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A GEOGRAPHIC THEORY OF A PUBLIC ECONOMY
IN SPATIAL EQUILIBRIUM

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Grant Ian Thrall, B.A., B.A., M.A.

* * * * *

The Ohio State University
1975

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ACKNOWLEDGMENTS

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1. Signs of the Partial Derivatives
CHAPTER I

INTRODUCTION AND PROBLEM STATEMENT

Problem Statement

A form of the property tax will be analyzed in a setting of an urban spatial equilibrium model. Changes in the spatial equilibrium land value surface, population density, quantity of land consumed per household, and size of urban area will be discussed given a change in the charge per unit of public good. The charge per unit of public good will be shown to behave analogous to a price for normal goods. Within the structure of this model it is hypothesised that land and the public good are substitutes. Increases in the public good charge will increase land consumption since normal good's price increases stimulate increases in demand for substitutes. Increases in the public good charge is expected to decrease equilibrium population density, increase the real areal extent of the urban environment, and decrease the spatial equilibrium land value surface. These relations provide a geographic rationale for the expansion of the number of incorporated municipalities lying about the central city.
Assumptions

The charge per unit of public good within this model will be shown to be analogous to the real world millage rate on land. The total expenditure by the household for public goods per unit of land is a function of this "millage rate" within the structure of this model.

This paper requires a simplifying approach similar to that presented by Alonso (Alonso, 1964; pp. 15-17). The city is located on a featureless plain, all land is equal in quality, and the land may be occupied without further capital improvements. Perfect knowledge by the buyers and sellers is required along with the omission of social and legal constraints such as discrimination or zoning. It is assumed that land buyers maximize satisfaction while land sellers maximize total revenue. All households are assumed to be identical in income. Identify two classes of land use, urban and agricultural; the agricultural rent gradient is independent of the urban rent gradient and, for the present, the agricultural rent gradient is perfectly elastic over space. Transform the origin of the agricultural rent distance gradient so that the intercept is zero.

Definitions

1. Property Tax

In theory, property tax levies can be structured in a variety of ways. This paper will analyze the property tax as levied analogous to a specific sales tax based upon quantity of land in one's possession and quantity of public good charged per unit of land.
Most property taxes are based upon an *ad valorem* tax; it will be shown within Chapter II that the *ad valorem* and specific tax behave in a similar manner. The specific tax has been chosen for brevity of exposition.

Empirically, the level of land taxation may be measured by either the nominal or the effective rate. The effective rate is the product of the nominal rate and the assessment ratio. For example,

1.01 \[ T = t_nAV \]

1.02 \[ AV = aMV \]

1.03 \[ t_e = T/MV \]

or

1.04 \[ t_e = t_n/a \]

The model to be presented in this paper is a theoretical derivation, not an empirical relation as in equations 1.01 through 1.04; however, the model to be developed does have similarities to real
world systems. Within this paper will be defined a charge per unit of public good, $t$, quantity of land, $q$, and the public good, $g$. $tg$ is the total expenditure per unit of land, and $tgq$ is the proportion of household income devoted to public good expenditures. Hence, the charge per unit of public good, $t$, is analogous to $tn$, where in lieu of the AV price relation there is a quantity relation. In the present model it will be assumed that $(a = 1)$ (see R. and P. Musgrave, 1973; p. 329).

2. Public Good

Public goods are often referred to as social goods in public finance literature. A public good is distinguished from a private good due to market failure resulting from conditions of (a) nonexcludability, and (b) nonrival consumption.

The market mechanism can function effectively only if the "exclusion principle" applies. For example, consumption of a good by A is contingent upon A paying the price, while those who do not pay the price are excluded. Hence, the well-functioning market allows the individual to reveal his preferences to the producer. Benefits are internalized. If consumption is not contingent upon payment, there is the case of nonexcludability, individuals are not required to reveal their preferences to the producer in the form of payment. Individuals have the incentive to become a "free rider" thereby breaking down the market system.

Market failure also arises where there is nonrival consumption. This arises where A's consumption of a good does not reduce the
benefits derived by others; hence, it would be inefficient to exclude all others from consumption of the good. The marginal cost of an additional person consuming the good is zero.

Richard and Peggy Musgrave provide a good discussion of public goods (R. and P. Musgrave, 1973; Ch. III).

The nonexcludability principle may break down when applied to spatial systems; many goods which are thought to be public are actually excludable due to location of the public good and the potential consumer. Such goods often occur on the local level; for example, city parks, police protection, and libraries. Also, many public goods which are available at a point in space are susceptible to rival consumption; for example, congestion of a park, highway, or school. Hence, geographically, it is the exception to find a pure public good on the local level. This complication, which is undoubtedly extremely important, will be assumed away for the present.

3. Price

The theory of rent has manifold definitional problems; hence, this model will emphasize the theoretical process by which land value is determined and how the exogeneous charge per unit of public good may alter the spatial equilibrium land values.

The land price represents the amount of money an occupant is willing to pay a landowner for the right to use a unit of land. The price of land multiplied by the quantity of land is the payment for the use of the site; it is assumed that this payment is for the same time period as is the payment for the public good.
4. **Featureless Plain**

The urban environment is assumed to rest upon a featureless plain. The plain does not allow for characteristics which may distort demanders away from or toward a land site. Such characteristics may include a beautiful view, altitude above smog level, or other externality producing phenomenon. The urban environment to be discussed is assumed to be circular thereby requiring an isotropic transportation surface. A star shaped city could have been chosen though with some loss of generality and brevity.

5. **Quantity of Land**

The quantity land is measured by the number of units of land consumed. The units may be in square inches, feet, yards, meters, or in acres, for example. The consumer of land is traditionally either an individual or a household. Conceptual problems do arise here since if the "individual" approach is assumed, realistically there may be more than one individual located upon a site requiring, therefore, discussions about fractions of sites per individual. Alternatively, all households may be required to have identical family size so that land may be allocated to uniform households. The latter approach will be used for its simplicity.

