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ELECTROMAGNETIC SCATTERING FROM NON-SPHERICAL HYDROMETEORS WITH APPLICATIONS IN RADAR METEOROLOGY

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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*****

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ELECTROMAGNETIC SCATTERING FROM NON-SPHERICAL HYDROMETEORS

WITH APPLICATIONS IN RADAR METEOROLOGY

By

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ABSTRACT

The theoretical problem of electromagnetic scattering from perfect conductors or dielectrics imbedded in a dielectric body is analyzed using an extended integral equation method. This technique is exact and provides a matrix solution to the scattering problem. The scattering properties of the body are characterized by the so-called transition or T-matrix which involves solutions of the vector Helmholtz equation in terms of spherical vector wave functions. Computer programs were developed to solve for the T-matrix of axisymmetric imbedded perfect conductors and dielectrics, the far-zone fields and scattering cross-sections. Solutions for the back-scattering cross-sections of concentric, spherical scatterers using the T-matrix method were found to be in excellent agreement with extended Mie theory solutions. In addition, solutions of radar cross-sections of a resonant-sized, non-concentric dielectric-clad spherical perfect conductor
compared favorably with available experimental data.

An important application of the method is in the field of radar meteorology where cross-sections of non-spherical hydrometeors have to be computed. Calculations of back-scattering cross-sections for incident orthogonal polarizations aligned along the principal axes were performed for (a) spheroidal raindrops, (b) spheroidal ice-stones, and (c) water-coated and dry ice-stones with surface perturbations. These calculations demonstrated the potential usefulness of a proposed radar measurement technique, based on the concept of differential reflectivity, for acquiring qualitative and/or quantitative information regarding rainfall, hail and storm structure.
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CHAPTER I
INTRODUCTION

1.1. General Nature of the Problem

This investigation is concerned with the problem of developing a general, systematic method of calculating the scattering cross-sections of axisymmetric perfect conductors or dielectrics imbedded in an axisymmetric dielectric body. The research is motivated by the need for a suitable scattering theory of non-spherical homogenous and imbedded dielectric bodies which may be applied to the field of radar meteorology. Radar methods are increasingly becoming an important tool for remote sensing of atmospheric phenomena. A fundamental problem which arises in this regard is the interaction between electromagnetic waves and atmospheric scatterers and how effectively a scatterer re-radiates energy incident upon it. In radar meteorological applications the principal atmospheric scatterers are hydrometeors, e.g., raindrops, hail, snowflakes. These hydrometeors can often be modelled as non-spherical, homogenous or imbedded dielectric scatterers, and their scattering cross-sections have to be computed using classical electromagnetic theory.

Many of the previous theoretical computations of the scattering properties of hydrometeors have relied primarily on the Rayleigh and Mie theories for spheres and their extensions to spheroidal and
concentric multi-layered spherical bodies, Kerker (1969), Battan (1970), Deirmendjian (1963), Van de Hulst (1957). These methods are frequently utilized by radar meteorologists due to their ease of application and also because of the lack of a general theory to handle non-spherical scatterers.

Meteorological radars are commonly operated at wavelengths of 10cm (S-band), 5cm (C-band) and 3cm (X-band). At these wavelengths hydrometeor sizes are such that they fall in the Rayleigh and Mie regions of electromagnetic scattering. In the Mie or resonance regions, there is strong interaction between the scatterer and the incident wave and the shape factor becomes very important in determining scattering behavior. In addition, exact theories have to be used for resonance-sized scatterers making the situation more difficult to handle especially when numerical values are desired.

A relatively recent scattering theory based on Waterman's (1965, 1969) extended integral equation technique is used in the present work to handle non-spherical, homogenous or imbedded dielectric scatterers. This theory is an exact matrix approach to the scattering problem. The non-spherical, homogenous or imbedded scatterer is characterized by the so-called Transition or T-matrix which depends on scatterer shape, relative size and dielectric constants. The T-matrix refers to expansions in spherical wave solutions of the vector Helmholtz equation. The T-matrix method is very amenable to solution by numerical techniques, resulting in extensive use of digital computers for calculating scattering properties of model hydrometeors. The results of such computations are useful for testing and developing hypotheses in radar meteorology.
1.2. Previous Related Studies

The theory of electromagnetic scattering has had a long history since Lord Rayleigh's (1871, 1881) classical treatise on light scattering by small particles. It is not the intent here to provide a comprehensive review of electromagnetic scattering theories. Instead, emphasis will be on scattering methods useful for homogenous, nonspherical and imbedded dielectric bodies with particular reference to hydrometeors.

1.2.1 Mie Theory for Spherical Bodies

The first thorough treatment of scattering from a spherical particle of arbitrary size and electrical properties was due to Mie (1908). The complete theory is given concisely by Stratton (1941), among others, and extensive applications are to be found in Van de Hulst (1957). Numerical results and comparisons with experiment on spherical water drops are to be found in Aden (1952). He also derived the theory of scattering from two concentric spheres with different dielectric constants using an extended Mie theory formulism but no numerical results were given. Systematic numerical results for scattering cross-sections of spherical water drops over a wide range of sizes and dielectric constants have been published by Herman, Browning, and Battan (1961). Battan, et al. (1970) have also published scattering cross-sections of spherical ice-stones covered with a concentric layer of water, again for a wide range of sizes, dielectric constants and water thicknesses. These comprehensive tabulations form an important reference source for scattering cross-sections of spherical hydrometeors.
1.2.2 Low Frequency Methods

For objects very small in comparison to the wavelength (Rayleigh region), the Mie theory reduces to the famous Rayleigh scattering law where the scattered intensity is proportional to the square of the particle volume and inversely proportional to \( \lambda^4 \) (\( \lambda \) being the wavelength). This result is a useful approximation for small hydrometeors at S- or C-band.

An approximate solution for homogenous ellipsoidal dielectrics in the Rayleigh region has been given by Gans (1912). Stevenson (1953a) has extended Rayleigh's theory by formulating the scattered field as a power series in \( ka, k = 2\pi/\lambda \) and \( a \) is the representative scatterer size. Successive terms in the series require the solution of standard problems in potential theory. The first three terms in this series have been calculated for the penetrable ellipsoid, Stevenson (1953b). More recently, Senior (1976) has formulated the scattered field from small dielectric bodies in terms of a single polarizability tensor which is a function of only the geometry of the body and the relative dielectric constant. Of the above methods only Gans' theory seems to have been used for spheroidal hydrometeors, e.g., see Atlas, et al. (1953), Humphries (1974), Seliga and Bringi (1976).

1.2.3 Geometrical Optics Methods

When the scatterer is very large compared to the wavelength, modified geometrical optics methods are quite useful in determining, approximately, the back-scattering cross-section, Kouyoumjian, et al. (1963), Van de Hulst (1957), Swarner and Peters (1963), Thomas (1962).
The major advantages in using ray techniques are (a) the ray trajectories enable the physical processes of scattering to be followed and (b) the resulting solution is quite simple and can be improved by considering additional classes of rays. A major limitation is that these methods cannot be realistically used to systematically handle lossy, non-spherical homogenous and imbedded bodies over a wide range of dielectric constants. In addition, these methods are often polarization insensitive and evaluation of bi-static scattering functions is a non-trivial problem.

1.2.4 The Integral Equation Method

Exact solutions to the scattering problem are required when the object size is of the order of the incident wavelength. The exact solutions when properly formulated, can be solved quite efficiently using numerical methods. One such technique that has found widespread application is the integral equation method, Harrington (1968). The integral equation formulation for the unknown surface currents on the scatterer is a general formulation in which the current distributions on the entire surface of the scatterer must be solved before the far-field can be constructed. The integral equation has the advantage of being entirely self-consistent as the boundary condition is already built into the approach. The solution of the integral equation is typically based on the moment method which transforms the original equation into a matrix form that is numerically inverted on a computer, see, for example, Richmond (1965), Miller and Poggio (1976), Harrington (1968). In principle, the integral equation technique is valid for the
entire frequency range, however, its application has been restricted to frequencies corresponding to the resonance region. This restriction results from the matrix methods employed, the matrices becoming very large and numerically unwieldy as the body becomes large compared to wavelength. The integral equation technique via moment methods has been used by Tsai, et al. (1976) on composite dielectric-dielectric and dielectric-metal scatterers with rotational symmetry.

Oguchi (1973a,b, 1974, 1975) has done extensive work on scattering from oblate spheroidal raindrops using (a) Point Matching and (b) Spheroidal function expansions. In the point matching technique, the incident field is expanded in vector spherical harmonics with known expansion coefficients, and scattered and transmitted fields are also expanded in spherical harmonics with unknown coefficients. If the infinite modal summations in the field expansions are truncated at some modal index (say m, n) and if boundary conditions are satisfied for representative points on the spheroid the number of which is appropriate to the index (m, n), a matrix equation is generated for the unknown coefficients. Oguchi (1975) has extended this technique whereby the boundary conditions are forced to hold at a large number of points on the spheroid surface (larger than the index at which fields were truncated) and a solution generated by using the method of least squares. It is important to note that Oguchi's point matching method does not use expansions of the surface or volume currents. In addition, the exterior Rayleigh hypothesis [Millar (1969)] is invoked implying that the vector Fourier series expansion of the scattered field in
terms of "outgoing" wave functions is valid on the non-spherical surface of the scatterer. The implications of these assumptions and the improvements due to Oguchi's least squares fitting procedure have been discussed by Bates (1975). The spheroidal function technique is similar, but theoretically exact. The incident, scattered and transmitted fields are expanded in vector spheroidal harmonics with known and unknown expansion coefficients. Application of the boundary conditions yields a system of simultaneous linear equations of infinite order for the unknown coefficients. The matrix equation is then solved after truncation at a suitable modal index, resulting in solution of the scattered fields.

1.2.5 Extended Integral Equation Method

Previous related studies on the technique used in this research will now be discussed. This technique can be classified under integral equation methods.

The extended integral equation technique was first formulated by Waterman for rotationally symmetric perfect conductors (1965) and later extended to lossless dielectrics (1969). The integral equation for the surface currents is formulated by requiring that the scattered field (due to these currents) precisely cancel the incident field everywhere within the scatterer. A unique feature is that the scattered and incident field expansion coefficients (these fields are expanded in vector spherical harmonics) are directly related through the transition or T-matrix of the body without explicit reference to the surface currents. An extensive study of the structure of the T-matrix was made
by Waterman (1971), including comparison with theory and measurements for a large class of rotationally symmetric perfect conductors. A rigorous formulation of scattering by arbitrarily shaped perfect conductors using the extended integral equation method has also been given by Hizal and Marincic (1970). Barber and Yeh (1974, 1975) have extended Waterman's technique to arbitrarily shaped lossy dielectric bodies and computations have been performed on a large class of non-spherical, rotationally symmetric, dielectric bodies. Warner and Hizal (1975) have independently used Waterman's method to calculate the scattering characteristics of axisymmetric raindrops. Recently, Peterson and Ström (1974) and Ström (1974) have extended Waterman's technique to theoretically formulate the T-matrix for an arbitrary number of multi-layered scatterers. Numerical results were, however, given for the hollow, lossless dielectric shell, Peterson (1974), and for oblate spheroidal raindrops, Peterson (1976).

The theoretical aspects of this work reconsiders Peterson and Ström's T-matrix formulation using the conceptually simple equivalence principle as compared to the Poincaré-Huygens principle used by them.

1.3. Problems for Investigation

The specific problems considered in this work are outlined below:

A. Develop the theory of scattering from axisymmetric perfect conductors or dielectrics imbedded within an axisymmetric dielectric using the extended integral equation method. A unique feature of this work is the
derivation of the T-matrix using the conceptually simple equivalence principle and presentation of the theory in a concise, systematic manner.

B. Numerically solve for the T-matrix and, hence, the scattering cross-sections for incident orthogonal polarizations aligned along the principal axes of the scatterer. A new method using an extended Gauss-Legendre quadrature scheme has been utilized to fill in the elements of the T-matrix. This numerical technique has proven to be accurate, stable and efficient.

C. Compare T-matrix calculations with extended Mie theory calculations for spherical concentric dielectrics and dielectric-clad perfect conductors. The first comparisons of T-matrix calculations with measurements made on a non-concentric dielectric-clad spherical perfect conductor were performed.

D. Calculate the scattering cross-sections of (a) oblate-spheroidal raindrops, (b) oblate-spheroidal ice-stones, (c) "rough" ice-stones with surface perturbations, and (d) water-coated "rough" ice-stones.

E. Investigate the polarization characteristics of such hydrometeors with special reference to potential applications of the differential reflectivity concept proposed by Seliga and Bringi (1976).
The theoretical formulation of the T-matrix is presented in Chapter II followed by a description of the numerical method and computer program flow chart in Chapter III. This chapter also contains several sample computations for comparison with extended Mie theory solutions for layered concentric spherical scatterers and a direct comparison between T-matrix solutions and experimental results for a non-concentric dielectric-clad spherical perfect conductor. Chapter IV is concerned with the application of the T-matrix method in the field of radar meteorology. Computations of back-scattering cross-sections are presented for a variety of model hydrometeor shapes and sizes. Chapter V summarizes major results and suggestions for future research.
CHAPTER II
THEORETICAL FORMULATION OF THE T(1,2)-MATRIX

The purpose of this chapter is to formulate the theory for calculating scattering cross-sections of axisymmetric perfect conductors or dielectrics imbedded in an axisymmetric dielectric when illuminated by a plane, monochromatic electromagnetic wave.

