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VITA

April 29, 1948 . . . . . Born - Minneapolis, Minnesota

June, 1971 . . . . . . . B.S., Applied Mathematics and Computer Science, Washington University, St. Louis, Missouri; M.B.A., Washington University

1971-1972. . . . . . . Financial Analyst, Washington University School of Medicine, St. Louis, Missouri

1972-1974. . . . . . . Teaching Associate, Department of Finance, The Ohio State University, Columbus, Ohio

1974-1975. . . . . . . Teaching Associate, Department of Management Sciences, The Ohio State University, Columbus, Ohio

1974-1975. . . . . . . Instructor of Quantitative Analysis, Capital University, Columbus, Ohio

1975-1976. . . . . . . Instructor of Management, University of Missouri-Columbia, Columbia, Missouri

FIELDS OF STUDY

Major Field: Quantitative Methods in Business

Advisor: Professor James A. Bartos

Minor Field: Finance

Advisor: Professor Peter L. Mullins
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CHAPTER I

INTRODUCTION

Long-term capital structure planning and dividend policy are vital aspects of a firm's existence, yet analytical treatment of both, particularly the former, have been limited. The literature to this time has dealt primarily with the Modigliani-Miller vs. traditional theory controversy (1,2). In essence the M-M position holds that the cost of debt and the cost of equity are identical except for tax effects, that the cost of equity is not influenced by the proportion of debt present in the capital structure, and that the firm's overall discount rate is a constant irrespective of the capital structure. The traditional writers claim that the firm's cost of capital is lowered by the acquisition of moderate amounts of debt, but "too much debt" increases the risk of insolvency and forces up the cost of any new fixed-return securities, the cost of equity and the firm's overall discount rate. However, these efforts have paid little attention to the timing and nature of the additions to the capital structure, although of course this is a relevant problem only under traditional
theory considerations. Similarly, the theorists have been segregated into two schools of thought on dividend policy, with David Durand and Myron J. Gordon defending the importance of paying dividends, and Modigliani and Miller speaking for the "irrelevance of dividends" group or "earnings theorists" (3,4,5,6). Walter (7) proposed a model in which the dividend decision is a function of the firm's discount rate and the rate of return available to investors on investments of similar risk. Still, there lacks a view of dividend policy as a financing decision for the firm over time. If the firm withholds earnings rather than paying cash dividends, the resultant retained earnings is a source of funds for investment. If, however, alternative sources of funds are cheaper and readily available, and if the cash position of the firm allows, then a large percentage of earnings can be paid out in dividends. Further, the rationale for these decisions can vary from year to year.

Specifically, the problem is this. The focus of attention is on the right side of the balance sheet in the area of long term, permanent financing of the firm. A firm typically has the following choices for satisfying its permanent financing needs: debt (bank loans, bonds, debentures), preferred stock, common stock and retained earnings. The debt requires a fixed interest payment every year which is tax-deductible and eventual repayment of
principal. Preferred stock is usually a perpetual obligation with annual, not tax-deductible dividends at a fixed rate. New common stock is issued into perpetuity also, with no specified annual monetary obligation attached. However, there is a cost of common stock to the firm because stockholders do expect a return of either dividends or capital gains in the market commensurate with the risk they have assumed. Retained earnings costs are considered to be the same as that of new common stock, except for the effects of having to sell the stock issue.

The firm wishes to choose the arrangement of financing alternatives which will maximize the stockholders' return on their investment. Since the earnings flows to the firm are independent of the financing decisions, for the most part, it is satisfactory to concentrate on obtaining a minimum cost solution to the problem of capital structure determination. Indeed, J. Fred Arditti (8) has argued in favor of using this approach and has shown that minimizing the before tax average cost of capital is an optimal strategy.

It is the position of this paper that the traditional writers had the proper perspective on capital structure issues and that the problem of interest here is to find an optimal capital structure for the firm. Indeed, if Modigliani and Miller are right in their assertions,
there is no point to this effort. More will be said of the
collision between these schools of thought in Chapter II.

The model which will be proposed seeks to minimize
the firm's weighted average cost of capital, measured at
the end of the planning horizon, subject to several types
of constraints. There is a constraint on the amount of
funds to be raised in each year during the planning period,
depending on the rate of asset expansion of the firm. A
constraint must also be applied to the debt ratio so as to
control business risk and to maintain the firm's current
bond rating. Similarly, constraints will exist on the
current ratio (for dividend policy), the times-interest-
earned ratio (insolvency), and the dividend payout ratio.

The contribution of this paper is seen in several
ways. First, preferred stock will be included in the model
explicitly, where others have not dealt with this source at
all. Its inclusion is important because preferred stock
has different characteristics from pure debt and common
stock, and thus would have a differing effect on the
optimal financial plan. In addition, preferred stock is a
very important source of funds to industries such as the
public utilities, so the model would have applicability to
such firms. Second, a greater variety of constraints is
considered here than is seen in the previous work. Debt
rationing has been examined, but ignored have been ratios
such as the dividend payout, times interest earned, and the current ratio, all of considerable interest to potential users of the work. Third, the minimization-of-cost approach is to be taken. The non-stochastic work in the literature leaves future earnings as constants [see for example, the Krouse articles (38,39)]. Since these efforts must leave future earnings as constants, the issue does become one of optimizing over the available sources of funds according to cost considerations.

Chapter II will present a review of some relevant literature. It is divided into three sections. The first will give a synopsis of the M-M vs. traditional theory controversy, while part two looks at a sampling of works on multiperiod investment models which have triggered interest in corporate finance problems of that nature. Finally, two articles which have the greatest bearing on the topic explored in this paper are reviewed in the second chapter. Chapter III will present the model of the paper formally. Chapter IV then will discuss the mechanics of solution, detail the solution and present an analysis of it. It will be shown that given initial conditions of the variables involved, optimal levels for them can be derived from the model for any time during the planning horizon.
CHAPTER II

REVIEW OF THE LITERATURE

A. The Capital Structure Controversy

Modigliani and Miller present two propositions on capital structure theory. The propositions assume a world of no corporate income tax.

Proposition I assumes that all firms can be divided into a number of homogeneous risk classes. The risk class would determine the size of the firm's discount rate ($K$), the larger the perceived risk the higher would be this rate. The value of a firm would then be the annual stream of economic inflows (net operating income) divided by the discount rate. Symbolically,

$$V = \frac{\text{NOI}}{K} \quad (1)$$

where $V =$ value of the firm.

The important point here is that value is independent of the composition of the firm's capital structure.

The discount rate is also not influenced by the relative proportions of debt and equity in the firm's financial structure (9). Thus, a firm would always want to
use exclusively the one method of financing deemed to be the cheapest, according to the M-M treatise.

Suppose the market value of the firm's debt is denoted by $D$ and the market value of its equity by $E$. Then $V = D + E$. It is seen that

$$\text{NOI} = K_i D + K_e E$$

where $K_i$ = interest yield on debt,

$K_e$ = earnings yield on equity.

Dividing this equation by $V$, we obtain

$$K = K_i P_d + K_e P_e$$

where $P_d$ and $P_e$ are the proportions of debt and equity, respectively, in the firm's financial structure.

If $P_d = 0$, the result is that $K = K_e P_e$, and firm's cost of capital is exactly the discount rate for all equity inflows (applicable to all firms in a given risk class).

Modigliani and Miller contend further that $K = K_e$ no matter what the value of $P_d$, and so the firm's cost of capital is independent of capital structure proportions.

Explanation of Proposition II requires rearrangement of equation (2) above, so that

$$K_e = K + (K - K_i) \delta$$

where $\delta$ = the firm's debt-equity ratio.
From Proposition I, K is assumed constant. Therefore the cost of equity capital equals this constant average cost of capital plus a premium for the additional financial risk implicit in \( \delta \). While few would contest equation (3), it is the constant K which distinguishes Modigliani and Miller's theory from traditional theory.