6. **Composite Good**

The composite good is assumed to contain all factors of consumption except the local public good and quantity of land. For example, the composite good contains the housing capital expenditure.
7. **Distance**

Distance is measured in real travel mileage from the central business district (henceforth, C.B.D.). Thus, such complications as a transportation network is easily incorporated into the model; however, integration over a discontinuous surface loses some generality and is more complex. As in the above evaluation of the featureless plain, let there be an isotropic transportation surface so that distance may be measured as a vector in every direction about the C.B.D.

8. **Income**

Real income within a time period consistent with the public good charge and the land payment is assumed. It is assumed that households maximize their welfare based upon current income rather than expected or life time projected income. Such complications may be easily incorporated into the models framework.

---

**Conclusions to the Chapter**

A specific site tax will be evaluated in a setting of an urban spatial equilibrium model. A number of geographic and economic relations will be derived; the effect of the specific site tax upon these relations will be shown. The usual geographic assumptions are required for the model to be presented as are simplifying economic and geographic definitions for the variables employed.
State of the Art Knowledge

Discussions about the property tax have been heated for the past three-quarters of a century, dating from Henry George's *Progress and Poverty*, 1879, and Edwin Seligman's *Essays in Taxation*, 1895. A controversy that has filtered down through economic literature from these early analyses, and has not been resolved, concerns the burden of the property tax; will the tax be passed onto the purchaser or renter of property or, alternatively, does the owner of the property absorb the cost of the tax (Edgeworth, 1925; Simon, 1943; Gillespie, 1965; Harberger, 1966). Richard Netzer summarized the state-of-the-art of property tax theory in *Economics of the Property Tax*, 1966, as it existed through the mid-twentieth century; it was thought that regeneratable and nonregeneratable capital would behave differently if taxed (Marshall, 1902; Edgeworth, 1925; Simon, 1943; Musgrave, et. al., 1951; Rolph, 1952; Harberger, 1966). If regeneratable capital (say buildings) were taxed, the cost of the tax would be passed onto the consumer of the services of this capital. Alternatively, if nonregeneratable capital (say land) were taxed, the cost would be
born by, absorbed in the price of, that factor. The central feature of this analysis was whether a factor had a perfectly inelastic supply schedule or not, and if so, how was such a relation to be treated if taxed.

"It is generally agreed that taxes on the value of bare land — the sites themselves exclusive of applications of reproducible capital in the form of grading, fertilizer, and the like — rest on the owners of the sites at the time the tax is initially levied or increased. The tax cannot be shifted, because shifting is possible, under reasonably competitive conditions, only if the supply of sites is reduced. But the supply of land is, for all practical purposes, perfectly inelastic. Individual landowners will not respond to an increase in land taxes by withdrawing their sites from the market, since doing so will not affect their tax liability. Indeed, their only chance of reducing the burdensomeness of the tax relative to their income streams is to seek to raise the latter by encouraging more intensive use of the sites they own. Collectively, landowners cannot reduce the stock of land; if individual landowners wish to liquidate in the face of higher taxes, they must sell the sites to other owners" (Netzer, 1966; p. 33).

When placed into a spatial context, contradictions to this "traditional thought" will arise, partly due to viewing the supply curve from a different perspective, as shown in this dissertation.

Given a typical income distribution, this traditional camp has felt that a tax on residential housing would be regressive, be a greater burden on low income individuals, while a tax on land would be neutral since the market price of land would decline by the amount of the tax. In part this is due to the propensity of low income households to have a relatively high proportion of income devoted to housing.
It has been often questioned whether economic's public finance community can present a general theory of the property tax that supports the conventional wisdom since the models, up to now, have not shown how factor prices may change with respect to the property tax, factor mobility across space has not been defined in these models, and the impact of the property tax has been seen to vary with respect to the geographic scale in which the tax has been levied (Mieszkowski, 1969, 1972; Gaffney, 1971; Berry, 1975). Most important, apart from regional variations in taxes due to assessment or tax history, these economic models have not considered the geographic structure of the city, such as distance decay of the rent gradient, optimal population, size of the urban area, nor have they considered the location or geographic scale of the distributor of the public good (Netzer, 1966, Ch. V; Orr, 1968; Oakland, 1972; Paglin and Fogarty, 1973). Recently there has been an attempt to incorporate property taxation into urban planning tools (Becker, 1970; Becker, 1970; Hahn, 1973; Raybeck, 1974; Hubbard, 1974; Bentick, 1974; Otte, 1974; Deaton and Mundy, 1975).

Provision of Public Goods

There are two hypotheses on the variation of the property tax rates within a metropolitan region. First is the relationship of the tax base to tax rates. Relatively high income communities have high property values per household generating greater gross taxes with low nominal rates. This encourages regressivity and great variations in the quality of public goods.
An alternative hypothesis for differential provisions of public goods in municipalities, and consequently property tax rates had its germination in a model by Charles M. Tiebout (Tiebout, 1956; see also Samuelson, 1954, 1955, 1958; Musgrave, 1959, Ch. IV; Coase, 1960; Head, 1962; Laif, 1963; Buchanan, 1963, 1965; Buchanan and Kafogolis, 1963; Thompson, 1968; Aaron and McGuire, 1969). In Tiebout's model, urban municipalities offer various baskets of public goods whose cost is reflected in local property taxes, while the population selects that composite good, implicitly through their location, which maximizes personal utility. Tiebout's model does require strong assumptions of perfect mobility of the population, zero transportation cost, and either a near infinite number of municipalities or homogeneous populations to map consumer demand to the location of the public good supply; in addition is the strong assumption of perfect knowledge on the part of the consumer about the location and composition of the public good and upon the supplier for knowledge of the public good cost function and demand. All revenue supported public services must be derived from taxes imposed upon the residents, in Tiebout's model; no aid or other revenues generated outside of the municipality may be drawn upon in this ideal world.

Though Tiebout's model has influenced public finance literature for the past twenty years, empirical verification does not exist. Oates has presented the most acclaimed empirical support for the Tiebout hypothesis; regretfully, Oates' model is difficult to interpret resulting from boundary problems in his data set and the lack of
structural specification of the model tested. This dissertation will provide both a pure geographic rational for a Tiebout like world without the restrictions required to derive Tiebout's model and will be presented in an easily verifiable form (see Oates, 1969).