2.1. The Scattering Problem

The scatterer geometry is illustrated in Fig. 2.1. The imbedded body can be either a lossy dielectric (with constitutive parameters \( \epsilon_2, \mu_0 \)) or a perfect conductor; the outer body is assumed to be a lossy dielectric characterized by constitutive parameters \( \epsilon_1, \mu_0 \). The surrounding medium is free space. The theoretical scattering problem is developed using Waterman's (1969) extended integral equation method.

Consider a plane electromagnetic wave \( E^i(r), H^i(r) \) incident on a two-layered scatterer defined by closed surfaces \( S_1 \) and \( S_2 \) where \( S_1 \) encloses \( S_2 \) as shown in Fig. 2.1. The surfaces are assumed to be sufficiently regular that the divergence theorem is applicable, and continuous, single-valued normals \( n_1 \) and \( n_2 \) are defined. Simple harmonic time dependence at angular frequency \( \omega \) is assumed, with the factor \( \exp(-j\omega t) \) suppressed in all field quantities. The boundary values of fields on a surface will be denoted by subscripts + and - where + refers
Fig. 2.1. Illustrating the geometrical configuration of the two-layered scatterer.

to fields external to a surface and - to fields internal to a surface.

The entire analysis can be summarized by the following steps:

(A) The external problem is first analyzed using the equivalence principle [Harrington (1961)] for the region exterior to surface $S_1$. Electric and magnetic surface currents on $S_1$ are forced to radiate the correct scattered fields external to $S_1$, but cancel the incident field everywhere within $S_1$. This condition serves to partially determine the surface currents on $S_1$.

(B) The field in the layered region between $S_1$ and $S_2$ is next expanded into vector spherical harmonics with unknown coefficients. Boundary conditions on $S_1$ and the condition
described in (A) are applied leading to a matrix equation relating the unknown coefficients to the known incident field coefficients.

(C) The equivalence principle for the layered region is now utilized to force surface currents on both $S_1$ and $S_2$. These surface currents generate the correct field in the layered region but null fields elsewhere (i.e., outside $S_1$ and within $S_2$).

(D) The surface fields on $S_2$ are expanded into vector spherical harmonics with unknown coefficients. The procedure will differ depending on the nature of the imbedded body, i.e., whether it is a perfect conductor or a dielectric. By making use of the null condition described in (C) relevant matrix equations can be derived for these unknown coefficients.

(E) Finally, the scattered field is expanded into vector spherical harmonics with unknown coefficients. These are related to the incident field coefficients through the Transition [or $T(1,2)$] matrix without explicit reference to the surface fields. The scattering cross-section, being proportional to the square of the far field amplitude, follows directly.

The above steps will now be explained in greater detail resulting in the formulation of the $T(1,2)$-matrix which characterizes the scattering behavior of the composite body. This is followed by a brief discussion of the theoretical aspects of the formulation.
2.2. The External Problem

The scattering problem is illustrated in Fig. 2.2. The total field \( \mathbf{E} \) is defined as the sum of the incident and scattered fields, \( \mathbf{E} = \mathbf{E}^i + \mathbf{E}^s \). The equivalence principle applied to the exterior region results in surface currents \( \mathbf{\hat{n}}_1 \times \mathbf{H}_1 \) and \( \mathbf{E}_1 \times \mathbf{\hat{n}}_1 \) on \( S_1 \) and a null field within \( S_1 \). This is shown in Fig. 2.2b. Using superposition the incident field and its sources are next removed resulting in the situation depicted in Fig. 2.2c. The surface currents generate the correct

![Fig. 2.2. Formulation of the equivalence principle for the exterior problem, (a) the scattering problem, (b) null fields forced within \( S_1 \), (c) incident field and its sources removed.](image)
scattered fields $E^S, H^S$ outside $S_1$ but precisely cancel the incident field everywhere within $S_1$. For convenience the medium within $S_1$ is assumed to be free space. Since the currents $\hat{n}_1 \times H_+ \text{ and } E_+ \times \hat{n}_1$ now radiate in unbounded, free space the vector potential formulation can be used to evaluate the fields everywhere. The scattered fields $E^S, H^S$ can be expressed as

$$E^S = -\nabla \times F - \frac{1}{j \omega \varepsilon_0} (\nabla \times \nabla \times A)$$

(2.1a)

$$H^S = \nabla \times A - \frac{1}{j \omega \mu_0} (\nabla \times \nabla \times F)$$

(2.1b)

where $A$ and $F$ are the magnetic and electric vector potentials, respectively, given by

$$A = \frac{1}{4\pi} \int_{S_1} (\hat{n}_1 \times H_+) \frac{\exp(jkR)}{R} \, dS$$

(2.2a)

and

$$F = \frac{1}{4\pi} \int_{S_1} (E_+ \times \hat{n}_1) \frac{\exp(jkR)}{R} \, dS$$

(2.2b)

where $R = |\mathbf{r} - \mathbf{r}'|$, the distance between source point and field point, and $k = \omega \sqrt{\mu_0 \varepsilon_0}$, the free-space wave number. Substituting (2.2) in (2.1), the total field $E$ can be expressed as

$$\begin{cases}
E(r) \bigg|_{\mathbf{r}} = E_0 (\mathbf{r}) + \nabla \times \int_{S_1} (\hat{n}_1 \times E_+) g(kR) \, dS \\
\quad - \nabla \times \nabla \times \int_{S_1} \frac{1}{j \omega \varepsilon_0} (\hat{n}_1 \times H_+) g(kR) \, dS \quad \mathbf{r} \text{ outside } S_1
\end{cases}$$

(2.3)
In the above, an origin has been chosen in the interior volume and
\( g(kR) = (4\pi R)^{-1} \exp(jkR) \) is the free space scalar Green's function.

In the interior region (\( \mathbf{r} \) inside \( S_1 \)), (2.3) partially determines the
surface currents \( \mathbf{n}_1 \times \mathbf{H}_+ \) and \( \mathbf{E}_+ \times \mathbf{n}_1 \) by requiring that the scattered
field precisely cancel the incident field. Hence, for \( \mathbf{r} \) inside \( S_1 \)

\[
\nabla \times \int_{S_1} (\mathbf{n}_1 \times \mathbf{E}_+) g(kR) ds - \nabla \times \int_{S_1} \frac{1}{j \omega \varepsilon_0} (\mathbf{n}_1 \times \mathbf{H}_+) g(kR) ds = -\mathbf{E}_i^i(\mathbf{r}) \quad (2.4)
\]

This equation can be expanded by making use of vector spherical
harmonics \( M \) and \( N \) [see Stratton (1940)]

\[
M_{\sigma mn}(\mathbf{r}) = \nabla \times \left[ \frac{\cos m \phi}{\sin m \phi} \right] P_n^m(\cos \theta) z_n(kr) \quad (2.5)
\]

\[
N_{\sigma mn}(\mathbf{r}) = \frac{1}{k} \nabla \times M_{\sigma mn}(\mathbf{r}) \quad (2.6)
\]

where \( \sigma = \text{even or odd} \) refers to the choice of the trigonometric function,
\( P_n^m(\cos \theta) \) are the associated Legendre functions,
and \( z_n(kr) \) is an appropriate spherical Bessel function.

Superscripts \( ^1 \) or \( ^3 \) on \( M \) will refer to the use of a regular or outgoing
type of Bessel function for \( z_n(kr) \), respectively.

The incident field (finite at the origin) can be expressed as

\[
\mathbf{E}_i^i(\mathbf{r}) = \sum_{\nu=1}^{\infty} D_{\nu} \left[ a_{\nu} M_\nu^1(\mathbf{r}) + b_{\nu} N_\nu^1(\mathbf{r}) \right] \quad (2.7)
\]
where \( \nu \) is a combined index incorporating \( \sigma, m, \) and \( n \). \( D_{\nu} \) is a normalization constant:

\[
D_{\nu} = \varepsilon \frac{(2n+1)(n-m)!}{m 4n(n+1)(n+m)!}, \quad \varepsilon = \begin{cases} 1 & \text{if } m = 0, \\ \frac{1}{2} & \text{if } m > 0. \end{cases}
\tag{2.8}
\]

and \( a_{\nu} \) and \( b_{\nu} \) are the assumed known incident field expansion coefficients.

The free space dyadic Green's function can be expressed as [Morse and Feshback (1953)],

\[
\frac{jk}{\pi} \sum_{\nu=1}^{\infty} D_{\nu} \left[ M_{\nu}^3(r) M_{\nu}^3(r') + N_{\nu}^3(r) N_{\nu}^3(r') \right]; \quad r > r'
\tag{2.9a}
\]

\[
\frac{jk}{\pi} \sum_{\nu=1}^{\infty} D_{\nu} \left[ M_{\nu}^1(r) N_{\nu}^3(r') + N_{\nu}^3(r) M_{\nu}^3(r') \right]; \quad r < r'
\tag{2.9b}
\]

By using (2.9b) for \( r > r' \) and (2.7) in (2.4) the following set of equations for the surface currents results:

\[
\frac{jk}{\pi} \int_{S_1} \left[ \frac{N_{\nu}^3(kr')}{(\mathbf{n}_1 \times \mathbf{E}_0)} + j \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{M_{\nu}^3(kr')}{(\mathbf{n}_1 \times \mathbf{H}_0)} \right] ds = -a_{\nu} \tag{2.10a}
\]

\[
\frac{jk}{\pi} \int_{S_1} \left[ \frac{M_{\nu}^3(kr')}{(\mathbf{n}_1 \times \mathbf{E}_0)} + j \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{N_{\nu}^3(kr')}{(\mathbf{n}_1 \times \mathbf{H}_0)} \right] ds = -b_{\nu} \tag{2.10b}
\]

where \( \nu = 1, 2, 3, \ldots \).

The solution of (2.10) ensures that the total field will vanish within an inscribed sphere \( \rho_1 \) (see Fig. 2.1). Waterman (1965) and Barber and Yeh (1975) have shown, using analytic continuation arguments, that this is sufficient to guarantee that the total field vanishes
throughout the interior volume $V_1$ bounded by the surface $S_1$. Complex variable theory indicates that if an analytic function can be expanded in a subregion of its region of analyticity, then the function can also be expanded around any origin within that sub-region out to the nearest singularity of the function. Referring to Fig. 2.3, the vector spherical harmonics of (2.10) can be expanded around a new origin $O'$ which is within the original region of analyticity, viz., the sphere $\rho_1$. Using the translational properties of vector harmonics this would involve a matrix operation on (2.10). The resulting set of equations would force the total field to vanish in a region with center $O'$ and radius equal to the nearest singularity point which is on the surface $S_1$, where $r = r'$. The extended region is shown shaded in Fig. 2.3 with radius $r_m$. Thus, the region of applicability of (2.10) has been extended; by repeating this procedure it is clear that (2.10) is sufficient to force the total field to zero everywhere within $S_1$. This procedure and formulation summarizes Waterman's extended integral equation principle as applied to the exterior problem of a single homogeneous scatterer.

Fig. 2.3. Illustrating analytic continuation. Shaded area shows extended region.
2.3. **The Layered Region**

Next, note that the fields in the layered region (i.e., the region between $S_1$ and $S_2$) can be approximated by a linear combination of vector spherical harmonics which are (considered) complete solutions of the vector Helmholtz equation for $E^1$:

$$E^1(r) = \sum_{\mu=1}^{N} \left[ \gamma_{\mu} M^1_{\mu}(k_1 r) + \alpha_{\mu} M^3_{\mu}(k_1 r) + \delta_{\mu} N^1_{\mu}(k_1 r) + \beta_{\mu} N^3_{\mu}(k_1 r) \right]$$  \(2.11\)

where $\gamma_{\mu}, \alpha_{\mu}, \delta_{\mu}, \beta_{\mu}$ are $4N$ unknown coefficients and $k_1 = \omega \sqrt{\mu_0 \varepsilon_1}$.

Note that the expansion in (2.11) consists of both "regular" and "outgoing" types of spherical harmonics. Using boundary conditions on the surface $S_1$ we obtain

$$\hat{n}_1 \times E^+ = \hat{n}_1 \times E^-$$  \(2.12a\)

$$\hat{n}_1 \times H^+ = \hat{n}_1 \times H^-$$  \(2.12b\)

The new boundary fields $\hat{n}_1 \times E^\pm$ and $\hat{n}_1 \times H^\pm$ can be readily derived from (2.11). Using boundary conditions (2.12) and substituting the boundary fields into the first $2N$ equations of (2.10) yeilds the simplified matrix equation,

$$\begin{bmatrix} Q^1(\text{Out, Re}) \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \end{bmatrix} + \begin{bmatrix} Q^1(\text{Out, Out}) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -ja \\ -jb \end{bmatrix}$$  \(2.13\)

where $a$ and $b$ are column vectors defining the incident field coefficients.

