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DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Kamal Kanti Chakravarti, M.S.

* * * * *

The Ohio State University

1975

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INTRODUCTION

From Whitney's theorem on graphs and matroids we know that a graph $G$ is nonseparable if and only if every pair of its edges belongs to a circuit of $G$ or dually to a bond of $G$. So naturally a question was in the air, "When does an edge triple of a nonseparable graph $G$ belong to a circuit of $G$, i.e. can be covered by a circuit of $G$, or to a bond of $G$, i.e. can be covered by a bond of $G"?" The first part of this question was dealt with very nicely in the paper, Traversing edge triples by simple circular paths, by Professor G. N. Robertson, my advisor.

He proved the theorem: In order that no circular path in a nonseparable graph $G$ traverse $\{a, b, c\} \subseteq E(G)$ it is necessary and sufficient that $G$ have a polar structure $R$ which strongly separates $\{a, b, c\}$ and is either degenerate or has only three parallel classes of $R$-components. Under these conditions $R$ is uniquely determined.

The above result can be restated as follows if we define a bead and a bond-necklace. Let $N$ be a nonseparable graph with a partition $E(N) = E_1 \cup E_2 \cup ... \cup E_k$ into subsets $E_i$ such that

(i) $N \times E_i$ is nonseparable, for all $i = 1, 2, ..., k$,
(ii) $|W(N, N \times E_i)| = 2$, for all $i = 1, 2, ..., k$, and
(iii) if $K$ is a bond in $N$, then $K \subseteq E_i$ for some $i$, or
K meets every \( E_j \), for \( j = 1, 2, \ldots, k \).

Then \( N \) is called a bond-necklace and each \( N \times E_j \) is called a bead of the bond-necklace \( N \).

If \( G \) is nonseparable and (i) applies then conditions (ii) and (iii) are equivalent to the statement there exists \( (x, y) \subseteq V(G) \) such that \( W(G, G \cdot E_i) = \{x, y\} \), for all \( i = 1, 2, \ldots, k \). These conditions, however, more easily dualize and generalize to concepts in matroid theory.

Suppose \( G \) is a nonseparable graph and \( a, b, c \) are distinct members of \( E(G) \). Also, suppose that \( G \) has a decomposition (the subgraphs in any decomposition of a graph are assumed to be pairwise edge-disjoint)

\[
G = N \cup R
\]

where \( G \times E(N) \) is a bond-necklace. Let the corresponding partition of \( E(N) \) be given by

\[
E(N) = E_1 \cup E_2 \cup \ldots \cup E_k
\]

where each \( G \times E_i \) is a bead of \( G \times E(N) \). Each \( G \cdot E_i \) has two vertices of attachment in \( G \). The union of all \( G \cdot E_i \) with a fixed pair of vertices of attachment is called a segment of \( N \). \( R \) will be said to separate \( a, b, c \) if and only if \( a, b, c \) belong to distinct segments of \( N \).
**Modified statement of the above theorem:** Let $G$ be a nonseparable graph and $a$, $b$, $c$ be distinct edges of $G$. Then $(a, b, c) \notin E(P)$, for any polygon $P$ in $G$, if and only if there exists a decomposition $G = N \cup R$

such that

1. $(a, b, c) \subseteq E(N)$,
2. $G \times E(N)$ is a bond-necklace with $R$ separating $a$, $b$, $c$,
3. $R$ consists of two connected components each of which is nonseparable,
4. if $S_i$ is a bead of the bond-necklace $G \times E(N)$ then $G \cdot E(S_i)$ has two vertices of attachment, one in each component of $R$,
5. if two beads do not have the same vertices of attachment then their vertices of attachment are distinct in any component of $R$ which is not a vertex-graph,
6. if both components of $R$ are not vertex-graphs then they have three vertices of attachment each, and finally
7. such a decomposition is unique.

We thought first of generalizing this result for a connected matroid. In so doing we sought to answer the immediate question, "When is an edge triple $(a, b, c)$ of a nonseparable graph $G$ contained in a bond of $G$?" This dissertation gives a reasonably complete answer to this latter question.
In later chapters, some of the definitions are numbered consecutively with the propositions to emphasize their relative importance.

In Chapter 1 we first prove that if \( \{a, b, c\} \notin K \), where \( K \) is any bond of \( G \), then there exists a polygon \( P \) containing the edges \( a, b, c \). Next we see that there is no pair of skew diagonals of such a \( P \) separating \( a, b, c \) (definition 1.6.3). The non-existence of a pair of skew diagonals of such a \( P \) leads us to the formation of a polygon-necklace (definition 1.6.5. This is the matroid dual concept to bond-necklace) containing \( a, b, c \) in three distinct residual segments (definition 1.6.10), and ultimately at the end of the Chapter 2 we prove the Necklace Decomposition Theorem, which can be stated as follows:

Theorem I. Suppose \( G \) is a nonseparable graph and \( a, b, c \) are three distinct edges of \( G \). Then in order that \( \{a, b, c\} \) is not contained in any bond of \( G \) it is necessary and sufficient that there exists a polygon-necklace decomposition (definition 1.6.8)

\[
D = (N, R)
\]

of \( G \) such that exactly one of the following conditions applies:

1. \( R = \emptyset \), and \( a, b, c \) belong to distinct beads of \( N \).
2. \( |W(G, R)| = 3 \), and \( R \) consists of one or more 3-bridges of \( N \) which separate \( a, b, c \).
3. \( |W(G, R)| \geq 4 \), and \( R \) is a bridge of \( N \) separating
a, b, c which contains no pair of skew diagonals of N.

(4) \(|W(G, R)| \geq 4\), and R is a bridge of N separating a
which contains a pair of skew diagonals of N,
but no such pair which separates a, b, c.

Moreover, if D = (N, R) satisfies conditions (1), (2) or (3), then
the decomposition is unique.

Chapter 3 is an analysis of condition (4) in Theorem I. Using
the existence of a pair of skew diagonals of N and the nonexistence
of such a pair which separates a, b, c we deduce the existence of
a cone (definition 1.9.2). Given such a cone, two necklace decompo-
sitions can be formed which agree with the segment of N (containing
a, b, c) outside the cone, and whose two segments inside the cone
meet only in their endvertices. These two segments form a rim
(definition 1.9.5) of the cone. Thus the decomposition of Theorem I
is never unique when condition (4) applies. We then define a new
decomposition of G, in terms of the cones and the beads contained
in no cones (which are common to all the necklace decompositions in
this case), and finally prove the uniqueness of this Bead-cone
Decomposition (definition 1.9.7). The main theorem can be stated as
follows.

Theorem II. Let G be a nonseparable graph and a, b, c be distinct
edges of G. Suppose

(1) \([a, b, c] \notin K\), for K any bond of G,

(2) there exists a necklace decomposition G = N \cup R where
R is a bridge of N separating a, b, c, and
(3) there exists a pair of skew diagonals of \( N \).

Then there exist cones \( C_1, C_2, \ldots, C_n \) of \( G \) and beads \( S_1, S_2, \ldots, S_m \) which determine a bead-cone decomposition of \( G \). This bead-cone decomposition is unique. Moreover, for any cone \( C \), with \( W(G, C) = \{u, v, w\} \) and apex \( w \), there exists a pair of necklace decompositions \( G = N_1 \cup R_1 \) and \( G = N_2 \cup R_2 \) such that

\[
(4) \quad N_1 \cup N_2 = (N_1 \cap N_2) \cup ((N_1 \cup N_2) \cap C), \quad \text{and}
\]

\[
(5) \quad [u, v] = (N_1 \cap N_2) \cap ((N_1 \cup N_2) \cap C),
\]

where \( N_1 \cap N_2 = N'(u, v) \) and \( (N_1 \cup N_2) \cap C \) is a rim of \( C \).
1.1 Graphs and their subgraphs.

We follow W. T. Tutte's book Connectivity in Graphs [1] very closely for our definitions and notation. A graph $G$ consists of a set $V(G)$ of elements called vertices and a set $E(G)$ of elements called edges, together with a relation of incidence, which associates with each edge two vertices called its ends or endvertices. The two ends need not be distinct. An edge with distinct ends is called a link and an edge with coincident vertices is called a loop. Generally speaking lower case letters at the beginning of the alphabet will denote edges and lower case letters at the end of the alphabet will denote vertices. All figures are introduced in Chapter 1, and are collected at its end.

We write $|S|$ for the number of members of the set $S$. For what follows, assume that $V(G) \cap E(G) = \emptyset$ and $G$ is finite, i.e. $|V(G) \cup E(G)| < \infty$. We define the null graph $\Omega$ as the graph with $|V(G) \cup E(G)| = 0$ and denote this $G$ by $\Omega$. $H$ is a subgraph of a graph $G$, denoted by $H \subseteq G$, if and only if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and each edge of $H$ has the same ends in $H$ as in $G$. If $H$ is not identical with $G$, then we write $H \subsetneq G$. Given a subset $\{v_1, v_2, \ldots\}$ of $V(G)$, we denote by $[v_1, v_2, \ldots]$ the edgeless subgraph of $G$ with this set of vertices.
Let $H_1, H_2, \ldots, H_k$ be subgraphs of the graph $G$. We define the union $H_1 \cup H_2 \cup \ldots \cup H_k = H$ as the subgraph $H$ of $G$ where

$$V(H) = \bigcup_{i=1}^{k} V(H_i), \quad E(H) = \bigcup_{i=1}^{k} E(H_i)$$

and the intersection $H_1 \cap H_2 \cap \ldots \cap H_k = H'$ as the subgraph $H'$ of $G$ where

$$V(H') = \bigcap_{i=1}^{k} V(H_i), \quad E(H') = \bigcap_{i=1}^{k} E(H_i).$$

1.2 Vertices of attachment of a subgraph $H$ in $G$.

Let $H \subseteq G$. A vertex of attachment of $H$ in $G$ is a vertex of $H$ that is incident in $G$ with an edge not belonging to $H$. We denote the set of vertices of attachment of $H$ in $G$ by $W(G, H)$. Then for subgraphs $H$ and $J$ of $G$ we quote the following results from [1].

**Proposition 1.2.1.** $W(G, G) = W(G, \Omega) = \emptyset$.

**Proposition 1.2.2.** $W(G, H) \subseteq V(H)$.

**Proposition 1.2.3.** $W(G, H) = W(J, H \cap J) \cup \{V(H) \cap W(G, H \cup J)\}$.

**Proposition 1.2.4.** $W(G, H \cup J) \cup W(G, H \cap J) = W(G, H) \cup W(G, J)$.

**Proposition 1.2.5.** $W(G, H \cup J) \cap V(H \cap J) \subseteq W(G, H) \cap W(G, J)$.

**Proposition 1.2.6.** Let $q(H, J)$ denote the number of vertices $x$ of $G$ such that $x$ belongs to $(W(G, H) \cup W(G, J)) \setminus W(G, H \cup J)$. Then

1.3 Bridges of a subgraph $J$ of $G$.

Let us now suppose that $J$ is a subgraph of $G$. Then $G \setminus V(J)$ is the maximal subgraph of $G$ with vertex set $V(G) \setminus V(J)$ (i.e. the subgraph induced by $V(G) \setminus V(J)$). Following Tutte [1], we generalize the notion of a component of $G$ to that of a $J$-component of $G$. The $J$-components of $G$ are often called the $J$-bridges of $G$, or the bridges of $J$ in $G$. An inner $J$-component is a loop-graph or a link-graph of $G$ which is not in $J$ but whose incident vertices are in $J$. An outer $J$-component is the union of a component of $G \setminus V(J)$ with all the link-graphs of $G$ joining a vertex of the component to a vertex of $J$. The set $C_J(G)$ of $J$-components of $G$ is the union of the sets of inner and outer $J$-components of $G$. This specializes to the set of components $C(G)$ of $G$ when $J$ is the null graph $\emptyset$. The $J$-components of $G$ have no edge in $J$. Distinct $J$-components of $G$ intersect only in elements of $W(G, J)$. The union of $J$ and all the $J$-components of $G$ is the graph $G$. Figure 1.3A depicts a typical decomposition of a graph $G$ with its $J$-components. The subgraph $J$ is made up of all vertices and edges inside the circle drawn with broken lines.

A path in a graph $G$ is a finite sequence $A = (x_0, e_1, x_1, e_2, \ldots, e_n, x_n)$ whose terms are alternately vertices $x_i$ and edges $e_j$ of $G$, and which satisfies the following condition: if $1 \leq i \leq n$ then $x_{i-1}$ and $x_i$ are the ends of $e_i$ in $G$. The vertices $x_0$ and $x_n$ are called the ends of $A$ and the other vertices $x_1, \ldots, x_{n-1}$ internal vertices are called the internal vertices of $A$. We
admit the case \( n = 0 \), in which \( A \) has the single term \( x_0 \), but
we do not allow \( A \) to be a null sequence. Any edge or vertex of
\( G \) may appear more than once as a term of \( A \). We write \( E(A) \) for
the set of all edges of \( G \) occurring in \( A \) and \( V(A) \) for the set
of all vertices of \( G \) occurring in \( A \). The sets \( E(A) \) and \( V(A) \)
define the subgraph \( G(A) \) called the graph of the path \( A \). If
\( A = (x_0) \) we call \( A \) a degenerate path.

A path \( A \) in a graph \( G \) is a simple open path if no edge and
no vertex of \( G \) appear more than once in \( A \). Thus degenerate paths
are simple. A nondegenerate simple open path \( A \) has distinct end
vertices, i.e. \( x_0 \neq x_n \), and \( G(A) \) is then called an arc with ends
\( x_0 \) and \( x_n \). If, on the other hand, a nondegenerate path in \( G
\) with \( e_1, \ldots, e_n \) distinct, and \( x_1, \ldots, x_n \) distinct, but \( x_0 = x_n \),
then \( A \) is called a simple closed path of \( G \) and \( G(A) \) is called
a polygon of \( G \).

A path \( A \) then is defined to avoid \( J \) when its internal
vertices and all its edges are not contained in \( J \). Each such path
is contained in a unique \( J \)-component of \( G \). We can see from the above
definitions and the familiar properties of connected graphs, that any
two distinct vertices \( u, v \) in a \( J \)-component are joined by some
\( \{u, v\} \)-path avoiding \( J \) in that \( J \)-component.

Lastly, the remark on \( J \)-avoiding paths can be somewhat
strengthened by the following bridging lemma.
Proposition 1.3.1. Suppose \( H \in C_1(G) \), and that \( H_1, H_2 \subseteq H \) satisfy \( H_1 \cap H_2 \subseteq J \) and \( V(H_1) \neq V(H_1 \cup H_2) \neq V(H_2) \). Then there exists a path \( A \) in \( H \), avoiding \( H_1 \cup H_2 \), with ends \( h_1 \in V(H_1) \setminus V(H_2) \) and \( h_2 \in V(H_2) \setminus V(H_1) \). Moreover, \( A \) can be chosen so that \( h_1 \notin V(J) \) if \( H_1 \notin J \), and \( h_2 \notin V(J) \) if \( H_2 \notin J \) (Fig. 1.3B).

The proof of proposition 1.3.1 depends on the facts that there exist \( h_1 \in V(H_1) \setminus V(H_2) \) and \( h_2 \in V(H_2) \setminus V(H_1) \), with \( h_1 \notin V(J) \) if \( H_1 \notin J \), and \( h_2 \notin V(J) \) if \( H_2 \notin J \), and there exists a \( J \)-avoiding \( \{h_1, h_2\} \)-path \( A \subseteq H \). Any such path of minimal length also avoids \( H_1 \cup H_2 \), and thus satisfies the conditions of the proposition 1.3.1.

1.4 Nonseparable graphs.

We now define a graph \( G \) to be nonseparable when \( G=\varnothing \) or \( C_{[v]}(G)=[G] \), for all \( v \in V(G) \), and otherwise to be separable. A vertex \( v \) of \( G \) is a cut-vertex when it is contained in two or more \( [v] \)-components of \( G \). Nonseparable graphs are then the connected graphs without any cut-vertices. The maximal non-null nonseparable subgraphs of \( G \) are its blocks or cyclic elements. In Chapter 9 of [1] the elementary theory of separable and nonseparable graphs is developed. We want to mention the following important results from Chapter 9 of [1].

Proposition 1.4.1. Let \( H \) be a nonseparable subgraph of a graph \( G \), and let \( L \) be an arc in \( G \) whose two ends \( x \) and \( y \) are in \( V(H) \). Then \( H \cup L \) is nonseparable.
Proposition 1.4.2. Let $G$ be a graph and $H, J$ be distinct cyclic elements of $G$. Then $H \cap J$ is either null or a vertex-graph.

Proposition 1.4.3. Let $H$ be a nonseparable subgraph of a graph $G$, having at least one edge. Then $H$ is a cyclic element of $G$ if and only if no $H$-component of $G$ has more than one vertex of attachment.

Proposition 1.4.4. Let $G$ be a nonseparable graph. Let $H$ be a subgraph of $G$ such that $E(G) \setminus E(H)$ and $E(H)$ are both non-null. Then

$$|W(G, H)| > 2.$$ 

Proposition 1.4.5. Let $x$ and $y$ be distinct vertices, and $e$ be an edge, of a nonseparable graph $G$. Then there is an arc $L$ of $G$ that joins $x$ and $y$ and traverses $e$ (i.e. $e \in E(L)$).

1.5 Matroids

Suppose we are given a set $M$ of non-null subsets of a fixed finite set $E$, such that

$\text{M(1)}$ No member of $M$ is a proper subset of another.

$\text{M(2)}$ Let $a$ and $b$ be two members of $E$. Let $X$ and $Y$ be members of $M$ such that $a \in X \cap Y$ and $b \in X \setminus Y$. Then there exists a member $Z$ of $M$ such that $a \notin Z$, $b \in Z$ and $Z \subseteq X \cup Y$.

Then $M$ is called a matroid on $E$. We will refer to the members of $E$ as the cells and the members of $M$ as the atoms of the matroid $M$. 
Let $G$ be a graph with $E(G)$ as its edge set. A polygon $P$ of $G$ is a connected non-null graph in which the valency of each vertex is 2. We define a circuit to be the edge set of a polygon of $G$. Dually, a bond is a minimal set of edges of $G$ whose removal disconnects $G$. Hence $K$ is a bond of a graph $G$ if and only if there are components $K_1$ and $K_2$ of $G$: $(E(G) \setminus K)$ such that each edge of $K$ has one end in $V(K_1)$ and the other end in $V(K_2)$. The subgraphs $K_1$ and $K_2$ are called the endgraphs of $K$ in $G$.

Let $P(G)$ and $B(G)$ be the set of circuits and the set of bonds of $G$, respectively. Then $P(G)$ and $B(G)$ both satisfy $M(1)$ and $M(2)$ and are respectively called the polygon-matroid and bond-matroid of $G$. The circuits and bonds of $G$ are then called the atoms of $P(G)$ and $B(G)$, respectively.

We next define a dendroid $D$ of $M$ as a subset of $E$ which meets every atom of $M$ and is minimal with respect to this property. Following Lectures on matroids [2] by W. T. Tutte we have

**Proposition 1.5.1.** All dendroids $D$ of $M$ have the same number of cells. A matroid $M$ on a given set $E$ is uniquely determined by its dendroids.

We define the rank of $M$, denoted by $r(M)$, as the common cardinality of its dendroids.

**Definition 1.5.2 (Dual Matroids).**

Two subsets $T$ and $U$ of a finite set $E$ are called orthogonal if $|T \cap U| \neq 1$. Given a matroid on $E$, we denote the set of all non-null subsets of $E$ which are orthogonal to every member of $M$
by $L(M)$. Then $L(M)$ satisfies $M(2)$. Also, the minimal members of $L(M)$ satisfy both $M(1)$ and $M(2)$ and hence they form a matroid on $E$. We denote this matroid on $E$ by $M^*$ and call it the dual of $M$.

From [2] we have the following propositions.

**Proposition 1.5.3.** The dendroids of $M^*$ are the complements in $E$ of the dendroids of $M$.

**Remark.** From proposition 1.5.3 we see that 
$$(M^*)^* = M \quad \text{and} \quad r(M) + r(M^*) = |E|.$$  

**Proposition 1.5.4.** For any finite graph $(P(G))^* = B(G)$ and $(B(G))^* = P(G)$.

Let $M$ be a matroid on a finite set $E$ and suppose $U \subseteq E$.

Let $L$ be the set of non-null intersections with $U$ of atoms of $M$ and let $M.U$ be the set of minimal members of $L$. $L$ satisfies $M(1)$ and $M(2)$ so $M.U$ is a matroid on $U$. We call it the reduction of $M$ to $U$.

Let $M \times U$ be the set of all atoms $X$ of $M$ such that $X \subseteq U$.

Then $M \times U$ satisfies both $M(1)$ and $M(2)$. $M \times U$ is thus a matroid on $U$. We call it the contraction of $M$ to $U$. Then we have

**Proposition 1.5.5.** $(M.U)^* = M^* \times U$ and $(M \times U)^* = M^*$. $U$ for any subset $U \subseteq E$.

To draw a parallel with graphs we define $G \cdot U$ and $G \times U$ for a graph $G$, where $U \subseteq E(G)$. Then $G \cdot U$ is called a reduced subgraph of $G$, and $G \times U$ is called a reduced contraction of $G$, where
\[ E(G \cdot U) = U = E(G \times U), \]
\[ V(G \cdot U) = \{ v \in V(G); v \text{ is incident with a member of } U \}, \]
\[ V(G \times U) = \{ c_1 \in C(G \setminus (E(G) \setminus U)); \text{ a vertex of } C_1 \text{ is incident with a member of } U \} \]

Remark. The graphs \( G \cdot U \) and \( G \times U \) have no isolated vertex. \( G \cdot U \) and \( G \times U \) have no isolated vertex.

Remark. Define a bond-graph to be a connected loopless graph with exactly two vertices. Referring to the definition of a bond in the third paragraph of 1.5 we see that a bond of \( G \) is the edge set of a bond-graph contraction of \( G \). This parallels a circuit, which is the edge set of a polygon subgraph of \( G \).

Proposition 1.5.6. (i) \( B(G \cdot U) = B(G) \cdot U \)
(ii) \( B(G \times U) = B(G) \times U \)
(iii) \( P(G \cdot U) = P(G) \times U \)
(iv) \( P(G \times U) = P(G) \cdot U \).

Remark. In proposition 1.5.6 statements (iii) and (iv) follow from statements (i) and (ii) and proposition 1.5.5. This accounts for the reversal of operations.

Definition 1.5.7 (Connected Matroids).

Let us consider a matroid \( M \) on a set \( E \). We define a separator of \( M \) as a subset \( U \) of \( E \) such that each atom of \( M \) is contained either in \( U \) or in \( E \setminus U \). Evidently, any union or intersection of separators of \( M \) is a separator of \( M \) and complement of a separator of \( M \) is also a separator of \( M \). We refer to the minimal non-null separators of \( M \) as its elementary separators. A matroid \( M \) is
connected if and only if its only separators are $E$ and $\emptyset$. Thus $M$ is connected if and only if $E$ is $\emptyset$ or $M$ has just the one elementary separator $E$.

The following propositions can be found in [2].

**Proposition 1.5.8.** The elementary separators of $M$ are disjoint non-null subsets of $E$ whose union is $E$.

**Proposition 1.5.9.** Suppose $U \subseteq E$. Then $U$ is a separator of $M$ if and only if $M \cdot U = M \times U$.

**Proposition 1.5.10.** The dual matroids $M$ and $M^*$ have the same separators.

If $U$ is an elementary separator of $M$, then we refer to the matroid $M \cdot U = M \times U$ as a component of $M$.

**Proposition 1.5.11.** Any component of $M$ is a connected matroid.

**Proposition 1.5.12.** The dual matroids $M$ and $M^*$ have the same components.

**Remark.** Suppose $G$ is a graph without any isolated vertices. Then $G$ is nonseparable if and only if $F(G)$ is connected, or equivalently if and only if $B(G)$ is connected.

Thus we can conclude that $G$ is nonseparable if and only if, for any two complementary non-null subsets $T$ and $U$ of $E(G)$, there exists a polygon of $G$ whose edge set meets both $T$ and $U$.

We now state some properties of the rank of a matroid.
Proposition 1.5.13. Let $M$ be a matroid. Then

1. $0 \leq r(M) \leq |E|$, 
2. $r(M) = 0$ if and only if $M$ has no atom, 
3. $r(M) = 1$ if and only if $M$ has one atom, 
4. $r(M) = |E|$ if and only if the singleton subsets of $E$ are atoms of $M$.

Proposition 1.5.14. $r(M \times U) + r(M \cdot (E \setminus U)) = r(M)$. 

Proposition 1.5.15. If $T, U \subseteq E$ then 

$$r(M \times (T \cup U)) + r(M \times (T \cap U)) \geq r(M \times T) + r(M \times U)$$

for every pair of subsets $T$ and $U$ of $E$.

Consider the geometry of matroids. Here we study the system of contractions $M \times U$ of $M$. We abbreviate the expression $r(M \times U)$ by $rU$. We define the dimension of $M \times U$ by $d(M \times U) = r(M \times U) - 1$ and simply write $dU = rU - 1$. A subset $U$ of $E$ is called a flat of $M$ if it is a union of atoms of $M$. The null subset of $E$ is counted as a null union of atoms and therefore is a flat of dimension $-1$. The flats of dimension 0 are the atoms of $M$. For any subset $U$ of $E$ there is an associated flat $\langle U \rangle$, defined as the union of atoms of $M$ contained in $U$, that is the union of atoms of $M \times U$. Hence

$$d\langle U \rangle = dU = rU - 1.$$ 

Next, we say that the flat $T$ is on the flat $U$ of $M$ if either $T \subseteq U$ or $U \subseteq T$. The atoms of $M$ are also called its points and flats of dimension 1 and 2 are called lines and planes of $M$, respectively. More generally, a flat of dimension $k$ is called a
k-flat of $M$. The above notation and the following propositions are from [2].

**Proposition 1.5.16.** If $U$ is a flat of $M$ and $a \in U$, then

$$d(U \setminus \{a\}) = dU - 1.$$ 

**Proposition 1.5.17.** If $T$ and $U$ are flats of $M$ such that $T \subseteq U$ then $dT < dU$ and there exists a flat $T'$ of $M$ such that $T \subseteq T' \subseteq U$ with $dT' = dT + 1$.

**Proposition 1.5.18.** Let $T$ and $U$ be flats of $M$. Then

$$d(T \cup U) + d(T \cap U) \geq d(T) + d(U).$$

This proposition is a characteristic property of flats of $M$. For example, if $L_1$ and $L_2$ are distinct lines of $M$ on the same plane, then $L_1$ and $L_2$ intersect at a point of $M$.

A flat $U$ of $M$ is called connected if $M \times U$ is a connected matroid. We refer to the separators of $M \times U$ as the separators of $U$. Here we record some of the interesting properties of connected flats of $M$ from [2].

**Proposition 1.5.19.** Let $L$ be a line of $M$ and suppose $a \in L$. Then $\langle L - \{a\} \rangle$ is the only point on $L$ which does not include $a$.

**Proposition 1.5.20.** Any line $L$ on $M$ is on two distinct points of $M$. If $X$ and $Y$ are distinct points on $L$ then $L = X \cup Y$. Moreover, $X \cap Y \neq \emptyset$ if and only if $L$ is connected.

**Proposition 1.5.21.** A disconnected line is on just two points of $M$ and a connected line is on at least three points of $M$. 
The following are two important and useful propositions concerning connected flats of a matroid $M$. For example, proposition 1.5.22 is the matroid version of the handle theorem for nonseparable graphs.

**Proposition 1.5.22.** Let $T$ and $U$ be connected flats of $M$ such that $T \nsubseteq U$. Then there exists a connected $(d(T)+1)$-flat $T'$ of $M$ such that $T \nsubseteq T' \subseteq U$.

**Proposition 1.5.23.** Let $T$ be a connected $d$-flat on a connected $(d+2)$-flat $U$ of $M$. Then there exist distinct connected $(d+1)$-flats $T_1$ and $T_2$ of $M$ such that

$$T = \langle T_1 \cap T_2 \rangle \text{ and } U = T_1 \cup T_2.$$ 

**Proposition 1.5.24.** Let $a$, $b$ be distinct cells of a connected matroid. Then there is an atom of $M$ which includes both $a$ and $b$.

Proposition 1.5.24 can be derived from the Proposition 1.5.23. It is the matroid analogue of the following proposition in graph theory. Every pair of distinct edges in a nonseparable graph $G$ belongs to a polygon of $G$. Proposition 1.5.24 and the proposition just mentioned are usually ascribed to H. Whitney.

1.6 Notation and some preliminary definitions.

Let $G$ be a graph and $a$, $b$, $c \in E(G)$. We will denote any polygon $P$ of $G$ containing $a$, $b$, $c$ by $P_{abc}$, and the end-vertices of $a$, $b$, $c$ on $P$ by $P_{ab'}$, $P_{ac'}$, $P_{ba'}$, $P_{bc'}$, $P_{cb'}$, $P_{ca'}$. 
respectively, as in figure 1.6A. Similarly any bond $K$ of $G$ containing $a, b, c$ will be denoted by $K_{abc}$.

The segment on $P$ with extreme vertices $x$ and $y$ and containing the least number of edges from the set \{a, b, c\} will be denoted by $P(x, y)$ while the segment with the most number of edges from the set \{a, b, c\} will be denoted by $P'(x, y)$. For any pair of vertices $x, y$ on $P$, $P(x, y)$ can have at most one edge from \{a, b, c\}. For example $P(p_a, p_b)$ is the segment of $P$ between $a$ and $b$, and it does not contain any of the members belonging to \{a, b, c\}.

Also, suppose $u, v, w$ are distinct vertices of a polygon $P$. If $x, y \in V(P)$ let $P(x, y)$ be the segment of $P$ with endvertices $x$ and $y$ and the fewest of the vertices $u, v, w$ and $P'(x, y)$ be the segment of $P$ containing the most of the vertices $u, v, w$ and $x, y$.

Definition 1.6.1 (Skew diagonals of a polygon $P$). Call any $P$-avoiding arc $L$ with endvertices $u, v$ in $P$ a diagonal of $P$, where $P$ is a polygon of a graph $G$. Let $L_1$ be the diagonals of $P$ having endvertices $u_i, v_i$ on $P$, for $i = 1, 2$. The edge sets of the $[u_i, v_i]$-components of $P$ constitute a partition of $E(P)$, say $\{T_i, U_i\}$, for $i = 1, 2$. We say that the partition $\{T_i, U_i\}$ of $E(P)$ is induced by $L_i$, for $i = 1, 2$.

The two diagonals $L_1$ and $L_2$ of $P$ are said to be skew with respect to $P$ if and only if the partitions $\{T_1, U_1\}$ and $\{T_2, U_2\}$
of $E(P)$ induced by $L_1$ and $L_2$, respectively, satisfy the conditions:

\[ T_1 \cap T_2 \neq \emptyset, \quad T_1 \cap U_2 \neq \emptyset, \]
\[ U_1 \cap T_2 \neq \emptyset, \quad U_1 \cap U_2 \neq \emptyset. \]

**Proposition 1.6.2.** Two arcs $L_i$ with endvertices $u_i, v_i \in V(P)$, for $i = 1, 2$, are skew with respect to $P$ if and only if

(i) $u_1, u_2, v_1, v_2$ are distinct, and

(ii) $u_2, v_2$ belong to two distinct $[u_1, v_1]$-components of $P$.

**Proof.** Obvious.

**Definition 1.6.3** (Skew diagonals separating $a, b, c$).

Let $P_{abc}$ be a polygon of $G$ containing three distinct edges $a, b, c \in E(G)$. Then two skew diagonals $L_1$ and $L_2$ with respect to $P_{abc}$ will be said to separate $a, b, c$ if and only if each of the arcs $P_{ab}, P_{ba}, P_{bc}, P_{cb}, P_{ca}, P_{ac}$ contain an endvertex of $L_1$ or $L_2$ (Fig. 1.6A).

**Definition 1.6.4** (A bridge of $P$ separating $a, b, c$).

The $P$-components of $G$ are called the bridges of a polygon $P$ in $G$. Let $B$ be a bridge of $P_{abc}$ in $G$. Then $B$ will be said to be separating $a, b, c$ if and only if $W(G, B)$ meets vertices of each segment $P_{ab}, P_{ba}, P_{bc}, P_{cb}$ and $P_{ca}, P_{ac}$ of $P$ (Fig. 1.6B).

**Definition 1.6.5** (Polygon-Necklace).

A polygon-necklace is a nonseparable graph $N$ with a decomposition
\[ N = S_1 \cup S_2 \cup \ldots \cup S_m \]
such that

(i) Each \( S_i \) is a nonseparable subgraph of \( N \)

(ii) \( E(S_1) \cup E(S_2) \cup \ldots \cup E(S_m) \) is a partition of \( E(N) \)

(iii) \(|W(N, S_i)| = 2 \quad \forall i = 1, 2, \ldots, m\)

and

(iv) Every polygon of \( N \) is either contained in a unique \( S_i \),

for some \( i \) or its edge set meets all the \( S_i \)'s

\( i = 1, 2, \ldots, m \) (Fig. 1.6C)

Each such \( S_i \) will be called a polygon-bead of \( N \).

When the context is clear, we will drop the adjective polygon

used in the definitions. The last condition in the definition
generalizes easily to matroids. The equivalent condition which we

derive below for graphs is much more indicative of the graph-structure

of necklaces, which resembles a polygon very closely.

Proposition 1.6.6. Suppose \( G \) is a nonseparable graph with proper

nonseparable subgraphs \( S_1, S_2, \ldots, S_m, m \geq 3 \), where

\[ E(G) = E(S_1) \cup E(S_2) \cup \ldots \cup E(S_m) \]

is a partition of \( E(G) \). Let \( P \) be a polygon of \( G \). Then following

are equivalent (Fig. 1.6C).

1) Either there exists a unique \( i \) such that \( E(P) \subseteq E(S_i) \) or

\[ E(P) \cap E(S_j) \neq 0 \quad \text{for all } j = 1, 2, \ldots, m. \]

2) (i) \( |W(G, S_i)| = 2 \quad \text{for all } i = 1, 2, \ldots, m \)

(ii) \( v \in V(G) \) implies \( v \) belongs to at most two of the \( S_i \)'s.
Proof. We will show that (1) implies (2).

\[ G \text{ is nonseparable, } E(S_i) \neq \emptyset \text{ implies that } |W(G, S_i)| \geq 2 \]
by proposition 1.4.4, for all \( i = 1, 2, \ldots, m \). We claim that
\[ |W(G, S_i) \cap W(G, S_j)| \leq 1, \text{ for all } i, j \in \{1, 2, \ldots, m\}, \]
distinct. ..................................................(I).

For, if \( v \) and \( w \in W(G, S_i) \cap W(G, S_j) \) then there exist arcs
in \( S_i \) and \( S_j \) containing the vertices \( v \) and \( w \). Also
\[ E(S_i) \cap E(S_j) = \emptyset. \] Hence, these two arcs form a polygon of \( G \)
whose edge set meets only two members \( S_i \) and \( S_j \) and, not a third
one from the remaining \( S \)'s. This contradicts (1). Hence the claim.

Next, let \( v \in W(G, S_i) \). We will prove that there exists a
unique \( j, j \neq i \) such that \( v \in S_j \). .........................(II).

If \( v \in W(G, S_i) \cap W(G, S_j) \cap W(G, S_\ell) \) where \( i, j, \ell \) are
distinct, then
\[
\{v\} = W(G, S_i) \cap W(G, S_j) = W(G, S_j) \cap W(G, S_\ell) \\
= W(G, S_\ell) \cap W(G, S_i),
\]
by the result above. \( E(S_i) \neq \emptyset, E(S_j) \neq \emptyset \neq E(S_\ell) \) and \( G \) is
nonseparable implies that there exists a polygon containing one edge
from \( S_i \) and another edge either from \( S_\ell \) or \( S_j \). So there exists
an arc \( A \) joining a vertex \( w \in S_i \setminus \{v\} \) and \( t \in S_\ell \setminus \{v\} \) or \( t \in S_j \setminus \{v\} \),
according as the arc \( A \) meets \( S_j \) first or \( S_\ell \) first. Accordingly,
the union of the arcs joining \( w \) and \( v \) in \( S_i \), \( v \) and \( t \) in
\( S_j \) or \( S_\ell \) and \( A \) form a polygon in \( G \) whose edge set meets two of
\( \{S_i, S_j, S_\ell\} \) and not the remaining one. This is a contradiction to
our hypothesis. So for any \( v \in V(G) \) either \( v \in V(S_i) \backslash W(G, S_i) \) and hence belongs to exactly one \( S_i \), or \( v \in W(G, S_i) \) for some \( i \) and hence \( v \) belongs to exactly two of the \( S_i \)'s by (II). To prove \( |W(G, S_i)| \leq 2 \), let us start with the assumption \( |W(G, S_i)| = n \geq 3 \). Then for a fixed \( i = 1, 2, \ldots, m \) define
\[ \{S_j/x \in V(S_j) \text{ for some } x \in W(G, S_j)\} . \]
Also let \( x, y \) and \( z \) be distinct members of \( W(G, S_i) \). Then \( S_x, S_y, S_z \) are distinct by (II).

Consider the edge \( c \) incident with \( z \) in \( S_z \) and \( d \) incident with \( y \) in \( S_y \). There exists a polygon of \( G \) containing \( c \) and \( d \). So there is an \( S_i \)-avoiding arc \( L \) from \( z \) to \( y \) in \( G \) containing \( c \) and \( d \). Amongst \( z, y \in W(G, S_i) \), let \( L \) be the \( S_i \)-avoiding arc from \( z \) to \( y \) which contains edges from minimum number of members of the set \( \{S_1, S_2, \ldots, S_m\} \).

If there exists a member \( S_j \neq S_z, S_y \) meeting \( L \) in an edge and \( S_i \) in a vertex (by I), then we have an \( S_i \)-avoiding arc \( L' \) from a vertex of \( W(G, S_j) \) to either \( z \) or \( y \) containing edges from fewer number of members of the set \( \{S_1, S_2, \ldots, S_m\} \) than \( L \), contradicting the definition of \( L \). Hence there is no member \( S_j \) which meets \( L \) in an edge and has a vertex in \( S_i \) other than \( S_z \) and \( S_y \).

The arc \( L \) and any arc from \( z \) to \( y \) in \( S_i \) form a polygon of \( G \) and this polygon has no edge in \( S_x \), which is a contradiction to the assumption that \( n \geq 3 \). So \( |W(G, S_i)| \leq 2 \), and finally \( |W(G, S_i)| = 2 \).

Now we show (2) implies (1).
Let $S_1 \in \{S_1, S_2, \ldots, S_m\}$ and $W(G, S_1) = \{u, v\}$. If $u$ or $v \in W(G, S_1) \cap W(G, S_j)$ and $i, j, \ell$ are distinct then $u$ or $v$ belong to at least 3 of $S_j$'s contradicting 2(ii). Hence $u$ or $v$ belong to at most one $S_j$, $j \neq i$. Again if both $u$ and $v$ belong to the same $S_j$ then $W(G, S_i \cup S_j) = \emptyset$ since $u$ or $v$ can belong to at most one $S_j$, $j \neq i$. Hence $G$ is not connected, since $m \geq 3$, which contradicts the fact that $G$ is nonseparable. So $u$ and $v$ belong to distinct members of $\{S_1, S_2, \ldots, S_m\}\{S_i\}$ and this is true $\forall i = 1, 2, \ldots, m$.

Let $W(G, S_1) = \{v_0, v_1\}$. We now start with $S_1$ and put together the distinct members of $\{S_2, \ldots, S_m\}$ which have $v_0$ and $v_1$ as a vertex of attachment in $G$ with $S_1$. Continuing the process and if necessary, by applying a permutation of the suffixes we get

$$W(G, S_i) = \{v_{i-1}, v_i\} \quad i = 1, 2, \ldots, m$$

where $v_m = v_0$ and Figure 1.6C.

Let $P$ be a polygon of $G$ and $e \in E(P) \cap E(S_i)$. If $v_{i-1}, v_i \notin P$ then $P \subseteq S_i$. Suppose $v_i \in P$ and $E(P) \cap E(S_i)$ contains two edges incident with $v_i$. If $v_{i-1} \notin P$ then $P \subseteq S_i$ since the degree of $v_i$ in $P \cap S_i$ is 2. If $v_{i-1} \in P$ and $E(P) \cap E(S_i)$ contains two edges incident with $v_{i-1}$, then degree of $v_i$ in $P \cap S_i = 2 =$ degree of $v_{i-1}$ in $P \cap S_i$. So, both the
[\nu_{i-1}, \nu_i]-components of \(P\) are contained in \(S_i\) i.e. \(P \subseteq S_i\).

If \(\nu_{i-1} \in P\) and \(E(P) \cap E(S_i)\) contains just one edge incident with \(\nu_{i-1}\), then one of the \([\nu_{i-1}, \nu_i]-components of \(P\) is in \(S_i\).

Also, the other \([\nu_{i-1}, \nu_i]\)-component of \(P\) cannot have any edge from \(S_{i+1}\) since degree of \(\nu_i\) in \(P \cap S_i\) is 2. If this component contains an edge from \(S_{i-1}\), then it must have an edge in \(S_{i-1}\) incident with \(\nu_{i-1}\). This implies a contradiction. Hence, by symmetry \(E(P) \cap E(S_i) \neq \emptyset\) and \(E(P) \not\subseteq E(S_i)\) implies that \(E(P) \cap E(S_i)\) contains one edge incident with each of \(\nu_{i-1}\) and \(\nu_i\). The degree of \(\nu_{i-1}\) and \(\nu_i\) in \(P\) is 2. So, \(P\) has one edge in each of \(S_{i-1}\) and \(S_{i+1}\) incident with \(\nu_{i-1}\) and \(\nu_i\) respectively.

Hence \(E(P) \cap E(S_{i-1}) \neq \emptyset \neq E(P) \cap E(S_{i+1})\) and \(E(P) \not\subseteq E(S_{i-1} \cup S_{i+1})\)

implying that \(P\) has one edge in each of \(S_{i-1}\) and \(S_{i+1}\) incident with \(\nu_{i-2}\) and \(\nu_{i+1}\) respectively. Hence by induction,

\(E(P) \cap E(S_j) \neq \emptyset \forall j = 1, 2, \ldots, m\). Hence the proof of the converse.

**Proposition 1.6.7.** Suppose \(N\) is a necklace with beads \(S_1, S_2, \ldots, S_m\) and \(X \subseteq \bigcup_{i=1}^{m} W(N, S_i)\). Then the following are true.

1. If \(|X| \leq 1\) then there is one \([X]\)-component of \(N\), namely \(N\) itself.
2. If \(|X| > 2\) then the number of \([X]\)-components of \(V\) is \(|X|\).
Moreover, each such \([X]\)-component of \(N\) is a union of beads of \(N\) and has exactly two vertices of attachment (in \(X\)). Each member of \(X\) is in exactly two \([X]\)-components of \(N\).

Proof. \(X\) is a subset of the set of vertices of attachment of the beads of \(N\) (Fig. 1.6C). Let

\[ W(G, S_i) = \{v_{i-1}, v_i\}, \text{ for all } i = 1, 2, \ldots, m, \]

where \(v_m = v_0\). Then \(X \subseteq \{v_0, v_1, \ldots, v_{m-1}\}\).