The traditional writers contend that the value of K in equation (1) is not a constant, but that its graph is a flat concave curve when shown as a function of \( P_d \). The comparison between the two schools of thought is shown in the following graph:

![Graph showing curves M and T]

Curve M shows a constant value for K, where curve T demonstrates that the average cost of capital will decrease with moderate amounts of debt but that "too much" debt will cause K to rise again. The traditional writers acknowledge the presence of risk associated with an increasing proportion of fixed-return securities in the capital structure.

David Durand suggests further how the M-M hypothesis cannot hold for actual firms with his super premium
argument. Large financial corporations like banks and insurance companies are required to invest entirely in the highest grade of securities, and Durand hypothesizes that with such demand for the least risky issues, prices are always going to be bid up and yields will drop below what would be expected for securities of this risk level. Hence, firms would be encouraged to take advantage of this super premium in price and consequent lower interest on bonds and would issue as many securities as possible, as long as maintenance of the high rating is assured.

In a later article, Modigliani and Miller revised their assertions to take tax effects into consideration (11). Since the interest on debt is tax-deductible, they admitted that debt financing would enhance the value of the firm most and lower its cost of capital. But the distinction remains since M-M regard the tax advantage as the only point which favors debt over equity financing. As seen above, the traditional writers provide more justification for borrowing as opposed to selling shares in the entity, as long as the risk of the firm is tolerable to the stockholders.

Four empirical tests of the theory are representative of the efforts in this area, but even this work leaves the basic questions unanswered (12,13,14,15). Other
major investigations of capital structure theory have been by Bierman (16), Robichek and Myers (17), Schwartz (18), and Solomon (19).

B. A Survey of Multi-Period Models

An initial exploratory effort was made in the area of multi-period modeling by Edmund Phelps in a 1962 article (20). The problem attacked in this publication was in the realm of personal saving, where previous efforts had produced deterministic models (21,22). Phelps introduced the possibility of capital gain or loss, i.e., risk, in order to examine the applicability of the conventional theory of saving when capital risk is considered and to discover the effects of such risk on the level of personal consumption. The technique used was stochastic dynamic programming in a discrete time framework. The objective was to maximize expected utility over the individual's lifetime, where utility is defined as follows:

\[
U = \sum_{n=1}^{N} a^{n-1} u(c_n)
\]

where \( u(c_n) \) is utility derived from consumption level \( c_n \) in period \( n \), and \( a^{n-1} \) is a discount factor, \( 0 < a < 1 \). The utility function is subject to the traditional economic assumptions
of separability, stationarity and independence of the N consumption levels. Separability refers to a characteristic of a mathematical function which allows the unknown variables to be "segregated," so that optimization may occur one variable at a time. Stationarity refers to the maintenance of a constant scale of measurement of utility. Independence of the N consumption levels means that consumption in year i is not a function of consumption in year j. Capital available for consumption in the (n+1)st period is given by

\[ x_{n+1} = B_n(x_n - c_n) + y \]

where

- \( x_n \) = capital available in the \( n^{th} \) period
- \( B_{n-1} \) = yield on invested capital in the \( n^{th} \) period
- \( y \) = nonstochastic nonwealth income received at the end of each period.

Recursion equations are developed to facilitate the dynamic programming solution to \( \text{Max}(U) \), and an important result is revealed by that solution: that optimal consumption is an increasing function of both age and capital.
of separability, stationarity and independence of the $N$ consumption levels. Separability refers to a characteristic of a mathematical function which allows the unknown variables to be "segregated," so that optimization may occur one variable at a time. Stationarity refers to the maintenance of a constant scale of measurement of utility. Independence of the $N$ consumption levels means that consumption in year $i$ is not a function of consumption in year $j$. Capital available for consumption in the $(n+1)$ st period is given by

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- $y = \text{nonstochastic nonwealth income received at the end of each period.}$

Recursion equations are developed to facilitate the dynamic programming solution to Max $(U)$, and an important result is revealed by that solution: that optimal consumption is an increasing function of both age and capital.

Phelps then examines several monomial functions, namely
\[
\begin{align*}
  u(c_n) &= u_o - \lambda c_n^{-\delta} & u_o, \delta > 0, \lambda > 1 \\
  u(c_N) &= \lambda c^\delta & \lambda > 0, 0 < \delta < 1 \\
  u(c_N) &= \log c_N
\end{align*}
\]

These functions do not provide any general results, although the log function yields an optimal consumption schedule which is independent of expected return and riskiness of capital.

All of the utility functions make consumption linear homogeneous in capital and nonwealth income, but they fail to support the straight-line consumption function assumption of academia. The reason behind this is that wage income and capital income have different variances, and it is shown that the marginal propensity to consume out of risky income is smaller than out of certain income.

Phelps discovered, as one might expect, that the effects of riskiness and expected rate of return on consumption depend entirely on the type of utility function. However, in all cases if an increase in the rate of return raises the propensity to consume, then an increase in risk reduces the MPC, and vice versa. If expected return has no effect, neither does risk.

Mossin (23) takes the traditional one-period portfolio selection problem and extends it to a multi-period
framework. He examines the approaches of Arrow (24) and Tobin (25), which differ only in the argument of the utility function. Both authors assume quadratic utility functions in their formulations. The critical difference appears in the solutions, where Arrow's optimal investment in the risky security is dependent on initial wealth, while Tobin's optimum is completely independent of this wealth variable. As Mossin points out, the Tobin approach has serious limitations for multi-period work.

Mossin employs backward recursion to solve his dynamic programming problem, which is stated in the following manner. An investor begins with wealth $A_0$ at time $0$ and makes a decision to allocate this wealth among several assets. At the end of period 1 this collection of assets will have a value $A_1$, which then will have to be reallocated among the available assets and held for the second period. This process continues until the end of the investor's planning horizon, at which time he consumes all of his wealth, $W_n$. At the beginning of the last period ($n$), the investor must divide his wealth among various assets, as indicated above. In terms of recursion equations, his optimal decision, $d_n$, is that which satisfies the following:

$$\max_{d_n} E_n[U(A_n)] = P_{n-1}(A_{n-1})$$
which requires maximization of expected utility of terminal wealth, which in turn is said to be equal to the derived utility provided by the wealth level, $A_{n-1}$. Then the optimal decision $d_{n-1}$ to make at the beginning of period $n-1$ is the one which optimizes

$$E_{n-1}[P_{n-1}(A_{n-1})] = E_{n-1}\{\text{Max}_{d_n} E_n[U(A_n)]\},$$

and the recursion is initiated. The function $P$ used above is known as the derived utility function because it is an expression for utility as a function of wealth derived from maximization of the expected utility in the previous stage of the recursion analysis.

Another contribution of the paper was an attempt to discover what types of utility functions of terminal wealth induce myopic utility functions of intermediate wealth levels. Myopia occurs when utility functions of wealth are independent of yields in periods other than the current period. The benefit of the study is an improved ability to explain and predict consumption/investment behavior, given that one can classify persons by utility function. Mossin reached the following conclusions:

1. Logarithmic and power utility functions produce complete myopia.
2. If interest rates in all periods are zero, all functions such that the risk tolerance index 
\[-U'(X)/U''(X)\] is linear in X induce completely myopic utility functions of short-term wealth; 
partial myopia will occur if future rates are known to be nonzero.

3. Previous results in 1 and 2 hold even if yields are serially correlated.

Mossin, as was just seen, extended the portfolio selection optimization problem to two periods. Now, Paul Samuelson (26) approaches the global problem of lifetime portfolio selection through stochastic dynamic programming. The study examines only utility functions which exhibit the same relative risk aversion \[-U''(X)X/U'(X)\] for all levels of wealth \(X\). The objective function of the Samuelson model is straightforward and traditional:

\[
\text{Max } \int_0^T e^{-pt} U[C(t)] \, dt
\]

where

\[C(t) = rW(t) - W(t)\]

and

\[C = \text{consumption}, \ W = \text{wealth}, \ \text{and} \ r = \text{rate of return on invested wealth, all at time } t,\]

and the objective is to maximize discounted utility over a
lifetime. This leads to a statement of the basic problem in discrete form, from which backward recursion equations are derived and a solution is revealed.

\[ J_T(W_0) = \max_{(C_t, w_t)} \mathbb{E} \sum_{t=0}^{T} (1 + p)^{-t} U(C_t) \]

subject to \[C_t = W_t - W_{t+1} [(1 - w_t)(1 + r) + w_t Z_t]^{-1} \]

where the decision variables are consumption, \(C_t\), and investment in the risky security, \(w_t\), \(Z_t\) is the expected return on the risky asset, and \(p\) is a discount factor equal to the desired rate of return. The optimal decisions at any time are functions of current wealth and all prior expected returns on the risky asset.