The $2N \times 2N$ matrix $Q^1$ consists of four $N \times N$ sub-matrices:
20

\[ Q^1(\text{Out, Re}) = \begin{bmatrix} K + \sqrt{\varepsilon_{1,1}} & J & L + \sqrt{\varepsilon_{1,1}} \\ I + \sqrt{\varepsilon_{1,1}} & L & J + \sqrt{\varepsilon_{1,1}} \end{bmatrix} \]  

(2.14)

where \( \varepsilon_{1,1} = \varepsilon_1 / \varepsilon_0 \). I, J, K and L are each \( N \times N \) matrices which implicitly assume the same arguments as \( Q^1 \) and are given by

\[
\begin{pmatrix}
I_{\gamma\mu} \\
J_{\gamma\mu} \\
K_{\gamma\mu} \\
L_{\gamma\mu}
\end{pmatrix} = \frac{k^2}{\pi} \int_{S_1} \hat{n}_1 \cdot \begin{pmatrix}
M_3^3(kr') \times M_1^1(k_1 r') \\
M_3^1(kr') \times N_1^1(k_1 r') \\
N_3^3(kr') \times M_1^1(k_1 r') \\
N_3^1(kr') \times N_1^1(k_1 r')
\end{pmatrix} \, dS
\]  

(2.15)

The superscripts 3 or 1 on \( M \) (and \( N \)) refer to the use of an outgoing (Out) or regular (Re) form of the spherical Bessel function, respectively. This dependence is also shown through the use of the combination (Out,Re) in the arguments of \( Q^1 \). The superscript 1 on \( Q \) indicates that integration is over \( S_1 \) in (2.15). \( Q^1(\text{Out, Out}) \) is given in an analogous manner, i.e., using superscripts 3 and 3 in (2.15).

2.4. Equivalence Principle for the Layered Region

The next step consists of applying the equivalence principle to the volume region between \( S_1 \) and \( S_2 \). Fig. 2.4 illustrates the relevant details. Null fields are forced inside \( S_2 \) and outside \( S_1 \). In general, surface currents \( \hat{n}_2 \times H_1^1 \), \( E_1^1 \times \hat{n}_2 \) on \( S_2 \) and \( \hat{n}_1 \times H_1^1 \), \( E_1^1 \times n_1 \) on \( S_1 \) are required to produce the correct field \( E^1 \), \( H^1 \) in the layered region.

For convenience the medium within \( S_2 \) and outside \( S_1 \) is assumed to have constitutive parameters \( \varepsilon_1, \mu_0 \). Again we have the situation where
surface currents radiate in an unbounded, homogenous media characterized by $\varepsilon_1, \mu_0$. Hence, the fields can be derived using the vector potential formulation. This leads to

$$
\mathbf{E}^1(\mathbf{r}) = \nabla \times \int_{S_1} \left( \mathbf{\hat{n}}_1 \times \mathbf{E}_1^1 \right) g(k_1 R) dS - \nabla \times \nabla \times \int_{S_1} \frac{1}{j\omega \varepsilon_1} \left( \mathbf{\hat{n}}_1 \times \mathbf{H}_1^1 \right) g(k_1 R) dS
$$

$$
+ \nabla \times \int_{S_2} \left( \mathbf{\hat{n}}_2 \times \mathbf{E}_2^1 \right) g(k_1 R) dS - \nabla \times \nabla \times \int_{S_2} \frac{1}{j\omega \varepsilon_1} \left( \mathbf{\hat{n}}_2 \times \mathbf{H}_2^1 \right) g(k_1 R) dS;
$$

$$
\mathbf{F} \begin{cases} 
\text{between } S_1 \text{ and } S_2 \\
\text{outside } S_1 \text{ or inside } S_2
\end{cases}
$$

(2.16)

where $\mathbf{\hat{n}}_1 = -\mathbf{\hat{n}}_1$ and $g(k_1 R) = (4\pi R)^{-1} \exp(jk_1 R)$, the appropriate Green's function for use in an unbounded, homogenous media characterized by $\varepsilon_1, \mu_0$. 

Fig. 2.4. Illustrating equivalence principle for the layered region.
2.5. The Imbedded Body

Before proceeding further the nature of the imbedded body must be considered. Recall that the imbedded body can be either a perfect conductor or a lossy dielectric with \( \varepsilon_2, \mu_0 \). The perfect conductor case is simpler and is treated first.

2.5.1 Perfect Conductor

Since \( \mathbf{\hat{n}}_2 \times \mathbf{E}^1_+ = 0 \) on the surface \( S_2 \) of a perfect conductor, (2.16) reduces to

\[
\mathbf{E}^1_+(\mathbf{r}) = \nabla \times \left( \mathbf{\hat{n}}_1 \times \mathbf{E}^1_- \right) g(k_1 R) dS - \nabla \times \nabla \times \left( \mathbf{\hat{n}}_1 \times \mathbf{H}^1_- \right) g(k_1 R) dS
\]

\[
- \nabla \times \nabla \times \int_{S_2} \frac{1}{j \omega \varepsilon_1} (\mathbf{\hat{n}}_2 \times \mathbf{H}^1_+) g(k_1 R) dS, \quad \mathbf{r} \in \text{between } S_1 \text{ and } S_2
\]

\[
\quad \text{or } \mathbf{r} \in \text{outside } S_1 \text{ or inside } S_2
\]

Now note that the electric surface current \( \mathbf{\hat{n}}_2 \times \mathbf{H}^1_+ \) can be expanded into vector spherical harmonics which form a complete set and approximate the current in a mean square sense [Waterman (1971)]:

\[
\mathbf{\hat{n}}_2 \times \mathbf{H}^1_+(\mathbf{r}') = -j \sqrt{\varepsilon_1 \mu_0} \mathbf{\hat{n}}_2 \times \sum_{\mu=1}^{N} \left[ \mathbf{s}_\mu, N^1_\mu(k_1 \xi') + \mathbf{l}_\mu M^1_\mu(k_1 \xi') \right]
\]

(2.18)

where \( \mathbf{s}_\mu \) and \( \mathbf{l}_\mu \) are 2N unknown expansion coefficients. The surface current \( \mathbf{\hat{n}}_2 \times \mathbf{H}^1_+(\mathbf{r}') \) alternatively could be expanded using "outgoing" vector harmonics \( \mathbf{M}_\mu^3(k_1 \xi') \) and \( \mathbf{N}_\mu^3(k_1 \xi') \). Waterman (1965, 1971) has used both types of expansions for the surface current; his numerical results support the choice of "regular" harmonics as in (2.18) for scattering from a single perfect conductor.
2.5.2 Dielectric

For the case where the imbedded body is a lossy, dielectric characterized by \( \varepsilon_2, \mu_0 \), Waterman (1969) showed that the field, \( E^2 \), inside the dielectric body (see Fig. 2.1) can be approximated by

\[
E^2(r) = \sum_{\mu=1}^{N} \left[ c_{\mu}(k_2 r) + d_{\mu}(k_2 r) \right] \quad (2.19)
\]

where \( c_{\mu} \) and \( d_{\mu} \) are again the unknown expansion coefficients and \( k_2 = \omega \sqrt{\varepsilon_2 \mu_0} \). "Regular" vector wave functions \( M^1_{\mu} \) and \( N^1_{\mu} \) have been used in (2.19) since \( E^2(r) \) must be finite at the origin which is within \( S_2 \). Application of boundary conditions on \( S_2 \) results in

\[
\hat{n}_2 \times E^2 = \hat{n}_2 \times E^1 \quad (2.20a)
\]

\[
\hat{n}_2 \times H^2 = \hat{n}_2 \times H^1 \quad (2.20b)
\]

where \( E^2 \) and \( H^2 \) are the values of \( E^2 \) on the surface \( S_2 \). Using (2.19) the surface currents \( \hat{n}_2 \times E^2(r') \) and \( \hat{n}_2 \times H^2(r') \) can be expressed, respectively, as

\[
\hat{n}_2 \times E^2(r') = \hat{n}_2 \times \sum_{\mu=1}^{N} \left[ c_{\mu}(k_2 r') + d_{\mu}(k_2 r') \right] \quad (2.21a)
\]

\[
\hat{n}_2 \times H^2(r') = -j \sqrt{\frac{\varepsilon_2}{\mu_0}} \hat{n}_2 \times \sum_{\mu=1}^{N} \left[ c_{\mu}(k_2 r') + d_{\mu}(k_2 r') \right] \quad (2.21b)
\]

In the above, it is assumed that the vector spherical harmonics \( M^1_{\mu} \) and \( N^1_{\mu} \) form a complete set for the representation of the surface currents on \( S_2 \).
2.5.3 The Null Field Conditions

The surface currents on \( S_2 \) in (2.16) have now been expanded on a spherical basis for perfect conductors (2.18) or dielectrics (2.21). The remaining surface currents on \( S_1 \), \( \hat{n}_1 \times \vec{E}_1 \) and \( \hat{n}_1 \times \vec{H}_1 \), can be obtained from (2.11). The next step is to substitute these expanded surface currents into (2.16) and make use of the null conditions, i.e., restricting \( \vec{r} \) to lie outside \( S_1 \) and inside \( S_2 \) consecutively. In each case the proper dyadic Green's function must be used, e.g., when \( \vec{r} \) is restricted to lie outside \( S_1 (r>r') \), the form given in (2.9a) is applicable with \( k \) replaced by \( k_1 \). Similarly, when \( \vec{r} \) is restricted to lie inside \( S_2 (r<r') \), the form given in (2.9b) is applicable. After extensive simplifications the condition \( \vec{r} \) inside \( S_2 \) in (2.16) yields the matrix equation:

\[
\begin{bmatrix}
\gamma \\
\delta
\end{bmatrix}
= j
\begin{bmatrix}
D & 0 \\
0 & D
\end{bmatrix}
\begin{bmatrix}
Q^2(\text{Out, Re}) \\
0
\end{bmatrix}
\begin{bmatrix}
c \\
d
\end{bmatrix}
\tag{2.22}
\]

where (2.21) has been used for the surface currents on \( S_2 \). \( D \) is a diagonal sub-matrix composed of elements \( D_v, v = 1, 2, \ldots N \). The 2\( N \times 2N \) matrix \( Q^2(\text{Out, Re}) \) is similar in form to \( Q^1(\text{Out, Re}) \) [refer (2.14)] except that integration is now over surface \( S_2 \) with \( \hat{n}_1, k, k_1 \) replaced by \( \hat{n}_2, k_1, k_2 \), respectively in (2.15). Similarly, the condition \( \vec{r} \) outside \( S_2 \) in (2.16) yields

\[
\begin{bmatrix}
a \\
b
\end{bmatrix}
= -j
\begin{bmatrix}
D & 0 \\
0 & D
\end{bmatrix}
\begin{bmatrix}
Q^2(\text{Re, Re}) \\
0
\end{bmatrix}
\begin{bmatrix}
c \\
d
\end{bmatrix}
\tag{2.23}
\]
Matrix equations (2.22) and (2.23) force null field conditions to hold in (2.16) for spherical regions interior to \( p_3 \) and exterior to \( p_2 \) (these regions are shown in Fig. 2.1), respectively. Analytic continuation arguments as given in Section 2.2 can again be invoked to show that (2.22) and (2.23) are sufficient to guarantee that null field conditions hold everywhere interior to \( S_2 \) and exterior to \( S_1 \). This is Waterman's extended integral equation technique as applied to layered scatterers.

When the imbedded body is a perfect conductor, the form of (2.22) and (2.23) remains the same. However the composition of \( Q^2(\text{Out}, \text{Re}) \) and \( Q^2(\text{Re}, \text{Re}) \) changes; \( Q^2(\text{Out}, \text{Re}) \) is expressed as

\[
Q^2(\text{Out}, \text{Re}) = \begin{bmatrix} I^\sigma & I^\sigma \\ L^\sigma & K^\sigma \end{bmatrix}
\]  \hspace{1cm} (2.24)

where the elements of \( I^\sigma \) are given by

\[
I^\sigma_{\nu\mu} = \frac{k_1^2}{\pi} \int_{S_2} \hat{n}_2 \cdot \frac{M^3(k_1 r')}{\rho_\nu} \times \frac{M^1(k_1 r')}{\rho_\mu} \, dS
\]  \hspace{1cm} (2.25)

\( Q^2(\text{Re}, \text{Re}) \) is expressed in a similar manner, i.e., using superscripts \( 1 \) and \( l \) in (2.25).

2.6. The Scattered Field

The final step in the formulation of the scattering problem is to express the far-zone scattered fields in terms of the surface fields \( \hat{n}_1 \times E^1 \) and \( \hat{n}_1 \times H^1 \) on \( S_1 \). Since these in turn are related to the incident field, the resultant expression gives the scattered field...
coefficients directly in terms of the incident field coefficients. The matrix which relates the two sets of coefficients is commonly referred to as the transition or T-matrix and is derived below.

The scattered field is expanded into spherical harmonics:

\[ E^S(r) = \sum_{\nu=1}^{N} 4D_{\nu} \left[ f_{\nu} N_{\nu}^3(kr) + g_{\nu} N_{\nu}^3(kr) \right]; r \text{ outside } \rho_2 \]  \hspace{1cm} (2.26)

where \( f_{\nu} \) and \( g_{\nu} \) are the scattered field coefficients.

