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DISSE rtA TION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Mark Joseph Van de Walle, B.A., B.S., M.S.

* * * * * *

The Ohio State University
1976

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Department of Electrical Engineering
ACKNOWLEDGMENTS

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CHAPTER I
INTRODUCTION

One major advantage an antenna array has over a continuous aperture antenna is that considerable control can be exercised over the pattern. This control can be in the form of main beam steering in which case there is real-time pattern control, or possibly in increasing the directivity of the array where the control is somewhat static.

The control over the array pattern is achieved by modifying the weighting or excitation coefficients of each element in the array. The adaptive antenna array is a system that takes advantage of the pattern control capability of an array with closed loop control.

An adaptive antenna array is composed of an array of antenna elements connected to variable weighting coefficients controlled by a signal processor. The processor is configured to act in a feedback loop arrangement in such a way that the array output is continuously optimized in some sense.

The most notable characteristic of an adaptive antenna array is its capability to produce pattern nulls in the direction of interfering or undesired signal sources while still receiving some desired signal from another direction. In this manner, then, it can be said that the adaptive array acts as a spatial filter by attenuating signals arriving at the antenna array from some directions and accepting those from
another direction. It is in this way that the adaptive antenna optimizes the array output.

Generally speaking an optimization of this type requires some a priori knowledge of the desired array output or knowledge about the signal it is desired to receive. This might be the frequency or direction of arrival of the desired signal. In most adaptive arrays the array output is compared with some desired array output and appropriate adjustments to the array weights are made until some optimum array output is obtained.

A. History

Adaptive signal processors have been applied to many different types of electronic systems to optimize or adjust some performance index. These include radar systems, arrays of hydrophones, hardwired communication systems and general R.F. antenna systems. In this paper we will only be concerned with an R.F. receiving antenna system.

The relevant research in this area includes an early paper by Shor[1] who dealt with an array of hydrophones. It was shown that the output S/N ratio could be improved by using an iterative technique to adjust the hydrophone weighting coefficients. The adjustments were based on direct calculation of correlations between signal and noise in each hydrophone. The mathematical method of steepest descent was used to maximize the S/N ratio. It was assumed that a reasonable approximation of the desired signal was available from some source. This requirement is a characteristic of most adaptive arrays.

Appelbaum and Howells[2] presented an analysis of an R.F. antenna array with weights that were derived from the optimization of an
arbitrary S/N ratio. A beam steering signal was assumed in the analysis that would keep a main beam in the direction of the desired signal source.

An algorithm that minimized the Least Mean Square error between the array output and some desired signal was presented by Widrow, et al [3]. In this case, the Least Mean Square (LMS) algorithm was implemented using the method of steepest descent. Widrow, et al presented a feedback rule for control of the element weights that in an analog manner made the adaptive system a real time system. He also showed an equivalent digital feedback control processing scheme utilizing data samples from the antenna elements.

In a subsequent paper, Griffiths[4] presented a modification to the LMS algorithm requiring a knowledge only of the direction of arrival of the desired signal and its spectral density for implementation of the feedback rule.

With the basic theory for the adaptive array available including the control laws, Brennan, et al[5] published an analysis of control loop noise or weight variance in two feedback algorithms that optimize S/N ratio.


Compton[7] and Zahm[8] noted a power equalization phenomena in a modified LMS algorithm permitting the acquisition of weak signals in the presence of strong interfering signals.
Others have contributed significantly to the understanding of adaptive arrays, especially in the communications area. Huff[9] and Reinhard[10] addressed the problem of integrating an adaptive array into a specific type of communication system.

Work is being undertaken on recursive algorithms utilizing small minicomputers for optimization calculations in real time[11]. Most recently a review and introductory paper following the basic Howell-Appelbaum development was published by Gabriel[12].

B. **Statement of Problem**

Adaptive array analysis presented in the current literature concentrates on arrays with conventionally spaced isotropic elements, i.e., spacings on the order of one-half wavelength. This provides for large element-to-element phase differences and makes spatial recognition of a source by the adaptive processor easier. In an analysis of this type with isotropic elements, the antenna array is essentially treated as a means to obtain a linearly changing phase difference on the element signals. With this as a set of input signals, the adaptive processor then generates the respective element weights or, in a sense, the array feed network. Analysis of adaptive arrays of this type requires little knowledge of the antenna array itself and somewhat simplifies the problem. It also restricts the application of adaptive antenna arrays to those antennas having relatively large apertures.

However, applications exist where aperture size is limited but it is still desirous to use an adaptive processor. In an application of this type, where element spacing is closer, element coupling starts to
become significant. There exists then mutual impedances between antenna elements that cannot be neglected in the analysis. Thus the problem that is addressed in this study is that determining the performance of adaptive arrays with closely spaced elements with mutual coupling effects taken into account.

C. Scope of Study

To analyze adaptive antenna arrays that have closely spaced elements two basic questions need to be investigated. One question is, what effect does large mutual impedance terms have on an adaptive array. The second is, given a small aperture, what is an optimum choice of the antenna array with regard to number of elements and element spacing. These are the two basic questions to be investigated in this paper.

As stated earlier when element spacings are large, mutual impedance effects can usually be neglected. The adaptive array analysis employed when mutual impedance effects are neglected generally starts with the element signals being treated as open circuit voltages. These signals are then the input to the adaptive processor. The effects of mutual impedance and the impedance match between the antenna and processor are neglected.

An analysis of an adaptive antenna array will be developed in this paper that includes the effects of mutual coupling. To accomplish this analysis, the antenna array and its external stimulus in the far field of the array will be modeled as an L+1 terminal network with coupling between all elements included. With this as a model the element voltages or the adaptive processor input can be expressed in terms of the mutual coupling and element load impedances.
The relationships for the steady state element weights for the LMS algorithm using the above model are then developed. Comparisons of the steady state weights and received power for tightly and loosely coupled arrays are made.

Utilizing the steady state weights for the LMS algorithms, the field pattern is calculated to give some understanding of the aperture size and performance tradeoffs in adaptive systems.

Some calculations for various length arrays having different numbers of elements are presented to clarify the effects of mutual coupling and limited aperture size on the performance of adaptive antenna systems. It is shown how additional elements in a fixed aperture improve performance in some cases.

The transient analysis for the LMS algorithm for coupled antenna elements is also presented. By presenting a number of calculated results, the effects of mutual coupling is shown on the transient response.

For comparison, a similar analysis of a modified LMS algorithm is presented. Both the steady state and transient solutions are presented along with sample numerical results.
CHAPTER II
THE ANTENNA ARRAY AS AN L-TERMINAL NETWORK

What is initially required is an expression for the element voltage at the element output when mutual coupling is taken into account. These voltages can then be used as the input signals to the adaptive processor. The required expression can be obtained by considering the L element antenna array and an outside stimulus as an L+1 terminal linear, passive, bilateral network. This is shown functionally in Fig. 1.

Fig. 1--Antenna array as L+1 terminal network.
Referring to Fig. 1, each port of the L-element array is shown terminated in a known load impedance, Z_L. The array has as its driving source a generator with open circuit voltage V_g and internal impedance Z_g. The source is located in the far field of the antenna array such that

\[ i_k \ll i_s \quad k=1, \ldots, L \]

Using standard notation, one can write the Kirchhoff relations for the L+1 terminal network as

\[
\begin{align*}
\mathbf{v}^1 &= i_1z_{11} + i_2z_{12} + \cdots + i_Lz_{1L} + i_sz_{1s} \\
\mathbf{v}^2 &= i_1z_{21} + i_2z_{22} + \cdots + i_Lz_{2L} + i_sz_{2s} \\
\vdots &= \vdots \\
\mathbf{v}^L &= i_1z_{L1} + i_2z_{L2} + \cdots + i_Lz_{LL} + i_sz_{Ls} \\
\mathbf{v}^s &= i_1z_{s1} + i_2z_{s2} + \cdots + i_Lz_{sl} + i_sz_{ss}
\end{align*}
\]

It is clear that if open circuit all elements in the array that

\[
\begin{align*}
i^1 &= 0 \\
\mathbf{v}^i &= \mathbf{v}_{oi} = z_{is}i_s
\end{align*}
\]

Making use of the relationship between terminal current and load impedance,

\[ i_i = -\frac{\mathbf{v}^i}{Z_L} \quad i = 1, \ldots, L \]

The first L equations in Eq. (2) can be arranged in matrix form to be
\begin{equation}
\begin{pmatrix}
\left(1 + \frac{z_{11}}{z_L}\right) & \frac{z_{12}}{z_L} & \cdots & \frac{z_{1L}}{z_L} \\
\frac{z_{21}}{z_L} & \left(1 + \frac{z_{22}}{z_L}\right) & \cdots & \frac{z_{2L}}{z_L} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{z_{LL}}{z_L} & \cdots & \left(1 + \frac{z_{LL}}{z_L}\right) & \frac{z_{LL}}{z_L}
\end{pmatrix}
\begin{bmatrix}
v \\
v \\
\vdots \\
v \\
v_0 \\
v_0 \end{bmatrix}
= \begin{bmatrix}
v_0 \\
v_0 \\
\vdots \\
v_0 \\
v_0 \end{bmatrix}
\end{equation}

or more compactly

\begin{equation}
Z_0 \mathbf{v} = \mathbf{v}_0
\end{equation}

Assuming $Z_0$ is non-singular, we can solve for the antenna element voltage column vector $\mathbf{v}$ as

\begin{equation}
\mathbf{v} = Z_0^{-1} \mathbf{v}_0
\end{equation}

Details of this formulation can be found in Appendix A.

What is normally done in analyzing adaptive antenna systems is to make the element spacing large enough so that mutual coupling between the elements is small and consequently the matrix $Z_0$ becomes diagonal. If one makes the assumption that the self impedance terms are all equal, then the input signal vector is just the open circuit voltage vector multiplied by a trivial scaling factor involving the self and load impedance terms. The case that will be considered here, however, is where the mutual impedance terms are not small compared to the self impedance values. To this end, arrays with spacings of one-half wavelength and less will be considered.

It should be noted here that the matrix $Z_0$ is a normalized impedance matrix, normalized to the load impedance. It acts like a
transformation matrix, transforming the open circuit element voltages to the terminated voltages.

If we consider an array of isotropic elements we can determine the open circuit voltage vector \( V_0 \) with the far field plane wave approximation to array analysis. If the array is not composed of isotropic elements but all elements are similar, then the column vector \( V_0 \) would be the element pattern multiplied by the isotropic response. However under any assumptions, the column vector \( V \) represents the generalized antenna element signals and as such represents the input signals to the adaptive processor. Before the isotropic response is calculated, the definition of the coordinate system and the formulation for an incident plane wave must be made.
CHAPTER III

ANTENNA ARRAY AND SIGNAL ENVIRONMENT MODELS

To derive the open circuit voltage at the output of each antenna element, the signal incident on the array will be approximated by a plane wave. The basic terminology will be defined for handling the plane wave. Once the basic definitions have been established, the open circuit voltages for the array under consideration will be derived.

A. Coordinate System Definition

For simplicity, the antenna array will be a linear array along the x-axis and the plane wave will contain the z-axis. That is, a one-dimensional array is being considered. The plane wave can be represented graphically as shown in Fig. 2. Utilizing a conventional coordinate system, the incoming plane wave makes an angle of $\phi$ with the x-axis and 90° with the z-axis. The vector $\vec{r}$ is chosen to lie along the x-axis, while $\hat{n}$ is a unit vector in the direction of the incoming plane wave. For a uniformly spaced array of isotropic elements along the x-axis with element spacing $d$ and with the geometrical center of the array at the center of the coordinate axis as shown in Fig. 2, we have for the open circuit voltage in the $k$th element

$$v_{ok}(t) = a_o(t) e^{j\omega \left( t - \frac{\vec{n} \cdot \vec{r}_k}{v} \right)}$$

(8)
Fig. 2—L element antenna array in coordinate system.

where

$a_0(t)$ represents the time varying amplitude of the plane wave,

$\omega$ represents the radian frequency of the plane wave,

$v$ represents the velocity of wavefront,

$r_k = \left(\frac{k - 1}{2}\right) d \hat{x}$ \quad k = 1, \ldots, L

$\hat{N} = \cos \phi \hat{x} + \sin \phi \hat{y}$,

$L$ = number of elements,
and

\begin{equation}
\vec{N} \cdot \vec{r}_k = \left( k - \frac{k+1}{2} \right) d \cos \phi.
\end{equation}

For the case shown in Fig. 2, the generalized column vector representing the open circuit voltages at each element of the array is

\begin{equation}
V_0 = \begin{bmatrix}
\frac{j\omega}{N} \left( t - \frac{\vec{N} \cdot \vec{r}_1}{v} \right) \\
a_0(t) e^{-j\omega \left( t - \frac{\vec{N} \cdot \vec{r}_1}{v} \right)} \\
\vdots \\
a_0(t) e^{-j\omega \left( t - \frac{\vec{N} \cdot \vec{r}_k}{v} \right)}
\end{bmatrix}
\end{equation}

All signals in the array environment can be defined using the general form of Eq. (10) and then Eq. (7) to arrive at the processor input voltages.

B. Signal Environment Definition

To evaluate the performance of different antenna arrays we must define a signal environment for the various systems to operate on. For the purposes of this study, we shall define the signal environment to contain one desired signal \(d(t)\) and \(M\) interfering or undesired signals \(i_k(t)\) \(k=1, \ldots, M\). These signals are incident upon the \(L\) element array shown in Fig. 3. The desired signal is defined as

\begin{equation}
d(t) = a_d(t) \cos(\omega_d t + \phi_d(t)).
\end{equation}
The desired signal has both amplitude and phase information where

\[ a_d(t) = \text{the amplitude modulation assumed to be wide sense stationary}, \]
\[ \omega_d = \text{the radian frequency of the desired signal}, \]
\[ \phi_d(t) = \text{the random phase modulation and which is uniformly distributed from } [0, 2\pi] \text{ and is of zero mean}. \]

The \( i \)th interfering signal is given by

\[ i_i(t) = b_i(t) \cos(\omega_i t + \phi_i(t)) \]

where

\[ b_i(t) = \text{the arbitrary amplitude modulation assumed to be wide sense stationary}, \]
\[ \omega_i = \text{the radian frequency of the interfering source}, \]
\[ \tilde{\phi}_i(t) = \text{the random phase modulation which is uniformly distributed } [0, 2\pi] \text{ and is of zero mean.} \]

In addition to the desired and interfering sources, we will assume a thermal noise voltage in each antenna element to be \( n_k(t) \) at the kth element. This voltage is defined to be normally distributed, zero mean and independent element to element.

For the purposes of this study we will assume narrow band signals and noise terms so that the impedance terms in \( Z_0 \) do not change appreciably over the signal bandwidths.

Thus the open circuit voltage at each element is then the sum of the desired signal and M interfering signals, modified by the appropriate time delay associated with the element spacing and angle of incidence plus the thermal noise voltage. Then using the notation set forth earlier the open circuit voltage at the kth element can be expressed as

\[ V_{ok}(t) = a_0 \left( t - \frac{\overrightarrow{N}_d \cdot \overrightarrow{r}_{kd}}{v} \right) e^{j\omega_d \left( t + \frac{\tilde{\phi}_d(t)}{\omega_d} - \frac{\overrightarrow{N}_d \cdot \overrightarrow{r}_{kd}}{v} \right)} \]

\[ + \sum_{i=1}^{M} b_i \left( t - \frac{\overrightarrow{N}_i \cdot \overrightarrow{r}_{ki}}{v} \right) e^{j\omega_i \left( t + \frac{\tilde{\phi}_i(t)}{\omega_i} - \frac{\overrightarrow{N}_i \cdot \overrightarrow{r}_{ki}}{v} \right)} \]

\[ + n_k(t) \]

where

\[ \frac{\overrightarrow{N}_d \cdot \overrightarrow{r}_{kd}}{v} = \text{the time delay of the desired signal at the kth element with respect to the geometrical center of the array, due to the element spacing,} \]

and
\[ \frac{\vec{N}_i \cdot \vec{r}_{ki}}{v} = \text{the time delay described just above but for the } \]
\[ \text{ith interfering source,} \]
\[ v = \text{the velocity of light.} \]

If we further assume that the element time delays are small compared to the rate of change of the modulation envelopes, then Eq. (13) can be approximated as

\[ V_{ok}(t) = d(t)e^{-j \frac{2\pi}{\lambda_d} \vec{N}_d \cdot \vec{r}_{kd}} + \sum_{i=1}^{M} i_i(t)e^{-j \frac{2\pi}{\lambda_i} \vec{N}_i \cdot \vec{r}_{ki}} + n_k(t) \]

where \( \lambda_d \) and \( \lambda_i \) are the wavelengths of the desired and ith interfering sources, respectively.

Thus Eq. (14) is the kth element of a column vector describing the open circuit voltage at the kth terminal due to a desired signal source, the interfering sources and a noise voltage. Since each of the column vectors is made up of a time varying signal multiplied by a column vector representing the associated phase delays, it is convenient to define a column vector for the phase delays associated with each signal. Thus we define

\[ \alpha_d = \begin{bmatrix}
- \frac{2\pi}{\lambda_d} \vec{N}_d \cdot \vec{r}_{ld} \\
\vdots \\
- \frac{2\pi}{\lambda_d} \vec{N}_d \cdot \vec{r}_{kd} \\
e \\
e \\
\vdots \\
e \\
e \\
e
\end{bmatrix} \quad \alpha_i = \begin{bmatrix}
- \frac{2\pi}{\lambda_i} \vec{N}_i \cdot \vec{r}_{ki} \\
\vdots \\
- \frac{2\pi}{\lambda_i} \vec{N}_i \cdot \vec{r}_{li} \\
e \\
e \\
\vdots \\
e \\
e \\
e \\
e
\end{bmatrix} \]

where \( \alpha_d \) is the column vector representing the phase delay of the desired signal and \( \alpha_i \) represents the phase delay for the ith interfering signal.
With these definitions, the open circuit voltage column vector can be written as

\[ V_0(t) = d(t) \alpha_d + \sum_{i=1}^{M} i_i(t) \alpha_i + n(t) \]

Now utilizing Eq. (7) with Eq. (16) we have defined the generalized input signals to the adaptive processor when mutual coupling is taken into account.
CHAPTER IV
LEAST MEAN SQUARE (LMS) ADAPTIVE ALGORITHM

To determine what effect mutual coupling and antenna array geometry have on an adaptive antenna system, it is helpful to choose several adaptive algorithms and analyze the adaptive array when mutual coupling effects are taken into account. The algorithm that will be studied initially is the Least Mean Square (LMS) algorithm that has been used for many analyses of adaptive antenna systems. The details of this algorithm will not be presented here as they are readily available in the literature, for example, see Widrow[3].

The LMS algorithm minimizes the mean squared difference (error) between the array output and some desired output by adaptively changing a complex weighting coefficient associated with each element in the array. It has been shown that this algorithm will converge, for realistic signal conditions, provided a convergence constant is chosen properly. The LMS algorithm is implemented using the method of steepest descent[3], that is, the element complex weighting coefficients are adjusted so that they change in the direction of the negative gradient of the squared error.

Figure 4 shows a functional representation of the LMS algorithm for one element in the array. Referring to this figure, the array...
output, $v_T(t)$, can be represented as the normal Euclidian inner product of the element weights $w^i(t)$ and the element signals $v^i(t)$ or

$$V_T(t) = (W(t), V(t))$$

where both $V(t)$ and $W(t)$ are complex column vectors. If we let $r(t)$ represent the desired array output, then the error signal, $e(t)$, as shown in Fig. 4 is

$$e(t) = (V_T(t) - r(t))$$
It can be shown[3] by setting the derivative of the complex weighting vector equal to the negative of the gradient of the squared error signal that the differential equation governing the behavior of the element weights is given by

\begin{equation}
\frac{dw(t)}{dt} + k_1 V(t) V(t)^* W(t) = k_1 \overline{r(t)} V(t)
\end{equation}

where \( k_1 \) is a constant chosen to ensure convergence. The \(^*\) notation indicates complex conjugate transpose and \( \overline{\cdot} \) indicates complex conjugate.

The quantity \( v(t)v(t)^* \) is a square matrix representing products of all signals in each of the elements. If the element signals are characterized as stochastic processes, then the differential equation for the weights has stochastic coefficients which makes the solution a formidable one. The assumption that is normally made is that the bandwidth of the adaptive processor is small compared to the bandwidth of the element signals and thus the average weight behavior adequately describes the element weight behavior. The validity of this assumption has been demonstrated by Koleszar[13].

By taking the expected values of Eq. (19) we have then a coupled differential equation of constant coefficients for the average weight behavior. If we define the expected value of the signal product matrix to be

\begin{equation}
\hat{R}_v = E[V(t)V^*(t)]
\end{equation}

and use the notation \(^\wedge\) to indicate expected value has been taken, then Eq. (19) becomes
(21) \[ \frac{d\hat{W}}{dt} + k_1 \hat{R}_V \hat{t} = k_1 (\hat{r}\hat{V}) \]

where the time dependence notation has been dropped.