Proof of (1): If \(|X| \leq 1\) then \(C_{[X]}(N)\). Since \(N\) is a nonseparable graph.

Proof of (2): We prove by induction. Let \(X = \{v_i, v_{i+k}\}\), where

\[ 0 \leq i < i + k \leq m - 1. \]

Then \(S_i \cap S_{i+1} = [v_i]\) and \(S_{i+k} \cap S_{i+k+1} = [v_{i+k}]\) and the \(S_j\)'s are nonseparable subgraphs of \(N\). Hence, by construction of \(N\),

\[ S_{i+k+1} \cup S_{i+k+2} \cup \ldots \cup S_{i+k+\lambda} \cup S_{i+\lambda} \cup S_{i+\lambda+1} \cup \ldots \cup S_{i+k} \]

are two \([v_i, v_{i+k}]\)-components of \(N\) satisfying all the conditions laid down in (2). So we assume the result (2) to be true for all proper subsets \(Y\) of \(X\). Let \(|X| = n + 1 = |Y| + 1\) for a given \(Y\).

Let \(x \in X \setminus Y\). This implies that \(x\) belongs to a \([Y]\)-component \(J\) of \(N\). Without loss of generality let the vertices of attachment of \(J\) in \(N\) be \(v_i\) and \(v_{i+k}\) for \(0 \leq i < i+k < m\), and let \(x = v_{i+l}\) such that \(i < i+l < i+k\). Then \(J\) is the subgraph

\[ S_{i+1} \cup S_{i+2} \cup \ldots \cup S_{i+\lambda} \cup S_{i+\lambda+1} \cup \ldots \cup S_{i+k} \]
satisfying all the conditions in the statement (2). The \([v_{i+2}]\)-components of \(J\) consist of the subgraphs
\[
S_{i+1} \cup S_{i+2} \cup \ldots \cup S_{i+\ell} \quad \text{and} \quad S_{i+\ell+1} \cup S_{i+\ell+2} \cup \ldots \cup S_{i+k}
\]
of \(N\) and satisfy the properties as stated in the statement (2). Hence
\[
C_{[X]}(N) = (C_{[Y]}(N) \setminus [J]) \cup (S_{i+1} \cup S_{i+2} \cup \ldots \cup S_{i+\ell}, S_{i+\ell+1} \cup S_{i+\ell+2} \cup \ldots \cup S_{i+k}).
\]
Hence \(|C_{[X]}(N)| = |C_{[Y]}(N)| - 1 + 2 = n - 1 + 2 = n + 1\), and each of the members of \(C_{[X]}(N)\) and of \(X\) satisfy the conditions of the proposition. Hence, by induction on \(|X|\), we have proved the proposition.

We will refer to the \([X]\)-components of \(N\) as the \([X]\)-segments of \(N\), and also to the vertex graph \([x]\), for \(x = \{x\}\), as a (degenerate) \([x]\)-segment.

**Definition 1.6.8** (Decomposition of a nonseparable graph into a polygon-necklace and its residue).

Let \(G\) be a nonseparable graph. Then a decomposition of
\[
G = N \cup R
\]
into edge-disjoint subgraphs \(N\) and \(R\) of \(G\) such that
(i) \(N\) is a polygon-necklace with beads \(S_i\), for \(i = 1, 2, \ldots, m\), and
(ii) \(W(G, R) \subseteq \bigcup_{i=1}^{m} W(N, S_i)\), will be called a polygon-necklace decomposition with residue \(R\). When \(R = \Omega\) the graph \(G\) is a polygon necklace with beads \(S_i\), for \(i = 1, 2, \ldots, m\).
Definition 1.6.9 (Bridge of a necklace decomposition).

Suppose \( G \) is a nonseparable graph with a polygon-necklace decomposition \( D = (N, R) \). Then each \( N \)-component of \( G \) will be called a bridge of \( N \).

If \( B \) is such an \( N \)-component of \( G \) then

\[(G \times (E(G)\setminus E(N))) \cdot E(B)\]

will be a nonseparable graph with

\[W(G, B) \subseteq W(G, R) \subseteq \bigcup_{i=1}^{m} W(N, S_i)\]

where the \( S_i \), for \( i = 1, 2, \ldots, m \), are the beads of \( N \) (Fig. 1.6D).

Definition 1.6.10 (Residual segments in a polygon-necklace decomposition).

Suppose \( G \) is a nonseparable graph with a polygon-necklace decomposition \( D = (N, R) \). Then the \( W(G, R) \)-segments of \( N \) will be called the residual segments of \( N \) in the decomposition \( D = (N, R) \) of \( G \) and each such residual segment will be denoted by \( T_j \) for some subscript \( j \) (Fig. 1.6D).

Suppose \( G \) is a nonseparable graph with three distinct edges \( a, b, c \) and having a polygon-necklace decomposition \( D = (N, R) \).

Also, suppose that \( a, b, \) and \( c \) belong to distinct beads of the polygon-necklace \( N \) and \( \{x, y\} \subseteq \bigcup_{i=1}^{m} W(N, S_i) \), where \( S_i \)'s are the beads of \( N \) and \( x \neq y \). Then \( N(x, y) \) will denote that \( [x, y] \)-segment of \( N \) which contains the least number of edges from the set \( \{a, b, c\} \).
while the \([x,y]-segment\) of \(N\) with most number of edges from \(\{a, b, c\}\) will be denoted by \(N'(x, y)\).

For the sake of brevity, whenever possible we will denote by \(S_a, S_b\) and \(S_c\) the beads of \(N\) containing \(a, b\) and \(c\) respectively. Then for any pair of vertices \(x, y \in \bigcup_{i=1}^{m} W(N, S_i)\), the segment \(N(x,y)\) can have at most one bead from \(\{S_a', S_b', S_c'\}\) and hence at most one edge from \(\{a, b, c\}\).

If \(G\) is a polygon-necklace and \(s_{ab}, s_{ac}, s_{ba}, s_{bc}\) and \(s_{cb}, s_{ca}\) are the vertices of attachment of \(S_a, S_b\) and \(S_c\), respectively, in their circular order around the necklace, then \(N(s_{ab}', s_{ba}')\) is the segment of \(N\) between \(S_a\) and \(S_b\) and it does not contain any of \(a, b\) or \(c\). If \(R \neq \emptyset\), let \(T_a, T_b, T_c\) be the \(W(G, R)\)-segment of \(N\) in the decomposition \(D = (N, R)\) of \(G\) containing the beads \(S_a, S_b\) and \(S_c\) of \(N\), respectively. When \(T_a, T_b\) and \(T_c\) are distinct we will denote the vertices of attachment of \(T_a, T_b\) and \(T_c\) in \(N\) by \(t_{ab}, t_{ac}; t_{ba}, t_{bc}; t_{cb}, t_{ca}\), respectively, in their circular order around the necklace. Then \(N(t_{ab}', t_{ba}')\) will denote the segment of \(N\) between \(T_a\) and \(T_b\) not containing any of \(a, b\) or \(c\), while \(N'(t_{ab}', t_{ba}')\) will be the other \([t_{ab}', t_{ba}]-component\) of \(N\), containing all the three edges \(a, b,\) and \(c\).

**Definition 1.6.11** (Skew diagonals of a necklace \(N\)).

Let \((N, R)\) be a necklace decomposition of \(G\). An \(N\)-avoiding arc \(L\) with endvertices \(u, v \in V(N)\) is called a diagonal of \(N\).
Let $L_i$ be diagonals of $N$ having endvertices $u_i, v_i$ on $N$, for $i = 1, 2$. The edge sets of the $[u_i, v_i]$-components $N(u_i, v_i)$ and $N'(u_i, v_i)$ form a partition of $E(N)$, say $(E(N(u_i, v_i)), E(N'(u_i, v_i)))$, $i = 1, 2$. We say that this partition is induced by the $L_i$, for $i = 1, 2$.

The two diagonals $L_1$ and $L_2$ of $N$ are said to be skew with respect to $N$ (Fig. 1.6E) if and only if the respective partitions induced by $L_1$ and $L_2$ satisfy the conditions

\[
E(N(u_1, v_1)) \cap E(N(u_2, v_2)) \neq \emptyset \neq E(N'(u_1, v_1)) \cap E(N(u_2, v_2)) \\
E(N(u_1, v_1) \cap E(N'(u_2, v_2)) \neq \emptyset \neq E(N'(u_1, v_1)) \cap E(N'(u_2, v_2)).
\]

**Proposition 1.6.12.**

Two diagonals $L_1$ and $L_2$ are skew with respect to $N$ if and only if

(i) $u_1, u_2, v_1, v_2$ are distinct

(ii) $u_2$ and $v_2$ belong to two distinct members of the set

\[
(V(N(u_1, v_1), V(N'(u_1, v_1)),
\]

where $u_1, v_1$ and $u_2, v_2$ are the end vertices of $L_1$ and $L_2$, respectively, on $N$ (Fig. 1.6E).

**Definition 1.6.13** (Skew diagonals of a necklace separating $a, b$ and $c$).

Suppose $G$ is a nonseparable graph with three distinct edges $a, b, c$ and having a polygon-necklace decomposition.
Also, suppose that \( a, b \) and \( c \) belong to distinct beads, say \( S_a, S_b \) and \( S_c \), respectively, on the polygon-necklace \( N \). Let
\[
W(N, S_a) = \{s_{ab}, s_{ac}\}, \quad W(N, S_b) = \{s_{ba}, s_{bc}\} \quad \text{and} \quad W(N, S_c) = \{s_{cb}, s_{ca}\},
\]
as usual.

Then two skew diagonals \( L_1 \) and \( L_2 \) of \( N \) will be said to separate \( a, b, c \) if and only if each of the segments \( N(s_{ab}, s_{ba}) \), \( N(s_{bc}, s_{cb}) \) and \( N(s_{ca}, s_{ac}) \) contains an endvertex of \( L_1 \) or \( L_2 \) (Fig. 1.6E).

**Definition 1.6.14 (Bridges of \( N \) separating \( a, b, c \)).**

Suppose \( G \) is a nonseparable graph with three distinct edges \( a, b, c \) and \( E(G) \) and having a polygon-necklace decomposition
\[
D = (N, R),
\]
where \( N \) has three distinct beads \( S_a, S_b \) and \( S_c \) containing the three edges \( a, b \) and \( c \), respectively. If \( s_{ab}, s_{ac}, s_{ba}, s_{bc} \) and \( s_{cb}, s_{ca} \) are the vertices of attachment of \( S_a, S_b \) and \( S_c \) in \( N \), in standard order, then an \( N \)-component \( B \) of \( G \) will be called a bridge of \( N \) separating \( a, b, c \) if and only if \( W(G, B) \) contains vertices from each of the segments
\[
N(s_{ab}, s_{ba}), N(s_{bc}, s_{cb}) \quad \text{and} \quad N(s_{ca}, s_{ac}).
\]
1.7 Skew bridges.

Let $P$ be a polygon of a nonseparable graph $G$. Then the $P$-components of $G$ are called the bridges of $P$ in $G$. If $B$ is a bridge of $P$ in $G$ then $(G \times (E(G)\setminus E(P))) \cdot E(B)$ is a nonseparable graph. We know from proposition 1.5.6 that

$$P(G) \cdot (E(G)\setminus E(P)) = P(G \times (E(G)\setminus E(P))).$$

Let $Q$ be a polygon of the graph $G \times (E(G)\setminus E(P))$. If $Q' = G \cdot E(Q)$ is a polygon of $G$ and $|V(Q') \cap V(P)| \leq 1$, then we define the partition of $E(P)$ induced by $Q$ to be $\{E(P), \emptyset\}$. If $Q'$ is not a polygon of $G$ then $Q'$ is a diagonal of $P$. Hence the endvertices of $Q'$ induce a partition of $E(P)$. Let this partition be $\{X_Q, Y_Q\}$.

Suppose $B$ is a bridge of $P$ in $G$. Then, for all polygons $Q$ of $G \times (E(G)\setminus E(P))$, for $Q \subseteq B$, find $\{X_Q, Y_Q\}$, the partition of $E(P)$ induced by $Q$. Let $\pi(P(G), B, P)$ be the set of minimal non-null intersections of the $X_Q$ and $Y_Q$ for all $Q \in P(G \times (E(G)\setminus E(P)))$ such that $Q \subseteq B$. Obviously the members of $\pi(P(G), B, P)$ form a partition of $E(P)$.

Two bridges $B_1$ and $B_2$ of $P$ will be said to avoid each other if there exist $Z_1 \in \pi(P(G), B_1, P)$ and $Z_2 \in \pi(P(G), B_2, P)$ such that $Z_1 \cup Z_2 = E(P)$, and to overlap one another in the contrary case.

Suppose $Q_1$ and $Q_2$ are polygons of $G \times (E(G)\setminus E(P))$ such that $Q'_1 = G \cdot E(Q_1)$ and $Q'_2 = G \cdot E(Q_2)$ are two diagonals of $P$. Then, by definition 1.6.1, $Q'_1$ and $Q'_2$ are skew with respect to $P$ if
and only if

\[ X_{Q_1} \cap X_{Q_2} \neq \emptyset \neq X_{Q_1} \cap Y_{Q_2} \]
\[ Y_{Q_1} \cap X_{Q_2} \neq \emptyset \neq Y_{Q_1} \cap Y_{Q_2} . \]

We define two bridges \( B_1 \) and \( B_2 \) of \( P \) to be skew with respect to the polygon \( P \) if and only if there are polygons \( Q_1 \) and \( Q_2 \) of \( G \times (E(G) \setminus E(P)) \) on \( B_1 \) and \( B_2 \) respectively such that \( Q_1' \) and \( Q_2' \) are skew diagonals with respect to \( P \).

Let \( B \) be any bridge of \( P \) in \( G \). We call it an \( n \)-bridge of \( P \) if and only if

\[ |\pi(P(G), B, P)| = n . \]

Two \( n \)-bridges \( B_1 \) and \( B_2 \) of \( P \) in \( G \) are called equipartite if and only if

\[ \pi(P(G), B_1, P) = \pi(P(G), B_2, P) . \]

Remark. \( |\pi(P(G), B, P)| = n \) if and only if \( |W(G, B)| = n \), for any bridge \( B \) of \( P \) in a connected graph \( G \).

**Proposition 1.7.1.** Let \( B_1 \) and \( B_2 \) be two overlapping bridges of \( P \) in \( G \). Then either \( B_1 \) and \( B_2 \) are skew or they are equipartite 3-bridges.

**Proof.** Suppose first that \( B_1 \) and \( B_2 \) are equipartite \( n \)-bridges of \( P \) in \( G \). If \( n \leq 2 \), they clearly avoid one another. If \( n \geq 4 \),
they are equally clearly skew bridges. Thus if $B_1$ and $B_2$ overlap and are not skew then they are equipartite 3-bridges. Secondly, let us assume that $B_1$ and $B_2$ are not equipartite. Then some $Z \in \pi(G, B_1, P)$ exists such that $P \cdot Z$ contains a vertex of attachment $v$ of $B_2$ as an internal vertex. If $W(G, B_2) \subseteq V(P \cdot Z)$ then $B_1$ and $B_2$ avoid each other. Otherwise $B_2$ has a vertex of attachment $u$ in $V(P) \setminus V(P \cdot Z)$. Then any $P$-avoiding arc in $B_2$ joining $u$ and $v$, and any $P$-avoiding arc in $B_1$ joining the extreme vertices of $P \cdot Z$ form a pair of skew diagonals of $P$. Hence in this case $B_1$ and $B_2$ are skew.

1.8 Polygons and bonds.

Let $G$ be a nonseparable graph with three distinct edges $a, b, c$. Suppose $\{a, b, c\} \not\subseteq K$ for any bond $K$ of $G$.

A polygon $P$ (a bond $K$) of $G$ containing the edges $a, b, c, \ldots$ will be denoted by $P_{a, b, c, \ldots} (K_{a, b, c, \ldots})$. For convenience we will drop the commas in between the edges used as the subscripts in either $P$ or $K$. Let $\{f, g, h\} = \{a, b, c\}$. Then we define

$$E_{abc} = \{e \in E(G); \text{there exists a polygon } P_{abc}; \text{in } G\},$$

$$E_{fg} = \{e \in E(G); \text{there exists a polygon } P_{fge} \text{ but no } P_{abc} \text{ or } P_{fhe} \text{ in } G\},$$

$$E_{fg, gh} = \{e \in E(G); \text{there exists polygons } P_{fge} \text{ and } P_{ghe} \text{ but no polygon } P_{abc} \text{ or } P_{fhe} \text{ in } G\}.$$
$E_{ab, bc, ca} = \{e \in E(G) : \text{there exist polygons } P_{abc}, P_{bce} \text{ and } P_{cae}
\text{but no polygon } P_{abce} \text{ in } G\}.$

$E^1_{abc} = \{e \in E_{abc} : \text{at least one of the bonds } K_{abc}, K_{bce} \text{ and } K_{cae}
\text{does not exist in } G\}, \text{ and }$

$E^2_{abc} = \{e \in E_{abc} : \text{all bonds } K_{abc}, K_{bce} \text{ and } K_{cae}
\text{exist in } G\}.$

Proposition 1.8.1.

(i) $E^1_{abc} \cup E^2_{abc} = E_{abc}$ is a partition of $E_{abc}.$

(ii) $E^1_{abc} \cup E^2_{abc} \cup E_{bc} \cup E_{ca} \cup E_{ab} \cup E_{ab, bc} \cup E_{bc, ca} \cup E_{ca, ab}
\cup E_{ab, bc, ca} = E(G)$ is a partition of $E(G).$

1.9 Bead-cone decompositions.

Throughout this section we will suppose that $G$ is a nonseparable graph having a necklace decomposition $D = (N, R)$ such that

(1) $\{a, b, c\} \subseteq E(N),$

(2) $|W(G, R)| \geq 4,$

(3) $R$ is a single bridge of $N$ separating $a, b, c,$ and

(4) $\{a, b, c\} \nsubseteq K,$ for any bond $K$ of $G.$

Remark. In Chapter 2, it is shown that under conditions (1), (2), (3) condition (4) above is equivalent to the existence of a pair of skew diagonals of $N$ separating $a, b, c.$

Definition 1.9.1 (Singularity in a necklace decomposition).

Any pair $L_1, L_2$ of skew diagonals of $N,$ whose endvertices
are not all in a single member of $\{N(t_{ab'}, t_{ba}), N(t_{bc'}, t_{cb}), N(t_{ca'}, t_{ac'})\},$
is called a singularity of the decomposition $D$.

It will be seen in Chapter 2, that under conditions (1), (2), (3) above, the existence of a pair of skew diagonals of $N$ in $G$ leads either to a pair of skew diagonals separating $a$, $b$, $c$, and hence to a bond $K_{abc}$, or under (4) to a singularity of $D$. Moreover, in a singularity, one of the diagonals, say $L_1$, must have both of its endvertices $u$, $v$ in $N(t_{fg}, t_{gf})$, for some $\{f, g\} \subseteq \{a, b, c\}$, and the other diagonal has exactly one endvertex in $N(t_{fg}, t_{gf})$. We will also see that $E(N(u, v)) \subseteq E_{abc}$, which is a property characteristic of the existence of singularities.

By definition, $u$, $v \in W(G, R)$. This implies that there exists a residual segment, say $T$, of $N$ contained in $N(u, v)$. Such a pair $L_1, L_2$, with endvertices of $L_1$ in $N(t_{fg}, t_{gf})$, will be called a singularity spanning $T$ or spanning any segment of $N(u, v)$ with respect to $D$, and will be denoted by $S(N, R, T)$.

**Definition 1.9.2** (Partial cone and cone with respect to a necklace decomposition of $G$).

Any nonseparable subgraph $C'$ of $G$ such that

1. $W(G, C') = \{u', v', w'\}$, for distinct $u', v' \in W(G, N)$ and $w' \notin V(N),$
2. there exists a $\theta$-graph $\theta \subseteq C'$, with $u'$, $v'$, $w'$ as internal vertices of distinct branches of $\theta$,
3. $C'$ is a union of some 3-bridges of $[u', v', w']$ in $G$, with none of $a$, $b$, $c$ belonging to such bridges, and
(4) \( N(u', v') \subseteq C' \),
is called a partial cone of \( D \).

A maximal partial cone of \( D \) is called a cone of \( D \). A
typical cone is shown in Fig. 1.9.B. In Chapter 3 cones are shown
to be edge-disjoint.

In the above notation \( u', v' \) are called the extremes and \( w' \) the
apex of the partial cone or the cone.

**Definition 1.9.3** (Meeting of a singularity and a partial cone of a
necklace decomposition).

Suppose \( S(N, R, T) \) is a singularity of \( D \) spanning \( N(u, v) \)
and \( C' \) is a partial cone of \( D \) with extremes \( u', v' \). Then
\( S(N, R, T) \) meets \( C' \) if and only if \( N(u, v) \cap N(u', v') \) contains
a bead of \( N \).

**Definition 1.9.4** (Engulfing a singularity in a partial cone of a
necklace decomposition).

Suppose \( S(N, R, T) \) is a singularity of a necklace decomposition
\( D \) and \( C' \) is a partial cone of \( D \).

Then \( S(N, R, T) \) is engulfed in the partial cone \( C' \) when the
arc \( L \in S(N, R, T) \) with both endvertices in some \( N(t_{fg}, t_{gf}) \), for
\( \{f, g\} \subseteq \{a, b, c\} \), is contained in \( C' \).

**Remark.** A singularity is always engulfed in some partial cone, by
results obtained in Chapter 2. Hence all singularities of a necklace
decomposition on \( D \) are engulfed by the cones of \( D \).
**Definition 1.9.5** (Rim of a partial cone of a necklace decomposition).

Let \( C' \) be a partial cone of \( G \) with respect to the necklace decomposition \( D \), with \( W(G, C') = \{u', v', w'\} \), where \( u' \), \( v' \) are the extremes and \( w' \) is the apex. Suppose \( G \) has another necklace decomposition \( D_{C'} = (N_{C'}, R_{C'}) \), such that \( N_{C'} \subseteq C' \) with \( u', v' \in V(N_{C'}) \) and \( w' \notin V(N_{C'}) \). Then \( N_{C'} \) is called a rim of the partial cone \( C' \) when \( C' \) has exactly one bridge with respect to each of the two segments of \( N_{C'} \) determined by \([u', v']\).

**Remark.** Every partial cone has a rim. It is readily seen that \( N'(u', v') \), together with either of the \([u', v']\)-segments of \( N_{C'} \), forms a necklace decomposition. Hence, when partial cones exist, necklace decompositions are not unique.

**1.9.6. Essential beads of a necklace.**

In Chapter 2, we will prove that \( E_{ab} \cup E_{bc} \cup E_{ca} \cup E_{abc}^1 \subseteq E(N) \), where \( N \) is the necklace in any decomposition \( D = (N, R) \) of \( G \) satisfying conditions (1), (2), (3), and (4) of this section. Again, it is proved in Chapter 2 that any residual segment \( T \) of \( N \) satisfies exactly one of the conditions

(i) \( E(T) \subseteq E_{ab} \cup E_{bc} \cup E_{ca} \cup E_{abc}^1 \), or

(ii) \( E(T) \subseteq E_{abc}^2 \).
Later, in Chapter 3, we prove that a residual segment $T$ of $N$ is spanned by a singularity if and only if $E(T) \subseteq E_{abc}^2$. This implies that $E(T) \subseteq E_{abc}^2$ if and only if $T$ is contained in a partial cone and hence a cone of $D$. Therefore $N \cap (G \cdot (E_{ab} \cup E_{bc} \cup E_{ca} \cup E_{abc}^1))$ consists of all beads of $G$ which are not contained in any cone of $G$, and these beads are common to all necklaces of all necklace decompositions of $G$ satisfying our condition. We call them the essential beads of a necklace in a necklace decomposition.

**Definition 1.9.7 (Bead-cone decomposition).**

Let $\{S_1, S_2, \ldots, S_k\}$ be the set of beads of $G$ which are essential to each necklace of all necklace decompositions of $G$. Also, let $C_1, C_2, \ldots, C_n$ be the cones in $G$ and $R'$ be the minimal subgraph of $G$ such that

$$G = S_1 \cup S_2 \cup \ldots \cup S_k \cup C_1 \cup C_2 \cup \ldots \cup C_n \cup R'.$$

Then the set $\{S_1, S_2, \ldots, S_k, C_1, C_2, \ldots, C_n, R'\}$ of edge-disjoint subgraphs of $G$ will be said to form a bead-cone decomposition of $G$ (Fig. 1.9C).

**Remark.** For a fixed $D = (N, R)$ of $G$, the bead-cone decomposition is unique, by the uniqueness of the cones, the essential beads, and the fact that all other beads are contained in cones. The main result of chapter 3 shows that the set of cones associated with $D$ does not depend on the choice of necklace-decomposition $D$. 
1.10 **Background.**

Covering an edge triple by an edge set of a bond of \( G \) owes its origin to the paper of my advisor entitled, *Traversing edge triples by simple circular paths*. This research was in turn initiated by the papers (1) *On path properties versus connectivity*, by M. D. Plummer, and (2) *Path traversibility in planar graphs*, coauthored by M. D. Plummer and G. N. Robertson. Because of this I would like to mention the results published in these two papers after defining the proper terminology.

From a very well-known theorem of Whitney we have that a graph \( G \) is nonseparable if and only if every pair of its edges (i) belongs to a circuit of \( G \) or dually (ii) belongs to a bond of \( G \). A graph \( G \) with more than one edge is nonseparable if and only if it is vertex-2-connected (or 2-connected, see [1], Chapter 10). A graph \( G \) with more than one edge is vertex-\( n \)-connected if and only if every pair of vertices in \( G \) can be joined by at least \( n \) internally vertex-disjoint paths. This shows that the study of paths in a graph \( G \) is very much interrelated with connectivity of \( G \). Almost all graphs appearing in this dissertation are assumed to be nonseparable and hence vertex-2-connected.

In the paper [3] "On path properties versus connectivity, the author started with two definitions given below. Let \( G \) be a graph with at least \( n+2 \) vertices. Then \( G \) is \((2,n^+)\)-connected if given
any sequence of \( n^2 \) vertices \( p, q; v_1, v_2, \ldots, v_n \), there is a path with endvertices \( p \) and \( q \) and containing all of \( v_1, v_2, \ldots, v_n \).

\( K_{3,3} \) is an example of a graph which is 3-connected but not \((2,3^+)\)-connected. Flummer shows that \((2,3^+)\)-connectivity is a concept properly intermediate to those of vertex-3-connectivity and vertex-4-connectivity. This intermediate concept leads to the main result of the paper.

**Theorem 1.10.1** Let \( G \) be a planar graph. Then \( G \) is \((2,3^+)\)-connected if and only if \( G \) is vertex-3-connected.

The above result cannot be improved upon in the following sense. For \( n \geq 8 \), the family of graphs on \( n \) points pictured in Fig. 1.10.A at page are planar and vertex-3-connected and hence \((2,3^+)\)-connected, but none is \((2,4^+)\)-connected. It is easy to see that there is no \((p,q; t,u,v,w)\)-path in any graph in the above family, or the family due to Eric Wilson obtained from the above family by adding lines \( pq, ps, ph, qv, qh \) and \( sh \).

The next paper [4], Path traversibility in planar graphs, starts from this point and answers the question: When is a vertex-3-connected planar graph \((2,4^+)\)-connected? We now define the terminology of paper [4]. The Cube \( C \) with bipartition (Fig. 1.10.B)

\[ \{[p,q,r,s], [t,u,v,w] \} \]

is an example of a vertex-3-connected planar graph in which no \((p,q)\)-path contains \((t,u,v,w)\). In the following, we shall call \( p, q, t, u, v, w \) principal vertices of the cube \( C \). Let \( \tilde{C} \) be any graph
formed by adding edges to \( C \) with both endvertices in \( \{p, q, r, s\} \). Then \( \tilde{C} \) also is vertex-3-connected and it is planar.

A graph mapping \( \Psi: G \rightarrow H \) is contractive if \( \Psi(G) = H \) and \( \Psi^{-1}(v) \) is connected for all \( v \in V(H) \). Suppose that \( H \) is a subgraph of \( G \). We define

\[
\text{Ad}_G(H) = \{x \in V(G) \setminus V(H) : v \text{ is adjacent to some } v \in V(H)\}.
\]

Then for any \( \tilde{C} \) as defined above, with principal vertices \( \{p, q, t, u, v, w\} \), a contractive mapping \( \Psi: G \rightarrow \tilde{C} \) is singular with respect to \( \{t, u, v, w\} \subseteq V(G) \setminus \{p, q\} \) if

1. \( \Psi \) permutes \( \{t, u, v, w\} \),
2. \( \Psi^{-1}(p) = [p], \Psi^{-1}(q) = [q] \),
3. if \( x \in \{r, s\} \) and \( y \in \{t, u, v, w\} \), then at most one vertex of \( \Psi^{-1}(x) \) is adjacent to any vertex of \( \Psi^{-1}(y) \), and
4. if \( \{x\} \not\subseteq V(\Psi^{-1}(r)) \) and if \( \{y\} \not\subseteq V(\Psi^{-1}(s)) \) and \( x \) and \( y \) are adjacent in \( G \) then
   \[
   \{x, y\} \subseteq \text{Ad}_G(\Psi^{-1}(v)) \text{ or } \{x, y\} \subseteq \text{Ad}_G(\Psi^{-1}(w))
   \]
   (i.e. \( x \) and \( y \) are both adjacent to vertices all of which are mapped to \( v \) or all of which are mapped to \( w \)).

Under these conditions, \( \tilde{C} \) is called a singular contraction of \( G \). It is readily verified that if \( \tilde{C} \) is a singular contraction of \( G \) with principal vertices \( p, q, t, u, v, w \) then no \( \{p, q\} \)-path in \( G \) contains \( \{t, u, v, w\} \). The theorem can now be stated.
Theorem 1.10.2. Let \( p,q \) be distinct vertices of a vertex-3-connected planar graph \( G \). Then there is a set of four vertices \( \{t,u,v,w\} \subseteq V(G)\setminus\{p,q\} \) which is not in any \( (p,q) \)-path in \( G \) if and only if there is a \( \tilde{C} \) with principal vertices \( \{p,q; t,u,v,w\} \), and a contractive mapping \( \gamma: G \to \tilde{C} \) which is singular with respect to \( \{t,u,v,w\} \subseteq V(G)\setminus\{p,q\} \).

The configuration obtained from the graph \( G \), Fig. 1.10.B, by deleting the vertices \( p,q \) gave my advisor the basic idea for his paper entitled, Traversing edge triples by simple circular paths. There he establishes necessary and sufficient conditions for any edge triple \( \{a,b,c\} \subseteq E(G) \) not to be traversible by the edge set of a polygon \( P \) of a nonseparable graph \( G \). The results of his paper can be restated as follows.

Theorem 1.10.3 (Fig. 1.10.C). Let \( G \) be a nonseparable graph and \( a,b \) and \( c \) be distinct edges of \( G \). Then \( \{a,b,c\} \not\subseteq E(P) \), for any polygon \( P \) in \( G \), if and only if there exists a decomposition

\[ G = N \cup R \]

into edge-disjoint subgraphs \( N \) and \( R \) such that

(i) \( \{a,b,c\} \subseteq E(N) \),

(ii) \( G \times E(N) \) is a bond-necklace with \( R \) separating \( a,b \) and \( c \),

(iii) \( R \) consists of two connected components each of which is nonseparable,

(iv) if \( S_i \) is a bead of the bond-necklace \( G \times E(N) \) then \( G \cdot E(S_i) \) has two vertices of attachment, one in each
component of $R$,

(v) if two beads do not have the same vertices of attachment then their vertices of attachment are distinct in any component of $R$ which is not a vertex-graph,

(vi) if both components of $R$ are not vertex-graphs, then they have three vertices of attachment each, and finally

(vii) such a decomposition is unique.

1.11 Statement of Theorems.

Here, in this dissertation, we solve the dual problem: Find necessary and sufficient conditions for covering an edge triple $\{a, b, c\}$ by a bond $K$ of a nonseparable graph $G$. In that direction, we derive the following two theorems.

Theorem I. Suppose $G$ is a nonseparable graph and $a, b, c$ are three distinct edges of $G$. Then in order that $\{a, b, c\}$ is not contained in any bond of $G$ it is necessary and sufficient that there exists a polygon-necklace decomposition (definition 1.6.8)

$$D = (N, R)$$

of $G$ such that exactly one of the following conditions applies:

1. $R = \emptyset$, and $a, b, c$ belong to distinct beads of $N$.

2. $|W(G, R)| = 3$, and $R$ consists of one or more 3-bridges of $N$ which separate $a, b, c$.

3. $|W(G, R)| \geq \frac{h}{2}$, and $R$ is a bridge of $N$ separating $a, b, c$ which contains no pair of skew diagonals of $N$. 
Moreover, if $D = (N, R)$ satisfies conditions (1), (2) or (3), then the decomposition is unique.

Remark. There are other necklace decompositions $D = (N, R)$, $R$ separating $a$, $b$, and $c$ from which we can deduce the non-existence of a bond $K$ of $G$ covering $a$, $b$, and $c$. However, if we restrict ourselves to the above four types, then we get the uniqueness for (1), (2) and (3).

Theorem II. Let $G$ be a nonseparable graph and $a$, $b$, $c$ be distinct edges of $G$. Suppose

1. $(a, b, c) \notin K$, for $K$ any bond of $G$,
2. there exists a necklace decomposition $G = N \cup R$ where $R$ is a bridge of $N$ separating $a$, $b$, $c$, and
3. there exists a pair of skew diagonals of $N$.

Then there exist cones $C_1, C_2, \ldots, C_n$ of $G$ and beads $S_1, S_2, \ldots, S_m$ which determine a bead-cone decomposition of $G$. This bead-cone decomposition is unique. Moreover, for any cone $C$ with $V(G, C) = \{u, v, w\}$ and apex $w$, there exists a pair of necklace decompositions $G = N_1 \cup R_1$ and $G = N_2 \cup R_2$ such that

4. $N_1 \cup N_2 = (N_1 \cap N_2) \cup ((N_1 \cup N_2) \cap C)$, and
5. $[u, v] = (N_1 \cap N_2) \cap ((N_1 \cup N_2) \cap C)$,
where $N_1 \cap N_2 = N'(u, v)$ and $(N_1 \cup N_2) \cap C$ is a rim of $C$.

Remark. Nonuniqueness in case (4) of theorem I is restricted to the reasonably regular behaviour within the uniquely determined cones of this second decomposition viz., the bead-cone decomposition.
The J-components of a graph $G$  

Figure 1.3A

The bridging lemma  

Figure 1.3B
Polygon-Necklace
Figure 1.6C

Bridge of $P_{abc}$ separating $a$, $b$ and $c$
Figure 1.6B

Polygon $P_{abc}$ with skew diagonals
$L_1$ and $L_2$ separating $a$, $b$ and $c$.
Figure 1.6A
Skew diagonals of a necklace separating a, b and c, in the Necklace decomposition of G.

Figure 1.6E

Necklace decomposition with residual segments and bridges separating a, b and c.

Figure 1.6D
Bead-Cone decomposition
Figure 1.9C

A cone of the Necklace decomposition
Figure 1.9B

Singularity of a Necklace decomposition $D=(N,R)$
Figure 1.9A
A $(2,3^+)$ connected graph but not $(2,4^+)$ connected.

Singular contraction

Figure 1.10A

Figure 1.10B
Non-containment of an edge triple \((a,b,c)\) in a polygon of a nonseparable graph \(G\)

Figure 1.10C
A necklace
Figure I(i)

A necklace with equipartite 3-bridges
Figure I(ii)

A necklace with a residue $R$, separating $a$, $b$, $c$ and $|W(G,R)| > 4$
Figure I(iii)

Bead-cone decomposition of $G$
Figure II
CHAPTER 2

We devote this Chapter completely to proving Theorem I. The results we get here also pave the way for the proof of Theorem II in Chapter 3.

2.1. Theory of Necklaces.

This section develops further the theory of polygon-necklaces started in 1.6 of the previous Chapter.

Proposition 2.1.1.

Suppose $S$ is a nonseparable subgraph of a nonseparable graph $G$, with $|W(G, S)| = 2$. Then $S$ is in a unique necklace decomposition of $G$.

Proof. Let $G = S \cup \overline{S}$, where $\overline{S}$ is the unique minimal subgraph of $G$ whose union with $S$ is $G$. Then $W(G, S) = W(G, \overline{S})$ and, say, $\{u, v\} = W(G, S)$. $\overline{S}$ is connected since $G$ is nonseparable.

Case 1. $\overline{S}$ is nonseparable: Then $S$ and $\overline{S}$ form a necklace of $G$. If there exists another necklace decomposition with $S$ as a bead, then $\overline{S}$ has at least two beads in this second necklace. Hence $\overline{S}$ has a cut-vertex, i.e. $\overline{S}$ is separable, contrary to our assumption. So in this case $S$ is in a unique necklace decomposition of $G$. 55
Case 2. $\overline{S}$ is separable: First we find the cyclic elements of $\overline{S}$.

Let $H$ be a cyclic element of $\overline{S}$. Then, by proposition 1.4.3, each H-component of $\overline{S}$ has exactly one vertex on $H$, since $\overline{S}$ is connected.

To obtain the result in this case we prove the following propositions.

**Proposition 2.1.1.1.** The vertices $u, v$ do not belong to the same cyclic element of $\overline{S}$.

**Proof.** Let $u$ and $v$ belong to the same cyclic element $H$ of $\overline{S}$. If there exists an H-component of $\overline{S}$, then the vertex of attachment of this H-component is a cut-vertex of $G$, which is a contradiction, for $G$ is nonseparable. Hence $H = \overline{S}$, i.e. $\overline{S}$ is nonseparable, which is again a contradiction for case 2.

**Proposition 2.1.1.2.** If $H$ is a cyclic element of $\overline{S}$ and $\{u, v\} \cap V(H) = \emptyset$ then there exist exactly two H-components of $\overline{S}$, one containing the vertex $u$ and the other $v$, and they have distinct vertices of attachment on $H$.

**Proof.** Here $\{u, v\} \cap V(H) = \emptyset$ and $u, v \in \overline{S}$ imply there exists at least three H-components of $\overline{S}$. If there exist at least 3 H-components of $\overline{S}$ then $u$ and $v$ can belong to at most two of them. In that case, the vertex of attachment of the H-component of $\overline{S}$ not containing $u$ and $v$ will be a cut-vertex of $G$, which is a contradiction to the nonseparability of $G$. Hence the number of H-components of $\overline{S}$ is at most two. Also, if there exists exactly one H-component of $\overline{S}$, or two H-components of $\overline{S}$ with the same vertex of attachment, the vertex of
attachment of these components on \( H \) of \( G \).

Hence there are exactly 2 \( H \)-components of \( \overline{S} \), one containing \( u \) and the other containing \( v \), and they have distinct vertices of attachment.

**Proposition 2.1.1.3.** If \( w \notin \{u,v\} \) and \( w \) is a cut-vertex of \( \overline{S} \), then there exist exactly two \([w]\)-components of \( \overline{S} \), one containing the vertex \( u \) and the other containing \( v \).

**Proof.** Since \( w \) is a cut-vertex of \( \overline{S} \), the number of \([w]\)-components of \( \overline{S} \) is at least two. If there exist at least three \([w]\)-components of \( \overline{S} \), then \( u \) and \( v \) can belong to at most two of them. Hence the \([w]\)-component containing neither of \( u \) and \( v \) makes \( w \) a cut-vertex of \( G \), a contradiction. Hence, the number of \([w]\)-components of \( \overline{S} \) is at most two. So, the number of \([w]\)-components is exactly two. Again \( u \) and \( v \) belong to the two distinct \([w]\)-components of \( \overline{S} \), for otherwise \( w \) will be a cut-vertex of \( G \). Hence the proposition is proved.

**Proposition 2.1.1.4.** The vertices \( u \) and \( v \) are not cut-vertices of \( \overline{S} \).

**Proof.** If \( u \) is a cut-vertex of \( \overline{S} \), then there exist at least two \([u]\)-components of \( \overline{S} \), of which at most one contains \( v \). Hence \( u \) will be a cut-vertex of \( G \), a contradiction.

**Proposition 2.1.1.5.** If \( H \) is a cyclic element of \( \overline{S} \), containing say \( u \), then there is only one \( H \)-component of \( \overline{S} \), and this component contains \( v \).
Proof. Here $v \notin V(H)$, by proposition 2.1.1.1, and so there exists at least one $H$-component of $\bar{S}$. If there are at least two $H$-components of $\bar{S}$, then $v$ belongs to exactly one of them. Hence the vertex of attachment of the $H$-component of $\bar{S}$ not containing $v$ is a cut-vertex of $G$. This is contrary to hypothesis. Hence, there exists exactly one $H$-component of $\bar{S}$ and it contains $v$.

Now we prove first the existence of a necklace and then its uniqueness in case 2.

Let us denote the cyclic elements of $\bar{S}$ by $S_2, S_3, \ldots S_m$. We will denote $S$ by $S_1$. If $u$ and $v$ do not belong to $S_i$, when $i \geq 2$, then by proposition 2.1.1.2

$$|W(\bar{S}, S_i)| = 2.$$  

Also $W(G, S_i) = W(\bar{S}, S_i)$, by proposition 1.2.3, and so $|W(G, S_i)| = 2$.

None of the $S_i$'s, $i \geq 2$, contain both $u$ and $v$, by the proposition 2.1.1.1. Hence if $u \in V(S_i)$ or $v \in V(S_i)$, for some $S_i$, $i \geq 2$, then $|W(\bar{S}, S_i)| = 1$, by proposition 2.1.1.5,

and hence $W(G, S_i) = W(\bar{S}, S_i) \cup \{x\}$, for $x \in \{u, v\}$, by proposition 1.2.3. Hence $|W(G, S_i)| = 2$. Also $|W(G, S_i)| = |W(G, S)| = 2$. So $|W(G, S_i)| = 2$, for all $i = 1, 2, \ldots, m$. Next, let $x \in V(G)$. Then either $x \in V(S_i) \setminus W(G, S_i)$ or $x \in W(G, S_i)$, for some $i$. If $x \in W(G, S_i)$ and $x \notin \{u, v\}$, then $x \in W(G, S_i)$, for some $i \geq 2$. Then $x$ is a cut-vertex of $\bar{S}$ and $x$ belongs to exactly one $S_j$, for $j \neq i$ and
If \( j \geq 2 \), by the proposition 2.1.1.3. If \( x \notin \{u,v\} \), then \( x \) is not a cut-vertex of \( \overline{S} \), by proposition 2.1.1.4. But \( u,v \) belong to distinct \( S_i \), for \( i \geq 2 \), by proposition 2.1.1. Hence \( x \) belongs to a unique member \( S_i \), \( i \geq 2 \), and also to \( S_1 \). Hence for all \( x \in V(G) \), \( x \) belongs to at most two of the \( S_i \)'s, \( i = 1, 2, \ldots, m \). Also \( G = S_1 \cup S_2 \cup \ldots \cup S_m \), where each \( S_i \) is a nonseparable subgraph of \( G \) with

\[
E(G) = E(S_1) \cup E(S_2) \cup \ldots \cup E(S_m)
\]

a partition of \( E(G) \). Hence, by the proposition 1.6.6, condition 2, \( G \) is a necklace.

