=KNS(K1)*DEL
5111 Y=YS*AT=OGEN(TH)
DDMN=CMNS(NUM+I1)*YS
1001 CONTINUE
IF(NUM.GT.20) DATAU=0.00
IF(NUM.GT.20) DDW=0.00
IF(I.LT.5) DDMN=0.00
IF(I.LT.5) DDWS=0.00
IF(I.LT.5) DATAS=0.00
Y(KJ)=Y(KJ)+DMN+DDM+DDW
C
IF(I.LT.5.AND.NUM.LT.5) WRITE(3,7000) Y(KJ),DDMN,DDMS,DDW,TAU
C
1006 CONTINUE
1003 CONTINUE
1004 CONTINUE
XX(KJ)=TAU
Y(KJ)=0.00
IF(KJ.GT.2) GO TO 1009
DEL2=DEL/2.00
Y(KJ)=Y(KJ)/2.00
Y(K)=Y(KJ)
CALL D0ET3(DEL2,Y(KJ),Z,IER)
IF(S1(KJ).LT.0.0) COERS=-1015K1(3)+D03131
CALL D0ET3(DEL2,D03131,3,IER)
D25(KJ)=D25+11
DO 1021 I=1,3
1021 D2Z(I)=D2Z(I)-51
CALL D0ET3(DFL2,D03131,3,IER)
D55(KJ)=O12(3)
CALL D0ET3(DEL2, D12, D24, 3, IER)
D4S(KJ) = D24(3)

1009 CONTINUE
IF(KJ.LT.3) GO TO 1020
IF(KJ.EQ.3.OR.KJ.GT.4) GO TO 1012
MM = KJ - 5
WRITE(7000) YW(I1), I = 1, KJ
DO 1010 I1 = 1, 5

1010 YW(I1) = YW(I1) + MM
CALL D0ET5(DEL, YW2, D25, 5, IER)
IF(IS(KJ).LE.O) COERS = -(IS(KJ) + D23(5))
CALL D0ET5(DEL, D23, D22, 5, IER)
D25(KJ) = D22(5) - 51
DO 1016 I1 = 1, 5

1016 D25(I1) = D25(I1) + MM
CALL D0ET5(DEL, D25, D24, 5, IER)
D35(KJ) = D24(5)
DO 1017 I1 = 1, 5

1017 D26(I1) = D35(I1)
CALL D0ET5(DEL, D26, D27, 5, IER)
D45(KJ) = D24(5)
GO TO 1020

1012 NN = KJ - 3
WRITE(7000) YW(I1), I = 1, KJ
DO 1014 I1 = 1, 3

1014 YW(I1) = YW(I1) + NN
CALL D0ET5(DEL, YW2, D25, 3, IER)
D35(KJ) = D25(3)
DO 1019 I1 = 1, 3

1019 D26(I1) = D35(I1)
CALL D0ET5(DEL, D26, D27, 3, IER)
D45(KJ) = D24(3)

1020 CONTINUE
PRINT 7000, IS(KJ), D25(KJ), TAU1, ASS, ARNS
IF(IS(KJ).LT.0) VFINAL = COERS*D01*425
IF(IS(KJ).EQ.0) WRITE(7000) COERS, VFINAL, D25(I1), I = 3, JCI
IF(KJ.EQ.20 AND 0) D5S(KJ) = 011(D11) = D1S(KJ)
IF(KJ.EQ.20 AND 0) D2S(KJ) = D5S(KJ)
IF(KJ.EQ.20 AND 0) D3S(KJ) = D2S(KJ)
IF(KJ.EQ.20 AND 0) D4S(KJ) = D3S(KJ)
IF(KJ.EQ.20 AND 0) KJ = KJ - 19
IF(KJ.EQ.20) GO TO 205
IF(IS(KJ).LT.0) GO TO 40
IF(IS(KJ).LT.0) STOP
IF(IS(KJ).LT.0) STOP
IF(IS(KJ).LT.0) STOP

2000 CONTINUE
IF(IS(KJ).LT.0) GO TO 200
IF(KJ.EQ.20) DEL1 = DEL1*10.00
IF(IS(KJ).LT.0 AND 0) DEL1 = DEL1*10.00
IF(IS(KJ).LT.0) GO TO 200
IF(IS(KJ).LT.0) STOP