The Samuelson model is applied to a log utility function and a specific solution derived for this case, but the result is too unwieldy to present here.

In another section of the paper, Samuelson provides an interesting result for the special case of isoelastic utility functions. The optimal portfolio solution is independent of wealth and independent of the consumption-saving decision at all times, so the amount invested in the risky asset is constant. The optimal consumption schedule depends upon current wealth, current
risky investment, and last period's marginal propensity to consume out of wealth.

Nils Hakansson provides a generalization of the Phelps model in his 1970 article (27). The objective of the individual is to maximize expected utility from consumption over time. He starts with an initial quantity of wealth and has opportunities to borrow and to lend during his planning horizon. Investment opportunities are assumed to exist with stochastic return whose probability distributions are known and which satisfy the "no easy money condition," which means that borrowing occurs at an interest rate which exceeds the existing lending rate. The primary advantage of the Hakansson approach over all others is "that the portfolio composition decision, the financing decision, and the consumption decision are all analyzed simultaneously in one model." Discrete time dynamic programming is utilized.

Hakansson models tend to be complex and elaborate, so a word description will be provided here. The basic model of the paper seeks to maximize utility from consumption plus the expected proceeds from nonwealth income, savings, and investment in risky and risk-free assets. The decision variables are the level of consumption and the amount of investment in each of the risky and
non-risky opportunities. The constraints require consumption and investment to be in non-negative quantities and that the investor/consumer be solvent at all times. The model is solved for the following utility functions:

1. \( u(c) = \frac{1}{g}c^g \quad 0 < g < 1 \)
2. \( u(c) = \frac{1}{g}c^g \quad g < 0 \)
3. \( u(c) = \log c \)
4. \( u(c) = -e^{-gc} \quad g > 0 \)

These functions were of interest because their risk-tolerance measures are linear functions, and they follow the conventional assumptions of \( U' > 0 \) and \( U'' < 0 \).

The first two functions are exponentials. Function 1 shows a strong positive relationship between utility and consumption, while the second one produces negative utility for all levels of consumption. Still, function 2 suggests it is better to consume more than less.

The log function describes a positive marginal utility, like the first function, but it is not as great as the exponential's marginal utility. Function 4 describes an individual who derives negative utility from consumption, but the marginal utility of this function is much smaller than that of the second one. Hence, the paper examines both positive and negative utility functions, and further
divides these categories into two divergent behavior patterns for study.

Since the general solutions are complicated, some observations and conclusions will be furnished instead. For each of the four models optimal consumption was found to be an increasing function of capital \( x \) and nonwealth income \( y \). The individual will consume even if \( x < 0 \) as long as \( y > 0 \) and the solvency constraint is satisfied. (This holds for Models 1-3 only). Also consumption is found to be proportional to permanent income in Models 1-3, which confirms the Friedman hypothesis (28). In all models optimal consumption decreases as the discount rate increases, which does not show conflict with intuition.

In Model 4, it was found that the optimal investment strategy was invariant over levels of wealth. Models 1-3 yielded the same mix and proportion of risky assets in the portfolio no matter what level of wealth, but the amount depended on permanent income. In all cases, however, lending was linear in wealth and proportional to wealth.

A companion article by Hakansson (29) extends the basic model to include three changes. The individual's lifetime is considered now to be a random variable with a known probability distribution. Also, a utility function representing the individual's bequest motive is introduced.
Finally, the opportunity to purchase life insurance is developed. The assumptions of the revised model are the same as before, but the objective becomes the maximization of expected utility from consumption before death and from the bequest left at death.

The problem is solved for the following utility functions:

1. \( u(c) = c^g \) \quad 0 < g < 1
2. \( u(c) = -c^{-g} \) \quad g > 0
3. \( u(c) = \log c \)

The following four cases are investigated, with the three models of utility being used in solution of each:

A. No bequest motive, no insurance
B. Bequest motive, insurance unavailable
C. No bequest motive, insurance available
D. Bequest motive, insurance available

Case A essentially reduces to the problem solved previously by Hakansson, and the solution has the same structure and implications. Case B produces a solution similar to that of Case A, but with one important difference. With the bequest motive intact, the optimal strategy calls for a reduction in the percentage of permanent income spent on consumption at all decision points. The solution to
Case C is also very similar to that of the basic model. It is discovered through Case C that the purchase of insurance is justified only when no bequest motive exists and when the individual's capital position (x) falls below a certain level. The optimal strategy for Case D then is the same as for Case B since the availability of insurance is unimportant when a bequest motive exists.

A later Hakansson effort took issue with conclusion 2 drawn by Mossin concerning myopic utility functions (30). He proved that conclusion 2 was true only with strict qualifications if serial returns are independent and that only the log function provides myopic utility short-run utility functions when serial dependence is present. The basic model used to refute the Mossin thesis is the following:

\[
f_j(x_j) = \max \mathbb{E} \left( f_{j+1} \left[ (B_{2j} - 1)z_{2j} + x_j \right] \right)
\]

where

\[
z_{ij} = \text{amount to be invested in opportunity } i \text{ at time } j
\]
\[
x_j = \text{amount of investment capital available at time } j
\]
\[
B_{ij} = \text{yield on investment opportunity } i \text{ at time } j
\]
Hakansson solves the model for the case of one risky opportunity whose proceeds are expressed with the following uncertainty:

\[ B_{2j} = 0 \text{ with probability } 1/2 \]
\[ 3 \text{ with probability } 1/2 \]

The optimal value for \( z_{2j} \) is shown to be dependent upon decisions and expectations about periods \( j+1, j+2 \) and so on, which demonstrates that myopic behavior is not present in the induced functions \( f_j \). The idea of borrowing (where \( z_{2j} \) is not constrained by \( x_j \)) is then introduced. Three cases are developed, two with borrowing limits imposed and one specifying a solvency constraint, and it is demonstrated again that myopia is not optimal.

On the issue of serially correlated yields, Hakansson draws the following two conclusions. The optimal investment policy is myopic when returns are stochastically independent and the terminal utility function is \( x^{1/2} \), as Mossin has suggested. When the terminal utility function is logarithmic, the induced functions are myopic regardless of the nature of the correlation among yields.

Eugene F. Fama integrates the consumption and investment decisions in a 1972 paper (31). He reviews the results of Phelps, Mossin and Hakansson, and then proposes
to present a more general model since he will place no restrictions on the form of the consumer's utility function. Even with this generalization Fama found that if the consumer is risk averse and if consumption and capital markets are efficient, his behavior will be no different from that of a one-period risk averse expected utility maximizer. Another important result of this effort was to incorporate the one period, two parameter wealth allocation models of Markowitz (32), Sharpe (33), Tobin (34), et al. into a multiperiod setting.