By proposition 1.6.6, condition 1, each polygon of \( \overline{S} \) is contained in a unique \( S_i \), \( i \geq 2 \). The \( S_i \), \( i \geq 2 \), on the other hand are all nonseparable, whence \( \overline{S} = S_2 \cup S_3 \cup \ldots \cup S_m \) is the cyclic element decomposition of \( \overline{S} \), which is unique. Hence there is at most one necklace decomposition of \( G \) containing \( S_1 = S \) as a bead. Hence proposition 2.1.1 is valid.

**Proposition 2.1.2.** Suppose \( G \) is a necklace having two distinct necklace decompositions with at least 3 beads in one decomposition. Then there exist beads \( S_1 \) and \( S_2 \) in the first and second decompositions, respectively, such that

\[
S_1 \cup S_2 = G.
\]
Remark. A bond graph $G$ with 4 edges provides a counterexample if the above proposition is modified so that each necklace decomposition has 2 beads.

Let

\[ G = S_1 \cup S_2 \cup \ldots \cup S_h, \text{for } h \geq 2 \]

\[ = S'_1 \cup S'_2 \cup \ldots \cup S'_k, \text{for } k \geq 3 \]

be two distinct necklace decompositions of $G$. Then

\[ |W(G, S_i)| = 2 = |W(G, S'_j)|, \text{for all } i = 1, 2, \ldots, h \text{ and all } j = 1, 2, \ldots, k, \]

and

\[ E(G) = E(S_1) \cup E(S_2) \cup \ldots \cup E(S_h) \]

\[ = E(S'_1) \cup E(S'_2) \cup \ldots \cup E(S'_k) \]

are two partitions of $E(G)$.

**Proposition 2.1.2.1.** Each $E(S_i)$, for $i = 1, 2, \ldots, h$, meets either exactly one $E(S'_j)$ or all $E(S'_j)$, for $j = 1, 2, \ldots, k$.

**Proof.** By a permutation of the subscripts, let $E(S_1)$ meet both $E(S'_1)$ and $E(S'_2)$. Also, let $e_1 \in E(S_1) \cap E(S'_1)$ and $e_2 \in E(S_1) \cap E(S'_2)$. $S_1$ is a nonseparable graph, hence there exists a polygon $P$ in $S_1$ containing both $e_1$ and $e_2$. But $E(P)$ meets both the edge sets $E(S'_1)$ and $E(S'_2)$. So $E(P)$ meets all $E(S'_j)$'s, $j = 1, 2, \ldots, k$, by the definition of a necklace. Hence $E(S_1)$ meets all the $E(S'_j)$'s, $j = 1, 2, \ldots, k$. Hence the result of proposition 2.1.2.1 is valid.
Remark. If one $E(S_1)$ meets two of $\{E(S'_1), E(S'_2), \ldots, E(S'_k)\}$ then $E(S_1)$ meets all of them.

**Proposition 2.1.2.2.** At most one of $E(S_1), E(S_2), \ldots, E(S_h)$ can meet two members of the set $\{E(S'_1), E(S'_2), \ldots, E(S'_k)\}$.

Proof. By a permutation of the subscripts let each of $E(S_1)$ and $E(S_2)$ meet two members of $\{E(S'_1), E(S'_2), \ldots, E(S'_k)\}$. Then, by the method of proof in proposition 2.1.2.1, we have two polygons $P_1$ and $P_2$ in $S_1$ and $S_2$, respectively, such that $E(P_1)$ and $E(P_2)$ meet each of $E(S'_1), E(S'_2), \ldots, E(S'_k)$. Hence

$$V(P_1) \cap V(P_2) = \bigcup_{j=1}^{k} W(G, S'_j)$$

and

$$|W(G, S_1)| \geq \bigcup_{j=1}^{k} (W(G, S'_j)) = k \geq 3,$$

a contradiction to the fact that $|W(G, S_1)| = 2$. This proves proposition 2.1.2.2.

**Proposition 2.1.2.3.** If one member of $\{E(S_1), \ldots, E(S_h)\}$ meets two members of $\{E(S'_1), E(S'_2), \ldots, E(S'_k)\}$ then one member of $\{E(S'_1), \ldots, E(S'_k)\}$ also meets two members of $\{E(S_1), \ldots, E(S_h)\}$.

Proof. By a permutation of the subscripts let $E(S_1)$ meet $E(S'_1)$ and $E(S'_2)$. Then $E(S_1)$ meets all the $E(S'_j)$'s, $j = 1, 2, \ldots, k$. If every member of $\{E(S'_1), \ldots, E(S'_k)\}$ meets only one member of
\{E(S_1), \ldots, E(S_h)\} \text{ then } E(S'_1) \cup \ldots \cup E(S'_k) = E(G) \subseteq E(S_1), \text{ which is a contradiction since } E(G) = E(S_1) \cup E(S_2) \cup \ldots \cup E(S_h) \text{ is a partition of } E(G) \text{ with } h \geq 2. \text{ Hence there exists a member, say } E(S'_3), \text{ belonging to } \{E(S'_1), \ldots, E(S'_k)\}, \text{ which meets two members from } \{E(S_1), E(S_2), \ldots, E(S_h)\}. \text{ Hence } E(S'_3) \text{ meets every } E(S_i), \text{ for } i = 1, \ldots, h, \text{ by the remark in proposition 2.1.2.1.}

Proposition 2.1.2.4. If there exists a member of 
\{E(S_1), E(S_2), \ldots, E(S_h)\} \text{ meeting two members of } \{E(S'_1), E(S'_2), \ldots, E(S'_k)\},
then there exist } S_i \text{ and } S'_j, \text{ for some } i = 1, 2, \ldots, h \text{ and } j = 1, 2, \ldots, k, \text{ such that }
S_i \cup S'_j = G.

Proof. Let } E(S_1) \text{ meet two of the } E(S_j)'s, j = 1, 2, \ldots, k. \text{ Then, by proposition 2.1.2.3, there exists } E(S'_1), \text{ say, which meets two of the } E(S_i)'s, i = 1, 2, \ldots, h. \text{ Then, by the remark of proposition 2.1.2.1, } E(S_1) \text{ meets all the } E(S_j)'s, j = 1, 2, \ldots, k, \text{ and } E(S'_1) \text{ meets all the } E(S_i)'s, i = 1, 2, \ldots, h. \text{ Now, by proposition 2.1.2.1, each } E(S_1) \text{ meets either one } E(S'_j) \text{ or all the } E(S'_j)'s. \text{ Hence }
E(S_2) \cup E(S_3) \cup \ldots \cup E(S_h) \subseteq E(S'_1)
\text{ and }
E(S'_1) \cup E(S'_1) = E(G).

Hence } S_1 \cup S'_1 = G. \text{ This proves the proposition.}
Proposition 2.1.2.5. If none of $E(S_1),\ldots,E(S_h)$ meets two members of \{\(E(S'_1), E(S'_2),\ldots,E(S'_k)\}\} then $h \geq k \geq 3$.

Proof.

\[
E(G) = E(S_1) \cup E(S_2) \cup \ldots \cup E(S_h) = E(S'_1) \cup E(S'_2) \cup \ldots \cup E(S'_k)
\]

are two partitions of $E(G)$. If $h < k$, then $E(S'_\lambda) = \emptyset$, for some $\lambda = 1,2,\ldots,k$, since, by proposition 2.1.2.1, each $E(S_i)$, for $i = 1,2,\ldots,h$, is contained in exactly one $E(S'_j)$, $j = 1,2,\ldots,k$. But $E(S'_\lambda) = \emptyset$ is impossible in a partition. Hence the result $h \geq k > 3$.

Proposition 2.1.2.6. If none of $E(S_1),\ldots,E(S_h)$ meets two members of \{\(E(S'_1),\ldots,E(S'_k)\)\} and if none of $E(S'_1),\ldots,E(S'_k)$ meets two members of \{\(E(S'_1),\ldots,E(S'_h)\)\}, then the two necklace decompositions are the same.

Proof. By proposition 2.1.2.5, $h \geq k \geq 3$, and, by symmetry, $k \geq h$. Hence $h = k$. By proposition 2.1.2.1, each $E(S_i)$ is contained in exactly one $E(S'_j)$ and each $E(S'_j)$ is contained in exactly one $E(S_i)$, for $i = 1,2,\ldots,h$ and $j = 1,2,\ldots,k$. Hence, by a permutation of the subscripts,

\[
E(S_i) = E(S'_i).
\]

Hence the two decompositions are the same.

Proof of proposition 2.1.2.
Case 1. \( h = 2 \) and \( k \geq 3 \): The result follows from propositions 2.1.2.4 and 2.1.2.5.

Case 2. \( h \geq 3 \) and \( k \geq 3 \): By proposition 2.1.2.1, exactly one \( E(S_j') \), or all \( E(S_j') \), for \( j = 1, 2, \ldots, k \). By symmetry in case (2), each \( E(S_j') \) meets either exactly one \( E(S_i) \), or all \( E(S_i) \), for \( i = 1, 2, \ldots, h \).

By proposition 2.1.2.2, at most one \( E(S_i) \), \( i = 1, 2, \ldots, h \) can meet all the \( E(S_j') \), and similarly at most one \( E(S_j') \), \( j = 1, 2, \ldots, k \), can meet all the \( E(S_i) \)’s.

By lemma 2.1.2.3 either there exist one \( E(S_i) \) and one \( E(S_j') \) meeting all \( E(S_j') \) and all \( E(S_i) \), respectively, or none of \( E(S_i) \), \( i = 1, 2, \ldots, h \), and none of \( E(S_j') \), \( j = 1, 2, \ldots, k \), meet all \( E(S_j') \) and all \( E(S_i) \), respectively. In the first case mentioned above, we apply the proposition 2.1.2.4 and get the required result. In the alternative case we apply the propositions 2.1.2.1 and 2.1.2.6 successively and arrive at the conclusion, contradicting the distinctness of the necklace decompositions. This completes the proof.

Proposition 2.1.3. Suppose \( G \) is a necklace and \( a, b, c \) are distinct edges belonging to distinct beads of \( G \). Then this necklace decomposition of \( G \) is unique.
Proof. If $G$ has two distinct necklace decompositions with each of $a, b, c$ belonging to distinct beads in each of the decompositions then, by proposition 2.1.2, there exist beads $S_1$ and $S_2$ in the first and the second decompositions, respectively, such that $S_1 \cup S_2 = G$.

Hence at most two of $a, b, c$ can belong to $G$. This is a contradiction.

Proposition 2.1.4. Suppose $P$ is a polygon of a nonseparable graph $G$ and that $u, v$ and $w$ are distinct vertices of $P$. Also, suppose that for each bridge $B$ of $P$ there exists $(x, y) \subseteq \{u, v, w\}$ such that $W(G, B) \subseteq P(x, y)$. Then $G$ is a necklace with $u, v, w$ as vertices of attachment of some beads.

Proof. We move from $u$ to $v$ on $P$ along the segment $P(u, v)$.

If there exists no bridge of $P$, having vertices of attachment in $V(P(u, v))$ and including $u$ as a vertex of attachment, then the edge incident with $u$ in $P(u, v)$ forms a nonseparable subgraph with two vertices of attachment in $G$, i.e. a bead with its two vertices of attachment in $V(P)$. Let $S$ be the bead arising in this case. When there exists a bridge $B$ of $P$, having vertices of attachment in $V(P(u, v))$ and $u$ as a vertex of attachment, then we form the following recessional sequence.

$B_0 = \{B\}$,

$\mathcal{B}_1 = \{B_1 \mid B_1$ is a bridge of $P$ in $G$ and $B_1$ is skew to $B\}$ and

$\mathcal{B}_j = \{B_j \mid B_j$ is a bridge of $P$ in $G$, $B_j \notin \bigcup \mathcal{B}_0 \cup \bigcup \mathcal{B}_{j-1}$ and $B_j$ is skew to $B_{j-1}$ for some $B_{j-1} \in \mathcal{B}_{j-1}\}$.
By finiteness there exists \( n \) such that \( \emptyset \neq \emptyset_1 = \emptyset_2 = \cdots \).

We define

\[
H = \bigcup_{j=0}^{n} \bigcup_{B_j \in \emptyset_j} U(B_j)
\]

the subgraph of \( G \) containing all such bridges in this recessional sequence. Then

\[
W(G, H) = \bigcup_{j=0}^{n} \tilde{W}(G, B_j) \subseteq V(P(u, v)),
\]

by our hypothesis. Let the extreme vertices of \( W(G, H) \) on \( P(u, v) \) be \( v_1 \) and \( v_2 \). Then \( u_1 \in \{v_1, v_2\} \) and \( u_1 = v_1 \) say.

Then

\[
P(u_1, v_2) \cup \bigcup_{B' \text{ is a bridge of } P} \tilde{W}(G, B') \subseteq P(u_1, v_2)
\]

is a nonseparable subgraph of \( G \) with two vertices of attachment \( u_1, v_2 \) in \( G \), for \( u_1, v_2 \in V(P(u, v)) \). Hence this is a bead, say \( S \), of \( G \) in these two cases. Now, in all these cases there exists a unique necklace containing this \( S \) as a bead and \( u \) is a vertex of attachment of this bead.

We intend to prove that \( \{u, v, w\} \subseteq \bigcup_{S' \text{ is a bead of } G} W(G, S') \).

We already know that \( u \in W(G, S) \), by our construction, and \( S \) is a bead of \( G \). We will prove that \( v \) and \( w \) are vertices of attachment of some bead, by contradiction. If possible, let \( r \in \{v, w\} \) and \( r \notin \bigcup_{S' \text{ is a bead of } G} W(G, S') \). Then, by condition 2 of proposition 1.6.6, we

\( S' \) is a bead of \( G \).
have \( r \in V(S') \setminus W(G, S') \), for some unique bead \( S' \) of the necklace \( G \). Let \( W(G, S') = \{s, t\} \) with \( u, s, t \) in the orientation of \( u, v, w \). Then \( \{s, t\} \subseteq V(P) \), by condition 1 of proposition 1.6.6. Also \( r \in V(P) \). Hence \( P(s, t) \) contains \( r \) as an internal vertex. This arc \( P(s, t) \) has at least two edges. We claim that there exists a bridge \( B' \) of \( P(s, t) \) in \( S' \) which has a vertex in \( P(s, r) \setminus \{r\} \) and a vertex in \( P(r, t) \setminus \{r\} \), since \( S' \) is a bead of \( G \). For, otherwise all the bridges of \( P(s, t) \) in \( S' \) will have vertices of attachment either in \( P(s, r) \) or in \( P(r, t) \). Then \( r \) will be a cut-vertex of \( S' \), contradicting the nonseparability of \( S' \).

Hence \( B' \) is a bridge of \( P(s, t) \) in \( S' \) having a vertex \( s' \in P(s, r) \setminus \{r\} \) and also a vertex \( t' \in P(r, t) \setminus \{r\} \). But this is a bridge of \( P \) in \( G \) having a vertex in \( P(r, p) \setminus \{r\} \) and a vertex in \( P(r, q) \setminus \{r\} \), where \( \{p, q, r\} = \{u, v, w\} \). If \( \{s', t'\} \neq \{u, v, w\} \setminus \{r\} \) or this bridge contains a vertex of attachment not in \( \{u, v, w\} \setminus \{r\} \) then we get a contradiction to our assumption that if \( B \) is any bridge of \( P \) then \( W(G, B) \subseteq P(x, y) \), for some \( \{x, y\} \subseteq \{u, v, w\} \).

If \( \{s', t'\} = \{p, q\} \) and \( W(G, B') = \{p, q\} \) then either \( r = v \) and \( s = u \) contrary to the construction of the necklace from \( S \), or \( r = w \), \( W(G, B') = \{u, v\} \) whence \( B' \subseteq S \) contrary to \( S \neq S' \). Hence \( \{u, v, w\} \subseteq U \subseteq W(G, S') \) and the proof of the proposition is complete.

**Proposition 2.1.5.** Suppose \( N \) is a necklace with edges \( a, b \) and \( c \) belonging to distinct beads of \( N \). Then there exist \( u, v \) and \( w \in V(P_{abc}) \), for each polygon \( P_{abc} \) contained in \( N \), such that for
every bridge \( B \) of \( P_{abc} \) in \( N \) there exist distinct \( x, y \in \{u, v, w\} \) satisfying the condition \( W(G, B) \subseteq P_{abc}(x, y) \).

Proof. The vertices of attachment of the beads containing \( a, b \) and \( c \) provide us with at least three such vertices \( u, v \) and \( w \).

2.2 Two beads in a nonseparable graph.

Proposition 2.2.1. Given a nonseparable graph \( G \), with two distinct beads \( H_1 \) and \( H_2 \) such that \( H_1 \cup H_2 \subseteq G \), exactly one of the following occurs.

(i) \( H_1 \cap H_2 \subseteq [x, y] \), for some \( x, y \in W(G, H_1) \cap W(G, H_2) \),

(ii) \( H_1 \subseteq H_2 \),

(iii) \( H_2 \subseteq H_1 \), or

(iv) \( W(G, H_1) = W(G, H_2) = W(G, H_1 \cup H_2) = W(G, H_1 \cap H_2) \) and \( H_1 \cap H_2 \) is connected.

Proof. If (i), (ii) and (iii) do not occur, then we have

\[ E(H_1) \cap E(H_2) \neq \emptyset \quad \text{and} \quad E(H_1) \setminus E(H_2) \neq \emptyset \neq E(H_2) \setminus E(H_1) . \]

Let \( e \in E(H_1) \cap E(H_2) \), \( e_1 \in E(H_1) \setminus E(H_2) \) and \( e_2 \in E(H_2) \setminus E(H_1) \),

\( W(G, H_1) = \{u_1, v_1\} \) and \( W(G, H_2) = \{u_2, v_2\} \). There exists a polygon \( P_1 \) in \( H_1 \) containing \( e \) and \( e_1 \), since \( H_1 \) is nonseparable. But \( P_1 \) contains \( e \in E(H_2) \) and \( W(G, H_2) = \{u_2, v_2\} \). This implies that \( u_2, v_2 \in V(P_1) \) and hence in \( V(H_1) \). So \( V(H_1) \supseteq \{u_1, v_1, u_2, v_2\} \) and similarly \( V(H_2) \supseteq \{u_1, v_1, u_2, v_2\} \). Finally \( \{u_1, v_1, u_2, v_2\} \subseteq \)
\( V(H_1) \cap V(H_2) = V(H_1 \cap H_2) \). Now \( G \) is nonseparable, and \( H_1 \cup H_2 \subseteq G \) and \( H_1 \cap H_2 \) has at least one edge. Thus, by proposition 1.4.4,

\[
|W(G, H_1 \cup H_2)| \geq 2, \quad |W(G, H_1 \cap H_2)| \geq 2
\]

and therefore

\[
4 \leq |W(G, H_1 \cup H_2)| + |W(G, H_1 \cap H_2)|
= |W(G, H_1)| + |W(G, H_2)| - q(H_1, H_2)
= 4 - q(H_1, H_2) \leq 4 .
\]

This implies \( |W(G, H_1 \cup H_2)| = 2 = |W(G, H_1 \cap H_2)| \) and \( q(H_1, H_2) = 0 \), where

\[
q(H_1, H_2) = |(W(G, H_1) \cap W(G, H_2)) \setminus W(G, H_1 \cup H_2)| .
\]

Using the proposition 1.2.4,

\[
W(G, H_1) \cup W(G, H_2) = W(G, H_1 \cup H_2) \cup W(G, H_1 \cap H_2) .
\]

This implies \( W(G, H_1 \cup H_2) \cup W(G, H_1 \cap H_2) = \{u_1, v_1, u_2, v_2\} \). But

\[
|W(G, H_1 \cup H_2)| = 2 .
\]

Hence \( W(G, H_1 \cup H_2) = \{u, v\} \subseteq \{u_1, v_1, u_2, v_2\} \)

\( \subseteq V(H_1 \cap H_2) \), where \( u \) and \( v \) are distinct. Again, by proposition 1.2.5,

\[
W(G, H_1 \cup H_2) \cap V(H_1 \cap H_2) \subseteq W(G, H_1) \cap W(G, H_2) .
\]

Hence \( \{u, v\} \subseteq W(G, H_1) \cap W(G, H_2) \). Now \( |W(G, H_1)| = 2 = |W(G, H_2)| \)
implies that \( W(G, H_1) = W(G, H_2) = \{u, v\} \) and hence \( \{u, v\} = W(G, H_1) \cap W(G, H_2) \subseteq W(G; H_1 \cap H_2) \), by proposition 1.2.6. But \( |W(G, H_1 \cap H_2)| = 2 \). So finally \( W(G, H_1) = W(G, H_2) = W(G, H_1 \cup H_2) \)
= \text{W}(G, H_1 \cap H_2). Also, we have \( H_1 \not\subset H_2 \) and \( H_2 \not\subset H_1 \) and \( H_1 \cap H_2 \) has at least one edge.

If possible, let \( H_1 \cap H_2 \) be not connected. Then

\[
\text{W}(G, H_1 \cap H_2) = \bigcup_{C \in C(H_1 \cap H_2)} \text{W}(G, C).
\]

\( G \) is nonseparable and \( C \) has at least one edge, hence \(|\text{W}(G, C)| \geq 2\). But \(|\text{W}(G, H_1 \cap H_2)| = 2\) and \( \text{W}(G, C) = \text{W}(G, H_1 \cap H_2) \), Hence \( \text{W}(G, C) = \text{W}(G, H_1 \cap H_2) \).

which implies \( H_1 \cap H_2 = C \). Hence \( H_1 \cap H_2 \) is connected. Hence \( \text{W}(G, H_1) = \text{W}(G, H_2) = \text{W}(G, H_1 \cup H_2) = \text{W}(G, H_1 \cap H_2) \) and \( H_1 \cap H_2 \) is connected.

\textbf{Proposition 2.2.2.} Suppose \( G \) is a nonseparable graph with two beads \( H_1 \) and \( H_2 \) such that

\[
H_1 \cup H_2 \not\subset G \quad \text{and} \quad E(H_1) \cap E(H_2) \neq \emptyset.
\]

Then either 1) \( H_i \subsetneq H_j \) for distinct \( i \) and \( j \) and \( \{i,j\} = \{1,2\} \)

or \( \text{W}(G, H_1) = \text{W}(G, H_2) = \text{W}(G, H_1 \cup H_2) = \text{W}(G, H_1 \cap H_2) \) and \( H_1 \cap H_2 \) is connected.

\textbf{2.3 Existence of a \( P_{abc} \) or a \( K_{abc} \).}

In this section we consider a connected matroid \( M \) on \( E = E(M) \), its set of cells. We will call the atoms of \( M \) bonds and the atoms of \( M^* \) circuits of \( M \) and denote the set of bonds of \( M \) and the set of circuits of \( M \) by \( B(M) \) and \( P(M) \), respectively. Suppose \( a, b, c \in E(M) \) are distinct and \( \{a,b,c\} \notin K \) for any \( K \in E(M) \).
In the following proposition we prove that \(\{a, b, c\} \subseteq K^*,\) where \(K^*\) is a circuit of \(M.\)

**Proposition 2.3.1.** Suppose \(M\) is a connected matroid on \(E(M),\) that \(a, b, c \in E(M)\) are distinct, and there exists no bond \(K \in B(M)\) such that \(\{a, b, c\} \subseteq K.\) Then \(\{a, b, c\} \subseteq K^*,\) where \(K^*\) is a circuit of \(M.\)

**Proof.** \(M\) is a connected matroid implies that there exist \(K_1, K_2 \in B(M)\) such that \(a, b \in K_1\) and \(b, c \in K_2.\) Then \(K_1\) and \(K_2\) are connected flats and \(b \in K_1 \cap K_2,\) hence \(K_1 \cup K_2\) is connected.

Also, by the hypothesis, \(K_1 \neq K_2.\) Hence

\[r(K_1 \cup K_2) \geq 2\] or the dimension of \(K_1 \cup K_2\) is at least one (i.e. \(d(K_1 \cup K_2) \geq 1).\) If \(d(K_1 \cup K_2) = 1\) then \(K_1 \cup K_2\) is a line of \(M\) containing \(a, b\) and \(c.\) If \(d(K_1 \cup K_2) = n \geq 2,\) then we construct a sequence of connected flats of \(M\) given by

\[K_1 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_n = K_1 \cup K_2,\] such that \(d(F_0) = 0, d(F_1) = d(F_1) + 1, i = 1, 2, \ldots, n-1,\) and \(d(K_1 \cup K_2) = d(F_{n-1}) + 1 = n.\) Also \(a, b\) and \(c\) belong to \(K_1 \cup K_2.\) The above sequence gives us two connected flats \(F_{n-2}\) and \(K_1 \cup K_2\) such that \(d(F_{n-2}) = n-2\) and \(d(K_1 \cup K_2) = n.\) Hence, by proposition 1.5.23, we have two connected flats \(F_{n-1}\) and \(F_{n-1}^2\) of \(M\) such that

\[\langle F_{n-1}^1 \cap F_{n-1}^2 \rangle = F_{n-2}\] and \(F_{n-1}^1 \cup F_{n-1}^2 = K_1 \cup K_2.\)

But \(K_1 \subseteq F_{n-2} \subseteq F_{n-1}^1\) and \(K_1 \subseteq F_{n-2} \subseteq F_{n-1}^2,\) and hence
\(\{a,b\} \subseteq \left<F_{n-1}^1 \cap F_{n-1}^2\right>\) and \(\{a,b,c\} \subseteq F_{n-1}^1 \cup F_{n-1}^2 = K_1 \cup K_2\). Hence one of \(F_{n-1}^1\) and \(F_{n-1}^2\) contains \(c\), say \(\{a,b,c\} \subseteq F_{n-1}^1\). \(M \times F_{n-1}^1\) is a connected matroid and \(\{b,c\} \subseteq F_{n-1}^1\). Therefore there exists a bond, say \(K_2\), of \(M \times F_{n-1}^1\) containing \(b\) and \(c\). Now \(K_1 \cup K_2 \subseteq F_{n-1}^1\), \(K_1 \cup K_2\) contains \(a\), \(b\) and \(c\), and \(K_1 \cup K_2\) is a connected flat of \(M\) with \(d(K_1 \cup K_2) \leq n-1\). Hence, by induction on \(d(K_1 \cup K_2)\), we get a connected flat \(L\) of \(M\) such that 
\[\{a,b,c\} \subseteq L\] and \(d(L) = 1\).

Therefore \(L\) is a connected line of \(M\) and hence \(L = X_1 \cup X_2 \cup \ldots \cup X_t\), for distinct \(X_i \in E(M)\) and \(t \geq 3\). \(M \times L\) is connected and 
\(\{a,b,c\} \subseteq L\), therefore there exist bonds of \(M \times L\) containing the pairs \(\{a,b\}, \{b,c\}\) and \(\{c,a\}\), say \(X_1, X_2\) and \(X_3\), respectively.

Also these bonds are distinct, for otherwise there exists a bond of \(M \times L\) containing \(a\), \(b\) and \(c\) and it will be a bond of \(M\) containing \(a\), \(b\) and \(c\). We apply proposition 1.5.19 to the line \(L\) above and obtain 
\[X_1 = \langle L \setminus \{c\}\rangle,\] the only point on \(L\) not containing \(c\), 
\[X_2 = \langle L \setminus \{a\}\rangle,\] the only point on \(L\) not containing \(a\), and 
\[X_3 = \langle L \setminus \{b\}\rangle,\] the only point on \(L\) not containing \(b\).

So if there exists another point \(X\) on \(L\) distinct from 
\(X_1, X_2, X_3\) then \(\{a,b,c\} \subseteq X\), implying that there exists a bond of
M containing \(a, b\) and \(c\). This is contrary to our assumption and hence \(t = 3\) and \(L = X_1 \cup X_2 \cup X_3\). By proposition 1.5.5, 
\((M \times L)^* = M^* \cdot L\). Again, by the orthogonality of dual matroids

\[
B(M^* \cdot L) = B((M \times L)^*)
\]

\[
= \{S \subseteq L : S \neq \emptyset, |S \cap X_i| \neq 1, \text{for } i = 1, 2, 3, \}
\]

and \(S\) is minimal with respect to these properties.

If \(S = \{a, b, c\}\) then \(S \in B(M^* \cdot L)\) and, by the definition of atoms of \(M^* \cdot L\), there exists a bond \(K^*\) of \(M^*\) such that

\[
S = K^* \cap L.
\]

This implies that there exists a bond \(K^*\) of \(M^*\) containing \(\{a, b, c\}\), i.e. a circuit \(K^*\) of \(M\) containing \(\{a, b, c\}\).

The following proposition follows from proposition 2.3.1.

**Proposition 2.3.2.** Suppose \(G\) is a nonseparable graph with distinct edges \(a, b\) and \(c\) such that \(\{a, b, c\} \not\subseteq K\), for any bond \(K\) of \(G\). Then there exists a circuit of \(G\) and hence a polygon of \(G\) containing \(a, b\) and \(c\).

2.4 Skew diagonals of polygons and necklaces.

This section gives conditions under which existence of a pair of skew diagonals, for a polygon \(P_{abc}\) or a necklace with \(a, b, c\) in distinct beads, implies existence of a bond \(K_{abc}\).
Proposition 2.4.1. Suppose \( G \) is a finite nonseparable graph and \( a, b, c \in E(G) \) are distinct. Also, suppose \( P_{abc} \) is a polygon of \( G \) and there exists a pair of skew diagonals of \( P_{abc} \) separating \( a, b, \) and \( c \). Then there exists a bond \( K_{abc} \) of \( G \).

Proof. Let \( L_1 \) and \( L_2 \) be the pair of skew diagonals of \( P_{abc} \). Without loss of generality we can assume

\[
V(L_1) \cap V(P_{abc}(P_{ab}, P_{ba})) = \{u_1\}, \text{ for } i = 1, 2,
\]

\[
V(L_2) \cap V(P_{abc}(P_{cb}, P_{bc})) = \{v_2\} \text{ and }
\]

\[
V(L_2) \cap V(P_{abc}(P_{ac}, P_{ca})) = \{v_1\},
\]

where \( \{u_i, v_i\} \) are the endvertices of \( L_i \) on \( P_{abc} \), for \( i = 1, 2 \). \( L_1 \) and \( L_2 \) are skew with respect to \( P_{abc} \), hence \( P_{abc}(u_1, u_2) \) has at least one edge \( e \). Also, the arcs \( L_1, L_2, P_{abc}(u_1, u_2) \) and \( P_{abc}(v_1, v_2) \) form a polygon in \( G \).

We form a spanning tree \( H \) of \( G \) such that

1. \( e, a, b \notin E(H) \),
2. \( E(L_1) \cup E(L_2) \subseteq E(H) \)

and (iii) \( E(F) \setminus \{e, a, b\} \subseteq E(H) \). Then \( c \in E(H) \), and the removal of \( c \) from \( H \) gives us two connected components such that \( a, b \) are incident with vertices of each. Whence \( \{a, b, c\} \subseteq K \), for some bond \( K \) of \( G \), i.e. the set of edges with endvertices in both these connected components.

Proposition 2.4.2. Suppose \( G \) is a finite nonseparable graph and
$a, b, c \in E(G)$ are distinct. Suppose also that $P_{abc}$ is a polygon of $G$ having a pair of skew diagonals $L_1$ and $L_2$ in a bridge $B$ of $P_{abc}$ such that

(i) $B$ separates $a, b$ and $c$, and

(ii) there exist two segments in the set

$$\{ P_{abc}(P_{ab}, P_{ba}), P_{abc}(P_{bc}, P_{cb}), P_{abc}(P_{ac}, P_{ca}) \}$$

such that each of $L_1$ and $L_2$ has a single endvertex in each of the segments. Then there exists a bond $K_{abc}$ in $G$.

Proof. Let the endvertices of $L_1$ and $L_2$ belong to the pair $P_{abc}(P_{ab}, P_{ba})$ and $P_{abc}(P_{ac}, P_{ca})$. Now $B$ separates $a, b$ and $c$. Therefore there exists an arc $L$ in $B$ from a vertex $v$ of $P_{abc}(P_{bc}, P_{cb})$ to a vertex $w$ of $(L_1 \cup L_2) \setminus V(P_{abc})$ which avoids $P_{abc} \cup L_1 \cup L_2$, by proposition 1.3.1. Without loss of generality suppose $w \in V(L_1) \setminus V(P_{abc})$. If

$$V(P_{abc}(P_{ab}, P_{ba})) \cap V(L_1) = \{u_i\} \text{ and}$$

$$V(P_{abc}(P_{ac}, P_{ca})) \cap V(L_1) = \{v_i\}; \text{ for}$$

$i = 1, 2$, where $u_2$ follows $u_1$ on the arc $P_{abc}(P_{ab}, P_{ba})$, then $L \cup L_1(w, u_1)$ and $L_2$ form a pair of skew diagonals of $P_{abc}$ separating $a, b$ and $c$. Hence, by proposition 2.4.1, there exists a $K_{abc}$ in $G$. 
Proposition 2.4.3. Suppose a nonseparable graph $G$ with distinct edges $a$, $b$, $c$ is given. Let $D = (N, R)$ be a necklace decomposition of $G$ satisfying the conditions

(i) $|W(G, R)| \geq 4$,
(ii) $R$ is a single bridge of $N$ separating $a$, $b$ and $c$, and
(iii) there exists a pair of skew diagonals of $N$ separating $a$, $b$, $c$.

Then there exists a $K_{abc}$ in $G$.

Proof. There exists a pair of skew diagonals of $N$ separating $a$, $b$ and $c$. Therefore there exists a pair of skew diagonals of $P_{abc}$, for any polygon containing $a$, $b$, $c$ and contained in $N$.

Also, there exists at least one polygon $P_{abc} \subseteq N$, thus there exists a $K_{abc}$ in $G$, by proposition 2.4.1.

Proposition 2.4.4. Suppose $G$ is a nonseparable graph with distinct $a$, $b$, $c$ belonging to $E(G)$. Let $D = (N, R)$ be a necklace decomposition of $G$ satisfying the conditions

(i) $|W(G, R)| \geq 4$,
(ii) $R$ is a single bridge of $N$ separating $a$, $b$, $c$ and containing two skew diagonals $L_1$, $L_2$, and
(iii) there exist two segments in the set

\[ \{N(t_{ab}, t_{ba}), N(t_{ac}, t_{ca}), N(t_{cb}, t_{bc})\} \]

such that each of $L_1$ and $L_2$ has a single endvertex in each of the segments. Then there exists a bond $K_{abc}$ in $G$. 
Proof. By our hypothesis, there exists at least one $P_{abc}$ contained in $N$. For this $P_{abc}$ the diagonals $L_1$ and $L_2$ satisfy the hypothesis of proposition 2.4.2, since

$$W(G, N) \subseteq W(G, P_{abc}).$$

Hence there exists a bond $K_{abc}$ in $G$.

2.5 Algorithms involving skew bridges.

Here we describe how to enlarge a bridge of a polygon by incorporating a part of another bridge which is skew to it. Also, some algorithms involving sequences of such enlargements, e.g.

(1) Clearing bridges from a segment of the polygon,

(2) Clearing skew bridges altogether and

(3) Clearing skew bridges and producing skew diagonals of a polygon, are formulated in the section.

Definition 2.5.1 (Types of bridges).

Suppose $G$ is a nonseparable graph with distinct edges $a, b, c$ and a polygon $P_{abc}$. A bridge $B$ of $P_{abc}$ in $G$ is of type I, II or III according as it meets exactly one, two or three, respectively, of the segments $P_{abc}(P_{ab}, P_{ba}), P_{abc}(P_{bc}, P_{cb})$ and $P_{abc}(P_{ca}, P_{ac})$.

Moreover, type I bridges with vertices of attachment in $P_{abc}(P_{ab}, P_{ba})$, $P_{abc}(P_{bc}, P_{cb})$ and $P_{abc}(P_{ca}, P_{ac})$ are respectively called type I($a, b$), type I($b, c$) and type I($c, a$) bridges. Also, type II bridges with vertices of attachment not in $P_{abc}(P_{ab}, P_{ba})$ are called type II($c$) bridges, and similarly for type II($a$) and type II($b$) bridges.
**Proposition 2.5.2.** Suppose $G$ is a nonseparable graph with distinct edges $a$, $b$ and $c$. Suppose also that $(P_1)_{abc}$ is a polygon of $G$ and $B_1$ is a bridge of $(P_1)_{abc}$ skew to a type I, say type I(a,b), bridge $C_1$ of $(P_1)_{abc}$. Then there exists a polygon $(P_2)_{abc}$ in $G$ and a bridge $B_2$ of $(P_2)_{abc}$ in $G$, such that

$$B_1 \nsubseteq B_2.$$

**Proof.** For convenience we will write $P_i$ for $(P_i)_{abc}$ when the context is clear.

By hypothesis, $W(G, C_1) \subseteq V(P_1((P_1)_{ab}, (P_1)_{ba}))$. Choose $u_1, v_1 \in W(G, C_1)$ to be as close as possible to $(P_1)_{ab}, (P_1)_{ba}$, respectively. There exists $w \in W(G, B_1) \cap V(P_1(u_1, v_1)) \setminus \{u_1, v_1\}$, since $B_1$ and $C_1$ are skew with respect to $P_1$. We now form the polygon $P_2$ whose set of edges is given by

$$E(P_2) = (E(P_1) \setminus E(P_1(u_1, v_1))) \cup E(L_1),$$

where $L_1$ is any diagonal of $P_1$ joining $u_1$ and $v_1$ in $C_1$.

Then $P_2(u_1, v_1) = P_1(u_1, v_1)$ and $P_2(u_1, v_1) = L_1$. Hence $P_2$ contains $a$, $b$ and $c$, since they belong to $P_1(u_1, v_1)$. Let $B_2$ be the bridge of $P_2$ containing $B_1 \cup P_1(u_1, v_1)$. Evidently

$$B_1 \nsubseteq B_2.$$ This completes the proof.
Proposition 2.5.2.1. Suppose the statement of the proposition 2.5.2 holds. Also, suppose that we have the polygon $P_2$ and the bridge $B_2$ of $P_2$ in $G$ obtained in the conclusion of proposition 2.5.2. Then $(V(P_1'(^u_1, v_1)) \cap W(G, B_1)) \cup \{u_1, v_1\} \subseteq W(G, B_2)$.

Proof. By the construction of $P_2$ in proposition 2.5.2 we have $P_1'(^u_1, v_1) = P_2'(^u_1, v_1)$. Hence $(V(P_1'(^u_1, v_1))) \cap W(G, B_1) \subseteq W(G, B_2)$, since $B_1 \subseteq B_2$. Also, there exist edges incident with $u_1$ and $v_1$ on $P_1$ belonging to $P_1'(^u_1, v_1) = P_2'(^u_1, v_1)$ and hence not contained in $B_2$, while the arc $P_1(^u_1, v_1) \subseteq B_2$. This implies that $u_1, v_1 \in W(G, B_2)$. Hence $((V(P_1'(^u_1, v_1))) \cap W(G, B_1)) \cup \{u_1, v_1\} \subseteq W(G, B_2)$, proving the proposition.

Proposition 2.5.2.2. Suppose the statements of proposition 2.5.2.1 hold. Let $H$ be a bridge of $(P_1)_{1, 2}$ such that $W(G, H) \subseteq V(P_1'(^u_1, v_1))$. Then $H \subseteq B_2$.

Proof. $H$ has a vertex $v \in V(P_1'(^u_1, v_1)) \setminus \{u_1, v_1\}$, by hypothesis.

Also, $P_1(^u_1, v_1) \subseteq B_2$. Hence $H \subseteq B_2$.

Proposition 2.5.2.3. Suppose the statements of proposition 2.5.2.1 hold. Then $H$ is a bridge of $(P_1)_{1, 2}$ with $W(G, H) \subseteq V(P_1'(^u_1, v_1))$ if and only if $H$ is a bridge of $(P_2)_{1, 2}$ with $W(G, H) \subseteq V(P_2'(^u_1, v_1))$. 
Proof. This follows from proposition 2.5.2.2 and the fact that 
\( P_2'(u_1, v_1) = P_1'(u_1, v_1) \).

Proposition 2.5.2.4. Suppose the statement of proposition 2.5.2 holds. Then one of the following holds.

1. \( B_2 \) is a type I bridge of \( (P_2)_{abc} \) if all bridges of \( (P_1)_{abc} \) skew to \( C_1 \) are type I bridges.

2. \( B_2 \) is a type II bridge of \( (P_2)_{abc} \) if all the bridges of \( (P_1)_{abc} \) skew to \( C_1 \) are type I bridges or the same kind of type II bridges, there being at least one type II bridge.

3. \( B_2 \) is a type III bridge of \( (P_2)_{abc} \) if there exists a type III bridge of \( (P_1)_{abc} \) skew to \( C_1 \), or two type II bridges of different kinds skew to \( C_1 \).

Proof. Follows from propositions 2.5.2.1 and 2.5.2.2.

Remark. If \( B_1 \) is a type III bridge of \( (P_1)_{abc} \) in \( G \) then \( B_2 \) is a type III bridge of \( (P_2)_{abc} \) in \( G \) with \( |W(G, B_2)| \geq 4 \). This follows from propositions 2.5.2.1 and 2.5.2.4.

Algorithm 2.5.3. (Clearing all type I bridges of \( P_{abc} \) skew to a type III bridge).

Suppose \( B_1 \) is a type III bridge of a polygon \( (P_1)_{abc} \) in a nonseparable graph \( G \), which is skew to a type I bridge \( C_1 \). We describe how to replace \( (P_1)_{abc} \) by an alternate polygon \( (P_n)_{abc} \), for some integer \( n \geq 2 \), which has a type III bridge \( B_n \) in \( G \).
such that \( B_i \subseteq B_n \) and there exists no type I bridge of \((P_i)_{abc}\) in \(G\) skew to \(B_n\). Suppose, as inductive hypothesis, that \( B_i \) is a type III bridge of a polygon \((P_i)_{abc}\) with \( B_i \subseteq B_i' \). Define \( C_i = \{C_i': C_i' \text{ is a type I bridge of } P_i \text{ in } G \text{ skew to } B_i\} \). If \( C_i \neq \emptyset \), then, by proposition 2.5.2, we obtain a polygon \( P_{i+1} \) with a bridge \( B_{i+1} \) such that \( B_i \subseteq B_{i+1} \). Beginning with \( P_1 \), \( B_1 \) this procedure can be repeated until stage \( P_n \), \( B_n \) with \( C_n = \emptyset \), for some \( n \geq 2 \), in which case \( B_1 \subseteq B_2 \subseteq B_3 \subseteq \ldots \subseteq B_n \) and no type I bridge \( C_n \) skew to \( B_n \) on \( P_n \) exists, as required.

This process is called enlarging \( B_1 \) by clearing all type I bridges of \((P_i)_{abc}\) skew to \( B_1 \).

**Proposition 2.5.4.** Suppose \( G \) is a nonseparable graph and \( a, b, c \in E(G) \) are distinct. Let \( B_1 \) be a type III bridge of a polygon \( Q_{abc} \) in \( G \). Then there exists a polygon \( P_{abc} \) and a type III bridge \( B \) of \( P_{abc} \) in \( G \), such that \( B \supseteq B_1 \) and \( B \) is not skew to any type I bridge of \( P_{abc} \) in \( G \).