40 CONTINUE
IF(TE = 20 AND 0) D02(T) = D02(T)
IF(IS(KJ).LT.0) CALL PRPLOT(TA3, ARNS2, PY, J6, 1)
*RIMPLE FORI*15)
PRINT 7000, (ARNST2(KJ), SS1(KJ), KJ=1, JE)
JC = JE
JC = JC + 2
SS1(JE) = 0.0
IF (SS1(JE) GT 0.0) STOP
FE = 0.0
TA = 0.0
FEE = PI/180
DFEE = 0.0
DTAS = PI/180
TAS = 0.0
COUNT = 1
IF COUNT GT 2 TAS = TAS + DTAS
CONTINUE
!
Z = COS(FEE)
TAU = 0.0
Y0NW2 = 0.0
Y0NY1 = 0.0
Y0NG2 = 0.0
Y0NG1 = 0.0
Y0NH1 = 0.0
Y0NH2 = 0.0
Y0NH3 = 0.0
Y0NH4 = 0.0
Y0NH5 = 0.0
DNHS0 = 0.0
YNS1 = 0.0
YNMC = 0.0
YN9C = 0.0
YN8S = 0.0
Y011 = 0.0
Y022 = 0.0
Y033 = 0.0
Y044 = 0.0
Y055 = 0.0
Y066 = 0.0
TAUL = 0.0
DO 3000 KJ = 1, JC
IF (KJ LE 1) DEL = 0.00
IF (KJ GT 1) DEL = PI/200.00
IF (KJ GT 1) DEL = 0.05
Y11JKJ = 0.0
Y0P1(KJ) = 0.0
YCP1(KJ) = 0.0
YCP1(KJ) = 0.0
YCP2(KJ) = 0.0
Y21JKJ = 0.0
Y31JKJ = 0.0
Y41JKJ = 0.0
Y51JKJ = 0.0
Y61JKJ = 0.0
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620
Y16(KJ)=Y16(KJ)+YONGS  
Y17(KJ)=Y17(KJ)+DNHC0  
Y18(KJ)=Y18(KJ)+DNHS0  
Y19(KJ)=Y19(KJ)+YONG2  
Y20(KJ)=Y20(KJ)+YONG2  
Y21(KJ)=Y21(KJ)+YONG2  
YC=A31(1,J)*GDS184(ATAU)  
YP1=DNWJ1(NUM+I1+1)*FL-DNWJ1(NUM+I1+1)*FL  
YP2=DNWJ2(NUM+I2+1)*FL-DNWJ2(NUM+I2+1)*FL  
YP3=DNWJ3(NUM+I3+1)*FL-DNWJ3(NUM+I3+1)*FL  
YP4=DNWJ4(NUM+I4+1)*FL-DNWJ4(NUM+I4+1)*FL  
YP5=DNWJ5(NUM+I5+1)*FL-DNWJ5(NUM+I5+1)*FL  
YP6=DNWJ6(NUM+I6+1)*FL-DNWJ6(NUM+I6+1)*FL  
YCP1(KJ)=YCP1(KJ)+YP1  
YCP2(KJ)=YCP2(KJ)+YP2  
YCP3(KJ)=YCP3(KJ)+YP3  
YCP4(KJ)=YCP4(KJ)+YP4  
YCP5(KJ)=YCP5(KJ)+YP5  
YCP6(KJ)=YCP6(KJ)+YP6  
20 CONTINUE  
IF(I.LT.31) K=1  
IF1(I.GT.2) GO TO 4001  
IF2=OGL2(NUM+K) DDEL  
AK=DSIN(AT2)  
Y2=A41(1,J)*AK  
IF(KJ.GT.11) GO TO 1122  
DO 1122 L=1,2  
NPSISI(NUM+K+L)=NPSISI(L)  
NLAMCI(NUM+K+L)=NLAMCI(L)  
NPSICI(NUM+K+L)=NPSICI(L)  
NLAMCI(NUM+K+L)=NLAMCI(L)  
1122 CONTINUE  
FR=GA41A*YNS  
YNS1=NPSIS1(NUM+K+2)*FL-NPSIS1(NUM+K+2)*FR  
YNS2=NPSIS2(NUM+K+2)*FL-NPSIS2(NUM+K+2)*FR  
YNS3=NPSIS3(NUM+K+2)*FL-NPSIS3(NUM+K+2)*FR  
YNS4=NPSIS4(NUM+K+2)*FL-NPSIS4(NUM+K+2)*FR  
YNS5=NPSIS5(NUM+K+2)*FL-NPSIS5(NUM+K+2)*FR  
YNS6=NPSIS6(NUM+K+2)*FL-NPSIS6(NUM+K+2)*FR  
Y1(KJ)=Y1(KJ)+YLAML  
Y2(KJ)=Y2(KJ)+YNS1  
Y3(KJ)=Y3(KJ)+YNSCI  
Y4(KJ)=Y4(KJ)+YNNS1  
Y5(KJ)=Y5(KJ)+YNLAMC  
Y6(KJ)=Y6(KJ)+YNAMS  
4001 CONTINUE  
IF1(EQ.1) GO TO 1008  
IF1(I.GT.1) GO TO 4008  
XDEL=5*DMEG2111*DEL  
YD0=5*GMEGL111*TEL  
XY=5*GMEGL111*TEL  
YO=5*GMEGL111*TEL  
YCP1(KJ)=YCP1(KJ)+YOD8BJJ01(1)  
YCP2(KJ)=YCP2(KJ)+YOD8BJJ02(1)  
YCP3(KJ)=YCP3(KJ)+YOD8BJJ03(1)  
YCP4(KJ)=YCP4(KJ)+YOD8BJJ04(1)  
YCP5(KJ)=YCP5(KJ)+YOD8BJJ05(1)  
YCP6(KJ)=YCP6(KJ)+YOD8BJJ06(1)  
Y8(KJ)=Y8(KJ)+(BJJ(6)+BWO(6))*OSINIXDEL  
4008 CONTINUE
CONTINUE 683
CONTINUE 684
CONTINUE 685
CP = (5*Q4*GAMMA*NUM,1)**2 686
IF(KJ.EQ.1) PRESR1=YP1(KJ)*RAP*CP 687
IF(KJ.EQ.1) PRESR2=YP2(KJ)*RAP*CP 688
IF(KJ.EQ.1) PRESR3=YP3(KJ)*RAP*CP 689
IF(KJ.EQ.1) PRESR4=YP4(KJ)*RAP*CP 690
IF(KJ.EQ.1) PRESR5=YP5(KJ)*RAP*CP 691
IF(KJ.EQ.1) PRESR6=YP6(KJ)*RAP*CP 692
IF(KJ.EQ.1) WRITE(12,7001) PRESR1,PRESR2,PRESR3,PRESR4,PRESR5,PRESR6 693
7001 FORMAT(6E20.6) 694
CONTINUE 695
3000 CONTINUE 696
IF(JC.GT.2) GO TO 4006 697
YP1(1)=YP1(1) 698
YP1(2)=YP1(1)+YP1(2)**2 699
YP2(3)=YP2(2) 700
YP2(4)=YP2(3)+YP2(4)**2 701
YP3(1)=YP3(1) 702
YP3(2)=YP3(1)+YP3(2)**2 703
YP4(3)=YP4(2) 704
YP4(4)=YP4(3)+YP4(4)**2 705
YP5(3)=YP5(2) 706
YP5(4)=YP5(3)+YP5(4)**2 707
YP6(3)=YP6(2) 708
YP6(4)=YP6(3)+YP6(4)**2 709
DEL2=DEL2/2. 710
CALL DDETD2(DEL2,YP1,TS5,3,IER) 711
CALL DDETD2(DEL2,TS5,PR1,3,IER) 712
CAL DDETD2(DEL2,TS5,PR2,3,IER) 713
CALL DDETD2(DEL2,TS5,PR3,3,IER) 714
CALL DDETD2(DEL2,TS5,PR4,3,IER) 715
CALL DDETD2(DEL2,TS5,PR5,3,IER) 716
CALL DDETD2(DEL2,TS5,PR6,3,IER) 717
CALL DDETD2(DEL2,TS5,PR7,3,IER) 718
CALL DDETD2(DEL2,TS5,PR8,3,IER) 719
CALL DDETD2(DEL2,TS5,PR9,3,IER) 720
CALL DDETD2(DEL2,TS5,PR10,3,IER) 721
CALL DDETD2(DEL2,TS5,PR11,3,IER) 722
4006 IF(JC.GT.3 .OR. JC.GE.4) GO TO 1112 723
IF(JC.GT.3 .OR. JC.GE.4) GO TO 1112 724
DO 1111 KK=1,5 725
D1(KK)=YP1(KK)+LL 726
P1(KK)=YP1(KK)+LL 727
P2(KK)=YP1(KK)+LL 728
P3(KK)=YP1(KK)+LL 729
P4(KK)=YP1(KK)+LL 730
P5(KK)=YP1(KK)+LL 731
P6(KK)=YP1(KK)+LL 732
1111 CONTINUE 733
CALL DDETD5(DEL+P1,TS5,5,IER) 734
CALL DDETD5(DEL+TS5,PR1,5,IER) 735
CALL DDETD5(DEL+TS5,PR2,5,IER) 736
CALL DDETD5(DEL+TS5,PR3,5,IER) 737
CALL DDETD5(DEL+TS5,PR4,5,IER) 738
CALL DDETD5(DEL+TS5,PR5,5,IER) 739
CALL DDETD5(DEL+TS5,PR6,5,IER) 740
CALL DDETD5(DEL+TS5,PR7,5,IER) 741
CALL DDETD5(DEL+TS5,PR8,5,IER) 742
CALL DDETD5(DEL+TS5,PR9,5,IER) 743
CALL DDETD5(DEL+TS5,PR10,5,IER) 744
CALL DDET3(DEL+TSS+PR6+3,IER)
GO TO 1120
1112 NK=KJ-3
DO 1114 NI=1,3
P1(NI)=YCPI(NI+NK)
P2(NI)=YCPI(NI+NK)
P3(NI)=YCPI(NI+NK)
P4(NI)=YCPI(NI+NK)
P5(NI)=YCPI(NI+NK)
1114 P6(NI)=YCPI(NI+NK)
1115 CALL DDET3(DEL+P1,TSS+3,IER)
CALL DDET3(DEL+P3,TSS+3,IER)
CALL DDET3(DEL+P3,TSS+3,IER)
CALL DDET3(DEL+P5,TSS+3,IER)
CALL DDET3(DEL+P5,TSS+3,IER)
CALL DDET3(DEL+P6,TSS+3,IER)
CALL DDET3(DEL+TSS+3,IER)
CALL DDET3(DEL+TSS+3,IER)
1120 CONTINUE
4007 CONTINUE
DO 300 JK=1,JC
HFIA=0.
GFIA=0.
WFIA=0.
HTIA=0.
CGMFI=0.
HFIS=0.
CGMTA=0.
WF=0.
GF=0.
HF=0.
WF=0.
GF=0.
WT=0.
GTA=0.
MTS=0.
CGM=0.
CFE=0.
CFETF=0.
WFIA*WFIA=WFIA(JK)
HFIA=HFIA+HFIA1(JK)
GFIA=GFIA+GFIA1(JK)
WF=WF+WFEL(JK)
GF=GF+GFIL(JK)
WF=WF*WFEL(JK)
HF=HF+HF1(JK)
WF=WF*WFIL(JK)
GFI=GFIL(JK)
HF=HF+HF1(JK)
WTA=WTA+WTA1(JK)
GTA=GTA+GTA1(JK)
MTA+HTA+HTAI(LJK)
VF1=VF1+USIN(FEE)*HTA/OTAN(FEE)/OSIN(FEE)
HFIS=HFIS+HF IS1(JK)
MTAS=MTAS+HTAS1(JK)
WTA+HTAS/OSIN(FEE)
CGM=CGM+CGML1(JK)
C Fee=CFE+CFE(JK)
CFEI=CFEI+CFEI(JK)
CFETA=CFETA+CFETA(JK)
CGO=CGO+CGO(JK)
SA=SA+SA+D(IN(FEE)-CFE+DCS(FEE)+WF
SB=VTA/USIN(FEE)
SD=HFI/DAN(FEE)-CFE+DCS(FEE)+WF
STRE=SSA
STRT=SST
STRT=STRT*(1+HTA/DTAN(FEE)/USIN(FEE)-CFE)
TAUF1=TAUF1*(E1/HF)*CFE+GFI*(CFE-CGM)*TAN(FEE)
TAUF2=TAUF2*(E1/HTA)*HFI+GFI*(CFE-CGM)*TAN(FEE)
TAUF3=TAUF3*(HTA/HF)*GFI*(CFE-CGM)*TAN(FEE)
UD*HFI+CFE+DSIN(FEE)
VD=HTA/DSIN(FEE)
BITA*GFI+GFI+DSIN(FEE)
BITA=HTA+GTA1/DSIN(FEE)
WD=WF
UD=UD+A1
VD=VD+A1
WD=WD+A1
SN=SN+DO
SK=SK
SSN=SN*1.0
PD*FEE+180.00/PI
TA=TAS+180.00/PI
WRITE(10,10) FEE+THA+TAS+TAUF1,TAUF2,TAUF3
WRITE(11,11) FEE+THA+TAS+TAUF1,TAUF2,TAUF3
WRITE(12,12) FEE+THA+TAS+TAUF1,TAUF2,TAUF3
C PUNCH 7, FEE+THA+TAS+TAUF1,TAUF2,TAUF3
C 300 CONTINUE
DD 800 I1=1,J1
PRE1=0.0
PRE2=0.0
PRE3=0.0
PRE4=0.0
PRE5=0.0
PRE6=0.0
PRE1=PRE1+PR1
PRE2=PRE2+PR2
PRE3=PRE3+PR3
PRE4=PRE4+PR4
PRE5=PRE5+PR5
PRE6=PRE6+PR6
WRITE(7,7) FEE+THA+PRE1,PRE2,PRE3,PRE4,PRE5,PRE6
100 CONTINUE
WRITE(7,7) FEE+THA+PRE1,PRE2,PRE3,PRE4,PRE5,PRE6,PN
C 800 CONTINUE
IF(TA=347.0 AND TA=355.1) TA=350.00
IF(TA=355.0 AND TA=365.1) TA=360.00
C PUNCH 7, FEE+THA+TAS+TAUF1,TAUF2,TAUF3
C 300 CONTINUE
DD 800 I1=1,J1
PRE1=0.0
PRE2=0.0
PRE3=0.0
PRE4=0.0
PRE5=0.0
PRE6=0.0
PRE1=PRE1+PR1
PRE2=PRE2+PR2
PRE3=PRE3+PR3
PRE4=PRE4+PR4
PRE5=PRE5+PR5
PRE6=PRE6+PR6
WRITE(7,7) FEE+THA+PRE1,PRE2,PRE3,PRE4,PRE5,PRE6,PN
C 800 CONTINUE
IF(TA=347.0 AND TA=355.1) TA=350.00
IF(TA=355.0 AND TA=365.1) TA=360.00
CALL DDETS1(DEL2,P2,P3,3,IER1)  
CALL DDETS1(DEL2,P3,P4,3,IER1)  
1009 CONTINUE  
IF(KJ.LT.3) GO TO 1010  
U11=TAU/4.*  
U12=TAU/2.*DO  
U13=DO*TAU/4.*DO  
DEL3=TAU/4.*  
IF(KJ.LT.4) GO TO 1012  
CALL TLAG(X,Y,U,F,SYM,KJ,3)  
P211=0.DO  
P212=SYM(1)  
P213=SYM(2)  
P214=SYM(3)  
P215=P1(KJ)  
IF(KJ.GT.3) GO TO 1013  
1012 CONTINUE  
P211=0.DO  
P212=P1(1)/2.0  
P213=P1(2)  
P214=P1(2)+P1(3)/2.  
P215=P1(3)  
1013 CONTINUE  
CALL DDETS1(DEL3,P2,P3,5,IER1)  
CALL DDETS1(DEL3,P3,P4,5,IER1)  
1010 RETURN  
END  
SUBROUTINE TLAG(X,Y,U,F,SYM,N,P)  
C LAGRANGIAN INTERPOLATION  
IMPLICIT REAL*8 (A-H,O-Z)  
INTEGER P  
DIMENSION X(N),Y(N),U(P)  
DIMENSION SUM(P)  
LL=P-1  
SUMM=0.0  
WRITE(13,21)  
2 FORMAT(I2X,'N*',5X,'ARGUMENT '*10X,'VALUE OF FUNCTION*')  
WRITE(13,89)  
DO 30 KK=1,P  
ARG=UKK  
SUM(KK)=0.DO  
DO 6 I=1,N  
PROD=Y(I)  
DO 7 J=1,N  
IF(I-J.EQ.0) SUMM=SUMM*PROD  
7 CONTINUE  
6 PROD=PROD*(ARG-X(J))/((X(I)-X(J)))  
30 CONTINUE  
WRITE(3,91) N,ARG,SYM(KK)  
WRITE(12,160)  
C WRITE(6,901) UKK,DIF  
WRITE(3,901) UKK,SYM(KK)  
C SUMM=SUMM+DIF*DIF  
30 CONTINUE  
C VAR=SUM/LL  
C STD=OSQRTIVAR  
C WRITE(6,861) VAR,STD  
86 FORMAT('F19.10','52X','VAR =*F19.10','52X','SIGMA =*F19.10')  
89 FORMAT('F19.10','52X','DIF =*F19.10')
SUBROUTINE ROOT
IMPLICIT REAL*8(A-H,O-Z)
COMPLEX*16 P,C
REAL*8 INT,INPUT1,INPUT2
REAL*8 T(201),X0(201),X1(201),X2(201),X3(201),X4(201),X5(201)
REAL*8 K1,K2,KR,K5
LOGICAL TT
COMMON/CONT/EP
COMMON/TIN/D,E
COMMON/PJTIN/AK,V,E,H,P,ROHOS
COMMON/GREEX/BETA,BETANU,GAMMA
COMMON/AK/K1,K2,KR,K5,S
COMMON/GRCON/ALPHA,F
COMMON/CDNST/AS,B,ES,F,G,S,XS,CNT/B
DIMENSION PTOLE(2),OMEGAL(2,31)
COMMON/OMEGAL(2,31)
WRITE(U,80001)
WRITE(1,90001)
H = H/2.
KS*AK
GRAVTY=386.9
HROS=RHOS/GRAVTY
CS2=E/HROS/(1.0+V*V)
CS=DSRT(CS2)
S=C/CS
ALPHA = (H/A)=92/12.00
B = 2.00/(1.0+V)
GS = 1.00/(BK5)
AS = GS*V-1.00
BS = GS*V+1.00
F = RHOF*A/(RHOS*H)
FS = 1.00+(1.00+V)/GS
K1 = 1.00*ALPHA
K2 = 2.00*ALPHA
KR = 1.00+1.00*ALPHA
BETANU=2.00+(1.00+V)/A/E/H/KS
THETAS = GS+ALPHA*(1.00-V)
ES = 1.00/K5
XS = ES-2.00*ALPHA
DS = 2.00+1.00-ES)
YS = 2.00+(1.00+V)/GS
S2 = C*CHROS*(1.00-V)/E
S = DSRT(S2)
WRITE(2,15)
PRINT 2,ALPHA,AS,B,DS,FS,GS,K1,K2,KR,K5,S2,THETAS,VS,YS
C FORMAT(3F15.6)
C THE ABOVE VARIABLES ARE SOLELY FOR COMMON USED ONLY IN FUNC
II=31
LL=30
READ(7) (T0MEGL1(I,L),L=1,LL),I=1,II)
1 FORMAT(14,F9.6)
7 FORMAT(5F16.10)
VZ=1.+V
VI=1.+V
GS1=K1+KR-2.*K2
GS2=2.*ALPHA/V2
GS = GS1 * GS2 * V2