The basic wealth allocation model of Fama is summarized by the following recursion equation of the backward optimization scheme:

\[
U_t(C_{t-1}, w_t | B_t) = \max_{C_t, H_t} \int_{B_{t+1}} U_{t+1}(C_t, H_t | B_{t+1}) \times dF_{B_t}(B_{t+1})
\]

subject to \(0 \leq C_t \leq w_t\) and \(H = w_t - C_t\)

\(H\) all positive or zero

where

\(U_t\) = utility function at time \(t\)

\(C_t\) = consumption at time \(t\)

\(w_t\) = wealth at time \(t\)

\(B_t\) = state of the world at time \(t\)

\(H\) = vector representing dollars of investment in various risky assets

\(R\) = yield on risky assets from \(t\) to \(t+1\)

\(F_{B_t}\) = distribution function of \(B_{t+1}\) given \(B_t\)
This formulation is the maximization of utility (no restrictions on form other than additivity) given a current state of nature and as a function of consumption at time $t$ and condition of the portfolio at time $t+1$ (assumed to be predictable). The constraints provide assurance that one will consume no more than he has in wealth and that he will invest all of what remains after consumption. The modification required to integrate the one-period, two-parameter model results is dropping the $B_t$ subscripts and conditional references. The reason for this is that Fama shows that the consumer will act in the current period irrespective of expectations about future states of the world. Also, the index of integration becomes $dF(R_{t+1})$, as integration is made over possible changes in the values of portfolio assets, not over future states of nature.

Another model of the consumption/investment decision was developed by Robert C. Merton (35). This model is a continuous-time analogue to the one presented by Samuelson in a discrete-time form. Merton investigates primarily a two-asset problem (one risky) where returns are random and where constant relative risk aversion is assumed of the investor. The basic problem is to maximize the sum of future discounted utility from consumption until death plus the value of a bequest function which is dependent on
the level of wealth at death and the time of death. The constraints are a budget equation, a specified initial sum of wealth (positive), and requirements that consumption and wealth always remain positive. The recursion equations are given in the article, but only the results need be examined here.

For the case of constant relative risk aversion, the optimal levels of consumption and investment, respectively, at any time $t$ were discovered to be the following:

$$c^*(t) = \left[\frac{1}{(T - t + \varepsilon)}\right] W(t) \text{ for } \nu = 0$$
$$= \left[\frac{\nu}{1 - (\nu \varepsilon - 1)e^{\nu(t - T)}}\right] W(t) \text{ for } \nu \neq 0$$

$$w^*(t) = \frac{(a - r)}{S^2 (1 - g)} = w^* \text{ (independent of } W \text{ or } t)$$

where

$$\nu = \frac{p - g}{1 - g}$$

and

$$1 - g = - \frac{U''(C)C}{U'(C)}$$

$T$ = time at the end of the planning horizon
$a$ = expected rate of return from risky assets
$s^2$ = variance of risky asset return
$r$ = rate of return on risk-free assets
$0 < \varepsilon \ll 1$
It is interesting to note that the portfolio selection result is the same regardless of wealth level or time; indeed the individual is exhibiting constant relative risk aversion, and the magnitude of this aversion affects the size of his risky investment inversely. When the bequest function equals zero, the marginal propensity to consume out of current wealth is highest (implied by $\varepsilon = 0$). As $\varepsilon$ grows in value, the consumption rate decreases, showing the need to save for the ultimate bequest. It is also found that the saving impulse increases in intensity with increasing expected portfolio returns.

A later article by Merton expanded the results of the first work (36). The author dealt with a variety of utility functions and several stock price behavior assumptions in deriving his general results. The objective function of the second paper is identical to that of the initial paper, i.e., maximize utility from consumption and the bequest function. Some of the more important conclusions are to be noted here. Merton finds that when the assumption of perfectly random prices holds, the classical mean-variance portfolio rules are applicable without requiring the utility function to be quadratic. Particular attention is paid to the utility functions exhibiting hyperbolic absolute risk aversion properties.
because this allows explicit solutions for optimal consumption and portfolio investment to be derived. The effects of allowing prices to be serially dependent are examined in terms of their influence on the optimal solution. Then, wage income, uncertain death and default on supposedly risk-free assets are studied to determine their effects on the optimum.

Kare P. Hagen has produced a general model of the consumption-investment decision (37). The study is initiated by considering the pure saving problem. The formulation is almost identical to the one used by Phelps in 1962. The recursion equations require maximization of utility from consumption plus the discounted utility of saving at the prevailing interest rate. Hagen then states the one-period portfolio selection problem in terms of maximizing utility of return on the risky investment plus the utility of return on the risk-free investment, assuming a decreasing marginal utility for money on the investor's part. Hagen discovers that if the measure of relative risk aversion \[-U''(Y)Y/U'(Y)\] is increasing in final wealth, then the elasticity of risky investment with respect to wealth is a number less than unity; if the measure of RRA is decreasing in final wealth, then this elasticity exceeds one. The multiperiod portfolio model is then produced as
an extension of the one-period formulation, followed by the ultimate consumption-investment model which follows:

\[
R_{n-j}(W_j) = \max_{0 \leq c_j \leq W_j} \left[ u(c_j) + \int_{-\infty}^{\infty} \left[ R_{n-j-1} a_j(X-r) \right] + (W_j - c_j)(1+r) \ dF(x) \right] \\
\text{s. t. } \Pr(W_j > 0) = 1 \quad j = 1, 2, \ldots, N
\]

where \( u, c_j, \) and \( W_j \) have traditional definitions of utility, consumption and wealth at time \( j, \)

- \( r \) = rate of return on riskless assets
- \( x \) = rate of return on risky assets
- \( a_j \) = amount of investment in the risky asset
- \( F(x) \) = probability distribution of returns on risky asset
- \( R_{N-j} \) = total discounted utility at time \( j, \) given \( N-j \) periods until end of the planning horizon.

If constant relative risk aversion is assumed, optimal values for the model can be derived. They are

\[
c_j^* = m_{n-j} W_j \\
a_j^* = n(W_j - c_j)
\]

\( n \) is a constant proportion, while \( m_{n-j} \) is the marginal propensity to consume out of wealth and depends upon the
time remaining to the horizon. Unfortunately, the general solution to the Hagen model is not so explicit or uncomplicated.

C. A Major Paper on the Problem Being Investigated

Two articles by Clement Krouse (38,39) have dealt with the problem which motivates this paper. His 1972 work (38) is the more comprehensive of the two since it concerns the entire capital structure, while the latter article explores the internal equity vs. external equity financing decision only.

The initial Krouse effort has as its decision variables the following:

\[ q(t) = \text{earnings retained at time } t \]
\[ p(t) = \text{the dollar amount of external equity acquired at time } t \]
\[ b(t) = \text{the dollar amount of debt capital acquired at time } t \]

The Krouse model seeks the optimal values of these variables at each point in time by maximizing an expression for initial stockholder wealth. This expression is the sum of future earnings flows to the stockholders continuously discounted at rate \( k \). With the time subscripts removed the objective of the dynamic programming problem is as follows.
Max \[ p, q, b \int_{0}^{\infty} (E - q - p)e^{-kt} \, dt \]

where

\[ E = \text{net earnings at time } t. \]

The constraints then follow

\[ E = rS + (r - i) B \quad (a) \]

where

\[ S = \text{the state of equity capital employed by the firm at time } t \]

\[ B = \text{the state of debt capital employed by the firm at time } t. \]

This constraint defines net earnings as the sum of two factors: the internal rate of return \((r)\) times equity funds applied, and the constraint \((r)\) less the fixed cost of debt times debt capital applied. The second constraint gives the differential equation defining rate of change inequity.

\[ \frac{dS}{dt} = q + cp \quad (b) \]

where

\[ 1 - c = \text{fractional flotation costs of equity.} \]

Constraint (b) shows that the change in the state of equity consists of new common stock \((cp)\) and new earnings retained \((q)\). Constraint (c) provides the definition of
change in the state of debt capital. It is simply

\[ \frac{dB}{dt} = b \quad (c) \]

The fourth constraint is the following:

\[ q \leq rS + (r - i) B = E \quad (d) \]

This merely requires that internal financing at point t not exceed net earnings at that instant. The last two constraints impose limits on the acquisition of debt and external equity capital at time t. In the following equations the \( z_{ij} \) are market-determined coefficients:

\[ z_{11p} + z_{12b} \leq S \quad (e) \]

\[ z_{21p} + z_{22b} \leq B \quad (f) \]

Remembering that \( p \) and \( b \) are equal to linear functions of the time derivatives of \( S \) and \( B \), respectively, indicates that these constraints restrain instantaneous rates of acquisition rather than absolute levels of debt and equity. Finally, the decision variables are constrained to be positive or zero and initial levels of equity, \( S(0) \), and debt, \( B(0) \), are specified as given conditions.