We can now solve Eq. (21) with conventional techniques. To do this, first consider the homogeneous equation

(22) \[ \frac{d\hat{W}_h}{dt} + k_1 \hat{R}_V \hat{W}_h = 0 \]

which has as its solution

(23) \[ \hat{W}_h = a_1 e^{-k_1 \hat{R}_V t} \]

where \(a_1\) is an arbitrary constant. To find the forced response, Eq. (21) can be written in the form of an exact differential as

(24) \[ \frac{d}{dt} \left[ \hat{W}_f e^{k_1 \hat{R}_V t} \right] = k_1 e^{k_1 \hat{R}_V t} (\hat{r}\hat{V}) \]

and then integration of both sides yields

(25) \[ \hat{W}_f = \left[ \hat{R}_V^{-1} + a_2 k_1 e^{-k_1 \hat{R}_V t} \right] (\hat{r}\hat{V}) \]

where the matrix \(\hat{R}_V\) is assumed non-singular. Thus the total solution to Eq. (21) is the sum of the homogeneous and forced solutions or

(26) \[ \hat{W} = a_1 e^{-k_1 \hat{R}_V t} + \left[ \hat{R}_V^{-1} + a_2 k_1 e^{-k_1 \hat{R}_V t} \right] (\hat{r}\hat{V}) \]

By imposing the initial condition that

(27) \[ \hat{W}(0) = W_0 = \begin{bmatrix} w_{01} \\ w_{02} \\ \vdots \\ w_{0L} \end{bmatrix} \]
the complete solution for the average weights is given by

\[ \hat{W} = e^{-k_1 \hat{R}_v t} W_0 + \hat{R}_v^{-1} \left[ I - e^{-k_1 \hat{R}_v t} \right] \left( \hat{r} \hat{V} \right) \]

where \( I \) is the identity matrix.

It is easily seen that the steady-state solution \( w_{ss} \) to the weight equation is

\[ \hat{w}_{ss} = \hat{R}_v^{-1} \left( \hat{r} \hat{V} \right) \]

It can be shown[3] that this set of weights also maximizes the signal-to-noise power ratio. This solution is also the solution to the Wiener-Hopf equation minimizing the mean square error when the reference signal is made identical to the desired array response[3].

With the total solution to the weight equation developed we can investigate first the steady-state response and then the transient response.
CHAPTER V

STEADY-STATE ANALYSIS OF ADAPTIVE ARRAYS USING LMS ALGORITHM

In this section we will investigate the steady-state performance of adaptive systems utilizing the LMS algorithm and make comparisons of performance using coupled and uncoupled antenna arrays.

Equation (28) gives the steady-state solution for the differential equation (21) as

\[ \hat{W}_{ss} = \hat{R}_V^{-1} \hat{r} V \]

We will now concentrate our attention on just the steady-state performance.

Up to this point we have not considered whether mutual coupling terms were significant or not. All that has been said is that the voltage applied to the adaptive processor is represented by the following

\[ V = Z_0^{-1} V_0 \]

where \( Z_0^{-1} \) is a square dimensionless matrix and \( V_0 \) is the open circuit voltage vector with a typical term being of the form

\[ V_{ok}(t) = d(t)e^{-j \frac{2\pi}{\lambda_d} N_d \cdot \bar{r}_d} + \sum_{j=1}^{M} i_j(t)e^{-j \frac{2\pi}{\lambda_j} N_j \cdot \bar{r}_{kj}} + n_k(t) \]

The elements of the vector \( V_0 \) were defined earlier in Eqs. (14) and (15).
A. **Uncoupled Element Case**

For the first case we will consider the uncoupled array where we can say for the $Z_0$ matrix that

\[
\begin{cases}
i = 1, \ldots, L \\
\end{cases}
\]

(33) $z_{ij} \neq 0$ for $i \neq j$

\[
\begin{cases}
    j = 1, \ldots, L \\
\end{cases}
\]

Thus for this assumption $Z_0^{-1}$ has a diagonal form and will be denoted as $Z_{ou}$. Its inverse is given by

(34) 
\[
Z_{ou}^{-1} = \begin{bmatrix}
\frac{Z_L}{Z_L + Z_{LL}} & 0 \\
0 & \frac{Z_L}{Z_L + Z_{LL}}
\end{bmatrix}
\]

Let the uncoupled input voltage vector be given by $V_u$. The covariance matrix $\hat{R}_v$ for the uncoupled case we will denote $\hat{R}_{vu}$. Thus $\hat{R}_{vu}$ is

(35) 
\[
\hat{R}_{vu} = E[V_u V_u^*]
\]

where the * notation represents complex conjugate transpose. Then combining Eqs. (31) and (35) and carrying out the calculation we have

(36) 
\[
\hat{R}_{vu} = E[(Z_{ou}^{-1} V_0)(Z_{ou}^{-1} V_0)^*]
\]

and
Using the signal model developed earlier in Eq. (16) we can calculate the signal covariance matrix, $E(V_0 V_0^*)$, shown in Eq. (37). Thus with the help of Eq. (16) we have

\[
E[V_0 V_0^*] = E \left[ \left( d(t) \alpha_d + \sum_{i=1}^{M} i_i(t) \alpha_i + n(t) \right) \left( d(t) \alpha_d + \sum_{j=1}^{M} i_j(t) \alpha_j + n(t) \right)^* \right].
\]

If we now make certain assumptions and definitions, Eq. (38) can be expanded. We will assume that the desired signal, the interference signal, and the noise signal are all statistically independent and of zero mean value. This means that

\[
E[d(t) i_i^*(t)] = 0, \quad E[d(t) n^*(t)] = 0, \quad E[i_i(t) n^*(t)] = 0
\]

and for all similar terms. Also, we further define the average power in each signal and the thermal noise power to be:

\[
E[d(t) d(t)^*] = D
\]

\[
E[i_i(t) i_j^*(t)] = \begin{cases} I_i & i = j \\ 0 & i \neq j \end{cases}, \quad E[n(t) n(t)^*] = \begin{cases} \sigma^2 & i = j \\ 0 & i \neq j \end{cases}
\]
Using Eq. (40) we can define signal to thermal noise power ratios which are essentially the per element or input power ratios for the various signals. Thus we define

\[
\begin{align*}
\frac{p_D}{\sigma^2} &= \frac{D}{\sigma^2} \\
\frac{p_{I_i}}{\sigma^2} &= \frac{I_i}{\sigma^2}
\end{align*}
\]

(41)

to be the desired signal to thermal power ratio and the ith interference to thermal noise power ratio, respectively. Thus Eq. (41) defines the power ratios in each element.

With these definitions and assumptions, Eq. (38) can be reduced to

\[
E[V_0V_0^*] = D \alpha_d\alpha_d^* \sum_{i=1}^{M} I_i \alpha_i\alpha_i^* + \sigma^2 I
\]

(42)

where Eq. (42) is the sum of M+2, L x L matrices with \( \sigma^2 I \) being the identity matrix I, multiplied by the noise power in each element \( \sigma^2 \).

For the cross-correlation of the reference signal, \( r \), and the input signal vector, \( V \), in Eq. (30), it need only be noted that \( r \) is not a vector quantity and we can then write \( rV \) to be

\[
\hat{rV} = Z_{0u}^{-1} \hat{rV}_0
\]

(43)

We can further define the cross-correlations \( \hat{rV}_0 \) to be

\[
\hat{rV}_0 = r_V
\]

(44)

For all cases in this study the reference signal was chosen to be an exact replica of the desired signal as if it were received by the array at broadside and have peak amplitude \( \frac{R}{\sqrt{M}} \). Thus,
The physical interpretation of these statistical assumptions is not explicitly defined here. However it could be a frequency or modulation difference to cause the correlations to be as defined.

Combining Eqs. (44), (37), and (30) we have \( \mathbf{W}_{\text{ss}} \) for the uncoupled case denoted by \( \mathbf{W}_{\text{ssu}} \) to be

\[
\hat{W}_{\text{ssu}} = (Z_{ou}^{-1} \mathbf{E}[V_o V_o^*] Z_{ou}^{-1*})^{-1} Z_{ou}^{-1} \hat{r}_v
\]

and carrying out the indicated operations

\[
\hat{W}_{\text{ssu}} = Z_{ou}^* [\mathbf{E}(V_o V_o^*)]^{-1} Z_{ou} Z_{ou}^{-1} \hat{r}_v
\]

This further simplifies to

\[
\hat{W}_{\text{ssu}} = Z_{ou}^* (\mathbf{E}[V_o V_o^*])^{-1} \hat{r}_v
\]

which simplifies even further if we assume as in Eq. (34) that all array elements have equal self impedances. In this case \( Z_{ou} \) will be the identity matrix multiplied by the complex number \((Z_L + z_{ii})/Z_L\). Substituting this result in Eq. (34) we get

\[
\hat{W}_{\text{ssu}} = \left( \frac{Z_L + z_{ii}}{Z_L} \right) (\mathbf{E}[V_o V_o^*])^{-1} \hat{r}_v
\]

With the steady-state weights for the uncoupled case now derived we next find the total power out of the array for the uncoupled case using Eq. (49).
The total voltage out of the array will be the sum of the product of the element weight and the respective element voltages. In vector notation, this will be

\[ V_{TU} = W_{ssu}^* V_u \]  

where \( V_{TU} \) represents the total signal (voltage) at the array output.

The total average power at the output of the array is defined to be \( \hat{P}_{TU} \) and given by

\[ \hat{P}_{TU} = E[V_{Tu} V_{Tu}^*] \]

where again \( E[\ ] \) represents the expected value.

Combining Eqs. (49) and (50) we get

\[ \hat{P}_{TU} = E[W_{ssu}^* V_u V_u^*] \]  
\[ \hat{P}_{TU} = W_{ssu}^* E[V_u V_u^*] \hat{W}_{ssu} \]

where use has been made of the assumption that the weight variance is small.

We recognize the expected value of the term \( V_u V_u^* \) has been defined in Eq. (35). Thus combining Eqs. (35) and (53) we have

\[ \hat{P}_{TU} = W_{ssu}^* R_{vu} \hat{W}_{ssu} \]

Expanding Eq. (54) with the aid of Eq. (37) we get

\[ \hat{P}_{TU} = W_{ssu}^* Z_{ou}^{-1} E[V_o V_o^*] Z_{ou}^{-1} \hat{W}_{ssu} \]

and using the assumption of equal self impedances outlined earlier we have
Equation (56) is the final expression for the total average power out of the array at steady state for the uncoupled case.

B. Coupled Element Case

We will now consider the case for the coupled antenna array where mutual impedance terms cannot be neglected. Thus starting again with Eq. (30) we define the steady-state weights for the coupled case to be denoted by \( \hat{W}_{ssc} \) which is given by

\[
(57) \quad \hat{W}_{ssc} = \hat{R}_{vc}^{-1} \hat{r}_c
\]

where the subscript \( c \) has been added to distinguish the coupled case from the uncoupled case. Proceeding as in the previous case, the impedance or transformation matrix is represented by a square matrix of the form

\[
Z_{oc} = \begin{bmatrix}
\frac{Z_{11} + Z_L}{Z_L} & \frac{Z_{12}}{Z_L} & \cdots & \frac{Z_{1L}}{Z_L} \\
\frac{Z_{21}}{Z_L} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\frac{Z_{L1}}{Z_L} & \cdots & \cdots & \frac{Z_{LL} + Z_L}{Z_L}
\end{bmatrix}
\]

(58) $Z_{oc}$

where the \( c \) subscript again denotes the coupled case. If we also let the coupled case input voltage vector be denoted by \( V_c \), and the corresponding covariance matrix by \( \hat{R}_{vc} \), we have
or similar to the uncoupled case we have

\[ R_{vc} = Z_{oc}^{-1} E[V_o V_o^*] Z_{oc}^{-1} \]

The reference signal with input signal cross-correlation vector can similarly be represented for the coupled case to be

\[ \hat{r}_{vc} = Z_{oc}^{-1} \hat{r}_v \]

Thus combining Eqs. (57), (60), and (61) we have the expression for the average steady-state weight for the coupled case,

\[ \hat{w}_{ssc} = (Z_{oc}^{-1} E[V_o V_o^*] Z_{oc}^{-1})^{-1} Z_{oc}^{-1} \hat{r}_v \]

Carrying out the indicated inverse operations we have

\[ \hat{w}_{ssc} = Z_{oc}^* (E[V_o V_o^*])^{-1} Z_{oc} Z_{oc}^{-1} \hat{r}_v \]

which simplifies to

\[ \hat{w}_{ssc} = Z_{oc}^* (E[V_o V_o^*])^{-1} \hat{r}_v \]

Now combining Eq. (49) with Eq. (64) we have

\[ \hat{w}_{ssc} = \left( \frac{Z_L}{Z_L + z_{ii}} \right) Z_{oc}^* \hat{w}_{ssu} \]

Thus the average steady-state weights for the coupled case are the product of the impedance matrix \( Z_{oc}^* \), the weights that would occur if the mutual coupling terms were very small, and the proportionality constant \( (Z_L/(Z_L + z_{ii})) \). The weights for the coupled case are related to those for the uncoupled case in a straightforward manner.
We may now proceed to calculate $\hat{P}_{TC}$, the total average power out of the array. $\hat{P}_{TC}$ is defined similar to the previous case to be

$$\hat{P}_{TC} = E[V_{TC}V_{TC}^*]$$

where

$$V_{TC} = W_{SSC}^* V_c$$

Combining Eqs. (59), (66) and (67) we have

$$\hat{P}_{TC} = \hat{W}_{SSC}^* R_{VC} \hat{W}_{SSC}$$

Inserting Eq. (60) into Eq. (68) we have

$$\hat{P}_{TC} = \hat{W}_{SSC}^* Z_{oc}^{-1} E[V_o V_o^*] Z_{oc}^{-1} * \hat{W}_{SSC}$$

which is the total average power out of the array for the coupled case.

Using Eq. (65), Eq. (69) can be written as

$$\hat{P}_{TC} = \left[ \begin{array}{c} \frac{Z_L}{Z_L + z_{ii}} \\ \frac{Z_L}{Z_L + z_{ii}} \end{array} \right]^* Z_{oc}^{-1} E[V_o V_o^*] Z_{oc}^{-1} * \hat{W}_{SSC}$$

Performing the indicated transpose operation with

$$\left| \frac{Z_L}{Z_L + z_{ii}} \right|^2 = \left( \frac{Z_L}{Z_L + z_{ii}} \right) \left( \frac{Z_L}{Z_L + z_{ii}} \right)$$

we have

$$\hat{P}_{TC} = \left| \frac{Z_L}{Z_L + z_{ii}} \right|^2 \hat{W}_{SSU}^* E[V_o V_o^*] \hat{W}_{SSU}$$

Comparing Eq. (72) with Eq. (56) we see that
which indicates that the total power out of the array at steady state is not a function of the mutual impedance terms of the matrix $Z_0$. Since this is possibly not the intuitive, expected result some discussion is necessary.

The relationship between the total received powers for the coupled and uncoupled cases given by Eq. (73) may seem somewhat contradictory. However if we consider in detail the relationships between the element voltages, weights and mutual coupling we can get a physical understanding of the causes for Eq. (73) to be as such.

With this in mind, consider the three element antenna array shown in Fig. 5. We will assume that the elements of this array are dipoles of fixed spacing, $d$, but variable height $h$. Referring further to Fig. 5, the $v^i_C$ represent the element voltages determined by Eqs. (31) and (32). In general these voltages will change as the element coupling is varied. Now let each element voltage be fed to an arbitrary amplitude and phase adjustment device, (element weighting coefficients) noted as $\bar{w}^i_C$ on Fig. 5. We assume for this discussion that the weights $\bar{w}^i_C$ can either amplify or attenuate the signal $v^i_C$. The array output, $V_T$, is then the sum of the products of the element voltages and weights as shown in Fig. 5. The corresponding array output power is then $V_T \bar{V}_T^*$. 

We are now in a position to determine some characteristics of the array that we have defined above. Let the weight $w^i_C$ be set to some arbitrary value along with the element height $h$, and the far field pattern or gain of the antenna array be measured and recorded. It is well

$$p_{TC} = p_{Tu}$$
known then that with knowledge of the transmitting antenna in the far field of the receiving array, the total received power can be calculated. The point to be made here is that the received power from an antenna array can be determined without explicit knowledge of the coupling between elements even though the element voltages $v^i_C$ will change as the coupling changes as pointed out earlier.

To demonstrate this point, let the transmitter mentioned earlier be maintained in the far field of the array, but change the element height at the receive array to a new value $h$. When this is done, obviously the element voltages $v^i_C$, and the received power will be changed. However, it appears reasonable to assume that one could readjust the
weights to bring the received power back to its original value. Thus it appears that with the proper adjustment of the weights \( w^i_c \) the received power can be maintained constant with respect to the element coupling. The LMS algorithm gives that choice of weights. This is the physical interpretation of Eq. (73).

To investigate further the relationship between the LMS choice of weights and element coupling, let the weights shown in Fig. 5 be determined by the adaptive LMS algorithm. Thus for the case under consideration, the \( w^i_c \) are determined by Eq. (64). For this example let the desired signal be incident on the array at broadside producing a signal to thermal power in each element of 10 dB. In addition, let there be one interfering source, incident on the array at an angle \( \phi_1 \) and producing an interfering signal to thermal power ratio of 20 dB. With these definitions the \( w^i_c \) (from Eq. (64)) can be calculated for any \( \phi_1 \). After the weights have been determined a test source of unit amplitude can be placed at \( \phi_1 \) and using Eq. (31) the \( v^i_c \) can be calculated. Then the products, \( v^i_c w^i_c \), can be calculated and plotted as vectors. The sum of these products represents the relative value of the field strength in the direction of the test source. The results of these calculations for \( \phi_1 = 20^\circ, 40^\circ, 60^\circ \) and \( 80^\circ \) are shown in Figs. 6a-d.

For this example and all others in this study, all signals are chosen to be the same frequency for convenience in comparison. This is not confining or limiting since a difference in frequency only represents a difference in phase shift across the array.

The mutual and self impedance terms were calculated using programs written by Richmond[19]. This program utilizes the Moment Method
Fig. 6a--$v_{wc}^i$ vectors with test source located at $\phi_1$ and $w_i^c$ determined by adapting to $\phi_d = 90^\circ$, $\phi_1=20^\circ$ including mutual coupling effects.

$$V_T = (w_c, v_c)$$
$$= \sum_{i=1}^{3} v_{wc}^{i-i}$$
$$= 0.00150 + j0$$

Fig. 6b--$v_{wc}^i$ vectors with test source located at $\phi_1$ and $w_i^c$ determined by adapting to $\phi_d = 90^\circ$, $\phi_1=40^\circ$ including mutual coupling effects.

$$V_T = (w_c, v_c)$$
$$= \sum_{i=1}^{3} v_{wc}^{i-i}$$
$$= 0.00270 + j0$$
Fig. 6c—\( \vec{v}^{i}_{C} \) vectors with test source located at \( \phi_1 \) and \( \vec{w}^{i}_{C} \) determined by adapting to \( \phi_d = 90^\circ, \phi_1 = 60^\circ \) including mutual coupling effects.

\[
V_T = (\vec{W}_C, \vec{V}_C) = \sum_{i=1}^{3} \vec{v}^{i}_{C} \vec{w}^{i}_{C} = 0.00689 + j0
\]

Fig. 6d—\( \vec{v}^{i}_{C} \) vectors with test source located at \( \phi_1 \) and \( \vec{w}^{i}_{C} \) determined by adapting to \( \phi_d = 90^\circ, \phi_1 = 80^\circ \) including mutual coupling effects.

\[
V_T = (\vec{W}_C, \vec{V}_C) = \sum_{i=1}^{3} \vec{v}^{i}_{C} \vec{w}^{i}_{C} = 0.03804 + j0
\]
applied to thin wire antennas. This program was used extensively throughout this study.

These same calculations were repeated with $Z_0$ replaced by the identity matrix in Eqs. (64) and (31). This represents the more conventional approach where mutual coupling is neglected. The results of these calculations are shown in Figs. 7a-d. Since our test source is placed in the same locations for both sets of calculations, according to Eq. (73) we should get the same total received power or voltage, and indeed if the respective vectors shown in Figs. 6 and 7 are added up the resultant sum is the same, even though the magnitudes and phase angles of the corresponding vectors are different. Note that since the test source is in the direction of the interfering source, the vector sum is at the null made by the adaptive array. Thus the conclusion is that the LMS choice of weights modifies the element voltages in such a way as to force the received power to be independent of mutual coupling in the array array. Analytically, if we combine Eqs. (64), (31) and the average value of Eq. (67) we get

$$\hat{V}_{TC} = \hat{r}_V \ E[V_O V^*_O]^{-1} V_O$$

which shows that the total average received voltage is also independent of the mutual coupling terms.

To illustrate in another way the effect of mutual coupling on the element voltages and LMS adaptive weights, the same calculations as shown in the previous two examples were again made but with $\psi_1$ held fixed at $60^\circ$ and the element height varied. This is a convenient way to change the element coupling while holding the spacing fixed.
\[ V_T = (w_c, v_c) \]
\[ = \sum_{i=1}^{3} v_i^{w_i c} \]
\[ = .00149 + j0 \]

Fig. 7a—\( v_i^{w_i c} \) vectors with test source located at \( \phi_1 \) and \( w_i \) determined by adapting to \( \phi_d = 90^\circ, \phi_1 = 20^\circ \), neglecting mutual coupling effects.

\[ V_T = (w_c, v_c) \]
\[ = \sum_{i=1}^{3} v_i^{w_i c} \]
\[ = .0027 + j0 \]

Fig. 7b—\( v_i^{w_i c} \) vectors with test source located at \( \phi_1 \) and \( w_i \) determined by adapting to \( \phi_d = 90^\circ, \phi_1 = 40^\circ \), neglecting mutual coupling effects.
\[ V_T = (W_c, V_c) \]
\[ = \sum_{i=1}^{3} v_i w^i_{-i} \]
\[ = 0.00689 + j0 \]

Fig. 7c—\( v_i w^i_{-i} \) vectors with test source located at \( \phi_1 \) and \( w^i_{-i} \) determined by adapting to \( \phi_d = 90^\circ, \phi_1 = 60^\circ \), neglecting mutual coupling effects.

\[ V_T = (W_c, V_c) \]
\[ = \sum_{i=1}^{3} v_i w^i_{-i} \]
\[ = 0.03804 + j0 \]

Fig. 7d—\( v_i w^i_{-i} \) vectors with test source located at \( \phi_1 \) and \( w^i_{-i} \) determined by adapting to \( \phi_d = 90^\circ, \phi_1 = 80^\circ \), neglecting mutual coupling effects.
The results of these calculations are shown in Table 1. The resultant received power in the direction of \( \phi_1 = 60^\circ \) is shown constant for all element lengths. As can be seen from this table as the element height decreases and coupling decreases, the element voltage \( v^i_1 \) in general decrease in magnitude while the LMS weights increase in magnitude. This is partly due to the mismatch generated when the load impedance is held fixed and the real part of self impedance gets very small as the element height is decreased.