DO 199 NUM = 1, 31
N = NUM - 1
LN2 = N + 1
LN2 = LN2 + 2
G1 = LN2 * (ALPHA * LN2 + (1 + K2) / KS) + 1 / KS / KS
G3 = 2 * (ALPHA / LN2 + (1 + K2) / KS) + 1 / KS / KS / V2
PC1 = DCMLX(G3, 0, 0, 0, 0)
PC2 = DCMLX(G5, 0, 0, 0, 0)
CALL MULLER(Z, 3, 1, D - 5, 10, PC, PRT, PTOL, IPFLAG)
DO 1000 I = 1, 2
1000 PKI(I) = PKI(PRT(I))
DO 1001 I = 1, 2
1001 PRI(I) = COSCST(PRT(I))
DO 1002 I = 1, 2
IF (NUM .EQ. 1) PRI(I) = (0.0, 0.0)
OMEL(1) = OMED2(NUM, 1)
DO 1003 NUM = 1, 2
1003 OMELI = OMEL2(NUM, 1)
DO 1004 I = 1, 32
1004 OMEL = OMEL2(NUM, 1)
WRITE(1, 1) (OMEL(NUM + K1 + K = 1, 32)
I = 1, 32
SCORE = 5 * OMELI = (5 * OMELI(NUM, I))
I = 1, 32
SCORE = 5 * OMELI = (5 * OMELI(NUM, I))
199 Continue
C)
1000 FORMAT(20X, 'TABLE = 1')
9000 FORMAT(130X, 'ROOTS CAPITAL OMEGA')
2000 RETURN
END
REAL FUNCTION Func8 (X, I)
IMPLICIT REAL* 8, *8
REAL* 8 X, A, H, D, Z-
REAL* 8 K1, K2, KR, KS-
LOGICAL TT-
COMMON /PUTIN/ AK, V, A, H, C, E, RHOF, RHOS-
COMMON /GACKN/ ALPHA, F-
COMMON /GREER/ BETA, BETANU, GAMMA, THETAS-
COMMON /A<1/K1, K2, KR, KS, S-
COMMON /CON/ T/AS, BS, ES, FS, GS, XS, KS, CNT/B-
DIMENSION U(451)
COMMON TT-
TT = . TRUE.
TT = . FALSE.-
S2 = 5*S-
Z = X**2-
LAM = |X**1|+
P2 = -S2*X**2-
M1 = K1*P2 + LAM*AS-
M2 = GS*ALPHA + P2 - GS-
M3 = -FS*P1-
G1 = K2*P2 - GS-
G2 = ALPHA*KR*P2 + LAM*ALPHA + THETAS-
G3 = LAM-
W1 = B5-
W2 = GS-
SX = DSIN(X)-
EX = DCOS(X)-
19 IF (I.EQ.0) WJ = SX/X
IF (I.EQ.0) GO TO 19
CALL DSLRA(X*X+1.0+2.0)
IF (I.GT.0) WJ = 1.0/(X*R2)
W3 = X*K*(1.0+K) + F*WJ + P2 + LAM * YS
RETURN
END
RETURN
END
FUNCTION FFUNC = 68(I)
IMPLICIT REAL 68 A-H, O-Z
DIMENSION U(51)
N = I+1
NUM = 3
IF (I.EQ.3) GO TO 5
SX = SX*IN(I)
CX = DCSKX(I)
IF (I.EQ.0) YA = SX/X
IF (I.EQ.1 OR N.EQ.1) YC = (SX/X-CXI)/X
IF (N.EQ.1) YB = YC
IF (N.EQ.2) AB = 3.00*15/SX-CXI)/X
IF (I.EQ.3) GoTo 5
IF (I.EQ.0) FFUNC = YA
IF (I.EQ.1) FFUNC = YC
IF (I.EQ.2 AND N.EQ.3) GO TO 5
IF (I.EQ.3) RETURN
5 CALL SBF(X*I+2*I+I2+U,NUM,450)
FFUNC = U(I+1)
RETURN
ENTRY FFC(I)
IF (I.EQ.2) FFC = Y
IF (I.EQ.1) FFC = YB
RETURN
END
SUBROUTINE BSLRA(I2*I2*N,J)
IMPLICIT REAL 68 A-Z
INTEGER I,K,L,H,N
IF (I2 > 10 OR Z2 > 10) GO TO 500
M = N
Z = Z2
5 ZZ = Z2/Z2
TERM = 1.0
TERM1 = 1.0
DO 100 K = 1, 5000
TERM = TERM*(ZZ/1.000+1.000)/K
IF (ABS(TERM) < 1.0) GO TO 150
100 TERM = TERM1 + (-1.0)*K*TERM
150 IF (I.EQ.11) GO TO 200
JZ2N = TERM1
Z = Z1
GO TO 5
200 JZ2N = TERM1
IF (I.EQ.22) GO TO 250
JR = JZ2N/JZ2N
JRT = (Z2/Z2)*N*JR
GO TO 290
250 IF (K < N) GO TO 350
JZ2N = TERM1
JR = 1.0
290 IF (I.EQ.2) GO TO 300
RETURN
300 IF (M < N) GO TO 400
SUBROUTINE BESSEL (Z1, N, J, JP)
IMPLICIT REAL*8 (A-Z)
INTEGER I, K, L, M, N
LOGICAL LIGHT