The model is solved through use of the Maximum Principle of Pontryagin (10). The Hamiltonian function
is developed from the objective function and constraints (a-c):

\[ H = [(rS^* + (r-i)B^* - q - p)]e^{-kt} + (q - cp) e^{-kt}v_s + be^{-kt}v_b \]

where \( v_s \) and \( v_b \) are variables analogous to Lagrange multipliers in static problems. For optimal solution of the problem at hand, the Hamiltonian function must be maximized at all points in time, subject to the constraints in the original formulation (d-f). After some further mathematical maneuvering, it turns out that \( H \) will be maximized through solution of the following linear program:

\[
\begin{align*}
\text{Max} & \quad [q(v_s - v_d) + p(cv_s - v_s) + bv_b] \\
p, q, b
\end{align*}
\]

where

\( v_d \) is defined as "1"

subject to constraints d, e, and f above and the non-negativity conditions on the decision variables.

Hence, solution of the model becomes a matter of looking at a variety of cases determined by relative values of \( v_d, v_s, v_b \) and \( cv_s \). Nine cases are examined in the paper and the results displayed. It turns out that the optimal action is dictated in a manner shown below:
\begin{align*}
\text{Condition} & \quad \text{Action} \\
v_b & \geq v_d \quad \text{Acquire debt} \\
cv_s & > v_d \quad \text{Acquire external equity} \\
v_s & > v_b \quad \text{Retain earnings} \\
\end{align*}

and so on.

The \( v \)'s are preference indicators. Of course, in a number of cases two or three methods of financing may be acceptable.

Finally, Krouse shows how the linear program above can be converted to its dual, which is the minimization of cost of capital problem. This involves minimizing the weighted average cost of capital subject to several requirements constraints.

The second effort in this area posed the following problem:

\[ \text{Find } V_o^* = \max_{I_t, E_t} \int_0^\infty D_t e^{-kt} \, dt \]

where

\begin{align*}
I_t & = \text{amount of earnings retained by the firm at time } t \text{ as a fraction of earnings} \\
& \quad 0 \leq I_t \leq 1 \\
E_t & = \text{amount of external financing by the firm at time } t, \ E_t > 0
\end{align*}
\( D_t^T \) = the portion of time-\( t \) dividends paid to the stockholders at instant \( T, T \geq t \)

\( D_t \) = the total amount of dividends distributed at time \( t \)

\( V_t \) = the value of the corporation at time \( t \).

The expression above states the value of the firm as the maximized value of future discounted dividend payments.

Two constraints appear:

\[
\frac{dX_t}{dt} = r(I_t + cE_t)X_t
\]

where

\( X_t \) = amount of earnings at time \( t \)

\( c \) = one minus flotation costs for equity

\( r \) = the firm's internal rate of return on invested capital.

This constraint shows the instantaneous growth in earnings to be equal to growth in net capital investment. The expression \((I_t + cE_t)X_t\) is net investments, and when multiplied by \( r \), it will give a rate of growth of assets.

The second constraint is as follows:

\[
\frac{dA_t}{dt} = wA_t
\]

where

\( A_t \) = market value of the firm's assets at time \( t \)

\( w \) = a constant.
This restriction places an upper bound on the firm's instantaneous asset expansion. That is, the rate of change in assets is to be no more than a constant times $A_t$.

Solution of the model will give the optimal dividend policy of the firm so as to maximize current value of the corporation to the stockholders.

In a manner similar to the previous article, this model is solved by use of the Maximum Principle. The Hamiltonian is the following:

\[ H = (1 - I_t - E_t)X_t e^{-kt} + q_t r(I_t + cE_t)X_t \]

Optimization again becomes a matter of solving a linear program.

Four cases are examined with parameters allowed to vary:

- Case A: $w \leq r$, $0 \leq c \leq 1$
- Case B: $w \leq r$, $c = 1$
- Case C: $w > r$, $0 \leq c \leq 1$
- Case D: $w > r$, $c = 1$

Cases A and C admit flotation costs for equity, while B and D do not. Cases A and B provide a higher return on investments than is required by the constraint on asset expansion. The remaining cases, however, show a higher allowable expansion rate than internal sources can produce.
The cases are displayed along with the optimal solutions in the article.
CHAPTER III

THE MODEL

As indicated in Chapter I, the objective will be to minimize the before-tax average cost of capital to the firm. Let us define the levels of the various sources of funds at time $t$ as follows:

\[
\begin{align*}
    b(t) &= \text{amount of debt present at time } t \\
    p(t) &= \text{amount of preferred stock present at time } t \\
    e(t) &= \text{amount of common stock present at time } t \\
    r(t) &= \text{amount of retained earnings present at time } t.
\end{align*}
\]

Each type of financing will have an associated cost at time $t$. This cost may be thought of as having two parts: a stochastic one and a deterministic one. A convenient (but by no means restrictive) form is the following:

\[
y = n + mx
\]

where

- $y$ is the cost at time $t$
- $n$ is a stochastic variable, completely dependent on the general level of interest rates. It would correspond to the interest rate on the so-called "riskless security" in the marketplace.
m is the marginal increase in cost due to the riskiness of the firm. The more risky the firm, the higher would be m, indicating a severe penalty by the market for increased riskiness.

x is a quantification of risk. An appropriate measure might be the debt-equity ratio. In any case, x would have to be a function of b, p, e, and r at instant t.

Each cost function follows this general form, but the parameters would vary from one to another. The following notation will be used in identifying cost functions:

<table>
<thead>
<tr>
<th>Source of Funds</th>
<th>Cost Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt</td>
<td>f(t)</td>
</tr>
<tr>
<td>Preferred Stock</td>
<td>g(t)</td>
</tr>
<tr>
<td>Retained Earnings</td>
<td>h(t)</td>
</tr>
<tr>
<td>Common Stock</td>
<td>ch(t)</td>
</tr>
</tbody>
</table>

where c is the flotation cost.

In general n > 0 and m > 0, if the functions are as above. The effects of increasing fixed obligations on the various costs are shown in the following first derivatives:

\[
\frac{\delta f}{\delta b} > 0 \quad \frac{\delta g}{\delta b} > 0 \quad \frac{\delta h}{\delta b} > 0
\]

\[
\frac{\delta f}{\delta p} < 0 \quad \frac{\delta g}{\delta p} > 0 \quad \frac{\delta h}{\delta p} > 0
\]

These relations indicate that costs will tend to increase with increasing levels of debt and preferred stock. That is, increasing riskiness leads to higher costs. On the
other hand, increases in equity are perceived as reducing risk, so

\[
\frac{\delta f}{\delta e} < 0 \quad \frac{\delta f}{\delta t} < 0 \quad \frac{\delta g}{\delta e} < 0
\]

\[
\frac{\delta g}{\delta t} < 0 \quad \frac{\delta h}{\delta e} < 0 \quad \frac{\delta h}{\delta t} < 0
\]

Thus, costs of financing should be reduced when equity is being created.

If the objective of minimizing costs is taken over continuous time, it can be expressed as follows:

\[
\text{Min} \int_{0}^{T} \left[ b(t)f(t) + p(t)g(t) + r(t)h(t) + ce(t)h(t) \right] \, dt
\]

Recall that \( f, g, \) and \( h \) are functions of \( b, p, e, \) and \( r \) implicitly, despite the notation used. The integral is taken from the beginning of the planning horizon ( \( t = 0 \)) to the end at time \( t = T \). Each cost is weighted by the associated level of financing present at each instant in time.

The first constraint appears as a differential equation. It states that the rate of acquisition of funds from all sources must satisfy a lower bound at each instant. That is,
The parameter $A$ is determined by the firm on the basis of its perceived capital needs from period to period. This constraint ensures that enough new capital will be produced at each point in time.