**Proof.** This follows from proposition 2.5.2 and algorithm 2.5.3.

**Algorithm 2.5.5** (Clearing bridges from \( L_1 \)).

Suppose that the statement of proposition 2.5.2 holds and that \( L_1 \) is any arc of \( C_1 \) joining the vertices \( u_1, v_1 \in W(G, C_1) \), where
Then the previous algorithm can be refined so that all bridges of $L_1$ in $C_1$ are cleared in the first stage. This can be repeated for all type I bridges of $P_1$ skew to $B_1$ not absorbed in the extension of $B_1$ obtained by this process, i.e. not skew to $C_1$ on $P_1$. Let $C_1 = \{C_1: C_1$ is a type I bridge of $(P_1)_{abc}$ in $G$ skew to $B_1\}$.

We now start with a bridge $C_1 \in C_1$, and $P_1$ and $B_1$ satisfying the hypothesis of proposition 2.5.2. We thus obtain a bridge $B_2$ of a polygon $(P_2)_{abc}$ in $G$ such that $B_1 \subseteq B_2$, as stated in the conclusion of proposition 2.5.2. Now define

$$C_2 = \{C_2: C_2 \text{ is a bridge of } (P_2)_{abc} \text{ in } G, \quad C_2 \subseteq C_1, \text{ and } C_2 \text{ is skew to } B_2 \text{ on } P_2\}.$$  

If $C_2 = \emptyset$, we stop. Otherwise, there exists a $C_2 \in C_2$. We then apply proposition 2.5.2 on the triple $P_2, B_2$ and $C_2$. Hence, inductively, we have $(P_k)_{abc}$ and $B_k$ with $B_{k-1} \subseteq B_k$. Therefore, we can define. Therefore

$$C_k = \{C_k: C_k \text{ is a bridge of } (P_k)_{abc} \text{ in } G, \quad C_k \subseteq C_1, \text{ and } C_k \text{ is skew to } B_k \text{ on } P_k\}.$$  

Again we stop if $C_k = \emptyset$, otherwise the process terminates with $C_n = \emptyset$, for some smallest positive integer $n$, since $G$ is a finite
The arc $L_n = P_n(u_1, v_1)$, where $u_1, v_1$ are the endvertices of $L_1$, is an arc of $C_1$. All bridges of $P_n$ with vertices of attachment not contained in $P_n(u_1, v_1)$ are bridges of $L_n$ in $C_1$, by 2.5.2.3, and are not skew to $B_n$ on $P_n$. The above process is called clearing the bridge $C_1$ by replacing $P_n(u_1, v_1)$ by $L_n$.

**Proposition 2.5.6.** Suppose $G$ is a nonseparable graph and $a, b, c \in E(G)$ are distinct. Let there be a $P_{abc}$ in $G$ and a sequence of bridges $B_1, B_2, ..., B_n$ of $P_{abc}$ in $G$ such that

1. $B_1$ is a type II (a) bridge and $B_n$ is a type II (b) bridge, and
2. $B_i$ for $1 < i < n$ are all type I(a,b) bridges skew to $B_{i-1}$ and $B_{i+1}$, and to no other bridge of the sequence.

Then one of the following occurs:

1. $n = 2$, in which case $B_1$ and $B_n$ are skew with respect to $P_{abc}$,
2. $n = 3$, in which case there exists a polygon $Q_{abc}$ in $G$ having a type III bridge $B_1'$ in $G$, with $B_1 \cup B_3 \subseteq B_1'$, or
3. $n \geq 4$, in which case there exists a polygon $Q_{abc}$ in $G$ having a bridge $B_1'$, with $B_1 \cup B_3 \subseteq B_1'$, which is either of type III, or is a type II(a) bridge when $B_1'$ is a type II(a) bridge. The sequence of bridges $B_1', B_4, B_5, ..., B_n$ of $Q_{abc}$ in $G$ satisfies conditions analogous to (1) and (2) above for $B_1, B_2, ..., B_n$ with
respect to $P_{abc}$.

Proof. For $n = 2$, the result follows from the statement of the proposition. Suppose $n \geq 3$. Let the extreme vertices of $B_i$, for $1 \leq i \leq n$, be $\{u_i, v_i\}$ with $u_i < v_i$ in the linear order from $P_{ca}$ to $P_{cb}$ on $P'(P_{ca}, P_{cb})$. By our assumption, $B_{i+1}$ is skew to both $B_i$ and $B_{i+2}$, and $B_i$ and $B_{i+2}$ are not skew with respect to $P_{abc}$ for all $i = 1, 2, \ldots, n-2$. Considering the possibilities for $B_i$ and $B_{i+2}$ we have two cases, namely $B_i$ and $B_{i+2}$ overlap or avoid each other on $P_{abc}$. In the first case, $B_i$ and $B_{i+2}$ overlap but are not skew, implying that they are equipartite 3-bridges of $P_{abc}$. Then $i > 1$, and $B_{i-1}$ and $B_{i+2}$ are skew with respect to $P_{abc}$, since $B_{i-1}$ and $B_i$ are skew, which is a contradiction to our assumption.

So we are left with the alternative case: $B_i$ and $B_{i+2}$ avoid each other with respect to $P_{abc}$. Now, two possibilities arise. Firstly, it may happen that

$$P_{abc}(u_{i+2}, v_{i+2}) \subseteq P_{abc}(u_i, v_i) \subseteq P_{abc}(u_i, v_i),$$

for some pair of consecutive vertices $u, v$ of $B_i$ on $P_{abc}$. Then $W(G, B_j) \subseteq V(P_{abc}(u, v))$ for all $j = i+3, i+4, \ldots, n-1$, for otherwise some such $B_j$ will be skew to $B_i$ with respect to $P_{abc}$. It
follows that $B_n$ has a vertex of attachment in $P_{abc}(u,v)$, since $B_{n-1}$ and $B_n$ are skew. But $B_n$ is a type II(b) bridge of $P_{abc}$.

Hence $B_n$ and $B_1$ are skew with respect to $P_{abc}$. This is a contradiction if $i \leq n-2$, by our hypothesis about the sequence. Therefore, the second possibility holds, namely

$$P_{abc}(u_{i-1},v_i) \cap P_{abc}(u_{i+2},v_{i+2}) = \emptyset,$$

for all $i = 1, 2, \ldots, n-2$.

We know that $B_1$ is a type II(a) bridge and $B_2$ is a type I(a,b) bridge skew to $B_1$ in the sequence $B_1, B_2, B_3, \ldots, B_n$, for $n \geq 3$.

Moreover, $B_3$ is a bridge skew to $B_2$ with $P(u_1,v_1)$ and $P(u_3,v_3)$ disjoint, while $u_1 < u_2 < v_1 \leq u_3 < v_2 < v_3$. If $B_3$ is a type I(a,b) bridge then $B_4$ is a bridge with $u_2 < u_3 < v_2 \leq u_4 < v_3 < v_4$, and so on. Ultimately, we reach $B_n$, which is a type II(b) bridge skew to $B_{n-1}$ and with $P(u_{n-2},v_{n-2})$ disjoint from $P(u_n,v_n)$, so that

$$u_{n-2} < u_{n-1} < v_{n-2} \leq u_n < v_{n-1} < v_n.$$

This argument rules out the possibility that some $B_{i+2}$ is closer to $B_1$ than $B_1$ on $P_{abc}(P_{ab}, P_{ba})$.

Continuing the proof of the proposition, we have $u_2 < v_1 \leq u_3 < v_2$.

We define a polygon $Q$ with the edge set

$$E(Q) = (E(P_{abc}) \setminus E(P_{abc}(u_2,v_2))) \cup E(L))$$
where $L$ is any arc in $B_2$ joining $u_2$ and $v_2$. Now

$P_{abc}(u_2, v_2) \subseteq P_{abc}(p_{ab}, p_{ba})$, since $B_2$ is a type I(a,b) bridge.

Hence $a, b, c \in E(Q)$. Also $P_{abc}(u_2, v_2) \cup B_1 \cup B_3$ is contained in a bridge, say $B_1'$, of $Q$, by proposition 2.5.2, using the fact $u_2 < v_1 < u_3 < v_2$ on $P_{abc}(p_{ab}, p_{ba})$ as indicated above. By proposition 2.5.2.4, $B_1'$ has vertices of attachment on both $Q_{abc}(q_{ab}, q_{ba})$ and $Q_{abc}(q_{ac}, q_{ac})$. As $u_1 < u_2$, $B_2$ cannot be a link-graph joining $p_{ab}$ and $p_{ba}$, which would be the only possible exception to this assertion. Therefore, if $n = 3$ then $B_1'$ is a type III bridge of $Q_{abc}$ since $B_1 \cup B_3 \subseteq B_1'$. If $n > 4$ and if there exists a type II(b) bridge $B$ of $P_{abc}$ in $G$, $B \neq B_n$, $B$ having a vertex $w \in V(P_{abc}(u_2, v_2))$ with $w \notin \{u_2, v_2\}$, then

$B \cup B_1 \cup B_3 \subseteq B_1'$, and hence $B_1'$ is a type III bridge of $Q_{abc}$.

Otherwise, when $n > 4$, $B_1'$ is a type II(a) bridge of $Q_{abc}$ and $B_1'B_{k}B_{k+1}B_{k+2}...B_{n-k+4}$ are bridges of $Q_{abc}$ with the same vertices of attachment in $Q_{abc}$ as in $P_{abc}$, by proposition 2.5.2.3. Then

$Q_{abc}(u_i, v_i) \cap Q_{abc}(u_{i+2}, v_{i+2}) = \Omega$,

for all $i = k, k+1, \ldots, n-2$. This implies that $B_{i+1}$ is skew to both $B_1$ and $B_{i+2}$ with respect to $Q_{abc}$ for all $i = k, k+1, \ldots, n-1$

and to no other bridge in this sequence. Also $B_1'$ is skew to $B_k$ and
not skew to any of the bridges $B_k, B_{k+1}, \ldots, B_n$, $k > 4$. The proposition now follows.

**Remark.** In the above notation $u_1 < u_2 < v_1 < u_3 < v_2 < u_4 < \ldots
< v_{n-3} < u_{n-1} < v_{n-2} < u_n < v_{n-1} < v_n$.

**Proposition 2.5.7.** Suppose $G$ is a nonseparable graph with distinct edges $a, b, c$. Let $(P_{abc})$ be a polygon of $G$ and $B_1, B_2, \ldots, B_n$ be bridges of $(P_{abc})$ in $G$ such that

1. $B_1$ is a type II(a) bridge and $B_n$ is a type II(b) bridge,
2. $B_i$ for $1 < i < n$ are all type I(a,b) bridges of $(P_{abc})$ and
3. $B_i$ for $1 < i < n$ is skew to $B_{i-1}$ and $B_{i+1}$ and no other of the $B_j$'s for $1 \leq j \leq n$.

Then, there exists a polygon $P_{abc}$ and a bridge $B$ of $P_{abc}$ in $G$ such that $B_1 \subseteq B$ and either

4. $B$ is a type III bridge of $P_{abc}$ in $G$ or
5. $B$ is a type II(a) bridge, and $B_n$ is a type II(b) bridge of $P_{abc}$ in $G$ skew to $B$.

**Remark 1.** Conditions (4) and (5) are not exclusive.

**Remark 2.** Condition (5) implies, by proposition 2.4.1, that $G$ has a bond $K_{abc}$.

**Proof.** We apply induction on $n$. When $n = 2$ conditions (2) and
(3) are vacuously valid and condition (5) applies. If \( n = 3 \) then conclusion (4) of proposition 2.5.7 applies by proposition 2.5.6. Therefore, we consider the case when \( n \geq 4 \). In this case we obtain the sequence \( B', B_k, B_{k+1}, \ldots, B_{n', k>4} \) of bridges of a polygon \((P_2)_{abc}\) in \( G \).

Hence, we consider the case when \( n > b \).

In this case we obtain the sequence \( B'_1, B_3 \subseteq B'_1 \), and the hypotheses (1), (2), (3) of proposition 2.5.7 are satisfied, for this sequence, by proposition 2.5.6. By inductive hypothesis, therefore, there exists a polygon \( P_{abc} \) and a bridge \( B \) of \( P_{abc} \) in \( G \) such that \( B'_1 \subseteq B \) and either (4) or (5) of proposition 2.5.7 apply, by proposition 2.5.6. As \( B'_1 \subseteq B' \), the proposition follows.

2.6 Existence of a necklace decomposition.

From this section onward by \( H(I) \) we will mean that

1. \( G \) is a finite nonseparable graph,
2. \( a, b, c \) are distinct edges of \( G \), and
3. \( \{a, b, c\} \nsubseteq K \), for any bond \( K \) of \( G \).

These conditions will be assumed to apply in what follows, unless the contrary is explicitly stated.

Proposition 2.6.1. If \( H(I) \) holds then there exists a polygon \( P_{abc} \) of \( G \).

Proof. Follows from proposition 2.3.2.

Definition 2.6.2 (Polygons \( P_{abc} \) relative to type III bridges).
Considering the set $P$ of all polygons $P_{abc}$ of $G$ and the bridges of each $P_{abc}$ in $G$ we get the following two exclusive cases:

(S): There exists a polygon $P_{abc} \in P$ such that $P_{abc}$ has a bridge in $G$ separating $a, b$ and $c$.

(NS): There exists no bridge of $P_{abc}$ separating $a, b$ and $c$, for all $P_{abc} \in P$.

The case (NS) splits up into three cases, namely:

(1) there exists a polygon $P_{abc}$ such that $P_{abc}$ has no bridge in $G$,

(2) there exists a polygon $P_{abc}$ such that the bridges of $P_{abc}$ in $G$ are all of type $I$, and

(3) there exists a polygon $P_{abc}$ and a bridge $B$ of $P_{abc}$ in $G$ of type $II$.

Remark. The significance of conditions (S) and (NS) lies in the fact that when (NS) applies $G$ is a necklace with $a, b, c$ in distinct beads. We prove this fact in the next three propositions, which also imply that cases (1), (2) and (3) of (NS) are exclusive. When (S) applies we obtain a necklace decomposition

$$D = (N, R)$$

with non-null residue $R$.

**Proposition 2.6.3.** If $H(I)$ and case (1) of (NS) hold, then $G$ is a necklace and $a, b, c$ belong to distinct beads of $G$. 
Proof. In this case \( G = P_{abc} \), and hence the proposition follows.

**Proposition 2.6.4.** If \( H(I) \) and case (2) of (NS) hold, then \( G \) is a necklace and \( a, b, c \) belong to distinct beads of \( G \).

Proof. The triple \( \{p_{ab}, p_{bc}, p_{ac}\} \) satisfies the conditions of the proposition 2.1.4. Obviously \( G\{a\}, G\{b\}, \) and \( G\{c\} \) are distinct beads of this necklace.

**Proposition 2.6.5.** If \( H(I) \) and case (3) of (NS) hold, then \( G \) is a necklace and \( a, b, c \) belong to distinct beads of the necklace.

Proof. By hypothesis, there exists a polygon \( P_{abc} \) of \( G \) and a type II bridge, say, type II(a) bridge \( B \) of \( P_{abc} \) in \( G \). Also, by hypothesis there is no type III bridge of \( P_{abc} \) in \( G \).

Again, if there exists either a type II(b) or a type II(c) bridge of \( P_{abc} \) skew to \( B \) then there will be a pair of skew diagonals of \( P_{abc} \) separating \( a, b \) and \( c \). So, by proposition 2.4.1, there will be a \( K_{abc} \) in \( G \), contradicting \( H(I) \). Hence there are no type II(b) or type II(c) bridges of \( P_{abc} \) skew to \( B \). We now form the recessional sequence

\[
\mathcal{S}_1 = \{B_1: \text{\( B_1 \) is a type II(a) bridge of \( P_{abc} \) in \( G \)}\}.
\]

\[
\mathcal{S}_2 = \{B_2: \text{\( B_2 \) is a bridge of \( P_{abc} \) in \( G \), \( B_2 \notin \mathcal{S}_1 \) and \( B_2 \) is skew to \( B_1 \), for some \( B_1 \in \mathcal{S}_1 \)}\}.
\]

.................................
$\mathcal{B}_j = \{ B_j : B_j \text{ is a bridge of } P_{abc} \text{ in } G, B_j \notin \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_{j-1} \text{ and } B_j \text{ is skew to } B_{j-1}, \text{ for some } B_{j-1} \in \mathcal{B}_{j-1} \}$.

Then, by finiteness, we get a positive integer $n$ such that

$\mathcal{B}_n \neq \emptyset, \mathcal{B}_{n+1} = \emptyset = \mathcal{B}_{n+2} = \ldots$, since the $\mathcal{B}_j$'s are disjoint. We claim that no member of $\mathcal{B}_j$, $j \geq 2$ is a type II(b) bridge of $P_{abc}$ in $G$. We prove this claim by a contradiction. If possible, let $i$ be the first subscript such that there is a type II(b) bridge $B_i \in \mathcal{B}_i$.

Then there exists a sequence $B_1, B_2, \ldots, B_i$ of bridges of $P_{abc}$ in $G$ such that $B_k \in \mathcal{B}_k$, $k = 1, 2, \ldots, i$ where

1. $B_1$ is a type II(a) bridge, $B_1$ is a type II(b) bridge of $P_{abc}$ in $G$,
2. $B_k$, $k = 2, \ldots, i-1$ are all type I bridges of $P_{abc}$ in $G$, by choice of the subscript $i$. (In fact, they are type $I(a,b)$ bridges because $B_1$ is a type II(a) and $B_2$ is a type II(b) bridge.), and
3. $B_k$ is skew only to the pair $B_{k-1}$ and $B_{k+1}$ in this sequence, $k = 2, 3, \ldots, i-1$. Hence, by proposition 2.5.7 there exists a polygon say $Q_{abc}$ and a bridge $B$ of $Q_{abc}$ in $G$, $B \sqsupseteq B_1$, such that either
4. $B$ is a type III bridge of $Q_{abc}$ in $G$, or
5. $B$ is a type II(a) bridge, $B_1$ is a type II(b) bridge of $Q_{abc}$ and they are skew with respect to $Q_{abc}$.

Case (4) above contradicts the hypothesis (NS) that there is no polygon in $G$ having a type III bridge in $G$. In case (5) above, we
have a pair of skew diagonals one in each of $B$ and $B_i$. Also they separate $a$, $b$ and $c$, since $B$ is a type II(a) bridge and $B_i$ is a type II(b) bridge of $Q_{abc}$ in $G$, and they are skew with respect to $Q_{abc}$. Hence, by proposition 2.4.1, there exists a $K_{abc}$ in $G$, which contradicts $H(I)$, and establishes our claim.

Similarly, no member of $\mathcal{A}_{j}$, $j \geq 2$ is a type II(c) bridge of $P_{abc}$ in $G$. Taking into consideration the definition of $\mathcal{A}_{1}$ we have that no member of $\mathcal{A}_{j}$, $j \geq 1$ is a type II(b) or a type II(c) bridge of $P_{abc}$ in $G$.

Let $S' = \bigcup_{j=1}^{n} (\bigcup_{B_j} B_j)$ be the subgraph of $G$ containing the bridges of the recessional sequence above. Let the vertices of $W(G, S'_a)$ on $P_{abc}$ nearest to $p_{ba}$ and to $p_{ca}$ be $s_{ab}$ and $s_{ac}$ respectively. Also, there cannot be any bridge $B'$ of $P_{abc}$ which has vertices $v, w$, $\{v, w\} \cap \{s_{ab}, s_{ac}\} = \emptyset$ and $v \in P_{abc}(s_{ab}, s_{ac})$, $w \in P_{abc}'(s_{ab}, s_{ac})$. For, otherwise $B'$ will be skew to a member $B_j$ of the set $\bigcup_{j=1}^{n} \mathcal{A}_j$. The existence of such a $B_j$ follows, by the remark following proposition 2.5.6. Then $B'$ will belong to $\bigcup_{j=1}^{n} \mathcal{A}_j$, which is impossible.

Now, there may or may not exist a bridge, say $C$, of type II(b) of $P_{abc}$ in $G$. If there exists such a $C$ then exactly as above we
form $S'_b$. Let the vertices of $W(G, S'_b)$ on $P_{abc}$ nearest to $P_{ab}$ and $P_{cb}$ be $s_{ba}$ and $s_{bc}$, respectively. If there exists no such $C$, then $P_{ba} = s_{ba}$ and $s_{bc} = P_{bc}$.

In both the cases $s_{ab}$, $s_{ac}$ and $s_{bc}$ are distinct. We claim that there exists \( q, r \subseteq \{ s_{ab}, s_{ac}, s_{bc} \} \) such that $W(G, C') \subseteq P_{abc}(q, r)$ for each bridge $C'$ of $P_{abc}$ in $G$.

We have already proved that there exists no bridge of $P_{abc}$ having vertices $v \in P_{abc}(s_{ab}, s_{ac})$ and $w \in P_{abc}(s_{ab}, s_{ac})$ where $v, w \notin \{ s_{ab}, s_{ac} \}$. Also, the similar result is true for $P_{abc}(s_{ba}, s_{bc})$ and $P_{abc}(s_{ba}, s_{bc})$.

If possible, let there be a bridge $C'$ of $P_{abc}$ in $G$ having vertices $u'$ and $w'$ in $P_{abc}(s_{ac}, s_{bc})$ and $P_{abc}(s_{bc}, s_{ab})$ respectively such that \( \{ u', w' \} \cap \{ s_{ab}, s_{bc}, s_{ac} \} = \emptyset \). Then $w' \in V(P_{abc}(s_{ab}, s_{ba}))$, by the results stated in the last paragraph. Therefore, exactly two cases arise, namely where $u' \in V(P_{abc}(s_{ac}, P_{ca}))$ or $u' \in V(P_{abc}(P_{cb}, s_{bc}))$. In the first case $C' \subseteq S'_a$ because $C'$ is a type II(a) bridge. Similarly, in the second case $C' \subseteq S'_b$.

Hence there exists no such $C'$. This proves our claim that there exists \( q, r \subseteq \{ s_{ab}, s_{ac}, s_{bc} \} \) such that $W(G, C') \subseteq P_{abc}(q, r)$ for each bridge $C'$ of $P_{abc}$ in $G$. Hence, by the proposition 2.1.4,
G is a necklace with \( s_{ab}, s_{ac}, \) and \( s_{bc} \) vertices of attachment of some beads. Also, the edges \( a, b \) and \( c \) belong to three distinct \([s_{ab}, s_{ac}, s_{bc}]\)-components of this necklace \( G \). Hence \( a, b \) and \( c \) belong to distinct beads of this necklace \( G \). This completes the proof of the proposition.

In the following paragraphs we consider the case

\((S)\) : there exists a polygon \( P_{abc} \subseteq \mathcal{P} \) such that \( P_{abc} \) has a bridge \( B \) in \( G \) separating \( a, b \) and \( c \). Here we will prove that if \( H(I) \) and \( (S) \) hold then \( G \) has a necklace decomposition \( D = (N, R) \) where \( N \) is a necklace and \( R \) is a single bridge of \( N \) separating \( a, b \) and \( c \), or \( R \) is the union of two or more equipartite 3-bridges of \( N \) separating \( a, b \) and \( c \).

**Proposition 2.6.6.** Suppose \( G \) is a nonseparable graph with a polygon \( P_{abc} \) in \( G \), where \( a, b, c \) are distinct edges of \( G \). Also, suppose \( B_1 \) and \( B_2 \) are type III bridges of \( P_{abc} \) in \( G \). Then either

(i) \( B_1 \) and \( B_2 \) are equipartite 3-bridges, or

(ii) there exists a pair of skew diagonals of \( P_{abc} \) separating \( a, b \) and \( c \).

**Proof.** Each of \( B_1 \) and \( B_2 \) separate \( a, b \) and \( c \). Therefore, they overlap and then, by proposition 1.7.1, we obtain the required result.

**Proposition 2.6.7.** Suppose \( G \) is a nonseparable graph with a polygon \( P_{abc} \), where \( a, b, \) and \( c \) are distinct edges of \( G \). Also, suppose that \( B_1 \) is a type III bridge and \( B_2 \) is a type II bridge of \( P_{abc} \) in \( G \) such that \( B_1 \) and \( B_2 \) are skew with respect to \( P_{abc} \).
Then there exists a pair of skew diagonals $P_{abc}$ separating $a$, $b$, and $c$.

Proof. Suppose $B_2$ is a type-II(b) bridge of $P_{abc}$ in $G$. Let the vertices of $W(G, B_2)$ nearest to $p_{ab}$ and $p_{cb}$ be $u_2$ and $v_2$ respectively. Then there exists a vertex $u_1 \in W(G, B_1)$ such that $u_1 \in V(P_{abc}(u_2, v_2))$ and $u_1 \notin \{u_2, v_2\}$. Let $v_1 \in W(G, B_1) \cap V(P_{abc}(p_{ac}, p_{ca}))$. Also the arcs joining $u_2$ and $v_2$ in $B_2$ and $u_1$ and $v_1$ in $B_1$ form a pair of skew diagonals of $P_{abc}$ separating $a$, $b$, and $c$.

**Definition 2.6.8** (Polygons $P_{abc}$ relative to the case (S)).

We define $P_1 = \{P_{abc}: P_{abc} \in P\}$ and there exists a type III bridge of $P_{abc}$ in $G\}$,

$P_{1,1} = \{P_{abc}: P_{abc} \in P_1 \text{ and each type III bridge of } P_{abc} \text{ in } G \text{ has exactly three vertices of attachment in } G\}$, and

$P_{1,2} = \{P_{abc}: P_{abc} \in P_1 \text{ and there exists a type III bridge of } P_{abc} \text{ in } G \text{ which has four or more vertices of attachment in } G\}$.

**Remark.** $P_{1,1} \cup P_{1,2} = P_1$ and $P_{1,1} \cap P_{1,2} = \emptyset$.

**Proposition 2.6.9.** If $H(I)$ and (S) hold, then

1. for all $P_{abc} \in P_{1,1}$, $P_{abc}$ has one type III 3-bridge, or two or more equipartite type III 3-bridges and in both cases there does not
exist any type II bridge of $P_{abc}$ in $G$ which is skew to a 3-bridge.

(2) for all $P_{abc} \in P_{1,2}$, $P_{abc}$ has exactly one type III bridge $B$ and there does not exist any type II bridge skew to $B$ with respect $P_{abc}$.

Proof. By condition (S) there exists $P_{abc} \in P_1$. By definition of $P_1$ there exists a type III bridge $B$ of $P_{abc}$ in $G$. If $P_{abc}$ has a pair of skew diagonals separating $a$, $b$ and $c$ then proposition 2.4.1 implies that there exists a bond $K_{abc}$ contrary to $H(I)$. Thus, $P_{abc}$ has no pair of skew diagonals separating $a$, $b$ and $c$. This also rules out, by proposition 2.6.7, any type II bridge $B'$ skew to $B$. Again, if there exists a type III bridge $B'$ then proposition 2.6.6 implies that $B$ and $B'$ are equipartite 3-bridges of $P_{abc}$ in $G$. Hence, if $P_{abc} \in P_{1,1}$ then either $P_{abc}$ has exactly one type III 3-bridge or all the type III bridges are equipartite 3-bridges. Also, if $P_{abc} \in P_{1,2}$ then the bridge $B$ is unique. This completes the proof of the proposition.

Proposition 2.6.10. Suppose $H(I)$ and (S) hold. Also, suppose $P_{1,2} = \emptyset$. Then, there exists a necklace decomposition

$$D = (N, R)$$

of $G$ such that $a$, $b$, $c$ are in distinct beads of $N$, and $R$ is a 3-bridge or $R$ is the union of two or more equipartite 3-bridges of $N$, and $R$ separates $a$, $b$, $c$. 

Proof. The condition (S) and $P_{1,2} = \emptyset$ imply that $P_{1,1} \neq \emptyset$. Let $P_{abc} \in P_{1,1}'$. Then, by the definition of $P_{1,1}'$, there exists at least one type III 3-bridge $B$ of $P_{abc}$. Again $P_{1,2} = \emptyset$ implies that there exists no type I bridge $B'$ of $P_{abc}$ skew to $B$. Also, by proposition 2.6.9, there does not exist any type II or type III bridge of $P_{abc}$ skew to $B$. Hence other type III bridges of $P_{abc}$ are equipartite with $B$.

Let $R$ be the union of the type III bridges of $P_{abc}$ and $N = G \cdot (E(G) \setminus E(R))$. Then $R \neq \emptyset$ since $B \subseteq R$. By definition of $R$, $P_{abc} \subseteq N$. Therefore $N$ is the union of $P_{abc}$ and all its type I and type II bridges. Hence $N$ is a nonseparable graph, by proposition 1.4.1.

Because no type I or type II bridges of $P_{abc}$ skew to $B$ exist, and $W(G, R) = W(G, B)$, we have for each bridge $B'$ of $P_{abc}$ in $N$, the existence of $(x, y) \subseteq W(G, R)$ such that $W(N, B') \subseteq W(P_{abc}(x, y))$.

Hence, by proposition 2.1.4, $N$ is a necklace with the members of $W(G, R)$ as vertices of attachment of some beads of $N$. Also $a$, $b$, $c$ belong to distinct beads of $N$ since they belong to distinct $[W(G, R)]$-components of $N$.

Thus $D = (N, R)$ is a necklace decomposition of $G$ with the
required properties.

**Proposition 2.6.11.** Suppose $H(I)$ and $(S)$ hold. Also, suppose $\mathcal{P}_{1,2} \neq \emptyset$. Then, there exists a necklace decomposition $D = (N, R)$ of $G$ such that $R$ is a single bridge of $N$ separating $a, b$ and $c$ and $|W(G, R)| \geq 4$.

**Proof.** $\mathcal{P}_{1,2} \neq \emptyset$ implies that there exists a polygon, say $Q_{abc}$, with a type III bridge $B_1$ having $|W(G, B_1)| \geq 4$.

By proposition 2.6.9, there does not exist any type III bridge except for $B_1$, and there does not exist any type II bridge of $Q_{abc}$ which is skew to $B_1$.

Suppose $C$ is a bridge of $Q_{abc}$ in $G$ and is skew to $B_1$. Then $C$ is a type I bridge of $Q_{abc}$. Then, by proposition 2.5.4, there exists a polygon $P_{abc}$ and a type III bridge $B$ of $P_{abc}$ in $G$ such that $B \subseteq B_1$ and $B$ is not skew to any type I bridge of $P_{abc}$ in $G$. By the remark of proposition 2.5.2, $|W(G, B)| \geq 4$.

Again, by proposition 2.6.9, there exists no other type III bridge of $P_{abc}$ except $B$ and no type II bridge of $P_{abc}$ in $G$ which are skew to $B$. Also, by definition of $B$, there exists no type I bridge of $P_{abc}$ which is skew to $B$. Hence all other bridges $B'$ of $P_{abc}$ avoid $B$. Let $N = G \cdot (E(G) \setminus E(B))$ and $R = B$. Then $P_{abc} = N$, and $N$ is the union of $P_{abc}$ and all bridges of $P_{abc}$ in $G$ except
B. Therefore $N$ is a nonseparable graph, by proposition 1.4.1, since $P_{abc}$ is nonseparable.

Let $x, y, z$ be distinct members of $W(G, R)$ and $B'$ be any bridge of $P_{abc}$ in $N$. Then $W(G, B') = W(N, B')$. If possible, let

$$v' \in (W(G, B') \cap V(P_{abc}(u,v)))\backslash\{u,v\}$$

$$u' \in (W(G, B') \cap V(P_{abc}(u,v)))\backslash\{u,v\}$$

where $\{u,v\} \subseteq \{x,y,z\}$. Then, by proposition 1.3.1, there exist arcs $L$ in $B$ joining $u$ and $v$ and $L'$ in $B'$ joining $u'$ and $v'$ such that

(i) $u, v, u', v'$ are distinct

(ii) $u', v'$ belong to two distinct $[u,v]$-components of $P_{abc}$. Hence, by proposition 1.6.2, $L$ and $L'$ are skew with respect to $P_{abc}$. Therefore, by definition 1.7, the bridges $B$ and $B'$ are skew with respect to $P_{abc}$.

This contradicts our earlier assertion in the proof that all bridges $B'$ of $P_{abc}$ avoid $B$. Hence, for each bridge $B'$ of $P_{abc}$ in $G$ there exists $\{u,v\} \subseteq \{x,y,z\}$ such that $W(N,B') \subseteq \{u,v\}$.

Therefore $N$ is a necklace, by proposition 2.1.4. Also, $x, y, z$ are vertices of attachment of beads of this necklace $N$. Again, any vertex $w \in W(G, R)$ is contained in some such triple $\{x,y,z\}$. This implies that every vertex of $W(G, R)$ is a vertex of attachment of some bead of $N$. But $R = B$ separates $a, b, c$. Hence $a, b, c$ and $c$
belong to distinct beads of \( N \) or more specifically in distinct \\
\( W(G, R) \)-segments, i.e. in distinct residual segments of \( N \).

Finally \( D = (N, R) \) is a necklace decomposition of \( G \) as \\
required. This completes the proof.

2.7. Edges in a graph which has a necklace decomposition.

Given that \( H(I) \) holds we have proved the existence of a necklace \\
decomposition \( D = (N, R) \) of \( G \) in the last section. In a necklace \\
decomposition thus obtained the following three exclusive conditions \\
hold.

(1) \( R = \Omega \) and \( a, b, c \) belong to distinct beads of \( N \),

(2) \( |W(G, R)| = 3 \) and \( R \) consists of one or more 3-bridges of \\
\( N \) in \( G \) separating \( a, b \) and \( c \), or

(3) \( |W(G, R)| \geq 4 \) and \( R \) is a single bridge of \( N \) in \( G \) separating \\
\( a, b, c \).

From now on by \( H(II) \) we will mean that \( G \) has a necklace decomposition \\
with one of the above three exclusive conditions (1), (2) and (3).

Given different necklace decompositions \( D_i = (N_i, R_i) \) for \\
i = 1, 2, ..., \( n \), of a nonseparable graph \( G \) with edges \( a, b, c \) in \\
distinct beads of each necklace we will prove in this section that \\
there are certain edges which are common to all the necklace \( N_i \) and \\
also certain edges which are common to all residues \( R_i \). Furthermore, \\
for all necklace decompositions \( D_i \) and \( D_j \),

\[
E(N_i) \cap E(R_j) \subseteq E^2_{abc}.
\]
Proposition 2.7.1. Suppose $G$ is a nonseparable graph with distinct edges $a$, $b$ and $c$. Also, suppose $G$ has a necklace decomposition $D = (N, R)$ with $e \in E(B)$, for a bridge $B$ of $N$ separating $a$, $b$ and $c$.

Then there exists a Y-graph, $Y \subseteq B$ such that $e \in E(Y)$ and $Y \cap N = \{y_a, y_b, y_c\}$, where $y_a$, $y_b$ and $y_c$ belong to $V(N(t_{bc}', t_{cb})), V(N(t_{ac}', t_{ca}))$ and $V(N(t_{ab}', t_{ba}))$, respectively.

Proof. Let $\{f, g, h\} = \{a, b, c\}$. By hypothesis $B$ is a bridge of $N$ and $e \in E(B)$. There exists an arc $L$ in $B$ containing $e$ with endvertices in $N$, as $G$ is nonseparable and $|W(G, B)| \geq 3$.

If this arc $L$ has its endvertices $x, y$ on one $N$-segment, say $N(t_{fg}', t_{gf}')$, then there exists an arc from a vertex, say $y_g \in V(N(t_{fh}', t_{hf}'))$, to a vertex, say $s$, on $L \setminus \{x, y\}$. Then, we get an arc $L'$ containing $e$ with endvertices $y_g \in V(N(t_{fh}', t_{hf}'))$ and $y_h \in V(N(t_{fg}', t_{gf}'))$, where $y_h \in \{x, y\}$.

Alternatively, if the arc $L$ has its endvertices in two distinct such $N$-segments then we can write $L = L'$ at this point and continue the following argument. There exists another arc $L''$ from a vertex $y_f \in V(N(t_{gh}', t_{hg}'))$ to a vertex $t$ of $L \setminus \{y_g, y_h\}$ since $B$ is a bridge of $N$ separating $a$, $b$ and $c$. Then $L' \cup L''$ is a Y-graph,
say Y, with three arms of Y branching out in three directions from t, and e ∈ E(Y). This completes the proof of the proposition.

Remark. The edge e can belong to any of the three arms of the Y-graph Y.

Proposition 2.7.2. Suppose H(I) and H(II) hold and e ∈ E(R). Then there exist \( K_{eab}, K_{ebc} \) and \( K_{eca} \).

Proof. First we prove the existence of \( K_{eab} \). The others will follow from symmetry in the conclusion of proposition 2.7.1.

Let \( S_a \) and \( S_b \) be the beads of R containing a and b respectively. As usual we write

\[
W(G, S_a) = \{s_{ab}, s_{ac}\} \quad \text{and} \\
W(G, S_b) = \{s_{ba}, s_{bc}\}.
\]

Proposition 2.7.2.1. There exists a bond \( K_{e_a} \) in \( S_a \) with end graphs \( H_{ab} \) and \( H_{ac} \) in \( S_a \) such that \( H_{ab} \) and \( H_{ac} \) contain the vertices \( s_{ab} \) and \( s_{ac} \), respectively.

Proof. We augment the graph \( S_a \) to \( S'_a \) such that

\[
V(S'_a) = V(S_a), E(S'_a) = E(S_a) \cup \{a'\}
\]

where \( a' \notin E(S_a) \) and \( a' \) is incident with vertices \( s_{ab} \) and \( s_{ac} \) of \( S_a \). \( S'_a \) is nonseparable by construction and proposition 1.4.1, since \( S \) is nonseparable. Hence, there exists a bond \( K_{a,a'} \) of \( S'_a \)
containing a and a'. This implies that each of the endgraphs of 
$K_{a,a'}$ in $S'$ contains exactly one vertex from $s_{ab}$ and $s_{ac}$, since
$a'$ is incident with both of them.

These two endgraphs then form the endgraphs of $K_{a,a'} \setminus \{a'\}$ in 
$S_a$. Hence $K_a = K_{a,a'} \setminus \{a'\}$ is a bond of $S_a$ containing a with 
the properties demanded by the proposition.

Now we return to the proof of proposition 2.7.2. We have, by 
proposition 2.7.2.1, two bonds $K_a$ and $K_b$ in $S_a$ and $S_b$, respectively, 
such that the endgraphs $H_{ab}$, $H_{ac}$ of $K_a$ in $S_a$ contain the vertices 
$s_{ab}$ and $s_{ac}$, while the endgraphs $H_{ba}$ and $H_{bc}$ of $K_b$ in $S_b$
contain the vertices $s_{ba}$ and $s_{bc'}$, respectively.

Let $u$ and $v$ be the vertices incident with $e \in E(R)$. By
proposition 2.7.1, there exists an arc $L \subseteq R$, $L$ containing the edge 
e with endvertices $y_c \in V(N(s_{ab}, s_{ba}))$ and $y_b \in V(N(s_{ac}, s_{ca}))$, 
chosen without loss of generality, where $y_c$, $u$, $v$, $y_b$ is the order 
of vertices as we move from $y_c$ to $y_b$ on $L$. Then $y_c$ and $u$ may 
or may not be distinct. Similarly $v$ and $y_b$ may or may not be 
distinct.

We define

$$J = N(s_{ab}, s_{ba}) \cup L(y_c, u) \cup H_{ab} \cup H_{ba}$$

and $H$ to be the outer $J$-component of $G$ containing the subgraph
Then the set $K_{eab} = \{ f \in E(G) : f$ is incident with a vertex of $H$ and a vertex of $J \}$, is a bond of $G$ since $J$ and $H \setminus V(J)$ are connected. Evidently $K_{eab}$ contains $a$, $b$ and $e$. We note that $H \setminus V(J)$ is an endgraph of $K_{eab}$ and $J$ is contained in the other endgraph. The proposition now follows.

Proposition 2.7.3. Suppose $H(I)$ and $H(II)$ hold. Then for every necklace decomposition

$D = (N, R)$ of $G$

\[
(1) \quad E_{bc} \cup E_{ca} \cup E_{ab} \subseteq E(N), \text{ and}
\]

\[
(2) \quad E(R) \subseteq E_{abc} \cup E_{ab, bc} \cup E_{bc, ca} \cup E_{ca, ab} \cup E_{ab, bc, ca} \text{ hold.}
\]

Proof. Let $e \in E(R)$. Then $R \not\subseteq \Omega$ and hence conditions (2) or (3) of $H(II)$ hold, i.e., $|W(G, R)| \geq 3$ and $R$ separates $a$, $b$ and $c$. Therefore $e$ belongs to a bridge $B$ of $N$ separating $a$, $b$ and $c$. Hence, by proposition 2.7.1, there exists a $Y$-graph $Y, Y \subseteq B$, such that $e \in E(Y)$ and the three arms of $Y$ meet the three $N$-segments $N(t_{ab}, t_{ba}), N(t_{ac}, t_{ca})$ and $N(t_{bc}, t_{cb})$ at vertices, say $y_a$, $y_b$ and $y_c$, respectively. This implies that if $e \not\in E_{abc}$ then there exist at least two polygons in $G$ from the set

\[\{P_{eab}, P_{ebc}, P_{eca}\}.
\]

Hence $e \in E_{abc} \cup E_{bc, ca} \cup E_{ca, ab} \cup E_{ab, bc} \cup E_{ab, bc, ca}$. Then
$E(R) \cap E_{bc} = \emptyset = E(R) \cap E_{ca} = E(R) \cap E_{ab}$. Hence $E_{bc} \cup E_{ca} \cup E_{ab} \subseteq E(N)$, since $E(N) \cup E(R) = E(G)$ is a partition of $E(G)$.

**Proposition 2.7.4.** Suppose $H(I)$ and $H(II)$ hold. Then for any necklace decomposition

$$D = (N, R)$$

(1) $E(N) \subseteq E_{abc} \cup E_{bc} \cup E_{ca} \cup E_{ab}$, and

(2) $E_{ab, bc} \cup E_{bc, ca} \cup E_{ca, ab} \cup E_{ab, bc, ca} \subseteq E(R)$ hold.

**Proof.** Let $N = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_k$ be the decomposition of $N$ into beads. By hypothesis $a$, $b$, and $c$ belong to distinct beads $N$, say $S_1$, $S_2$, and $S_3$, respectively. Now, there exists a polygon $P_{abc}$ in $G$ containing $a$, $b$, and $c$. By definition of a necklace and the fact that $a$, $b$, $c$ belong to distinct beads of the given necklace $N$, $P_{abc}$ meets all the beads of $N$ in edges.

Let $e \in E(S_1)$ and $W(G, S_1) = \{x, y\}$ for $i \geq 4$. Then, there are exactly two $[x, y]$-components of $P_{abc}$, one of which, say $L_1$, is contained in $S_1$ and the other, say $L_2$, contains the edges $a$, $b$, and $c$. But $S_1$ is a nonseparable graph. Hence there exists an arc, say $L$, in $S_1$ containing the edge $e$ and the two vertices $x$ and $y$. Then $L_2 \cup L$ is a polygon of $G$ containing $a$, $b$, $c$ and $e$. This implies that $e \in E_{abc}$. Therefore $E(S_1) \subseteq E_{abc}$ for all $i \geq 4$.

Let $e \in E(S_1)$. Then $a \in E(S_1)$ and $W(G, S_1) = \{s_{ab}, s_{ac}\}$. The
If every arc \( L' \) in \( S_1 \) containing the edge \( e \) and vertices \( s_{ab} \) and \( s_{ac} \) does not contain \( a \), then \( e \notin E_{abc} \) but \( e \in E_{bc}' \), as shown in the above paragraph.