Spatial Literature

Recently, the property tax has emerged in spatial literature through seminal articles by Robert O. Harvey and W. A. V. Clark, William Vickery, Jonathan J. Lu, and J. A. Mirrlees, among others (Harvey and Clark, 1965; Lu, 1967; Vickery, 1970; Mirrlees, 1972). Harvey and Clark suggest that tax advantages to owner occupied dwellings and real costs to farmers on the periphery whose land is taxed at a differentially higher rate than alternative agricultural land may force agrarians to surrender their fields to subdivision development earlier than the market would alternatively dictate leading to "urban sprawl".

Vickery has developed a model relating a tradeoff between horizontality and verticality; building height at any distance from the city center may be found, in this model, by equating the marginal cost of adding space through increases in height to the price of rented space; as taxes are levied on improvements, the price of height becomes relatively greater than horizontal land use, again encouraging urban sprawl. Vickery, in agreement with critics of the conventional wisdom of property tax, asserted that the classicists' belief in the incidence of a tax on capital improvements, which was thought to be on
the tenant, and that a land value tax, which was thought to be absorbed through depressed land market prices, must be modified; he did not show explicitly how this modification should be fully carried through (see also Meiszkowski, 1969, 1972).

Jonathan Lu, in a linear programming analysis, sought to allocate land-use so that the government would receive a maximum return from the property tax levies (Lu, 1967). Lu implicitly assumed that the land tax was passed forward by simulating tax increases via assigning relatively higher assessed land values to taxed regions. Hence, farmers in Lu's model were forced to relocate since near-urban locations suffer greater costs in his model through taxes while farm productivity is everywhere constant.

With some simplification, the spatial literature may be divided into two camps. Analyses analogous to that by Clark and later by Lu, require that land taxes applied at differential rates depending on land use, for example taxing urban fringe farm land as suburban property, create an exodus of farmers and an expansion of the urban area; call this camp the "pull camp" since a vacuum is created at the fringe pulling the urban area outward.

The "push camp", characterized by Vickrey, taxes capital at a different rate than land. Taxing the former at a greater rate will push development outward from the inner to outer portions of the city.

Mirrlees implicitly uses both the push and pull processes where the spatial structure of the optimal town deviates from the town derived through market forces. Taxes, both positive and negative, may
be applied to equate the two city types through altering consumer preferences via price.

Spatial and economic models have treated property taxes on land or structures as costs; increases in the property tax may be thought of in this body of literature as an increase in the charge per unit of a constant public good. The models have not reflected property taxes as a price per unit of an optimal quantity of public good. The impact of this omission will be discussed in the following chapters.

As property tax literature developed, the population density literature — Clark, Casetti, Papageorgiou among others — and modern theories of urban land use of the von Thünen variety — for example, Alonso, Wingo, Muth, Casetti — were presented (C. Clark, 1951; Casetti, Jan. 1970, Aug. 1970, 1967; Papageorgiou, 1971; Wingo, 1961; Alonso, 1964; Muth, 1969).

Alonso, in Location and Land Use (1964; p. 116) hypothesized that a tax on land as a percentage of rent will not affect land prices or the pattern of land use. A flat rate tax, however, would be born by tenants at the urban fringe since there, land would become non-zero in price; however, equilibrium price-locations would not be altered due to changes in a tax levy.

Muth discusses the effect of the income tax upon the spatial structure of the urban environment. Deductions one is allowed on income taxes which accrue from home ownership lower the real price of owner occupied dwellings by 25% less than renter occupied dwellings (Muth, 1969; p. 104). Muth attributes this to be one of the major causes of suburban expansion and the resulting central city decay
during the post-World War II redistribution of population.

Even though tax structures were not fully developed in either books by Alonso or Muth, two opposing analyses may be extracted. Alonso argues that taxation will not alter the spatial structure of the urban environment such as the population density gradient. Alternatively, Muth suggests that tax systems play a major role in determining population density and rent gradients among other spatial phenomenon.

It will be argued in the following analysis that at least one form of a property tax on land may be used as a land use determinant. Personal utility will be shown to remain invariant even though population densities increase as gross optimal taxes increase; this is not the case when population densities increase due to zoning (Casetti, 1970).

Theoretical Tax Forms

Taxes may be levied in various forms, each form affecting that which is taxed in a different manner. (The theoretical effects have been presented in Henderson and Quandt, 1971; Chs. IV, VI, and VII.) A brief discussion of taxes in the form of a specific sales tax, an ad valorem sales tax, a lump sum tax, and a profit tax follows. In the traditional manner, the discussion will be with respect to a single firm, first as a monopoly then in a perfectly competitive setting.
A "lump sum" tax acts as a constant added to a function. A profit function in a monopoly market with a lump sum tax may be formulated

\[ 2.01 \quad \pi = R(q) - C(q) - t_1 \]

where \( \pi \) is profit, \( R(q) \) is revenue, \( C(q) \) is cost, and \( t_1 \) is a lump sum tax. Setting the derivative of 2.01 equal to zero

\[ 2.02 \quad \frac{d\pi}{dq} = \frac{\partial R(q)}{\partial q} - \frac{\partial C(q)}{\partial q} = 0 \]

which upon simplification shows the firm should operate, in spite of the lump sum tax, where marginal revenue equals marginal cost. Price of the good and quantity of the item will not be affected by the lump sum tax.

A "profit tax" in a monopoly market requires payment of a given proportion of the difference between total revenue and total cost.

\[ 2.03 \quad \pi = R(q) - C(q) - t_2[R(q) - C(q)] \]

or

\[ 2.04 \quad \pi = (1 - t_2)[R(q) - c(q)] \quad 0 < t_2 < 1 \]

Setting the derivative of 2.04 equal to zero
\[ \frac{d\pi}{dq} = (1 - t_2)(MR - MC) = 0 \]

since

\[ \frac{\partial R(q)}{\partial q} = MR \]

and

\[ \frac{\partial C(q)}{\partial q} = MC \]

where MR and MC represent marginal revenue and marginal cost respectively. Since the value \((1 - t_2)\) has been restricted to not equal zero, 2.05 may be simplified to the condition that marginal revenue equal marginal cost which is identical to the result in 2.02 for the lump sum tax. The equality of marginal revenue and marginal cost determines the optimal quantity and price; hence, this optimal position will not change with respect to a change in a profit tax.