From (2.3) we note that

\[ E^S(r) = \nabla \times \int_{S_1} \left( \hat{n}_1 \times E_1 \right) g(kr) dS - \nabla \times \nabla \times \int_{S_1} \left( \hat{n}_1 \times H_1 \right) g(kr) dS; r \text{ outside } S_1 \]  \hspace{1cm} (2.27)

Using the free-space dyadic Green's function as given in (2.9a) and substituting (2.11) and (2.12) in (2.27) results in the matrix equation

\[ \begin{bmatrix} f \\ g \end{bmatrix} = \frac{-i}{4} \begin{bmatrix} Q^1(Re, Re) \\ \gamma \end{bmatrix} - \frac{i}{4} \begin{bmatrix} Q^1(Re, Out) \end{bmatrix} \begin{bmatrix} \alpha \\ \delta \end{bmatrix} \]  \hspace{1cm} (2.28)

This equation relates the scattered fields coefficients to the \( E^1 \) field expansion coefficients given by (2.11) within the layered region. This is a valid representation for the scattered field coefficients as long as \( r \) is restricted to lie outside \( \rho_2 \) (see Fig. 2.1). Since the far-zone fields are of interest, this restriction does not pose any problem. The principal matrix equations form a compact set and are grouped together here for convenience:
\[
\begin{pmatrix}
Q^1(\text{Out},\text{Re}) \\
\gamma
\end{pmatrix}
+ \begin{pmatrix}
Q^1(\text{Out},\text{Out}) \\
\alpha
\end{pmatrix}
= \begin{pmatrix}
-j a \\
-j b
\end{pmatrix}
\]
(2.29)

\[
\begin{pmatrix}
\gamma \\
\delta
\end{pmatrix}
= j \begin{pmatrix}
D & 0 \\
0 & D
\end{pmatrix}
\begin{pmatrix}
Q^2(\text{Out},\text{Re}) \\
\beta
\end{pmatrix}
\]
(2.30)

\[
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
= -j \begin{pmatrix}
D & 0 \\
0 & D
\end{pmatrix}
\begin{pmatrix}
Q^2(\text{Re},\text{Re}) \\
c
\end{pmatrix}
\]
(2.31)

\[
\begin{pmatrix}
f \\
g
\end{pmatrix}
= -\frac{j}{4} \begin{pmatrix}
Q^1(\text{Re},\text{Re}) \\
\gamma
\end{pmatrix}
-\frac{j}{4} \begin{pmatrix}
Q^1(\text{Re},\text{Out}) \\
\beta
\end{pmatrix}
\]
(2.32)

From the above, the scattered field coefficients \(f\) and \(g\) can be directly related to the incident field expansion coefficients \(a\) and \(b\) by eliminating \(\alpha\), \(\beta\), \(\gamma\), \(\delta\). The result is

\[
\begin{pmatrix}
f \\
g
\end{pmatrix}
= \left[ Q^1(\text{Re},\text{Re}) - Q^1(\text{Re},\text{Out})T(2) \right]
\left[ Q^1(\text{Out},\text{Re}) - Q^1(\text{Out},\text{Out})T(2) \right]^{-1}
\begin{pmatrix}
a_* \\
b_*
\end{pmatrix}
\]
(2.33)

where \(T(2) = \left[ Q^2(\text{Re},\text{Re}) \right] \left[ Q^2(\text{Out},\text{Re}) \right]^{-1}\) and \(a_*, b_*\) are modified incident field coefficients, \(\begin{pmatrix} a \\ b \end{pmatrix} = -4 \begin{pmatrix} a_* \\ b_* \end{pmatrix}\). (2.33) can be expressed in compact form as

\[
\begin{pmatrix}
f \\
g
\end{pmatrix}
= \begin{pmatrix} T(1,2) \end{pmatrix}
\begin{pmatrix}
a_* \\
b_*
\end{pmatrix}
\]
(2.34)

where \(T(1,2)\) is the transition matrix of the composite body.
2.7. **The Transition Matrix**

The transition matrix explicitly relates the scattered field coefficients to the incident field coefficients without directly evaluating the surface currents. If there is no inner body present, $T(2) = 0$ and (2.33) reduces to the form applicable to scattering from a single, homogenous scatterer:

\[
\begin{bmatrix}
  f \\
  g
\end{bmatrix} = \left[ Q^1(Re,Re) \right] \left[ Q^1(Out,Re) \right]^{-1} \begin{bmatrix} a^* \\ b^* \end{bmatrix} \tag{2.35}
\]

or

\[
\begin{bmatrix}
  f \\
  g
\end{bmatrix} = \left[ T(1) \right] \begin{bmatrix} a^* \\ b^* \end{bmatrix} \tag{2.36}
\]

where $T(1)$ is the transition matrix for a single scatterer. $T(2)$ can be interpreted as the transition matrix of the imbedded body (either a perfect conductor or a dielectric) immersed in an infinite, homogeneous medium characterized by $\varepsilon_r, \mu_0$.

(2.36) is in the form given by Barber and Yeh (1975). (2.34) was first derived by Peterson and Ström (1974) using the Poincaré-Huygens principle instead of the conceptually simpler equivalence approach used here. Their development requires detailed reasons for using the "outgoing" Green's function $g(k_1 R)$ in (2.16) instead of the general Green's function

\[
G(k_1 R) = \lambda_1 g(k_1 R) + \lambda_2 \overline{g}(k_1 R), \quad \lambda_1 + \lambda_2 = 1
\]

where $\overline{g}$ (bar denotes complex conjugate) is an "ingoing" Green's function.

(2.16) is a key step in the T-matrix formulation and the procedure used here has the further advantage that the choice of the proper Green's
function is obvious and needs no explanation. Additionally, the notation and field expansions [e.g., see (2.11)] used here yields a more systematic and concise approach to the T-matrix formulation than Peterson and Ström's and to greater facility for subsequent computer programming operations. Peterson and Ström have also shown that a recursive relation for the T-matrix of a multilayered scatterer, consisting of an arbitrary number of consecutively enclosing surfaces, can be obtained.

The transition matrix of the composite body can be expressed as

\[ T(1,2) = \begin{bmatrix} Q^1(\text{Re, Re}) - Q^1(\text{Re, Out})T(2) \\ Q^1(\text{Out, Re}) - Q^1(\text{Out, Out})T(2) \end{bmatrix}^{-1} \]

or

\[ T(1,2) = \left[ T(1) - Q^1(\text{Re, Out})T(2) \right] \left[ Q^1(\text{Out, Re})^{-1} \right] \times \left[ I + Q^1(\text{Out, Out})T(2)Q^1(\text{Out, Re})^{-1} \right]^{-1} \]

where \( I \) is the identity matrix.

A formal expansion of the inverse in (2.38) can be interpreted as various multiple scattering contributions to the total \( T(1,2) \)-matrix. The first three terms in the expansion of (2.38) are shown in Fig. 2.5. The first term \( T(1) \) corresponds to scattering from a homogenous dielectric body enclosed by \( S_1 \). The second contribution \( Q^1T(2)Q^1^{-1} \) isolates the scattering from the imbedded body. Successive terms represent higher order interactions or multiple scattering components.
at the inner and outer surfaces. This interpretation of the scattering components of $T(1,2)$ is useful for testing the accuracy and applicability of geometrical optics-type theories of scattering to imbedded bodies.

2.8. Theoretical Aspects

Since Waterman's (1965) first paper on scattering by axisymmetric perfect conductors was published, a number of theoretical questions have been raised by several researchers [e.g., Burrows (1969), Millar (1969, 1973), Bates (1969, 1975), Hizal and Marincic (1970)] on such areas as the validity of the Rayleigh hypothesis and completeness of spherical vector wave functions on a non-spherical surface. Therefore,
a number of questions concerning the validity of expansions of fields and surface currents on a spherical basis are appropriately discussed here.

Consider the problem of scattering from a single perfect conductor using the extended integral equation method. A key step in this formulation is the expansion of the surface current in vector spherical harmonics (see Section 2.5.1) which are assumed to be a valid representation over the non-spherical surface. Waterman (1971) has shown that such a representation is complete and approximates the surface current in a mean square sense. In addition, Millar (1973) has discussed the completeness of such sets of functions and given rigorous proofs for the case of cylindrical harmonics. On the other hand, Hizal and Marincic (1970) claim that the Rayleigh hypothesis has been invoked by Waterman (1965). The Rayleigh hypothesis [e.g., see Millar (1969)] assumes that an expansion for the scattered field of the type given in (2.26) is valid at all points exterior to the scatterer; a rigorous region of validity, however, is restricted to all points exterior to the smallest sphere circumscribing the body about the origin (e.g., outside the sphere $r_2$ in Fig. 2.1). Hizal and Marincic claim that the near-zone fields (region between surface $S_1$ and $r_2$ in Fig. 2.1) and hence the surface current has to be expanded using both "regular" and "outgoing" type functions in contrast to (2.18). If in fact the expansion in (2.18) is complete for the representation of the surface current, Waterman's method is exact since the near-zone regions are completely avoided. The completeness proofs have been given by Waterman (1969a) for the (scalar) acoustic case and a similar point
has been discussed by Bates (1975). Bates states that any complete set of basis functions is satisfactory for expressing the surface current. However, if an expansion of the type (2.26) does not converge at all points exterior to the body, the near-zone boundary conditions cannot be invoked (e.g., as in the case of point matching). Such near-zone boundary conditions are not invoked in Waterman's formulation.

The above discussion forms the justification for using an expansion as in (2.18) for the surface current on the imbedded scatterer. Similar reasoning applies to the case of an imbedded dielectric. In this case Waterman (1969) has shown that the assumption of (2.19) is not necessary and that the expansions of (2.21) for the electric and magnetic surface currents can be written down directly. A direct result of expanding the surface currents on the imbedded body in this manner gives rise to the recursive nature of the $T(1,2)$-matrix as in (2.37) where $T(2)$ is the transition matrix of the imbedded body.

It should be recognized that the near-zone fields and hence the surface currents can be rigorously expanded using both "regular" and "outgoing" wave functions and a solution generated using Waterman's method for both homogenous and imbedded bodies [Hizal and Marincic (1970), Hizal and Yasa (1973), Aydin (1976)]. Such solutions require substantially larger matrix sizes and do not yield a simple recursive relation for the $T(1,2)$-matrix. Their utility enables near-zone fields to be calculated and may provide improved functional representation of the surface currents.
CHAPTER III
NUMERICAL SOLUTION AND SAMPLE COMPUTATIONS

Waterman (1971) has developed computer programs to calculate the T(1)-matrix for axisymmetric perfect conductors and lossless dielectrics. Barber and Yeh (1974) have similarly developed computer programs to calculate the T(1)-matrix for an arbitrarily shaped, lossy dielectric body. This basic program was provided to us by Dr. Peter Barber and has been modified to evaluate the full T(1,2)-matrix for both axisymmetric perfect conductors or dielectrics imbedded within another dielectric body. Once the T-matrix has been calculated for a specific scatterer configuration, the far-zone fields and scattering cross-sections are readily evaluated. Considerable simplifications were achieved by assuming that the composite body be rotationally symmetric (i.e., the radius vector r(θ,ϕ) describing surfaces S₁ or S₂ be ϕ-independent, ϕ being the azimuthal angle). This restriction is not necessary, but was made because of practical limitations on computer resources.

3.1. Outline of the Numerical Solution

3.1.1 The T(2)-Matrix

The basic step in the numerical solution is the computation of the T(1,2)-matrix for a specific, imbedded scatterer using (2.37)
repeated here for reference:

\[
T(1,2) = \left[ Q^1(Re,Re) - Q^1(Re,Out)T(2) \right] \left[ Q^1(Out,Re) - Q^1(Out,Out)T(2) \right]^{-1}
\]  (3.1)

Recall that an exact solution to the theoretical scattering problem requires the solution of an infinite set of equations; in other words, the various matrix components of \( T(1,2) \) are all infinite dimensional matrices. An approximate solution to the scattering problem requires truncation of the various matrices involved. This truncation is evident when the expansions of the various fields on a spherical basis are examined [see for example (2.11) or (2.21)]. From the results of the previous chapter it is readily seen that each component matrix in (3.1) is a square matrix of dimension \( 2N \times 2N \). The scattering results will be obtained numerically by solving the complete set of equations for successive values of \( N \) until the final results (the far-zone amplitudes or scattering cross-sections) converge to a specified accuracy. This will ensure that enough of the expansion terms have been retained to guarantee the correct final result.

The first step involves the calculations of the \( T(2) \)-matrix which involves the computation of two matrices \( Q^2(Re,Re) \) and \( Q^2(Out,Re) \):

\[
T(2) = \left[ Q^2(Re,Re) \right] \left[ Q^2(Out,Re) \right]^{-1}
\]  (3.2)

\( T(2) \) has been interpreted as the transition matrix of the imbedded body (perfect conductor or dielectric) when immersed in an infinite, homogenous medium characterized by \( \varepsilon_1, \mu_0 \). The structure of the
$Q^2$-matrix has been examined in great detail by Waterman (1971) for perfect conductors and by Barber and Yeh (1974) for dielectrics. Hence, these results are not repeated here. The $Q^2$-matrix depends on the shape, size and dielectric constant of the imbedded body as well as the nature of the surrounding medium. The $Q^2(\text{Out,Re})$ matrix for an imbedded dielectric is given by

$$Q^2(\text{Out,Re}) = \left\{ \begin{array}{ccc}
K_{12}^{12} + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} & J_{12}^{12} & L_{12}^{12} + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \\
I_{12}^{12} + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} & J_{12}^{12} & K_{12}^{12}
\end{array} \right\}$$

(3.3)

where

$$I_{12}^{12} = \frac{k_1^2}{\nu q} \int_{S_2} \hat{n}_2 \cdot \hat{M}(k_1 \mathbf{x}') \times \hat{M}(k_2 \mathbf{x}') \, dS$$

(3.4)

The surface integral in (3.4) reduces to an integration over $\theta$ (the elevation angle in spherical coordinates) since the radius vector $r(\theta)$ is independent of the azimuthal angle $\phi$. This represents substantial savings in computer time since only integration over $\theta$ is required. Additionally, the number of matrix elements are reduced significantly due to many of the integrals vanishing over axisymmetric integrations. Computation of the $I_{12}^{12}$, $J_{12}^{12}$, $K_{12}^{12}$, $L_{12}^{12}$ matrices and their combination to generate $Q^2(\text{Out,Re})$ follows directly. Inversion of $Q^2(\text{Out,Re})$ followed by multiplication with $Q^2(\text{Re,Re})$ as in (3.2) gives the $T(2)$-matrix.
3.1.2 The $Q^1$-Matrices and $T(1,2)$

The $Q^1$-matrices for the outer surface are now calculated. These are four in number, $Q^1(Re,Re)$, $Q^1(Out,Re)$, $Q^1(Re,Out)$ and $Q^1(Out,Out)$. These matrices depend on the shape of the outer body, its size and dielectric constant. Integration is over surface $S_1$ and rotational symmetry forces integration over the $\hat{s}$-coordinate alone.