**TABLE 1**

**ELEMENT VOLTAGES AND WEIGHTS VS ELEMENT HEIGHT FOR L=3**

\[ d=.25\lambda, \; Z_L=50+j0, \; \phi_1=60^\circ \]

<table>
<thead>
<tr>
<th>( h(\lambda) )</th>
<th>.5</th>
<th>.4</th>
<th>.3</th>
<th>.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \begin{bmatrix} v^1_1 \ v^2_1 \ v^3_1 \end{bmatrix} )</td>
<td>.416/34.4°</td>
<td>.115/125°</td>
<td>.053/131°</td>
<td>.029/133°</td>
</tr>
<tr>
<td>( \begin{bmatrix} v^1_2 \ v^2_2 \ v^3_2 \end{bmatrix} )</td>
<td>.237/1.1°</td>
<td>.111/74.4°</td>
<td>.052/84.8°</td>
<td>.029/87.8°</td>
</tr>
<tr>
<td>( \begin{bmatrix} v^1_3 \ v^2_3 \ v^3_3 \end{bmatrix} )</td>
<td>.212/-53.4°</td>
<td>.116/26.9°</td>
<td>.053/39.3°</td>
<td>.029/42.7°</td>
</tr>
<tr>
<td>( \begin{bmatrix} w^1_1 \ w^2_1 \ w^3_1 \end{bmatrix} )</td>
<td>1.34/-79.0°</td>
<td>5.08/26.8°</td>
<td>11.4/33.0°</td>
<td>21.2/35.5°</td>
</tr>
<tr>
<td>( \begin{bmatrix} w^1_2 \ w^2_2 \ w^3_2 \end{bmatrix} )</td>
<td>1.06/15.1°</td>
<td>1.77/68.8°</td>
<td>3.3/83.0°</td>
<td>5.9/87.4°</td>
</tr>
<tr>
<td>( \begin{bmatrix} w^1_3 \ w^2_3 \ w^3_3 \end{bmatrix} )</td>
<td>2.13/38.5°</td>
<td>5.2/127.0°</td>
<td>11.5/137.0°</td>
<td>21.2/140°</td>
</tr>
<tr>
<td>( \begin{bmatrix} v'^1_{cw1} \ v'^2_{cw2} \ v'^3_{cw3} \end{bmatrix} )</td>
<td>.56/113.4°</td>
<td>.583/98.2°</td>
<td>.605/98.0°</td>
<td>.611/97.6°</td>
</tr>
<tr>
<td>( \begin{bmatrix} v'^1_{cw2} \ v'^2_{cw3} \ v'^3_{cw3} \end{bmatrix} )</td>
<td>.25/-14.0°</td>
<td>.197/5.6°</td>
<td>.176/1.8°</td>
<td>.171/-4.4°</td>
</tr>
<tr>
<td>( \begin{bmatrix} v'^1_{cw3} \ v'^2_{cw3} \ v'^3_{cw3} \end{bmatrix} )</td>
<td>.45/-91.9°</td>
<td>.606/-100.1°</td>
<td>.612/-97.7°</td>
<td>.612/-97.8°</td>
</tr>
<tr>
<td>( VTV^* )</td>
<td>.47 x 10(^{-4})</td>
<td>.47 x 10(^{-4})</td>
<td>.47 x 10(^{-4})</td>
<td>.47 x 10(^{-4})</td>
</tr>
</tbody>
</table>
To illustrate the effect of the load impedance on these voltages, the same calculations were made as shown in Table 1 except the elements were conjugate matched to the self impedance. The results of these calculations are shown in Table 2. When the element height is decreased to .2λ the real part of the self impedance becomes small, in the neighborhood of 8 ohms. However, the correspondingly high element voltages in turn cause the LMS weights to be small and to hold the total received power constant.

**TABLE 2**

**ELEMENT VOLTAGES AND WEIGHTS VS ELEMENT HEIGHT FOR L=3**

\[ d=-25\lambda, Z_L=Z_{11}^*, \phi_1=60^\circ \]

<table>
<thead>
<tr>
<th>h(\lambda)</th>
<th>.5</th>
<th>.4</th>
<th>.3</th>
<th>.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ V = \begin{bmatrix} v_1^C \ v_2^C \ v_3^C \ w_1^C \ w_2^C \ w_3^C \end{bmatrix} ]</td>
<td>.616/23.2°</td>
<td>5.51/138.2°</td>
<td>24.9/143.0°</td>
<td>111.2/144.6°</td>
</tr>
<tr>
<td>[ V = \begin{bmatrix} v_1^C \ v_2^C \ v_3^C \ w_1^C \ w_2^C \ w_3^C \end{bmatrix} ]</td>
<td>.364/-23.4°</td>
<td>3.21/92.4°</td>
<td>14.2/98.4°</td>
<td>62.4/101.5°</td>
</tr>
<tr>
<td>[ V = \begin{bmatrix} v_1^C \ v_2^C \ v_3^C \ w_1^C \ w_2^C \ w_3^C \end{bmatrix} ]</td>
<td>.328/-75.6°</td>
<td>2.88/38.7°</td>
<td>12.7/42.7°</td>
<td>55.3/43.3°</td>
</tr>
<tr>
<td>[ V = \begin{bmatrix} v_1^C \ v_2^C \ v_3^C \ w_1^C \ w_2^C \ w_3^C \end{bmatrix} ]</td>
<td>.880/-85.8°</td>
<td>.098/29.3°</td>
<td>.021/34.0°</td>
<td>.005/35.4°</td>
</tr>
<tr>
<td>[ V = \begin{bmatrix} v_1^C \ v_2^C \ v_3^C \ w_1^C \ w_2^C \ w_3^C \end{bmatrix} ]</td>
<td>.699/-9.3°</td>
<td>.079/107.0°</td>
<td>.018/113.6°</td>
<td>.004/117.2°</td>
</tr>
<tr>
<td>[ V = \begin{bmatrix} v_1^C \ v_2^C \ v_3^C \ w_1^C \ w_2^C \ w_3^C \end{bmatrix} ]</td>
<td>1.39/22.5°</td>
<td>.156/137.0°</td>
<td>.035/141/3°</td>
<td>.008/142.4°</td>
</tr>
<tr>
<td>[ V = \begin{bmatrix} v_1^C \ v_2^C \ v_3^C \ w_1^C \ w_2^C \ w_3^C \end{bmatrix} ]</td>
<td>.542/-98.0°</td>
<td>.538/109.0°</td>
<td>.534/109.0°</td>
<td>.530/109.2°</td>
</tr>
<tr>
<td>[ V = \begin{bmatrix} v_1^C \ v_2^C \ v_3^C \ w_1^C \ w_2^C \ w_3^C \end{bmatrix} ]</td>
<td>.254/-14.1°</td>
<td>.255/-14.6°</td>
<td>.257/-15.2°</td>
<td>.259/-15.7°</td>
</tr>
<tr>
<td>[ V = \begin{bmatrix} v_1^C \ v_2^C \ v_3^C \ w_1^C \ w_2^C \ w_3^C \end{bmatrix} ]</td>
<td>.455/108.9°</td>
<td>.450/-98.3°</td>
<td>.443/-98.6°</td>
<td>.436/-99.0°</td>
</tr>
</tbody>
</table>

\[ V_TV^\dagger \]

\[ .47 \times 10^{-4} \quad .47 \times 10^{-4} \quad .47 \times 10^{-4} \quad .47 \times 10^{-4} \]
We return now to the calculation of the total received power in terms of the power associated with the desired signal, the interfering signals and thermal noise. Inserting Eq. (42) into Eq. (72) we get

\[ P_{TC} = \frac{Z_L}{Z_L + Z_{ii}}^2 \hat{W}_{ssu}^* \left( D \alpha_d \alpha_d^* + \sum_{i=1}^{M} I_i \alpha_i \alpha_i^* + \sigma^2 I \right) \cdot \hat{W}_{ssu} \]

Expanding Eq. (75) we can identify the power out of the array due to the desired signal \( \hat{P}_{Dc} \), the interfering signals \( \hat{P}_{Ii} \), and thermal noise \( \hat{P}_{nc} \). These are given by

\[ \hat{P}_{Dc} = D \left( \frac{Z_L}{Z_L + Z_{ii}}^2 \hat{W}_{ssu}^* \alpha_d \alpha_d^* \hat{W}_{ssu} \right) \]
\[ \hat{P}_{Ii} = I_i \left( \frac{Z_L}{Z_L + Z_{ii}}^2 \hat{W}_{ssu}^* \alpha_i \alpha_i^* \hat{W}_{ssu} \right) \]
\[ \hat{P}_{nc} = \sigma^2 \left( \frac{Z_L}{Z_L + Z_{ii}}^2 \hat{W}_{ssu} \hat{W}_{ssu} \right) \]

Based on our previous discussions we note that the powers are equal for the coupled and uncoupled cases. That is,

\[ \hat{P}_{Dc} = \hat{P}_{Du} \]
\[ \hat{P}_{Ii} = \hat{P}_{Ii} \]
\[ \hat{P}_{nc} = \hat{P}_{nu} \]

For convenience in calculation, we can normalize with respect to \( \sigma^2 \) and the normalized powers of Eq. (76) are given by
In addition to the above relations, probably the most useful quantity for evaluation of adaptive arrays is the received signal-to-noise ratio. For this study it is defined to be

\[
\frac{\hat{P}_{Dc}}{\sigma^2} = \frac{D}{\sigma^2} \left| \frac{Z_L}{Z_L + Z_{ii}} \right|^2 \hat{w}_{ssu}^* \alpha \hat{d} \hat{w}_{ssu}
\]

\[
\frac{P_{I_i c}}{\sigma^2} = \frac{I_i}{\sigma^2} \left| \frac{Z_L}{Z_L + Z_{ii}} \right|^2 \hat{w}_{ssu}^* \alpha_i \hat{w}_{ssu}
\]

\[
\frac{\hat{P}_{nc}}{\sigma^2} = \left| \frac{Z_L}{Z_L + Z_{ii}} \right|^2 \hat{w}_{ssu}^* \hat{w}_{ssu}
\]

(78)

In addition to the above relations, probably the most useful quantity for evaluation of adaptive arrays is the received signal-to-noise ratio. For this study it is defined to be

\[
S/N = \frac{\hat{P}_{Dc}}{\hat{P}_{nc} + \sum_{i=1}^{M} P_{I_i c}}
\]

(79)

C. **Field Intensity of Adaptive Array for LMS Algorithm**

Since an adaptive antenna system is a spatial filter, placing a beam maximum near the desired signal direction and pattern nulls near the directions of interfering signals, considerable insight into the operation of adaptive systems can be obtained by looking at the field patterns of the adaptive array. In dealing with conventional antenna arrays effort is generally concentrated on finding amplitude and phasing coefficients or weights for each element to produce a field pattern of some desirable shape. The adaptive antenna array, in a sense, automatically selects these coefficients for each element to adjust the field pattern
to the desired shape depending on the signal environment in which the array operates. In this section, the field patterns for the adaptive array in steady state will be investigated for both loosely and tightly coupled arrays based on the adaptive weights developed earlier.

The field pattern or field intensity, $E_T(\phi)$, in the horizontal or x-y plane of the adaptive array under consideration can be written for the steady state case as

\begin{equation}
E_T(\phi) = (V(\phi), \hat{W}_{ss})
\end{equation}

where $V(\phi)$ is a column vector representing the terminal voltage at each element of the array resulting from a distant test source making an angle $\phi$ with the linear array. The terminal voltage $V(\phi)$ can be represented in terms of the terminal open circuit voltage and the impedance matrix for the array. Thus

\begin{equation}
V(\phi) = Z_0^{-1} V_0(\phi)
\end{equation}

where $Z_0^{-1}$ is defined as before and $V_0(\phi)$ is the open circuit voltage vector of element voltages due to the distant test source. We will assume unit amplitude for $V_0(\phi)$. Thus $V_0(\phi)$ can be represented as

\begin{equation}
V_0(\phi) = \begin{bmatrix}
-e^{j(\frac{2\pi}{\lambda} )} (1 - \frac{L+1}{2} d \cos \phi) \\
e^{j(\frac{2\pi}{\lambda} )} (k - \frac{L+1}{2} d \cos \phi) \\
\vdots \\
e^{j(\frac{2\pi}{\lambda} )} (L - \frac{L+1}{2} d \cos \phi)
\end{bmatrix}
\end{equation}
where $\lambda_T$ is the wavelength of the test source and the other terms are as defined earlier.

The steady-state weight for the general case is given in Eq. (30) as

$$\hat{W}_{SS} = \hat{R}_V^{-1} \hat{r}_V$$  \hspace{1cm} (83)

If we make the same choice for the reference signal as previously in Eq. (45), then

$$\hat{r}_V = \sqrt{R_D} Z_0^{-1} \alpha_d$$  \hspace{1cm} (84)

Similarly, for the inverse of the average signal variance matrix

$$\hat{R}_V^{-1} = Z_0^* (E[V_0V_0^*])^{-1} Z_0$$  \hspace{1cm} (85)

and the generalized steady-state weight is then

$$\hat{W}_{SS} = \sqrt{R_D} Z_0^* (E[V_0V_0^*])^{-1} \alpha_d$$  \hspace{1cm} (86)

Substituting Eqs. (86) and (81) into Eq. (80) we find the expression for the field pattern to be

$$E_T(\phi) = \sqrt{R_D} V_0^*(\phi) (E[V_0V_0^*])^{-1} \alpha_d$$  \hspace{1cm} (87)

In its present form little insight can be gained about the field pattern from Eq. (87). However, by suitable manipulation of Eq. (87) we can achieve a greater understanding of the adaptive array performance.

Let us first consider the product of the average signal covariance matrix and the desired signal vector $\alpha_d$. If we define the signal
environment to be composed of one desired signal and one interfering signal plus the thermal noise in each element, then we have

\[(88) \quad E[V_0 V_0^*] = D \alpha_d \alpha_d^* + I_1 \alpha_1 \alpha_1^* + \sigma^2 I\]

and therefore

\[(89) \quad (E[V_0 V_0^*])^{-1} \alpha_d = [D \alpha_d \alpha_d^* + I_1 \alpha_1 \alpha_1^* + \sigma^2 I]^{-1} \alpha_d .\]

If the inverse shown in Eq. (89) were carried out, we would have a square L x L matrix multiplying an (L x 1) column vector with the result being another column vector. Thus it seems reasonable that Eq. (89) might be expanded as a sum of column vectors \(\alpha_1\) and \(\alpha_d\) with appropriate multiplying coefficients. Let us then define

\[(90) \quad [D \alpha_d \alpha_d^* + I_1 \alpha_1 \alpha_1^* + \sigma^2 I]^{-1} \alpha_d = \alpha_1 \alpha_d + \alpha_2 \alpha_1\]

where \(\alpha_1\) and \(\alpha_2\) are scaler quantities to be determined such that the equality holds.

To determine the coefficients \(\alpha_1\) and \(\alpha_2\) we multiply Eq. (90) by the non-inverted signal covariance matrix giving

\[(91) \quad \alpha_d = [D \alpha_d \alpha_d^* + I_1 \alpha_1 \alpha_1^* + \sigma^2 I] [\alpha_1 \alpha_d + \alpha_2 \alpha_1] .\]

Carrying out the indicated multiplication and collecting like terms we have

\[(92) \quad \alpha_d = [a_1 (D |\alpha_d|^2 + \sigma^2) + a_2 D(\alpha_d, \alpha_1)] \alpha_d \]

\[+ [a_1 I_1(\alpha_1, \alpha_d) + a_2 (I_1 |\alpha_1|^2 + \sigma^2)] \alpha_1 .\]
Equating coefficients of like terms in Eq. (92) and noting that 
($a_d, \alpha_1$) terms and the like are scalers we have

\[
\begin{align*}
    a_1(D|a_d|^2 + \sigma^2) + a_2 D(a_d, \alpha_1) &= 1 \\
    a_1 I_1(a_1, a_d) + a_2(I_1|\alpha_1|^2 + \sigma^2) &= 0
\end{align*}
\]

Rewriting Eq. (93) in matrix form we have

\[
\begin{pmatrix}
    D|a_d|^2 + \sigma^2 & D(a_d, \alpha_1) \\
    I_1(a_1, a_d) & I_1|\alpha_1|^2 + \sigma^2
\end{pmatrix}
\begin{pmatrix}
    a_1 \\
    a_2
\end{pmatrix} =
\begin{pmatrix}
    1 \\
    0
\end{pmatrix}
\]

Solving for $a_1$ and $a_2$ gives

\[
\begin{pmatrix}
    a_1 \\
    a_2
\end{pmatrix} = \frac{1}{\Delta}
\begin{pmatrix}
    I_1|\alpha_1|^2 + \sigma^2 & -D(a_1, a_d) \\
    -I_1(a_d, \alpha_1) & D|a_d|^2 + \sigma^2
\end{pmatrix}
\begin{pmatrix}
    1 \\
    0
\end{pmatrix}
\]

where $\Delta = \text{the determinant of the matrix to be inverted which is given by}$

\[
\Delta = (D|a_d|^2 + \sigma^2)(I_1|\alpha_1|^2 + \sigma^2) - D I_1(a_d, \alpha_1)(a_1, a_d)
\]

Expanding the first term in the expression for $\Delta$ we have

\[
D I_1|\alpha_d|^2 |\alpha_1|^2 + \sigma^2(I_1|\alpha_1|^2 + D |a_d|^2) + \sigma^4
\]

To determine the sign of $\Delta$, we know that since $I_1$, $D$, $\sigma^2$, and the vector magnitudes are all positive quantities, we can apply Schwarz's inequality to the first part of Eq. (97) and the second term in Eq. (96). This indicates that
and we conclude that $\Delta > 0$.

Now simplifying Eq. (95) to solve for $a_1$ and $a_2$ we have

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} I_1|a_1|^2 + \sigma^2 \\ -I_1(a_d,a_1) \end{pmatrix}.$$  

Using the restriction on $\Delta$ we find that

$$\begin{cases} a_1 > 0 \\ a_2 < 0 \end{cases}$$

Now if we combine Eqs. (100), (90), and (87) we have the following expression for the field pattern

$$E_T(\phi) = \frac{\sqrt{RD}}{\Delta} V_0(\phi) \left[ (I_1|a_1|^2 + \sigma^2)a_d - I_1(a_d,a_1)a_1 \right]$$

Expanding Eq. (101) we have

$$E_T(\phi) = \frac{\sqrt{RD}}{\Delta} \left[ (I_1|a_1|^2 + \sigma^2)(V_0(\phi),a_d) \\ - I_1(a_d,a_1)(V_0(\phi),a_1) \right]$$

or

$$E_T(\phi) = \frac{\sqrt{RD}}{\Delta} \left( a_1 E_D(\phi) + a_2 E_1(\phi) \right)$$

where

$E_D(\phi) =$ pattern associated with the desired signal,

$E_1(\phi) =$ pattern associated with the first interfering signal.
It can further be shown that

\[
E_D(\phi) = \sin \left[ \frac{L}{2} \left( \frac{2\pi}{\lambda_T} d \cos \phi - \frac{2\pi}{\lambda_d} d \cos \phi_d \right) \right] \\
\sin \left[ \frac{1}{2} \left( \frac{2\pi}{\lambda_T} d \cos \phi - \frac{2\pi}{\lambda_d} d \cos \phi_d \right) \right]
\]

and similarly

\[
E_I(\phi) = \sin \left[ \frac{L}{2} \left( \frac{2\pi}{\lambda_T} d \cos \phi - \frac{2\pi}{\lambda_I} d \cos \phi_I \right) \right] \\
\sin \left[ \frac{1}{2} \left( \frac{2\pi}{\lambda_T} d \cos \phi - \frac{2\pi}{\lambda_I} d \cos \phi_I \right) \right]
\]

Thus we can clearly see that the field pattern for the steady-state adaptive array for an environment of one desired and one interfering source is made up of two beams of the form \( \frac{\sin \frac{L}{2} z}{\sin \frac{z}{2}} \). One of the beams is associated with the desired signal with weight \( |\alpha_1|^2 + \sigma^2 \) and a second beam of weight \( I_1(\alpha_d, \alpha_1) \) is associated with the undesired signal which is subtracted from the desired signal beam. The beams have maxima centered about the angle of arrival of the desired and interfering signals, respectively. However, since the total pattern is the sum of these two beams (the interfering beam being negative), the deepest null in the pattern may not be in the direction of the interfering source. This phenomenon has been noted by others[12].