A = .5
B = 50.
LIGHT = .TRUE.
IF (Z1 .GE. 0.) GO TO 1
LIGHT = .FALSE.
Z = Z1
IF (IN .GT. 0.) GO TO 2
J = DSIN(Z1)/Z
JP = DCOS(Z1)-DSIN(Z1)/Z
GO TO 4
2 IF((Z .GE. A) .AND. (Z .LE. B)) GO TO 3
IF(Z .GT. B) GO TO 31
CALL SERIES(Z, N, J, JP)
GO TO 4
3 CALL BRECJR(Z, N, J, JP)
GO TO 4
31 CALL FRECJR(Z, N, J, JP)
4 IF (.NOT.LIGHT) RETURN
Z = Z1
IF ((N/2)*Z .EQ. N) GO TO 5
J = J + 1
RETURN
5 JP = JP + 1
RETURN

SUBROUTINE SERIES(Z, N, J, JP)
IMPLICIT REAL*8 (A-Z)
INTEGER I, K, L, M, N
LOGICAL TEST / .FALSE. /
REAL*8 LAMDA1, GAMMA1
X = Z/2.
IF (TEST) GO TO 300
LAMDA1(1) = 1.
GAMMA1(1) = 1.32934500
DO 100 K=2.55
LAMBDA(K) = K* LAMBDA(K-1)
DO 200 K=2.56
GAMMA(K) = (K+0.5)*GAMMA(K-1)
TEST = TRUE
300 J=1/GAMMA(N)
JP=J*N/2
L=N
M=56-N
DO 400 K=1,N
L=L+1
IF (K*LOG10(1+ABS(X))) GT 53.1 GO TO 500
600 A=K*(K+1)/LAMBDA(K)*GAMMA(K)*GAMMA(L)
IF (A+LT.10.-E-10) TEST = FALSE
IF (1(K+2)*GAMMA(K) A=-A
J=J+A
JP=JP+A*(K+2)*
IF (NOT TEST) GO TO 500
400 CONTINUE
500 J=J*0.866227**N
JP=JP*0.866227**N(N-1)
TEST = TRUE
RETURN
END
SUBROUTINE BRECUR(Z,J,F)
IMPLICIT REAL(A-Z)
INTEGER I,K,L,M,N
IF(N.GT.0) GO TO 100
J=DSIN(Z)/Z
JP=(DCOS(Z)-OSIN(Z)/Z
RETURN
100 J=10.-60
JB=10.-59
IF(Z.LE.2.) GO TO 150
K=69
GO TO 175
150 K=29
175 DO 200 LL=1,K
L=L+1-LL
JC=(2*L+1)-JB/Z-JA
JCP=L/Z*JB-JA
M=N+1
IF(L.EQ.M) J=JC
IF(L.EQ.N) JP=JCP
JA=JB
200 JB=JC
JP=DCOS(Z1)/Z1
JP1=DCOS(Z1/Z-JO/Z)*Z+Z*JO
F1=JC/J0
F2=JCP/J1P
JA=JF
JP=JP/F2
RETURN
END
SUBROUTINE FRECUR(Z,J,F)
IMPLICIT REAL(A-Z)
INTEGER I,K,L,M,N
IF(I.GT.0) GO TO 199
J=DSIN(Z)/Z
JP=(DCOS(Z)-OSIN(Z)/Z1/Z
RETURN
199 DO 100 I=1,N

J_A = DSIN(Z)/Z

J_B = (J_A - DCOS(Z) )/Z

J_A_P = IDCOS(ZI) 

J_B_P = I*P2*J_A -C*P2 ) « J_C I/ ( 2.*P2 « 1. )

J_A = JB

J_B = JC

RETURN

END

SUBROUTINE ROOT2I(KOUNT)

IMPLICIT REAL*3(A-H,O-Z)

REAL*8 LW,LMN,LMNS

REAL*8 MINUS

REAL*8 BINT,INPUT1,INPUT2

REAL*8 WMN,WMNS,LLMNS

REAL*8 K1,K2,KR,LV,LANA,KS

COMPLEX*16 W,P4,W6,PRT12),QRT(3),PCK(3),QCK(3)

DIMENSION PTOL(2),Q_TOL(3)

COMMON/PUTIN2/A,H,V,KS,E,RHS

COMMON/TIN/D1H25*2,D2H25*CS

COMMON/PUTIN1/Y1,Y2,Y3,Y4,Y5,Y6,Y7

COMMON/GREEK/Delta, BetaU, Gamma+TH

COMMON/ASLEG,ASLEG*,ASLEG

COMMON/ANGLE/PI,P12,ATN,CMU,CTEE,X

COMMON/GMN(31,5),HMNS(31,5),HMNS(31,5),LLMNS(31,2)

COMMON/EGS(31,5),OMEGS(31,3),OMEGS2(31,2)

COMMON/F/ FIRST

EQUIVALENCE ( LN+LM ), ( GAMMA+GAMA )

EQUIVALENCE ( CTAS+CTETAC1 ), ( CTEE, CPHIC )

EQUIVALENCE ( DLAMG+LMN ), ( GAMM+GAMMN )

EQUIVALENCE ( N1+NUM )

COMMON /F/ FIRST

P4WJ=C1+C3*CS*10W

Q6WJ=D1+D3*10S+07W1*10W

PI = 3.141592653589790D0

CTETA0=PI/8.