On the other hand, if there exists an arc \( L' \) in \( S_1 \) containing the edges \( e \) and \( a \) and the vertices \( s_{ab} \) and \( s_{ac} \), then \( e \in E_{abc} \).

Hence \( E(S_1) \subseteq E_{abc} \cup E_{bc} \) and by symmetry \( E(S_2) \subseteq E_{abc} \cup E_{ca} \), and \( E(S_3) \subseteq E_{abc} \cup E_{ab} \). Therefore

\[
E(N) = E(S_1) \cup E(S_2) \cup \ldots \cup E(S_k)
\]

\[
\subseteq E_{abc} \cup E_{bc} \cup E_{ca} \cup E_{ab}.
\]

Also \( E_{bc,ca} \cup E_{ca,ab} \cup E_{ab,bc} \cup E_{ca,ab,bc} \subseteq E(R) \) since by proposition 1.8.1 \( E(G) = E_{abc} \cup E_{bc} \cup E_{ca} \cup E_{ab} \cup E_{bc,ca} \cup E_{ca,ab} \cup E_{ab,bc} \) is a partition of \( E(G) \). This completes the proof of the proposition.

**Proposition 2.7.5.** Suppose \( H(I) \) and \( H(II) \) hold. Then, for every necklace decomposition

\[
D = (N, R) \text{ of } G
\]

(1) \( E_{abc} \cup E_{bc} \cup E_{ca} \cup E_{ab} \subseteq E(N) \subseteq E_{abc} \cup E_{bc} \cup E_{ca} \cup E_{ab} \) and
(2) \( E_{ab,bc} \cup E_{bc,ca} \cup E_{ca,ab} \cup E_{ab,be,ca} \subseteq E(R) \subseteq E_{abc}^1 \cup E_{bc,ca} \)

Proof. Follows from proposition 2.7.2, 2.7.3, and 2.7.4, and the definitions of \( E_{abc}^1 \) and \( E_{abc}^2 \) in 1.8.

Proposition 2.7.6. Suppose \( H(I) \) and \( H(II) \) hold. Then \( E(R) \cap E_{abc} \neq \emptyset \) if and only if \( R \) is a bridge of \( N \) separating \( a, b \) and \( c \) and \( |W(G, R)| \geq 4 \).

Remark. The second condition of the equivalence is condition (3) of \( H(II) \).

Proof. Suppose condition (3) of \( H(II) \) holds. Then \( R \) is a bridge of \( N \) separating \( a, b, c \), and \( |W(G, R)| \geq 4 \). This implies that \( R \) has at least two vertices, say \( u, v \), on one of the subgraphs belonging to \( \{N(t_{ab},t_{ba}), N(t_{bc},t_{cb}), N(t_{ca},t_{ac})\} \), say \( N(t_{ab},t_{ba}) \). Therefore, there exists an arc \( L_1 \) in \( R \) with endvertices \( u \) and \( v \) in \( V(N(t_{ab},t_{ba})) \). Then, any arc \( L_2 \) in the subgraph \( N'(u,v) \) containing \( a, b, c \) with endvertices \( u, v \) along with \( L_1 \) makes a polygon \( P_{e abc} \) for some \( e \in E(R) \). This implies \( e \in E_{abc} \) and hence \( E(R) \cap E_{abc} \neq \emptyset \). Next, suppose that \( E(R) \cap E_{abc} \neq \emptyset \). Then \( E(R) \neq \emptyset \). Therefore, by \( H(II) \), \( R \) separates \( a, b \) and \( c \) and \( |W(G, R)| \geq 3 \). If possible, let \( |W(G, R)| \leq 3 \).
Then $|W(G, R)| = 3$ and hence $R$ is either a 3-bridge of $N$ or is the union of two or more equipartite 3-bridges of $N$. Let $e \in E(R)$. Then any arc $L$ in $G$ containing $e$ can have at most two edges from $\{a, b, c\}$. Hence $e \notin E_{abc}$ and $E(R) \cap E_{abc} = \emptyset$, a contradiction to $E(R) \cap E_{abc} \neq \emptyset$. This completes the proof of the proposition.

2.8 Bonds and skew diagonals in necklace decompositions.

Suppose $D = (N, R)$ is a necklace decomposition of a nonseparable graph $G$ and case (3) of $H(II)$ holds. We will see that the existence of a bond $K_{abc}$ in $G$ implies the existence of a pair of skew diagonals of the necklace $N$ separating $a$, $b$ and $c$. Hence we do not assume $H(I)$ in this section. The proof requires the existence of certain nonseparable subgraphs of $G$, called partial cones, with exactly three vertices of attachment, containing a nondegenerate segment of the necklace $N$. These are analyzed more fully in Chapter 3, where the maximal partial cones are called cones. The consequences of the existence of skew diagonals separating $a$, $b$ and $c$ above are the uniqueness in case (3) of theorem I and the non-existence of a bond $K_{abc}$ in cases (3) and (4) of the theorem I.

Proposition 2.8.1. Suppose $G$ is a nonseparable graph with distinct edges $a$, $b$ and $c$. Also, suppose that $P_{abc}$ is a polygon in $G$ having a pair of skew diagonals $L$ and $M$ in a type III bridge $B$ of $P_{abc}$ in $G$. Then there exists a pair of skew diagonals $L_1$, $M_1$ such that either
(1) $L_1$ and $M_1$ separate $a$, $b$ and $c$, or

(2) the endvertices $u_1$, $v_1$ of $L_1$ are in $V(P_{abc}(f, g', g'))$, for $\{f, g\} \subset \{a, b, c\}$, and $M_1$ has only one vertex, $s_1$, in the same segment and the other vertex, $y_1$, in another segment of $P_{abc}$. Moreover, in either of the cases $L_1$ can be chosen to be one of the arcs $L$ and $M$, say $L$, and in case (2) the edge of $M_1$ incident with $s_1$ is also in $M$.

**Remark.** The pair of skew diagonals $L_1$ and $M_1$ of $P_{abc}$ in case (2) is said to form a singularity of $P_{abc}$.

**Proof.** By hypothesis we have a polygon $P_{abc}$ and a pair of skew diagonals $L$ and $M$ of $P_{abc}$ in $G$. Orient the polygon $P_{abc}$ in the sense determined by the edge sequence $a$, $b$, $c$.

Let the endvertices of $L$ and $M$ on $P_{abc}$ be $u, v$ and $s, y$, respectively, in the order $u, s, v, y$ on $P_{abc}$. By proposition 1.6.2, the vertices $u, v, s, y$ are distinct and $s, y$ belong to two distinct $[u, v]$-components of $P_{abc}$. Let $\{f, g, h\} = \{a, b, c\}$. The following four cases are then possible, namely

(1) $u \in V(P_{abc}(f, g', g'))$, $v \in V(P_{abc}(g, h', h'))$, $s \in V(P_{abc}(u, v')) \subseteq V(P_{abc}(f, g'))$, and $y \in V(P_{abc}(n, f, f'))$. In this case $L_1 = L$, $M_1 = M$ and $L_1$, $M_1$ separate $a$, $b$, $c$. 


(2) \( u \in V(P_{abc}(P_{fg},P_{gf})), \; v \in V(P_{abc}(P_{gh},P_{hg})) \),
\( s \in V(P_{abc}(u,v)) \subseteq V(P_{abc}(P_{fg},P_{gf})) \) and \( y \in V(P_{abc}(P_{gh},P_{hg})) \). Here \( L \) and \( M \) satisfy the hypotheses of proposition 2.4.2 and, by the proof of 2.4.2, there exist skew diagonals \( L_1 \) and \( M_1 \) separating \( a, b, c \) with \( \{L, M\} \cap \{L_1, M_1\} \neq \emptyset \).

(3) \( u, v \in V(P_{abc}(P_{fg},P_{gf})), \)
\( s \in V(P_{abc}(u,v)) \subseteq V(P_{abc}(P_{fg},P_{gf})) \) and \( y \in V(P_{abc}(P_{fg},P_{gf})). \) In this case we define \( L_1 = L \) and \( M_1 = M \), and writing \( s = s_1, u = u_1, v = v_1, y = y_1 \) we get the conclusion as required.

(4) \( u, v \in V(P_{abc}(P_{fg},P_{gf})), \)
\( s \in V(P_{abc}(u,v)) \) and \( y \in V(P_{abc}(u,v)) \cap V(P_{abc}(P_{fg},P_{gf})). \)

Now \( B \) is a type III bridge and \( L \cup M \subseteq B \). Hence, by proposition 1.3.1, there exists an arc \( L_0 \) from a vertex \( y \in V(P_{abc}(P_{fg},P_{gf})) \) to an internal vertex, say \( x \), of one of \( L \) and \( M \), say \( M \), such that \( L_0 \) does not meet \( L \). Then we define \( M_1 = L_0(y,x) \cup M(x,s) \) and \( L_1 = L \). Writing \( s = s_1, u = u_1, v = v_1, y = y_1 \) and noting that \( s \neq x \), we get the conclusion as required.

**Definition 2.8.2.** Let \( P_i = (P_i)_{abc} \) be a polygon of a nonseparable graph \( G \) having distinct edges \( a, b \) and \( c \).

Suppose \( L_1 \) and \( M_1 \) are two skew diagonals of \( P_i \) in \( G \). Also, suppose that the endvertices \( u_1, v_1 \) of \( L_1 \) and \( s_1 \) of \( M_1 \)
belong to \( P_i((p_i)_{ab}, (p_i)_{ba}) \) in the linear order 
\( (p_i)_{ab} = u_i < s_i < v_i = (p_i)_{ba}, \) and the endvertex \( y_i \) of \( M \) is such that

\[
(p_i)_{ca} < y_i < (p_i)_{ac} \quad \text{on } V(P_i((p_i)_{ca}, (p_i)_{ac})).
\]

Let \( W_i \) and \( Z_i \) be internally disjoint arcs of \( G \) avoiding 
\( P_i \cup L_i \cup M_i \), and with endvertices \( t_i, w_i \) and \( x_i, z_i \), respectively, such that

\[
u_i < t_i < v_i \quad \text{on } L_i,
\]

\[
s_i < w_i < x_i < y_i \quad \text{on } M_i \quad \text{and } \quad (p_i)_{bc} < Z_i < (p_i)_{cb}
\]

on \( P_i((p_i)_{bc}, (p_i)_{cb}) \).

Then define

\[
Q_i = P_i(u_i, v_i) \cup L_i,
\]

\[
\theta_i = Q_i \cup M_i(s_i, w_i) \cup W_i,
\]

\[
\bar{\theta}_i = P'_i(u_i, v_i) \cup M_i(w_i, y_i) \cup Z_i.
\]

\[
U_i = \theta_i \cup P'_i((p_i)_{ab}, u_i) \cup P'_i(v_i, (p_i)_{ba})
\]

\[
\cup M_i(w_i, x_i), \quad \text{and}
\]

\[
F_i = (P_i)_{abc} \cup L_i \cup M_i \cup W_i \cup Z_i = \theta_i \cup \bar{\theta}_i.
\]

An augmenting arc \( A_i \) (when it exists) is an arc in \( G \) avoiding 
\( F_i \) which has endvertices \( q_i \in V(\theta_i) \setminus \{u_i, v_i, w_i\} \) and \( r_i \in V(G) \setminus V(\theta_i). \)

Also define \( \Theta_i \) to be the set of bridges of \( F_i \) which have a vertex of attachment in \( V(\theta_i) \setminus \{u_i, v_i, w_i\} \), and then write
Proposition 2.8.3. Let $G$ be a nonseparable graph with distinct edges $a$, $b$ and $c$. Also, let $P_1 = P_{abc}$ be a polygon in $G$ having a pair of skew diagonals $L_1$, $M_1$ contained in a type III bridge $B$ of $P_{abc}$ in $G$ and forming a singularity of $P_{abc}$. Suppose no pair of skew diagonals of $P_{abc} = P_1$ separate $a$, $b$ and $c$. Then arcs $W_1$ and $Z_1$ exist with endvertices $t_1$, $w_1$ and $x_1$, $z_1$, respectively, such that the notation of figure $F_1$ applies to $P_1 \cup W_1 \cup Z_1 \cup L_1 \cup M_1$, after an appropriate permutation of the symbols for the edges $a, b, c$.

Proof. Without loss of generality let us suppose that

$$u_1, v_1 \in V(P_1((P_1)_{ab}, (P_1)_{ba})) \text{ and } s_1 \in V(P_1(u_1, v_1)),$$

$$y_1 \in V(P_1((P_1)_{ac}, (P_1)_{ca})).$$

We have skew diagonals $L_1$, $M_1$ in a bridge $B$ of $P_1$. By proposition 1.3.1, there exists an arc $W_1$, with an endvertex $t_1$ internal to $L_1$ and $w_1$ internal to $M_1$, which avoids $P_1 \cup L_1 \cup M_1$. There exists $Z_1 \in W(G, B) \cap V(P_1((P_1)_{bc}, (P_1)_{cb})), \text{ as } B \text{ is a type III bridge of } P_1$. Applying proposition 1.3.1 to $[z_1]$ and $L_1 \cup M_1 \cup W_1$ in the bridge $B$, there exists an arc $Z_1$ with endvertices $z_1$ and some $x_1 \in V(L_1 \cup M_1 \cup W_1) \setminus V(P_1)$. If $x_1 \in V(L_1 \cup W_1) \setminus \{w_1\}$ write $X_1 = L_1(x_1, u_1)$ if $x_1 \in V(L_1)$ and
$X_1 = W_1(x_1, t_1) \cup L_1(t_1, u_1)$ if $x_1 \in V(W_1)$. Then $Z_1 \cup X_1$ is an arc from $z_1$ to $u_1$ skew to $M_1$ such that the pair $M_1$, $Z_1 \cup X_1$ separates $a$, $b$, $c$. If $x_1 \in V(M_1(s_1, w_1) \setminus \{s_1, w_1\}$ then $Z_1 \cup (M_1(x_1, s_1))$ is skew to $L_1(v_1, t_1) \cup W_1 \cup M_1(w_1, y_1)$, and the pair separates $a$, $b$, $c$.

As no pair of skew diagonals separate $a$, $b$, $c$ we conclude that $x_1 \in V(M_1(y_1, w_1)) \setminus \{y_1\}$ as required.

**Proposition 2.8.4.** Suppose $G$ is a nonseparable graph with distinct edges $a$, $b$ and $c$. Also, suppose that $F_i$ is a subgraph of $G$ of the type defined in 2.8.2. Then one of the following three alternatives applies.

1. There does not exist any augmenting arc $A_i$, in which case

   $$C' = B_i^\infty \cup \bigcup_{B_i^\infty} B_i$$

   is a nonseparable subgraph of $G$ with

   $$W(G, C') = \{u_i, v_i, w_i\}.$$  

2. There exists an augmenting arc $A_i$ with $r_i \in V(U_i)$, in which case there exists a subgraph $F_{i+1}$ in the notation of definition 2.8.2, with $\theta_i \neq \theta_i$ and $H_i \subseteq H_{i+1}$.

3. There exists an augmenting arc $A_i$, with $r_i \notin V(U_i)$, in which case there exists a pair of skew diagonals of $(P_i)_{abc}$ which separate $a$, $b$ and $c$. 

Remark. There exists an augmenting arc $A_i$ if and only if there exists a $B_i \in \mathcal{B}_i$ such that $W(G, B_i) \subseteq V(\theta_i)$.

Proposition 2.8.4.1. Suppose the hypotheses of proposition 2.8.4 hold. Also, suppose that there does not exist any augmenting arc $A_i$ as defined in 2.8.2. Then

$$C' = \theta_i \cup \left( \bigcup_{B_i \in \mathcal{B}_i} B_i \right)$$

is a nonseparable subgraph of $G$ with

$$W(G, C') = \{u_i, v_i, w_i\}.$$

Proof. By hypothesis $W(G, B_i) \subseteq V(\theta_i)$, for all $B_i \in \mathcal{B}_i$. Hence $B_i$ is a bridge of $\theta_i$ in $G$, for all $B_i \in \mathcal{B}_i$. Therefore, by proposition 1.4.1,

$$C' = \theta_i \cup \left( \bigcup_{B_i \in \mathcal{B}_i} B_i \right)$$

is a nonseparable subgraph of $G$. There are edges of $(P_i)_{abc}$ incident with $u_i$, $v_i$ and not belonging to $E(C')$. Also, there exists an edge $M_i$ incident with $w_i$ which does not belong to $E(C')$. Hence $\{u_i, v_i, w_i\} \subseteq W(G, C')$.

By hypothesis $\bigcup_{B_i \in \mathcal{B}_i} W(G, B_i) \subseteq V(\theta_i)$. Hence $W(G, C') \subseteq V(\theta_i)$.

If there exists an edge $e$ belonging to $E(G) \setminus E(C')$ incident with a vertex $v \in W(G, C')$, then $e \notin E(F_i)$. Therefore there exists a
bridge $B'$ of $F_i$ containing $e$ such that $v \in W(G, B')$. But then $B' \subseteq C'$, by the definitions of $G_i$ and $C'$. This is contrary to the statement $e \notin E(C')$. Hence

$$W(G, C') = \{u_i, v_i, w_i\}.$$  

This completes the proof of the proposition.

**Proposition 2.8.4.2.** Suppose the hypotheses of proposition 2.8.4 hold. Also, suppose that there exists an augmenting arc $A_i$ as defined in 2.8.2 with $r_i$ in $V(U_i)$. Then there exists $F_{i+1}$, in the notation of definition 2.8.2, with $\overline{e}_{i+1} \subseteq \overline{e}_i$.

**Proof.** Notice that when $r_i \in V(U_i)$ there are three cases

$r_i \in V(P_i(((P_i)_{ab}, u_i))\{u_i\}$, $r_i \in V(P_i(v_i, (P_i)_{ba}))\{v_i\}$ and $r_i \in V(M_i(w_i, x_i))\{w_i\}$. The second case is similar to the first one. Hence without loss of generality we need only prove the proposition for the first and third cases.

The procedure in going from $F_i$ to $F_{i+1}$ consists of adding $A_i$ and deleting the edges and internal vertices of an arc contained in a branch of $\theta_i$. The new $\overline{e}_{i+1}$ consists of $\overline{e}_i$ with the arc $P_i(r_i, u_i)$ deleted (except for $r_i$) in the first case above, and $\overline{e}_i$ with the arc $M_i(r_i, w_i)$ deleted (except for $r_i$) in the second case above, so that $\overline{e}_{i+1} \subseteq \overline{e}_i$, by definition. We set $r_i = u_{i+1}$.
\( v_i = v_{i+1}, \quad w_i = w_{i+1} \) and \( r_i = w_{i+1}, \quad u_i = u_{i+1}, \quad v_i = v_{i+1}, \)

respectively, and write \( s_i = s_{i+1}, \quad y_i = y_{i+1}, \quad z_i = z_{i+1} \) in \( \theta_{i+1}. \)

In either case we can restrict \( q_i \) to be in \( V(P_i(u_i, v_i)) \cup M_i(s_i, w_i), \)

without loss of generality. In the following construction \( t_{i+1} = t_i \)

by the above assumptions. The above two cases fall into five subcases

in each of which \( \theta_{i+1} \) is defined. Given such a \( \theta_{i+1} \) we define

\[
F_{i+1} = \theta_{i+1} \cup \overline{\theta}_{i+1}
\]

and define \( L_{i+1}, M_{i+1}, Z_{i+1}, W_{i+1} \) to be consistent with the notation for \( F_{i+1} \) previously established. It

remains to consider how \( \theta_{i+1} \) is obtained from \( \theta_i. \)

Case 1) Suppose \( r_i \in V(P_i((p_i)_{ab}, u_i)) \setminus \{u_i\}. \) There are three

subcases, namely

(i) \( q_i \in V(P_i(u_i, s_i)) \setminus \{u_i\}, \)

(ii) \( q_i \in V(P_i(v_i, s_i)) \setminus \{v_i\} \) and

(iii) \( q_i \in V(M_i(s_i, w_i)) \setminus \{v_i\}. \)

In subcase (i) set

\[
\theta_{i+1} = A_i \cup P_i(r_i, u_i) \cup L_i \cup P_i(v_i, q_i) \cup W_i \cup M_i(s_i, w_i)
\]

with \( s_{i+1} = s_i. \) In subcase (ii) set

\[
\theta_{i+1} = A_i \cup P_i(r_i, u_i) \cup L_i \cup P_i(v_i, q_i) \cup W_i \cup M_i(s_i, w_i) \cup P_i(s_i, q_i)
\]

with \( s_{i+1} = q_i. \) In subcase (iii) set

\[
\theta_{i+1} = A_i \cup P_i(r_i, u_i) \cup L_i \cup P_i(v_i, s_i) \cup M_i(s_i, w_i) \cup W_i
\]

with \( s_{i+1} = q_i. \)
Case 2) Suppose \( r_i \in \mathcal{V}(M_1(w_i, x_i)) \setminus \{w_i\} \). There are two essentially different subcases, namely

i) \( q_i \in \mathcal{V}(P_i(v_i, s_i)) \setminus \{v_i\} \) and

ii) \( q_i \in \mathcal{V}(M_1(s_i, w_i)) \setminus \{w_i\} \). In both subcases \( Q_{i+1} = Q_i \). In subcase (i) set

\[
\theta_{i+1} = Q_{i+1} \cup W_i \cup M_1(w_i, r_i) \cup A_i \quad \text{with} \quad s_{i+1} = q_i.
\]

In subcase (ii) set

\[
\theta_{i+1} = Q_{i+1} \cup A_i \cup W_i \cup M_1(s_i, r_i) \quad \text{with} \quad s_{i+1} = q_i.
\]

The proposition now follows.

Proposition 2.8.4.3. If the statement of proposition 2.8.4.2 holds, then

\[
H_i \subset H_{i+1}.
\]

Proof. First, we claim that \( \theta_i \subset H_{i+1} \). To prove this we go back to the proof of the last proposition. In all the five subcases in the last proposition, the arc \( A_i' \) of \( \theta_i \) not contained in \( \theta_{i+1} \) has both of its endpoints in \( \mathcal{V}(\theta_{i+1}) \setminus \{u_{i+1}, v_{i+1}, w_{i+1}\} \). Hence \( A_i' \subset H_{i+1} \) and therefore \( \theta_i \subset H_{i+1} \). Now, let \( B_i \in \mathcal{W}_i \). Then \( B_i \) is a bridge of \( F_i \) in \( G \). Two cases arise, namely

(1) \( A_i \nsubseteq B_i \) and

(2) \( A_i \subseteq B_i \).

In case (1) \( B_i \) is a \( F_i \cup A_i \)-bridge in \( G \) and \( A_i \cap B_i \subseteq [W(G, B_i)] \).
But $B_i$ is a $F_i$-bridge in $G$. Hence $W(G, B_i) \subseteq V(F_i)$, and there exists at least one $x \in W(G, B_i) \cap (V(\theta_i) \setminus \{u_i, v_i, w_i\})$. If $x_i$ is not an internal vertex of $A_i$, then $x_i \in V(\theta_i+1) \setminus \{u_{i+1}, v_{i+1}, w_{i+1}\}$.

Alternatively, $x$ is an internal vertex of $A_i$. In this subcase let $B_{i+1}$ be the bridge of $F_{i+1}$, where $F_{i+1} \subseteq F_i \cup A_i$ containing $B_i$. Then $B_i \cup A_i \subseteq B_{i+1}$. But then the endvertices of $A_i$ are vertices of attachment of $B_{i+1}$ in $G$, and both are in $V(\theta_{i+1}) \setminus \{u_{i+1}, v_{i+1}, w_{i+1}\}$. In case (2) $A_i = B_i$. Let $e \in E(B_i)$. If $e \in E(A_i)$, then $e \in E(\theta_i+1)$. Therefore $e \in E(H_i+1)$. Otherwise, $e \notin E(A_i)$. In this case $e \in E(B_i')$, where $B_i'$ is a bridge of $F_i \cup A_i$ in $B_i$. Then we claim $W(G, B_i') \subseteq W(G, B_i) \cup V(A_i)$. If $x \in W(G, B_i')$ then $x \in V(B_i')$ and $x$ is incident with an edge $e'$ of $G$ not in $B_i'$. Again two possibilities arise, according as $e' \notin E(B_i')$ or $e' \in E(B_i')$. In the case where $e' \notin E(B_i')$, we conclude that $x \in W(G, B_i)$. If $e' \in E(B_i')$ then $x \in W(B_i, B_i') \subseteq V(A_i)$, because $(F_i \cap A_i) \cap B_i = A_i$. Hence we have established the claim. Now, if $W(G, B_i') \subseteq W(G, B_i)$ then $B_i' \subseteq [W(G, B_i')]$ or $B_i = B_{i+1}$ contrary to $e \in E(B_i')$ and $B_i \neq B_i$. We conclude that $B_i$ has a vertex of attachment $x$ in $G$ which is an internal vertex of $A_i$. This implies that the bridge $B_{i+1}$ of $F_{i+1}$ which contains $B_i'$ also has this vertex of attachment, which is in
Now, we have proved that $\theta_i \subseteq H_i$, and $B_i \subseteq H_i$, for all $B_i \in \mathcal{B}_i$. Therefore $H_i \subseteq H_i + 1$. Furthermore $\theta_{i+1} \subseteq \theta_i$. This implies that there exists $e'' \in E(\theta_i) \setminus E(\theta_{i+1})$, which belongs to $E(\theta_{i+1})$, by construction of $F_{i+1}$. But $\theta_i \cap H_i$ is edgeless. Hence $H_i \subseteq H_i + 1$. This completes the proof of the proposition.

**Proposition 2.8.4.** Suppose the hypotheses of proposition 2.8.4 hold. Also, suppose that there exists an augmenting arc $A_i$ as defined in 2.8.2, with $r_i \notin V(U_i)$. Then there exists a pair of skew diagonals of $(P_i)_{abc}$ separating $a$, $b$ and $c$.

**Proof.** Because of symmetry, we will only need to consider the following three cases, in each of which

$$r_i \in V(P_i((p_i)_{bc}, (p_i)_{cb}) \cup Z_i) \setminus \{x_i\},$$

namely

1. $q_i \in V(P_i(s_i, v_i)) \setminus \{v_i\}$,

2. $q_i \in V(L_i(s_i, u_i)) \setminus \{u_i\}$ or

3. $q_i \in V(M_i(s_i, w_i)) \setminus \{w_i\}$.

In all cases there is an arc $A_i' \supset A_i$ which forms with $L_i(v_i, t_i) \cup W_i \cup M_i(w_i, y_i)$ a pair of skew diagonals separating $a$, $b$, $c$. In cases (1) and (2) define $A_i' = A_i$ if $r_i \in V(P_i((p_i)_{bc}, (p_i)_{cb}))$,
and $A'_i = A_1 \cup Z_i(r_i, z_i)$ if $r_i \in V(Z_i) \setminus \{x_i, z_i\}$. In case (3)
$A'_i = M_i(q_i, s_i) \cup A_1 \cup Z_i(r_i, z_i)$ if $r_i \in V(Z_i) \setminus \{x_i, z_i\}$. This
$A'_i = M_i(q_i, s_i) \cup A_1 \cup Z_i(r_i, z_i)$ if $r_i \in V(Z_i) \setminus \{x_i, z_i\}$. This
completes the proof of the proposition.

**Proof of the proposition 2.8.4.** Follows from propositions 2.8.4.1, 2.8.4.2, 2.8.4.3 and 2.8.4.4.

**Definition 2.8.5** (Partial cone of $G$ with respect to a necklace decomposition of $G$).

Suppose $G$ is a graph satisfying $H(I)$ with a decomposition $D = (N, R)$ satisfying $H(II)$. Then any nonseparable subgraph $C'$ such that

1. $W(G, C') = \{u,v,w\}$ for distinct $u,v \in W(G, N(t_{fg}, t_{gf}))$ and $w \in V(R) \setminus W(G, N)$, where $f, g \in \{a,b,c\}$ and $f \neq g$,
2. $C'$ is a union of some $[u,v,w]$-components $H$ of $G$, with $W(G, H) = \{u,v,w\}$ and $a,b,c \notin E(H)$,
3. there exists a $\theta$-graph $\theta \subseteq C'$, with $u,v,w$ internal vertices of distinct branches of $\theta$, and
4. $N(u,v) \subseteq C'$ is called a partial cone of $G$ with respect to $(N, R)$.

**Proposition 2.8.6** (Existence of a partial cone). Suppose $G$ has a necklace decomposition $D = (N, R)$ which satisfies $H(II)$, and no pair of skew diagonals of $N$ exist which separate $a,b,c$. Suppose $N$ has a singularity (definition 1.9.1) formed by skew diagonals $L_1$
and \( M_1 \) with endvertices \( u_1, v_1 \) and \( s_1, y_1 \), respectively, with
\[
\{u_1, v_1, s_1\} \subseteq V(N(t_{fg}, t_{gf})), \text{ for } \{f, g\} \subseteq \{a, b, c\}.
\]

Then there exists a partial cone \( C' \) of \( G \) with respect to the decomposition \( D = (N, R) \), which contains \( L_1, N(u_1, v_1) \), and the edge of \( M_1 \) incident with \( s_1 \).

**Proof.** The skew diagonals \( L_1 \) and \( M_1 \) of \( N \) have distinct endvertices. Hence \( |W(G, R)| \geq 4 \). By H(II), \( N \) has exactly one bridge \( R \) and it separates \( a, b, c \). Therefore for all \( P_{abc} \subseteq N, R \) is the single type III bridge of \( P_{abc} \) in \( G \), with \( L_1 \cup M_1 \subseteq R \). Again, there exists no pair of skew diagonals of \( P_{abc} \) for \( P_{abc} \subseteq N \), separating \( a, b, c \), by hypothesis. Hence proposition 2.8.3 applies and we get \( F_1 \) as defined in 2.8.2. Next the hypotheses of proposition 2.8.4 hold. The conclusion (3) of proposition 2.8.4 is ruled out, by hypothesis. Therefore either conclusion (1) or conclusion (2) of proposition 2.8.4 apply. In case of conclusion (2) of proposition 2.8.4, we apply induction on \( i \) and obtain a sequence \( F_1, F_2, \ldots, F_i, \ldots \) of subgraphs of \( G \). By finiteness and the fact that
\[
\overline{e}_{n+1} \subseteq \overline{e}_i
\]
we get an \( n \) such that conclusion (1) applies to \( F_n \). Thus
\[
C' = \bigcup_{n} (\bigcup_{B_n} B_n)_{B_n \in \mathbb{N}}
\]
is a nonseparable subgraph of $G$, by 1.4.1, with $W(G, C') = \{u_n, v_n, w_n\}$.

By construction, there exists a $\theta$-graph $\theta_n \subseteq C'$ with $u_n, v_n, w_n$ as internal vertices of distinct branches of $\theta_n$. Also

$W(G, C') = \{u_n, v_n, w_n\}$; by construction, where

$$u_n, v_n \in V(N(t_{fg}, t_{gf})),$$

and $w_n \in V(R) \setminus W(G, N)$, for some $\{f, g\} \subseteq \{a, b, c\}$. Here $w_n$ is an internal vertex of the arc $M_1$. We claim that $L_1 \cup M_1(s_1, w_1) \cup N(u_1, v_1) \subseteq C'$. By definition of $M_1$, $N(u_1, v_1)$ has at least two beads. Also, $N(u_1, v_1)$ is the union of $P(u_1, v_1)$ and its bridges $B$ in $N(u_1, v_1)$. These $B$'s are also bridges of $P(u_1, v_1)$ in $G$.

Each $B$ must have at least two vertices of attachment in $P(u_1, v_1)$, since $G$ is a nonseparable graph. Also, each $B$ is contained in a bead of $N(u_1, v_1)$, and at least two such beads exist, hence no $B$ can have both $u_1, v_1$ as vertices of attachment. Therefore, each $B$ has a vertex of attachment in $V(P_1(u_1, v_1)) \setminus \{u_1, v_1\} \subseteq V(\theta_1) \setminus \{u_1, v_1, w_1\}$.

Also, by construction of $F_1$, each such bridge is a bridge of $F_1$ in $G$. Hence $N(u_1, v_1)$ is the union of some bridges of $F_1$ in $G$ having vertices of attachment in $V(\theta_1) \setminus \{u_1, v_1, w_1\}$. This implies that

$$N(u_1, v_1) \subseteq H_1.$$

Also by construction
Hence $U \subseteq M(s^w) \cup N(u^w) \subseteq C$. Next, we claim that $N(u_n, v_n) \subseteq C$. We will prove this by induction. Let $N(u_{n-1}, v_{n-1}) \subseteq H_{n-1} \subseteq C$ hold.

If $\{u_{n-1}, v_{n-1}\} = \{u_n, v_n\}$, then we have nothing to prove. Hence we may suppose that $u_n \neq u_{n-1}$, $v_n = v_{n-1}$, by symmetry. Then there exists an arc $p_n(u_n, u_{n-1}) \subseteq N(u_n, u_{n-1})$ such that $p_n(u_n, u_{n-1}) \subseteq \emptyset$.

By a similar argument to that given above, we can prove that $N(u_n, u_{n-1}) \subseteq H_n$.

Hence

$$N(u_n, v_n) = N(u_n, u_{n-1}) \cup N(u_{n-1}, v_n)$$

$$= N(u_n, u_{n-1}) \cup N(u_{n-1}, v_{n-1})$$

$$\subseteq H_n \cup H_{n-1} \subseteq C$$.

We show finally that $C$ is a union of some $[u_n, v_n, w_n]$-components of $G$, not containing $a$, $b$ or $c$.

Now $[u_n, v_n, w_n]$-components of $G$ have at least two vertices of attachment, since $G$ is a nonseparable graph. If a $[u_n, v_n, w_n]$-component of $G$ has exactly two vertices of attachment, then it cannot have any vertex of attachment in $V(\emptyset_n \setminus \{u_n, v_n, w_n\})$. But this $[u_n, v_n, w_n]$-component of $G$ is a bridge of $F_n$. Therefore this
component will not belong to $\mathcal{H}_n'$, and hence will not be contained in $\mathcal{H}_n$.

$C'$ is therefore a partial cone of $G$ with respect to the given necklace decomposition with the required properties. This completes the proof of the proposition.

**Proposition 2.8.7.** Suppose $G$ is a nonseparable graph, and $a, b, c$ are distinct edges of $G$. Also suppose $B$ is a type III bridge of a polygon $P_{abc}$ in $G$ such that

(i) $|W(G, B)| > k$,

(ii) there is no bridge of $P_{abc}$ in $G$ skew to $B$, and

(iii) $\{a, b, c\} \subseteq K$, for some bond $K$ of $G$.

Then there exists a pair of skew diagonals of $P_{abc}$ in $G$ separating $a, b$ and $c$.

Let the extreme vertices of $B$ on $P(p_{fg}, p_{gf})$ be $t_{fg}$ and $t_{gf}$, where $\{f, g\} \subseteq \{a, b, c\}$ and $f \neq g$.

**Proposition 2.8.7.1.** Let $G$ be a nonseparable graph. Suppose $G = H \cup F$ where $H \cap F = [x, y]$, for some $x, y \in V(G)$. Let $h \in E(H)$ and $f \in E(F)$. Suppose there exists a bond $K_{hf}$ in $G$. Then $x, y$ are in distinct endgraphs $K_1$ and $K_2$, respectively of $K_{hf}$ in $G$.

**Proof.** Let us assume the contrary i.e. $x, y$ belong to the same endgraph, say $K_1$. Let $u, v$ be the endvertices of $h, f$, respectively, in $K_{hf}$. Then $u \neq v$ because $h \in E(H)$, $f \in E(F)$ and $\{u, v\} \cap \{x, y\} = \emptyset$. 


Any path joining \( u, v \) in \( G \) contain either \( x \) or \( y \) as \( H \cap F = [x, y] \), and so \( K_2 \) is not connected, contrary to the definition of an endgraph of a bond of \( G \).

**Proposition 2.8.7.2.** Suppose the hypotheses of proposition 2.8.7 hold. Then the members of each pair \( t_{ca}, t_{cb}; t_{ba}, t_{bc}; \) and \( t_{ab}, t_{ac} \) are in different endgraphs of \( K \).

**Proof.** Follows from proposition 2.8.7.1 above.

**Proposition 2.8.7.3.** Suppose the hypotheses of proposition 2.8.7 hold. Then there exists a pair of disjoint arcs \( L, M \) of \( G \) with endvertices \( t_{fg}, t_{gh} \) and \( t_{gf}, t_{fh} \), respectively, which do not contain \( a, b \) or \( c \), where \( \{f, g, h\} = \{a, b, c\} \).

**Proof.** By proposition 2.8.7.2, the members of each pair \( t_{ca}, t_{cb}; t_{ab}, t_{ac} \) and \( t_{ba}, t_{bc} \) belong to different endgraphs of \( K \) in \( G \). Let \( K_1 \) and \( K_2 \) be the endgraphs of \( K \) in \( G \). The following two possible cases then occur.

(1) The vertices \( t_{fg}, t_{gf} \) are in distinct endgraphs of \( K \), for some \( \{f, g\} \subset \{a, b, c\} \), or
(2) the vertices $t_{fg}, t_{gf}$ are in the same endgraph of $K$, for each $\{f,g\} \subseteq \{a,b,c\}$.

In case (1), applying proposition 2.8.7.2, we may suppose, without loss of generality, that

$$\{t_{fg}, t_{gh}\} \subseteq V(K_1)$$

and

$$\{t_{gf}, t_{fh}\} \subseteq V(K_2),$$

where $\{f,g,h\} = \{a,b,c\}$.

There exist arcs $L$ and $M$ in $K_1$ and $K_2$, respectively, joining the vertices of these pairs. Clearly $L$ and $M$ are disjoint, do not contain any of the edges $a, b$ or $c$, and hence satisfy the requirements of the proposition.

In case (2) we may suppose, without loss of generality, that

$$\{t_{fg}, t_{gf}\} \cup \{t_{gh}, t_{hg}\} \subseteq V(K_1),$$

for some $\{f,g,h\} = \{a,b,c\}$. But then the pair $t_{gf}, t_{gh} \in V(K_1)$, contrary to proposition 2.8.7.2. This case cannot arise, and so the proposition follows.

Proposition 2.8.7.4. Suppose the hypotheses of proposition 2.8.7 hold. Let there exist a pair of disjoint arcs $L$ and $M$ with endvertices $t_{fg}, t_{gh}$ and $t_{gf}, t_{fh}$, respectively, on $P_{abc}$ which do not contain $a, b$ or $c$, where $\{f,g,h\} = \{a,b,c\}$. Then there exists a pair $L_1, M_1$ of skew diagonals of $P_{abc}$ with endvertices $u_1, v_1$ and $s_1, y_1$, respectively, such that $L_1 \subseteq L, M_1 \subseteq M$.

Furthermore, if no pair of skew diagonals of $P_{abc}$ separate $a,b,c$
then \( L_1 \) and \( M_1 \) form a singularity of \( P_{abc} \).

**Proof.** By hypothesis, \( t_{gf} \) and \( t_{fh} \) are endvertices of \( M_1 \) and \( M \) does not contain the edges \( a, b, c \) or the vertices \( t_{fg}, t_{gh} \).

This implies that there exists an arc \( M_1 \subset M_1 \) avoiding \( P_{abc} \) from a vertex

\[
s \in V(P_{abc}(t_{fg}, t_{gf})) \setminus \{t_{gf}\}
\]

to a vertex

\[
y \in V(P_{abc}(t_{gh}, t_{fh})) \setminus \{t_{fh}\}.
\]

Similarly, there exists a \( P_{abc} \)-avoiding arc \( L_1 \subset L_1 \) from a vertex

\[
u \in V(P_{abc}(s,y)) \setminus \{s,y\}
\]
to a vertex \( v \in V(P_{abc}(s,y)) \setminus \{s,y\} \). The arcs \( L_1 \) and \( M_1 \) are skew diagonals of \( P_{abc} \) with endvertices \( u, v \) and \( s, y \), respectively. \( M_1 \) has its endvertices in distinct segments \( P_{abc}(t_{ab}, t_{ba}), P_{abc}(t_{bc}, t_{cb}), P_{abc}(t_{ca}, t_{ac}) \). When \( L_1 \) has its endvertices in distinct segments then \( L_1 \) and \( M_1 \) either form a pair of skew diagonals separating \( a, b, c \) or they can be replaced, by proposition 2.4.2 by such a pair. When \( L_1 \) has its endvertices in just one segment of \( P_{abc} \) listed above, then obtain \( L_1, M_1 \) from a singularity of \( P_{abc} \) by skewness. In case of a singularity we rename the endvertices of \( L_1, M_1 \) by \( u_1, v_1 \) and \( s_1, y_1 \) with \( u_1, v_1, s_1 \) in one \( P_{abc} \)-segment, possibly switching \( s \)
and \( y \) if required.

Therefore in all cases the conclusion of the proposition apply.

This ends the proof.

**Proposition 2.8.7.5.** Suppose the hypotheses of proposition 2.8.7 apply for a graph \( G \), and that arcs \( L \) and \( M \) exist under the conditions of proposition 2.8.7.3. Then for any decomposition \( G = C' \cup H \) such that

\[
E(C' \cap L) \neq \emptyset \neq E(C' \cap M)
\]

and

\[
\{ t_{fg}, t_{gh}, t_{gf}, t_{fh} \} \subseteq V(H)
\]

it necessarily follows that \( |W(G, C')| \geq 4 \).

Proof. Suppose \( e \) and \( e' \) belong to \( E(C' \cap L) \) and \( E(C' \cap M) \), respectively. Let the vertices incident with \( e \) and \( e' \) be the pairs \( x,y \) and \( x',y' \) respectively. The arcs \( L(t_{fg}, x) \), \( L(y, t_{gh}) \), \( M(t_{gf}, x') \) and \( M(y', t_{fh}) \) are disjoint, possibly after permuting \( x,y \) or \( x',y' \). Each of the above four arc segments contains a vertex of attachment of \( C' \) because it joins a vertex of \( C' \) to one of \( H \), hence \( |W(G, C')| \geq 4 \). This proves the proposition.

**Proof of the proposition 2.8.7.** If possible let there exist a \( K_{abc} \) in \( G \) and also no pair of skew diagonals of \( P_{abc} \) in \( G \), separating \( a,b,c \).

By propositions 2.8.7.2, 2.8.7.3, 2.8.7.4, we have the existence of a pair of disjoint arcs \( L,M \) in \( G \) along with a singularity of
$P_{abc'}$ defined by two skew diagonals $L_1 \subseteq L$, $M_1 \subseteq M$. By the same propositions, the endvertices $u_1, v_1$ of $L_1$ and one endvertex $s_1$ of $M_1$ belong to the same segment, say $P_{abc}(P_{ab}, P_{ba})$, and the other endvertex $y_1$ of $M_1$ belongs to another segment, say $P_{abc}(P_{bc}, P_{ac})$. Under the hypotheses of 2.8.7 the latter part of the proof of proposition 2.6.11 applies, without change, from the definition of the decomposition $(N, R)$, to show that $G$ has a necklace decomposition $D = (N, R)$ which satisfies $H(II)$. Hence, by proposition 2.8.6 we obtain a partial cone $C'$ as defined in 2.8.5 with $W(G, C') = \{u, v, w\}$, where $L_1 \cup M_1(s_1, w_1) \subseteq C'$. We define $H$ to be the minimal subgraph of $G$ such that $H \cup C' = G$. The vertices $t_{gh'}, t_{fh}$ do not belong to $C'$ since $u, v \in V(P_{abc}(t_{fg}, t_{gf}))$ and $w \in V(B \setminus V(P_{abc}))$. Again $t_{gf}, t_{fg}$ are the extreme vertices of $B$ on $P_{abc}(P_{ab}, P_{ba})$. Hence

$\{t_{gf}, t_{fg}, t_{gh'}, t_{fh}\} \subseteq V(H)$

Now $C', L$, and $M$ satisfy the conditions of proposition 2.8.7.5, since $L_1 \cup M_1(s_1, w_1) \subseteq C'$. Therefore $|W(G, C')| \geq 4$, contradicting a part of the definition of $C'$, i.e. that $|W(G, C')| = 3$. This completes the proof of the proposition 2.8.7.