These individual patterns appear to come from uniform arrays with linear phase progression along the array. Arrays of the same length having patterns of this type have essentially the same patterns even as the number of elements is increased and the spacing is decreased. As the number of elements increase the pattern approaches the \( \sin \frac{z}{z} \) pattern of a continuous source.
The expansion for the inverted covariance matrix times the desired signal vector can easily be generalized for cases involving more than one interfering signal. Thus for \( M \) interfering sources and a single desired signal we have

\[
(D \alpha_d \alpha_d^* + \sum_{i=1}^{M} I_i \alpha_i \alpha_i^* + \sigma^2 I)^{-1} \alpha_d
\]

\[= a_1 \alpha_d + \sum_{i=1}^{M} a_{i+1} \alpha_i\]

and the \( a_i \)'s can be determined from the set of linear equations

\[
\begin{bmatrix}
\sigma^2 + |\alpha_d|^2 & \cdots & I_M(\alpha_d, \alpha_M) \\
I(\alpha, \alpha_d) & \sigma^2 + |\alpha_1|^2 & \cdots & I_M(\alpha_1, \alpha_M) \\
\vdots & \ddots & \ddots & \vdots \\
I_M(\alpha_M, \alpha_d) & \cdots & \sigma^2 + |\alpha_M|^2
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_M
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

Thus, for each interfering source one additional beam is formed and in general the total field pattern at steady state can be written as

\[
E_T(\phi) = K_1 \left[ a_1 E_d(\phi) + \sum_{i=1}^{M} a_{i+1} E_i(\phi) \right]
\]

where \( K_1 \) is a constant resulting from the solution of Eq. (107).

To illustrate the effect of aperture size on the beam of a uniform array, Eq. (104) has been plotted for the case where the desired signal is incident on the array at broadside for two and six-element arrays at spacings of \( .1\lambda \), \( .25\lambda \), and \( .5\lambda \). The results of these calculations are shown in Figs. 8 and 9. As can be seen from these two plots, small
Fig. 8—\( \frac{\sin Lz}{\sin z} \) vs \( \phi \) for \( L=2, \ d=0.1\lambda, \ 0.25\lambda \) and \( 0.5\lambda \).
Fig. 9—\(\frac{\sin Lz}{\sin Z}\) vs \(\phi\) for \(L=6, d=0.1\lambda, 0.25\lambda\) and \(0.5\lambda\).
apertures severely limit the resolution of the array when one considers that the steady-state adaptive pattern is made up of a weighted sum of patterns of this shape. The beam maxima occur in the direction of the incoming source.

To illustrate how the total steady-state pattern is made up, four examples were chosen utilizing Eqs. (104), (105), and (108). For these examples, two arrays were chosen, one a two-element array with .5λ spacing and the other a six-element array with a .25λ spacing. The signal environment has a single desired signal at broadside and an interfering source at 40° off endfire. The two beams associated with desired and interfering beams were calculated and plotted separately along with the total resultant beam. The results of these calculations are shown in Figs. 10 and 11. All of the plots are normalized to the desired signal maximum value $E_{D_{\text{max}}}$ occurring at broadside. The normalized patterns are denoted by the superscript $N$.

Looking at Fig. 10, the interfering beam is quite broad and has considerable magnitude in the direction of the desired signal. It can be seen also that this situation gets worse as the interfering beam gets closer to the desired signal. However, looking at Fig. 11, the wider aperture having the sharper beams allows greater array resolution.

With this particular formulation one of the aspects of aperture limitation can be illustrated by carrying out the same computations as above but with two interfering sources in the environment. This was done for a six element array with a spacing of .1λ and also for .15λ with interfering sources at 50° and 130° from endfire. The desired signal was maintained at broadside. For this case $p_D/\sigma^2=10$ dB and $p_{I_1,2}/\sigma^2=17$ dB.
Fig. 10—Components of steady state field for $L=2$, $d=0.5\lambda$, $P_{I_1}/\sigma^2 = 20$ dB, $P_{D}/\sigma^2 = 10$ dB.
Fig. 11—Components of steady state field for $L=6$, $d=.25\lambda$, $p_{I_1}/\sigma^2 = 20$ dB, $p_D/\sigma^2 = 10$ dB.
Figure 12 shows the relative field patterns for the three signals involved and the total resultant field for the .1λ spacing. The total field was scaled by an additional factor of ten for clarity. The three points of interest in this plot are at the locations of the three signals. Looking at the total field pattern at these points we note that the nulls do not occur at the exact location of the interfering sources. For this particular geometry the pattern coefficients for the two interfering signals are both negative. So to produce a null at precisely the 50° and 130° points, $a_2$ and $a_3$ would have to be slightly increased, however this would then decrease the total field in the direction of the desired signal. Thus the LMS algorithm provides the best compromise to maximizing noise rejection and desired signal maximization, i.e., best signal-to-noise ratio.

Now let us keep the same signal environment, but widen the spacing to .15λ thereby increasing the aperture. The results of these calculations are shown in Fig. 13. With this size aperture the pattern nulls are directly located in line with the interfering sources. Because of the larger aperture, the two interfering patterns have less contribution in the direction of the desired signal.

Thus a study of the preceding six plots sheds some light on the basic resolution problem of the adaptive antenna system. However, it should be kept in mind that the resultant pattern also is a function of the relative powers of the interfering sources, desired signal and thermal noise.

It should be recognized here that the field pattern of an adaptive antenna system is somewhat of an artificial quantity since to actually
Fig. 12—Composite field pattern for $L=6$, $d=.1\lambda$, $P_{I_1}/\sigma^2 = 17$ dB, $P_d/\sigma^2 = 10$ dB.
Fig. 13--Composite field pattern for $L=6$, $d=0.15\lambda$, $p_{I_1}/\sigma^2 = 17$ dB, $p_{D}/\sigma^2 = 10$ dB.
measure this pattern for a real system the adaptive loop has to be broken or in some way modified so that the weights are frozen. Nonetheless, the patterns do exist in reality but they are a function of time and change as soon as the environment changes. Consequently, to analyze these patterns in a conventional way could be misleading. That is, if we look at sidelobe levels and front-to-back ratios, these are things that in general involve S/N ratio considerations, and as such in the adaptive case the pattern will be changing to adapt to the prevailing environment. However, the field pattern representation for a fixed environment is of interest and provides valuable insight into adaptive array behavior.

D. Calculated Field Intensity Results

In this section the steady state field intensity of two arrays of the same physical aperture are compared. One array has two elements spaced .5λ, the other has six elements spaced .1λ apart. We first consider only one interfering signal in the environment at an angle of arrival of 40 degrees from endfire and with $p_{1}/\sigma^2 = 20$ dB. Comparing the two plots, Figs. 14 and 15, for the two and six element arrays, respectively, indicates very similar performance indicating no advantage for the six-element array or the two element array.

Next, two interfering signals were put in the environment and the steady-state pattern calculated. The interfering sources were of equal magnitude of $p_{1}/\sigma^2 = 20$ dB and placed at 20° and 60° from endfire. Comparing these two plots, Figs. 16 and 17, indicates the six-element array placed nulls of approximately 28 dB and 30 dB in the direction
Fig. 14—Power pattern for $L=2$, $d=0.5\lambda$, $p_{I_1}/\sigma^2 = 20$ dB, $p_D/\sigma^2 = 10$ dB.

Fig. 15—Power pattern for $L=6$, $d=0.1\lambda$, $p_{I_1}/\sigma^2 = 20$ dB, $p_D/\sigma^2 = 10$ dB.
Fig. 16--Power pattern for $L=2$, $d=.5\lambda$ and $p_{I_1}/\sigma^2 = p_{I_2}/\sigma^2 = 20$ dB, $p_D/\sigma^2 = 10$ dB.

Fig. 17--Power pattern for $L=6$, $d=.1\lambda$ and $p_{I_1}/\sigma^2 = p_{I_2}/\sigma^2 = 20$ dB, $p_D/\sigma^2 = 10$ dB.
of the two interfering sources, while the two-element array placed nulls of approximately 8 dB and 11 dB in the directions of the two interfering sources. This illustrates the advantage of additional elements in a fixed aperture. This aspect will be considered in more detail later. Figures 18 and 19 show the same conditions except the interfering sources are located at 40° and 20° from endfire. Here the six element array placed only one null in the pattern however the magnitude of the pattern in the general direction of the interfering sources is reduced compared to the two element array.

Next three interfering sources of equal amplitude were placed in the environment and the steady-state pattern again calculated. The three sources were at 20 degrees, 40 degrees, and 60 degrees from endfire. These results are shown in Figs. 20 and 21. In this case the six-element array put only two nulls in the pattern while attenuating the interfering sources by approximately 21 dB, 25 dB, and 50 dB. The two-element array attenuated these signals by approximately 7 dB, 43 dB, and 11 dB.

This same three-signal interference environment was used again with a different geometrical configuration. For this case the interfering sources were placed at 75°, 50°, and 25° from endfire, again of equal amplitude. These results are shown in Figs. 22 and 23. For this configuration the six-element array attenuated the interfering sources by approximately 22 dB, 24 dB, and 36 dB, while the two-element array attenuated the interfering sources by approximately 6 dB, 22 dB, and 11 dB, respectively. Again in this case, the six-element array only placed
Fig. 18--Power pattern for $L=2$, $d=.5\lambda$ and $p_{I_1}/\sigma^2 = p_{I_2}/\sigma^2 = 20$ dB, $p_D/\sigma^2 = 10$ dB.

Fig. 19--Power pattern for $L=6$, $d=.1\lambda$ and $p_{I_1}/\sigma^2 = p_{I_2}/\sigma^2 = 20$ dB, $p_D/\sigma^2 = 10$ dB.
Fig. 20--Power pattern for L=2, d=.5λ, $p_{I1}/\sigma^2 = p_{I2}/\sigma^2 = p_{I3}/\sigma^2 = 20$ dB, $p_D/\sigma^2 = 10$ dB.

Fig. 21--Power pattern for L=6, d=1λ and $p_{I1}/\sigma^2 = p_{I2}/\sigma^2 = p_{I3}/\sigma^2 = 20$ dB, $p_D/\sigma^2 = 10$ dB.
Fig. 22--Power pattern for $L=2$, $d=.5\lambda$, $p_1/\sigma^2 = p_2/\sigma^2 = p_3/\sigma^2 = 20$ dB, $p_D/\sigma^2 = 10$ dB.

Fig. 23--Power pattern for $L=6$, $d=.1\lambda$, $p_1/\sigma^2 = p_2/\sigma^2 = p_3/\sigma^2 = 20$ dB, $p_D/\sigma^2 = 10$ dB.
two nulls in the pattern. However, if the interfering signal power is increased to 40 dB above the thermal noise, a third null occurs in the pattern of the six element array. This pattern is shown in Fig. 24. As can be seen, the pattern nulls do not fall exactly on the interfering sources. If the interference power were increased further, the nulls would move closer to the interfering sources.

![Power pattern for L=6, d=0.5λ, $\frac{p_{I_1}}{\sigma^2} = \frac{p_{I_2}}{\sigma^2} = \frac{p_{I_3}}{\sigma^2} = 40$ dB, $\frac{p_D}{\sigma^2} = 10$ dB.](image)

Fig. 24--Power pattern for L=6, d=0.5λ, $\frac{p_{I_1}}{\sigma^2} = \frac{p_{I_2}}{\sigma^2} = \frac{p_{I_3}}{\sigma^2} = 40$ dB, $\frac{p_D}{\sigma^2} = 10$ dB.

For the test cases shown, if only a 0.5λ aperture is available for the array, the lack of aperture can be compensated to some extent by using more elements in that same aperture. As can be seen from the example just presented, the 0.1λ six-element array is clearly superior to the 0.5λ two-element array for multiple interfering signal environments.
E. **Array Power Output Calculation**

The field intensity results presented in the previous section show the nulls and maxima of the pattern in the vicinity of the interfering and desired sources. The nulls represent the amount of reduction in power of the interfering sources; however, because of normalizing factors it is difficult to determine the precise amount of received interference power and desired signal power at the array output, i.e., determine the (S/N) ratio at the array output. To determine these parameters more precisely, these various quantities are calculated and some test cases presented in this section.

Using Eq. (78) the steady-state output of the desired and interfering signal powers was calculated for two, three, four and six element arrays with spacings for these four arrays set at \(0.05\lambda, 0.1\lambda, 0.25\lambda\) and \(0.5\lambda\). A family of curves for each array is presented as the thermal noise power is varied causing the input interference power \(p_{I_1}/\sigma^2\) to vary from 10 dB to 50 dB while \(p_D/\sigma^2\), the input desired signal power, was kept constant at 10 dB.

The angle of arrival of the interfering signal, \(\phi_1\), was varied from 0° (endfire) to 90° (broadside). The desired signal was held at broadside. The frequency of the interfering source was chosen equal to that of the desired signal. Thus if a steady state array output interference power, \(\hat{p}_{I_1}/\sigma^2\), is calculated at -20 dB for \(p_{I_1}/\sigma^2\) of 10 dB, this indicates a pattern null in the direction of the interfering signal of approximately 30 dB relative to the pattern maximum. The same calculation can be made for the desired signal power. It is in this way that the calculations presented here can be correlated with the radiation patterns presented
earlier. The results of these calculations are shown in Figs. 25 to 40. These curves serve to give a general feeling for the performance of adaptive arrays with closely spaced elements.

Refering to Figs. 25 to 28 for the two-element array, the performance of the .5λ spacing is improved considerably over that of the smaller apertures. It should be noted for this particular spacing and number of elements that when the interfering signal is at endfire and the desired signal is at broadside they produce signals that are orthonogonal thus giving a perfect null for this case.

If we look at the two-element case for a spacing of .05λ, Fig. 25, the performance of the adaptive system is not very desirable. There is serious degradation of the desired signal-to-thermal noise ratio with only moderate interference rejection. However, if we compare the six-element array at a .05λ spacing, Fig. 37, to the two-element array at .25λ spacing, Fig. 27, the results are very similar. Likewise, if we compare the six-element array with .1λ spacing, Fig. 38, to the three-element array with .25λ spacing, Fig. 31, and the two-element array with .5λ spacing, we see similar performance, except for the unique orthogonal signal condition pointed out earlier for the latter case.

The similarity in these three arrays just compared and the two before is that they all have the same physical aperture. This suggests that the basic limitation to the performance of an adaptive antenna array is the aperture size. This is not necessarily surprising since most antenna performance factors are limited by physical aperture size. The present results show the aperture limitation from the adaptive array or spatial filter viewpoint.
Figure 26. $P_d/\rho_0^2$ and $P_{d}^{\prime}/\rho_0^2$ vs $\phi$ for $L=2$, $d=0.5\lambda$.

Figure 27. $P_d/\rho_0^2$ and $P_{d}^{\prime}/\rho_0^2$ vs $\phi$ for $L=2$, $d=1\lambda$. 

$P_d/\rho_0^2 = 10$ dB. 

$P_{d}^{\prime}/\rho_0^2 = 50$ dB.
Fig. 27—$\hat{P}_D/\sigma^2$ and $\hat{P}_{I_1}/\sigma^2$ vs $\phi_1$ for $L=2$, $d=.25\lambda$, $p_D/\sigma^2 = 10$ dB.

Fig. 28—$\hat{P}_D/\sigma^2$ and $\hat{P}_{I_1}/\sigma^2$ vs $\phi_1$ for $L=2$, $d=.5\lambda$, $p_D/\sigma^2 = 10$ dB.
Fig. 29 -- $\hat{P}_D/\sigma^2$ and $\hat{P}_{I_1}/\sigma^2$ vs $\phi_1$ for $L=3$, $d=.05\lambda$, $p_D/\sigma^2 = 10$ dB.

Fig. 30 -- $\hat{P}_D/\sigma^2$ and $\hat{P}_{I_1}/\sigma^2$ vs $\phi_1$ for $L=3$, $d=.1\lambda$, $p_D/\sigma^2 = 10$ dB.
Fig. 31—$\hat{P}_D/\sigma^2$ and $\hat{P}_{I_1}/\sigma^2$ vs $\phi_1$ for $L=3$, $d=0.25\lambda$, $p_{D}/\sigma^2 = 10$ dB.

Fig. 32—$\hat{P}_D/\sigma^2$ and $\hat{P}_{I_1}/\sigma^2$ vs $\phi_1$ for $L=3$, $d=0.5\lambda$, $p_{D}/\sigma^2 = 10$ dB.
Fig. 33—$\frac{P_D}{\sigma^2}$ and $\frac{\hat{P}_{I_1}}{\sigma^2}$ vs $\phi_1$ for $L=4$, $d=.05\lambda$, $P_D/\sigma^2 = 10$ dB.

Fig. 34—$\frac{P_D}{\sigma^2}$ and $\frac{\hat{P}_{I_1}}{\sigma^2}$ vs $\phi_1$ for $L=4$, $d=.1\lambda$, $P_D/\sigma^2 = 10$ dB.
Fig. 35—$P_D/\sigma^2$ and $P_{I_1}/\sigma^2$ vs $\phi_1$ for $L=4$, $d=.25\lambda$, $P_D/\sigma^2 = 10$ dB.

Fig. 36—$P_D/\sigma^2$ and $P_{I_1}/\sigma^2$ vs $\phi_1$ for $L=4$, $d=.5\lambda$, $P_D/\sigma^2 = 10$ dB.
Fig. 39—\( \hat{p}_D / \sigma^2 \) and \( \hat{p}_{I_1} / \sigma^2 \) vs \( \phi_1 \) for \( L=6 \), \( d=.25\lambda \), \( p_D / \sigma^2 = 10 \) dB.

Fig. 40—\( \hat{p}_D / \sigma^2 \) and \( \hat{p}_{I_1} / \sigma^2 \) vs \( \phi_1 \) for \( L=6 \), \( d=.5\lambda \), \( p_D / \sigma^2 = 10 \) dB.
F. Array Tradeoffs

Probably the single most important parameter in choosing an antenna array for a communications application is the signal-to-noise (S/N) ratio at the array output. The (S/N) ratio as defined for this paper does not consider preamplifier or receiver characteristics such as bandwidth and demodulator performance as parameters. For the purpose of comparison of different types of antenna arrays, these effects are held constant.

As in most engineering problems tradeoffs are involved, and in the case of adaptive arrays tradeoffs can be made with respect to gain or loss of (S/N) ratio. However, since the performance of an adaptive array is largely dependent on the external signal environment in which it is operating some assumptions must be made about the environment before comparisons can be made between different arrays. The following set of figures shows the performance of adaptive antenna arrays for various aperture sizes in terms of signal-to-noise ratio.

The simplest environment is the case where only the desired signal is present in the environment. Table 3 shows the steady-state results of (S/N) for arrays having two through six elements in a 0.5λ aperture for a desired signal-to-thermal noise power ratio (p₀/σ²) of 10 dB incident on the array at broadside. For this particular algorithm, this result is independent of spacing, but dependent on the number of elements as shown. Since 10 dB also represents the (S/N) that would be received by one isotropic element, then referring to Table 1 the two-element array gives a 3 dB improvement over isotropic, while the six-element array gives 7.8 dB improvement over isotropic. This shows the obvious advantage in adding elements to improve (S/N) for this particular environment.
The next somewhat more complex environment is one having a desired signal and an interfering signal with a wide spatial separation. Figure 41 shows the steady-state (S/N) for two, four, and six-element arrays versus element spacing for an interfering source at 60° from endfire with 20 dB signal-to-thermal power ratio and the desired signal at broadside with a 10 dB signal-to-thermal power ratio. The obvious advantage of larger aperture size can be seen. Referring also to Table 3 and Fig. 41, the two-element array at .1λ spacing gives a steady-state (S/N) almost 10 dB below optimum while the two-element array spaced at .5λ has only a 3 dB degradation below optimum. The four and six-element arrays give nearly optimum (S/N) ratios (i.e., almost total interference rejection) for apertures of 1λ corresponding to spacing of .33λ and .2λ, respectively. It should be noted that if lines of constant aperture size were drawn on this plot, they would be nearly horizontal.
The next case to consider is again one desired signal and one interfering signal, but allowing the interfering source to come closer to the desired source. The same signal levels used in the previous example are used here again; however, the interfering source is placed at 75°, 80°, 85°, and 87° from endfire while the desired signal is at broadside. Figures 42, 43, and 44 show (S/N) versus element spacing for the two, four, and six-element arrays, respectively. Analysis of these three figures indicates that for interfering signals of this magnitude or greater that some (S/N) degradation will be experienced for interfering sources closer than 15° to the desired signal. For a one-half wavelength aperture, a (S/N) ratio degradation of approximately 12 dB can be expected for the 80° case. However,
doubling the aperture to $1\lambda$ corresponding to four elements and spacing of $0.33\lambda$, reduces the (S/N) ratio degradation to approximately 8 dB. Lengthening the aperture to $2.5\lambda$, corresponding to the six-element $0.5\lambda$-spaced array, still has approximately a 2 dB (S/N) ratio degradation for the $80^\circ$ case. Thus, rather large apertures are required for resolution of strong interfering signals closer than $15^\circ$. If this particular signal environment is expected, it may be necessary to overdesign for the non-interfering environment to compensate for the degradation expected for small, desired-to-interference signal spatial separations.
Fig. 43—S/N vs d(λ) for L=4, p_D/σ^2 = 10 dB, p_I/σ^2 = 20 dB and Φ_1 = 75°, 80°, 85°, 87°.