CPI0=PI/12.

IF (KOUNT.GT.1) GO TO 10002

WRITE(2,9000)

WRITE(2,9000)

WRITE(2,9000)

WRITE(18,1200)

WRITE(18,1300)

10002 CONTINUE

ALPHA=PI/(12.*DO*A*DO)

K1=ALPHA+1.*DO

K2=2.*DO*ALPHA

KR=1.*DO+1.*DO*ALPHA

V1 = V1

V2 = 1.*DO-V

BETA=2.*KS*V1*A/E/H

C51=K1*K2-2.*DO*K2

C52=2.*DO*ALPHA/V2

C5=C51*C52+2.*DO/V2

DS=2.*DO*(1.-1./KS)

ES=1./KS

AS=1./KS-2.*ALPHA
B = 2 + 0 0 / (1 + 0 0 - V 1 )
G S = 1 + 0 0 / (B * K S)
A S = G S * V - 1 . D 0
D = V 1 / 2 
B S = G S * V 1 - D O
F S = 1 + 0 0 + (1 + 0 0 + V 1 ) / G S
T H E T A S = G S - A L P H A *(1 + 0 0 - V 1 )
Y S = 2 + 0 0 + (1 + 0 0 + V 1 ) / G S
D 7 1 = A L P H A * B * K S / K 1
D 7 2 = K 1 * K R - 2 * 0 0 * K 2
D 7 = D 7 1 + D 7 2
D 3 2 = B * K S * G S * G S * K 1
D 0 = 1 5 0 * N U M = 1 + Z 9
N 1 = N U M
N = N U M - 1
1 7 0
L N = N * (1 + 1)
L N 2 = L N - 2
L V = L N - V 2
A N O = L N + A L P H A - T H E T A S
L N A = L N - A S
Y S A = L N - Y S
F S A = F S / L N
C 1 = L N 2 * (A L P H A * L N 2 + (1 + K 2) / K S) + 1 + 0 0 / K S
C 3 = 2 + 0 0 * A L P H A * L N 2 * (K R + K 1) + 2 + 0 0 * A L P H A * K R + 1 + 0 0 / K S + (K 1 + K 2)
2 / K S / V 2
V S A = L N - V 2 + G S
D 5 1 = 4 + 0 0 + D T 7 1 * G S
D 5 2 = A L P H A * Y S A + D T 7 2
D 5 3 = D T 7 1 * K R + L N A
D 5 4 = B * K S * K 1 * A N O * K 1
D 5 = D 5 1 + D 5 2 + D 5 3 + D 5 4
D 3 1 = 2 + 0 0 * K 2 * G S * Y S A
D 3 3 = K 2 * G S * F S A
D 3 4 = B * K S * L N A * A N O * K 1
D 3 5 = A L P H A * K R * L N A * Y S A
D 3 6 = K 1 * G S * L N
D 3 7 = K 1 * A N O * Y S A
D 3 8 = B * S * L N * (K 2 + A L P H A * K R)
D 3 9 = D 3 1 + D 3 2 + D 3 3 + D 3 4 + D 3 5 + D 3 6 + D 3 7 + D 3 8
D 1 1 = G S * G S * Y S A + F S A
D 1 2 = G S * L N * L N A
D 1 3 = L N A + A N O * Y S A
D 1 4 = B * S * G S * L N
D 1 5 = B * S * F S A * A N O
D 1 6 = D 1 1 + D 1 2 + D 1 3 + D 1 4 + D 1 5
1 0 0 0 1
I F (K O U N T . G T . 1) G O T O 1 0 0 0
P C 1 1 = D C M P L X (C 1 + 0 . 0 0 )
P C 1 2 = D C M P L X (C 3 + 0 . 0 0 )
P C 1 3 = D C M P L X (C 5 + 0 . 0 0 )
Q C 1 1 = D C M P L X (O 1 + 0 . 0 0 )
Q C 1 2 = D C M P L X (O 3 + 0 . 0 0 )
Q C 1 3 = D C M P L X (O 5 + 0 . 0 0 )
Q C 1 4 = D C M P L X (O 7 + 0 . 0 0 )
C A L L M U L L E R (1 2 + 3 + 1 + 0 + 5 + 1 0 + P C * P R T + P T O L + I P F L A G)
C A L L M U L L E R (1 3 + 4 + 1 + 0 + 5 + 6 + G S + Q K T + Q T O L + I Q F L A G)
D 0 3 = I = 1 + 1 2
D 0 4 = I = 1 + 1 3
P C K (1) = P 4 (P R T (1))
D 0 5 = I = 1 + 1 2
D 0 6 = I = 1 + 1 3
Q C K (1) = Q 6 (Q R T (1))
D 0 7 = I = 1 + 1 2
P R I (1) = C O S (P R T (1))
D 0 8 = I = 1 + 1 2
D 0 9 = I = 1 + 1 2
Q 1 0 = I = 1 + 1 2
D 1 1 = I = 1 + 1 2
D 1 2 = I = 1 + 1 2
IF (NUM.EQ.1) PRI(I)=0.0,0.0,0.0
90 OMEG51(NUM+1)=CAOBS1(PR1(I))
DO 16 I=1,3
16 QK1(I)=CSQRT(1/RT(I))
90 00 I=1,3
1000 CONTINUE
DO 155 J=1,5
155 Q0=0.0
DO0=0.0
BO=0.0
W1=0.0
LAM = N*(N+1)
MNINL+J = 0*
MNINL+J = 0*
MNINL+J = 0*
MNINL+J = 0*
MNINL+J = 0*
MININL+J = 0*
MININL+J = 0*
IF (J.GT.2) GO TO 40
LLMN(S1,J) = 0*
PSINNS1,J) = 0*
15 CONTINUE
IF (N1.EQ.1.AND.(J.EQ.1.OR.J.EQ.3)) GO TO 10
W1=OMEG51(N1+J)
W2=WN1+W1
DO = (CS=W2+C3)*W2+C1
BO = T4+DD0+CSS*W2+2+DD0+CSS*W1
DO0 = (CSS=W2+D5)*W2+O3*W2+D1
BOO = (D6+CSS+CSS憋+4+CSS)*W2+2+DD0+D3I*W1
110 PRINT 11,N,J+D1+D0+1000+BOO
11 FORMAT (25,5E16.6)
10 CONTINUE
IF (N1.EQ.1.AND.(J.EQ.1.OR.J.EQ.3)) GO TO 155
DEL00 = 300+BC00+800+BOO
12 IF (N1.EQ.1.AND.J.EQ.3) ACSW0=0.0
12 FORMAT (2,F16.10)
C C THE ABOVE POLYNOMIAL COEFFICIENTS ARE NEXT USED BELOW FOR G,H,W ET
C 140 M1=1,N1
140 M = M1-1
C C COEF IS (2N+1)/(2*(N-M+1)/(N+1))
C C ANG = -1500*PI*CPI/CPI
C CALL ASLEG1(N1,M1,CPI)
C ASLEG = ASLEG1
C PRINT 900,ASG,ASLEG,ASLEG
800 FORMAT (F16.10)
C COEF = N+1500
FIRST = 1.0D0
PLUS = FCTRL(N+1)
MINUS = FCTRL(N-M)
COEF = (2**N+1)*MINUS/PLUS
DENOM = (2*N-1)+(2*N+3)*(2*N+1)
UPPER = 2*N**3+N**2*(1+2*M)-4*N-3*DENOM-2*M+1
Y = FS*PLUS+2.0D0*UPPER/PLUS/DENOM
SLAM = Y
DENOM = (2*N+1)*(2*N+3)*MINUS
TERM = 2.0D0*(1+N)*DO-2.0
DGAMMA = PLUS*TERM/DENOM
DELTA = 2*FCTRL(N+1)*TERM/DENOM
PRINT 32, N, N**3, LAS, CHU, X
FORMAT(1255, 16, 10 )
IF (M.EQ.0) GO TO 31
C

COEF = COEF*DECTAO*CPI/100000000000
GAMMA = 2.0D0*CPI
IF (M.EQ.0) COEF = COEF*4.0D0*DECTAC*CPI/PI*ASG

DELMS = DELMNS/2.0D0*CPI

ANNS = ANNS*COEF

DMN = COEF/(2.0D0*PI)