**Proposition 2.8.8.** Suppose $G$ is a nonseparable graph and $a, b, c \in E(G)$ are distinct. Let there be a polygon $P_{abc}$ and two skew bridges $B, B'$ of $P_{abc}$ in $G$, $B$ being a type III-bridge.
Also let there exist a bond $K_{abc}$ in $G$. Then there exists a pair of skew diagonals of $(P_{1})_{abc}$ in $G$, separating $a, b, c$, for some polygon $(P_{1})_{abc}$ in $G$.

Proof. Consider three cases:

(1) $B'$ is a type III-bridge of $P_{abc}$ in $G$.

(2) $B'$ is a type II-bridge of $P_{abc}$ in $G$.

(3) $B'$ is a type I-bridge of $P_{abc}$ in $G$, and $B$ has no type III or type II-bridge skew to it with respect to $P_{abc}$.

In case (1), by proposition 2.6.6, there exists a pair of skew diagonals of $P_{abc}$ in $G$ separating $a, b, c$, since $B$ and $B'$ are skew. In case (2), by proposition 2.6.7, there exists a pair of skew diagonals of $P_{abc}$ in $G$ separating $a, b, c$. In case (3), by proposition 2.5.4 there exists a polygon $(P_{1})_{abc}$ and a bridge $B_{1}$ of $(P_{1})_{abc}$ in $G$ such that $B_{1} \supset B$, $|W(G, B_{1})| \geq 4$ and $B_{1}$ is not skew to any type I bridge of $(P_{1})_{abc}$ in $G$. Now proposition 2.8.7 applies, and we obtain a pair of skew diagonals of $(P_{1})_{abc}$ in $G$ separating $a, b, c$. This completes the proof.

**Proposition 2.8.9.** Suppose $G$ is a nonseparable graph and $a, b, c \in E(G)$ are distinct. Also, let there be a type III bridge $B$ of a polygon $P_{abc}$ in $G$ such that $|W(G, B)| \geq 4$. Then the following two statements are equivalent.
(1) There exists a bond \( K_{abc} \) in \( G \).

(2) There exists a pair of skew diagonals of \( (P_l)_{abc} \) in \( G \) separating \( a, b, c \), for some polygon \( (P_l)_{abc} \) in \( G \).

Proof. Statement (1) implies statement (2), by propositions 2.8.7 and 2.8.8. Statement (2) implies statement (1), by proposition 2.4.1.

Proposition 2.8.10. Let \( G \) be a nonseparable graph with distinct edges \( a, b, c \). Also, let

\[ D = (N, R) \]

be a necklace decomposition of \( G \) such that

(1) \( R \) is the single bridge of \( N \) separating \( a, b, c \) and

(2) \( |W(G, R)| \geq 4 \).

Then the following two statements are equivalent.

(3) There exists a bond \( K_{abc} \) in \( G \).

(4) There exists a pair of skew diagonals of \( N \) separating \( a, b, c \).

Proof. Assume statement (3). By hypotheses, \( R \) is a type III bridge of \( P_{abc} \), for all \( P_{abc} \subseteq N \), with \( |W(G, R)| \geq 4 \). Also \( R \) is not skew to any other bridge of \( P_{abc} \) in \( G \), for all \( P_{abc} \subseteq N \). We fix one such \( P_{abc} \). Then, by proposition 2.8.7, we obtain condition (4), since \( P_{abc} \subseteq N \). Statement (4) implies statement (3), by proposition 2.4.3.
Proposition 2.8.11. Suppose H(I) and H(II) hold. Also suppose $|W(G, R)| \geq 4$ and $T_a', T_b', T_c'$ are the residual segments of $N$ containing $a, b, c$, respectively. Then

$$E(T_a') \cup E(T_b') \cup E(T_c') \subseteq E^1 \cup E_{bc} \cup E_{ca} \cup E_{ab}.$$ 

Proof. Let $e \in E(T_a') \cap E_{abc}$. We will prove that there exists no $K_{ebc}$.

If possible, let $K_{ebc}$ exist. This implies there exists a pair of skew diagonals $L_1, L_2$ of $N$ separating $e, b, c$, by proposition 2.8.10.

The edges $e$ and $a$ belong to the same residual segment of $N$. Therefore $L_1$ and $L_2$ separate $a, b, c$ since they separate $e, b, c$. Hence, by proposition 2.4.1, there exists a $K_{abc}$, contrary to H(I).

Thus no $K_{ebc}$ exists, and hence $e \in E_{abc}$. The proposition now follows from the symmetry of $T_a', T_b', T_c'$ and the fact that

$$E(T_a') \cup E(T_b') \cup E(T_c') \subseteq E_{abc} \cup E_{bc} \cup E_{ca} \cup E_{ab}.$$ 

Proposition 2.8.12. Suppose H(I) and H(II) hold. Also, suppose $|W(G, R)| \geq 4$ and $T$ is any residual segment of $N$ distinct from $T_a, T_b, T_c$, the segments of $N$ containing $a, b, c$, respectively. Then, for all $\{f, g\} \subseteq \{a, b, c\}$, $f, g$ distinct, there exists $K_{efg}$ if and only if there exists $K_{e'fg'}$ for all $\{e, e'\} \subseteq E(T)$. 
Proof. By proposition 2.8.10 there exists $K_{eab}$ if and only if there exists a pair of skew diagonals of $N$ separating $a$, $b$ and $e$. But this is true if and only if there exists a pair of skew diagonals of $N$ separating $a$, $b$ and $e'$, since $e, e' \in E(T)$. But, again this is true if and only if there exists $K_{e'ab}$. This proves the proposition.

Proposition 2.8.12.1. Suppose $H(I)$ and $H(II)$ hold and $|W(G,R)| \geq b$, with $T$ any residual segment of $N$, such that $T \notin \{T_a, T_b, T_c\}$.

Then $E(T) \subseteq E_{abc}^1$ or $E(T) \subseteq E_{abc}^2$.

Proof. Follows from proposition 2.8.12.

2.9 Uniqueness of necklace decompositions.

We will suppose that $H(I)$ and $H(II)$ hold throughout this section. Under these assumptions we will prove the uniqueness of the decomposition $D = (N, R)$ of $G$ into a necklace and its residue, given that

$$E(T) \subseteq E_{abc}^1$$

for all residual segments $T$ of $N$ not containing $a$, $b$ and $c$.

In the contrary case, where there exists a residual segment $T$ of $N$ such that $E(T) \subseteq E_{abc}^2$, the decomposition is not unique. This case is the subject of chapter 3 of the thesis.

Proposition 2.9.1. Suppose $H(I)$ holds. Let there exist two decompositions $D_1 = (N_1, R_1)$ and $D_2 = (N_2, R_2)$ of $G$ into
necklaces and residues, each of the decompositions satisfying one of the conditions in \( H(I) \). Then

1. \( R_1 = \emptyset \) if and only if \( R_2 = \emptyset \),
2. \( |W(G, R_1)| = 3 \) if and only if \( |W(G, R_2)| = 3 \), and
3. \( |W(G, R_1)| > 4 \) if and only if \( |W(G, R_2)| > 4 \).

Proof. (1) If \( R_1 = \emptyset \) then \( E_{\text{abc}}^2 = \emptyset \). Therefore proposition 2.7.5 implies \( E(N_1) = E_{\text{abc}}^{1} \cup E_{\text{bc}} \cup E_{\text{ca}} \cup E_{\text{ab}} \). Hence \( R_1 = \emptyset \) if and only if \( E_{\text{bc}}, E_{\text{ca}}, E_{\text{ab}}, E_{\text{bc}, \text{ca}}, E_{\text{ab}}, E_{\text{bc}}, E_{\text{ca}}, E_{\text{ab}} \). This is also the condition under which \( R_2 = \emptyset \).

(2) If possible, let \( |W(G, R_1)| = 3 \) and \( |W(G, R_2)| > 4 \). Then \( |W(G, R_1)| = 3 \) implies that \( E_{\text{abc}} \cap E(R_1) = \emptyset \). Therefore \( E(R_1) = E_{\text{bc}}, E_{\text{ca}}, E_{\text{ab}} \cup E_{\text{bc}, \text{ca}}, E_{\text{ab}}, E_{\text{bc}}, E_{\text{ca}}, E_{\text{ab}} \). Also there does not exist any pair of skew diagonals of \( N_1 \), since \( |W(G, R_1)| = 3 \). Hence \( E(N_1) = E_{\text{abc}}^{1} \cup E_{\text{bc}} \cup E_{\text{ca}} \cup E_{\text{ab}} \), and therefore \( E_{\text{abc}}^2 = \emptyset \). On the other hand \( E_{\text{abc}} \cap E(R_2) \neq \emptyset \) by proposition 2.7.6, since \( |W(G, R_2)| \geq 4 \). This implies that

\[
E_{\text{bc}}, E_{\text{ca}}, E_{\text{ab}}, E_{\text{bc}}, E_{\text{ca}}, E_{\text{ab}} \subseteq E(R_2)
\]

and

\[
E(R_2) \subseteq E_{\text{abc}}^{2} \cup \cup E_{\text{bc}, \text{ca}}, E_{\text{ab}}, E_{\text{bc}}, E_{\text{ca}}, E_{\text{ab}}, E_{\text{bc}}, E_{\text{ca}}.
\]
which contradicts $E_{abc}^2 = \emptyset$. Hence by symmetry,

$$|W(G, R_1)| = 3 \text{ if and only if } |W(G, R_2)| = 3.$$  

(3) Finally, (1) and (2) together imply that

$$|W(G, R_1)| \geq 4 \text{ if and only if } |W(G, R_2)| \geq 4.$$  

**Proposition 2.9.2.** Suppose $H(I)$ holds and let $D_1 = (N_1, \Omega)$ and $D_2 = (N_2, \Omega)$ be two decompositions of $G$. Then the two necklaces $N_1$ and $N_2$ are the same.

Proof. Follows from proposition 2.1.3.

**Proposition 2.9.3.** Suppose $H(I)$ and $H(II)$ hold. Also, suppose that there are two necklace decompositions $D_1 = (N_1, R_1)$ and $D_2 = (N_2, R_2)$ of $G$ such that

$$|W(G, R_1)| = 3 = |W(G, R_2)|.$$  

Then the two decompositions are the same.

Proof. We have

$$E(N_1) = E_{abc} \cup E_{bc} \cup E_{ca} \cup E_{ab} = E(N_2)$$

and

$$E(R_1) = E_{bc,ca} \cup E_{ca,ab} \cup E_{ab,bc} \cup E_{ab,bc,ca} = E(R_2)$$

by propositions 2.7.5 and 2.7.6 the result now follows.

**Proposition 2.9.4.** Suppose $H(I)$ and $H(II)$ hold. Also, suppose that there are two necklace decompositions $D_1 = (N_1, R_1)$ and $D_2 = (N_2, R_2)$ of $G$ such that
Then \( E(N_1) \cap E^2_{\text{abc}} = \emptyset \) if and only if \( E(N_2) \cap E^2_{\text{abc}} = \emptyset \).

Proof. Assume \( E(N_1) \cap E^2_{\text{abc}} = \emptyset \). Then

\[
E(N_1) = E^1_{\text{abc}} \cup E_{bc} \cup E_{ca} \cup E_{ab} \quad \text{and} \quad E(R_1) = E^2_{\text{abc}} \cup E_{bc,ca} \cup E_{ca,ab} \cup E_{ab,bc} \cup E_{ab,bc,ca}
\]

by proposition 2.7.5. If possible, assume the contrary for \( D_2 \), i.e.

\[
E_{bc} \cup E_{ca} \cup E_{ab} \cup E^1_{\text{abc}} \subseteq E(N_2) \subseteq E_{bc} \cup E_{ca} \cup E_{ab} \cup E^1_{\text{abc}} \cup E^2_{\text{abc}},
\]

\[
E_{bc,ca} \cup E_{ca,ab} \cup E_{ab,bc} \cup E_{ab,bc,ca} \subseteq E(R_2) \quad \text{and} \quad E(R_2) \subseteq E_{bc} \cup E_{ca} \cup E_{ab} \cup E_{ab,bc} \cup E_{ab,bc,ca} \cup E^2_{\text{abc}}.
\]

By proposition 2.8.12.1, there exists a residual segment \( T \) of \( N_2 \) in \( D_2 = (N_2, R_2) \) such that \( E(T) \subseteq E^2_{\text{abc}} \). Hence every polygon \( P_{\text{abc}} \) of \( G \) contains an edge \( e \in E^2_{\text{abc}} \), where either \( e \in E(T) \) or \( e \in E(R_2) \). But there exists a polygon \( P_{\text{abc}} \subseteq N_1 \) which does not contain any edge belonging to \( E^2_{\text{abc}} \). The above two statements about polygons in \( G \) contradict each other, and hence

\( E(N_2) \cap E^2_{\text{abc}} = \emptyset \).

By symmetry, the converse follows, and

\( E(N_1) \cap E^2_{\text{abc}} = \emptyset \) if and only if \( E(N_2) \cap E^2_{\text{abc}} = \emptyset \).
Proposition 2.9.1. Suppose the hypotheses of proposition 2.9.4 hold. Then the decomposition is unique.

Proof. We have

\[ E(N_1) = E_{abc} U E_{bc} U E_{ca} U E_{ab} = E(N_2) \]

and

\[ E(R_1) = E_{abc} U E_{bc,ca} U E_{ca,ab} U E_{ab,bc} U E_{ab,ca} \]

\[ = E(R_2). \]

The proposition now follows.

The uniqueness of the decomposition \( D = (N, R) \) when statements (1), (2) or (3) of theorem I apply has now been established.

2.10. Non-existence of \( K_{abc} \) when condition \( H(II) \) applies.

In this section we prove the converse part of theorem I.

Proposition 2.10.1. Suppose \( G = S_1 U S_2 U \ldots U S_m \) is a necklace with beads \( S_i \) for \( i = 1, 2, \ldots, m \), and edges \( a, b, c \) belonging to distinct beads of \( G \). Then \( \{a, b, c\} \notin K \), for any bond \( K \) of \( G \).

Proof. If possible, let there be a bond \( K_{abc} \) of \( G \). We write \( N \) for \( G \) since \( G \) is a necklace. Then we apply proposition 2.8.7.1 to \( N(s_{ab}', s_{ca}) U N'(s_{ab}', s_{ca}) \) and \( N(s_{ab}', s_{cb}) U N'(s_{ab}', s_{cb}) \). Let \( K_1 \) and \( K_2 \) be the endgraphs of \( K_{abc} \) in \( G \) and assume \( s_{ab} \in V(K_1) \).

Then \( s_{ca} \in V(K_2), s_{cb} \in V(K_2) \). But this contradicts an application of proposition 2.8.7.1 to \( N(s_{ca}', s_{cb}) U N'(s_{ca}', s_{cb}) \).
The proposition now follows.

**Proposition 2.10.2.** Suppose $G$ is a nonseparable graph with distinct edges $a, b, c$. Also suppose $G$ has a necklace decomposition

$$D = (N, R),$$

where $R$ is a 3-bridge of $N$ or the union of two or more equipartite 3-bridges of $N$, and $R$ separates $a, b, c$. Then $(a, b, c) \not\in K$, for any bond $K$ of $G$.

**Proof.** By hypothesis, $N$ has exactly three residual segments, each of which contains a distinct member of $\{a, b, c\}$. We will denote the segments containing $a, b, c$ by $T_a, T_b, T_c$, respectively. Then we have $t_{ab} = t_{ba}, t_{ac} = t_{ca}$, and $t_{bc} = t_{cb}$. We denote these distinct vertices by $u_1, u_2, u_3$, respectively.

If possible, let there be a bond $K_{abc}$ of $G$. By an application of proposition 2.8.7 to $N'(u_i, u_j) \cup N'(u_i, u_j), \quad \{i, j\} \subseteq \{1, 2, 3\}$ we find that the members of the sets $\{u_1, u_2\}, \{u_2, u_3\}$ and $\{u_3, u_1\}$ belong to distinct endgraphs of $K_{abc}$ in $G$. Let $K_1$ and $K_2$ be the endgraphs of $K_{abc}$ in $G$ and assume $u_1 \in V(K_1)$. Then $u_2 \in V(K_2)$ and $u_3 \in V(K_2)$ contradicting the conclusion that $u_2$ and $u_3$ belong to distinct endgraphs of $K_{abc}$ in $G$. The proposition follows from this contradiction.

**Proposition 2.10.3.** Suppose $G$ is a nonseparable graph with distinct edges $a, b, c$. Also, suppose $G$ has a necklace decomposition
\[ D = (N, R) \]

where

(i) \( R \) is a single bridge of \( N \) in \( G \) separating \( a, b, c \) with \( |W(G, R)| \geq 4 \), and

(ii) there exists no pair of skew diagonals of \( N \) separating \( a, b, c \). Then \( \{a, b, c\} \not\subseteq K \) for any bond \( K \) of \( G \).

**Proof.** By hypothesis, \( |W(G, R)| \geq 4 \). \( R \) separates \( a, b, c \), and hence \( a, b, c \) belong to distinct residual segments of \( N \). Therefore, if there exists a bond \( K_{abc} \) in \( G \) then, by proposition 2.8.10, there will be a pair of skew diagonals of \( N \) separating \( a, b, c \). But this contradicts condition (ii) in the hypothesis of this proposition. The proof of the proposition is now complete.

2.11 **Proof of the theorem I.**

This section will be devoted to proving theorem I. We state it again.

**Theorem I.**

Suppose \( G \) is a nonseparable graph and \( a, b, c \) are distinct edges of \( G \). Then in order that \( \{a, b, c\} \) is not contained in any bond of \( G \) it is necessary and sufficient that there exists a polygon-necklace decomposition

\[ D = (N, R) \]

of \( G \) such that exactly one of the following conditions applies:

1. \( R = \Omega \), and \( a, b, c \) belong to distinct beads of \( N \).
2. \( |W(G, R)| = 3 \), and \( R \) consists of one or more 3-bridges of \( N \).
which separate \( a, b, c \).

(3) \(|W(G, R)| \geq 4\), and \( R \) is a bridge of \( N \) separating \( a, b, c \)
which contains a pair of skew diagonals of \( N \), but no such pair
which separates \( a, b, c \).

Moreover, if \( D = (N, R) \) satisfies conditions (1), (2) or (3),
then the decomposition is unique.

Proof. If there exists no \( K_{abc} \) in \( G \) then propositions 2.6.1 to
2.6.11 establish the existence of such a necklace decomposition. Again,
propositions 2.9.1 to 2.9.4.1, establishes the uniqueness of such a decom­
position. Conversely, if necklace decompositions, as described in
theorem I are given, then by propositions 2.10.1, 2.10.2 and 2.10.3,
we prove that there exists no bond \( K_{abc} \) of \( G \).

This completes the proof of theorem I.
CHAPTER 3

This Chapter is devoted to the discussion of case (4) of theorem I. There a nonseparable graph $G$ has a necklace decomposition $D = (N, R)$. The bridge $R$ of $N$ separates $a$, $b$, $c$ has $|W(G, R)| \geq 4$, and contains a pair $L, M$ of skew diagonals of $N$. No such pair of skew diagonals separates $a, b, c$. We find that the above decomposition is never unique. Also the cones are proved to be edge-disjoint and independent of the decomposition $D = (N, R)$ used in their definition. The residual segments of $N$ containing $a, b$ and $c$, together with those whose edgesets are contained in $E_{abc}$, are shown to contain exactly the beads of $G$ in all necklace decompositions arising in case (4) of theorem I. The cones and these essential beads form a unique bead-cone decomposition of $G$. Elementary properties of the cones, and relationships between the bead-cone decomposition and the various necklace decompositions are developed.

3.1 Singularities and $E_{abc}$-edges in a graph with a necklace decomposition.

Given a residual segment $T$ of $N$, where $N$ is the necklace in a decomposition $D = (N, R)$ of $G$, we know that
for all $T \in \{T_a, T_b, T_c\}$. Also $E(T) = E_{abc}^1 \cup E_{gh}$ where

$\{f, g, h\} = \{a, b, c\}$. Under $H(I)$ and condition (3) of $H(II)$ we will prove that $E(T) \subseteq E_{abc}^2$ if and only if there exists a singularity spanning $T$ in $D = (N, R)$. This singularity, which we denote by $S(N, R, T)$, then gives rise to a partial cone, and hence to a cone, as we saw in section 2.8. Given any necklace decomposition $D = (N, R)$ of a graph $G$ satisfying $H(I)$, and any $T \in \{T_a, T_b, T_c\}$, there does not exist any singularity $S(N, R, T)$.

**Proposition 3.1.1.** Suppose $H(I)$ and case (3) of $H(II)$ hold. Also suppose that $D_1 = (N_1, R_1)$ and $D_2 = (N_2, R_2)$ are two necklace decompositions of $G$. Then the beads of $N_1$ contained in $G \cdot (E(N_1) \setminus E_{abc}^2)$ are the same as the beads of $N_2$ contained in $G \cdot (E(N_2) \setminus E_{abc}^2)$.

**Proof.** By propositions 2.7.5, 2.8.11 and 2.8.12.1 we have

$$E(N_1) \setminus E_{abc}^2 = E(N_2) \setminus E_{abc}^2$$

since, for $i = 1, 2$,

1. $E_{abc}^1 \cup E_{bc} \cup E_{ca} \cup E_{ab} \subseteq E(N_i) \subseteq E_{abc}^1 \cup E_{bc} \cup E_{ca} \cup E_{ab}$

and

2. $E(T) \subseteq E_{abc}^1 \cup E_{bc} \cup E_{ca} \cup E_{ab}$ or $E(T) \subseteq E_{abc}^2$,

for any residual segment $T$ of $N_i$. Hence
G · (E(N_1) \setminus E^2_{abc}) = G · (E(N_2) \setminus E^2_{abc}).

But under the given hypothesis $E(N_1) \cap E^2_{abc} = \emptyset$ if and only if $E(N_2) \cap E^2_{abc} = \emptyset$, by proposition 2.9.4. Therefore, if $E(N_1) \cap E^2_{abc} = \emptyset$, then $G · (E(N_1) \setminus E^2_{abc}) = G · E(N_1) = N_i$ is a necklace with $a$, $b$, $c$ in distinct beads of $N_i$, for $i = 1, 2$. This implies that $N_1 = N_2$ and the bead decomposition of $N_1$ and $N_2$ are the same, by proposition 2.1.3. We obtain the required result in this case. On the other hand the graphs $G · (E(N_1) \setminus E^2_{abc}) = G · (E(N_2) \setminus E^2_{abc})$ have the same set of cyclic elements. But, by construction of beads of the necklaces $N_i$, the set of beads of $N_i$ contained in $G · (E(N_i) \setminus E^2_{abc})$ is the set of cyclic elements of the graph $G · (E(N_i) \setminus E^2_{abc})$, for $i = 1, 2$. This completes the proof of the proposition.

**Proposition 3.1.2.** Suppose $H(I)$ and condition (3) of $H(II)$ hold. Let $e \in E(N)$ be such that the residual segment $T_e$ of $N$ with $e \in E(T_e)$ satisfies

1. $E(T_e) \cap \{a, b, c\} = \emptyset$ and

2. there exists either $K^e_{abc}$, or $K^e_{bac}$, or $K^e_{cba}$.

Then there exists a singularity $S(N, R, T_e)$. 
Proof. Let $T_e \subseteq N(tac, ca)$ and $W(G, T_e) = \{tec, tea\}$. We will need to consider only two cases, namely (i) when there exists $K_{eab}$ or (ii) when there exists $K_{eca}$, by symmetry.

In case (i) there exists $K_{eab}$ if and only if there is a pair of skew diagonals $L_1, L_2$ of $N$ separating $a, b, e$, by proposition 2.8.10. Let the endvertices of $L_i$ be $u_i, v_i$ for $i = 1, 2$. Now

\[
3 \geq |V(N(tbc, tac)) \cap \{u_1, v_1, u_2, v_2\}| \geq 2 \quad \text{and} \\
2 \geq |V(N(tab, tba)) \cap \{u_1, v_1, u_2, v_2\}| \geq 1
\]

since $L_1, L_2$ separate $a, b, e$ with respect to $N$.

If $|V(N(tab, tba)) \cap \{u_1, v_1, u_2, v_2\}| = 2$ then without loss of generality we can take

\[
V(N(tab, tba)) \cap \{u_1, v_1, u_2, v_2\} = \{u_1, u_2\}
\]

in the order $t_{ab} \leq u_1 < u_2 \leq t_{ba}$ on $N(tab, tba)$, and

\[
v_1 \in V(N(tbc, tcb)) \cup V(N(tca, tec)), v_2 \in V(N(tea, tac)) \quad \text{We apply proposition 2.4.3 or 2.4.4 according as} \quad v_1 \in V(N(tbc, tcb)) \quad \text{or} \quad v_1 \in V(N(tca, tec)), \quad \text{respectively, and obtain a} \quad K_{abc}, \quad \text{contrary to} \quad H(I). \quad \text{Hence}
\]
We may assume without loss of generality that

\[ v_2 \in V(N(t_{ab}, t_{ba})) \cup \{u_1, v_1, u_2\} \]

and that

\[ v_1 \in V(N(t_{ea}, t_{ac})). \]

But

\[ 2 \geq |V(N(t_{ea}, t_{ac})) \cap \{u_1, u_2\}| \geq 1 \]

and

\[ 2 \geq |V(N(t_{ec}, t_{bc})) \cap \{u_1, u_2\}| \geq 1 \]

since \( L_1, L_2 \) separate \( a, b, c \) with respect to \( N \). If

\[ V(N(t_{bc}, t_{cb})) \cap \{u_1, u_2\} \neq \emptyset \] then \( L_1, L_2 \) separate \( a, b, c \),

contrary to \( H(I) \). Hence we obtain \( u_1 \in V(N(t_{ca}, t_{ec})) \) and either

\[ u_2 \in V(N(t_{ca}, t_{ec}) \text{ or } u_2 \in V(N(t_{ea}, t_{ac})), \text{ with } u_1 < u_2 < v_1 \leq t_{ac} < v_2 \text{ on } N(t_{ca}, t_{ba}). \]

Hence \( L_1, L_2 \) form a singularity with respect to the decomposition \( D = (N, R) \) of \( G \), and \( T_e \subseteq N(u_1, v_1) \),

which implies the existence of \( S(N, R, T_e) \).

In case (ii), without loss of generality, we can assume

that \( u_1 \in V(N(t_{ea}, t_{ac})), u_2 \in V(N(t_{ec}, t_{ca})) \)
and also $V(N(t_{ab}, t_{cb}))$ contains at least one of $v_1, v_2$. Let $v \in \{v_1, v_2\}$. If $v \in V(N(t_{ab}, t_{ba}))$ then there exists $K_{eab}$, as in case (i). If $v \in V(N(t_{bc}, t_{cb}))$ then there exists $K_{ebc}$, giving a case similar to case (i).

**Proposition 3.1.3.** Suppose $H(I)$ and case (3) of $H(II)$ hold.

Let $T$ be any residual segment of $N$ containing none of the edges $a, b, c$. Also, suppose that there exists a singularity $S(N, R, T)$ spanning $T$. Then

$$E(T) \subseteq E^{2}_{abc}.$$ 

**Proof.** As in proposition 3.1.2, let $T \subseteq N(t_{ac}, t_{ca})$ and $S(N, R, T) = \{L_1, L_2\}$, where the diagonals $L_1, L_2$ have endvertices $u_1, v_1$ and $u_2, v_2$, respectively. By symmetry, we may without loss of generality assume $u_1, v_1, u_2, v_2$ are in the segments $N(t_{ca}, t_{ec}), N(t_{ea}, t_{ac}), N(t_{ca}, t_{ec})$ and $N(t_{ab}, t_{ba})$, respectively.

If we now take any polygon $P_1 = P_{abc} \subseteq N$, then all the hypotheses of proposition 2.8.3 apply. Thus we get a configuration $F_1$, as defined in 2.8.2, where $W_1$ is an arc joining a vertex $t$ of $L_1 \setminus \{u_1, v_1\}$ to a vertex $w$ of $L_2 \setminus \{u_2, v_2\}$, and $Z_1$ is an arc joining a vertex $z$ of $V(N(t_{cb}, t_{bc})) \cap W(G, N)$ to a vertex $x$ of $L_2(w, v_2) \setminus \{v_2\}$. The skew diagonals $L_1, L_2$ separate $a, c, e$
and also a, b, e with respect to N. Hence there exist \( K_{eca} \) and \( K_{eab} \), by proposition 2.8.10. The skew diagonals \( L_1, L_2(u_2, x) U Z \) separate b, c, e. Hence there exists \( K_{e,b,c} \), by proposition 2.8.10. This implies \( e \in E^2_{abc} \) and therefore \( E(T) \subseteq E^2_{abc} \), by proposition 2.8.12.1.

**Proposition 3.1.4.** Suppose \( H(I) \) and condition (3) of \( H(II) \) hold. Also suppose \( T \) is any residual segment of \( N \) containing none of a, b or c. Then the following statements are equivalent:

1. there exists a \( S(N, R, T) \),
2. \( E(T) \subseteq E^2_{abc} \), and
3. there exists at least one of \( K_{eab} K_{ebc} K_{eca} \) in \( G \), where \( e \in E(T) \).

**Proof.** Follows from propositions 3.1.2, 3.1.3 and definition of \( E^2_{abc} \).

**Remark.** From 3.1.4 we conclude that any residual segment \( T \) not containing a, b, or c with \( E(T) \subseteq E^1_{abc} \) is such that \( K_{eab}, K_{ebc}, K_{eca} \) do not exist, for any \( e \in E(T) \).

**Proposition 3.1.5.** Suppose \( H(I) \) and condition (3) of \( H(II) \) hold. Let \( S_a \) and \( T_a \) be the bead and residual segment of \( N \), respectively, containing a. Suppose \( e \in E(S_e) \), for a bead \( S_e \subseteq T_a \). Then the following conclusions hold.
(1) If $S_e \neq S_a$ then none of the bonds $K_{eab}$, $K_{eca}$, $K_{ebc}$ exist in $G$.

(2) If $S_e = S_a$ and $e \neq a$ then either there exist $K_{eab}$, $K_{eca}$ and not $K_{ebc}$ or none of the bonds $K_{eab}$, $K_{eca}$ and $K_{ebc}$ exist in $G$.

Proof. In all cases above, existence of $K_{ebc}$ will imply the existence of $K_{abc}$, by proposition 2.8.10. This is contrary to $H(I)$. Hence there exists no $K_{ebc}$. In case (1) we define $W(G, S_e) = \{s_{ea}, s_{eb}\}$. As usual let $W(G, S_a) = \{s_{ac}, s_{ab}\}$ and $W(G, T_a) = \{t_{ab}, t_{ac}\}$. Without loss of generality we can assume the order of the above vertices as

$$t_{ac} < s_{ac} < s_{ab} < s_{ea} < s_{eb} = t_{ab}$$
on $N(t_{ac}, t_{ab})$.

Suppose there exists a bond $K_{eab}$ in $G$. By applying proposition 2.8.7.1 to $N(t_{ac}, s_{ab})$ and $N'(t_{ac}, s_{ab}) \cup R$ we conclude that $t_{ac}$ and $s_{ab}$ belong to different endgraphs of $K_{eab}$ in $G$. Let $t_{ac} \in V(K_1)$ and $s_{ab} \in V(K_2)$, where $K_1$, $K_2$ are the endgraphs of $K_{eab}$ in $G$. By another application of 2.8.7.1, to $N(t_{ab}, s_{ab})$ and $N'(t_{ab}, s_{ab}) \cup R$, we conclude that $t_{ab} \in V(K_1)$, since $s_{ab} \in V(K_2)$.

But a third application of 2.8.7.1, to $N(t_{ab}, t_{ac})$ and $N(t_{ab}, t_{ac}) \cup R$, implies that $t_{ab}$ and $t_{ac}$ belong to different
endgraphs \( K_1, K_2 \). This is a contradiction. Hence there exists no bond \( K_{eab} \) in \( G \). Similarly there exists no bond \( K_{eac} \) in \( G \). So in this case none of the bonds \( K_{eab}, K_{ebc}, K_{eca} \) exist in \( G \). In case (2) we have \( e \in E(S_a) \) and \( e \neq a \), by hypothesis. The following two subcases are then possible.

(i) All bonds \( K_{ae} \) in \( S_a \) are such that the vertices \( s_{ab} \) and \( s_{ac} \) belong to the same endgraph of \( K_{ae} \) in \( S_a \).

(ii) There exists a bond \( K_{ae} \) in \( S_a \) containing \( a \) and \( e \) such that the vertices \( s_{ab} \) and \( s_{ac} \) belong to different endgraphs of \( K_{ae} \) in \( S_a \).

In subcase (i) above, suppose there exists a \( K_{eba} \). Then, by proposition 2.8.7.1, \( s_{ab} \) and \( s_{ac} \) will belong to different endgraphs, say \( K_1, K_2 \), respectively, of \( K_{eab} \) in \( G \). Obviously \( G \cdot (E(S_a) \setminus K_{eab}) = (S_a \cap K_1) \cup (S_a \cap K_2) \), where \( s_{ab} \in V(S_a \cap K_1) \) and \( s_{ac} \in V(S_a \cap K_2) \). Also \( S_a \cap K_1 \) and \( S_a \cap K_2 \) are disjoint induced subgraphs of \( S_a \), which together span \( S_a \). We claim that \( S_a \cap K_2 \) are connected. Suppose \( C \) is a connected component of \( S_a \cap K_1 \). Then \( C \subset K_1 \), as \( K_1 \) contains a vertex incident with \( b \) and \( C \) does not, so there exists a vertex \( v \in W(K_1, C) \). By definition of vertices of attachment, there exists an edge \( e \in E(K_1) \setminus E(C) \) incident with \( v \). But \( v \in V(S_a) \) and \( e \in E(G) \setminus E(S_a) \), so \( v \in W(G, S_a) \cap V(K_1) \) and hence \( v = s_{ab} \). Thus \( C = S_a \cap K_1 \)
and \( S_a \cap K_1 \) is connected. Similarly \( S_a \cap K_2 \) is connected. The above claim establishes that \( E(S_a) \setminus K_{eab} \) is a bond of \( S_a \) with endgraphs \( S_a \cap K_1 \) and \( S_a \cap K_2 \). Furthermore \( e, a \in E(S_a) \cap K_{eab} \) and the vertices \( s_{ab} \) and \( s_{ac} \) are in distinct endgraphs of this bond. This contradicts the assumption of case (i) and we conclude that case (ii) must apply.

In case (ii) there exists a bond in \( S_b \) containing \( b \) such that the vertices \( s_{bc} \) and \( s_{ba} \) belong to distinct endgraphs, by proposition 2.7.2.1. A similar result also holds in \( S_c \). Now let \( K'_b \) be the bond in \( S_b \) containing \( b \), with endgraphs \( H'_b \) and \( H'_c \) containing \( s_{ba} \) and \( s_{bc} \), respectively. Also let \( H'_a \) and \( H'_a \) be the endgraphs of \( K'_e \) in \( S_a \) containing \( s_{ab} \) and \( s_{ac} \), respectively. Then

\[
K_1 = N(s_{ab}, s_{ba}) \cup H_a \cup H'_b \quad \text{and} \quad K_2 = N'(s_{ac}, s_{ba}) \cup (R \setminus K_1) \cup H'_a \cup H'_b
\]

are the endgraphs of a bond containing \( e, a, b \). Hence there exists a \( K_{eab} \) and similarly a \( K_{eca} \). This completes the proof of the proposition.
3.2 Cones and partial cones.

Let \( D = (N, R) \) be a fixed necklace decomposition of \( G \) satisfying the conditions of case (4) in theorem I. A cone \( C \) of \( G \) is defined to be a maximal partial cone of \( D \). In this section we prove that, given any two partial cones of \( D \), either one is contained in the other, or they are edge-disjoint or they have the same three vertices of attachment in \( G \). As a consequence distinct cones are edge-disjoint and meet at most in one extreme and the apex.

Also, here we now give two more definitions. Let \( \{L, M\} = S(N, R, T) \) be a singularity of \( D \) such that \( L \) has endvertices \( u, v \in V(N(t_{fg}, t_{gr})) \), where \( \{f, g\} \subseteq \{a, b, c\} \). Then \( S(N, R, T) \) is said to meet a partial cone \( C; \) of \( D \), with \( W(G, C') = \{u', v', w'\} \) and apex \( w' \) when \( N(u', v') \) and \( N(u, v) \) have a bead of \( N \) in common.

\( S(N, R, T) \) is said to be engulfed in a partial cone \( C' \) of \( D \) when \( L \) is contained in \( C' \). Obviously, if \( S(N, R, T) \) is engulfed in \( C' \) then it meets \( C' \). From the previous chapter and the last section, every singularity is engulfed in a partial cone
and hence engulfed in a cone. By the condition that cones are edge-disjoint, it follows that each singularity is engulfed in a unique cone and that a cone engulfs all the singularities it meets.

We begin this section by proving two propositions on bridges concerning two subgraphs of a given graph. They will be used several times in the propositions of the following sections.

**Proposition 3.2.1.** Suppose $G$ is a graph and $C, F$ are two subgraphs of $G$ such that $W(G, C) \subseteq V(F)$. Then any bridge $B$ of $F \cap C$ in $C$ is a bridge of $F$ in $G$ contained in $C$.

**Proof.** If $B$ is an inner bridge of $F \cap C$ in $C$ then, by the definition in section 1.3, $E(B) = \{e\}, V(B) \subseteq V(F \cap C) \subseteq V(F)$, and $e \in E(C) \setminus E(F)$. This implies $B$ is an inner bridge of $F$ in $G$.

If $B$ is an outer bridge of $F \cap C$ in $C$ then $B \setminus (F \cap C) = B \setminus V(F)$, since $B \subseteq C$. By the definition in section 1.3, $B \setminus V(F) = B \setminus (F \cap C)$ is a connected component of $C \setminus (F \cap C) = C \setminus V(F)$, and $W(C, B) \subseteq V(F \cap C)$.

We know that $W(G \setminus V(F), B \setminus V(F)) \subseteq W(G \setminus V(F), C \setminus V(F)) \cup W(C \setminus V(F), B \setminus V(F))$, since $B \setminus V(F) \subseteq C \setminus V(F)$. Again $W(G, C \setminus V(F)) = \emptyset$, since any vertex $x \in W(G, C \setminus V(F))$ implies that $W(G, C) \cap V(F)$, a contradiction to $W(G, C) \subseteq V(F)$. Hence $W(G \setminus V(F), C \setminus V(F)) = \emptyset$, since $W(G \setminus V(F), C \setminus V(F)) \subseteq W(G, C \setminus V(F))$. We have $W(C, B) \subseteq V(F \cap C) \subseteq V(F)$. But $W(C, B \setminus V(F)) = \emptyset$, since any vertex $y$ belonging to $W(C, B \setminus V(F))$ would imply that $W(C, B) \cap V(F)$. Therefore $W(C \setminus V(F), B \setminus V(F)) \subseteq W(C, B \setminus V(F)) = \emptyset$. 
Hence combining the above results we get \( W(G \setminus V(F), B \setminus V(F)) = \emptyset \).

By the remarks above, \( B \setminus V(F) \) is connected, hence \( B \setminus V(F) \) is a connected component of \( G \setminus V(F) \), with

\[
W(G, B) \subseteq W(G, C) \cup W(C, B) \subseteq V(F) \cup V(F \cap C) \\
\subseteq V(F).
\]

Therefore \( B \) is an outer bridge of \( F \) in \( G \). This completes the proof of the proposition.

The next proposition is the converse of this one.

**Proposition 3.2.2.** Suppose the hypotheses of proposition 3.2.1 hold. Let \( B \) be a bridge of \( F \) in \( G \) such that \( B \cap C \subseteq F \). Then \( B \) is a bridge of \( F \cap C \) in \( C \).

**Proof.** Suppose \( B \cap C \) is edgeless. Then \( B \cap C \subseteq F \) implies that there exists \( x \in V(B \cap C) \setminus V(F) \). All edges of \( G \) incident with \( x \) are in \( B \), since \( B \) is a bridge of \( F \) in \( G \). Thus no such edge is in \( C \). Also \( x \) is not incident with any edge of \( B \), as \( W(G, C) \subseteq V(F) \), and hence \( x \) must be isolated in \( G \). In fact \( B = [x] \subseteq C \) and therefore \( B \) is a bridge of \( F \cap C \) in \( C \).

Now suppose \( e \in E(B \cap C) \) exists. We claim that \( B \subseteq C \). If not, then \( f \in E(B) \setminus E(C) \) exists. Let the vertices incident with \( e \) and \( f \) be \( x, x' \) and \( y, y' \), respectively, with \( x, y \notin V(F) \). Such vertices \( x \) and \( y \) always exist because \( B \) is an outer bridge of \( F \) in \( G \). Let \( P \) be any \((x, y)\)-path in \( B \) having no vertex of \( F \). Then define the path \( P' = (x', e, x)P(y, f, y') \). This \((x', y')\)-path \( P' \) then avoids \( F \).
Now e in C and f not in C imply that there exist consecutive edges e', f' in P' such that e' ∈ E(C) and f' ∉ E(C). Then the common incident vertex of e' and f' does not belong to F, contrary to \( W(G, C) = V(F) \). Hence the claim \( B = C \) is valid.

Suppose \( B \) is an outer bridge of \( F \) in \( G \). Then \( H = B \setminus V(F) \) is a connected component of \( G \setminus V(F) \). But \( B \setminus (F \cap C) = B \setminus V(F) = H \), since \( B = C \). Hence \( H = C \setminus V(F \cap C) = C \setminus V(F) \). Also \( W(C, B) = W(G, B) = V(F) \).

Hence \( W(C, B) = V(F \cap C) \). Therefore \( B \) is a bridge of \( F \cap C \) in \( C \). This is evidently also true when \( B \) is an inner bridge of \( F \) in \( G \) contained in \( C \).

**Proposition 3.2.3.** Suppose \( C_1, C_2 \) are two partial cones of \( G \) with respect to a given necklace decomposition \( D = (N, R) \). Then one of the following three possibilities occurs.

1. Either \( C_1 \subseteq C_2 \) or \( C_2 \subseteq C_1 \).
2. \( C_1 \) and \( C_2 \) are edge-disjoint.
3. \( W(G, C_1) = W(G, C_2) = \{u, v, w\} \).

**Remark 1.** In case (2) \( C_1 \cap C_2 \subseteq [W(G, C_1) \cap W(G, C_2)]\), where \( W(G, C_1) \cap W(G, C_2) \) can have at most one vertex which is an extreme of the partial cones.

**Remark 2.** Consider the 3-bridges of \( \{u, v, w\} \) in \( G \). One such bridge containing \( N(u, v) \) is common to both \( C_1 \) and \( C_2 \) while the
3-bridge containing $N'(u, v)$ is in neither $C_1$ nor $C_2$.