The next environment to be studied is one where there is one desired signal but multiple interfering sources. First we consider two interfering signals of equal power (same as previous examples) and spaced at 60° and 40° from endfire. The desired signal is again at broadside. This particular signal environment is used for the two, four, and six-element arrays to calculate the steady-state (S/N) ratio for various element spacings as before. The results of these calculations are shown in Fig. 45. In this particular case we expect aperture size as well as the number of elements within that aperture to have an effect on the final value of steady-state (S/N) ratio. Observing Fig. 45, one notes only a 2 dB variation in (S/N) as the spacing is varied from .1λ to .5λ for the
two-element case. However, placing more elements in that same aperture improves the performance by approximately 5 dB for the six-element case and approximately 3 dB for the four-element case for a \(0.5\lambda\) aperture. For a one-wavelength aperture, both the four and six-element cases are approximately 4 dB from their optimum (S/N) ratios.

The next environment to be considered is the same as the previous except a third interfering signal of equal strength as the other two is added at 20° from endfire. Thus we have three interfering signals in the environment at 60°, 40° and 20° from endfire with the desired signal at broadside. The same calculations as before were made and the results presented in Fig. 46. Observing Fig. 46 for the two-element case, no
Fig. 45—S/N vs d(λ) for $p_0/\sigma^2 = 10$ dB, $\phi_1=60^\circ$, $\phi_2=40^\circ$, $p_{I1}/\sigma^2 = p_{I2}/\sigma^2 = 20$ dB, $L=2, 4, 6$.

A significant advantage is gained by widening the element spacing from .1λ to .5λ. However, placing two more elements in that same .5λ aperture represents a 5 dB improvement in (S/N) ratio while placing six elements in the .5λ aperture represents approximately an 8 dB improvement but still a degradation of 13 dB with respect to optimum. If the aperture is increased to 1λ, the four and six-element arrays have about the same (S/N) ratios with the six being approximately 7 dB from optimum.

Analyzing the preceding six figures, Figs. 41 to 46, one can gain some insight into the effects of aperture size and number of elements in the array. As pointed out earlier, the performance of the adaptive array for one interfering source is only limited by aperture size. However,
where there are multiple interfering sources in the environment, adding elements in the same fixed aperture does improve performance. More precise handling of this question is presented later. For any signal environment performance is improved by making the aperture larger, either by adding elements or increasing spacing. However, even for the relatively small aperture of .5λ, some (S/N) ratio protection can be realized.

Also of importance are systems where the thermal noise power is small compared to the desired and interfering signal powers. In systems of this type, the ratio of the desired signal power to total interference signal power is of concern. It is instructive to look at the effect of aperture size and number of elements on this type of a system. To do
this, two \(0.5\lambda\) apertures, one with three elements and one with two elements, were compared in the presence of two interfering signals at \(50^\circ\) and \(75^\circ\) from endfire. The desired signal was maintained at broadside and at 10 dB above the thermal noise power. The magnitude of the two interfering sources was varied from 0 dB to 50 dB above the desired signal power.

The results of these calculations are shown in Fig. 47. The advantage of the third element in the fixed aperture is clearly shown here. The additional degree of freedom provided by the third element allows for the second null to be formed even though the element spacing has been decreased. The respective power patterns for the two arrays are shown in Figs. 48 and 49, for \(p_{I_1,2}/p_D = 40\) dB. The two-element array places one

\[
\begin{align*}
L = 3 & \quad d = 0.25\lambda \\
L = 2 & \quad d = 0.5\lambda
\end{align*}
\]

Fig. 47--\(P_D/\sum_{i=1}^{2} \hat{P}_{I_i}\) vs \(p_{I_1,2}/p_D\) for \(L=2\), \(d=0.5\lambda\) and \(L=3\), \(d=0.25\lambda\), \(\phi_1=75^\circ\), \(\phi_2=50^\circ\), \(p_D/\sigma^2 = 10\) dB.
Fig. 48—Power pattern for $L=2$, $d=0.5\lambda$, $p_{I_1,2}/p_D = 40$ dB.

Fig. 49—Power pattern for $L=3$, $d=0.25\lambda$, $p_{I_1,2}/p_D = 40$ dB.
pattern null between the two interfering sources while the three-element array places two pattern nulls in the direction of the two interfering sources. This further illustrates the advantage of extra elements in an aperture of fixed size.

To clarify this point somewhat further Table 4 showing S/N ratio and $\hat{P}_D/\sum_{i=1}^{2} \hat{P}_{I_i}$ vs number of elements in a fixed aperture size of .75A for $P_{I_1,2}/P_D = 10$ dB to 50 dB in 10 dB increments has been constructed. The two interfering sources were placed at 50° and 75° from endfire while the desired signal was placed at broadside with $P_D/\sigma^2 = 20$ dB. Refering to Table 4 for the $L=2, d=.75\lambda$ case, the S/N and $\hat{P}_D/\sum_{i=1}^{2} \hat{P}_{I_i}$ values continue to decrease as $P_{I_1,2}/P_D$ increases; however, if the third element is placed in the same .75A aperture making $d=.375\lambda$, there is dramatic improvement in S/N and $\hat{P}_D/\sum_{i=1}^{2} \hat{P}_{I_i}$. For this case, the S/N reaches a minimum of 11.1 dB as $P_{I_1,2}/P_D$ increases. It can further be seen from this table that no appreciable increase is noted in S/N or $\hat{P}_D/\sum_{i=1}^{2} \hat{P}_{I_i}$ as the number of elements is increased beyond three, for any aperture size shown.

As shown in the previous data, aperture size does improve performance, especially for interfering sources close to the desired source. To illustrate this point, a plot of S/N vs number of elements in a fixed aperture of .5\lambda, .75\lambda, 1.0\lambda and 1.25\lambda is shown in Fig. 50. As can be seen from this plot, for the three-element case, approximately a 10 dB improvement in S/N is realized in going from a .5\lambda aperture to a 1.0\lambda aperture. Little improvement is seen for increasing the number of elements in the aperture beyond three, for any aperture size shown.
TABLE 4

(S/N) RATIO AND $\hat{P}_D/\sum_{i=1}^{2} \hat{P}_{I_i}$ VS NUMBER OF ELEMENTS IN FIXED APERTURE FOR $\phi_1=75^\circ$, $\phi_2=50^\circ$

<table>
<thead>
<tr>
<th>$P_{I_1,2}/P_D$</th>
<th>10 dB</th>
<th>20 dB</th>
<th>30 dB</th>
<th>40 dB</th>
<th>50 dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>L $d(\lambda)$</td>
<td>$S/N$</td>
<td>$\frac{\hat{P}<em>D}{\sum \hat{P}</em>{I_i}}$</td>
<td>$S/N$</td>
<td>$\frac{\hat{P}<em>D}{\sum \hat{P}</em>{I_i}}$</td>
<td>$S/N$</td>
</tr>
<tr>
<td>2 .75</td>
<td>-6.7</td>
<td>-6.7</td>
<td>-16.7</td>
<td>-16.7</td>
<td>-26.7</td>
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<td>3 .375</td>
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<td>38.8</td>
<td>11.1</td>
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<tr>
<td>4 .25</td>
<td>11.8</td>
<td>30.0</td>
<td>11.8</td>
<td>39.9</td>
<td>11.8</td>
</tr>
<tr>
<td>5 .187</td>
<td>12.3</td>
<td>30.7</td>
<td>12.2</td>
<td>40.6</td>
<td>12.2</td>
</tr>
<tr>
<td>6 .15</td>
<td>12.7</td>
<td>31.4</td>
<td>12.6</td>
<td>41.3</td>
<td>12.6</td>
</tr>
</tbody>
</table>
Fig. 50--S/N vs number of elements in fixed aperture size of 
.5λ, .75λ, 1.0λ and 1.25λ for $\phi_1=75^\circ$, $\phi_2=.50^\circ$, 
$P_{I1,2}/P_D = 30$ dB, $\phi_d=90^\circ$, $P_D/\sigma^2 = 20$ dB.

If a third interfering source is placed in the environment along 
with the desired signal, one would expect the minimum number of elements 
for the best performance to be four. To confirm this, a third interfering 
source was placed in the environment at 25° from endfire and the same 
data as calculated for Table 4 was calculated again and the results are 
shown in Table 5. As expected, four elements is the minimum number of 
elements for optimum performance. Also as before, minimal improvement 
is achieved by adding more elements to the array than equal to the number 
of signals in the environment. However as before, improvement can be
TABLE 5

(S/N) Ratio and $\hat{P}_D / \sum_{i=1}^{3} \hat{P}_{I_i}$ vs Number of Elements in Fixed Aperture for $\phi_1 = 75^\circ$, $\phi_2 = 50^\circ$, $\phi_3 = 25^\circ$

<table>
<thead>
<tr>
<th>$\frac{P_{I_1,2,3}}{P_d}$</th>
<th>10 dB</th>
<th>20 dB</th>
<th>30 dB</th>
<th>40 dB</th>
<th>50 dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>L d</td>
<td>S/N</td>
<td>$\frac{\hat{P}<em>D}{\sum \hat{P}</em>{I_i}}$</td>
<td>S/N</td>
<td>$\frac{\hat{P}<em>D}{\sum \hat{P}</em>{I_i}}$</td>
<td>S/N</td>
</tr>
<tr>
<td>2 .75</td>
<td>-9.8</td>
<td>-9.8</td>
<td>-19.8</td>
<td>-19.8</td>
<td>-29.8</td>
</tr>
<tr>
<td>3 .375</td>
<td>- .64</td>
<td>-.55</td>
<td>-10.6</td>
<td>-12.5</td>
<td>-20.6</td>
</tr>
<tr>
<td>4 .25</td>
<td>3.7</td>
<td>8.3</td>
<td>2.1</td>
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</tr>
<tr>
<td>5 .187</td>
<td>4.7</td>
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<td>17.5</td>
<td>3.3</td>
</tr>
<tr>
<td>6 .15</td>
<td>5.2</td>
<td>11.2</td>
<td>4.1</td>
<td>18.8</td>
<td>3.9</td>
</tr>
</tbody>
</table>
made for the four element case if the aperture size is increased. The effect of aperture size is shown in Fig. 51 again with significant improvement in performance being shown for the 1.0λ aperture with four elements over the .5λ aperture with four elements.

Thus the major conclusion here is that the array needs the same number of elements in the aperture as there are signals in the environment for optimum performance even though the elements are more closely spaced than the array with an insufficient number of elements.

Fig. 51--S/N vs number of elements in fixed aperture size of .5λ, .75λ, 1.0λ and 1.25λ for φ1 = 75°, φ2 = 50°, φ3 = 25°, p_{12,3}/p_D = 30 dB, φ_d = 90°, p_D/σ^2 = 20 dB.
G. Relationship to Directivity

It has been shown that small aperture adaptive arrays with closely spaced elements do not always place pattern nulls in the direction of the interfering sources when there are multiple interfering signals. This is because the basic resolution of the adaptive array or spatial filter is limited by the $\frac{\sin Lz}{\sin z}$ function. However, it has also been shown that the total steady-state field pattern is made up of a weighted sum of these functions with one term associated with each signal in the environment.

Since directivity can be defined as the ratio of the radiation intensity in the direction of the main beam to the average radiation intensity, one could expect that as the weighted sum of $\frac{\sin Lz}{\sin z}$ terms is increased there would be a corresponding increase in directivity. However, as the aperture size is increased and the individual pattern terms become more directive, one might not expect as great an increase in overall directivity.

The above observation points out several questions that will be addressed in this section. Basically the topic is the inter-relation between optimum directivity, the LMS algorithm and aperture size. This is not meant to be an exhaustive study of this topic, but to merely introduce it and show that there is a relationship between increased directivity arrays and adaptive arrays.

Optimization of the directivity can be carried out by searching for the set of array weights that maximizes the ratio described above. The optimization of directivity is usually carried out by expressing the directivity as a ratio of two Hermitian quadratic forms[14]. When this
is done, then the optimum weight distribution can be found by solving an eigenvalue problem with the largest eigenvalue being the value of the maximum directivity. The associated eigenvector is then the optimum or superdirectivity weight distribution. It also can be shown that the \((S/N)\) ratio can be expressed as a ratio of two Hermitian quadratic forms with the optimization for \((S/N)\) ratio being carried out as described above. The weight distribution is the same as obtained with the LMS algorithm. It has been shown that[14] the optimizations for directivity and \((S/N)\) ratio are equivalent when the only noise present in the system is uniformly distributed (in space) noise. This case approximates that of a background noise limited system, such as an H.F. receiving array.

In a heuristic sense, a \((S/N)\) ratio optimization with discrete noise sources in the environment only seeks to minimize the discrete points in the pattern where the noise sources are located and maximize the pattern in the direction of the desired signal. However, when the discrete noise sources approximate a uniform distributed noise source, and the element spacing is small enough, the average radiation is decreased in reducing the interference or noise thus increasing the directivity.

Using the optimization procedure outlined above it can be shown that the optimum directivity, \(D_M\), is given by

\[
D_M = \alpha_d^* B^{-1} \alpha_d
\]

where \(B\) is an \(L \times L\) square matrix with typical elements for vertical dipoles of
\begin{align}
\begin{bmatrix} b_{ij} \end{bmatrix} &= \frac{\sin[2\pi(i-j)d]}{2\pi(i-j)d} + \frac{\cos[2\pi(i-j)d]}{[2\pi(i-j)d]^2} \\
&\quad - \frac{\sin[2\pi(i-j)d]}{[2\pi(i-j)d]^3}
\end{align}

and \( \alpha_d \) is as defined earlier.

It can also be shown\cite{21} that the directivity for an arbitrary weight distribution is given by the ratio

\begin{equation}
D = \frac{\hat{W}^* A \hat{W}_{ss}}{\hat{W}^* B \hat{W}_{ss}}
\end{equation}

where \( A \) is an \( L \times L \) square matrix with typical elements of the form

\begin{equation}
[a_{ij}] = e^{-j2\pi d(i-j) \cos \phi_d}
\end{equation}

where \( \phi_d \) is the direction of the desired signal. Also, \( \hat{W}_{ss} \) is as defined earlier.

With the above relationships some comparisons can be made between the maximum directivity and the directivity obtained using the steady-state weights after adapting to some fixed environment. To make these comparisons, four different signal environments were chosen. The first has only a desired signal present from the endfire location. The resulting weight distribution is a uniform one. Thus the directivity calculated for this set of weights would correspond to the endfire, uniform directivity. The next three signal environments have the same desired signal but four, nine and eighteen interfering sources distributed uniformly in \( \phi \) around the array with the first signal always placed at endfire. Thus for the case where \( M = 4 \), or four interfering sources, the
sources were located at $\phi_1=0^\circ$, $\phi_2=90^\circ$, $\phi_3=180^\circ$ and $\phi_4=270^\circ$. For each environment all of the interfering sources were of the same power level.

Figures 52, 53, and 54 show the results of these calculations where the endfire directivity is plotted against element spacing for a four, six and ten element array for the four signal environments. The dashed curve represents the maximum directivity using the relationship shown in Eq. (109)

Fig. 52—Directivity vs $d(\lambda)$ for $L=4$, $p_{I_i}/\sigma^2 = 20$ dB, $\phi_d=0^\circ$, $p_D/\sigma^2 = 10$ dB.

Fig. 53—Directivity vs $d(\lambda)$ for $L=6$, $p_{I_i}/\sigma^2 = 20$ dB, $\phi_d=0^\circ$, $p_D/\sigma^2 = 10$ dB.
Fig. 54—Directivity vs $d(\lambda)$ for $L=10$, $p_1/\sigma^2 = 20$ dB, $\phi_d=0^\circ$, $p_D/\sigma^2 = 10$ dB.

Refering to Figs. 52 and 53 for the four- and six-element arrays and $M=12,18$, for spacings of $0.15\lambda$ and greater the LMS distribution very closely matches the distribution giving maximum directivity. This would be expected though because with $M=18$ and the aperture sizes involved, this would approach a background noise limited case. This is not quite the case for the ten element array where the aperture is significantly larger. It is interesting to note, however, that all three arrays for $M=9$ give improved directivity over a uniform distribution for the very close spacings.

Figures 55 and 56 show the same calculations for the six- and ten-element arrays, except the ratio of interfering signal to thermal noise at the array has been increased to 40 dB. This has the general effect of increasing the directivity. As can be seen for the six-element array and $M=18$, the LMS weights closely approximate the maximum directivity...
Fig. 55—Directivity vs d(x) for L = 1, for L = 6, P_l = 10 dB.
distribution. This also holds true for the ten-element array until the aperture size becomes too large for increased directivity.

The relative power pattern for the six- and ten-element arrays at .15λ spacing and M=18 with $p_{I_i}/\sigma^2 = 40$ dB are shown in Figs. 57 and 58, respectively.

There are two main conclusions that can be drawn from this discussion on directivity. The first is that in the case where a receiving system is background noise limited, an adaptive array may be an alternative in a superdirective array design. The apparent advantages are the ability to steer the desired beam, maintaining the weighting coefficients in a feedback loop arrangement and being able to provide some protection against discrete interfering sources.

The second conclusion that can be drawn is that even with as few a number of interfering sources as four and closely spaced elements, some directivity improvement is realized over that resulting from a conventional array with a uniform distribution.
Fig. 57—Power pattern for \(L=6\), \(d=.15\lambda\), \(M=18\) and \(P_{I_1}/\sigma^2 = 30\) dB.

Fig. 58—Power pattern for \(L=10\), \(d=.15\lambda\), \(M=18\) and \(P_{I_1}/\sigma^2 = 30\) dB.
CHAPTER VI
TRANSIENT ANALYSIS OF LMS ADAPTIVE SYSTEM

In this section we will investigate the transient performance of an adaptive system utilizing the LMS algorithm and made comparisons of performance using coupled and uncoupled antenna arrays.

A. Transient Solution for Coupled and Uncoupled Cases

Recalling Eq. (28), the complete solution for the average weights is

\[
\hat{w} = e^{-k_1 \hat{R}_v t} \hat{w}_0 + \hat{R}_v^{-1} \left( I - e^{-k_1 \hat{R}_v t} \right) (\hat{r}v)
\]

(113)

It is apparent that the form of this equation is not suitable for further analysis. The exponential terms involving the square constant matrix, \( \hat{R}_v \), must be expanded. To do this we will make use of some theorems from linear algebra presented by Zahm[8]. A detailed proof of the theorems used will not be shown here but the results merely stated. (See [15] or [16].) What we desire is a decomposition of the square matrix \( \hat{R}_v \) into its eigenvectors and eigenvalues.

If we denote \( x_i \) to be the \( i \)th eigenvector with \( \lambda_i \) its associated distinct eigenvalue for the matrix \( \hat{R}_v \), then

\[
\hat{R}_v x_i = \lambda_i x_i
\]

(114)
It can then be shown that the matrix $R_v$ can be written as

$$ (115) \quad \hat{R}_v = \sum_{i=1}^{m} \lambda_i E_i $$

where

$$ (116) \quad E_i = \frac{x_i x_i^*}{(x_i, x_i)} $$

and the index $m$ is the number of distinct eigenvalues of $\hat{R}_v$ determined by

$$ (117) \quad \det | \hat{R}_v - \lambda_i | = 0 $$

It can also be shown\(^{[15]}\) that

$$ (118) \quad \begin{cases} \sum_{i=1}^{m} E_i = I \\ E_i E_j = \begin{cases} E_i & i=j \\ 0 & i \neq j \end{cases} \end{cases} $$

and further that

$$ (119) \quad e^{-k_1 \hat{R}_v t} = \sum_{i=1}^{m} e^{-k_1 \lambda_i t} E_i $$

and for non-zero eigenvalues that

$$ (120) \quad \hat{R}_v^{-1} = \sum_{i=1}^{m} \frac{1}{\lambda_i} E_i $$

For details of the above see\(^{[15]}\) or\(^{[16]}\).
With these definitions established, the weight equation (113) can now be written in terms of the eigenvalues and eigenvectors or the matrix $\hat{R}_v$ as

$$\hat{w} = \sum_{i=1}^{m} e^{-k_1 \lambda_i t} E_i W_0 + \sum_{i=1}^{m} \frac{E_i}{\lambda_i} \left(1 - e^{-k_1 \lambda_i t}\right).$$

The column vector $W_0$ represents the weights at $t=0$. The choice for these weights is arbitrary and for the test cases shown here we will define $W_0$ as

$$W_0 = Z^* W_0'$$

where $W_0'$ is defined as

$$W_0' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus, with this choice of initial conditions, the array at $t=0$ will consist of one element with a unity weight and all other elements with weights of zero value. Thus, in effect, in the plane we are considering, the array at $t=0$ looks like a single isotropic element.

The eigenvectors $x_i$ as defined in Eq. (114) span the space and form a basis. Thus $W_0'$ can be expressed as a linear combination of the eigenvectors as

$$W_0' = \sum_{i=1}^{m} b^i x_i$$

and the $b^i$ can be shown to be
Thus, 

\[(126) \quad W_0 = Z_0^* \sum_{i=1}^{m} \frac{(x_i, W_0^{*})}{||x_i||^2} x_i \]

Equation (119) can now be written

\[(127) \quad W = \sum_{i=1}^{m} \frac{x_i x_i^*}{||x_i||^2} \frac{Z_0^*}{||x_j||^2} \sum_{j=1}^{m} \frac{(x_j, W_0^{*})}{||x_j||^2} x_j \]

\[+ \sum_{i=1}^{m} \frac{E_i}{\lambda_i} \left(1 - e^{-k_1 \lambda_i t}\right) (\hat{r}V)\]

As in the previous section two cases will be considered, the uncoupled array where

\[(128) \quad \hat{R}_{vu} = Z_{ou}^{-1} E[V_0 V_0^{*}] Z_{ou}^{-1*}\]

and

\[(129) \quad (\hat{r}V_u) = Z_{ou}^{-1} \hat{r}_v\]

and the coupled case where

\[(130) \quad \hat{R}_{vc} = Z_{oc}^{-1} E[V_0 V_0^{*}] Z_{oc}^{-1*}\]

and
(131) \( (\hat{\mathbf{r}}^T \mathbf{c}) = Z_{oc}^{-1} \hat{\mathbf{r}}_v \)

where for both cases

(132) \( \hat{\mathbf{r}}_v = \sqrt{R D} \alpha_d \)

B. Calculated Transient Adaptive Array Performance

To see how the transient response of the adaptive array depends on mutual coupling, we must solve the eigenvalue problems for the coupled and uncoupled cases. Specifically, we must find the eigenvalues and eigenvectors of the following two equations

\begin{align*}
(133) \quad Z_{ou}^{-1} E[V_0 V_0^*] Z_{ou}^{-1} x_i &= \lambda_i x_i \\
(134) \quad Z_{oc}^{-1} E[V_0 V_0^*] Z_{oc}^{-1} x_j &= \lambda_j x_j 
\end{align*}

As can be seen from Eq. (127), in general, the magnitude of \( \lambda_i \) will affect the speed of the transient response. Observing Eqs. (133) and (134), for fixed signal conditions or for \( E[V_0 V_0^*] \) constant, the eigenvalues will change when the elements in \( Z_0 \) change. In general, as the elements in \( Z_0^{-1} \) increase we should find eigenvalues of increasing value and a more rapid transient response.