E11 = -B*K1*W2 + LAM - DS
E12 = -B*K2*W2 - ES
E13 = 0
E14 = 0
E15 = 0
E21 = -2.0D0*ALPHA*K2*W2 - ES
E22 = -B*ALPHA*K2*W2 + ALPHA*LAM *KS
E23 = 0
E24 = 0
E25 = 0
E31 = -A3
E32 = 0
E33 = -K1*W2 + LAM + A5
E34 = -K2*W2 - GS
E35 = BS
E41 = 0
E42 = -ALPHA*A3
E43 = -2.0D0*ALPHA*K2*W2 - GS
E44 = ALPHA*(LAM + K2*W2) + THETAS
E45 = GS
E51 = FS*LMNS
E52 = A3
E53 = FS*LAM
E54 = LAM
E55 = LAM + YS - K1*BS*K5*W2

C
COMPUTE A'S, G'S, C'S'S

LAM = (1.0D0*W2 + ES1)/100*K1*W2 + DS - LAM
A1 = ALPHA*(LAM + K1*W2) + 5D0*KS*(1.0D0-V)
A2 = LW*FS + 1.0D0
A4 = K2*W2 - GS
A5 = 2.0D0*ALPHA*K2*W2 - GS
A6 = LAM - (1.0D0-V)*DS/2.0D0 + K1*W2

198
A7 = 2.00*BS*0.5 + ES
A8 = ALPHA*(LAM + BS*K0*0.5) + XS
G11 = -LAM*(A1*A2 + ALPHA*A3)
G12 = LAM?LW*A3 + A2*A4*0.5
G13 = A3*(ALPHA*A4 + LW*A1)
G21 = LAM*(ALPHA*A3*PS - A2*A5)
G22 = LAM*(FS*LW*A3 - A2*A6)
G23 = LAM*(FS*LW*A3 - A2*A6)
G31 = -BETAV*(GAMA*A7 + BMNS)
G32 = BETAV*(GAMA*A3)
G33 = BETAV*K0/ALPHA + AMNS*(FS*A7*BMNS)

100 FORMAT(E20.5,315)

COMPUTE G, H, W, W, PSE ETC
C11 = ACSWD*E11*E22
C12 = ACSWD*E12*E21
IF (J.LT.3) GO TO 151
IF (N.EQ.3) GO TO 903
GMNS(N1,J) = GMNS(N1,J) + C11*(G32*E45-E43*E35)
HMNS(N1,J) = HMNS(N1,J) + C11*(G34*E45-E43*E35)

C PRINT 801 #ASLEG ASLEG
801 FORMAT(6X,2F16.10)
WMNS(N1,J) = WMNS(N1,J) + (C11-C12)*(E33*E44-E34*E43)
ASLEG

HMNS(N1,J) = HMNS(N1,J) + DYN*(G31*G32*G32*G33*G33)/
0 DELOO ASLEG

103 CONTINUE
GMNS(N1,J) = GMNS(N1,J) + DYN*(G21*S1 + G22*S2 + G33*S3)/
ASLEG
HMNS(N1,J) = HMNS(N1,J) + DYN*(G11*S1 + G12*S2 + G13*S3)/
ASLEG

151 CONTINUE
IF (J.LT.3) LLMSH(N1,J) = LLMSH(N1,J) + DYN*BETAV*GAMA*A8/B0*ASLEG
IF (J.LT.3) PSMNS(N1,J) = PSMNS(N1,J) + DYN*BETAV*GAMA*A7/B0*ASLEG

140 CONTINUE

155 CONTINUE
C WRITE (8,6) (GMNS(N1,J) + HMNS(N1,J) + WMNS(N1,J) + HMNS(N1,J),
C DWMNS(N1,J) + J = 1,5)

150 CONTINUE
6 FORMAT(6X,6E19.6)
9 FORMAT(5X,2E20.6)
9 FORMAT(1NO)
8000 FORMAT(20X* TABLE 2 2 )
9000 FORMAT(30X*ROOTS SMALL OMEGA )
1200 FORMAT(20X* TABLE 2 3 )
1300 FORMAT(20X* LEGENDER SERIES COMPUTATION FOR SMALL SHELL )
RETURN
END
SUBROUTINE B4 (NUM,1)
IMPLICIT REAL(A-H,K-O-Z)
REAL*8 LANDO
COMMON/B,NUM/1,1
COMMON/B,NUM/1,1
COMMON/B,NUM/1,1
COMMON/B,NUM/1,1
CALL BFUNC(NUM,1)
CALL B2(NUM+1)
CALL B3(NUM+1)
N=NUM-1
LAMDA=N*(N+1)
CALL GROUP(NUM+1)
RETURN
END
SUBROUTINE BFUNC(NUM+1)
IMPLICIT REAL*8 (A-H,O-Z)
REAL*8 NU,JDERIV
INTEGER*4 LAMDA
REAL*8 LANO0
REAL*4 INTJU,INPUT2
INTEGER*4 J,K,J1,K1,K2,K3,K4
COMMON/PRE/UNJP(6),BJJO16,BW0(6)
COMMON/reek/BETA,BETANU,GAMA,GAMMA,
COMMON/GRCON/ALPHA,F
COMMON/ASL/ASLEG,ASLEGC
COMMON/BETAC/ASLEG,ASLGC
COMMON/OMEG1/OMEGAC1(31,30),OMEG2(31,21)
COMMON/GRCON/ALPHA,F
COMMON/DIMENSION(NUM,LAMDA)
COMMON/EQUIVALENCE(NU,V)
N=NUM-1
R=0.0
B*2=NU/(1.-NU)
W=OMEG(A(NUM+1))
AB=L00
W2=W*W
W3=W2*W
UNO=FUUNC(W,N)
UN1=FFUNC(W,NU)
JDERIV=NUO/W-UN1
DJ2=JDERIV*2
WD=W3*DJ2
UQJ=UNO/W/JDERIV
W4=W2*W2
DB2=UNO*JDERIV**2/UN0**W
DB3=W2*DJ2
DB4=UNO*UN0/(4.-LAMDA)
D31=(DB2+DB3+D54)/WD
C IF(NUM-LT.5.AND.1.LT.J) WRITE(8,12) UNO,UN1,JDERIV,D9
C X=R*W
UN=FUUNC(X,N)
UNJP1J=UN/JDERIV/W
BFUN(J)=JS(K1,F*U0J)
BDP1J(J)=JS(KS,F*00J)
IF(NUM. EQ.1.AND.1.LT.21) BDQ2=0.00
IF(NUM. EQ.1.AND.1.LT.21) BJ10(J)=0.00
IF(NUM. EQ.2.AND.1.LT.21) BJ10(J)=0.00
IF(NUM. EQ.3.AND.1.LT.21) BJ10(J)=0.00
IF(NUM. EQ.4.AND.1.LT.21) BJ10(J)=0.00
IF(NUM. EQ.5.AND.1.LT.21) BJ10(J)=0.00
IF (NUM.GE.1.AND.J.GT.1) BWO(JJ)=BJO(JJ)
CONTINUE
PRINT 12,BFUN(IJ),DDP(IJ),BJO(JJ),BD02
IF(J.EQ.1) R=0.0
R=R+0.2
CONTINUE
FORMAT(1X,4E20.6)
RETURN
END
SUBROUTINE B2(INUM,II)
IMPLICIT REAL*8(A-H,K-O,Z)
REAL LANOO
REAL MSINT,INPUT1,INPUT2
COMMON /GRCON/ ALPHA,F
COMMON/B/BFUN(61),BDP(61),BJO(61),LANOO,LAM0AN
COMMON/AK/K1,K2,KS,K5
COMMON/DEL/DELNP1,DELNP2(61)
COMMON/OEG/OEGL(31+30),OMEGA(31,32+)
B=OMEGA(31,32),SOMG(31,2)
COMMON/PUTIN/KS1,V1,A1,H1,Y,E1,RHOF,RHOS1
EQUIVALENCE(INU,V1)
XS=OMEGA(INUM,1)
XA=OMEGA(INUM,II)
LAMDO2=LAM0AN-2
A=4.*ALPHA*(K1*K2)*KA/2./II.-NU)*2
B=ALPHA*(K1*K2)*LAMDO2*(K1*K2)/K5/2.*ALPHA*(K1+1)/KS*X
C=LAMBDA2*ALPHA*LAMDO2*(1.+2.*ALPHA/K5)
DELN1=II.-NU1+C1./K5*2
DELNP1=II.-NU1+4./X0-2.+8./X0
FORMAT(1X,4E20.6)
RETURN
END
SUBROUTINE B3(INUM,II)
IMPLICIT REAL*8(A-H,K-O,Z)
REAL LANOO
REAL K1,K2,KS,K5
REAL MSINT,INPUT1,INPUT2
REAL INUM,LANOO
REAL MSINT,INPUT1,INPUT2
REAL K1,K2,KS,K5
REAL MSINT,INPUT1,INPUT2
COMMON /GRCON/ ALPHA,F
COMMON/B/BFUN(61),BDP(61),BJO(61),LANOO,LAM0AN
COMMON/AK/K1,K2,KS,K5
COMMON/DEL/DELNP1
COMMON/OEG/OEGL(31+30),OMEGA(31,32+)
SOMG(31,2)
COMMON/PUTIN/KS1,V1,A1,H1,Y,E1,RHOF,RHOS1
EQUIVALENCE(V1,NU1)
XS=OMEGA(INUM,1)
X=OMEGA(INUM,II)
P=S*P*X
P2=S2*P*X
ALAMDN=ALPHA*LAM0AN
ALPHAP=ALPHA0P
ALPHAP2=ALPHA0P2
VL=1.-NU1
KSNU=V1./2./KS
C=ALAMDN*KSNU*(1.-2.*K5*ALPHA)
C1=K*K2*ALPHAP2*C
C2=K*K2*P2*C
O2*Z=II.-NU1/KSNU
DIM 10 J=1,6
DB1=DBDP(J)
BU=BFUN(J)
BF=BFU0+O2+LAM0AN
DG=Z*BF/X+P2*DB1
C102=Z*KF1P(C1=BF-KSNU*LAM0AN)
GZ=Z*ALPHP2-KSNU
BS=1+NU*KNU
GSBS=Z*KF2P((G0=BF-LAM0AN+BS))
FS=LAM0AN+10J=012
C INCONSISTENT NOMENCLATURE: THIS IS NOT THE CONSTANT FS
C SPECIFIED IN COMMON STATEMENT IN OTHER ROUTINES
C3=ALAMON+KSNU(1-2*KF1)
C3JETC=(K1P2+C31(20=KREALPHP*BF-0*KREALPHP2+C31*DB1)
CFINAL=(K2P2-KSNU)+(G0=0*2*ALPHP*BF)
DELNP2(J)=C102+LSUS*DBS+C3JETC-FINAL
10 DELNP2(J)=C102+LSUS*DBS+C3JETC-FINAL
WRITE(8,12) 0ELNP2(J),DELN2(J),J=1,6
PRINT 12,0ELNP2(J),DELN2(J),J=1,6
12 FORMAT(I1X,6E18.6)
RETURN
END
SUBROUTINE GROUP1(INU,1)
IMPLICIT REAL*8(A-H,O-Z)
REAL*BINT,INPUT1,INPUT2
REAL*8 K1,K2,K2,K2,K2
REAL*8 LANDO
REAL*8 NU, NW(J=3:6), NG(J=3), NLAM(J=2), NPSI(J=2)
REAL*8 NLAM+KPS(J=1), NPSIC+KPSIC(J=1), NLAMC+NLAMS
COMMON/DELEN/CLENP2,DELP2(J=6),DELNP2(J=6)
COMMON/VART/DN1(J=3),DNR1(J=3),DNG1(J=3),DONC(J=3),DN=SF(J=3),DONC(J=3)
0NHS1(J=3),0NGS(J=3),DNW1(J=3),DNW2(J=3)
COMMON/ASL/ASLEG,ASLEGJ,ASLEGK,ASLEGK,ASLEGK,ASLEGK,ASLEGK,ASLEGK,ASLEGK
COMMON/AS3/AS3
COMMON/BFE/B1
COMMON/OMEG1/OMEG1,OMEG1
COMMON/T1/T1
COMMON/PUBL/IU(I),IU(I),IU(I)
COMMON/B/8FUN(J),DBDP(J),BFU0(J),LAM0AN
COMMON/GREEK/BErA,BETANU,GAMMA,THETAS
COMMON/COS/Z/OH/ONH10,0NH10
COMMON/PRE/UNJ(J),B1J(J),BWO(J)
COMMON/SOS/SSI201,SSI01,SSI01,TAU,SCC,CTAS0,HO
COMMON/CONST/A5,AS5,AS5,F5,GSY5,AX
COMMON/GRCON/ALPHA,F
COMMON/GOMP/P2,LAM0AN,ALPHOM
COMMON/AK/K1,K2,K2,K2,K2
COMMON/VAR/LAM0AN(J=2),NPSI(J=2),NPSIC(J=2),NPSIC(J=2),NLAMC(J=2),NLAMS(J=2)
0NHS(J=3),ONH1(J=3),DNW1(J=3),DNW2(J=3)
COMMON/EQUIVALENCE(NU,J1)
N=INU-1
ALL1=10
ALL2=KNS(K1)
ALL2=KNS
A(1,1) = A(1,2) = A(1,3)
A(2,1) = A(2,2) = A(2,3)
A(3,1) = A(3,2) = A(3,3)