All four matrices are calculated together within the program according to (2.14) and (2.15); interaction with $T(2)$ takes place according to (3.1) and the resulting $T(1,2)$ is properly formulated. Due to the axisymmetric scatterer configuration, it is possible to obtain the solution for both orthogonal polarizations (viz., $\hat{e}_\parallel$ and $\hat{e}_\perp$, see Fig. 3.1) from the same transition matrix.

Fig. 3.1. Scattering geometry showing the principal XZ and XY planes. Incident field polarizations $\hat{e}_\parallel$ and $\hat{e}_\perp$ are parallel and perpendicular to the principal planes respectively.
3.1.3 The Scattered Field

The T-matrix operates on the incident field coefficients $(a_*, b_*)$ to generate the scattered field coefficients $(f, g)$:

$$
\begin{bmatrix}
  f \\
  g
\end{bmatrix} =
\begin{bmatrix}
  T(1,2)
\end{bmatrix}
\begin{bmatrix}
  a_* \\
  b_*
\end{bmatrix}
$$

(3.5)

The scattered field is reconstructed according to

$$
E_s = \sum_{\nu=1}^{N} 4D_\nu \left[ f_{\nu} M_\nu(kr) + g_{\nu} N_\nu(kr) \right]
$$

(3.6)

The vector far-zone amplitude of the scattered field is defined by

$$
E_s(kr) = \frac{F(\theta_s, \phi_s | \theta_1^i, \phi_1^i) \exp(jkr)}{r}, \ r \to \infty
$$

(3.7)

where $F(\theta_s, \phi_s | \theta_1^i, \phi_1^i)$ is the far-field amplitude in the $(\theta_s, \phi_s)$ direction due to an incident field in the $(\theta_1^i, \phi_1^i)$ direction. The scattering cross-section for unit incident field is given by

$$
\sigma(\theta_s, \phi_s | \theta_1^i, \phi_1^i) = 4\pi |F(\theta_s, \phi_s | \theta_1^i, \phi_1^i)|^2
$$

(3.8)

The back-scatter or radar cross-section is computed by setting $(\theta_s, \phi_s) = (\theta_1^i, \phi_1^i)$.

This formulation conveniently yields the monostatic and bistatic scattered fields over two principal planes, the XZ- and XY-planes as shown in Fig. 3.1. For evaluation over the XZ-plane, the incident field is assumed to propagate along the positive Z axis and the bistatic
scattered field is calculated for elevation angle $\theta$ varying from 0 to $180^\circ$. The angle of incidence of the incident field may be changed in the XZ-plane as shown in Fig. 3.1 and the monostatic or back-scattering cross-section calculated for each angle of incidence (0 through $180^\circ$).

Two incident field polarizations are also shown in Fig. 3.1. $\hat{e}_L$ and $\hat{e}_T$ are unit vectors characterizing polarizations of the incident field perpendicular and parallel to the plane of incidence (i.e., the XZ-plane), respectively. For composite bodies having rotational symmetry about the Z-axis, depolarization of the scattered field will not occur for this geometry. For evaluation over the XY-plane, the incident field is assumed to propagate along the positive X-axis and the monostatic and bistatic scattered fields may be calculated for all angles in the XY-plane.

3.2. **The Computer Program**

Two separate computer programs have been written in FORTRAN to solve the scattering problem for (a) a perfect conductor imbedded in a dielectric and (b) a dielectric imbedded within another dielectric. Both programs have been extensively run on The Ohio State University's (OSU) IBM370/168 computer and on the National Center for Atmospheric Research (NCAR) CDC 7600/6600 computers. A block diagram of the program is given in Fig. 3.2 showing main routines only. A brief description of these routines follows.

**RDDATA** - This routine reads the input data. This consists of (a) the relative size and dielectric constant of both imbedded and outer bodies, (b) the matrix size (2N) and
Fig. 3.2. Block diagram of the computer program showing main routines only.

integration parameters, and (c) any special body-shape parameters.

MAIN - This routine executes proper computation of the four $Q^1$-matrices and two $Q^2$-matrices. The $T(2)$-matrix is first calculated followed by evaluation of the $T(1,2)$-matrix according to (3.1).

GENKR - Generates $r$, the distance from the origin to surfaces $S_1$ or $S_2$, and $\frac{dr}{d\theta}$ (derivative with respect to $\theta$). This routine defines the axisymmetric shape of the scatterer and is custom written for different shapes.

QUAD - This routine (in conjunction with GENER) calculates the elements of the six $Q$-matrices using an extended Gauss-Legendre integration procedure. Previous programs
by Waterman (1971) and Barber and Yeh (1974) have used a fourth order Newton-Cotes type integration formula. From previous experience with both methods, the Gauss-Legendre scheme has proven to be considerably more stable and accurate and more efficient in the use of computer time.

GENER - The integrand of each of the Q-matrix elements consists of a complicated combination of various spherical Bessel and Hankel functions and associated Legendre polynomials. This routine formulates these various combinations for use within QUAD.

GENBKRE - These routines generate the various spherical Bessel, Hankel and Neumann functions of both real and complex arguments for all orders from zero through N. GENBKRE uses Miller's algorithm for spherical Bessel function of complex arguments (see Handbook of Mathematical Functions edited by Abramowitz and Stegun) and GENBSL uses downward recursion to generate Bessel functions of real arguments.

GENLGP - Generates the associated Legendre functions \( P_n^m(\cos \theta) \). The angle \( \theta \) and the maximum indices \( m, n \) are input. For a given value of \( m \) all the Legendre functions \( P_n^m \) through \( P_n^m \) are generated using a recursive procedure.

PRCSSM - Performs matrix inversion using a Gauss-Jordan elimination scheme with iteration, if necessary.

ADDPRC - Computes the scattered field expansion coefficients by multiplying the Transition matrix with the incident field expansion coefficient. This is followed by calculation of the far-zone field amplitude and scattering cross-section.
OUTPUT - The T(1,2)-matrix, the complex far-zone scattered field amplitudes and back-scattering cross-sections at orthogonal polarizations are printed on output.

3.3. Sample Computations and Comparisons

3.3.1 Water-Coated Ice Sphere

Sample computations were first compared with extended Mie theory calculations of Battan, et al. (1970) for a water-coated ice sphere. The body configuration and relevant details are given in Fig. 3.3 and Table 3.1. Excellent agreement with Mie theory is obtained. The concentric, spherical configuration gives rise to the simplest form for the T-matrix, since the matrix becomes fully diagonal. For this case the T-matrix is of size 12 x 12. Execution time was about 30 seconds on the IBM 370/168 using double-precision arithmetic.

\[
\begin{align*}
\varepsilon_0/\mu_0/\lambda = 10\text{ cm} \\
2\pi r_1/\lambda = 1.885 \\
2\pi r_2/\lambda = 2.513 \\
\varepsilon_{\text{ice}} = 3.168 + j(0.00855) \\
\varepsilon_{\text{water}} = 78.646 + j(26.509)
\end{align*}
\]

Fig. 3.3. Concentric spherical scatterer composed of ice surrounded by water.
### TABLE 3.1

A Comparison of Extended Mie and T-matrix Methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Normalized Radar Cross-Section ($\sigma/\pi r_z^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Battan, et al.</td>
<td>0.89407</td>
</tr>
<tr>
<td>T(1,2)-matrix</td>
<td>0.894068</td>
</tr>
</tbody>
</table>

3.3.2 **Dielectric Clad Spherical Perfect Conductor**

Comparisons were next made with boundary value solutions obtained by Swarner and Peters (1963) for a concentric spherical perfect conductor surrounded by a spherical dielectric as shown in Fig. 3.4. Fig. 3.5 illustrates their calculated radar cross-sections.

![Fig. 3.4. Concentric dielectric-clad perfect conductor. Radar cross-section is calculated as a function of $r_2/\lambda$.](image)

\[ \epsilon_0, \mu_0 \]

\[ \epsilon_r, \mu_0 \]

\[ \epsilon_r = 0.75 \]

\[ \frac{r_1}{\lambda} = 0.05 \]
as a function of $r_2/\lambda$, where $r_2$ is the radius of the outer sphere.

The T-matrix calculations are also given in Fig. 3.5. Excellent agreement is obtained, particularly the location of the deep null at $r_2/\lambda = 0.134$. The maximum T-matrix size required for the $r_2 = 0.4\lambda$ case was 12 x 12. The total execution time required to generate the curve in Fig. 3.5 was about eight minutes, since each of the thirteen cases had to be run separately.

3.3.3 Non-concentric Dielectric Clad Perfect Conductor

Finally, comparisons were made with calculations and measurements made by Swarner and Peters on the non-concentric dielectric clad
perfect conductor shown in Fig. 3.6. Their calculations were based on a superposition approximation where the radar cross-section is obtained by considering the scattered field to be the phasor sum of two principal components, (a) the field scattered by the air-dielectric interface and (b) the field scattered by an equivalent conducting body which differs from the actual body because of the lens action of the dielectric shell.

![Diagram](image)

**Fig. 3.6.** Significant parameters for the non-concentric dielectric-clad spherical perfect conductor. The imbedded sphere is a perfect conductor whose angular orientation is $\alpha^\circ$. Parallel polarization is shown.

**Fig. 3.7** shows the T-matrix calculations of back-scattering cross-section as a function of the angular orientation, $\alpha$, of the inner conductor. Both parallel and perpendicular polarizations are shown assuming $\varepsilon_r = 1.8$; a slight dependency on polarization is observed.
as expected. Fig. 3.8 shows the calculations of Swarner and Peters (1963) together with the T-matrix calculations for $\varepsilon_r = 1.74$ (value chosen for a good fit to the measured null positions). The effect of adding a slight loss to the dielectric constant is also shown.

---

**Fig. 3.7.** Theoretical T-matrix echo areas versus angular orientation, $\alpha^0$, for the non-concentric dielectric-clad spherical perfect conductor shown in Fig. 3.6.

Again, in Fig. 3.9 T-matrix calculations are compared with measurements after matching $\varepsilon_r$ for an overall "best" fit. Using $\varepsilon_r = 1.7+j(0.0085)$ and considering that the measurements are probably at best accurate
Fig. 3.8. Comparison between theoretical and experimental echo areas versus angular orientation, $\alpha^\circ$, for the body configuration of Fig. 3.6.

Fig. 3.9. T-matrix calculations compared with experimental measurements for the body configuration shown in Fig. 3.6. Data points for measurements are taken from Swarner and Peters (1963), see Fig. 3.8.
to within about ±1.0dB, very good agreement with T-matrix calculations is obtained. It is noted that Swarner and Peters also obtained excellent agreement between theory and experiment. The size of the T-matrix for this case was 20 x 20. An execution time of about 13 minutes was required to generate the results shown in Fig. 3.9.

Using the T-matrix, the separate scattering components were isolated using (2.38). For simplicity only the air-dielectric component, \( T(1) \) and the air-dielectric-conductor component, \( Q^1 T(2) \left[ Q^1 \right]^{-1} \) were computed from which the component far-zone backscattered amplitudes were derived for \( \alpha = 0^\circ \). Table 3.2 shows the results of this calculation. It is seen that the inner conductor contributes significantly to the overall scattering amplitude. Also, the higher order ray contributions (see Fig. 2.5) are not insignificant. The calculations of Swarner and Peters represent an approximation to the first two

<table>
<thead>
<tr>
<th>Matrix Used</th>
<th>Scattered Far-Zone Amplitudes for Unit Plane Wave Incidence and ( \alpha = 0^\circ )</th>
<th>Normalized Scattering Cross-Sections dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T(1,2) )</td>
<td>(-0.10955 + j0.5604)</td>
<td>(-4.867)</td>
</tr>
<tr>
<td>( T(1) )</td>
<td>(0.4051 + j0.3296)</td>
<td>(-5.642)</td>
</tr>
<tr>
<td>( Q^1 T(2) \left[ Q^1 \right]^{-1} )</td>
<td>(0.6311 - j0.1605)</td>
<td>(-3.726)</td>
</tr>
<tr>
<td>( T(1) - Q^1 T(2) \left[ Q^1 \right]^{-1} )</td>
<td>(-0.225 + j0.491)</td>
<td>(-5.351)</td>
</tr>
</tbody>
</table>
scattering components illustrated in Fig. 2.5. A comparison between the actual scattering cross-section obtained from $T(1,2)$, and the approximation, obtained from $T(1) - Q^1 T(2) [Q^1]^{-1}$, indicates a difference of about 0.5 dB at $\alpha = 0^0$. This demonstrates the usefulness of comparing the transition matrix with its term by term expansion for testing proposed methods of calculating scattering cross-sections.
CHAPTER IV
APPLICATIONS TO RADAR METEOROLOGY

4.1. Basis for Applications

The problem of electromagnetic scattering from hydrometeors is of fundamental importance in radar meteorology. Knowledge of back-scattering or radar cross-sections of many of the common hydrometeors such as raindrops and hailstones is essential in order to derive quantitative information (e.g., hydrometeor size distributions) from radar measurements. An important application is in the estimation of rainfall rate or liquid water content using radar measurements, e.g., see Atlas and Ulbrich (1974), Eccles and Mueller (1971), Smith, et al. (1974). A related area is the propagation of microwaves through rainfilled media which causes rain-induced attenuation and phase shift. Such phenomena have been considered by Morrison, Cross and Chu (1973), Oguchi and Hosoya (1974) and Oguchi (1975). In addition, the polarization characteristics of hydrometeors have recently been recognized as being important. The Stormy Weather Group of McGill University, The National Research Council of Canada and the Alberta Hail Studies have pioneered in the practical application of polarization effects; these have been reported on by Hendry and McCormick (1972), McCormick, et al. (1972), McCormick and Hendry (1974) and Humphries (1974). Dual-wavelength radar techniques are being used by the CHILL (University of
Chicago and Illinois State Water Survey) group and the National Hail Research Experiment (NHRE) for hail detection and related weather modification studies, see, for example, Srivastava and Jameson (1975), Eccles and Atlas (1973) and Eccles (1976).