To illustrate this effect, a signal environment again was chosen having a desired signal at broadside and one interfering source at 60° from endfire. The array in this case has three elements with .25\( \lambda \) element spacing. The interfering signal level was maintained at 10 dB above the desired signal at the array input.
Utilizing Eqs. (127) and (134) the S/N ratio for the transient case was calculated for antenna element heights of .2λ, .3λ, .4λ and .5λ. The results of these calculations are shown in Fig. 59. Varying the element heights changes the mutual coupling hence changes the elements of Z₀ causing corresponding changes in the eigenvalues. For these calculations the load impedance Z_L was chosen to be Z_{11}. Thus for the .2λ element height case, the load must be very large to tune out the large reactance. Recalling that all elements of Z₀ have been normalized with respect to Z_L, Eq. (133) indicates that larger eigenvalues are obtained as Z_L becomes correspondingly larger. The result of the larger eigenvalues is seen in Fig. 59, where the .2λ element case has a significantly shorter transient response than does the .5λ case.

Fig. 59—S/N vs T for L=3, d=.25λ, Z_L=Z_{11}, \phi_1=60° and p_{I_1}/\sigma^2 = 20 dB.
For comparison, the same calculations were made again except the load impedance $Z_L$ was kept fixed at $50.+j0$. This would be the more likely case in practice. The results of these calculations are shown in Fig. 60. In this case while the magnitude of the elements of $Z_0$ are increasing as the element height is decreasing, the elements of $Z_0^{-1}$ are decreasing, causing smaller eigenvalues and a longer transient response. Thus as can be seen in Fig. 60, the transient response for the $0.2\lambda$ element case is longer than for the $0.5\lambda$ case.

Fig. 60—S/N vs $T$ for $L=3$, $d=.25\lambda$, $Z_L=50.+j0$, $\phi_1=60^\circ$ and $p_{I_1}/\sigma^2 = 20$ dB.
It also can be noted that in both cases, with only one interfering signal in the environment, all of the arrays converge to the same steady state S/N ratio regardless of the element height. This characteristic was noted in the previous section.

Also of interest is the case where the aperture size is fixed but the number of elements in that fixed aperture is increased. In effect, increasing the number of elements in the fixed aperture increases the coupling and increases the magnitude of the elements of $Z_0$. For this example, two interfering signals at 75° and 50° from endfire were placed in the signal environment along with the desired signal at broadside as before. The S/N ratio for the transient response was calculated for three arrays each having a $0.5\lambda$ aperture. The results of these calculations are shown in Fig. 61. It appears that the more tightly coupled array (more than two elements) with the closer element spacing has a slightly longer transient response than the more loosely coupled array (two elements).

It also can be observed that a six element array has several slopes to the transient curve because of additional degrees of freedom obtained with the larger number of elements.

The same calculations were repeated for the same antenna arrays as used in the above case; however, a third interfering signal was placed in the environment. The results of these calculations are shown in Fig. 62. In this case the six element array still has a slightly longer transient response which is not significantly different than the case shown in Fig. 61.
It also can be noted in Figs. 61 and 62 that the three different arrays converge to three different steady state S/N ratios. The numerical values show the advantage of the additional elements in the fixed aperture when multiple interfering signals are present.

For the case where the receiving system is not thermal noise limited, the ratio $\frac{\hat{P}_D}{\sum_{i=1}^{M} \hat{P}_{I_i}}$ is of interest. To investigate this case, the three element array with $.25\lambda$ spacing was again utilized. In
addition, two interfering sources at 75° and 50° from endfire were placed in the signal environment with the desired signal at broadside. Calculations were made for the cases when the $p_D/p_{I_1}$ ratio was taken to be 10 dB, 20 dB and then 30 dB. The transient behavior of $\hat{p}_D/\sum_{i=1}^{2} \hat{p}_{I_i}$ was calculated and the results are shown in Fig. 63. It can be seen that even though the steady state values are different for the three cases, the transient responses are the same.

The transient variations in the ratio $P_D/\sum_{i=1}^{M} \hat{P}_{I_i}$ correspond to a changing antenna pattern. To get an insight into the transient behavior of the array pattern, the element weights at various times during the
Fig. 63--$\frac{\hat{P}_D}{\sum \hat{P}_{I_i}}$ vs $T$ for $L=3$, $d=.25\lambda$, $h=.5\lambda$, $Z_L=Z_{11}$, $\phi_1=75^\circ$, $\phi_2=50^\circ$, $p_D/\sigma^2 = 10$ dB.

Transient period were calculated and the corresponding power pattern of the array was determined. The results of these calculations are shown in Fig. 64. The same array and signal conditions as used in Fig. 61 were used again. The $p_D/p_{I_1,2}$ ratio used was 30 dB.

As can be seen, from Fig. 62, the array first forms a null between the two interfering sources which tends to reduce the desired signal level. The next pattern shows a further reduction in the desired signal
level, and the final pattern shows two nulls over the two interfering sources.

Fig. 64—Transient power pattern for $L=3$, $d=.25\lambda$, $h=.5\lambda$, $Z_L\approx z_1$, $\phi_1=75^\circ$, $\phi_2=50^\circ$, $p_{I_1,2}/\sigma^2 = 30$ dB, $p_D/\sigma^2 = 10$ dB.
CHAPTER VII
STEADY-STATE ANALYSIS OF ADAPTIVE ARRAY WITH
MODIFIED LMS ALGORITHM

The LMS adaptive algorithm utilizes what is often referred to as a reference loop. By means of this loop, a reference signal is subtracted from the array output and an error signal is formed which is then correlated with the signal from each element. If the reference signal is not known or the replica is weakly correlated with any signal in the array output, the steady-state weights tend to zero and the array would have to be reinitialized if the signal were not acquired during the transient period. This, of course, may be undesirable. Another somewhat undesirable feature of the reference loop is that performance is highly dependent on the signal environment.

To counteract these somewhat undesirable features, a modified LMS algorithm has been suggested by several researchers[7,8]. This algorithm is shown functionally in Fig. 65. There is no reference signal used here but a steering vector $b$ is introduced. Presumably this vector can be chosen such that the desired signal is tracked. In addition to the steering vector, a bias signal $n$ also is added to each loop. This signal has the net effect of increasing the effective thermal noise in the system. By adding this signal $n$, some performance is sacrificed for
some control over the effect of the signal environment. This will become clearer later on. The performance of this modified algorithm for both the coupled and the uncoupled cases will be analyzed first for the steady state and then later for the transient state.

Referring to Fig. 65, the differential equation for the $i$th weight can be written as
(135) \[ \frac{dW^i}{dt} = k_1 \left( -n w^i + b^i - \sum_{j=1}^{L} v^i v^*_{ij} w^j \right) \]

and the vector equation for all loops can be written as

(136) \[ \frac{dW}{dt} + k_1 (V V^* + n I)W = k_1 b \]

Proceeding as before and utilizing the earlier definitions, the differential equation for the average weights can be determined by taking the expected value of both sides of Eq. (136). Thus, by doing this and using the earlier definition for the signal covariance matrix, Eq. (136) becomes

(137) \[ \frac{d\hat{W}}{dt} + k_1 (\hat{R}_v + n I)\hat{W} = k_1 b \]

where \( b \) has been assumed to be deterministic.

We can quickly arrive at a solution to Eq. (137) by comparing it to Eq. (21). Recognizing that if we replace \( R_v \) in Eq. (21) by \( \hat{R}_v + n I \) and replace \( \hat{V} \) in Eq. (21) by \( b \), we can utilize Eq. (28) and write down the total solution to Eq. (137). Thus, by making the above suggested replacements, the solution to Eq. (137) is

(138) \[ \hat{W} = e^{-k_1(\hat{R}_v+n I)t} W_0 + (\hat{R}_v+n I)^{-1} \]

\[ \cdot \left( I - e^{-k_1(\hat{R}_v+n I)t} \right) b \]

where all parameters have the same definitions as before.

The steady-state performance will now be analyzed for the coupled and uncoupled cases. The steady-state weights for this algorithm are given by
(139) \[ \hat{W}_{ss} = (R_v + \eta I)^{-1} b \]

where again \( \hat{W}_{ss} \) represents the average weight value after all transients have died out.

One can get some feel for the significance of the parameter \( \eta \) by looking at Eq. (139). If \( \eta \) is made large compared to the elements \( R_v \), then the weight distribution \( \hat{W}_{ss} \) will become a scaled version of the vector \( b \). Thus, the resulting nulls generated by the interfering sources will not be as deep. On the other hand, if \( \eta = 0 \) and if the desired signal direction is not known and \( b \) is chosen to give an omni-directional pattern, then for large \( P_D/\sigma^2 \) and \( P_{I_1}/\sigma^2 \) the output \( \hat{P}_D/\hat{P}_{I_1} \) is proportional to \( P_{I_1}/P_D \). However, the output signal power, \( \hat{P}_D \), is inversely proportional to \( P_D \). Thus, for this algorithm some tradeoff in performance must be made to establish the value of \( \eta \). This subject will not be addressed in this study. The reader is referred to \([8]\).

A. Uncoupled Element Case

We will consider first the uncoupled case where the impedance matrix \( Z_0 \) is diagonal. As before for this case, the signal covariance matrix becomes

(140) \[ \hat{R}_{vu} = Z_{ou}^{-1} E[V_0 V_0^*] Z_{ou}^{-1*} \]

and substituting Eq. (140) into Eq. (139) we have

(141) \[ \hat{W}_{ssu} = (Z_{ou}^{-1} E[V_0 V_0^*] Z_{ou}^{-1*} + \eta I)^{-1} b \]
If we assume as before that load impedances and self impedances are equal, then Eq. (141) becomes

\[
\hat{W}_{S\!S\!U} = \left( \left| \frac{Z_L}{Z_L + z_{ii}} \right|^2 E[V_O V_O^*] + n \right)^{-1} b
\]

If we define \( n' \) to be

\[
n' = \frac{n}{\left| \frac{Z_L}{Z_L + z_{ii}} \right|^2}
\]

then Eq. (142) becomes

\[
\hat{W}_{S\!S\!U} = \left( \frac{Z_L + z_{ii}}{Z_L} \right)^2 (E[V_O V_O^*] + n' I)^{-1} b
\]

The steering vector \( b \) can be chosen to track the desired signal.

To do this we make the following definition for the vector \( b \)

\[
b = Z_{0u}^{-1} \alpha_d
\]

or

\[
b = \left( \frac{Z_L}{Z_L + z_{ii}} \right) \alpha_d
\]

where it is recalled that \( \alpha_d \) is the direction vector for the desired signal. Then inserting Eq. (145) in Eq. (144) we have

\[
\hat{W}_{S\!S\!U} = \left( \frac{Z_L + z_{ii}}{Z_L} \right) (E[V_O V_O^*] + n' I)^{-1} \alpha_d
\]

Now utilizing Eq. (56) for the total received power for the uncoupled case and Eq. (147) for the uncoupled weight we have
\[ (148) \quad \hat{P}_{Tu} = \alpha_d^* \left( E[V_0 V_0^*] + n' I \right)^{-1} \alpha_d \]

For notational convenience let us define the complex matrix \( U \) to be

\[ (149) \quad U = E[V_0 V_0^*] + n' I \]

Then inserting Eq. (149) into Eq. (148) we have

\[ (150) \quad \hat{P}_{Tu} = \alpha_d^* U^{-1} \alpha_d \]

B. Coupled Element Case

We will now consider the coupled case where the impedance matrix \( Z_o \) is not diagonal. For this case the steady-state weights are given by

\[ (151) \quad \hat{w}_{ssc} = (Z_{oc}^{-1} E[V_o V_o^*] Z_{oc}^{-1} + n I)^{-1} b \]

which can be simplified somewhat by rearranging the terms. If we factor out the \( Z_{oc}^{-1} \) and \( Z_{oc}^{-1} \) terms from the covariance matrix, Eq. (149) can be rewritten as

\[ (152) \quad \hat{w}_{ssc} = Z_{oc}^* \left( E[V_o V_o^*] + n Z_{oc} Z_{oc}^* \right)^{-1} Z_{oc} b \]

The steering vector \( b \) can be chosen in the same manner as before to track the desired signal. Thus we let it be defined as

\[ (153) \quad b = Z_{oc}^{-1} \alpha_d \]
Then inserting Eq. (153) into Eq. (152) we have

$$(154) \quad \hat{w}_{ssc} = Z_{oc}^* (E[V_0^*V_0^*] + \eta Z_{oc} Z_{oc}^*)^{-1} \alpha_d$$

It can be noted at this point that no obvious relationship exists between the uncoupled and the coupled weights as existed in the analysis of the LMS algorithm presented previously in Eq. (65). Thus we proceed to calculate the total power out of the array for the coupled case. This quantity is given by Eq. (69) and is

$$(155) \quad P_{Tc} = \hat{w}_{ssc}^* Z_{oc}^{-1} E[V_0^*V_0^*] Z_{oc}^{-1} \hat{w}_{ssc}$$

Now inserting Eqs. (154) into Eq. (155) we have

$$(156) \quad \hat{P}_{Tc} = \alpha_d^* (E[V_0^*V_0^*] + \eta Z_{oc} Z_{oc}^*)^{-1}^* E[V_0^*V_0^*] (E[V_0^*V_0^*] + \eta Z_{oc} Z_{oc}^*)^{-1} \alpha_d$$

Comparing Eq. (148) and Eq. (156) it appears that for this particular algorithm $\hat{P}_{Tu} \neq \hat{P}_{Tc}$. It is also apparent that to calculate the performance of the adaptive system for this case, explicit knowledge of $Z_o$ is required, whereas it was not for the unmodified LMS algorithm. However, we note in Eq. (156) that if $\eta$ were chosen to be zero, Eq. (156) reduces to the unmodified LMS algorithm case.

For notational convenience let us define a complex matrix $Q$ to be

$$(157) \quad Q = E[V_0^*V_0^*] + \eta Z_{oc} Z_{oc}^*$$

then with Eq. (157), Eq. (156) can be rewritten as

$$(158) \quad \hat{P}_{Tc} = \alpha_d^* Q^{-1}^* E[V_0^*V_0^*] Q \alpha_d$$
It is instructive at this point to examine the relationship between the element voltages, the weights and the total received power as was done for the LMS algorithm. We will assume the same array configuration and notation as shown in Fig. 5. However, in this case the coupled weights will be determined by Eq. (154). The element voltages can be calculated using Eqs. (31) and (32). As before the desired signal will be incident on the array at broadside while the interfering source will be varied from 20° from endfire to 80° from endfire in 20° increments. The desired signal to thermal noise power ratio will be 10 dB at the array input, while the interference to thermal noise power ratio will be 20 dB at the same point. For this example we set \( n = .1 \) and \( Z_L = Z_{\|} \).

After calculating the steady state weights or letting the array adapt to the signal environment, a test source of unit amplitude will be placed at the same point as the interfering source, then the output of the element weights, or the \( v_C^i w_c^i \) products, can be calculated. As before, the sum of these products is the value of the relative field strength in the direction of the interfering source. The results of these product calculations are plotted as vectors for each respective element and the results are shown in Figs. 66a-d.

For comparison, the same calculations as above were repeated with the \( Z_0 \) matrix being replaced by the identity matrix. This represents the uncoupled case. The results of these calculations are shown in Figs. 67a-d.

Comparing Figs. 66a-d and 67a-d we see that, unlike before, the resultant vector sums in the two situations are not equal. We can see that the introduction of the mutual impedance terms has modified the element voltages and weights in such a way that the resultant sum for the
\[ V_T = (\bar{W}_c, \bar{V}_c) \]
\[ = \frac{3}{c} \sum_{i=1}^{\infty} \bar{v}_c^i \bar{w}_c^i \]
\[ = 0.00267 + j0 \]

**Fig. 66a**—Vectors with test source located at \( \phi_1 \) and \( \bar{w}_c^1 \) determined by adapting to \( \phi_d = 90^\circ \), \( \phi_1 = 20^\circ \), including mutual coupling effects.

\[ V_T = (\bar{W}_c, \bar{V}_c) \]
\[ = \frac{3}{c} \sum_{i=1}^{\infty} \bar{v}_c^i \bar{w}_c^i \]
\[ = 0.0071 + j0 \]

**Fig. 66b**—Vectors with test source located at \( \phi_1 \) and \( \bar{w}_c^1 \) determined by adapting to \( \phi_d = 90^\circ \), \( \phi_1 = 40^\circ \), including mutual coupling effects.
Fig. 66c—$v_{i}w_{j}$ vectors with test source located at $\phi_{1}$ and $\phi_{2}$, determined by adapting to $\phi_{d}=90^\circ$, $\phi_{1}=60^\circ$, including mutual coupling effects.

$$V_T = (W_c, V_c) = \sum_{i=1}^{3} v_{i}w_{j}$$
$$= \sum_{i=1}^{3} v_{i}w_{j}$$
$$= 0.02082 + j0$$

V.T. = (W_c, V_c)

Fig. 66d—$v_{i}w_{j}$ vectors with test source located at $\phi_{1}$ and $\phi_{2}$, determined by adapting to $\phi_{d}=90^\circ$, $\phi_{1}=80^\circ$, including mutual coupling effects.

$$V_T = (W_c, V_c) = \sum_{i=1}^{3} v_{i}w_{j}$$
$$= \sum_{i=1}^{3} v_{i}w_{j}$$
$$= 0.06722 + j0$$
Fig. 67a—$v_i^{w_i}$ vectors with test source located at $\phi_1$ and $w_i^c$ determined by adapting to $\phi_d=90^\circ$, $\phi_1=20^\circ$, neglecting mutual coupling effects.

$$V_T = (w_c, v_c)$$
$$= \frac{3}{3} \frac{v_i^{w_i}}{i=1}^{c c}$$
$$= 0.00288 + j0$$

Fig. 67b—$v_i^{w_i}$ vectors with test source located at $\phi_1$ and $w_i^c$ determined by adapting to $\phi_d=90^\circ$, $\phi_1=40^\circ$, neglecting mutual coupling effects.

$$V_T = (w_c, v_c)$$
$$= \frac{3}{3} \frac{v_i^{w_i}}{i=1}^{c c}$$
$$= 0.00512 + j0$$
Fig. 67c—$v_{i}^{c_{w_{c}}}$ vectors with test source located at $\phi_1$ and $w_{c}^{i}$ determined by adapting to $\phi_d=90^\circ$, $\phi_1=60^\circ$, neglecting mutual coupling effects.

$$V_T = (W_c, V_c)$$
$$= \sum_{i=1}^{3} v_{c_{w_{c}}}^{i-1}$$
$$= .01259+j0$$

Fig. 67d—$v_{i}^{c_{w_{c}}}$ vectors with test source located at $\phi_1$ and $w_{c}^{i}$ determined by adapting to $\phi_d=90^\circ$, $\phi_1=80^\circ$, neglecting mutual coupling effects.

$$V_T = (W_c, V_c)$$
$$= \sum_{i=1}^{3} v_{c_{w_{c}}}^{i-1}$$
$$= .05310+j0$$
case when mutual coupling is considered is not the same as that where mutual coupling effects have been neglected.

Now that it has been determined for the modified LMS algorithm that mutual impedance terms do have an effect on the steady state power received, we can gain some insight as to what the effect is if we vary the height of the elements to vary the amount of coupling while keeping the spacing fixed. The signal environment will have the desired source at broadside and one interfering source at 60° from endfire. The test source of unit amplitude will again be in the position of the interfering source. If we sum the products of the element voltages and weights and multiply the sum by its complex conjugate, this would be the received power from our test source.

As in the case for the LMS algorithm, computations were made for the coupled element voltages, the coupled weights, their products and the sum of the products for element heights of .5λ, .4λ, .3λ, and .2λ. The load impedance was fixed at $\bar{Z}_{ll}$ and η set equal to .1. The results of these calculations are shown in Table 6. Observing the total power received $V_{T}V_{*}$ from the test source located in the same position as the interfering source we see that it drops almost one order of magnitude when the element height is reduced to .4λ. What we see in this case is the effect of the relative values of $E[V_{o}\bar{V}_{o}]$ and the $\eta Z_{oc}Z_{oc}^{*}$ terms (see Eq. (156)). As the elements get shorter and $Z_{L}$ becomes larger the $E[V_{o}\bar{V}_{o}]$ matrix will become dominant and the modified LMS algorithm will approach the LMS algorithm. This can be seen by comparing Table 6 for an element height of .2λ and the corresponding case in Table 2 for the LMS algorithm.
These same calculations were again repeated for \( Z_L \) set equal to 50\,\text{ohms}. The results of these calculations are shown in Table 7. We see in this case that the coupled weights decrease much more slowly as does the test source received power. Thus we can see from these calculations that the received power from the array using the modified LMS algorithm is a function of the mutual impedance terms. In general, the magnitude of the effect can be determined by comparing the relative magnitudes of \( E[V_0V_0^*] \) and \( n Z_0Z_0^* \).