Remark 3. Condition (2) is exclusive of conditions (1) and (3), but it is possible that conditions (1) and (3) hold together.

Proposition 3.2.3.1. Suppose $C$ is a partial cone of $G$ with respect to a given necklace decomposition $D = (N, R)$. Also suppose that $W(G, C) = \{u, v, w\}$, where $w$ is the apex of $C$. Let $N(u, v) \subseteq N(t_{ab}', t_{ba})$. Then there exists an $N$-avoiding arc $A$ in $R$ joining $w$ to a vertex $y \in V(N(t_{bc}, t_{cb}))$, meeting $C$ only at $w$.

Proof. By definition, $w \in V(R) \setminus V(N)$. $R$ is a bridge of $N$ separating $a, b, c$. Hence there exists a vertex $y \in V(R) \cap V(N(t_{bc}, t_{cb}'))$. Therefore $y \neq w$ and $y, w \in V(R)$. Thus there is an arc $A$ in $R$, avoiding $N$, joining $w$ and $y$. Because $\{w\} = W(G, C) \setminus V(N)$ it follows that $A \cap C = [w]$. This completes the proof of the proposition.

Remark. The vertex $y$, clearly, can also be chosen to belong to $N(t_{ac}', t_{ca})$.

Proposition 3.2.3.2. Suppose the hypotheses of proposition 3.2.3.1 hold. Then either there exists an $N$-avoiding arc in $C$ containing $u, v$ and $w$ or there exists a $Y$-graph $Y, Y \subseteq C$, with ends $u, v, w$ such that $Y \cap N = [u, v]$.

Proof. We consider the bridges of $N(u, v)$ in $C$. There exists a unique bridge $B$ of $N(u, v)$ in $C$ containing the apex $w$, since $w \notin V(N(u, v))$. Now, if there exists at least one more bridge $B'$ of
$N(u, v)$ in $C$ then $B'$ has its vertices of attachment only in $V(N(u, v))$.

To apply proposition 3.2.1 we write $G = G$, $C = C$, $F = N \cup [w]$, $B = B'$. Then $W(G, C) = \{u, v, w\} \subseteq V(F)$, and $F \cap C = (N \cup [w]) \cap C = N(u, v) \cup [w]$. Therefore $B'$ is a bridge of $F \cap C$ in $C$. Hence $B'$ is a bridge of $N \cup [w]$ in $G$. But $w \notin V(B')$ and therefore $B'$ is a bridge of $N$ in $G$. This contradicts the fact that $R$ is the only bridge of $N$ in $G$.

Therefore $B$ is the only bridge of $N(u, v)$ in $C$. We claim that $u, v \in W(C, B) = W(C, N(u, v))$. If, say, $u \notin W(C, N(u, v))$ then the bead $S$ of $N(u, v)$ containing $u$ is a cyclic element of $C$. But $w \notin V(S)$. Hence $S \subset C$. This contradicts the nonseparability of $C$, and therefore establishes the claim.

Hence there is an arc $A$ in $B$ joining $u$ and $w$, where $u$ is the only vertex of $N$ belonging to $A$. Also there is an arc in $B$ avoiding $N \cup A$ from $v$ to a vertex $x$ of $A \backslash \{u\}$, by proposition 1.3.1, since $B$ is a bridge of $N(u, v)$ in $C$. If $x = w$ we obtain the first part of the conclusion, while $x \neq w$ gives us the alternative part of the conclusion.

**Proposition 3.2.3.3.** Suppose the hypotheses of proposition 3.2.3 hold. Also suppose that $W(G, C_i) = \{u_i, v_i, w_i\}$ with $w_i$ as apex of $C_i$, for $i = 1, 2$. Let $w_1 \in V(C_2) \backslash \{w_2\}$. Then $N(u_1, v_1) \subseteq N(u_2, v_2)$ and $w_2 \notin V(C_1) \backslash \{w_1\}$. 
Proof. Suppose $N(u_2, v_2) \subseteq N(t_{ab}, t_{ba})$ without loss of generality.

Then, by proposition 3.2.3.1, there exists an $N$-avoiding arc $A$ in $R$ joining $w_1$ to a vertex $y$ of $N(t_{bc}, t_{cb})$, say, meeting $C_1$ only at $w_1$ and $y$ being the only vertex on $A$ belonging to $V(N)$.

Therefore $w_2 \in A$, since $W(G, C_2) \setminus V(N) = \{w_2\}$, and $w_1 \in V(C_2) \setminus \{w_2\}$, by hypothesis. We see that $w_2 \notin V(C_1) \setminus \{w_1\}$.

Let $\{u_1, v_1\} \subseteq V(N(u_2, v_2))$, if possible. By proposition 3.2.3.2 we get either an arc $A'$ in $C_1$ containing $u_1, v_1, w_1$ or a $Y$-graph $Y \subseteq C_1$ containing the vertices $u_1, v_1, w_1$ as ends, where $A'$ and $Y$ meet $N$ at $u_1, v_1$. By propositions 3.2.3.1 and 3.2.3.2, $A' \cap A = \{w_1\}$ or $Y \cap A = \{w_1\}$. Hence $w_2 \notin V(A')$ and $w_2 \notin V(Y)$, respectively. Thus the arc $A'$ or the $Y$-graph $Y$ must pass through at least one of $u_2, v_2$ distinct from $u_1, v_1$, since $W(G, C_2) = \{u_2, v_2, w_2\}$. Finally we arrive at a contradiction since $u_1, v_1$ are the only vertices of $N$ on $A'$ or $Y$. The proposition now follows.

Proposition 3.2.3.4. Suppose the hypotheses of proposition 3.2.3 hold. Let $W(G, C_i) = \{u_i, v_i, w_i\}$, where $w_i$ is the apex of $C_i$, for $i = 1, 2$. Then $\{u_1, v_1, w_1\} \neq \{u_2, v_2, w_2\}$ and $\{u_2, v_2, w_2\} \subseteq V(C_1)$ imply $C_2 \subseteq C_1$.
Proof. By the definition of a partial cone $C_2$ is the union of 3-bridges of $[u_2, v_2, w_2]$ in $G$. Let $B$ be one such bridge.

Apply proposition 3.2.2 with respect to the graphs $G$, $[u_1, v_1, w_1, u_2, v_2, w_2] = F$, $C_1 = C$ and $B$. Then $W(G, C_1) \subseteq V(F)$. Also $u_1, v_1, w_1 \notin V(B) \setminus \{u_2, v_2, w_2\}$, by proposition 3.2.3.3. Hence $B$ is a bridge of $F$ in $G$. We conclude that $B$ is a bridge of $F \cap C_1 = [u_1, v_1, w_1, u_2, v_2, w_2]$ in $C_1$. Again using the fact that $u_1, v_1, w_1 \notin V(B) \setminus \{u_2, v_2, w_2\}$ we see that $B$ is a bridge of $[u_2, v_2, w_2]$ in $C_1$. As $B$ is arbitrary 3-bridge of $[u_2, v_2, w_2]$ in $G$ contained in $C$, and $W(G, C_1) \neq W(G, C_2)$, we have $C_2 \subset C_1$.

Proposition 3.2.3.5. Suppose the statement of proposition 3.2.3.3 holds. Then $C_2 \subset C_1$.

Proof. By proposition 3.2.3.3, $\{u_2, v_2\} \subseteq V(N(u_1, v_1))$. Hence $\{u_2, v_2, w_2\} \subseteq V(C_1)$ with $\{u_1, v_1, w_1\} \neq \{u_2, v_2, w_2\}$. The proposition now follows from proposition 3.2.3.4 and the fact that there exists an edge $e \in E(C_1) \setminus E(C_2)$ incident with $w_1$.

Proposition 3.2.3.6. Suppose the hypotheses of proposition 3.2.3 hold. Let $W(G, C_i) = \{u_i, v_i, w_i\}$, for $i = 1, 2$, be such that $u_1 < v_1, u_2 < v_2$ and $u_1 \leq u_2$ on $N(t_{fg}, t_{gf})$ for $(f, g) \subseteq \{a, b, c\}$. Then $N(u_1, v_1) \cap N(u_2, v_2)$ equals one of the following graphs:
(1) $\Omega,$

(2) $[v_1] = [u_2],$

(3) $N(v_1, u_2),$ or

(4) $N(u_2, v_2).$

Proof. Obvious.

Remark. Under the hypothesis of 3.2.3, the assumptions $u_1 < v_1,$ $u_2 < v_2$ and $u_1 < u_2$ on $N(t_{fg}, t_{gf})$ of proposition 3.2.3.6 can be made without any loss of generality.

Proposition 3.2.3.7. Suppose the hypotheses of 3.2.3.6 hold. Also suppose that $|E(C_1 \cap C_2)| \geq 1$ and $w_1 \notin V(C_2) \setminus \{w_2\}$, $w_2 \notin V(C_1) \setminus \{w_1\}.$

Then one of the following occurs.

(1) If $w_1 \neq w_2$ then

\[
[u_2] \subseteq W(G, C_1 \cap C_2) \subseteq \{u_2, v_1, v_2\}
\]

and

\[
|W(G, C_1 \cap C_2)| = 2.
\]

(2) If $w_1 = w_2 = w$ then

\[
[w, u_2] \subseteq W(G, C_1 \cap C_2) \subseteq \{v_1, u_2, v_2, w\}
\]

and

\[
2 \leq |W(G, C_1 \cap C_2)| \leq 3.
\]

Proof. We know that

\[
W(G, C_1 \cap C_2) = V(C_1 \cap C_2) \cap (W(G, C_1) \cup W(G, C_2)),
\]

by proposition 1.2.4. Therefore
Now, by hypothesis, $|E(C_1 \cap C_2)| \geq 1$. Therefore, by proposition 1.4.4, $|W(G, C_1 \cap C_2)| \geq 2$, since $G$ is nonseparable. By the last proposition, $|V(C_1 \cap C_2) \cap \{u_1, v_1, u_2, v_2\}| \leq 2$. If $w_1 \neq w_2$ then $w_1, w_2 \notin W(G, C_1 \cap C_2)$. Hence $W(G, C_1 \cap C_2) \subseteq V(C_1 \cap C_2) \cap \{u_1, v_2, u_2, v_2\}$. By the restrictions imposed on $|V(C_1 \cap C_2) \cap \{u_1, v_1, u_2, v_2\}|$ and $|W(G, C_1 \cap C_2)|$ above,

$W(G, C_1 \cap C_2) \subseteq \{u_1, v_1, u_2, v_2\}$

and

$|W(G, C_1 \cap C_2)| = 2$.

Then $u_2 \in W(G, C_1 \cap C_2)$, and if $u_1 \in W(G, C_1 \cap C_2)$ then $u_1 = u_2$.

Thus we can write $W(G, C_1 \cap C_2) \subseteq \{u_2, v_1, v_2\}$. If $w_1 = w_2 = w$,

then $w \in W(G, C_1 \cap C_2)$ and $W(G, C_1 \cap C_2) \subseteq V(C_1 \cap C_2) \cap \{u_1, v_1, u_2, v_2, w\}$. Also $2 \leq |W(G, C_1 \cap C_2)| \leq 3$, by the restrictions imposed on $|W(G, C_1 \cap C_2)|$ and $|V(C_1 \cap C_2) \cap \{u_1, v_1, u_2, v_2\}|$ above. As in the previous case, $u_2 \in V(C_1 \cap C_2)$ and $W(G, C_1 \cap C_2) \subseteq \{w, u_2, v_1, v_2\}$. This completes the proof.

Proposition 3.2.3.8. Suppose the statement of proposition 3.2.3.7 holds. Then

$|W(G, C_1 \cap C_2)| = 3$. 

Proof. By hypothesis and by proposition 3.2.3.6 we have either
\[ W(G, C_1 \cap C_2) = \{u_2, v_1\}, \]
\[ W(G, C_1 \cap C_2) = \{u_2, v_2\}, \text{ or} \]
\[ W(G, C_1 \cap C_2) = \{u_2, w\}. \]
In the first case we observe that \( v_1 \in V(N(u_2, v_2)) \) and hence every edge incident with \( v_1 \) in \( G \) belongs to \( C_2 \). Again \( C_1 \cap C_2 \subseteq C_1 \) and \( C_1 \) is nonseparable, and \(|E(C_1 \cap C_2)| \geq 1\), by hypothesis.
This implies that \(|W(C_1, C_1 \cap C_2)| \geq 2\). But \( W(C_1, C_1 \cap C_2) \leq W(G, C_1 \cap C_2) = \{u_2, v_1\} \). Therefore \( W(C_1, C_1 \cap C_2) = \{u_2, v_1\} \).
This implies that there exists an edge \( e \) incident with \( v_1\), \( e \in E(C_1) \setminus E(C_2) \) since \( v_1 \in W(C_1, C_1 \cap C_2) \). But \( e \in E(C_2) \), by our remark above. This is a contradiction. Hence this case does not arise.

In the remaining cases we observe that
\[ W(G, C_1 \cap C_2) \subseteq W(G, C_2), \text{ and} \]
\[ |W(G, C_1 \cap C_2)| = 2. \]
Hence \( C_1 \cap C_2 \) contains 2-bridges of \( C_2 \), contrary to the definition of a partial cone. This completes the proof of the proposition.

Proposition 3.2.3.9. Suppose the statement of proposition 3.2.3.8 holds. Then we get either case (1) or case (3) of the conclusion of proposition 3.2.3.
Proof. By hypothesis and propositions 3.2.3.7 and 3.2.3.8, we have either

(1) $W(G, C_1 \cap C_2) = \{u_2, v_2, w\}$, or

(2) $W(G, C_1 \cap C_2) = \{u_2, v_1, w\}$.

In case (1) we have $W(G, C_2) \subseteq V(C_1)$. If $W(G, C_2) = W(G, C_1)$ we have the conclusion (3) of proposition 3.2.3. If $W(G, C_2) \neq W(G, C_1)$ we have $C_2 \subseteq C_1$, by proposition 3.2.3.4.

In case (2), $W(G, C_1 \cap C_2) = \{u_2, v_1, w\}$. If $v_1 = v_2$ we have the previous case. If $u_1 = u_2$ we interchange $u$ and $v$ and obtain the previous case. Hence $u_1 \neq u_2$ and $v_1 \neq v_2$ can be assumed without loss of generality.

Every edge incident with $v_1$ belongs to $C_2$. Thus the polygon $Q$ of the $s$-graph for the partial cone $C_1$, which contains $u_1$ and $v_1$, has both its edges incident with $v_1$ contained in $C_1 \cap C_2$. As $w \notin V(Q)$ and $W(G, C_1 \cap C_2) = \{u_2, v_1, w\}$ we conclude that $Q \subseteq C_1 \cap C_2$. This is contrary to $u_1 \in V(Q)$ and $u_1 \notin V(C_1 \cap C_2)$.

Proof of the proposition 3.2.3. Let $W(G, C_1) = \{u_i, v_i, w_i\}$ where $w_i$ is the apex of $C_1$, for $i = 1, 2$. Then either

(A) $E(C_1) \cap E(C_2) = \emptyset$, or

(B) $|E(C_1) \cap E(C_2)| \geq 1$. 
In case (A) above we get the conclusion (2) of the proposition 3.2.3.

In case (B) we get either

(B1) \( w_2 \in V(C_1) \setminus \{w_1\} \),

(B2) \( w_1 \in V(C_2) \setminus \{w_2\} \), or

(B3) \( w_2 \notin V(C_1) \setminus \{w_1\} \), \( w_1 \notin V(C_2) \setminus \{w_2\} \).

In cases (B1) and (B2) we get \( C_2 \subseteq C_1 \) and \( C_1 \subseteq C_2 \), respectively, by proposition 3.2.3.5. In case (B3) we get either conclusion (1) or conclusion (3) of proposition 3.2.3, by proposition 3.2.3.9. The proof of the proposition is now complete.

3.3 Rim of a partial cone \( C \) of \( G \).

Suppose \( D = (N, R) \) is a necklace decomposition of a graph \( G \) satisfying the conditions of case (4) of theorem I. In section 3.1 it is shown that there exists a residual segment \( T \) of \( N \) with \( E(T) = E \setminus \{abc\} \), a singularity \( S(N, R, T) \) spanning \( T \), and a partial cone \( C \) of \( G \) with respect to \( D \) which engulfs \( S(N, R, T) \). In this section we will prove the existence of a rim of \( C \) (defined in 1.9.5) which will establish the non-uniqueness of the necklace decompositions of \( G \). The existence of such a rim will also prove the last part of theorem II.

Algorithm 3.3.1 (Formation of a rim in a given partial cone \( C \)).

The following notation will be used to describe the algorithm for forming a rim of a partial cone. Suppose we have a partial cone \( C \) of \( G \) with extremes \( u, v \) and apex \( w \), with respect to a fixed
necklace decomposition $D = (N, R)$ of $G$. By definition we can take $N(u, v) \subseteq N(t_{ab}, t_{ba})$, without loss of generality. We know that $N(u, v) \subseteq C$, by definition, and there exists a $\theta$-graph $\theta \subseteq C$ with $u, v, w$ as internal vertices of distinct branches of $\theta$. Let $Q$ be the polygon of $\theta$ containing both the vertices $u$ and $v$. Suppose the branch $A$ of $\theta$ containing $w$ as an internal vertex meets the two $[u, v]$-bridges of $Q$ at $s$ and $t$. We write $L$ and $L'$ for the $[u, v]$-bridges of $Q$ containing $s$ and $t$, respectively.

**Proposition 3.3.1.1.** Suppose the hypotheses and notation stated above hold. Let $B, B'$ be distinct bridges of $N'(u, v) \cup L$ in $G$, with $L' \subseteq B$. Then each of these bridges has a vertex of attachment in $G$, which is internal to $L$, and $B'$ is a bridge of $L$ in $C$. Also $B$ is a type III bridge of any polygon $P_{abc}$ containing the arc $L$.

**Proof.** Obviously $B$ contains the vertex $w$ and $L' \cup A$. Hence $B$ has the internal vertex $s$ of $L$ as a vertex of attachment in $G$. Now $B, B'$ are bridges of $N'(u, v) \cup L$ so that $w \in V(B)$ and $w \notin N'(u, v) \cup L$ imply that $w \notin V(B')$.

Suppose $B'$ has no vertex in $L \setminus \{u, v\}$ as a vertex of attachment in $G$. Then $W(G, B') \subseteq V(N'(u, v))$ and hence $B'$ is a bridge of $N$ in $G$. This implies that $B' = R$ by the definition of $D$. This is a contradiction since $w \in V(R) \setminus V(B')$. We therefore conclude that each $B'$ has some vertex of attachment $x$ in $G$, which is internal to $L$.

Next we note that $B'$ is a bridge of $N'(u, v) \cup L \cup \{w\}$ in $G$, because $w \notin V(B')$. Write $N'(u, v) \cup L \cup \{w\}$ for $F$, and $B'$ for
B in proposition 3.2.2. We have \( W(G, C) = \{u, v, w\} \subseteq V(F) \).

Again all edges incident with \( x \) belong to \( C \), since \( x \neq u, v, w \). If \( B' \) does not contain any edge of \( C \) then \( x \) will be an isolated vertex in \( B' \). Hence \( B' = [x] \), since \( B' \) is connected. But then

\[
B' = [x] \subseteq N'(u, v) \cup L
\]

contradicts the fact that \( B' \) is a bridge of \( N'(u, v) \cup L \) in \( G \). Hence \( B' \cap C \) has an edge incident with \( x \). This implies that

\[
B' \cap C \subseteq N'(u, v) \cup L \cup [w] = F.
\]

Therefore \( B' \) is a bridge of \( F \cap C = L \cup [w] \) in \( C \), by proposition 3.2.2. This implies that \( B' \) is a bridge of \( L \) in \( C \), since \( w \notin V(B') \).

We can easily construct a polygon \( P_{abc} \) in \( G \) containing \( L \). As \( P_{abc} \subseteq N'(u, v) \cup L \) and \( W(G, B) \subseteq V(P_{abc}) \) it follows that \( B \) is a bridge of \( P_{abc} \) in \( G \). A similar remark holds for the bridges \( B' \). However, \( B \) has all the vertices of attachment of \( R \) which are not on \( L \). Also \( s \in V(L) \subseteq V(P_{ab}, P_{ba}) \). Hence \( B \) is a type III bridge of \( P_{abc} \) in \( G \).

**Remark.** \( G \) is the union of \( N'(u, v) \cup L \), \( B \) and all bridges \( B' \), as defined in proposition 3.3.2.1.

**Remark.** The other bridges of \( P_{abc} \), i.e. those contained in beads of \( N'(u, v) \) are clearly not skew to the bridge \( B \).
Proposition 3.3.1.2. Suppose the statements of proposition 3.3.1.1 hold. Then there exists a polygon \( Q_{abc} \) in \( G \) and a type III bridge \( B_n \) of \( Q_{abc} \) in \( G \), such that \( B_n \supseteq B \) and \( B_n \) is not skew to any type I bridge of \( Q_{abc} \) in \( G \).

Proof. We know that there exists a polygon \( P_{abc} \subseteq N'(u,v) \cup L \). The bridge \( B \) of \( N'(u,v) \cup L \) containing \( L' \cup A \) is a type III bridge of \( P_{abc} \) in \( G \), by the last proposition. Therefore the conclusion of this proposition follows from proposition 2.5.4.

Remark. By the algorithm of 2.5.4 the polygon \( Q_{abc} \) of this proposition and \( P_{abc} \) of the last proposition have a common segment

\[ Q_{abc}'(u,v) = P_{abc}'(u,v) \subseteq N'(u,v). \]

Proposition 3.3.1.3. Suppose the statement of proposition 3.3.1.2 holds. Let \( B_i \) be the bridge of the polygon \( P_i = (P_i)_{abc} \) in \( G \) with \( B_i \supseteq B \), obtained at the \( i \)th stage in the process of clearing all type I bridges of \( P_{abc} = P_1 \) skew to \( B = B_1 \). Also suppose that we obtain \( Q_{abc} = P_n \) and \( B_n \) with \( B_n \supseteq B_n \) at the terminal stage \( n \) of the process. Then there exists a \( \theta \)-graph \( \theta_i \subseteq C \), such that

1. the polygon \( Q_i \) of \( \theta_i \) containing \( u \) and \( v \) is the union of two \( u, v \) arcs \( L_i, L' \) with \( L_i \subseteq P_i, L' \subseteq B_i \), and
(2) the branch $A_i$ of $\theta_i$ not in $Q_i$ contains $A$, is contained in $B_i$, and has endvertices $s_i$ and $t$ internal to $L_i$ and $L'$, respectively.

Proof. For $i = 1$, we write $P_i = P_{abc}, L_i = L, A_i = A$. By construction, $L_i \subseteq P_i$, $L' \subseteq B_i = B, Q_i = L_i \cup L'$, and $A_i$ has endvertices $s_1 = s, t$, as desired. Also, if $B_i'$ is a bridge of $N'(u, v) \cup L_i, B_i' \neq B_i$, then $B_i'$ is a bridge of $L_i$ in $G$, by proposition 3.3.2.1. Again $B_i'$ has a vertex of attachment $x_1$ in $G, x_1$ being an internal vertex of $L_i$.

Let the above comments be valid at the $(i-1)$th stage of our induction. Then we have $P_{i-1}, L_{i-1}, A_{i-1} = A, B_{i-1} = \beta, \theta_{i-1}, s_{i-1}$, and bridges $B_i'$ with vertices $x_{i-1}$ internal to $L_{i-1}$, etc., with the required properties.

Now two possibilities occur, namely,

(A1) there exists no $B_{i-1}'$ skew to $B_{i-1}$, and

(A2) some $B_{i-1}'$ exists such that $B_{i-1}'$ is skew to $B_{i-1}$.

In case (A1) the process terminates with $n = i - 1$. In case (A2), let $u_{i-1}$ and $v_{i-1}$ be the vertices of $B_{i-1}$ on $L_{i-1}$ nearest to $u$ and $v$, respectively. Then $u \leq u_{i-1} \leq x_{i-1} \leq v_{i-1} \leq v$ on $L_{i-1}$.

Let $X_{i-1}$ be any arc in $B_{i-1}$ joining the vertices $u_{i-1}$ and $v_{i-1}'$, and avoiding $L_{i-1}$.
We have now two possibilities in case (A2), namely

(A2.1) $B_{i-1}'$ is not skew to $L' \cup A_{i-1}'$, or

(A2.2) $B_{i-1}'$ is skew to $L' \cup A_{i-1}$.

In case (A2.1), we define $L_i = L_{i-1}(u, u_{i-1}) \cup X_{i-1} \cup L_{i-1}(v_{i-1}, u)$, $A_i = A_{i-1}, \theta_i, P_i, \text{ etc.}$, accordingly. Therefore there exists

$B_i \supset B_{i-1}$, by proposition 2.5.2. Hence $B_i \supset B$. In case (A2.2)

we have two subcases. They are

(A2.2.1) $x_{i-1} = u_{i-1}$ or $x_{i-1} = v_{i-1}$, and

(A2.2.2) $u_{i-1} < x_{i-1} < v_{i-1}$ on $L_{i-1}(u, v)$.

In case (A2.2.1) above, $v \neq v_{i-1}$ or $u \neq u_{i-1}$, according as

$x_{i-1} = u_{i-1}$ or $x_{i-1} = v_{i-1}$. We define $s_i = x_{i-1}$. Then we write

$L_i = L_{i-1}(u, u_{i-1}) \cup X_{i-1} \cup L_{i-1}(v_{i-1}, v)$, $A_i = A_{i-1} \cup L_{i-1}(s_{i-1}, x_{i-1})$.

Accordingly, we define $P_i, \theta_i, \text{ etc.}$ Also there exists a bridge $B_i$

of $P_i$ with $B_i \supset B_{i-1}$, by proposition 2.5.2. Hence $B_i \supset B$.

In case (A2.2.2) we have $u_{i-1} < x_{i-1} < v_{i-1}$. By proposition 1.3.1,

there exists an $L_{i-1} \cup X_{i-1}$-avoiding arc $Z_{i-1}$ from $x_{i-1}$ to an

internal vertex, say $s_i$, of $x_{i-1}$. We define $L_i = L_{i-1}(u, u_{i-1})$

$\cup X_{i-1} \cup L_{i-1}(v_{i-1}, v)$, and $A_i = A_{i-1} \cup L_{i-1}(s_{i-1}, x_{i-1}) \cup Z_{i-1}$.

Accordingly, we define $P_i, \theta_i, \text{ etc.}$ By proposition 2.5.2, there

exists a bridge $B_i$ of $P_i$, $B_i \supset B_{i-1}$. Hence $B_i \supset B$. 

Again, by construction, \( L' \cup A_1 \subseteq B_1 \) and \( A \subseteq A_1 \) in all cases above. This completes the proof of this proposition.

**Proposition 3.3.1.4.** Suppose the statement of proposition 3.3.1.3 holds. Then there exist two necklace decompositions \( D_1 = (N_1', R_1') \), \( D_2 = (N_2', R_2') \) of \( G \) with \( N_1'(u, v) \cup N_2'(u, v) \subseteq C \), where \( D_1 \) and \( D_2 \) are of the same type as \( D \). Also there exists an arc \( F \) in \( C \), avoiding \( N_1'(u, v) \cup N_2'(u, v) \) and containing \( w \) as an internal vertex which joins vertices \( s', t' \) where \( s' \in W(G, N_1'(u,v)) \backslash \{u,v\} \) and \( t' \in W(G, N_2'(u,v)) \backslash \{u,v\} \).

Proof. By proposition 3.3.1.3, \( P_n = Q_{abc} \) and therefore there exist \( L_n, L', \theta_n, B_n, \) etc., having the properties as described there. Therefore no bridge \( B'_n \) of \( Q_{abc} \) distinct from \( B_n \) is skew to \( B_n \). Let \( H' \) be the union of all bridges \( B'_n \) of \( Q_{abc} \) distinct from \( B_n \) and define \( H = L_n \cup L' \cup H' \cup A_n \). Then \( H \) is nonseparable, by proposition 1.4.1, since \( L_n \cup L' = P \) is a polygon. The vertices \( u, v, t \) and \( s_n \) are on \( P \) and are distinct.

Let \( H_1 \) be a bridge of \( P \) in \( H \). Then \( H_1 \) cannot have vertices of attachment in both of \( L_n(u, s_n) \backslash \{s_n\} \) and \( L_n(s_n, v) \backslash \{s_n\} \). For otherwise there would exist a bridge of \( Q_{abc} \) skew to \( B_n \). Thus \( W(H, H_1) \subseteq \{x, y\} \), for some \( \{x, y\} \subseteq \{u, v, s_n\} \). Hence, by proposition 2.1.4, \( H \) is a necklace with \( u, v, s_n \) vertices of attachment.
of some beads in $H$.

Let $N_1(u, v)$ be the segment of $H$ defined by $u, v$ complementary to $L'$. We define

$$N_1 = N'(u, v) \cup N_1(u, v)$$

and

$$R_1 = B_n.$$ 

Clearly $G = N_1 \cup R_1$. $N_1$ is a necklace with at least two beads in $N_1(u, v)$, because $s_n$ is incident with an arc $A_n \subseteq R$ which avoids $N_1(u, v)$. Therefore $s_n \in W(G, R_1)$. Also, the vertices of attachment of $R_1$ in $G$ contained in $N_1'(u, v)$ are the same as the vertices of attachment of $R$ in $G$ contained in $N'(u, v)$. Hence $R_1$ separates $a, b, c$. Therefore $D_1 = (N_1, R_1)$ is a necklace decomposition of the same type as $D$, by the uniqueness of necklace decompositions of alternative types, as obtained in theorem 1.

We now repeat the above argument, with $L'$ in place of $L$, $L_n$ in place of $L'$, $A_n$ in place of $A$, and the bridge $J$ of $N'(u, v) \cup L'$ which contains $L_n \cup A_n$ in place of $B$. Note that $N_1(u, v) \subseteq J$ with $N_1(u, v) \cap L' = [u, v]$. After repeating the above algorithm to its termination, a necklace decomposition $D_2 = (N_2, R_2)$ of $G$, of the same type as $D$ is obtained. Also $N_2 = N'(u, v)$ and $R_2$ is the bridge produced from $J$ as $R_1$ was produced from $B$. In the process the arc $A_n$ lengthens into an arc $F$.
of C avoiding $N_1(u, v) \cup N_2(u, v)$ and containing the original arc A. The endvertices $s', t'$ of F are such that

$$s' \in W(G, N_1(u, v)) \setminus \{u, v\}$$

and

$$t' \in W(G, N_2(u, v)) \setminus \{u, v\}.$$ 

**Proposition 3.3.1.5.** Suppose G satisfies H(I). Let $D = (N, R)$ be a necklace decomposition of G for which case (3) of H(II) applies, and C be a partial cone of D with extremes u, v and apex w. Suppose $D_1 = (N_1, R_1)$ is a necklace decomposition of G such that $N_1(u, v) = N'(u, v)$ and $N_1(u, v) \subseteq C \setminus \{w\}$. Then $N_1(u, v)$ has exactly one bridge in C if and only if case (3) of H(II) applies to $D_1$.

**Proof.** Suppose $D_1$ satisfies case (3) of H(II), under the above hypotheses. Then $R_1$ is a bridge of $N_1$. If $N_1(u, v)$ has a bridge in C not containing w then this bridge is a bridge of $N_1$, distinct from $R_1$, which is impossible.

By hypotheses, $R$ is a bridge of N in G. Let $G'$ be the unique minimal $H \subseteq G$ such that $G = C \cup H$. Let $R' = G' \cap R$. If a bridge of $N'(u, v)$ in $G'$ does not contain w then it is a bridge of N distinct from $R$, which is impossible. Thus $N'(u, v)$ has a single bridge in $G'$ and this must be $R'$. If $N_1(u, v)$ has a single bridge $B$ in C then $B \cup R' = R_1$, and hence $R_1$ is a
bridge of $N_1$ in $G$. Since $R_1$ has the same vertices of attachment in $N'(u, v)$ as $R$, this implies that case (3) of $H(II)$ applies to $D_1$. The proof is now complete.

Continuation of algorithm 3.3.1.

By propositions 3.3.1.1 to 3.3.1.4 we produce necklace decompositions $D_1 = (N_1, R_1)$ and $D_2 = (N_2, R_2)$ of the same type as $D = (N, R)$. Moreover, by 3.3.1.5, the segments $N_1(u, v)$ and $N_2(u, v)$ have one bridge only in $C$.

Let $N_C = N_1(u, v) \cup N_2(u, v)$ and $R_C$ be the smallest $H \subseteq G$ such that $N_C \cup H = G$. Then $(N_C, R_C)$ is a necklace decomposition of $G$ with $N_C \subseteq C$, and hence $N_C$ is a rim of $C$, by definition.

Proposition 3.3.2. Suppose $C$ is a cone with extremes $u$, $v$ and apex $w$, relative to a given necklace decomposition $D = (N, R)$ of $G$. Then there exists a pair of necklace decompositions $D_1 = (N_1, R_1)$ and $D_2 = (N_2, R_2)$ of $G$, of the same type as $D$, such that

1. $N_1 \cup N_2 = (N_1 \cap N_2) \cup ((N_1 \cup N_2) \cap C),
2. [u, v] = (N_1 \cap N_2) \cap ((N_1 \cup N_2) \cap C),

where $N_1 \cap N_2 = N'(u, v)$ and $(N_1 \cup N_2) \cap C$ is a rim of the cone $C$.

Proof. A cone $C$ is a partial cone of $G$, therefore, by algorithm 3.3.1, there exists a rim $N_C$ of $C$ with the properties stated above.
Proposition 3.3.3. Suppose there exists a partial cone $C$ with respect to a necklace decomposition $D = (N, R)$ of $G$ under the hypotheses $H(I)$ and case (3) of $H(II)$. Let $u, v$ be the extremes and $w$ the apex of $C$. Then $E(C) \subseteq E_{abc}^2$.

Proof. Let $N_C$ be a rim of $C$ and $e \in E(C)$. If $e \in E(N_C)$ then $e \in E_{abc}$. If $e \in E(C) \setminus E(N_C)$ then we claim that $e \in E_{abc}$. To prove this claim choose an edge $e' \in E(N_C)$. Then there exists a polygon $P$ in $C$ containing $e$ and $e'$, since $C$ is nonseparable. Now follow $P$ in both directions from $e$ until vertices on $N_C$ are uncountered for the first time. These vertices are distinct. The segment of $P$ containing $e$, thus determined, can clearly be extended to a polygon $P_{abc}$ containing $e$, establishing the claim.

Now $D = (N, R)$ and $N_C$ give rise to two necklace decompositions $D_1 = (N_1, R_1)$ and $D_2 = (N_2, R_2)$ of $G$.

If $e \notin E(N_C)$ then $e \in E(R_1 \cap C)$. If $e \in E(N_C)$ then $e \in E(R_1 \cap C)$ or $E(R_2 \cap C)$. Therefore in all these case $e \in E_{abc}^2$, by proposition 2.7.2. Thus $E(C) \subseteq E_{abc}^2$. This completes the proof of the proposition.

3.1 Relations between the necklaces of distinct necklace decompositions.

Suppose $H(I)$ and case (3) of $H(II)$ hold. Also, suppose that there exists a pair of skew diagonals of $N$, but no such pair can
separate a, b, c. From now on we will denote the above three conditions together by $H(\text{III}).$

By the results of the last section, $H(\text{III})$ implies that we have at least two distinct necklace decompositions $D_1 = (N_1, R_1)$ and $D_2 = (N_2, R_2)$ of $G$ each of which satisfies $H(\text{III}).$

The main result of this section can then be summed up as follows:

If $H(\text{III})$ holds for $D_1$ and $D_2$ then all the $N_1$-bridges of $N_1 \cup N_2$ are contained in cones obtained with respect to the necklace decomposition $D_1.$

**Proposition 3.4.1.** Suppose $D_i = (N_i, R_i)$ for $i = 1, 2,$ are two distinct necklace decompositions of $G$ under hypothesis $H(\text{III}).$

For $i = 1, 2,$ let $S_i$ be a bead of $N_i$ chosen so that $E(S_i) \cap E(S_2) \neq \emptyset.$ Then $S_1 = S_2.$

To prove this proposition, we first derive the following propositions. In these propositions we will assume that

$W(G, S_i) = \{x_i, y_i\},$ for $i = 1, 2,$

and $V_1$ is the set of vertices of attachment of beads of $N_1.$

**Proposition 3.4.1.1.** Suppose that the hypotheses of proposition 3.4.1 hold and that $S_1 \subset S_2.$ Then $\{x_2, y_2\} \subset V_1.$

If $S_1 = S_2$ then the conclusion is obvious. Suppose $S_1 \subsetneq S_2.$
Proof. $S_2$ contains at most one edge of $\{a, b, c\}$ since it is a bead of $N_2$. This implies that $N_1 \not\subseteq S_2$ and hence every polygon $P_{abc}$ in $G$ containing an edge $e \in E(S_1)$ must contain both the vertices $x_2, y_2$. Therefore $x_2, y_2 \in V(N_1)$ because such a $P_{abc}$ contained in $N_1$ exists.

We claim that $\{x_2, y_2\} \subseteq V_1$. To prove this by contradiction, let us suppose that $x_2 \notin V(N_1) \setminus V_1$. Then $x_2$ belongs to a unique bead of $N_1$, say $S_1'$, by proposition 1.6.6. Also $x_2 \notin W(G, S_1')$, since $x_2 \notin V_1$.

If $E(S_2) \setminus E(S_1') = \emptyset$ then $E(S_2) \subseteq E(S_1')$. However $E(S_1) \not\subseteq E(S_2)$, by hypothesis. Thus $E(S_1) \not\subseteq E(S_1')$, which is impossible for two beads of $N_1$. Hence $E(S_2) \setminus E(S_1') \neq \emptyset$.

Next, if $E(S_1' \cap S_2) = \emptyset$ then $x_2 \in W(G, S_1')$, a contradiction. Therefore $E(S_1' \cap S_2) \neq \emptyset$. Finally, if $E(S_1' \setminus E(S_2)) = \emptyset$ then $S_1' \subseteq S_2$. But $S_1' \subseteq S_2$, $x_2 \in W(G, S_2)$ and $x_2 \in V(S_1')$ imply that $x_2 \in W(G, S_1')$, which is a contradiction. Hence $E(S_1' \setminus E(S_2)) \neq \emptyset$.

Therefore $S_1' \subseteq S_2$. But $S_1' \subseteq S_2$, $x_2 \in W(G, S_2)$ and $x_2 \in V(S_1')$ imply that $x_2 \in W(G, S_1')$, which is a contradiction. Hence $E(S_1' \setminus E(S_2)) \neq \emptyset$.

Therefore $S_1' \subseteq S_2, S_2 \not\subseteq S_1', S_1' \cap S_2 \neq \emptyset$ and $S_1' \cup S_2 \not\subseteq G$. So, by proposition 2.2.1, $W(G, S_1') = W(G, S_2) = \{x_2, y_2\}$, implying that $x_2 \in W(G, S_1') \subseteq V_1$, which is contrary to our assumption. Thus we have established the claim. The same argument applies to $y_2$. The
Proposition 3.4.1.2. Suppose $D_1 = (N_1, R_1)$ is a necklace decomposition of $G$ and $H_1, H_2 \subseteq G$ are such that $G = H_1 \cup H_2$ and $H_1 \cap H_2 = [x, y]$, where $x, y \in W(G, N_1)$. If $Z$ is an $[x, y]$-segment of $N_1$ with at least two beads, and $Z \cap H_1 \subseteq [x, y]$ then $Z \subseteq H_1$ and $Z \cap H_2 = [x, y]$.

Proof. A bead of $N_1$ in $Z$ can meet $[x, y]$ in at most one vertex, and is nonseparable, hence is contained in $H_1$ or $H_2$. If $Z \cap H_1 \subseteq [x, y]$ there is an arc $L$ of $Z$ joining $x$ and $y$ with $L \cap H_1 \subseteq [x, y]$. Thus $L \subseteq H_1$, and all beads of $Z$ are in $H_1$, so that $Z \subseteq H_1$. As $[x, y] \subseteq Z \cap H_1 \subseteq H_2 \cap H_1 = [x, y]$, the last assertion is valid.

Proposition 3.4.1.3. Suppose the statement of proposition 3.4.1.1 holds. Then $S_1 = S_2$.

Proof. First we will prove that $\{x_1, y_1\} = \{x_2, y_2\}$. If possible, let $\{x_1, y_1\} \neq \{x_2, y_2\}$, so that $S_1 \neq S_2$. We can write $\{x_2, y_2\} \subseteq V_1$, by proposition 3.4.1.1.

Evidently $N_1$ contains at least two beads, by its definition.

Write $G = H_1 \cup H_2$, where $H_2 = S_2$ and $H_1 \cap H_2 = [x_2, y_2]$. At least two of the edges $a, b, c$ belong to $H_1$. By proposition 3.4.1.2,
it follows that \( N'_1(x_2, y_2) \subseteq H_1 \) and \( N'_1(x_2, y_2) \cap S_2 = [x_2, y_2] \).

But \( S_1 \subset S_2 \), and therefore \( S_1 \) is a bead of \( N_1(x_2, y_2) \). Also \( \{x_1, y_1\} \neq \{x_2, y_2\} \) implies that \( N_1(x_2, y_2) \) has at least two beads of \( N_1 \). So, by proposition 3.4.1.2, we get \( N_1(x_2, y_2) \subseteq S_2 \).

We have now a separable graph \( N_1(x_2, y_2) \) contained in a non-separable graph \( S_2 \). Therefore \( N_1(x_2, y_2) \subset S_2 \). Thus there exists a bridge \( B \) of \( N_1(x_2, y_2) \) in \( S_2 \). Write \( F = N_1 \) and \( C = S_2 \).

Then \( W(G, C) = W(G, S_2) \subseteq V(N_1) = V(F) \). Thus, by proposition 3.2.1, \( B \) is a bridge of \( N_1 \) in \( G \) distinct from \( R \), contrary to the definition of \( D_1 \). Hence \( \{x_1, y_1\} = \{x_2, y_2\} \).

By hypothesis \( S_1 \subseteq S_2 \). If \( S_1 \subseteq S_2 \), we repeat the above argument, with \( N_1(x_2, y_2) = N_1(x_1, y_1) = S_1 \subseteq S_2 \), and arrive at a contradiction. The proposition follows.

**Proof of proposition 3.4.1.** Since \( a, b, c \) belong to distinct beads of any necklace under our consideration, we find that at most two members of \( \{a, b, c\} \) can belong to \( S_1 \cup S_2 \). Therefore \( S_1 \cup S_2 \subset G \).

Also \( E(S_1) \cap E(S_2) \neq \emptyset \), by hypothesis. Hence, by proposition 2.2.1, exactly one of the following occurs, namely,

1. \( S_1 \subset S_2 \),
2. \( S_2 \subset S_1 \),
3. \( E(S_2) \setminus E(S_1) \neq \emptyset \neq E(S_1) \setminus E(S_2) \), \( W(G, S_1) = W(G, S_2) = W(G, S_1 \cup S_2) = W(G, S_1 \cap S_2) \), and \( S_1 \cap S_2 \) is connected.
In the last case \( \{x^1, y^1\} = \{x^2, y^2\} = W(G, S_1) = W(G, S_2) \).