Constraint (2) places a limit on the firm's debt to equity ratio at each instant. It is expressed as follows:

$$\frac{b(t) + p(t)}{e(t) + r(t)} \leq V(t)$$

$V$ is the upper limit on this ratio. The debt-equity relationship is a monitor on the riskiness of the firm, so to control it is to control risk to the satisfaction of stockholders and lineholders.

The third constraint is another attempt to prevent the firm from becoming debt-heavy or unable to cover fixed obligations. This is in the form of a limit on the times interest earned ratio, which compares earnings, before interest and taxes, $J(t)$, to fixed debt payments. That is,

$$\frac{J(t)}{f(t)b(t)} > D(t)$$

So, there is a minimum number of times that earnings before interest and taxes must cover the current cost of debt.
Constraint number 4 will prevent the firm from paying out too much in dividends when liquidity considerations make it unwise. The current ratio (current assets divided by current liabilities) is used to monitor liquidity. The constraint is constructed as follows:

$$\frac{S(t) - [zf(t)b(t) - g(t)p(t) - dr]}{L(t)} \geq Q(t)$$

where

- $S(t)$ = current assets prior to dividend declaration
- $L(t)$ = current liabilities prior to dividend declaration
- $z$ = $1 - \text{the firm's tax rate}$
- $dr$ = new retained earnings produced in an infinitesimal amount of time.

Current assets, $S(t)$, are reduced by dividends paid (quantity in parentheses). This amount divided by current liabilities must be no less than the figure $Q$.

Finally, a constraint is placed on the dividend payout ratio. It is assumed that a satisfactory (to the stockholders) dividend policy has been followed to this point in time and that it would be advantageous to the firm to maintain it at a minimum. This usually means placing a floor on the percentage of earnings paid in dividends. Here the constraint is shown by putting a
ceiling on the percentage of earnings retained.

\[
\frac{dr}{zJ(t) - zf(t)b(t) - g(t)p(t)} \leq M(t)
\]  

Here, new retained earnings is divided by earnings available to the owners, and the result can be no more than the fraction M.

Using the objective function and five constraints shown, an optimal financing plan can be developed for each point in time. The general solution to this problem will be derived in the next chapter.
CHAPTER IV

SOLUTION OF THE MODEL AND ANALYSIS

To summarize the discussion of the previous chapter, the model will be repeated here as follows:

\[ \text{Min } \int_0^T \left[ f(t)b(t) + g(t)p(t) + h(t)r(t) + ch(t)e(t) \right] dt \]

s.t. \[ \frac{db}{dt} + \frac{dp}{dt} + \frac{dr}{dt} + \frac{de}{dt} \geq A(t) \tag{1} \]

\[ \frac{b(t) + P(t)}{r(t) + e(t)} \leq V(t) \tag{2} \]

\[ \frac{J(t)}{f(t)b(t)} \geq D(t) \tag{3} \]

\[ S(t) - \left[ \frac{J(t)z - zf(t)b(t) - g(t)b(t)}{L(t)} - \frac{dr}{dt} \right] \geq Q(t) \tag{4} \]

\[ \frac{dr}{zJ(t) - zf(t)b(t) - g(t)b(t)} \leq M(t) \tag{5} \]

To draw insights into the solution of nonlinear optimization problems such as the one in this paper, the Kuhn-Tucker necessary conditions are derived. Good discussions of these conditions and their ramifications are
not hard to find [see Hadley (40), Cooper (41), Hillier and Lieberman (42), for example]. A brief statement of the Kuhn-Tucker technique will follow here.

Consider the following nonlinear program:

\[
\begin{align*}
\text{Min} & \quad f(x_1, x_2, \ldots, x_n) \\
\text{s.t.} & \quad h_1(x_1, x_2, \ldots, x_n) = 0 \\
& \quad h_2(x_1, x_2, \ldots, x_n) = 0 \\
& \quad \vdots \\
& \quad h_m(x_1, x_2, \ldots, x_n) = 0
\end{align*}
\]

Form the following function \( W \), often called the LaGrangian function:

\[
W = f(X) + \lambda_1 h_1(X) + \lambda_2 h_2(X) + \ldots + \lambda_m h_m(X)
\]

where the \( \lambda_i \) are known sometimes as LaGrange multipliers. The capital letter \( X \) denotes the vector \((x_1, x_2, \ldots, x_n)\).

Necessary conditions for a minimum are the following:

\[
\frac{\delta W}{\delta x_1} = 0
\]

\[
\frac{\delta W}{\delta x_2} = 0
\]

\[\vdots\]

\[\vdots\]
\[
\frac{\delta W}{\delta x_n} = 0 \\
\frac{\delta W}{\delta \lambda_1} = 0 \\
\frac{\delta W}{\delta \lambda_2} = 0 \\
\cdot \\
\cdot \\
\frac{\delta W}{\delta \lambda_m} = 0 \\
\]

\[
\frac{\delta^2 W}{\delta x_1^2} \geq 0 \\
\frac{\delta^2 W}{\delta x_2^2} \geq 0 \\
\cdot \\
\cdot \\
\frac{\delta^2 W}{\delta x_n^2} \geq 0 \\
\frac{\delta^2 W}{\delta \lambda_1^2} \geq 0 \\
\frac{\delta^2 W}{\delta \lambda_2^2} \geq 0 \\
\cdot \\
\cdot \\
\frac{\delta^2 W}{\delta \lambda_m^2} \geq 0 
\]
Solution of the n+m simultaneous equations for values of X and the λ_i's which satisfy the inequalities will yield candidates for the optimal solution. These trial solutions could actually be maxima or merely inflection points in the curve or surface of f. For a minimization problem, the Kuhn-Tucker conditions are sufficient for a global minimum if the objective function is convex and the constraint space formed by the h_i, (i = 1, ..., m), is convex. In a great many practical problems, the Kuhn-Tucker conditions are simple to formulate but extremely difficult to solve. Determination of the sufficient conditions is often nearly impossible as well. Both these problems are encountered in the analysis of the model presented above.

Several modifications must be made in the model prior to actual solution. Since equality constraints are required, time-dependent slack variables must be added to or subtracted from the left-hand side of each constraint. If the left-hand side of the i^{th} constraint is denoted by g_i(b, p, e, r, t) and the right-hand side as K_i, then the slack variables may be affixed as follows. Constraints 2 and 5 are "less than or equal to" constraints, so they become
\[ g_2(b, p, e, r, t) + S_2^2(t) - K_2 = 0 \]

and

\[ g_5(b, p, e, r, t) + S_5^2(t) - K_5 = 0 \]

It will be convenient later to express these equations as follows:

\[ -g_2(b, p, e, r, t) - S_2^2(t) + K_2 = 0 \quad (6) \]

\[ -g_5(b, p, e, r, t) - S_5^2(t) + K_5 = 0 \quad (7) \]

Expressions (1), (3), and (4) are "greater than or equal to" constraints, so the slack variables are subtracted from the left side to produce equalities. Thus we have

\[ g_1(b, p, e, r, t) - S_1^2(t) - K_1 = 0 \quad (8) \]

\[ g_3(b, p, e, r, t) - S_3^2(t) - K_3 = 0 \quad (9) \]

\[ g_4(b, p, e, r, t) - S_4^2(t) - K_4 = 0 \quad (10) \]

The slack variables are squared primarily to avoid placing any non-negativity restrictions on them. In the derivation of the Kuhn-Tucker conditions, the slack variables must be treated like any other variables, i.e., derivatives with respect to them are required.
The second modification is major and will allow some insight into this very difficult problem. In the time interval 0 to t, choose an arbitrary period, say t to t+\(\Delta\), where \(\Delta\) is very small but finite. Assume that values of the variables and parameters at time t are known. Now, for this small period of time, the objective function becomes

\[
\text{Min} \int_t^{t+\Delta} F(b, p, e, r, t) \, dt \quad (11)
\]

where \(F\) is shorthand notation for the integrand shown above. Initially, t = 0 and the objective would be optimized for the period of length \(\Delta\). Then, since certainty then would exist at time \(\Delta\), the model would be solved for the period \(\Delta\) to 2\(\Delta\). This process clearly could be repeated indefinitely. It can be shown by methods of the calculus [Kaplan (43), e.g.] that (11) can be approximated by

\[
\frac{\Delta}{2} [F(b, p, e, r, (t+\Delta)) + F(b, p, e, r, t)] \quad (12)
\]

That is, the average of \(F\) at time t and time (t+\(\Delta\)). The objective now is to minimize (12) for some small but finite \(\Delta\).