A "specific tax" is stated in terms of the number of dollars one must pay per unit of good. Levy a specific tax of \(t_3\) dollars per unit of output.

\[ \pi = R(q) - C(q) - t_3q \]
The first order conditions are

\[ 2.07 \quad \frac{d\pi}{dq} = \frac{\partial R(q)}{\partial q} - \frac{\partial C(q)}{\partial q} - t_3 = 0 \]

which upon simplification yields

\[ 2.08 \quad MR = MC + t_3 \]

The optimal position is found by equating marginal revenue to marginal cost plus the specific sales tax of \( t_3 \). The total differential of 2.08 is

\[ 2.09 \quad \frac{\partial^2 R(q)}{\partial q^2} dq = \frac{\partial^2 C(q)}{\partial q^2} dq + dt_3 \]

or

\[ 2.10 \quad \frac{dq}{dt_3} = \frac{1}{\frac{\partial MR}{\partial q} - \frac{\partial MC}{\partial q}} \]

Since first order conditions state that the rate of change or marginal revenue is less than that of marginal cost, 2.10 is negative. Hence, as the specific sales tax increases, a smaller quantity and higher price result.
A similar result may be shown if the sales tax is a proportion of the value of total revenue, say, sales, rather than quantity. In this case the tax is called *ad valorem*, say $t_4$. $t_3$ and $t_4$ are similar in their formulation and will be shown to behave similarly.

\[ \pi = R(q) - C(q) - t_4 R(q) \]

or

\[ \pi = (1 - t_4) R(q) - C(q) \]

then

\[ \frac{d\pi}{dq} = (1 - t_4) MR - MC = 0 \]

and solving 2.13

\[ MC = (1 - t_4) MR \quad 0 < t_4 < 1 \]

The optimal position is where marginal cost equals the proportion of marginal revenue one is allowed to retain after the *ad valorem* tax. The total differential yields
\[
2.15 \quad \frac{dq}{dt_4} = \frac{MR}{(1 - t_4) \frac{\partial MR}{\partial q} - \frac{\partial MC}{\partial q}}
\]

Since the numerator of 2.15 is positive and the denominator, as in 2.10, negative, 2.15 is negative. Analogous to the specific sales tax case, increases in the \textit{ad valorem} sales tax \(t_4\) will result in lower quantities and higher prices.

An imperfectly competitive market has been assumed in equations 2.01 through 2.15. In a competitive world, the cost of the tax will be shared by the producer and consumer of the good. In the case of the two sales taxes, the \textit{ad valorem} tax, \(t_4\), and the specific tax, \(t_3\), the proportion of the tax passed forward to the consumer is greater, the smaller are the slopes of the supply and demand curves.

In a competitive formulation, total costs, \(TC\), are found for the specific tax case to be

\[
2.16 \quad TC = FC(q) + VC(q) + t_3q
\]

and for the \textit{ad valorem} case

\[
2.17 \quad TC = FC(q) + VC(q) + t_4pq
\]

Take the derivative of 2.16 and 2.17 with respect to quantity. In the competitive formulation, the first order conditions require the result
to be set equal to the market price, \( p \). For the specific tax case

\[
2.18 \quad MC + t_3 = p
\]

and in the *ad valorem* case

\[
2.19 \quad MC + t_4p = p
\]

In both cases, marginal cost plus the unit tax charge are equal to the market price. Hence, both taxes are similar.

The resulting supply functions for the specific, \( S \), and *ad valorem*, \( S' \), cases in a competitive market are

\[
2.20 \quad S = S(p - t_3)
\]

and

\[
2.21 \quad S' = S'[p(1 - t_4)]
\]

Equations 2.01 through 2.21 represent a basis for the state of the art theory; the empirical verification may not be presented so neatly. Much of the empirical evidence does not support the above. For example, some empiricists argue that, in an imperfect market, not only is 100\% of the tax passed forward to the consumer, but, values as high as 135\% have been found for a corporation income tax
(Harberger, 1962; Cragg, Harberger, Mieszkowski, 1967; Gordon, 1967; Meiszowski, 1969). Debate as to who pays the tax, and the validity of the theoretical models in 2.01 through 2.21, is current in tax literature (Oakland, 1969, 1972).

**Conclusion to the Chapter**

Though equations 2.01 through 2.21 assume a firm rather than land, the form of these taxes may be applied to a tax on land. This paper will assume that land is taxed via a specific tax on land and public goods. It has been shown that the specific tax is one of the set of tax structures and is similar in form and effect to the *ad valorem* tax. Most land taxes today are levied *ad valorem*. The property tax may also be applied to tangible and intangible wealth, and to capital upon land. This paper will not deal with this form of property tax. Property taxes may be levied on the local, state, or national level; this paper will only consider local taxes.
CHAPTER III

THE MODEL

Introduction

A large body of literature has been shown to exist, in Chapter II of this paper, which has been devoted to analyses of the provision of public goods and to methods of revenue collection. Recent studies have incorporated spatial complexities into models of the public economy. Apart from collective decision-making and the provision of transportation, spatial equilibrium models have disregarded the public economy.

The model to be introduced within this chapter will include a specific land tax and public goods into a spatial equilibrium setting; Casetti's reformulation of Alonso's model will be used as a basis for this discussion. Implicit within this model are the assumptions listed in Chapter I of this paper; among the assumptions are that households be identical in composition, income and preferences, and that the restrictive geographic isotropic surface exist.
Review of Casetti's Model

In Casetti's operationalization of Alonso's model, the households' preferences are specified by an indifference map,

\[ u = u(z, q, s) \]

which identifies combinations of the composite good, \( z \), the quantity of residential land, \( q \), and distance from the city center, \( s \), that produces identical utility to the household. Land rent, transportation expenses, and purchases of the composite good are required to exhaust the households' real income; that is

\[ y = pz + rq + ks \]

where \( y \) represents real income; \( p \), \( r \), and \( k \) are the unit prices of the composite good, land, and transportation respectively.