In every application, a fundamental quantity is the radar cross-section of the hydrometeor. Meteorological radars are commonly operated at wavelengths of 10cm (S-band), 5cm (C-band) or 3cm (X-band). At these wavelengths, the sizes of the hydrometeors are such that they fall in the near-Rayleigh and Mie regions of electromagnetic scattering. These conditions usually imply a strong interaction between the incident wave and the scatterer and the shape of the particle becomes very important in determining its scattering properties.

It is commonly assumed that hydrometeors are spherical in shape. As a consequence, extensive use of the Rayleigh and Mie theories for spheres and extensions to concentric spherical multi-layered scatterers have been made to calculate scattering cross-sections. In general, however, hydrometeors are far from spherical. For example, raindrops can frequently be modelled as oblate spheroids, Pruppacher and Beard (1970), and hailstones can assume a variety of shapes (e.g., spheroidal, conical) with either smooth or rough surfaces. In addition, hail can be dry or wet, i.e., the stones may acquire a thin coating of water. Spongy hail which is a mixture of ice and unfrozen water and layered hailstones occur often as well. For a comprehensive review of the physical characteristics of hail see Gokhale (1975).
4.2. The Radar Equation and Scattering Cross-section

The basic equation relating radar measurements to hydrometeor characteristics is the radar equation [Probert-Jones (1962)] which is given by

\[
\langle P(r) \rangle = \left( \frac{C^*}{r^2} \right) \frac{1}{\text{Pulse Volume}} \int \sigma(D/\lambda, m) N(D) dD
\]

(4.1)

\( \langle P(r) \rangle \) is the average backscattered power from an ensemble of hydrometeors in a pulse volume centered at range \( r \). \( C^* \) takes into account the radar constants (i.e., antenna gain, beamwidths, etc.), \( \sigma(D/\lambda, m) \) is the radar cross-section of a hydrometeor of relative size \( D/\lambda \) and refractive index \( m \), and \( N(D) \) is the hydrometeor size-distribution (\( D \) being a representative size parameter of the scatterer). If the scatterers are spherical with diameter \( D \) and Rayleigh \( (D/\lambda << 1) \), \( \sigma(D/\lambda, m) \) reduces to, Stratton (1941):

\[
\sigma(D/\lambda, m) = \left( \frac{\pi^5}{\lambda^4} \right) \left| \frac{(m^2-1)}{(m^2+2)} \right| D^6
\]

(4.2)

The radar reflectivity factor, \( Z \), is defined as

\[
Z = \int_0^{\infty} D^6 N(D) \ dD
\]

(4.3)

If the scatterers fall in the resonance region, conventional Mie theory is used to establish \( \sigma(D/\lambda, m) \), see, for example, Battan, et al. (1970). Difficulties in applying the radar equation arise when the hydrometeors are non-spherical, since even small perturbations from sphericity may significantly alter \( \sigma \). For spheroidal shapes and small size \( (D/\lambda << 1) \), Gans' (1912) theory, which is outlined by Van de Hulst (1957), can be
used. This method is a useful approximation for oblate spheroidal raindrops at S-band as demonstrated later in Section 4.4.1 and was used by Atlas, et al. (1953) and Seliga and Bringi (1976) to calculate cross-sections of hydrometeors for radar meteorological applications. The transition or T-matrix technique developed here and given in Chapter 2 provides a more general approach for treating non-spherical, axisymmetric hydrometeors. Thus, $\sigma(D/\lambda, m)$ can be calculated for (a) raindrops of either spheroidal or other more complex shapes, (b) hailstones of axisymmetric shape and (c) axisymmetric hailstones covered with a layer of water. The T-matrix approach is realistically applicable to scatterers of sizes $D \sim 3\lambda$ (where $\lambda$ is the free-space wavelength) and for a wide range of dielectric constants and is used here to compute $\sigma(D/\lambda, m)$ for a number of hydrometeor shapes and sizes. The results are presented in this chapter for (a) raindrops modelled as oblate spheroids with vertically oriented rotational axis; T-matrix calculations are compared to values obtained from Gans' theory (1912), (b) oblate spheroidal ice-stones, and (c) dry and wet ice-stones with surface perturbations to model roughness. In addition, the polarization characteristics of such non-spherical hydrometeors are examined and potential applications of these to a radar measurement technique based on the concept of differential reflectivity [Seliga and Bringi (1976)] are considered.

4.3. Limitations

Before proceeding further, certain limitations must be noted. This refers to the fact that hydrometeors, in general, form an ensemble of non-spherical scatterers with a wide range of shapes, sizes,
orientations and size distributions which fill the radar pulse volume. As a result, the integration in (4.1) is non-trivial necessitating knowledge of the joint probability distributions of scatterer sizes, shapes and orientations. The techniques developed here form part of the overall problem and care must be taken in interpreting any scattering computations. However, considerable simplifications are often applicable. For example, all the scatterers in a pulse volume may be indentically shaped and oriented with an exponential drop size distribution. Such assumptions are consistent with many, but not all, observations. Consequently, the interpretations of computations presented here are restricted to the stated assumed conditions.

4.4. **Oblate Spheroidal Raindrops**

In this Section the scattering properties of oblate spheroidal raindrops are examined and results interpreted for possible radar meteorological purposes. Raindrops falling at terminal velocity are non-spherical. Purppacher and Beard (1970) have shown that drop sizes up to an equivalent diameter of 3.0mm may be assumed to be distorted into oblate spheroids. Larger drops assume a more complex shape with the raindrop base becoming more flattened and in some cases (~0.9cm dia) developing a concave depression. Warner and Hizal (1975) found that (using the T-matrix approach) the complex shapes of actual raindrops as formulated by Pruppacher and Beard produce cross-sections not very different from their equivalent oblate spheroids. Extensive scattering calculations have been performed by Oguchi (1973a, 1975) on oblate spheroidal raindrops using (a) the perturbation technique, (b) the
point matching technique and (c) spheroidal function expansions. A comprehensive review of the different scattering methods and calculations has been presented by Oguchi (1975); however, the T-matrix method has not been included. Peterson (1976) has compared T-matrix calculations with those of Oguchi (1973a,b) and Morrison and Cross (1974).

4.4.1 Gans' Theory

Let $\sigma_H$ and $\sigma_V$ be the horizontal and vertical radar cross-sections of the oblate spheroid ($\sigma_H$ refers to the radar cross-section at horizontal polarization, $\sigma_V$ at vertical polarization). For a horizontally polarized incident wave, the scattered field is due to the induction of an electric dipole aligned along the major axis of the oblate spheroid (see Fig. 4.1). Similarly, a vertically polarized incident wave induces an electric dipole along the minor axis of the oblate spheroid.

Fig. 4.1. An oblate spheroid is the body of revolution formed when an ellipse is rotated about its minor axis.
Let $E_{HO}$ and $E_{VO}$ be the complex, plane wave electric field intensities incident on the oblate spheroid due to transmission of horizontal and vertical polarized waves, respectively. The electric dipole moments induced in the oblate spheroid are given by

$$P_H = 4\pi \varepsilon_0 g' E_{HO}$$

$$P_V = 4\pi \varepsilon_0 g E_{VO}$$

where $P_H$ and $P_V$ are the dipole moments due to horizontal and vertical polarized waves, respectively. The factors $g$ and $g'$ are given in Van de Hulst (1957):

$$g = V(m^2 - 1)/(4\pi + (m^2 - 1)P)$$

$$g' = V(m^2 - 1)/(4\pi + (m^2 - 1)P')$$

where $P$ and $P'$ are geometrical factors defined below by (4.6) for oblate spheroids and $V$ is the volume:

$$P = 4\pi - 2P' = (4\pi/e^2) \left[ 1 - \{(a/b)/e\} \sin^{-1}e \right]$$

Here $e$ is the eccentricity of the principal elliptical cross-section [$e^2 = 1 - (a/b)^2$], and $a$ and $b$ are defined in Fig. 4.1. Using the above simple expressions for $\sigma_H$ and $\sigma_V$ result:

$$\sigma_H = (16/9)(\pi^7/\lambda^4)(D_{eq}^6) | (m^2 - 1)/(4\pi + (m^2 - 1)P')|^2$$

$$\sigma_V = (16/9)(\pi^7/\lambda^4)(D_{eq}^6) | (m^2 - 1)/(4\pi + (m^2 - 1)P')|^2$$

where $D_{eq}$ is the diameter of an equivolumic spherical drop.
The similarity of (4.7) to the Rayleigh cross-section in (4.2) is evident. Either (4.7a) or (4.7b) reduces to (4.2) using $P = P' = 4\pi/3$ and $D_{eq} = D$.

Back-scattering calculations of $\sigma_H$ and $\sigma_V$ were performed using Gans' theory and the T-matrix approach for oblate spheroidal raindrops. Fig. 4.2. and 4.3 show model calculations of $\sigma_H$ and $\sigma_V$ where $D_{eq}$ is the diameter of an equivolumic spherical drop. The distortion ratio, $a/b$, is a function of $D_{eq}$ and is given by $a/b = 1.03 - 0.62 D_{eq}$ for $0.1 < D_{eq} < 1.0$ cm, Pruppacher and Beard (1970). The refractive index of water at 10 cm wavelength and $\theta$ was $n = 8.99 + j1.475$, Ray (1972). The results indicate excellent agreement between Gans' theory and the T-matrix solution for $D_{eq} < 0.6$ cm. This indicates that Gans' theory can be used with confidence to calculate radar cross-sections of oblate spheroidal raindrops at S-band.

4.4.2 The Differential Reflectivity Concept

The average backscattered power from a common pulse volume centered at a range $r$, due to horizontal or vertical polarized incident waves, may be expressed, respectively, by

$$<P_H(r)> = \frac{C}{r^2} \left[ \int_{D_{eq}} \sigma_H(D_{eq}) N(D_{eq}) dD_{eq} \right]$$

$$<P_V(r)> = \frac{C}{r^2} \left[ \int_{D_{eq}} \sigma_V(D_{eq}) N(D_{eq}) dD_{eq} \right]$$

Integration is carried out over the pulse volume assuming (a) $N(D_{eq}) = N_0 \exp(-3.67 D_{eq}/D_0)$, where $N_0$ and $D_0$ are parameters describing the magnitude and median volume diameter of the raindrop size distribution, respectively, [Marshall and Palmer (1948)] and (b) the oblate
Fig. 4.2. Horizontal radar cross-section, $\sigma_H$, for oblate spheroidal raindrops versus equivolumic diameter, $D_{eq}$.

Fig. 4.3. Vertical radar cross-section, $\sigma_V$, for oblate spheroidal raindrops versus equivolumic diameter, $D_{eq}$. 
spheroidal drops fall with a vertically oriented minor axis (i.e., zero canting angle). This latter assumption is consistent with Humphries (1974) observations of microwave depolarizations, i.e., raindrops tend to fall as oblate spheroids with their rotational axes vertical. Although Oguchi (1975) found very few direct measurements of in-situ raindrop canting angles, canting angles inferred from propagation measurements tend to support Humphries' conclusions, Watson and Arbabi (1975), McCormick, et al. (1972).