We now return to calculate the total received power from the desired signal, the interfering sources and the thermal noise. We assume that the

\[
\begin{array}{cccccc}
\text{element} & \text{height} & .5\lambda & .4\lambda & .3\lambda & .2\lambda \\
\hline
\text{height} & .5\lambda & .4\lambda & .3\lambda & .2\lambda \\
\hline
\vspace{1cm}
\end{array}
\]

\[
V_C = \begin{bmatrix}
  v_1^C \\
v_2^C \\
v_3^C \\
\end{bmatrix}
\]

\[
W_C = \begin{bmatrix}
w_1^C \\
w_2^C \\
w_3^C \\
\end{bmatrix}
\]

\[
V_{\text{total}}^T = 0.433 \times 10^{-3} \quad 0.511 \times 10^{-4} \quad 0.477 \times 10^{-4} \quad 0.475 \times 10^{-4}
\]
TABLE 7

ELEMENT VOLTAGE AND WEIGHTS VS ELEMENT HEIGHT FOR L=3,  
\(d=.25\lambda, Z_L=50.+j0, \phi_1=60^\circ, n=.1\)

<table>
<thead>
<tr>
<th>element height</th>
<th>.5(\lambda)</th>
<th>.4(\lambda)</th>
<th>.3(\lambda)</th>
<th>.2(\lambda)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(V_C)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(v_1^C)</td>
<td>.416/34.4°</td>
<td>.115/125.0°</td>
<td>.053/131.1°</td>
<td>.029/133.2°</td>
</tr>
<tr>
<td>(v_2^C)</td>
<td>.237/1.1°</td>
<td>.111/74.4°</td>
<td>.052/84.8°</td>
<td>.029/87.9°</td>
</tr>
<tr>
<td>(v_3^C)</td>
<td>.212/-53.4°</td>
<td>.116/26.9°</td>
<td>.53/39.3°</td>
<td>.029/42.7°</td>
</tr>
<tr>
<td>(W_C)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(w_1^C)</td>
<td>.782/-67.8°</td>
<td>.679/39.3°</td>
<td>.390/66.2°</td>
<td>.252/80.6°</td>
</tr>
<tr>
<td>(w_2^C)</td>
<td>.431/54.1°</td>
<td>.362/70.2°</td>
<td>.314/84.2°</td>
<td>.236/87.7°</td>
</tr>
<tr>
<td>(w_3^C)</td>
<td>1.12/26.6°</td>
<td>.695/114.9°</td>
<td>.390/104.6°</td>
<td>.251/95.3°</td>
</tr>
<tr>
<td>(V_{T,V_i^\dagger})</td>
<td>1.16 x 10^{-2}</td>
<td>0.240 x 10^{-2}</td>
<td>1.14 x 10^{-2}</td>
<td>0.246 x 10^{-3}</td>
</tr>
</tbody>
</table>

signal environment has one desired signal and M interfering signals plus the uncorrelated thermal noise in each element. With this assumption, the covariance matrix is given in Eq. (42) as

\[
E[V_0 V_0^\dagger] = D a_d a_d^\dagger + \sum_{i=1}^{M} I_i a_i a_i^\dagger + \sigma^2 I
\]

Thus utilizing Eqs. (159) and (150) and proceeding as before, we have the various received signal powers for the uncoupled case as
\[
\begin{align*}
\hat{p}_{Du} &= D \alpha_d^* U^{-1*} \alpha_d \alpha_d^* U^{-1} \alpha_d \\
\hat{p}_{I_{iu}} &= I_i \alpha_d^* U^{-1*} \alpha_i \alpha_i^* U^{-1} \alpha_d \\
\hat{p}_{nu} &= \sigma^2 \alpha_d^* U^{-1*} U^{-1} \alpha_d
\end{align*}
\] 

and similarly for the coupled case as
\[
\begin{align*}
\hat{p}_{Dc} &= D \alpha_d^* Q^{-1*} \alpha_d \alpha_d^* Q^{-1} \alpha_d \\
\hat{p}_{I_{ic}} &= I_i \alpha_d^* Q^{-1*} \alpha_i \alpha_i^* Q^{-1} \alpha_d \\
\hat{p}_{nc} &= \sigma^2 \alpha_d^* Q^{-1*} Q^{-1} \alpha_d
\end{align*}
\]

Thus with the relationships developed in Eqs. (160) and (161) we can calculate the total received signal powers and make comparisons for various signal environments and degrees of coupling. However, before doing this, it is instructive to calculate the steady state field pattern as was done for the LMS case.

C. Field Intensity of Adaptive Array
For Modified LMS Algorithm

Calculating the field pattern for the modified LMS algorithm gives further insight into the performance of adaptive systems. Since there is no analytic relationship existing between the coupled and uncoupled cases, we shall treat these cases separately.

The field pattern for the uncoupled case as defined in Eq. (80) is given by
\[
E_T(\phi) = (V_U(\phi), \hat{W}_{SSU})
\]
where \( V(\phi) \) is defined in the same way as in Eqs. (81) and (82). For
the uncoupled case \( V_u(\phi) \) can be written

\[
V_u(\phi) = \left( \frac{Z_L}{Z_L + Z_{\text{d}}^*} \right) V_0(\phi)
\]

If we define \( b \) as in Eq. (145), then combining Eqs. (145), (163), and
(144) and substituting into Eq. (162) we have the field pattern for the
uncoupled case

\[
E_t(\phi) = V_0^*(\phi) \left[ \mathcal{E}[V_0V_0^*] + \eta' \mathcal{I} \right]^{-1} \alpha_d
\]

Using one desired and one interfering signal for the signal environ­
ment, Eq. (164) can be expanded to be

\[
E_t(\phi) = V_0^*(\phi) \left[ D(\alpha_d \alpha_d^*) + I_1 \alpha_1 \alpha_1^* + (\sigma^2 + \eta') \mathcal{I} \right]^{-1} \alpha_d
\]

We note that the right side of Eq. (165) has the same form as
Eq. (89) and consequently can be expanded in the same way as was done for
the LMS algorithm. To do this, we utilize Eq. (102) and replace all \( \sigma^2 \)
with \( \sigma^2 + \eta' \). After this is done, then Eq. (165) becomes

\[
E_t(\phi) = \frac{1}{\Delta} \left[ (I_1 |\alpha_1|^2 + \sigma^2 + \eta')(V_0(\phi), \alpha_d)
- I_1(\alpha_d, \alpha_1)(V_0(\phi), \alpha_1) \right]
\]

where \( \Delta \) is now given by

\[
\Delta = (D |\alpha_d|^2 + \sigma^2 + \eta')(I_1 |\alpha_1|^2 + \sigma^2 + \eta')
- D I_1(\alpha_d, \alpha_1)(\alpha_1, \alpha_d)
\]
and the same arguments that were made earlier about the sign of \( \Delta \) can also be made here.

The action of the adaptive array for this particular algorithm can be analyzed for any array configuration by looking at the rather simple expression for the field pattern of Eq. (166). It also can be generalized as before to handle multiple interfering sources. As in the previous case, the total pattern is made up of two beams, one for the desired signal and one for the undesired signal. The magnitudes of these beams and ultimately the null depths and desired signal magnitudes are controlled by the signal environment.

We will now compute the field pattern for the coupled case. Starting again with the definition for the field pattern we have,

\[
E_T(\phi) = (V_C(\phi), \hat{w}_{SSC})
\]

where

\[
V_C(\phi) = Z^{-1} V_0(\phi)
\]

The steady-state weight for the coupled case is given in Eq. (151). We will again use the same definition for the steering vector \( b \). With these definitions, the field pattern for the coupled case can be written as

\[
E_T(\phi) = V^*_0(\phi) Z^{-1} [Z^{-1}_0 E[V_0^* V_0^*] Z^{-1} + n I]^{-1} Z^{-1}_0 \alpha_d .
\]

If we again consider the same signal environment Eq. (169) can be rewritten as
\[ E_T(\phi) = V^*_0(\phi) Z^{-1*}_{oc} \left[ Z^{-1}_{oc} (D \, \alpha_d \alpha_d^* + I_1 \, \alpha_1 \alpha_1^*) \right] Z^{-1*}_{oc} \]
\[ . + \sigma^2 \left( \frac{Z^{-1}_{oc} Z^{-1*}_{oc}}{Z^{-1}_{oc} Z^{-1*}_{oc} + n \, I} \right) Z^{-1}_{oc} \alpha_d \]

For notational convenience, let us make the following definitions:

\[ \begin{align*}
Z^{-1}_{oc} \alpha_d &= \alpha_d' \\
Z^{-1}_{oc} \alpha_1 &= \alpha_1' \\
\alpha_d^* Z^{-1}_{oc} &= \alpha_d^{*'} \\
\alpha_1^* Z^{-1*}_{oc} &= \alpha_1^{*'} \\
V^*_0(\phi) Z^{-1*}_{oc} &= V^*_0'(\phi)
\end{align*} \]

and define the complex matrix \( A \) to be

\[ A = (\sigma^2 \frac{Z^{-1}_{oc} Z^{-1*}_{oc}}{Z^{-1}_{oc} Z^{-1*}_{oc} + n \, I}) \]

With these definitions, then Eq. (170) can be rewritten as

\[ E_T(\phi) = V^*_0(\phi) \left[ D \, \alpha_d \alpha_d^* + I_1 \, \alpha_1 \alpha_1^* + A \right]^{-1} \alpha_d \]

We now have the field pattern in the same form as Eq. (87) except the column vectors representing the signals have all been transformed by the complex matrix \( Z_0^{-1} \) and the additive matrix that was \( (\sigma^2 + n) \) times the identity matrix is now the complex matrix \( A \). To get Eq. (173) into a clearer form, consider an expansion for the last two factors in Eq. (173) of the form
Observing the form of Eq. (180) we have the two beams that would result from an array with linear spacing, but there are additional pattern terms involving the matrix $Z_0$. Because of the complex nature of these terms, it will help to examine some test cases.

D. Calculated Power Patterns for Modified LMS Algorithm

As in the previous calculations for the LMS algorithm, a two-element array with $0.5\lambda$ spacing is used to illustrate the uncoupled case and a six-element array with $1\lambda$ spacing is used as the coupled case. For all cases shown here the desired signal is at broadside.

Several test calculations were made to determine the value of $n$. As the magnitude was increased the interference rejection decreased, as expected. Thus only a single value of $n$ was used for the calculations presented, that being the inverse squared magnitude of the self impedance of one of the dipole elements or

$$n = \left| \frac{1}{Z_{ii}} \right|^2$$

This appears to give a value close to the range suggested by Zahm[8].
The first test environment has one interfering source at 40° from endfire with $p_{I_1}/\sigma^2 = 20$ dB. The results of these calculations from the two arrays are shown in Figs. 68 and 69. As can be seen from these plots, the performance of both arrays is very similar. Both arrays have the null a few degrees from the actual direction of the interfering source. This is the effect of the bias signal, $n$. Figures 70 and 71 show power patterns for the two- and six-element arrays, respectively, with interfering sources at 60° and 20° from endfire. For this particular environment, both arrays place only one null in the pattern about midway between the two interfering sources. Recall that the LMS algorithm placed two pattern nulls for the six-element array for the two-interfering signal case.

However, as one might expect, increasing the interference signal power improves the nulling capability. The pattern for the six element array places two nulls in the pattern for a 10 dB increase in interference signal as shown in Fig. 72. As can be seen the nulls do not yet appear over the interfering sources, however this would happen if the interference power were increased even further.

Three interfering sources were put in the signal environment at 75°, 50°, and 25° from endfire with $p_{I_1,2,3}/\sigma^2 = 20$ dB and power patterns for the two test arrays were calculated. These results are shown in Figs. 73 and 74. As can be seen from these two plots again only one pattern null was formed by the six-element array. However, if the interference power is increased to 50 dB, then the six-element array does form three nulls, but at some expense to the desired signal power. This plot is shown in Fig. 75.
Fig. 68—Power pattern for $L=2$, $d=.5\lambda$, $p_{I_1}/\sigma^2 = 20$ dB, $p_D/\sigma^2 = 10$ dB, $n = (|1/z_{11}|^2)$.

Fig. 69—Power pattern for $L=6$, $d=.1\lambda$, $p_{I_1}/\sigma^2 = 20$ dB, $p_D/\sigma^2 = 10$ dB, $n = (|1/z_{11}|^2)$. 
Fig. 70--Power pattern for $L=2$, $d=0.5\lambda$, $p_{I_1}/\sigma^2 = p_{I_2}/\sigma^2 = 20$ dB, $p_d/\sigma^2 = 10$ dB, $n = (|1/z_{11}|^2)$.

Fig. 71--Power pattern for $L=6$, $d=1\lambda$, $p_{I_1}/\sigma^2 = p_{I_2}/\sigma^2 = 20$ dB, $p_d/\sigma^2 = 10$ dB, $n = (|1/z_{11}|^2)$. 

Fig. 72--Power pattern for $L=6$, $d=1\lambda$, $p_{I_1}/\sigma^2 = p_{I_2}/\sigma^2 = 30 \text{ dB}$, $p_d/\sigma^2 = 10 \text{ dB}$, $n = |1/\zeta_1|^2$.

Fig. 73--Power pattern for $L=2$, $d=0.5\lambda$, $p_{I_1}/\sigma^2 = p_{I_2}/\sigma^2 = p_{I_3}/\sigma^2 = 20 \text{ dB}$, $p_d/\sigma^2 = 10 \text{ dB}$, $n = |1/\zeta_1|^2$. 
Fig. 74—Power pattern for L=6, d=.1λ, \( \frac{p_{I_1}}{\sigma^2} = \frac{p_{I_2}}{\sigma^2} = \frac{p_{I_3}}{\sigma^2} = 20 \text{ dB}, \frac{p_D}{\sigma^2} = 10 \text{ dB}, n = |1/z_1|^2 \).

Fig. 75—Power pattern for L=6, d=.1λ, \( \frac{p_{I_1}}{\sigma^2} = \frac{p_{I_2}}{\sigma^2} = \frac{p_{I_3}}{\sigma^2} = 50 \text{ dB}, \frac{p_D}{\sigma^2} = 10 \text{ dB}, n = |1/z_1|^2 \).
Thus it appears that as was the case for the LMS algorithm, increasing the number of elements in a fixed aperture gives additional interference rejection for multiple interfering sources.

E. Array Power Output Calculations

The power patterns shown in the previous section give an indication of the performance of the adaptive array. The depth of the pattern nulls can be compared to make relative evaluations of the rejection of interfering signals and the amount of reduction of the desired signal strength. However, as pointed out earlier, a more exact comparison of performance of different arrays and environments can be made by calculating the received signal powers due to the various signals in the environment. To do this, Eqs. (160) and (161) will be utilized.

By utilizing these equations more exact comparisons between the different array geometries can be made. As before these equations relate to the power patterns presented in the previous section in that the difference in the input and output signal powers represent the absolute gain of the array in that direction.

For comparison purposes, calculations were made for three different apertures \(0.5\lambda\) in extent; one with two elements, one with three elements and the third with six elements. These three arrays have element spacings of \(0.5\lambda\), \(0.25\lambda\) and \(0.1\lambda\), respectively. The signal environment has the desired source at broadside while the direction of arrival of the interfering source is varied from endfire to broadside. The input power of the desired signal is held fixed at \(P_D/\sigma^2 = 10\) dB while the interfering signal power is varied in 10 dB increments from \(P_I/\sigma^2 = 10\) dB to 50 dB.
As before \( n \) was chosen equal to \( |1/z_1|^2 \). The resulting output signal powers for these three arrays are shown in Figs. 76-78. Comparing the family of curves for these three half-wave apertures, we see that the performance of the six-element array is degraded approximately 8 dB when compared to the three-element array for \( \phi_1 \) near endfire.

When compared to the corresponding plot for the LMS algorithm; it appears there is approximately a 10 dB degradation in performance with the modified LMS algorithm. However, it does appear that, as with the LMS algorithm, aperture size is a limitation in this case also.

![Graph showing signal power vs phi_1](image-url)
Fig. 77—$\hat{P}_D/\sigma^2$ and $\hat{P}_{I_1}/\sigma^2$ vs $\phi_1$ for $L=3$, $d=.25\lambda$, $P_D/\sigma^2 = 10$ dB.

Fig. 78—$\hat{P}_D/\sigma^2$ and $\hat{P}_{I_1}/\sigma^2$ vs $\phi_1$ for $L=6$, $d=.1\lambda$, $P_D/\sigma^2 = 10$ dB.
CHAPTER VIII
TRANSIENT ANALYSIS OF ADAPTIVE ARRAY
WITH MODIFIED LMS ALGORITHM

In this section we will investigate the transient performance of the adaptive antenna system with the modified LMS algorithm for both coupled and uncoupled arrays.

A. Transient Solution For Coupled And Uncoupled Cases

Recalling the total solution for the weight equation given in Eq. (138) we have

\begin{equation}
\hat{W} = e^{-k_1(R_v+nI)t} W_0 + (R_v+nI)^{-1}
\end{equation}

\begin{equation}
\cdot \left( I - e^{-k_1(R_v+nI)t} \right) b
\end{equation}

Proceeding as before for the LMS algorithm, we define the eigenvectors and eigenvalues of the matrix \((\hat{R}_v+nI)\) by

\begin{equation}
(\hat{R}_v + nI) y_i = \lambda_i y_i
\end{equation}

and then the modified correlation matrix can be written as

\begin{equation}
(\hat{R}_v + nI) = \sum_{i=1}^{m} \lambda_i E_i
\end{equation}

where as before
(187) \[ E_i = \frac{y_i y_i^*}{(y_i, y_i)} \]

and \( m \) is the number of distinct eigenvalues. In addition, it can be shown that

(188) \[ e^{-k_1 (R_v + n I)t} = \sum_{i=1}^{m} e^{-k_1 \lambda_i t} E_i \]

and

(189) \[ (R_v + n I)^{-1} = \sum_{i=1}^{m} \frac{1}{\lambda_i} E_i \]

Using these results and those in Eq. (118), the total solution for the weights can be written as

(190) \[ \hat{W} = \sum_{i=1}^{m} e^{-k_1 \lambda_i t} E_i W_0 + \sum_{i=1}^{m} \frac{E_i}{\lambda_i} \left( 1 - e^{-k_1 \lambda_i t} \right) b \]

The initial conditions again will be chosen as defined in Eqs. (122), (123), (124), and (125). Consequently, for this algorithm we have

(191) \[ W_0 = Z_0^* \sum_{i=1}^{m} \frac{(y_i, W_i')}{{||y_i||}^2} \]

where

\[
W'_0 = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

Thus Eq. (190) can be rewritten as
We again compare the coupled and uncoupled cases. For the uncoupled case

\[ (192) \quad \hat{w} = \sum_{i=1}^{m} e^{-k_1 \lambda_i t} \frac{y_i y_i^*}{||y_i||^2} z_0^* \sum_{j=1}^{m} \frac{(y_j, w_j)}{||y_j||^2} y_j + \sum_{i=1}^{m} \frac{E_i}{\lambda_i} \left(1 - e^{-k_1 \lambda_i t}\right) b. \]

We again compare the coupled and uncoupled cases. For the uncoupled case

\[ (193) \quad (\hat{R}_{vu} + \eta I) = Z^{-1}_{ou} E[V_o V_o^*] Z^{-1*}_{ou} + \eta I \]

and we will choose \( b \) to track the desired signal and define it as before to be

\[ (194) \quad b = \sqrt{R} D Z^{-1}_{ou} \alpha_d \]

which is similar to Eqs. (129) and (132).

Similarly, for the coupled case

\[ (195) \quad (\hat{R}_{vc} + \eta I) = Z^{-1}_{oc} E[V_o V_o^*] Z^{-1*}_{oc} + \eta I \]

and

\[ b = \sqrt{R} D Z^{-1}_{oc} \alpha_d \]

B. **Calculated Transient Adaptive Array Performance**

To calculate the transient response of the modified LMS algorithm we must solve the eigenvalue problems for the coupled and uncoupled cases. More specifically we must solve for the eigenvalues and eigenvectors of
Comparing Eqs. (196) and (197) with Eqs. (133) and (134), the additional terms $\eta Z_{ou} Z_{ou}^*$ and $\eta Z_{oc} Z_{oc}^*$ act as additional signals in the environment. However, in terms of the eigenvalues there appears to be somewhat of a compensating effect in that as the elements of $Z_0$ are made smaller those of $Z_0^{-1}$ get larger causing larger eigenvalues. Also the reverse is true for the larger elements of $Z_0$ causing a longer transient time. This same characteristic was noted with the LMS algorithm.

To illustrate these effects, the same calculations that were made for the LMS case were again made here. That is, the transient response of the S/N ratio for a three element array with a $0.25\lambda$ element spacing was calculated for element heights of $0.2\lambda$, $0.3\lambda$, $0.4\lambda$ and $0.5\lambda$. This was done for $Z_L = \overline{Z_{11}}$ and $Z_L = 50.\pm j0$. For the purpose of comparison, we let $\eta = 0.1$.