DO 3 J = 1, 3
DO 3 L = 1, 2

3

DNC(JJ) = 0, 0
DHI(JJ) = 0, 0
DGC(JJ) = 0, 0
DNG(JJ) = 0, 0
DNH(JJ) = 0, 0
DNHCC(JJ) = 0, 0
DNHCS(JJ) = 0, 0
NLAMLL(JJ) = 0, 0
PSISL(JJ) = 0, 0
PSICL(JJ) = 0, 0
NLAMCL(JJ) = 0, 0
NLAMS(JJ) = 0, 0

DO 3 IK = 1, 6

DNC(JJI + IK) = 0, 0

IF (NUM + EQ.1, AND + EQ.1) GO TO 23

DO 23 J = 1, 3

IF (IK + LT + 3) GO TO 60

CDEL = CMN(NUM, M) TASIS DELNI
AP2GS = 2ALPHA + GS
P2GS = K * P2 + GS
GSDLAM = GS * LAM0AN
ALTH = ALPHMONKR, ALAMON, THETAS
P2LAM = K * P2 + LAM0AN + AS
BAX = ALPHMONKS, ALAMON, KS
FBAX = P5 * BAX
BPE5 = 26 * P2 - ES
DUMMY = ALAMDA(N + M)
AMNB = AMH(N + M) BETANU
GAMN = GSMAMNB
BAXE = B(ALPHA, KS - 2) P2 + ALAMON, XS + ES
DUMMY = DELTA(N + M)
DUMMY = GAMMIN(N + M)
BMNB = BMN(N + M) BETANU
BMNAX = BMNBPAK
ABPES = ALPHA, ABPES
ALBPE5 = A3P5 * LAM0AN
DO 14 IK = 6