Using (4.7) in (4.8) and defining \( C^* = C(16/9)(\pi^7/\lambda^4) \) gives

\[
\begin{align*}
\langle P_H(r) \rangle &= C^* \frac{Z_H}{r^2} \\ \langle P_V(r) \rangle &= C^* \frac{Z_V}{r^2}
\end{align*}
\]

(4.9a, 4.9b)

where \( Z_H \) and \( Z_V \) are the horizontal and vertical radar reflectivity factors, respectively, given by

\[
\begin{align*}
Z_H &= \int_0^\infty D_6 \left| \frac{m^2 - 1}{4\pi + (m^2-1)p} \right|^2 N(D_{eq}) \, dD_{eq} \\
Z_V &= \int_0^\infty D_6 \left| \frac{m^2 - 1}{4\pi + (m^2-1)p} \right|^2 N(D_{eq}) \, dD_{eq}
\end{align*}
\]

(4.10a, 4.10b)

Differential reflectivity is defined by

\[ Z_{DR} = 10 \log \left( \frac{Z_H}{Z_V} \right) \, \text{dB} \]  

(4.11)

For details concerning many of the assumptions used here and potential applications of \( Z_{DR} \) see Seliga and Bringi (1976). Briefly, \( Z_{DR} \) depends only on \( D_0 \) and is independent of radar constants for equal system
response at both polarizations. Hence, \( Z_{DR} \) should be precisely determined through relative power measurements, thereby yielding \( D_0 \) directly. It also may be combined with absolute power measurements to give \( N_0 \) and, hence, radar derived rainfall rates.

Model calculations were performed using (4.10) to generate \( Z_{DR} \) and compared with calculations of \( Z_{DR} \) using the T(1)-matrix method, \( D_0 \) being the independent parameter. This is shown in Fig. 4.4a at \( \lambda = 10\text{cm} \) and a temperature of \( 10^\circ\text{C} \) corresponding to \( m = 8.99 + j1.0 \). Very slight differences in the T(1)-matrix method and Gans' theory is noted insofar as \( Z_{DR} \) is concerned, supporting the use of Gans' theory to calculate \( Z_{DR} \) vs \( D_0 \) at S-band.

In order to determine the sensitivity of the \( Z_{DR} - D_0 \) relation to water temperature, the T(1)-matrix approach was used to generate two curves for temperatures of \( 0^\circ\text{C} \) and \( 20^\circ\text{C} \) at S-band. This is shown in Fig. 4.4b. The \( Z_{DR} - D_0 \) relationship appears to be relatively insensitive to temperature variations \( (0^\circ\text{C} - 20^\circ\text{C}) \). This property was anticipated, since \( Z_{DR} \) is a ratio of \( Z_H \) and \( Z_V \) which vary alike as temperature (or equivalently, refractive index) changes. This insensitivity of \( Z_{DR} \) to temperature is a very desirable property which removes water temperature as a possible parameter affecting measurement of \( Z_{DR} \).

4.5. Oblate Spheroidal Ice Stones

Consider now back-scattering calculations of oblate spheroidal ice stones of varying sizes and eccentricities using the T(1)-matrix method. The scattering geometry will be explained. The oblate spheroid is located at the origin of an XYZ coordinate system with its minor axis
Fig. 4.4. Curves of differential reflectivity, $Z_{DR}$, as a function of median volume diameter $D_0$. (a) Comparison of T(1)-matrix and Gans' theory, (b) sensitivity of $Z_{DR} - D_0$ relation to temperature using T-matrix method.
coinciding with the Z-axis, see Fig. 3.1. The incident field propagates along the positive Z-axis with polarization $E_\parallel$ and $E_\perp$ and back-scattering cross-sections are calculated over the XZ-plane as the angle of incidence varies from $0^\circ$ through $90^\circ$ in the XZ-plane. At angle of incidence equal to $90^\circ$, $E_\parallel$ is aligned along the Z-axis while $E_\perp$ is aligned along the Y-axis which corresponds to polarizations aligned along the minor and major axes of the oblate spheroid, respectively. Hence at $90^\circ$, $\sigma_\parallel$ corresponds to $\sigma_V$ and $\sigma_\perp$ corresponds to $\sigma_H$. Differential radar cross-section, $\sigma_{DR}$, is then defined (analogous to $Z_{DR}$ which corresponds to the integrated form) by $\sigma_{DR} = 10 \log \frac{\sigma_H}{\sigma_V}$.

Fig. 4.5 shows total back-scatter cross-section (cm$^2$) of five oblate spheroidal ice-stones (marked A, B, C, D, E) as a function of angle of incidence. Each stone has a fixed parameter $k_b = 2\pi b/\lambda = 4.12$ where $b$ is the semi-major axis, see Fig. 4.1. The dielectric constant of ice was taken as $\varepsilon_r = 3.1684 + j(0.00855)$ at $T=0^\circ C$, Battan et al. (1970). The ratio of semi-minor to semi-major axes, $a/b$, is varied for cases A through E ($A-a/b=0.45$, $B-a/b=0.65$, $C-a/b=0.75$, $D-a/b=0.85$, $E-a/b=0.95$). The curves show significant variations as a function of angle of incidence. $\sigma_\parallel$ and $\sigma_\perp$ differ significantly for all cases as soon as the angle of incidence increases beyond about $12^\circ$. At $90^\circ$, $\sigma_{DR}$ for each case was calculated and is shown in Fig. 4.5. It is interesting to note that even case E with $a/b=0.95$ generates a $\sigma_{DR}$ of $-1.36$dB. At $0^\circ$, $\sigma_\parallel$ and $\sigma_\perp$ coincide as they should; this would correspond to a vertically pointing radar situation. The near $90^\circ$ case would correspond to a radar operating at low elevation angles. Fig. 4.6 shows a plot of $\sigma_{DR}$ versus radar elevation angle (which is the complement of the
Fig. 4.5. Total back-scattering cross-sections of oblate ice-stones versus angle of incidence with $a/b$ as parameter.

A - $a/b = 0.45$, B - $a/b = 0.65$, C - $a/b = 0.75$, D - $a/b = 0.85$, E - $a/b = 0.95$.

Fig. 4.6. Differential radar cross-section, $\sigma_{DR}$, versus radar elevation angle for the oblate ice-stones of Fig. 4.5.
angle of incidence). The values of $\sigma_{\text{DR}}$ for cases A, B, and C change slightly for radar elevation angles less than $6^0$ while cases D and E do not produce significant changes out to about $20^0$.

Fig. 4.7 shows plots of normalized back-scattering cross-sections $\sigma_H/\pi b^2$ and $\sigma_V/\pi b^2$ at angle of incidence $90^0$ as a function of $(a/b)$ with $kb$ as parameter. Seven values of $kb$ were chosen, $kb = (1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.12)$, to correspond to a significant size range for the ice-stones. For $kb>1.5$, it is noted that $\sigma_V$ exceeds $\sigma_H$ significantly for any value of $(a/b)$ between 0.45 and 0.95. This would imply negative values of $\sigma_{\text{DR}}$ for the larger stones ($b>0.75$cm at X-band) and for a wide range of typically observed ice-stone oblateness. This result is shown more clearly as a scatter plot of $\sigma_{\text{DR}}$ versus $kb$ with $(a/b)$ as a parameter in Fig. 4.8. A value of $kb=1.5$ divides positive values of $\sigma_{\text{DR}}$ from negative ones. Since the ice-stones fall in the Mie-region, resonance effects cause negative values of $\sigma_{\text{DR}}$ to occur.

Consider now the case when oriented ice-oblates of varying sizes and eccentricities fill a radar pulse volume and $Z_{\text{DR}}$ is measured. Preferred orientations are thought to be consistent with the free-fall behavior of such spheroids. If the hail sizes are such that the contributions to $Z_{\text{DR}}$ from $\sigma_{\text{DR}}$ are dominated by ice-stones in the range where $\sigma_{\text{DR}}$ is negative, then negative $Z_{\text{DR}}$ may uniquely identify the presence of hail. The results of Fig. 4.8 indicate that this is a distinct possibility. This same method was independently suggested by Harper (1962) who based his results on radar cross-section measurements of spheroids made of perspex, the dielectric constant of which is near that of ice. Further examination of this hypothesis requires validation by experiment with radar.
Fig. 4.7. Back-scatter cross-sections at orthogonal polarizations versus (a/b) with $k_b$ as parameter for oblate ice-stones. Cases (a) and (b) are given here.
Fig. 4.7. (cont'd.) Cases (c) and (d) are depicted here.
4.6. Ice-Stones with Surface Perturbations

Experimental observations of naturally occurring hailstones indicate that irregular-shaped stones with surface perturbations or lobes often occur, particularly as they grow to the largest sizes, Gokhale (1975). In this section model ice-stones with surface perturbations are modelled and their radar cross-sections calculated for comparison with equivalent smooth spheres. For the present analysis, two types of models have been chosen as shown in Fig. 4.9. The sinusoidal surface perturbations of the first model in Fig. 4.9a
\[ \frac{r}{a} = 1 + \delta \cos \theta, \quad 0 \leq \theta \leq \pi/2 \]
\[ = \left[ \cos^2 \left( \frac{\theta}{1+\delta} \right) \sin^2 \theta \right]^{-1/2}, \quad \frac{\pi}{2} \leq \theta \leq \pi \]

Fig. 4.9. Models of ice-stones with surface perturbations. 
\( \delta \) is the amplitude of the sinusoidal perturbations and "a" is the radius of the unperturbed sphere. 
(a) elliptical bottom surface, (b) perturbations extend all around the surface.
occur over the upper portion of the stone, the bottom half being smooth and ellipsoidal. The perturbations in the second model (Fig. 4.9b) differ from the first in that they extend all around the stone. In Fig. 4.9, "a" refers to the unperturbed sphere radius and \( \delta \) to the amplitude of the sinusoidal perturbations. It should be noted that the solid body of revolution is obtained by rotation about the \( \theta = 0^\circ \) or Z-axis; hence, three-dimensional roughness (as on a golf ball) is not simulated. This is due to the limitations of the T-matrix computer code which handles only axisymmetric bodies. It was felt that these models represent reasonable first order approximations to the roughness problem, since the main purpose is to compare back-scattering cross-sections with equivalent smooth spherical ice-stones and to note significant changes in cross-sections and polarization effects, if any.

4.6.1 Irregular Ice-Stones

The radar cross-section of a number of ice-stones modelled as in Fig. 4.9a were calculated as a function of the angle of incidence. Scattering calculations were made over the XZ-plane, see Fig. 3.1, for incident polarizations \( \hat{e}_\parallel \) and \( \hat{e}_\perp \). Fig. 4.10 shows normalized back-scattering cross-sections \( q_\parallel/\pi a^2 \) and \( q_\perp/\pi a^2 \) for \( ka \) values of 1.0, 2.0, 2.5 and 3.0 respectively with \( \delta = 0.1a \). The 90\(^\circ\) case corresponds, again, to a radar with low elevation angles; the geometry is the same as described in the first paragraph of Section 4.5 with the oblate spheroid replaced by the rough ice-stone.

Fig. 4.10a shows the calculations for \( ka = 1.0 \). The normalized radar cross-section for the \( \delta = 0 \) case (no perturbations, only a smooth
Fig. 4.10. Normalized cross-sections at orthogonal polarizations versus angle of incidence for ice-stone model in Fig. 4.9a with $\delta = 0.10a$. (a) $\delta = 0$ value equals 0.39, (b) $\delta = 0$ value equals 0.67, (c) $\delta = 0$ value equals 1.12 and (d) $\delta = 0$ value equals 3.80.
spherical stone of radius \( a \) is \( \sigma/\pi a^2 = 0.39 \). It is seen that \( \sigma_\parallel/\pi a^2 \) and \( \sigma_\perp/\pi a^2 \) for the rough stone are close to this value even at \( 90^\circ \).

Again in Fig. 4.10b where \( ka = 2.0 \), the \( \delta = 0 \) corresponds to \( \sigma/\pi a^2 = 0.67 \). At \( 0^\circ \) there is some variation from this value while at \( 90^\circ \) the curves marked A and B are not very different from the value of 0.67.

The calculations of Fig. 4.10c where \( ka = 2.5 \) show a differing trend. The \( \delta = 0 \) case corresponds to \( \sigma/\pi a^2 = 1.12 \). Curves A and B differ from this value quite significantly for the whole range of the angles of incidence. The \( 90^\circ \) values of normalized \( \sigma_\parallel \) and \( \sigma_\perp \) produce a \( \sigma_{DR} (= 10 \log \sigma_\parallel/\sigma_\parallel) \) of about 3.9dB.

Fig. 4.10d corresponds to the case where \( ka = 3.0 \), the maximum value considered. The curves marked A and B show a sharp decrease in magnitude with angle of incidence with a minima appearing at an angle of about \( 70^\circ \). For \( \delta = 0 \), \( \sigma/\pi a^2 = 3.8 \); in comparison, the \( 90^\circ \) values of normalized \( \sigma_\parallel \) and \( \sigma_\perp \) are about 1.2. Significant changes from the equivalent spherical case are observed.

It is very difficult to draw any firm conclusions on the basis of these sample calculations. It seems that the smaller sized stones (\( \text{ka}^2 \)) produce back-scattering cross-sections not very different from their equivalent spherical cases. By equivalent spherical case is meant the following: the volume of the rough ice-stone is calculated for each \( ka \) and equated to \( 4/3 \pi a^3_{\text{eq}} \). The resulting \( \text{ka}_{\text{eq}} \) was found to be very close in value to \( ka \) (to within a few percent), hence all cross-section values can be compared to the \( \delta = 0 \) case of \( \sigma/\pi a^2 \) (the normalized radar cross-section for a sphere of radius \( a \)).
4.6.2 Water Coated Irregular Ice-Stones

Consider now the case where the ice-stone of Fig. 4.9a is coated with a spherical layer of water with radius \( a' = a + \delta \). Back-scattering calculations were performed as a function of the angle of incidence for \( ka \) values of 1.0, 2.0 and 2.5 with corresponding values of \( ka' \) being 1.05, 2.10 and 2.65 respectively as in Fig. 4.11. The value for \( \delta \) in each case was 0.05a corresponding to a sinusoidal peak-to-peak amplitude of ten percent of the unperturbed sphere radius \( a \). At S-band the unperturbed stone radius \( a \) corresponds to 1.6cm, 3.2cm and 4.0cm for \( ka = 1.0, 2.0 \) and 2.5 respectively, and \( \delta \) corresponds to 0.8mm, 1.6mm and 2.38mm respectively. At X-band the values of \( a \) and \( \delta \) are reduced by a factor of 10/3 = 3.334. For all cases the dielectric constant of ice was chosen to be \( \varepsilon_{\text{ice}} = 3.17 + j(0.0086) \) and that of water as \( \varepsilon_{\text{water}} = 78.65 + j(26.51) \); these values apply for \( \lambda = 10\text{cm} \) and temperature = 0°C, Battan, et al. (1970).