The results of these calculations for the respective load conditions are shown in Figs. 79 and 80, respectively. As would be expected, the same general trend as noted for the LMS case is noted here, that of the quicker transient response for the shorter elements with $Z_L = \overline{Z_{11}}$ and the reverse for the $Z_L = 50.\pm j0$ case. The peaks in the $Z_L = 50.\pm j0$ case for the shorter elements is caused by the small eigenvalues and the relative values of the terms in Eq. (190). That is, when $T$ and $\lambda_i$ are small, the first term in Eq. (190) dominates. Then as $T$ becomes larger, the $-k_1 \lambda_i t$ terms become effective and then as they become small the steady state is reached.
It is also interesting to note the differences in steady state values for the two cases. In the $Z_L = \bar{Z}_{11}$ case, as the elements get shorter $Z_L$ increases and the term $\eta Z_0 Z^*_0$ decreases in comparison to $E[V_0 V^*_0]$ and the algorithm looks like the LMS algorithm. However, when $Z_L = 50 + j0$, as the elements get shorter the term $\eta Z_0 Z^*_0$ tends to get larger and in effect increases the thermal noise and the steady state $S/N$ drops off.
Also of interest is the response of the array of fixed aperture size but differing number of elements. To investigate this case, the .5λ aperture having two, three and six elements with element spacings of .5λ, .25λ and .1λ was again chosen. Two signal environments were used, one with two interfering sources of 75° and 50° from endfire, the other with a third signal at 25° from endfire. The results of these calculations are shown in Figs. 81 and 82. The load impedance $Z_L$ was chosen to be $Z_L$ while we let $\eta = 1$. 

Fig. 80--$S/N$ vs $T$ for $L=3$, $d=.25\lambda$, $Z_L=50+j0$, $\phi_1=60^\circ$, $p_{I_1}/\sigma^2 = 20$ dB, $\eta = 1$. 

$h = 0.5\lambda$

$= 0.4$

$= 0.3$

$= 0.2$

$T = k_1 t$ (sec)
Fig. 81--S/N vs T for $h=.5\lambda$, $Z_L=Z_{11}$, $\phi_1=75^\circ$, $\phi_2=50^\circ$,

$$p_{I_1}/\sigma^2 = p_{I_2}/\sigma^2 = 20 \, \text{dB}, \quad n=.1.$$  

In these two environments, small difference is seen in the transient times of the three arrays. The three and six element arrays have slightly longer transient times than the two element array.

Also of interest is the $P_D/\sum_{i=1}^{M} P_{I_i}$ ratio. It has been shown that for sufficiently large interference powers, the three element array of .25\lambda element spacing will form two pattern nulls in the direction of the interfering sources. Consequently it is of value to make comparisons of the desired to interference power ratio transient response for different interference power levels. This was done for the three-
Fig. 82--S/N vs T for h=.5\lambda, Z_L=z_{11}, \phi_1=75^\circ, \phi_2=50^\circ, \phi_3=25^\circ, \frac{p_{I_1}}{\sigma^2} = \frac{p_{I_2}}{\sigma^2} = \frac{p_{I_3}}{\sigma^2} = 20 \text{ dB}, n=.1.

Element array with two interfering sources at 75^\circ and 50^\circ from endfire and \frac{p_{I_1}}{\sigma^2} = 20 \text{ dB}, 30 \text{ dB} and 40 \text{ dB}. The results of these calculations are shown in Fig. 83. For this case, the total transient times are very similar, but as the interference power is increased the major portion of the transient is over more quickly. We can also note the improvement in \hat{p}_D/\sum_{i=1}^{2} \hat{p}_{I_i} as the interference power is increased showing the advantage of the third element in the .5\lambda aperture.
Fig. 83 -- $\frac{1}{2} \sum_{i=1}^{2} \hat{P}_{I_i}$ vs $T$ for $L=3$, $d=0.25\lambda$, $h=0.5\lambda$, $Z_L=\bar{z}_{11}$, $\phi_1=75^\circ$, $\phi_2=50^\circ$, $P_D/\sigma^2 = 10$ dB.
CHAPTER IX
SUMMARY AND CONCLUSIONS

This study set out to analyze adaptive antenna arrays that are aperture limited. To complete the analysis, the effects of mutual coupling and aperture size had to be evaluated.

To begin with, an L-element antenna array was placed in the far field of a source antenna and the system modeled as an L+1 terminal network. Using this model, a generalized input voltage to the adaptive processor was derived that would take into account any coupling effects between antenna elements. This relationship is given by

\[ V = Z_0^{-1} V_0 \]

where \( V \) is the column vector of element voltages that are the input to the adaptive processor. The matrix \( Z_0^{-1} \) is essentially the inverse of the impedance matrix of the L-terminal network array model normalized to the element load impedances. The diagonal elements have the load impedance \( Z_L \) added to them before normalization. \( V_0 \) is the column vector containing the open circuit voltages of each of the antenna elements. These voltages, contained in the \( V_0 \) vector, are those obtained from the plane wave response of the array.

The steady-state solution for the Least Mean Square (LMS) Adaptive Algorithm was analyzed first. Using Eq. (198) as the generalized input
element voltages to the adaptive processor, the total power received by
the array was found to be

\begin{equation}
\hat{P}_T = \hat{r}_V^* \mathbb{E}[V_O^*V_O^{-1}] \hat{r}_V
\end{equation}

This result indicates that for the choice of weights the LMS algorithm
provides, the array power output is not a function of the matrix \( Z_0 \) con-
taining the mutual impedance terms. However, it was shown that the LMS
weights and element voltages are functions of the \( Z_0 \) matrix. In illustrat-
ing this point, the element voltages, weights and the associated
products of the element voltages and weights were compared in an array
first with mutual coupling taken into account and then with mutual coupl-
ing neglected. It was shown graphically how the mutual impedance terms
change the element voltages and weights. It was further shown that
even though the weights and voltages have different magnitudes and
phase angles for the case where mutual coupling is considered and where
mutual coupling is neglected, the total vector sum at the array output
is the same in both cases. Thus it is concluded that mutual coupling
between elements of an adaptive array when using the LMS algorithm does
not affect the output of the array, providing spacing is fixed.

The effect of the load impedance on the element voltages and
weights was also investigated and it was found that from a practical
point of view, the output of the LMS array is independent of load imped-
ance.

To analyze the limited aperture adaptive arrays more thoroughly,
the steady-state field pattern for the LMS adaptive system was calcu-
lated. It was shown that the field pattern of the steady-state adaptive
array can be expressed as a sum of terms of the form $\frac{\sin Lz}{\sin z}$. There is one term associated with each signal in the environment, with the maximum value of the $\frac{\sin Lz}{\sin z}$ occurring at the angle of arrival of that signal. For the case of one desired signal and one interfering signal, there is one beam associated with the desired signal with the beam maximum in the direction of arrival of the desired signal and the second beam with a beam maximum (as an out-of-phase beam) in the direction of the interfering signal.

Thus, the adaptive array output can be represented as a sum of patterns that would result from arrays of constant amplitude and linear phase taper. The constants multiplying the pattern terms are functions of the signal levels in the environment and the thermal noise associated with each element. With this type of an expansion for the field pattern one can see why for some cases the pattern null is not positioned precisely in line with the direction of arrival of the interfering signal. By looking at the shape of the $\frac{\sin Lz}{\sin z}$ function for various spacings and numbers of elements, one can better understand the basic array resolution problem. For example, as the spacing increases for a given number of elements the main lobe in the function gets narrower and the $\frac{\sin Lz}{\sin z}$ function illustrates this quantitatively.

Several plots of the magnitude of the fields associated with a desired signal and one or two interfering signals were shown to illustrate how the pattern nulls are formed.

To compare the performance of a tightly coupled array to a loosely coupled array with one interfering signal in the environment the power pattern of two $0.5\lambda$ apertures were compared. One aperture had two
elements, the second had six elements at \( 0.1\lambda \) spacing. With one interfering signal in the environment both arrays put pattern nulls in the direction of the interfering source.

To relate the pattern nulls and maxima in a more specific manner, expressions for the power out of the array due to the desired signal and one interfering signal were developed. Some comparison calculations were made for several arrays. The test environment had a desired signal at broadside at a constant power level while the power from the single interfering signal was varied over a 40 dB range. The angle of arrival of the interfering source was varied from endfire to broadside. Upon analyzing these results, it is concluded that if the array length is constant, the performance of the adaptive array for the LMS algorithm is essentially independent of number of elements for a single interfering signal. That is, the null depths of a six-element array spaced \( 0.1\lambda \) apart giving an array length of \( 0.5\lambda \) are very similar to those of a two-element array spaced \( 0.5\lambda \) apart. This result holds true for a single interfering signal. Thus, it is concluded that the adaptive antenna system is not inherently limited by close element-to-element spacing but rather by aperture size.

To compare the performance of the six-element array to the two element array with \( 0.5\lambda \) apertures when there was more than one interfering source in the environment, several power patterns were calculated for the two arrays when multiple interfering sources were put in the environment. It was found that the six-element array would place multiple nulls over the interfering sources if the interference power was large enough. On the other hand, the two-element array placed one null
in some average position between the interfering signals. Thus it was concluded that the additional degrees of freedom provided in an array by adding more elements to a fixed aperture will outperform an array with insufficient degrees of freedom despite the increased coupling with the more closely spaced elements.

To more precisely analyze the relationship between the number of elements in a fixed aperture and the number of signals in the environment, data were presented for S/N and \( \frac{\sum_{i=1}^{M} \hat{P}_i}{\hat{P}_0} \) vs number of elements in a fixed aperture utilizing several signal environments. From these data it was concluded that an L-element array will form up to L-1 nulls regardless of element spacing (for spacings \( .5\lambda \) or less) providing the interfering signals are strong enough.

Since (S/N) ratio is an extremely important parameter in the design of any communication link that is thermal noise limited, some data were presented of a tradeoff nature in terms of steady-state (S/N) ratio for different array configurations working against five basic signal environments. These signal environments were chosen to be representative of typical environments that adaptive arrays might work against. To illustrate how these data can be used, consider the following example. Suppose it is desired to design a communication receiving antenna with a 10 dB output (S/N) ratio for a given signal level at the antenna. Assume also that the antenna aperture is limited in extent to \( .5\lambda \) to \( 1.0\lambda \). As possible solutions consider three four-element arrays, with uniform spacing, having apertures of \( 0.5\lambda \), \( 0.75\lambda \) and \( 1.0\lambda \). Now using Table 3 and Figs. 41, 43, 45 and 46, Table 8 is constructed as shown below. Arrays one, two, and three elements are spaced such that they have apertures of
TABLE 8
S/N MARGIN VS SIGNAL ENVIRONMENT FOR FOUR ELEMENT, .5λ, .75λ and 1.0λ, APERTURE SIZES

<table>
<thead>
<tr>
<th>Signal Environment</th>
<th>Array 1</th>
<th></th>
<th>Array 2</th>
<th></th>
<th>Array 3</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L=4</td>
<td>d =.16λ</td>
<td>L=4</td>
<td>d =.25λ</td>
<td>L=4</td>
<td>d =.33λ</td>
</tr>
<tr>
<td>1. Desired signal only</td>
<td>+6</td>
<td></td>
<td>+6</td>
<td></td>
<td>+6</td>
<td></td>
</tr>
<tr>
<td>2. Desired signal +</td>
<td>+.6</td>
<td></td>
<td>+3.5</td>
<td></td>
<td>+5.2</td>
<td></td>
</tr>
<tr>
<td>interfering signal at 60°</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Desired signal +</td>
<td>-4.5</td>
<td>-1.2</td>
<td></td>
<td></td>
<td>+1</td>
<td></td>
</tr>
<tr>
<td>interfering signal at 75°</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Desired signal +</td>
<td>-6</td>
<td>-1.5</td>
<td></td>
<td></td>
<td>+1.7</td>
<td></td>
</tr>
<tr>
<td>interfering signals at 60°</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>and 40°</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Desired signal +</td>
<td>-6.5</td>
<td>-2.5</td>
<td></td>
<td></td>
<td>- .5</td>
<td></td>
</tr>
<tr>
<td>interfering signals at 60°</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>, 40° and 20°</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
0.5λ, 0.75λ and 1.0λ, respectively. The (S/N) margin is the excess above 10 dB (S/N) ratio as found on the appropriate figures. If all conditions are weighted equally (most likely they would not be), the 0.5λ aperture falls considerably short of meeting requirements while the 0.75λ aperture is marginal.

If we next consider three more arrays of the same aperture as before but with six elements, a broader picture of the tradeoff can be obtained. These data are presented in Table 9. The extra two elements improve the performance one to two dB in all cases, making the 0.75λ aperture somewhat more acceptable in performance and significantly better than the 0.5λ aperture.

Obviously, the signal environments chosen in this study are not meant to totally exhaust all possible conditions but they do represent a useful class of signal environments against which comparisons can be made.

From the examples presented above it appears that apertures of .75 or larger generally are required for satisfactory rejection of undesired signals. Obviously, the larger the aperture can be made, at least to 2.5λ, the better the performance. This is most true in cases where the interfering signals are very close spatially to the desired signal. It appears that this is the most stringent requirement on aperture size. Small improvements on the order of several dB can be realized by increasing the number of elements such as going from four to six elements with array length fixed. Apertures of 0.5λ and less could be used where the interference environment is not so severe or where degraded operations can be tolerated during periods of interference reception. Thus the
### TABLE 9
S/N MARGIN VS SIGNAL ENVIRONMENT FOR SIX ELEMENT .5λ, .75λ and 1.0λ APERTURE SIZES

<table>
<thead>
<tr>
<th>Signal Environment</th>
<th>Array 4 L=6 d =.1λ (S/N) margin (dB)</th>
<th>Array 5 L=6 d =.15λ (S/N) margin (dB)</th>
<th>Array 6 L=6 d =.2λ (S/N) margin (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Desired signal only</td>
<td>+7.8</td>
<td>+7.8</td>
<td>+7.8</td>
</tr>
<tr>
<td>Desired signal + interfering signal at 60°</td>
<td>+1.6</td>
<td>+4.9</td>
<td>+6.7</td>
</tr>
<tr>
<td>Desired signal + interfering signal at 75°</td>
<td>-3.4</td>
<td>0</td>
<td>+2.0</td>
</tr>
<tr>
<td>Desired signal + interfering signals at 60° and 40°</td>
<td>-5.6</td>
<td>- .6</td>
<td>+3.0</td>
</tr>
<tr>
<td>Desired signal + interfering signals at 60°, 40° and 20°</td>
<td>-5.6</td>
<td>-2.0</td>
<td>+0.5</td>
</tr>
</tbody>
</table>
smaller aperture arrays do provide a measure of protection against interfering signals, but if the interference environment is severe the (S/N) ratio will be degraded for the smaller aperture over that provided by a larger aperture array.

Since it was found that the adaptive array utilizing the LMS algorithm and in an environment of multiple interfering sources will produce multiple pattern nulls for closely spaced elements, its relationship to a superdirective array was investigated. The directivity of an array after adapting to interfering sources spaced uniformly around the array was calculated. It was found for element spacing of about \(0.15\lambda\) that the directivity of the array does improve and approach maximum directivity as the number of interfering sources and power level increase. This was expected since the weights that give maximum directivity also give maximum S/N ratio if the dominant noise source is background noise such as that encountered in an H.F. receiving system. It was pointed out that this may have application in the superdirective arrays where the accuracy of weight adjustment is a problem.

In addition to the steady state solution to the adaptive array, the transient solution was also investigated.

To find the effect of coupling on the transient response, a three element array with \(0.25\lambda\) element spacing was chosen. The S/N ratio transient response was then calculated for various element heights and load impedance conditions. It was found that as the element height decreased the transient time became shorter when \(Z_L = \overline{Z}_{11}\), the opposite held true when \(Z_L = 50.+j0\).
For comparison purposes, a modified LMS algorithm was investigated in the same way as the LMS algorithm. This algorithm has the advantage of having no reference loop in the system. The reference loop of the LMS algorithm is replaced by a generalized beam steering signal which can be used to track the desired signal. This modified algorithm also has the advantage of not having the weights shut down if the desired signal is not acquired.

For this particular study it was assumed a signal was available to exactly simulate the desired signal. With these conditions it was shown that the total average steady-state power out of the array is given by

\[
\hat{P}_T = (Z_0 b)[E(V_o V_o^*) + n Z_0 Z_o^*]^{-1} E(V_o V_o^*)
\]

and as can be seen for this particular algorithm the power out of the array is not independent of \(Z_0\). Consequently, a knowledge of the mutual impedance terms is necessary in order to compute the array performance when mutual coupling is significant.

As was done with the LMS algorithm, the element voltages and weights and these products were investigated for the case when mutual coupling is taken into account and when it is neglected. It was found that the array outputs are not equal in both cases. It was also pointed out that if the \(nZ_0 Z_o^*\) term in Eq. (200) is made small by making \(Z_L\) large, the modified LMS algorithm approaches the LMS results.

The steady-state power pattern also was calculated for the modified algorithm. The same general form as for the LMS algorithm was developed but could not be represented by \(\frac{\sin Lz}{\sin z}\) terms for the coupled case.
However, for the uncoupled case the field pattern can be reduced to a sum of \( \frac{\sin Lz}{\sin z} \) terms as with LMS algorithm. The only difference is in the coefficients of the \( \frac{\sin Lz}{\sin z} \) terms.

The power patterns for two .5\( \lambda \) apertures, one with two elements and one with six elements, were calculated for different signal environments. It was found as before that the added degrees of freedom with the additional elements in the fixed aperture gives improved performance in multiple signal environments. In general the requirement is for one more element than interfering sources.

The power out of the array with the modified algorithm due to the desired signal and due to the one interfering signal plus thermal noise was calculated for the same conditions as for the LMS algorithm case. In general, the same conclusions drawn previously for the LMS algorithm again hold. However the performance of the more tightly coupled array was somewhat degraded from that for the loosely coupled array.

The effects of element coupling on the transient solution was also investigated with the results being similar to those for the LMS algorithm.

In summary, the major contributions and conclusions of this study are the following:

1. The steady state and transient performance of an adaptive processor working with closely spaced arrays has been analyzed and some test data presented.

2. The steady-state field pattern for the adaptive antenna array can be expressed as a sum of \( \frac{\sin Lz}{\sin z} \) terms whose coefficients are related to thermal noise and the corresponding magnitudes of the desired and undesired signals.
This expression sheds some light on the basic aperture and signal resolution characteristics.

(3) Adaptive processors working with antenna arrays of equal aperture size will have similar performance for an environment containing a single desired signal and a single interfering source, independent of number of elements.

(4) Adaptive processors can be used for (S/N) ratio protection with apertures as small as $0.5\lambda$ with some degradation. Apertures of $0.75\lambda$ to $1.0\lambda$ and larger give significantly better performance, however.

(5) In systems that are not thermal noise limited, significant improvement in $\hat{P}_D/\sum_{i=1}^{m} \hat{P}_{I_i}$ can be made by adding elements to a fixed aperture when multiple interfering sources are present. The minimum number of elements equal to the number of signals in the environment.

(6) The adaptive array does appear to have application in superdirective array designs.

(7) Unlike the steady state response, the transient response of adaptive systems can be modified significantly by changing the element coupling and load impedance.
REFERENCES


APPENDIX A
L+1 TERMINAL NETWORK FORMULATION

This appendix describes in more detail the formulation of the L element antenna array and an outside stimulus as an L+1 terminal network.

The mutual impedance terms, $z_{ij}$, $i \neq j$ in Eq. (2) are defined to be the ratio of the induced voltage in the $i$th port, $V_{io}$, to the current in the $j$th port, $i_j$, with all ports except the $j$th open circuited. This implies that

$$z_{ij} = \frac{V_{oi}}{i_j} \quad i \neq j$$

$$i,j = 1, \cdots L$$

Thus the $z_{ij}$ are only a function of the element spacing and element type. However, the mutual impedance terms, $z_{is}$, representing the coupling between the source and the array are a function of the geometrical relationship between the array and the source. Notationally we have the same form as in Eq. (A-1)

$$z_{is} = \frac{V_{oi}}{i_s} \quad i = 1, \cdots L$$

where $i_s$ is the current in the source loop and $V_{oi}$ is as above.
Refering back to Fig. 1 we can get an expression for the source current \( i_s \), by writing a loop equation for the source loop. Doing this we have

\[
(A-3) \quad i_s = \frac{V_g - V_s}{Z_g}
\]

where \( V_g, V_s \) and \( Z_g \) are all quantities associated with the source generator. Then combining Eqs. (A-2) and (A-3) we have

\[
(A-4) \quad V_{oi} = \frac{V_g - V_s}{Z_g} z_{is} \quad i = 1, \ldots, L.
\]

Thus the \( z_{is} \) terms are directly proportional to the open circuit element patterns. We could further generalize this formulation and define \( V_{io} \) to be the product of the element pattern and the response of an isotropic element placed at the \( i \)th location. This modification would allow for analysis of arrays using different element patterns. However, for this study all element patterns were assumed to be identical.

The self impedance terms \( z_{ii} \) are defined as usual to be the ratio of the \( i \)th terminal voltage, \( V_i \), to the \( i \)th terminal current, \( i_i \), with all other terminals open circuited. Thus we have

\[
(A-5) \quad z_{ii} = \frac{V_i}{i_i} \quad i = 1, \ldots, L.
\]

With these definitions and the relationship between the terminal current and load impedance \( Z_L \) in Eq. (A-4), the loaded terminal voltages at each of the elements in the array can be expressed in terms of the self and mutual impedance terms and the open circuit element voltages. This is shown in Eq. (A-5).