All households at any distance \( s \) from the city center will choose the mix of \( z \) and \( q \) that maximizes \( u \) subject to the budget constraint; call the optimal values \( \bar{z} \) and \( \bar{q} \). The optimal values are functions of \( s \) and \( r(s) \) along with the constants which appear in the utility function.

\[ \bar{q} = \bar{q}(s, r) \]

---

The optimal utility level may be transformed to a function of distance, $s$, and the land price, $r$

$$ u(s,r) = \bar{u}(z,q,s) $$

Household relocation from less favored areas to more favored areas bid the land rent up in the latter areas and down in the remainder thereby generating a state of spatial equilibrium in which households are indifferent towards relocating within the urban environment. Casetti operationalized the spatial equilibrium condition that all household utility be identical by equating the optimal household utility to a spatially invariant constant $\bar{u}$.

$$ u(s,t) = \bar{u} = \text{constant} \quad \dot{s} \geq 0 $$

Casetti's constant $\bar{u}$ permits the derivation of the spatial equilibrium land value surface by solution of equation 3.06

$$ \bar{r} = \bar{r}(s,\bar{u}) $$

A few of the relations in which $\bar{r}$ may be found is in defining the spatial equilibrium quantity of land consumed per household, areal size of the urban environment, population density, and aggregate
An Extension of Casetti's Formulation

In this paper assume that individuals at distance \( s \) from the city center act as if they maximize a utility function involving a public good along with the traditional composite good and quantity of land. The public good has \( g \) units allocated per unit of land; the total public good consumed per household is \( gq \) at a cost of \( tgq \).

\[
3.08 \quad u = U(z,q,gq,s) = u(z,q,g,s)
\]

The same strategy of deriving the optimal values may be utilized as in the reformulation of Alonso's model summarized above since except for the inclusion of the public sector the models are similar. The optimal values \( z, q, \) and \( g \) are found such that

\[
3.09 \quad u(z, q, g, s) = \text{maximum}
\]

subject to exhausting the real income

\[
3.10 \quad p\bar{z} + r\bar{q} + t\bar{g}q + ks = y
\]

\( t \) is the charge per unit of public good. Since \( g \) is the number of public good units allocated per unit of land, \( tg \) is the total expenditure per unit of land for public goods. \( tgq \) is the aggregate
public good cost per household.

The optimum utility level $\tilde{u}$ may be transformed to be a function of $s$, $r$, and the exogeneous $t$

$$3.11 \quad \tilde{u}(s, \bar{q}, \bar{g}) = \tilde{u}(s, r, t)$$

Lastly, equate Casetti's constant $\tilde{u}$ to the optimum utility level

$$3.12 \quad \tilde{u}(s, r, t) = \tilde{u}$$

which implicitly identifies the spatial equilibrium land value surface

$$3.13 \quad \bar{r} = \bar{r}(s, t, \tilde{u})$$

Substituting equation 3.13 into the structural relation of $\bar{g}$ and $\bar{q}$, the spatial equilibrium public good and quantity of land consumed are derived.

$$3.14 \quad \bar{g} = \bar{g}(\bar{r})$$

$$3.15 \quad \bar{q} = \bar{q}(\bar{r})$$

Furthermore, the optimal gross "tax", or total expenditure per unit of land, is $t\bar{g}$. The optimal proportion of household income disposed of within the public sector is $t\bar{g}\bar{q}$. 
The implications of this model are best presented in the framework of a specific example.

Conclusion to the Chapter

Within this chapter Casetti's operationalization of Alonso's model was presented; Casetti's constant was reviewed. A public economy has been introduced into Casetti's format in this dissertation. On the local government's provision side the optimal quantity of public good dispensed per unit of land has been derived; on the revenue side an optimal tax rate dependent upon the optimal quantity of public good per unit of land has been presented along with the spatial equilibrium quantity of household income to be collected by the local government.
CHAPTER IV

AN EXAMPLE

Introduction

The previous chapter provided a discussion of the formulation of the generalized model of the public economy in a spatial equilibrium setting. This chapter will provide an example of the generalized model by assuming a specific utility relation and budget constraint. The first and second order conditions for utility maximization will be presented along with the optimal values of the public good, $\bar{g}$, the quantity of land, $\bar{q}$, and the composite good, $\bar{z}$, which maximizes utility subject to the income relation. A discussion of the spatial equilibrium land value surface, $\bar{r}$, will then be presented.

The Basic Model

Let the composite good, $z$, the quantity of land, $q$, the public good, $g$, and distance from the central city, $s$, be represented in a utility function

$$ u = z^\pi q^\rho (qg)^\alpha e^{-\gamma s} $$
where public good is defined per unit of land. Total public good consumed is \( q_g \). The utility function in 4.01 may be abbreviated to

\[
4.02 \quad u = a^T q^\rho + a^\alpha e^{\Omega s}
\]

The parameters \( \pi, \rho, \) and \( \alpha \), provide for a transformation of the utility function. A function \( F(u) \) is a monotonic transformation of \( u \) if

\[
F(u_1) > F(u_0) \text{ whenever } u_1 > u_0 \quad \text{(Henderson and Quandt, 1971; p. 20)}.
\]

For example, if \( w = u \), \( w \) is a monotonic transformation of \( u \); a monotonic transformation guarantees that \( F' \neq 0 \). \( \Omega \) depicts the magnitude by which utility declines over distance from the central city.

Require that one's real income be exhausted in the following manner:

\[
4.03 \quad y = pz + rq + tgg + ks
\]

where \( p \) is the price of the composite good, \( r \) is the price of land per unit, \( t \) is the specific tax per unit of public good charged per unit of land, \( k \) is transport cost per unit distance \( s \) from the central city.