BU = DFUN(X)
BAL = BU + P2 * LAM0AN
BALT = BAL + YS
BALTH = BAL + THETAS
I F I I . L T . 3 )  N P S I I I(L)+N P S I I I(L)+N P S I I I(L)+ A S L E G C  
I F I I . L T . 3 )  N P S I I I(L)+N P S I I I(L)+N P S I I I(L)+ A S L E G C  
24 CONTINUE  
C A L L A S L G Q I N ' J M . Z A S ) 1927  
D N W C I J ) = D N W C( J ) + O N W * A S L G D C 1928  
D N H C I J ) = D N H C( J ) + O N H * A S L G D C 1929  
D N W 2 I J ) = D N W 2( J ) - D N W * A S L G D S M 1931  
D N H 2 I J ) = D N H 2( J ) - D N H * A S L G D S M 1932  
D N G 2 I J ) = D N G 2( J ) - D N G * A S L G D S M 1933  
I F I I . L T . 3 )  L=J 1934  
I F I I . L T . 3 )  N L A M C I L I = N L A M C I L I * N L A M C I L I * A S L G D C 1935  
I F I I . L T . 3 )  N P S I C I L I = N P S I C I L I * N P S I C I L I * A S L G D C 1936  
I F I I . L T . 3 )  N L A M C I L I = N L A M C I L I * N L A M C I L I * A S L G D C 1937  
I F I I . L T . 3 )  N P S I C I L I = N P S I C I L I * N P S I C I L I * A S L G D C 1938  
21 CONTINUE  
22 CONTINUE  
23 CONTINUE  
I F ( N U N . G T . 3 . A N D . I . G T . 3 )  G O T O 7 0 1940  
C W R I T E ( L ) ( D N H 1( J ), D N H 2( J ), D N W 1( J ), D N W 2( J ), D N G 1( J ), D N G 2( J ), D N H C C I J ) 1941  
C 7 0 C O N T I N U E  
6 F O R M A T ( * X . 7 E 1 6 . 6 ) 1942  
7 F O R M A T ( * X . 6 E 1 6 . 6 ) 1943  
8 F O R M A T ( * X . 6 E 1 6 . 6 ) 1944  
8 0 0 0 F O R M A T ( * X . 6 E 1 6 . 6 ) 1945  
9 0 0 0 F O R M A T ( * X . 7 E 1 6 . 6 ) 1946  
R E T U R N  
E N D  
F U N C T I O N C M N ( N U M , H T A S )  
R E A L * B U S I N E S S  
C O M M O N / J C O N S T A S R S + E S + F S + G S + Y S + X S 1948  
C O M M O N / A - H . K . O - Z B 1953  
K E Y  
N U M = 1 1958  
P I = 3 1959  
N U N = 1 1960  
P L U S S = F C T R L M * M 1961  
M I N U S S = F C T R L ( N - M ) 1962  
D = 1 1963  
I N S U S = F C T R L N * N 1964  
D = 1 1965  
I N S U S = F C T R L N 1966  
I F I S S I J E ) G T . 0 . 0 D ) T T = T A S 1967  
I F I S S I J E ) G T . 0 . 0 D ) F F = F E E C 1968  
I F I S S I J E ) L E . 0 . 0 D ) F F = F E E C 1969  
I F I S S I J E ) L E . 0 . 0 D ) T T = T A S 1970  
I F I T A S G T . T T . T A S C = T T 1971  
I F ( F E E G T . F F ) F E E C = F F 1972
IF (FEE.LT.FEEC) FEEC=FEE
IF (TAS.LT.TASC) TASC=TAS
IF (SSJE.GT.0.00) FEEC=FEE
IF (SSJE.GT.0.00) TASC=TAS
6 CONTINUE
IF (M.EQ.0) GO TO 3
PI=4.O#02。
TETAN=A#02。
THETA=2#0TASC*2。
AL=(ATN DSIN(TETAN))/PI
AC=(TASC DSIN(THETA))/PI
CALL ASLEG(INUM,M,FEN,ATN)
ASGI=ASLEG
CALL ASLEG(INUM,M,FEEC,TASC)
ASG2=ASLEG
IF (AC.EQ.0.00) C2=0.00
IF (AC.EQ.0.00) GO TO 12
C2=DSIN(NPTASC*ASG2/AC/H)
12 CONTINUE
FE=PI/2.00
TA=PI/2.00
IF (AL.EQ.0.00) C1=0.00
IF (AL.EQ.0.00) GO TO 15
C1=DSIN(4.0#0/2.1*DSIN(M#ATN)/AL/M#ASGI
15 CONTINUE
GO TO 5
3 C1=0.00
C2=0.00
5 C=C1+C2
CMN= BETAN#K#B9C
RETURN
ENTRY DLAMDA(N,M)
DENUM=(2#N-1)*(2#N+1)*(2#N+3)*2#N+1)
UPPER=2#N+3#N+3+4#N+3#N+2#N+1
Y=FS#PLUS#S2#UPPER/MINUS/DENUM/PI
DLAMDA=Y
RETURN
ENTRY AMNIN(M)
AMN=0#B9C
RETURN
ENTRY DGAMM(N,M)
DENUM=(2#N+1)*(2#N+3)*PCTRL(N-M-1)
TERM=(2.0#0)(1#M#0)-2.0)
DGAMM=PLUS#TERM/DENUM/PI
YG=DGAMM
RETURN
ENTRY DELTA(N,M)
DENUM=(2#N+1)*(2#N+3)#MINUS
TERM=N#0#2.0
DELTA=2.0#PCTRL(M+1)#TERM/DENUM/PI
YG=DELTA
RETURN
ENTRY BMN(N,M)
RMN=0#B9C/2.0#3.1#159265350
RETURN
ENTRY BMNN(M,N)
C ENTRY BMN MUST BE CALLED SUBSEQUENT TO ENTRIES DELTA AND DGAMM
BMN=YG+YD#B9C
RETURN
END
SUBROUTINE ASLGO(INUM,M,Z,TAS)
IMPLICIT REAL*8(A-H,O-Z)
INTEGER*4 DEN,OP1,C1,OMNP1,OMM1,APN1,AMN2,AMN3
INTEGER*4 APN1,GMNP2,OMM2,GMN
COMMON/ASL/ASLEG,ASLEGC,ASLEG+ASLGS/ASLGD+ASLGD+ASLGS/ASDD/
COMMON/ASL2/ASLEG2,ASLEG2C,ASLEG2S
COMMON/ANGLE/ FEE,GT,FEN,ATN,CUM,CTAS,CFEE,X
N=NUM-1
K=NOTAS
N1=NUM+1
OME=DSIN(FEE)1#2
CSEC2=1.0D0/OME
CSEC2=N#CSEC2
LN=N#(N+1)
CALL ASLEG1(NUM,M,Z,TAS)
AB=-(N+1)2ASLEG
AB=NCSEC-LN#ASLEG
CALL ASLEG1(N1,M,Z,TAS)
AC=(N-M+1)#ASLEG
Y=(AB+AC)/DSIN(FEEn)
ASLEG=Y
ASLGD=Y#D(C(S1TM))
ASLGDS=Y#D(S1TM)
C PRINT 2+ASLGC+ASLGD
2 FORMAT(10X#E20.10)
N=NUM-1
DELTA=0.05
ALPH=DABS(ALPHA)
A=-ALPH
ASLEG=0
Z#DOSIS(A+5#DELTA)
CALL ASLEG1(NUM,M,Z,TAS)
ASLEG=ASLEG#DELTA+ASLEG
A*A#DELTA
IF (A .GT. ALPH) RETURN
GO TO 10
END
SUBROUTINE ASLEG1(NUM,M,Z,TAS)
IMPLICIT REAL*8(A-H,O-Z)
COMMON/ASL/ASLEG+ASLEG+ASLEG+ASLEG/ASLGD+ASLGD+ASLGS/ASDD/
ASLEGL2=ASLEG2C+ASLEG2S
N=NUM-1
C PRINT 5+NUM
ASLEG=0
IF(M .GT. N OR N .LT. 0) GO TO 200
ASLEG=1
IF(N .EQ. 0) GO TO 200
FC=FCTRL(N+1)
FR=FCTRL(N-N)
FM=FCTRL(NM)
C PRINT 10+2*#0+PH+FM
FIRST=FC/(Z**N#M#FM)
ZP=1.0D0-Z*Z
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IF (ZP.EQ.0.00) ZPM=0.00
IF (ZP.GT.0.00) ZPH=ZP**(1/2)
A=1
B=1
C=1
D=1
E=1
SUM=1
DO 100 I=1,100
A=A*(N-M-I+1)
B=B*(N+M+1)
C=C*I
D=D*(N+I)
E=E*(Z-1)/Z
TERM=A*B*E/C*M
SUM=SUM+TERM
100 IF (ABS(ITERM).LE.1.D-6) GO TO 101
CONTINUE
101 ASLEG=FIRST*ZPM*SUM
200 Y=ASLEG
TH=ATAS
ASLEGG=Y*DCOS(TM)
ASLEG=Y*DSIN(TM)
5 FORMAT (2I5)
10 FORMAT (215,3F20.0)
RETURN
END
REAL FUNCTION FCTRL(N)
FCTRL=1
IF (N.LE.0) RETURN
DO 10 I=1,N
FCTRL=FCTRL*I
RETURN
END
//LKD=SYSLIB DD DSN=SYS1,FORTLIB
// DD DSN=SYS2,FORTSSP,DISP=SHR
//LKD=SYSLIN DD DSN=CEL3AOSET
// DD DSN=TSO206,SBF=OBJ,DISP=SHR
// OVERLAY ONE
INSERT B4,CN=GROUP1,ASLГО,BFUNC,B2,B3
OVERLAY ONE
INSERT ROJ1,FUNC=BSLRA,BESSEL,SERIES,BRECUR,FRECUR
OVERLAY ONE
INSERT ROJ2.
OVERLAY ONE
INSERT PRPLDT,YO3PCK
//GG,FTO1FO01 DD SYSOUT=A
//GG,FTO2FO01 DD SYSOUT*A
//GG,FTO3FO01 DD SYSOUT*A
//GG,FTO4FO01 DD SYSOUT*A
//GG,FTO5FO02 DD SYSOUT=A
//GG,FTO7FO01 DD SYSOUT=B
//GG,FTO1FO01 DD SYSOUT=A
//GG,FTO1FO01 DD SYSOUT=A
//GG,FTO2FO01 DD SYSOUT=A
//GG,FTO1FO01 DD SYSOUT=A
//GG,FTO1FO01 DD SYSOUT=A
//GG,FTOSIN DD *