The constraints require some adjustment as well. The first constraint (8) can be simplified by solution of a
differential equation as follows:

\[
\int_{t}^{t+\Delta} db + \int_{t}^{t+\Delta} dp + \int_{t}^{t+\Delta} dr + \int_{t}^{t+\Delta} de \\
= \int_{t}^{t+\Delta} [A(t) + S_{1}^{2}(t)] \, dt
\]

which yields

\[
b(t+\Delta) - b(t) + p(t+\Delta) - p(t) + r(t+\Delta) - r(t) + e(t+\Delta) - e(t) \\
= A(t)\Delta + \frac{S_{1}^{2}(t+\Delta) + S_{1}^{2}(t)}{2} \cdot \Delta
\]

In proper form this expression becomes

\[
b(t+\Delta) - b(t) + p(t+\Delta) - p(t) + r(t+\Delta) - r(t) \\
+ e(t+\Delta) - e(t) - A(t)\Delta - \frac{\Delta}{2} S_{1}^{2}(t+\Delta) - \frac{\Delta}{2} S_{1}^{2}(t) \tag{13}
\]

\[
= 0
\]

The second constraint (6) is evaluated at time \( t+\Delta \) in order to consider the effects of capital structure changes in the interval \( \Delta \). Thus (6) is expressed as
\[-b(t+\Delta) - p(t+\Delta) + V(t)r(t+\Delta) - S_2^2(t+\Delta)r(t+\Delta) + V(t)e(t+\Delta) - S_2^2(t+\Delta)e(t+\Delta) = 0\] (14)

The third constraint (9) is easily rearranged to give

\[J(t) - S_3^2(t+\Delta) - D(t)f(t+\Delta)b(t+\Delta) = 0\] (15)

The fourth constraint (10) becomes when simplified,

\[zf(t+\Delta)b(t+\Delta) + g(t+\Delta)p(t+\Delta) + r(t+\Delta) - r(t) + S(t+\Delta) - Q(t)L(t+\Delta) - S_4^2(t+\Delta)L(t+\Delta) - zJ(t) = 0\] (16)

Finally, in rearrangement (7) looks as follows:

\[[M(t) - S_5^2(t+\Delta)][zJ(t) - zf(t+\Delta)b(t+\Delta) - g(t+\Delta)p(t+\Delta)] - r(t+\Delta) + r(t) = 0\] (17)

For our model the LaGrangian function is the following

\[W = \frac{\Lambda}{2} [F(t+\Delta) + F(t)] + \lambda_1[LHS(13)] + \lambda_2[LHS(14)] + \lambda_3[LHS(15)] + \lambda_4[LHS(16)] + \lambda_5[LHS(17)]\] (18)

where "LHS" denotes "left-hand side of." What will be given next are the Kuhn-Tucker necessary conditions for a
minimum for expression (18). In the following set of
equations and inequalities, the time notation will be
deleted from the original right-hand sides of the
constraints. For example, A(t) will simply be A now.
Also, letter subscripts will denote partial differentiation
with respect to the variable written as the subscript. So,
\( f_b(t+\Delta) \) is the partial derivative of \( f \) with respect to \( b \)
at time \( t+\Delta \). Similarly, \( f_{bb}(t+\Delta) \) would be the second
partial derivative of \( f \) with respect to \( b \) at time \( t+\Delta \).

The Kuhn-Tucker necessary conditions follow:

\[
\frac{\delta W}{\delta b} = \frac{\Delta}{2} F_b(t+\Delta) + \lambda_1 - \lambda_2 - \lambda_3 D[f(t+\Delta)] + f_b(t+\Delta) + \lambda_4 [zf(t+\Delta) + zf_b(t+\Delta)b(t+\Delta)]
+ g_b(t+\Delta)p(t+\Delta)] + \lambda_5 [M - S_5^2(t+\Delta)]
\]

\[
[-zf(t+\Delta) - zf_b(t+\Delta)b(t+\Delta) - g_b(t+\Delta)p(t+\Delta)] = 0
\]

\[
\frac{\delta W}{\delta p} = \frac{\Delta}{2} F_p(t+\Delta) + \lambda_1 - \lambda_2 - \lambda_3 Df_p(t+\Delta)b(t+\Delta)
+ \lambda_4 [zf_p(t+\Delta)b(t+\Delta) + g(t+\Delta) + g_p(t+\Delta)p(t+\Delta)]
+ \lambda_5 [M - S_5^2(t+\Delta)][-zf_p(t+\Delta)b(t+\Delta)
- g(t+\Delta) - g_p(t+\Delta)p(t+\Delta)] = 0
\]
\[
\frac{\delta W}{\delta r} = \frac{A}{2} F_r(t+\Delta) + \lambda_1 + \frac{\lambda_2}{2} V - S_2^2(t+\Delta) + \lambda_3 D_f r(t+\Delta) b(t+\Delta) + \lambda_4 z_f r(t+\Delta) b(t+\Delta) + g_r(t+\Delta) p(t+\Delta) + \lambda_5 [M - S_5^2(t+\Delta)]
\]
\]
\[
[-z_f r(t+\Delta) b(t+\Delta) - g_r(t+\Delta) p(t+\Delta)] - \lambda_5 = 0
\]
\[
\frac{\delta W}{\delta e} = \frac{A}{2} F_e(t+\Delta) + \lambda_1 + \frac{\lambda_2}{2} V - S_2^2(t+\Delta) - \lambda_3 D_f e(t+\Delta) b(t+\Delta) + \lambda_4 z_f e(t+\Delta) b(t+\Delta) + g_e(t+\Delta) p(t+\Delta) + \lambda_5 [M - S_5^2(t+\Delta)]
\]
\]
\[
[-z_f e(t+\Delta) b(t+\Delta) - g_e(t+\Delta) p(t+\Delta)] = 0
\]
\[
\frac{\delta W}{\delta \lambda_1} = b(t+\Delta) - b(t) + p(t+\Delta) - p(t) + r(t+\Delta) - r(t) + e(t+\Delta) - e(t) - \Delta A - \frac{A}{2} S_1^2(t+\Delta) - \frac{A}{2} S_1^2(t) = 0
\]
\[
\frac{\delta W}{\delta \lambda_2} = -b(t+\Delta) - p(t+\Delta) + Vr(t+\Delta)
\]

\[-S_2^2(t+\Delta)r(t+\Delta) + Ve(t+\Delta)
\]

\[-S_2^2(t+\Delta)e(t+\Delta) = 0
\]

(24)

\[
\frac{\delta W}{\delta \lambda_3} = J(t) - S_3^2(t+\Delta) - Df(t+\Delta)b(t+\Delta) = 0
\]

(25)

\[
\frac{\delta W}{\delta \lambda_4} = zf(t+\Delta)b(t+\Delta) + g(t+\Delta)p(t+\Delta) + r(t+\Delta)
\]

\[-r(t) + S(t+\Delta) - QL(t+\Delta) - S_4^2(t+\Delta)L(t+\Delta)
\]

\[-zJ(t) = 0
\]

(26)

\[
\frac{\delta W}{\delta \lambda_5} = [M - S_5^2(t+\Delta)][zJ(t) - zf(t+\Delta)b(t+\Delta)
\]

\[-g(t+\Delta)p(t+\Delta)] - r(t+\Delta) + r(t) = 0
\]

(27)

Utilization of the forms (6) - (10) simplifies the conditions involving the slack variables tremendously.
From the conditions developed so far, the following statements may be made:

\[
\begin{align*}
\lambda_1 &= 0 \text{ or } S_1 = 0 \quad \text{(or both)} \\
\lambda_2 &= 0 \text{ or } S_2 = 0 \quad \text{(or both)} \\
\lambda_3 &= 0 \text{ or } S_3 = 0 \quad \text{(or both)} \\
\lambda_4 &= 0 \text{ or } S_4 = 0 \quad \text{(or both)} \\
\lambda_5 &= 0 \text{ or } S_5 = 0 \quad \text{(or both)}
\end{align*}
\]

The second partial derivatives with respect to the \( S_i \) provide the following information:
Thus, if any $\lambda_i$ is not equal to zero, it must be negative. These facts just derived are extremely useful in the solution of Kuhn-Tucker problems.