Maximize the utility function, 4.02, subject to the income relation, 4.03; that is maximize

\[
4.04 \quad z = z^T q^{\rho+\alpha} e^{\Omega s} + \lambda(y - pz - rq - tgg - ks)
\]

The following system of four equations is found:
4.05 \[ \frac{\partial z}{\partial z} = \frac{\pi}{z} - \lambda p = 0 \]

4.06 \[ \frac{\partial z}{\partial q} = (\rho + \alpha) u - \lambda (r + tg) = 0 \]

4.07 \[ \frac{\partial z}{\partial g} = \frac{\alpha u}{g} - \lambda t q = 0 \]

4.08 \[ \frac{\partial z}{\partial \lambda} = y - pq - rq - t g q - ks = 0 \]

Let \(|\bar{H}_1|\) represent the \(i\)th degree bordered Hessian determinant; solve for the second order conditions.

4.09 \[ |\bar{H}_2| = \begin{vmatrix} 0 & -p \\ -p & Z_{zz} \end{vmatrix} < 0 \]

since

\[-(-p)^2 < 0\]

The subscripts represent partial derivatives.
Equation 4.11 will be greater than zero if

\[ z_{qq}, z_{zz} < 0 \]

and

\[ z_{zq}, z_{qz} > 0 \]

where

\[ z_{qq} = \frac{u(\rho + \alpha)}{q^2} (\rho + \alpha - 1) < 0 \]

\[ z_{zq} = \frac{(\rho + \alpha)}{q} \frac{\pi u}{z} > 0 \]

\[ z_{zz} = \frac{\pi u}{z^2} (\pi - 1) < 0 \]
Equations 4.11 through 4.14 will hold if

\[ 4.15 \quad 0 < (\rho + \alpha) < 1 \]

\[ 4.16 \quad 0 < \pi < 1 \]

Lastly,

\[ 4.17 \quad |H_4| = \begin{vmatrix}
0 & -p & -(r+tg) & -tq \\
-p & Z_{zz} & Z_{zq} & Z_{zg} \\
-(r+tg) & Z_{qz} & Z_{qq} & Z_{qg} \\
-tq & Z_{gz} & Z_{gq} & Z_{gg}
\end{vmatrix} \]

where

\[ 4.18 \quad Z_{qg} = \frac{\alpha(\rho + \alpha)u}{gq} - \lambda t \]

Since

\[ 4.19 \quad \lambda = \frac{au}{qgt} \]

\[ 4.20 \quad Z_{qg} = (\rho + \alpha - 1) \frac{au}{gq} < 0 \]
Equation 4.20 is negative since \((p + \alpha)\) has been restricted to lie between 0 and 1.

\[
4.21 \quad Z_{gg} = \frac{\alpha u}{g^2} (\alpha - 1) < 0
\]

which is negative since \(\alpha\) is bounded between 0 and 1. Lastly,

\[
4.22 \quad Z_{gz} = \frac{\alpha p u}{g_z} > 0
\]

Determinant 4.17 upon solution yields

\[
4.23 \quad p^2 (Z_{qg}^2 - Z_{qq} Z_{gg}) + (r+tg)^2 (Z_{zz}^2 - Z_{zz} Z_{gg})
+ (tq)^2 (Z_{zq}^2 - Z_{zz} Z_{qq}) + 2ptq (Z_{zg} Z_{qq} - Z_{zg} Z_{qg})
+ 2p(r+tg)(Z_{zq} Z_{gg} - Z_{zg} Z_{gg}) + 2(r+tg)(tq)(Z_{zz} Z_{qg} - Z_{zg} Z_{qz}) < 0
\]

The second order conditions for a maximum require that equation 4.23 be negative which implies that the following structural relations must be true:

\[
4.24 \quad Z_{qq} Z_{gg} > Z_{qg}^2
\]

which upon solution yields
4.25 \( \rho > 0 \)

4.26 \( Z_{zz}Z_{gg} > Z_{zg}^2 \)

which upon solution yields

4.27 \( (\pi + \alpha) < 1 \)

4.28 \( Z_{zz}Z_{qq} > Z_{zq}^2 \)

which simplified is

4.29 \( 0 < (\rho + \alpha + \pi) < 1 \)

4.30 \( Z_{zq}Z_{qg} > Z_{zg}Z_{qq} \)

or the trivial condition

4.31 \( 1 > 0 \)

4.32 \( Z_{zg}Z_{gg} > Z_{zq}Z_{gg} \)

which yields the same condition as equations 4.19 and 4.20.

4.33 \( Z_{zg}Z_{qz} > Z_{zz}Z_{qq} \)
Inequality 4.28 yields the same trivial condition as inequality 4.25.

The system of equations from 4.09 through 4.28 yield nontrivial conditions for a maximum which may be reduced to the following two sufficient conditions

\begin{equation}
0 < (\rho + \alpha + \pi) < 1
\end{equation}

and

\begin{equation}
\rho, \alpha, \pi > 0
\end{equation}

See Appendix A for a derivation of the Slutsky equations, and Appendix B for a discussion of the implications of 4.34 and 4.35.

**The Optimum Values, \( \bar{g}, \bar{q}, \bar{z} \)**

Solve for the value of \( g, q, \) and \( z \) which will maximize the utility function yet satisfy the income relation; call these values, \( \bar{g}, \bar{q}, \) and \( \bar{z}. \) \( \bar{g} \) is derived by dividing equation 4.06 by 4.07

\begin{equation}
\frac{(\rho + \alpha)u}{q(r + tg)} = \frac{au}{gqt}
\end{equation}

rather

\begin{equation}
\bar{g} = \frac{ar}{\rho t}
\end{equation}
Substitute the value for \( q \), 4.37 into 4.03

\[ 4.38 \quad y - pz - rq(1 + a/\rho) - ks = 0 \]

Also, substitute 4.37 into 4.06

\[ 4.39 \quad \frac{\partial \mathcal{E}}{\partial q} = \frac{(\rho + \alpha)u}{q} - \lambda r(1 + \alpha/\rho) = 0 \]

To solve for \( q \), solve equations 4.05 and 4.39 for \( \lambda \).