Fig. 4.11a shows the normalized radar cross-sections \( q_{\|}/\pi a'^{2} \) and \( q_{\perp}/\pi a'^{2} \) versus angle of incidence. The \( \delta = 0 \) case gives a cross-section of \( \sigma/\pi a'^{2} = 4.77 \). The curves marked A and B fall below this value for all angles of incidence; at 90° the differential radar cross-section, \( \sigma_{\text{DR}} = 6.6\text{dB} \).

Fig. 4.11b shows a sharp increase in cross-sections for 0° angle of incidence. The enhancement over the \( \delta = 0 \) value (≈5.58) at 0° is about 8.9dB. The radar cross-sections at both polarizations fall off very rapidly as incidence angle increases; at 90°, the values are about 5.7dB less than the \( \delta = 0 \) value of \( \sigma/\pi a'^{2} = 5.58 \). After examining this behavior for \( ka \) values close to 2.0, it is concluded that the sharp increase at 0° is due to resonance effects.
Fig. 4.11. Normalized radar cross-sections at orthogonal polarizations for the model of Fig. 4.9a with spherical water-coating of radius $a' = a + \delta$ and $\delta = 0.05a$. The unperturbed normalized radar cross-sections are, respectively, (a) 4.77, (b) 5.58, and (c) 0.85.
Again in Fig. 4.11c the $0^\circ$ values of normalized $\sigma_\parallel$ and $\sigma_\perp$ show an increase of about 8.6 dB over the $\delta = 0$ value of about 0.85. There is a sharp fall as incidence angle increases; the curves show a marked oscillatory behavior, probably due to the increased stone size relative to wavelength. At $90^\circ$, the values are about 4.5 dB less than the $\delta = 0$ value of 0.85.

Again, it is very difficult, if not impossible, to draw general conclusions about the behavior of the radar cross-sections of irregular water-coated ice-stones from these limited calculations. Nevertheless, it is important to note that the presence of water-coating on the irregular ice-stones produces significantly different cross-sections than non-coated stones and has greater variability with angle of incidence.

The last case considered is the ice-stone shown in Fig. 4.9b with a spherical water coating of radius $a' = a + \delta$. Note that the stone now possesses perturbations all around its surface.

Fig. 4.12 shows plots of radar cross-section versus angle of incidence for $ka$ values of 1.0, 2.0 and 3.0 with $\delta = 0.05a$. The corresponding values of $ka'$ were chosen as 1.05, 2.10 and 3.15 respectively. Fig. 4.12a shows the case when $ka = 1.0$ and $ka' = 1.05$. The $\delta = 0$ value corresponds to $\sigma/\pi a'^2 = 4.77$; this is very close to the $0^\circ$ value of 4.75. The curves marked A and B vary smoothly as angle of incidence increases. The $90^\circ$ value of $\sigma_\perp/\pi a'^2 = 3.55$ and $\sigma_\parallel/\pi a'^2 = 2.0$. On the other hand, the curves of Fig. 4.12b show an oscillatory behavior which can partly be ascribed to the symmetrical surface perturbations of the ice-stone. $0^\circ$ values of normalized $\sigma_\parallel$ and $\sigma_\perp$ are very close to
Fig. 4.12. Normalized radar cross-sections at orthogonal polarizations for the model of Fig. 4.9b with spherical water-coating of radius \( a' = a + \delta \) and \( \delta = 0.05a \). The unperturbed normalized radar cross-sections are, respectively, (a) 4.77, (b) 5.58, and (c) 3.42.
Fig. 4.12. Normalized radar cross-sections at orthogonal polarizations for the model of Fig. 4.9b with spherical water-coating of radius $a' = a + \delta$ and $\delta = 0.05a$. The unperturbed normalized radar cross-sections are, respectively, (a) 4.77, (b) 5.58, and (c) 3.42.
the unperturbed ($\delta = 0$) value of 5.58. The cross-sections then start decreasing as angle of incidence increases.

The oscillatory trend of the curves A and B increases in Fig. 4.12c where $ka = 3.0$ and $ka' = 3.15$. The $\delta = 0$ case corresponds to a normalized cross-section of 3.42. The curves A and B rise quickly to a maximum and then start falling off, with A attaining a secondary maximum at $90^\circ$. Differential radar cross-section $\sigma_{\text{DR}}$ equals -10.5dB at $90^\circ$.

As far as Fig. 4.12 is concerned, the $0^\circ$ value (corresponding to a vertically pointing radar) is quite close to the unperturbed spherical case for the ka sizes considered. The same cannot be said, however, for the cases shown in Fig. 4.11. For most of the cases, the $90^\circ$ values differ from the equivalent spherical case and do show polarization characteristics as given by $\sigma_{\text{DR}}$ values. This tendency increases with increasing the size of the stone (or ka) and/or increasing the perturbation parameter $\delta$. The dependence of back-scatter cross-sections on $\delta$ was investigated in only a few cases of the dry ice-stone models because of limitations on computer resources. The sample computations presented here serve the purpose of (a) demonstrating that the T(1,2)-matrix method is well suited for handling irregular and smooth water-coated ice-stones and (b) showing that smooth and rough water-coated ice-stones produce radar cross-sections quite different from the equivalent spherical cases (vertically pointing radars may not be as much affected as radars operating at low elevation angles). Other conclusions about the magnitude of cross-sections, polarization effects, etc., would require extension of the T-matrix method to arbitrarily shaped bodies.
CHAPTER V

SUMMARY AND CONCLUSIONS

5.1. Summary of Results

The specific objective of this work was to develop a theory of scattering from axisymmetric, imbedded perfect conductors or dielectrics using an extended integral equation method. The resulting matrix equations were solved numerically on a computer for the scattering cross-sections. Theoretical calculations were compared with extended Mie theory calculations for radar cross-sections of concentric spherical scatterers. Calculations were also compared with measurements of back-scattering cross-sections of a non-concentric, dielectric-clad spherical perfect conductor. A number of computations of scattering cross-sections of oblate spheroidal raindrops, oblate ice-stones and water-coated "rough" ice-stones of different sizes were performed and their application to radar meteorology briefly discussed. This chapter summarizes the major results and provides suggestions for future research.

5.1.1 Theoretical Solution

Electromagnetic scattering by axisymmetric perfect conductors or dielectrics imbedded in an axisymmetric dielectric body was investigated theoretically using an extended integral equation method. This resulted in the formulation of the transition or T(1,2)-matrix which
characterizes the scattering characteristics of the axisymmetric body. Once the T(1,2)-matrix is known for a specific scatterer geometry, the monostatic and bistatic far-zone scattered fields and scattering cross-sections for incident orthogonal polarizations aligned along the principal axes of the scatterer follow directly.

5.1.2 Numerical Solution

Computer programs which numerically solve for the T(1,2)-matrix and the resulting far-zone fields and scattering cross-sections have been developed for two scatterer configurations, (a) axisymmetric perfect conductor imbedded in an axisymmetric dielectric, and (b) axisymmetric dielectric imbedded in a dielectric body of differing dielectric constant. This formulation also yields computer programs for handling axisymmetric perfect conductors and dielectrics. The size of the T(1,2)-matrix required for a convergent solution depends on the relative size, shape and dielectric constants of the imbedded scatterer. The matrix size was found to increase with increases in both object size (relative to wavelength) and deviation from sphericity. Concentric spherical and spheroidal bodies were easiest to handle from a numerical viewpoint, since they generate a strongly diagonal T(1,2)-matrix. For other non-spherical scatterers such as cone-spheres or surfaces with sinusoidal perturbations, the T(1,2)-matrix method was found to be realistically applicable for sizes up to D-3λ where D is a representative size parameter and λ is the wavelength. The computer code at present can handle a maximum T(1,2)-matrix size of 40 x 40; however, this restriction is not necessary and larger matrix sizes can
be used by simply altering the computer code. The computer programs were extensively tested and run on the Ohio State University's IBM 370/168 system and the National Center for Atmospheric Research (NCAR) CDC 6600/7600 computers.

5.1.3 Sample Computations

Excellent agreement was obtained between T-matrix solutions and extended Mie theory solutions for concentric spherical scatterers. Experimental results of radar cross-section measurements on a non-concentric dielectric-clad spherical perfect conductor were compared directly to T-matrix solutions and very good agreement was observed considering that the measurements were at best probably accurate to within about ±1.0 dB.

Calculations of back-scattering cross-sections of oblate spheroidal raindrops at orthogonal polarizations demonstrated the applicability of Gans' theory for such scatterers at S-band wavelengths. As a consequence it was also demonstrated that Gans' theory was suitable for calculating $Z_{DR} - D_0$ curves for oblate drops with exponential size distributions. The $Z_{DR} - D_0$ relation was found to be relatively insensitive to temperature variations at S-band.

A series of calculations of back-scattering cross-sections of oblate ice-stones of various sizes and eccentricities were performed. The calculations indicate that under certain conditions (see Section 4.5) a radar measurement of negative $Z_{DR}$ would result, suggesting the possibility of uniquely detecting hail within a storm.

Radar cross-sections of water-coated ice-stones with sinusoidal
surface perturbations were also calculated for a variety of sizes and water thicknesses. The primary objective was to compare these cross-sections with their equivalent concentric spherical cases. The number of calculations performed were not sufficient to draw any firm conclusions other than to demonstrate that surface-perturbed hailstones with water-coatings produce cross-sections significantly different from their equivalent, smooth spherical cases, especially for the larger sized stones ($ka = 2\pi a/\lambda > 1.5$), "a" being the radius of the unperturbed stone. These calculations represent the first theoretical attempt at determining the effects of irregularities on the surface of ice-stones.

5.2. Suggestions for Future Research

5.2.1 Extension of the Theory

The $T(1,2)$-matrix formulated in Chapter II is composed of the $T(2)$-matrix for the imbedded body and certain $Q^1$-matrices associated with the outer surface assuming a common origin for both imbedded and outer surfaces. The $T(2)$-matrix and $Q^1$-matrices are independent so far as the surfaces are concerned, hence, they can be formulated with respect to different origins. Since the various matrices are composed of spherical wave solutions of the vector Helmholtz equation, the different origins can be related by using the translational properties of vector spherical harmonics. This method would be extremely valuable since the inner body could then be moved to any position inside the outer body and different origins specified for the inner and outer surfaces. Instead of requiring overall rotational symmetry, it would then be necessary only for the inner and outer surfaces to be axisymmetric with respect
to their corresponding origins. The entire operation would reduce to multiplying the $T(1,2)$-matrix by another $R$-matrix which would take into account the translation between origins. It is anticipated that such a formulation would greatly extend the scope of application of the $T$-matrix technique and substantially improve and extend the capability of the numerical solutions.

A further extension of the theory would be to treat arbitrarily shaped, imbedded scatterers. The assumption of rotational symmetry used here greatly simplifies the numerical solution and reduces the $T$-matrix size required for a convergent solution. Non-axisymmetric bodies would, however, require an additional integration over azimuthal angles; the $T$-matrix size, in general, would also increase significantly hence optimization of the numerical procedures would be of utmost importance. The ability to calculate scattering cross-sections of resonant-sized, arbitrarily-shaped imbedded perfect conductors or dielectrics would prove very useful in many areas, e.g., microwave scattering by hydrometeors (especially large, irregular hailstones), light scattering by bacteria, aerosols, etc., radar detection of complex shaped metal targets, etc.

5.2.2 Applications

Many of the applications in this work are concerned with the calculation of radar cross-sections of axisymmetric hydrometeors. The present theory and numerical methods and their extension to non-axisymmetric shapes can be used to further examine the validity of approximating complex-shaped hailstones by smooth spheroids. It would
also be important to assess other polarization effects, propagation effects and mean scattering from appropriate hail size distributions. The theoretical methods developed here can be directly extended to calculate forward-scattering amplitudes, the propagation constants of precipitation-filled media, attenuation cross-sections of hydrometeors, and the cross-polarized scattered fields.

5.2.3 Radar Measurement of Differential Reflectivity

The calculations of radar cross-sections of different hydrometeors, e.g., raindrops, hailstones, at orthogonal polarizations demonstrate the potential use of radar differential reflectivity measurements ($Z_{DR}$) for detecting and extracting quantitative information (e.g., size-distributions, rainfall rate) regarding such hydrometeors.

Future research should be aimed at testing the $Z_{DR}$-technique in the field using radars which can transmit and receive orthogonal polarizations either simultaneously or on a pulse-to-pulse basis. The main purpose of the experiments would be to test whether differential reflectivity ($Z_{DR}$) can be measured and whether it can be qualitatively or quantitatively related to rainfall, hail and storm structure.
LIST OF REFERENCES