Let \( H_2 = G \cdot (E(S_2) \setminus E(S_1)) \) and \( G = H_1 \cup H_2 \), where \( H_1 \cap H_2 = [x^1, y^1] \). Then \( H_2 \) has at least one edge, and hence \( |W(G, H_2)| \geq 2 \), since \( G \) is nonseparable. On the other hand, \( W(G, H_2) \subseteq \{x^1, y^1\} \) by condition (3) above. This implies that \( W(G, H_2) = \{x^1, y^1\} \). Now \( N_1(x^1, y^1) = S_1 \subseteq H_1 \), by definition of \( H_2 \). Also \( N'_1(x^1, y^1) \cap H_1 \cap [x^1, y^1] \), and so, by proposition 3.4.1.2,

\[ N'_1(x^1, y^1) \subseteq H_1. \]

Therefore \( N_1 \subseteq H_1 \). But \( H_2 \) is a union of bridges of \( [x^1, y^1] \) in \( G \), and hence a union of bridges of \( N_1 \) in \( G \).

This is impossible as \( H_2 \not\subseteq R_1 \).

The above contradiction removes the last possibility altogether.

We then revert to the first two cases and conclude that \( S_1 = S_2 \), by proposition 3.4.1.3. The proof of the proposition is now complete.

**Proposition 3.4.2.** Suppose \( D_i = (N_i, R_i) \), for \( i = 1, 2 \), are two distinct necklace decompositions of \( G \) under the hypothesis \( H(III) \). Also, suppose that \( S_i \) is a bead of \( N_i \), for \( i = 1, 2 \). Then either \( S_1 \) and \( S_2 \) have at most one vertex in common or they are the same.

**Proof.** As in the last proposition, \( S_1 \cup S_2 \) can have at most two of the three edges from \( a, b, c \). Therefore \( S_1 \cup S_2 \subseteq G \). Two possibilities now occur, namely

1. \( E(S_1) \cap E(S_2) = \emptyset \) and
2. \( E(S_1) \cap E(S_2) \neq \emptyset \).
In case (1), there exists \( x, y \in W(G, S_1) \cup W(G, S_2) \) such that 
\[ S_1 \cap S_2 \subseteq [x, y]. \] If \( S_1 \cap S_2 = [x, y] \) then \( W(G, S_1) = W(G, S_2) \).

Apply 3.4.1.2 to \( N_1'(x, y) \), with \( S_2 = H_2 \). We conclude that 
\[ N_1'(x, y) \subseteq H_1, \] since \( S_2 \) contains at most one of \( a, b, c \). Also 
\[ N_1(x, y) = S_1 \subseteq H_1, \] since \( S_1 \cap S_2 = [x, y] \). Hence \( N_1 \subseteq H_1 \) and 
\[ S_2 \subseteq R_1. \] Now \( S_2 \) is a union of bridges of \( N_1 \) in \( G \) because 
\[ S_2 \cap N_1 = [x, y]. \] This forces \( S_2 = R_1 \), which is contrary to the 
definition \( D_1 \). Alternatively, \( S_1 \cap S_2 \subseteq [x, y] \) and \( S_1, S_2 \) have 
at most one vertex in common.

In case (2) we apply proposition 3.4.1 and obtain \( S_1 = S_2 \).

This establishes the proposition.

**Proposition 3.4.3.** Suppose \( D_i = (N_i, R_i), \) for \( i = 1, 2 \) are 
distinct necklace decompositions of \( G \) under hypothesis \( H(III) \).
Then every \( N_1 \)-bridge of \( N_1 \cup N_2 \) is a segment \( N_2(u, v) \), for some 
distinct vertices \( u, v \in W(N_1 \cup N_2), N_2 \).

**Remark.** When we consider necklace decomposition \( D_1 \) and \( D_2 \), as in 
proposition 3.4.3, we will in what follows denote by \( X \) the set of 
vertices of \( N_1 \cup N_2 \) contained in three or four beads of \( N_1 \) and 
\( N_2 \). It is easily verified that \( X = W(N_1 \cup N_2), N_1) = W(N_1 \cup N_2), N_2). \)

Before proving proposition 3.4.3 we prove the following propositions.
**Proposition 3.4.3.1.** Suppose the hypotheses of proposition 3.4.3 hold. Then an \([X]\)-bridge of \(N_1\) is an \([X]\)-bridge of \(N_1 \cup N_2\), for \(i = 1, 2\).

Proof. Let \(B\) be an \([X]\)-bridge of \(N_1\). We write \(N_1 \cup N_2, N_1, [X]\), \(B\) for \(G, C, F\) and \(B\) of proposition 3.2.1, respectively. Then \(W(N_1 \cup N_2, N_1) \subseteq X\) and \([X] \cap N_1 = [X]\). This implies that \(B\) is an \([X \cap N_1]\)-bridge in \(N_1\). Therefore, by proposition 3.2.1, \(B\) is an \([X]\)-bridge of \(N_1 \cup N_2\) contained in \(N_1\). This completes the proof.

**Proposition 3.4.3.2.** Assume that the hypotheses of proposition 3.4.3 hold. Then the set of \([X]\)-bridges of \(N_1 \cup N_2\) is the union of the set of \([X]\)-bridges of \(N_1\) with the set of \([X]\)-bridges of \(N_2\).

Proof. By the last proposition each \([X]\)-bridge of \(N_1\) and each \([X]\)-bridge of \(N_2\) is an \([X]\)-bridge of \(N_1 \cup N_2\). However \(N_1 \cup N_2\) is clearly the union of all these bridges, and hence no other \([X]\)-bridge of \(N_1 \cup N_2\) exists.

**Proposition 3.4.3.3.** Assume that the hypotheses of proposition 3.4.3 hold. Then the connected components of \(N_1 \cap N_2\) are the \([X]\)-bridges of \(N_1 \cup N_2\) contained in both \(N_1\) and \(N_2\).

Proof. Suppose \(B\) and \(B'\) are distinct \([X]\)-bridges of both \(N_1\) and \(N_2\). Then \(B \cap B' \subseteq [X]\). Assume that there exists \(x \in V(B \cap B')\). Then \(x \in X\) and \(x\) is contained in a bead of \(B\) and a bead of \(B'\),
which are distinct. Also these beads are in both $N_1$ and $N_2$ and hence no other bead of $N_1$ or $N_2$ contains $x$. This contradicts the definition of $X$. We conclude that $B \cap B' = \emptyset$, i.e. the common $[X]$-bridges of $N_1$ and $N_2$ are pairwise disjoint.

We claim that $N_1 \cap N_2$ is the union of such bridges. Let $e \in E(N_1 \cap N_2)$. Then $e \in E(N_1)$ and $e \in E(N_2)$, so that $e$ is in an $[X]$-bridge $B_1$ of $N_1$ and an $[X]$-bridge $B_2$ of $N_2$. But $B_1$, $B_2$ are also $[X]$-bridges of $N_1 \cup N_2$ and have an edge in common. This implies that $B_1 = B_2$. The proposition now follows.

**Proposition 3.4.3.** Assume that the hypotheses of proposition 3.4.3 hold. Then an $[X]$-bridge $B$ of $N_1 \cup N_2$ which is an $[X]$-bridge of $N_1$ and not an $[X]$-bridge of $N_2$ is an $N_2$-bridge of $N_1 \cup N_2$, and all such bridges arise in this way.

**Proof.** We know that $N_2$ is the union of $[X]$-bridges of $N_1 \cup N_2$ which are contained in $N_2$. Thus $B \cap N_2 \subseteq [X]$, because $B$ is an $[X]$-bridge of $N_1 \cup N_2$ and not in $N_2$.

We then apply proposition 3.2.2 with $G = N_1$, $C = B$, $F = [X]$, and $B = B$. Then $W(G, B) \subseteq X$ and $B$ is an $[X]$-bridge of $N_1$ imply that $B$ is a $B \cap [X]$-bridge of $B$. Again $B \cap N_2 = B \cap [X]$, since $B \cap N_2 \subseteq [X]$. Hence $B$ is a $B \cap N_2$-bridge of $N_1$ contained in $B$, by proposition 3.2.2.
Next we write $N_1 \cup N_2$, $B$, $N_2$ and $B$, $G$, $C$, $F$, $B$ for as in proposition 3.2.1. Then $W(N_1 \cup N_2, B) \subseteq X$ and $B$ is a $N_2$-bridge of $B$ imply that $B$ is an $N_2$-bridge of $N_1 \cup N_2$.

$N_1 \cup N_2$ is the union of $N_2$ with the $[X]$-bridges of $N_1$, which are $[X]$-bridges of $N_2$ and not $[X]$-bridges of $N_1$. Therefore, all $N_2$-bridges of $N_1 \cup N_2$ are of this type. This completes the proof.

Proof of proposition 3.4.3. By proposition 3.4.3, any $N_1$-bridge $B$ is an $[X]$-bridge of $N_1 \cup N_2$ contained in $N_2$. Hence $B$ is an $[X]$-bridge of $N_2$. This implies, by proposition 3.6.7, that $B = N_2(x, y)$, for some $x, y \in X$. As $a, b, c \in E(N_1 \cap N_2)$ we know that $B \neq N_2'(x, y)$. The proof is now complete.

Proposition 3.4.4. Suppose $D_i = (N_i, R_i)$, for $i = 1, 2$, are distinct necklace decompositions of $G$ under hypothesis $H(III)$. Then every $N_1$-bridge of $N_1 \cup N_2$ is contained in a cone of $D_i$.

Moreover, no such bridge contains the apex of the cone.

In order to facilitate the proof of the above proposition we first prove the results stated below.

Proposition 3.4.4.1. Suppose the hypotheses of proposition 3.4.4 hold.

Let $C$ be a cone of $D_1$, where $u, v$ are the extremes and $w$ the apex of $C$. Then at most two of $u, v, w$ belong to $V(N_2)$. 
Proof. Assume $u, v, w \in V(N_z)$. We then claim that there exists a bridge $B$ of $N_z \cap C$ in $C$.

If not, then $C \subseteq N_z$. Then $N_1(u, v) \subseteq N_z$ and, by proposition 3.2.3, it follows that $N_1(u, v) = N_z(u, v)$. However, for cones there exists $x \in W(G, N_1(u, v)) \setminus \{u, v\}$, and thus an edge $e \in E(C) \setminus E(N_1(u, v))$ incident with $x$ but not belonging to $N_z$. This is contrary to $C \subseteq N_z$, and so the above claim is justified.

By proposition 3.2.1, $B$ is a bridge of $N_z$ in $G$, which implies $B = R_z$. However, either $t_{ab}$ or $t_{ac} \notin V(B)$, as $B \subseteq C$, although $t_{ab}, t_{ac} \in V(R_z)$. This contradiction proves the proposition.

**Proposition 3.4.4.2.** Suppose the hypotheses of proposition 3.4.4 hold. Let $C$ be a cone of $D_1$ such that $C \cap N_z \notin \{W(G, C)\}$. Then we can write

$$W(G, C) = \{x, y, z\}, \text{ where }$$

$$N_z(x, y) \subseteq C \setminus \{z\} \text{ and }$$

$$N_z(x, y) \cap C = [x, y].$$

Proof. We can write $W(G, C) = \{x, y, z\}$, where $z \notin V(N_z)$, by proposition 3.4.4.1. Let $B$ be any $[x, y, z]$-bridge of $N_z \cup [x, y, z]$. Then proposition 3.2.2 can be applied to $B$, with $G, C, F$ replaced by $N_z \cup [x, y, z], C \cap (N_z \cup [x, y, z]), [x, y, z]$, respectively.
Now $B \subseteq N_2$ and $B$ is a $B \cap [x, y]$-bridge of $N_2$. Thus $B = N_2$ or $x, y \in W(G, N_2)$ and $B$ is one of the two $[x, y]$-segments of $N_2$. Evidently

$$W(N_2 \cup [x, y, z]) = W(G, C) = V(x, y, z) = \{x, y, z\}.$$

The set inequality above can be easily verified from the definition of vertices of attachment.

Now $C \cap N_2 \subseteq [x, y, z]$, by hypothesis, and $N_2$ is a union of bridges $B$ as defined above. Thus $B$ can be chosen such that $C \cap B \subseteq [x, y, z]$. Then we have $$(C \cap (N_2 \cup [x, y, z])) \cap B = C \cap B \subseteq [x, y, z].$$ By proposition 3.2.2, $B$ is an $[x, y, z]$-bridge of $C \cap (N_2 \cup [x, y, z])$. In particular $B \subseteq C$. As $a, b, c \notin E(C)$, it follows that $B = N_2(x, y)$ and $N_2'(x, y) \cap C \subseteq [x, y, z]$. This implies that $N_2'(x, y) \cap C = [x, y]$, since $z \notin V(N_2)$, and $N_2(x, y) \subseteq C \setminus \{z\}$.

**Proposition 3.4.4.** Suppose the hypotheses of proposition 3.4.4 apply. Let $N_2(x, y)$ be an $N_1$-bridge of $N_1 \cup N_2$, for $x, y \in W(N_1 \cup N_2, N_2')$. Then the following statements are true.

1. $N_1(x, y) \nsubseteq N_2$
2. Let $w \in V(N_2(x, y)) \setminus \{x, y\}$ be such that $N_2(x, w)$ is contained
in a cone \( C \) of \( D_1 \) with \( W(G, C) = \{u, v, w\} \), where \( u \leq x \leq v \leq y \) on \( N_1(x, y) \). Then \( N_1(v, y) \subseteq N_2 \).

(3) If there exist \( w_1, w_2 \in V(N_2(x, y) \backslash \{x, y\}) \), and cones \( C_1, C_2 \) such that

\[
\begin{align*}
(\text{i}) & \quad W(G, C_i) = \{u_i, v_i, w_i\}, \text{ for } i = 1, 2, \\
(\text{ii}) & \quad N_2(x, w_1) \subseteq C_1 \\
(\text{iii}) & \quad N_2(w_2, y) \subseteq C_2, \text{ and} \\
(\text{iv}) & \quad u_1 \leq x \leq v_1 \leq u_2 \leq y < v_2 \text{ on } N_1(u_1, v_2), \text{ then} \\
& \quad N_1(v_1, u_2) \subseteq N_2.
\end{align*}
\]

Proof. Notice that a union of segments of \( N_2 \) which forms a necklace must be \( N_2 \) itself. Thus if we assume to the contrary in each of the cases mentioned above, we find that \( N_2 \) equals \( N_1(x, y) \cup N_2(x, y) \) in case (1), \( N_2(w, v) \cup N_1(v, y) \cup N_2(y, w) \) in case (2), and either

\[
N_2(w_1, v_1) \cup N_1(v_1, u_2) \cup N_2(u_2, w_2) \cup N_2(w_2, v_1), \text{ or}
\]

\[
N_2(w_1, v_1) \cup N_2(u_2, w_2) \cup N_2(v_2, w_1)
\]

in case (3). In each case above, one of \( a, b, c \) is not contained in \( N_2 \), contrary to the definition of \( D_2 \).

Remark. The above argument applies also to \( N_2'(x, y) \) when \( N_2(x, y) \) contains one of the residual segments \( T_a, T_b, T_c \).
Remark. In case (2) of the above proposition \( v = y \) is ruled out by the conclusion. In case (3) the subcases \( v_1 \neq u_2 \) and \( v_1 = u_2 \) will be treated slightly differently when the proposition is applied.

**Proposition 3.4.4.** Assume the hypotheses of proposition 3.4.4.

Then each \( N_1 \)-bridge of \( N_1 \cup N_2 \) is the union of two or more residual segments of \( N_2 \).

**Proof.** Any \( N_1 \)-bridge of \( N_1 \cup N_2 \) is of the form \( N_2(x, y) \), for some \( x, y \in W(N_1 \cup N_2, N_1) \), by proposition 3.4.3. But \( W(N_1 \cup N_2, N_1) \subseteq W(G, N_1) \), and hence \( N_2(x, y) \) is a union of residual segments of \( N_2 \).

If \( N_2(x, y) \) is just one residual segment of \( N_2 \) then 

\[
W(G, N_2(x, y)) = [x, y] \subseteq W(G, N_1).
\]

As \( N_2(x, y) \subseteq R_1 \), it follows that \( N_2(x, y) \) is a bridge of \( N_1 \) in \( G \), so that \( N_2(x, y) = R_1 \). This is contrary to \( |W(G, R_1)| > 4 \), a part of our hypothesis.

**Remark.** Each \( N_2 \)-bridge of \( N_1 \cup N_2 \) is the union of two or more residual segments of \( N_1 \).

**Proposition 3.4.5.** Assume the hypotheses of proposition 3.4.4 apply. Let \( N_2(x, y) \) be an \( N_1 \)-bridge of \( N_1 \cup N_2 \), with \( x, y \in W(N_1 \cup N_2, N_1) \). Suppose that the residual segments \( T_b, T_c \) are in
Let \( N'(x, y) \), and \( x < y \), \( t_{ab} \leq y \leq t_{ba} \) on \( N_1(t_{bc}, t_{ba}) \). Then there exist \( s' \in W(G, N_1(x, y)) \setminus V(N_2) \) and \( t' \in (W(G, N_2) \cap V(N_1'(x, y))) \setminus \{x, y\} \), together with an \( N_2 \)-avoiding arc \( M' \) in \( R_2 \) joining any such pair of vertices \( s', t' \). Moreover, there exists a segment \( M \) of \( M' \), \( M \subseteq R_1 \), avoiding \( N_1 \) and joining vertices \( s, t \) such that \( s \in V(N_1(x, y)) \setminus \{x, y\} \) and \( t \in V(N_1'(x, y)) \setminus \{x, y\} \).

Proof. The segment \( N_1(x, y) \) is not contained in \( N_2 \), by case (1) of proposition 3.4.4.3. This implies that there exists an \( N_2 \)-bridge of \( N_1 \cup N_2 \) contained in \( N_1(x, y) \). By proposition 3.4.4.4, such a bridge is the union of two or more residual segments of \( N_1 \). Also \( N_2 \)-bridges of \( N_1 \cup N_2 \) are contained in \( R_2 \). Hence there exists a vertex \( s' \in (V(R_2) \setminus V(N_2)) \cap W(G, N_1(x, y)) \) so that \( s' \in W(G, N_1(x, y)) \setminus V(N_2) \). Again \( t_{bc} \in W(G, N_2) \) and \( t_{bc} \in V(N_1'(x, y)) \setminus \{x, y\} \). So we can take \( t' = t_{bc} \). In general \( s', t' \in V(R_2) \), and therefore there exists an \( N_2 \)-avoiding arc \( M' \) from \( s' \) to \( t' \), where \( t' \) is the only vertex of \( N_2 \) on \( M' \).

Let us now consider the set of all arcs \( M'(s_1, t_1) \subseteq M' \), such that

\[
s_1 \in V(N_1(x, y)) \setminus \{x, y\} \quad \text{and} \quad \quad t_1 \in V(N_1'(x, y)) \setminus \{x, y\}.
\]
This set is a non-empty finite collection of arcs with
\([s_1, t_1] \cap \{x, y\} = \emptyset\), since \(M'\) is a member of this set. Hence
there exists one such arc, say \(M = M'(s, t)\), of minimal length.
None of the internal vertices of \(M\) are in \(N_1\). Therefore \(M\) is an
\(N_1\)-avoiding arc with the required properties. This completes the
proof of the proposition.

Proposition 3.4.4.6. Assume the statement of proposition 3.4.4.5.
Then exactly one of the following two alternatives occur.
(1) There exists a cone \(C\) of \(D_1\) such that \(N_2(x, y) \subseteq C\{w\}\),
where \(w\) is the apex of \(C\).
(2) There exists a cone \(C\) of \(D_1\) such that \(C \cap N_2(x, y)\)
\(\subseteq \{W(G, C)\}\), having apex \(w \in W(G, N_2(x, y))\{x, y\}\).
Proof. Let \(L\) be any arc joining \(x\) and \(y\) in \(N_2(x, y)\). Notice
that two possibilities occur according as \(x \in V(N_1(t_{ca}, t_{ac}))\) or
\(x \in V(N_1(t_{ab}, t_{ba}))\).

Suppose \(x \in V(N_1(t_{ca}, t_{ac}))\). We can assume, if necessary by
interchanging \(x\) and \(y\), and \(b\) and \(c\), respectively, that
\(s \in V(N_1(t_{ab}, y))\{y\}\). Then \(x < t_{ab} \leq s < y \leq t_{ba}\) on \(N_1(x, y)\).
Propositions 2.4.3 and 2.4.4 and the nonexistence of \(K_{abc}\) force \(t\)
to be in \(V(N_1(t_{ba}, y))\). Hence \(t \in V(N_1(t_{ba}, y))\{y\}\). Then \(\{L, M\}\)
defines a singularity of \(D_1\). Therefore there exists a cone \(C\) of
D₁ such that C ⊃ M, and w ∈ V(L)\{x, y}, by construction. Thus
C ∩ N₂(x, y) ⊆ [W(G, C)]. Using proposition 3.4.1.2, we can see that
w ∈ W(G, N₂(x, y)). Thus conclusion (2) is obtained.

Next, let x ∈ V(N₁(t_ab, t_ba)). Then t_ab < x < s < y < t_ba.
If t ∈ V(N₁(t_bc, t_ac)) then [L, M] defines a singularity of D₁
such that C ⊃ L and w ∈ V(M)\{t}. Now M ⊊ M' and M' avoids
N₂. Hence w ∉ V(N₂(x, y)). Therefore N₂(x, y) ⊆ C\{w}, and
conclusion (1) is obtained.

If t ∉ V(N₁(t_bc, t_ac)) then we can assume that t ∈
V(N₁(y, t_ba)\{y}, without loss of generality. Now, by proposition
1.3.1, there exists an M U N₁ U N₂(x, y)-avoiding arc X from t_bc
to a vertex q ∈ V(N₂(x, y) U M)\{s, t}. If q ∈ V(M), define
L₁ = L, M₁ = M(s, q) U X. Therefore {L₁, M₁} defines a singularity
of D₁. Hence there exists a cone C of D₁ such that C ⊃ L and
w ∈ V(M₁)\{s, t_bc}. Also M₁ avoids N₂, by the definitions of M'
and X. This implies N₂(x, y) ⊆ C\{w}, and conclusion (1) applies.

If on the other hand q ∈ V(L₁), in the last case, then define
L₁ = L(y, q) U X and M₁ = M. Then {L₁, M₁} defines a singularity
of D₁. Thus we obtain a cone C of D₁ containing M₁ with apex
w ∈ V(L₁)\{y, t_bc}. The apex w belongs to V(L)\{x, y}, for
otherwise C will have a vertex of attachment different from the
extremes and $w$. As $I(y, w) \subseteq C$, we have $C \cap N_2(x, y) \subseteq \{w(G, C)\}$. Hence $w \in \mathcal{W}(G, N_2(x, y)) \setminus \{x, y\}$, using proposition 3.4.4.2, and conclusion (2) is obtained. When both conclusions of this proposition apply, the two cones arising are distinct but not edge-disjoint, contrary to 3.2.3. This completes the proof of the proposition.

Proof of proposition 3.4.4. Let $N_2(x, y)$ be an $N_1$-bridge of $N_1 \cup N_2$. By definition $N_1(x, y)$ contains at most one of $a$, $b$, $c$. We can assume the hypotheses of proposition 3.4.5, without loss of generality, and hence also its conclusion.

Now, by proposition 3.4.6, there exists a cone $C$ of $D_1$ with apex $w$ such that either $N_2(x, y) \subseteq C \setminus \{w\}$ or $w \in \mathcal{W}(G, N_2) \setminus \{x, y\}$, with either $N_2(y, w) \subseteq C$ or $N_2(x, w) \subseteq C$. Also there can be at most two cones of $D_1$ with apex in $\mathcal{V}(N_2(x, y)) \setminus \{x, y\}$, containing, segments of $N_2$, by propositions 3.2.3 and 3.4.1.

Suppose $N_2(x, y) \nsubseteq C$, for any cone $C$ of $D_1$. Then we have the following two exclusive possibilities:

(A) There exists a cone $C$ of $D_1$ with apex $w$ such that

1. $w \in \mathcal{W}(G, N_2(x, y)) \setminus \{x, y\}$
2. $N_2(x, w) \subseteq C$ (without loss of generality), and
3. $N_2(w, y)$ does not meet any other cone of $D_1$ in an edge.

(B) There exist cones $C_1, C_2$ with $\mathcal{W}(G, C_1) = \{u_1, v_1, w_1\}$, for
1 = 1, 2, such that

1. \( w_1, w_2 \in W(G, N_2(x, y)) \setminus \{x, y\}, \)

2. \( N_2(x, w_1) \subseteq C_1, \)

3. \( N_2(w_2, y) \subseteq C_2, \)

4. \( u_1 \leq x \leq v_1 \leq u_2 \leq y \leq v_2 \) on \( N_1(u_1, v_2), \) and

5. \( x < w_1 \leq w_2 < y \) on \( N_2(x, y). \)

In case (A) above \( N_1(v, y) \nsubseteq N_2, \) by case (2) of proposition 3.4.4.3. This implies that \( N_1(v, y) \) contains an \( N_2 \)-bridge of \( N_1 \cup N_2. \) Hence, by proposition 3.4.4.4, there exists a vertex \( s' \in W(G, N_1(x, y)) \setminus \{x, y\} \) such that \( s' \in V(R_2 \setminus W(N_2)). \) Also \( t_{bc} \in W(G, N_2). \) Therefore there exists an \( N_2 \)-avoiding arc \( M' \) joining \( s' \) and \( t_{bc}, \) with \( t_{bc} \) being the only vertex of \( N_2 \) on \( M'. \) Again \( w \in V(C) \) and \( C \cap N_2(x, y) \nsubseteq \) \( [W(G, C)]. \) Hence either \( w \) and \( v \) belong to \( V(N_2) \) or \( w \) and \( u \in V(N_2), \) by proposition 3.4.4.1. So \( M' \) cannot have \( v \) and \( w \) or \( u \) and \( w \) as internal vertices, in the respective cases, since \( t_{bc} \) is the only vertex of \( N_2 \) on \( M'. \) Because of this \( M' \) cannot enter \( C \) even though \( v \) or \( u \) may belong to \( M', \) in the respective cases (see proposition 2.8.7.4).

By proposition 3.4.4.5, there exists \( M = M'(s, t), M \subseteq R_1, \)
\( \{s, t\} \cap \{x, y\} = \emptyset, \) with \( s \in V(N_1(x, y)) \) and \( t \in V(N_1(x, y)). \)
Assume $t_{ca} \leq x \leq t_{ac}$ on $N_{1}(t_{ca}, t_{ac})$, and $s \in V(N_{1}(t_{ab}, y))$. If $t \in V(N_{1}(t_{bc}, u))$ then there exists a bond $K_{abc}$, either by proposition 2.4.3 or by proposition 2.4.4. This contradicts our hypothesis. Hence $t \in V(N_{1}(y, t_{ba})) \setminus \{y\}$. Also $M$ avoids $N_{2}$, since $M \subseteq M'$. Therefore any arc $L \subseteq N_{2}(x, y)$, having end vertices $x, y$, together with $M$ define a singularity of $D_{1}$, spanning residual segments of $N_{1}$, incident with $y$. Thus, there exists a cone of $D_{1}$ containing $y$ and a part of $N_{2}(x, y)$ since $N_{2}(x, y)$ is not contained in any cone of $D_{1}$. But this is contrary to case (A).

Next assume $s \in V(N_{1}(v, t_{ac}))$ along with $t_{ca} \leq x \leq t_{ac}$ on $N_{1}(t_{ca}, t_{ac})$ along with $t_{ca} \leq x \leq t_{ac}$ on $N_{1}(t_{ca}, t_{ac})$. If $t \in V(N_{1}(t_{cb}, y))$ then there exists a bond $K_{abc}$, either by proposition 2.4.3 or by proposition 2.4.4. Therefore $t \in V(N_{1}(t_{ca}, u))$.

Now we have a singularity $\{L, M\}$, where $L$ is any arc in $N_{2}(x,y)$ joining $x$ and $y$. This singularity meets $C$ but is not engulfed by $C$, which is impossible.

Alternatively, $t_{ab} \leq x < s < y \leq t_{ba}$ on $N_{1}(t_{ab}, t_{ba})$. If $t \in V(N_{1}(t_{bc}, t_{ac}))$ then any arc $L \subseteq N_{2}(x, y)$ joining $x, y$ and $M$ form a singularity of $D_{1}$. This gives rise to a cone $C$ containing $L$ and hence $N_{2}(x, y)$. But this contradicts the assumption with which we have started the proof of this proposition. If
t ∈ V(N₁(t_{ab}, y)) \setminus \{y\}, then we get a cone containing N₂(x, y) or a new cone, by proposition 3.4.4.6, containing the vertex y. The first one contradicts the assumption with which we have started the proof, while the second one contradicts case (A).

So we assume that $t \in V(N₁(t_{ab}, u))$ along with $t_{ab} \leq x < s < y \leq t_{ba}$ on $N₁(t_{ab}, t_{ba})$. In this case, by proposition 1.3.1, there exists a vertex $q \in V(L \cup M) \setminus \{x, y, s, t\}$, and an $N₁ \cup L \cup M$-avoiding arc $X$ in $G$ joining $q$ and $t_{bc}$. If $q \in V(M)$ we obtain a singularity which implies a cone $C'$ containing $L$, and if $q \in V(L)$ we obtain a singularity which implies a cone $C'$ containing $M$. In either case $C \subset C'$, which is impossible. This completes case (A).

In case (B), we are assured of existence of $s'$, by case (3) of proposition 3.4.4.3, whether $v₁ = u₂$ or $v₁ \neq u₂$. Hence we obtain a contradiction, by arguments very similar to the ones given above.

The proposition now follows.

3.5 Uniqueness of bead-cone decompositions.

We know that, for a given necklace decomposition $D$ of $G$ under hypothesis $H(III)$, the cones are unique, since each of them is a maximal partial cone of $D$. In this section we will first prove that the set of cones of $G$ is independent of the necklace decomposition of $G$ used in their definition. This will imply that the essential
beads and cones of necklace decompositions determine a unique decompositions of $G$ into subgraphs, which we refer to as the bead-cone decomposition of $G$. Hence bead-cone decompositions of $G$ are unique under hypothesis $H(III)$. As $H(III)$ states the hypothesis of case (4) of theorem I, the above result, along with proposition 3.32, proves theorem II.

Proposition 3.5.1. Assume that $D_i = (N_i, R_i)$, for $i = 1, 2$, are distinct necklace decompositions of $G$ satisfying $H(III)$. Then the set of cones for $D_1$ is the same as the set of cones for $D_2$.

We prove the following propositions in order to prove proposition 3.5.1.

Proposition 3.5.1.1. Suppose the hypotheses of proposition 3.5.1 hold. Suppose there exists a cone $C$ of $D_1$ with extremes $u, v$ and apex $w$. Let $N_1(x, y)$ be a residual segment of $N_1$ contained in $N_1(u, v)$, but not in a residual segment of $N_2$. Then there exists an $N_1$-bridge $N_2(s, t)$ of $N_1 \cup N_2$ such that

$$u \leq s \leq x \leq y \leq t \leq v \text{ on } N_1(u, v).$$

Proof. Let $N_1(x, y) \subseteq N_1(t_{ab}, t_{ba})$. By hypothesis, $N_1(x, y)$ is contained in an $N_2$-bridge $N_1(x', y')$ of $N_1 \cup N_2$. By proposition 3.4.4, $N_1(x', y')$ is contained in a cone, say $C_2$, of $D_2$ where $u_2, v_2$ are its extremes and $v_2$ its apex. By the same proposition,
\[ N_1(x', y') \subseteq c_2 \setminus \{w_2\} \]. Also this is true for every \( N_2 \)-bridge of \( N_1 \cup N_2 \) contained in \( c_2 \). Applying proposition 3.4.4.2, we have \( N_1(u_2, v_2) \subseteq c_2 \setminus \{w_2\} \) and \( N_2(u_2, v_2) \cap c = [u_2, v_2] \). As \( N_1(u_2, v_2) \subseteq c_2 \), \( N_1(u_2, v_2) \) does not contain any of \( a, b, c \).

Therefore \( N_1(x', y') \subseteq N_1(u_2, v_2) \subseteq N_1(t_{ab}, t_{ba}) \), since \( N_1(x, y) \subseteq N_1(t_{ab}, t_{ba}) \).

Consider \( N_2(x', y') \). Obviously \( x', y' \in W(N_1 \cup N_2, N_2) = W(N_1 \cup N_2, N_1) \). Let

\[
V_1 = \{s' \in V(N_2(x', y')): s' \in W(N_1 \cup N_2, N_1) \text{ and } t_{ab} \leq s' \leq x\},
\]

\[
V_2 = \{t' \in V(N_2(x', y')): t' \in W(N_1 \cup N_2, N_1) \text{ and } y \leq t' \leq t_{ba}\},
\]

where the inequalities in \( V_1 \) and \( V_2 \) hold on \( N_1(t_{ab}, t_{ba}) \). Then \( V_1 \neq \emptyset \), \( V_2 \neq \emptyset \) since \( x' \in V_1 \) and \( y' \in V_2 \). Also \( W(N_1 \cup N_2, N_1) \cap V(N_2(x', y')) = V_1 \cup V_2 \). Therefore there exists a pair \( s, t, s \in V_1 \text{ and } t \in V_2 \) such that

1. \( s \neq t \), since \( s \leq x < y \leq t \), and
2. no \( m \in V_1 \cup V_2 \) exists with \( s < m < t \).

We claim that \( N_2(s, t) \) is an \( N_1 \)-bridge of \( N_1 \cup N_2 \). We know that \( t_{ab} \leq s \leq x' \leq x < y \leq y' \leq t \leq t_{ba} \) on \( N_1(t_{ab}, t_{ba}) \). Again,
by proposition 3.4.3, \( N_1(s, t) = N_2(s, t) \) or \( N_2(s, t) \) is an 
\( N_1 \)-bridge of \( N_1 \cup N_2 \), since \( N_2(s, t) \) is an \( W(N_1 \cup N_2, N_1) \)-
bridge of \( N_1 \cup N_2 \). Here \( N_1(s, t) \not\neq N_2(s, t) \), since \( N_1(x, y) \)
\( \subseteq N_1(s, t) \), and hence \( N_1(s, t) \) is a \( N_2 \)-bridge of \( N_1 \cup N_2 \) with
\( s \leq x < y \leq t \) on \( N_1(t_{ab}, t_{ba}) \). This establishes the claim.

It follows that \( N_2(s, t) \) is contained in a cone \( C_1 \) of \( D_1 \).
Thus \( N_1(s, t) \subseteq C_1 \). But \( N_1(x, y) \subseteq N_1(x', y') \subseteq N_1(s, t) \), by
construction. Hence \( C = C_1 \). This implies that
\[
 u \leq s \leq x < y \leq t \leq v.
\]
This completes the proof of the proposition.

**Proposition 3.5.1.2.** Assume the hypotheses of proposition 3.5.1.
Let \( C \) be a cone of \( D_1 \), where \( u, v \) are the extremes and \( w \) is
the apex of \( C \). Then \( C \) is a partial cone of \( D_2 \).

**Proof.** Without loss of generality we can assume
\[
 t_{ab} \leq u < v \leq t_{ba} \text{ on } N_1(t_{ab}, t_{ba}).
\]
By definition of \( C \), we have
(1) \( W(G, C) = \{u, v, w\}, u, v \in W(G, N_1), w \notin V(N_1) \),
(2) there exists a \( \theta \)-graph \( \theta \subseteq C \) such that \( u, v, w \) are internal
vertices of the three branches of \( \theta \),
(3) \( C \) is the union of some \( 3 \)-bridges of \( \{u, v, w\} \) in \( G \) not
containing any of \( a, b, c, \) and
Suppose \( N_g(u, v) = N \cup (u, v) \). By proposition 3.4.4.1, \( N_g(u, v) \subseteq C \setminus \{w\} \). Therefore (1), (2), (3) and (4) above define required conditions for \( C \) to be a partial cone of \( D_2 \).

Next suppose \( N_2(u, v) \neq N_1(u, v) \). By proposition 3.4.4.4, there exists a residual segment \( N_1(x, y) \subseteq N_1(u, v) \), where \( N_1(x, y) \) is contained in an \( N_2 \)-bridge of \( N_1 \cup N_2 \). By proposition 3.5.1.1, we have an \( N_1 \)-bridge \( N_2(s, t) \) of \( N_1 \cup N_2 \) such that \( u \leq s \leq x < y \leq t \leq v \) on \( N_1(t_{ab}, t_{ba}) \). Also the vertex \( w \notin V(N_2) \), by applying proposition 3.4.4 to the \( N_1 \)-bridges of \( N_1 \cup N_2 \) and \( N_1 \cap N_2 \). Again \( N_2(s, t) \subseteq [W(G, C)] \), since \( N_2(s, t) \) has at least one edge. Therefore \( N_2(u, v) \subseteq C \setminus \{w\} \), by proposition 3.4.4.2.

Hence \( u, v \in W(G, N_2) \), \( w \notin V(N_2) \), and \( N_2(u, v) \subseteq C \). These are exactly the conditions required to prove that \( C \) is a partial cone of \( D_2 \).

**Proof of proposition 3.5.1.** Let \( C_1 \) be a cone of \( D_1 \). Then, by proposition 3.5.1.2, \( C_1 \) is a partial cone of \( D_2 \). Thus \( C_1 \) is contained in a cone \( C_2 \) of \( D_2 \).

By symmetry, \( C_2 \) is contained in a cone \( C_1' \) of \( D_1 \). Therefore
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$C_1 \subseteq C_2 \subseteq C_1'$. As cones of $D_1$ are edge-disjoint, we have

$C_1 = C_1' = C_2$.

Thus any cone of $D_1$ is a cone of $D_2$ and, by symmetry, any cone of $D_2$ is a cone of $D_1$, i.e., the sets of cones of $D_1$ and $D_2$, respectively, are equal.

Now we state and prove theorem II.

Theorem II.

Let $G$ be a nonseparable graph and $a, b, c$ be distinct edges of $G$. Suppose

1. $\{a, b, c\} \notin K$, for any bond $K$ of $G$,

2. there exists a necklace decomposition $D = (N, R)$ of $G$, where $R$ is a bridge of $N$ in $G$ separating $a, b, c$, and

3. there exists a pair of skew diagonals of $N$.

Then there exist cones $C_1, C_2, \ldots, C_n$ of $G$ and beads $S_1, S_2, \ldots, S_m$ which determine a bead-cone decomposition of $G$. This bead-cone decomposition is unique. Moreover, for any cone $C$ of $G$ with $W(G, C) = \{u, v, w\}$ and apex $w$, there exists a pair of necklace decompositions $D_1 = (N_1, R_1)$ and $D_2 = (N_2, R_2)$ such that

4. $N_1 \cup N_2 = (N_1 \cap N_2) \cup ((N_1 \cup N_2) \cap C)$, and

5. $[u, v] = (N_1 \cap N_2) \cap ((N_1 \cup N_2) \cap C)$,

where $N_1 \cap N_2 = N'(u, v)$ and $(N_1 \cup N_2) \cap C$ is a rim of $C$. 
Proof. The hypotheses imply that $G$ satisfies $H(III)$.

Hence, for every pair of necklace decompositions $D_1 = (N_1, R_1)$, $D_2 = (N_2, R_2)$, the corresponding set of cones are equal, by proposition 3.5.1. Let the cones be $C_1, C_2, \ldots, C_K$. Then $E(C_j) \subseteq E_{abc}^2$ for all $j = 1, 2, \ldots, K$, by proposition 3.3.3. The cones are pairwise edge-disjoint, by proposition 3.2.3, and the definition that cones are maximal partial cones.

By proposition 3.1.1, the graph $G \cdot (E(N_1) \setminus E_{abc}^2) = G \
G \cdot (E(N_2) \setminus E_{abc}^2) = G \cdot (E_{ab} \cup E_{bc} \cup E_{ca} \cup E_{abc})$ is the union a set \{S_1, S_2, \ldots, S_m\} of pairwise edge-disjoint beads, common to the bead decompositions of $N_1$ and $N_2$, and called the essential beads of $G$.

Let $R'$ be the minimal subgraph of $G$ such that $G = S_1 \cup S_2 \cup \ldots \cup S_m \cup C_1 \cup C_2 \cup \ldots \cup C_k \cup R'$. Then $G$ has a decomposition into pairwise edge-disjoint subgraphs belonging to the set \{S_1, S_2, \ldots, S_m, C_1, C_2, \ldots, C_k, R'\},

where the $S_i$'s are beads and the $C_j$'s are cones of $G$. The decomposition is unique apart from the order in which the subgraphs are written.

The last part of the theorem follows from proposition 3.3.2. This completes the proof of theorem II.
Remark. \( E_{ab, bc} \cup E_{bc, ca} \cup E_{ca, ab} \cup E_{ab, bc, ca} \subseteq E(R') \) and

\[ E(R') \subseteq E_{ab, bc} \cup E_{bc, ca} \cup E_{ca, ab} \cup E_{ab, bc, ca} \cup E^{2}_{abc}. \]

Proof. \( E(G) = E_{ab} \cup E_{bc} \cup E_{ca} \cup E^{1}_{abc} \cup E^{2}_{abc} \cup E_{ab, bc} \cup E_{bc, ca} \)

\( U E_{ca, ab} \cup E_{ab, bc, ca} \) is a partition of \( E(G) \), by proposition 1.8.1.

Also \( E_{bc} \cup E_{ca} \cup E_{ab} \cup E^{1}_{abc} \subseteq E(N_i) \subseteq E_{bc} \cup E_{ca} \cup E_{ab} \cup E^{1}_{abc} \)

\( U E^{2}_{abc}, E_{ab, bc} \cup E_{bc, ca} \cup E_{ca, ab} \cup E_{ab, bc, ca} \subseteq E(R_i), \) and \( E(R_i) \)

\[ \subseteq E_{ab, bc} \cup E_{bc, ca} \cup E_{ca, ab} \cup E_{ab, bc, ca} \cup E^{2}_{abc}, \] for \( i = 1, 2, \) and \( E(C_j) \subseteq E^{2}_{abc}, \) for \( j = 1, 2, \ldots, k \) by propositions 2.7.5 and 3.3.3.

The remark now follows.

Remark. Suppose \( G \) is a graph with necklace decomposition \( D = (N, R) \) satisfying \( H(III) \). Let \( C \) be a cone of \( G \) with extremes \( u, v \) and apex \( w \).

Define \( S(C) \) to be the number of distinct necklace segments \( N(u, v) \), over all such necklace decompositions. By proposition 3.3.1.5, all necklaces \( N \) which are the union of essential beads, and one such necklace segment for each cone of \( D \), satisfy \( H(III) \).

Then, by proposition 3.5.1, we can conclude that all necklace decompositions arise in this manner. Thus, if \( G \) has cones \( C_1, C_2, \ldots, C_n \) then \( G \) has exactly \( \pi \sum_{i=1}^{n} S(C_i) \) distinct necklace decompositions \( D = (N, R) \), satisfying \( H(III) \). By the rim theorem, proposition 3.3.1, there will be at least \( 2^n \) such decompositions.
Apparently the nonuniqueness under $H(III)$ is accounted for very nicely by theorem II.
REFERENCES

1. W. T. Tutte, Connectivity in Graphs, Mathematical Expositions No. 15, University of Toronto Press.


5. F. Harary, Graph Theory, Addison-Wesley, Reading Mass., 1969.