\[ 4.40 \quad \frac{\pi u}{zp} = \lambda \]

\[ 4.41 \quad \frac{(\rho + \alpha)u}{rq(1 + \alpha/\rho)} = \lambda \]

Then set 4.40 equal to 4.41 and solve for \( z \)

\[ 4.42 \quad z = \frac{\pi qr(1 + \alpha/\rho)}{p(\rho + \alpha)} \]

Substitute 4.42 into 4.38 and solve the resulting relation for \( q \), say \( \bar{q} \).
4.43 \[ \bar{q} = \frac{(\rho + \alpha)(y - ks)}{rv} \]

where

4.44 \[ v = (1 + \alpha/\rho)(\pi + \rho + \alpha) \]

Similarly, to solve for the value of \( z \), say \( \bar{z} \), that will maximize 4.04, set equation 4.41 equal to 4.40 and solve for \( q \)

4.45 \[ q = \frac{zp(\rho + \alpha)}{r(1 + \alpha/\rho)\pi} \]

Substitute 4.45 into 4.38 and solve for \( z \), say \( \bar{z} \),

4.46 \[ \bar{z} = \frac{\pi(y - ks)}{pw} \]

Since

4.47 \[ y - ks = pz + rq + t\bar{q} \]

alternative forms of \( \bar{q} \) and \( \bar{z} \) may be derived so that cross effects may be observed. Substitute 4.47 into 4.43 and 4.46 and solve for \( \bar{q} \) and \( \bar{z} \).
\[ 4.48 \quad \bar{z} = \pi (r \rho + t \rho q) / (\rho + \alpha) \]

\[ 4.49 \quad \bar{q} = \frac{\rho p z}{\pi r} \]

The maximum utility obtainable, given the income relation, is derived by substituting the values of \( \bar{z} \), \( \bar{q} \) and \( \bar{z} \), found in equations 4.37, 4.43, and 4.46 respectively, into the utility function defined in equation 4.02

\[ 4.50 \quad \bar{u} = \bar{z}^\rho \bar{q}^{\rho + \alpha} \bar{\xi}^{\alpha} e^{-\Omega s} \]

that is,

\[ 4.51 \quad u = (y - ks)^{\bar{w}} \bar{r}^{-\rho} p^{-\pi} t^{-\alpha} e^{-\Omega s} H \]

where

\[ 4.52 \quad H = \pi (\rho + \alpha) \rho + \alpha \left(\frac{\alpha}{\rho}\right) \bar{w}^{\bar{\alpha}} \left(1 + \frac{\alpha}{\rho}\right)^{-(\rho + \alpha)} \]

\[ 4.53 \quad \bar{w} = \rho + \pi + \alpha \]

The spatial equilibrium relations may be solved for by setting \( \bar{u} \) in 4.51 equal to Casetti's constant \( \bar{u} \). \( \bar{u} \) is a dummy variable guaranteeing
that utility remains the same over space so that in equilibrium there will be no incentive for population movement.

The Spatial Equilibrium Land Value Surface, \( \bar{r} \)

Set \( \bar{u} = \bar{u} \) and solve for the spatial equilibrium price of land, say \( \bar{r} \), which will define the spatial equilibrium land value surface at every distance \( s \) from the central city.

\[
4.54 \quad \bar{r} = \left[ (y - ks)^{\bar{u} - 1} p^{-\pi} t^{-\alpha} e^{-\Omega s} H \right]^{(\rho)}
\]

As in other spatial rent models, say by Casetti, Muth and Alonso, \( \bar{r} \) is an inverse function of \( s \) (Alonso, 1964; Casetti, January 1970, August 1970; Muth, 1969). Figure 1 has been drawn by assuming constants for the parameters and allowing \( s \) to vary; it represents an inverse relation. Let

\[
4.55 \quad \bar{r} = V e^{-\Omega s} R
\]

where

\[
4.56 \quad R = \left( \bar{u}^{-1} p^{-\pi} t^{-\alpha} H \right)^{\rho}
\]

and

\[
4.57 \quad V = (y - ks)^{(w \rho)}
\]
FIGURE 1
RENTER VERSUS DISTANCE
The first and second derivatives of the spatial equilibrium land value function with respect to distance are:

\[
\frac{\partial^2 F}{\partial s^2} = \frac{w_p k}{V} e^{-\Omega s} \left[ \Omega + \frac{w_p k}{V} + \frac{k w_p}{V} \right] < 0
\]

Equations 4.58 and 4.59 verify the concave upward function in Figure 1. The change in the spatial equilibrium land value surface may be found with respect to the public good charge.

\[
\frac{\partial F}{\partial t} = -\frac{\alpha \rho}{t} < 0
\]

The second order conditions are

\[
\frac{\partial^2 F}{\partial t^2} = \alpha \rho \frac{t}{(1 + \alpha \rho)} > 0
\]

where

\[
0 < t, \frac{t}{r}
\]
Given the restrictions upon the charge per unit of public good and the spatial equilibrium land value surface, and the signs of equations 4.60 and 4.61, the spatial equilibrium land value surface may be seen to decrease as the public good charge increases; the function is concave upward and asymptotically approaches a minimum with increases in the charge per unit of public good. See Figure 2.

Figure 2 has been drawn using constants for the parameters in equation 4.54 while giving a range of values to the public good charge, t. The result, as equations 4.60 and 4.61 require, is a reversed J shaped function which decreases at a decreasing rate in the manner of a rectangular hyperbola. Analyses by Netzer, and earlier by George require that, *ceteris paribus*, a relation of r and t must map out a straight line function with a slope of -1 since land supply is perfectly inelastic and, consequently, 100% of the land tax has been thought to be absorbed in the land market value.

The percentage change in the land value surface with respect to the percentage change in the charge per unit of public good, in equation 4.60, is

\[
4.62 \quad \frac{dr}{dt} = \alpha \frac{r}{t}
\]

If 4.62 is less than one, the function is inelastic implying that land values will change proportionately greater than the charge per unit of public good; hence, as the charge for public goods increase, the
FIGURE 2

RENT VERSUS PUBLIC GOOD CHARGE
land values will absorb more than 100% of the charge in depressed prices. If the combination of the two parameters in 4.62 equals one, then $\bar{r}$ will change proportionately at the same rate as $t$ and 100% of the tax will be passed onto the land owners by suppressed land market values. If 4.62 is greater than one, the spatial equilibrium land value surface will change proportionately less than the charge per unit of public good; that is, less than 100% of the charge will be capitalized in depressed land market values.