Remaining conditions, not strictly considered to be Kuhn-Tucker conditions, are easily derived as follows:

$$\frac{\delta^2 W}{\delta S_i^2} = -2\lambda_i \geq 0$$

for all $i$

$$\frac{\delta^2 W}{\delta b^2} = \frac{\Delta}{2} F_{bb}(t+\Delta) - \lambda_2 D[2F_b(t+\Delta)$$
$$+ b(t+\Delta)f_{bb}(t+\Delta)] + \lambda_4 [2zf_b(t+\Delta)$$
$$+ f_{bb}(t+\Delta)b(t+\Delta) + g_{bb}(t+\Delta)p(t+\Delta)]$$
$$+ \lambda_5 [M - S_5^2(t+\Delta)][-2zf_b(t+\Delta)$$
$$- zf_{bb}(t+\Delta)b(t+\Delta) - g_{bb}(t+\Delta)p(t+\Delta)] \geq 0$$
\[
\frac{\delta^2 W}{\delta p^2} = \frac{\Delta}{2} F_{pp}(t+\Delta) - \lambda_3 Df_{pp}(t+\Delta)b(t+\Delta)
\]
\[
+ \lambda_4 [zF_{pp}(t+\Delta)b(t+\Delta) + 2g_p(t+\Delta)] + g_{pp}(t+\Delta)p(t+\Delta)] + \lambda_5 [M - S_5^2(t+\Delta)]
\]
\[
[-zF_{pp}(t+\Delta)b(t+\Delta) - 2g_p(t+\Delta)] + g_{pp}(t+\Delta)p(t+\Delta)] > 0
\]
\[
\frac{\delta^2 W}{\delta r^2} = \frac{\Delta}{2} F_{rr}(t+\Delta) + \lambda_3 Df_{rr}(t+\Delta)b(t+\Delta)
\]
\[
+ \lambda_4 [zF_{rr}b(t+\Delta) + g_{rr}(t+\Delta)p(t+\Delta)] + \lambda_5 [M - S_5^2(t+\Delta)] [-zF_{rr}(t+\Delta)b(t+\Delta)
\]
\[
- g_{rr}(t+\Delta)p(t+\Delta)] > 0
\]
\[
\frac{\delta^2 W}{\delta e^2} = \frac{\Delta}{2} F_{ee}(t+\Delta) - \lambda_2 Df_{ee}(t+\Delta) b(t+\Delta) + \lambda_4 [z f_{ee}(t+\Delta) b(t+\Delta) + g_{ee}(t+\Delta) p(t+\Delta)]
\]

\[
+ \lambda_5 [M - S_{n-2}(t+\Delta)] [-z f_{ee}(t+\Delta) b(t+\Delta) - g_{ee}(t+\Delta) p(t+\Delta)] \geq 0
\]

\[
\frac{\delta^2 W}{\delta \lambda_i^2} = 0 \quad \text{for all } i
\]

It will be readily observed that a general analytical solution is virtually impossible to obtain from these conditions since equations (19) - (27) constitute a massive system of nonlinear simultaneous equations. Conceivably, with a real-world problem, a numerical solution could be derived through an iterative procedure, but this certainly would be a major project on its own due to the unique nature of the system.

Assuming that this system could be made to produce values for the unknowns, \( b(t+\Delta), \ p(t+\Delta), \ r(t+\Delta) \) and \( e(t+\Delta) \), here is how one would generate trial feasible solutions from the Kuhn-Tucker conditions. There are 32 cases to be analyzed involving the \( \lambda_i \) and \( S_i \) as a result of conditions...
(28a-e). One by one the cases should be examined and solutions for the decision variables plus the specified nonzero $\lambda_i$'s and $S_i$'s generated from equations (19-27). The first case might let all $\lambda_i = 0$ with the slack variables free to take on nonzero values. Case 2 might specify $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4 = 0$ and $S_5 = 0$ with $\lambda_5$ and the other slacks left in the solution. Each case would force five of the ten variables involved in conditions (28a-e) to be zero so that all 32 possible solutions are generated if possible. A case would be removed from consideration if (a) a solution could not be found from equations (19-27), (b) any of the decision variables were forced negative, (c) any $\lambda_i$ were forced positive or (d) the solution derived from equations (19-27) failed to satisfy conditions (30-33). Condition (a) takes care of the possibility that the set of simultaneous equations would be insoluble due to inconsistency of constraints, for example. Conditions (b-d) are feasibility considerations, (b) derived from common sense, (c) from equation (29) and (d) from equations (30-33). The solutions which survive this analysis would then be compared by substituting their respective values into the objective function (12) and determining which gives the lowest cost. This will be the absolute minimum cost solution to the problem if the
objective function is convex and the constraint space is a convex set. In this problem even this determination is impossible to make.

For a solution of the type described, the matter of selecting the magnitude of \( \Delta \) is very important. If \( \Delta \) is too large, the approximation to the objective function becomes inaccurate, and the solution then derived will diverge from the true continuous solution. If \( \Delta \) is too small, the computational effort required for solution over most time horizons would prohibit the use of this method. These are qualitative considerations of which one must be aware, but there is no obvious way of determining what constitutes "too large" and "too small" in the context of problems such as the one discussed in this paper.

Despite the obvious analytical drawbacks, this method of examining the capital structure planning problem shows promise. A problem of smaller magnitude (i.e., fewer and simpler constraints) would be more amenable to general solution in all likelihood. However, the model chosen here was examined because of its reasonableness as a financial model. It is apparent that this area is a fertile one for further investigation and that models like the one in this paper will continue to be proposed in the future until the questions raised have been resolved.
CHAPTER V

SUMMARY AND CONCLUSIONS

This paper dealt with the problem of capital structure planning, given known future costs and initial states of all variables involved. A model seeking to minimize weighted average before-tax cost of capital subject to five constraints was introduced. The solution to this non-linear program yields the optimal solution to the problem of how to add funds to the firm's capital structure over a given time period. Specifically, the time period was broken into small regions denoted by $\Delta$ and Kuhn-Tucker optimality conditions were developed for the model during the small period $\Delta$. Thus, given initial values of the variables, a solution algorithm for the resultant set of simultaneous equations, and enough computations, the optimal levels of debt, preferred stock, retained earnings and common stock could be determined for any time covered by the model. Unfortunately, the Kuhn-Tucker conditions made a general analytical solution impossible to obtain, but the methodology is widely accepted and the hope remains that simpler, still rigorous, models will be developed which lend themselves to solution more readily.
The contributions of this work were noted in Chapter I. What was of great importance here was the inclusion of the various relevant constraints, representing considerations made by creditors and stockholders in their decisions to lend to or invest in the firm. The model did not provide a very practical solution to the problem at hand, but this is the next stage of development in such models. Now the solution presented must suffice. The minimized cost approach was taken, opposing the earlier works in the field. This proved to be of little concern in view of the reasonable logic involved.

Interesting extensions of the work in the future will include solution of a similar stochastic model, exact specification of the cost functions, and work toward a more practical solution to the problem. If the distributions assumed by such a model were symmetric, it is likely the results would not differ much from what was found here. Specification of the cost functions would probably complicate the model further than it is already. Hopefully, the practical analytical solution will be found soon since the ultimate payoff from research is in the success of its methods in the business community.
LIST OF REFERENCES


28. Friedman, op. cit.


34. Tobin, op. cit.