It is the unique case, therefore, that $r$ and $t$ will change at the same rate. Given the second order conditions in 4.34 and 4.35 the domain of $\alpha$ and $\rho$ lies between 0 and 1, in this model. Equation 4.62 is constrained to be relatively inelastic given the present utility function and budget constraint.

The 45° line's intersection in Figure 2, with $\bar{r}$ provides that point where the model is consistent with the notion that 100% of the tax is absorbed by depressed land values; the negatively sloped straight line "EC" maps out the schedule where $\bar{r}$ and $t$ change by the same absolute amount. The second order conditions restrict the discussion to that portion of $\bar{r}$ which lies above the intersection of the 45° line; hence, $t$ must be less than $t^*$ for this analysis to guarantee individual maximum utility.
CHAPTER V

SPATIAL EQUILIBRIUM QUANTITY OF LAND, $\bar{q}$

Introduction

In the previous chapter the values of land, $\bar{q}$, composite good, $\bar{z}$, and the public good, $\bar{g}$, which maximize utility subject to the budget constraint were derived. Next, these optimal values were substituted into the utility function which in turn was set equal to Casetti's constant thereby generating the condition for spatial equilibrium; the spatial equilibrium land value surface, $\bar{r}$, was derived from this relation.

In this chapter, $\bar{r}$ will be substituted into the structural equation defining $\bar{q}$; this will construct $\bar{q}$, the spatial equilibrium quantity of land consumed per household at distance $s$ from the C.B.D. given a value for the exogeneous parameter $t$. The structural relation of $\bar{q}$ will then be investigated.

$q$

The spatial equilibrium quantity of land consumed at distance $s$ from the C.B.D. may be derived by substituting the value of $\bar{r}$, from equation 4.54, for $r$, in equation 4.43.

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Equation 5.01 represents \( \bar{q} \) and \( \bar{r} \) as inversely proportional.

\[
5.01 \quad \frac{\partial \bar{q}}{\partial \bar{r}} = -\frac{(\rho + \alpha)(y - ks)/\bar{r}^2}{\bar{v}} < 0
\]

and the second order conditions are

\[
5.03 \quad \frac{\partial^2 \bar{q}}{\partial \bar{r}^2} = \frac{-2(\rho + \alpha)(y - ks)/\bar{r}^3}{\bar{v}} > 0
\]

where

\[
0 < \frac{\bar{r}}{\bar{v}}, v
\]

\( \bar{q} \) is seen to decrease and asymptotically approach a minimum with respect to increases in the spatial equilibrium land value surface, \( \bar{r} \). See Figure 3.

The change in the spatial equilibrium quantity of land resulting from a change in the cost per unit of public good may be solved for, given the above analysis. From equation 5.01

\[
5.04 \quad \frac{\partial \bar{q}}{\partial t} = -\frac{(\rho + \alpha)(y - ks)}{\bar{r}^2 v} \frac{\partial r}{\partial t} \quad \frac{\partial \bar{r}}{\partial t}
\]
FIGURE 3

QUANTITY OF LAND VERSUS RENT
Substituting the result from equation 4.60 and 5.04

\[
\frac{\partial q}{\partial t} = \frac{+ (\rho + \alpha)(y - ks)(\alpha p)}{tv} > 0
\]

or, more simply,

\[
\frac{\partial q}{\partial t} = \frac{J(y - ks)}{t}
\]

where

\[
J = (\rho + \alpha)(\alpha p)/v
\]

The second order conditions are

\[
\frac{\partial^2 q}{\partial t^2} = -\frac{J(y - ks)}{t^2} < 0
\]

The spatial equilibrium quantity of land is concave downward with respect to the charge per unit of public good. See Figure 4. This result is consistent with theorems of substitution in economic literature; as the price of one good increases, say the public good, the consumption of its substitute, say land, will increase. To be consistent with theories of marginal utility, the latter good should
FIGURE 4

QUANTITY OF LAND VERSUS PUBLIC GOOD CHARGE
increase at a decreasing rate as in Figure 4. This analysis may be more fully explored by deriving the relevant elasticities.

Elasticities of $q$

Let the cost of providing the public good to residents at distance $s$ from the central city increase at a constant rate. Then,

$$\frac{dt}{ds} = \omega = \text{a constant} > 0$$

The following relation may be derived:

$$\frac{d\ln \bar{r}}{ds} = -(\alpha/\rho) \frac{d\ln t}{ds}$$

Solve 5.10 for the elasticity, $\eta_{\bar{r},t}$

$$\frac{t}{\bar{r}} \frac{d\bar{r}}{dt} = -(\alpha/\rho) = \eta_{\bar{r},t} < 0$$

Next derive the price cross elasticity of demand between land, $q$, and the public good, $g$, say $\eta_{q,t}$

$$\eta_{q,t} = \frac{\partial q}{\partial t}$$
Rearranging terms in 5.13

\[
\eta_{q,t} = -\frac{(\rho + \alpha)(y - ks) \frac{\partial r}{\partial t}}{\frac{v}{q} \frac{\partial r}{\partial t}}
\]

Equation 5.14 may be simplified by substituting the derivations of \( \bar{r} \), equation 4.61, and into \( \eta_{\bar{r},t} \), equation 5.11, yielding

\[
\eta_{q,t} = -\eta_{\bar{r},t}
\]

Hence,

\[
\eta_{q,t} > 0
\]

Conclusion to the Chapter

\( \bar{r} \) has been shown to be inversely related to \( t \) in proportion to the ratio of the two parameters \( (\alpha/\rho) \); also, a one unit increase in \( t \) will increase the optimal consumption of \( q \) by the value of this ratio. \( t \) has a negative effect upon \( \bar{r} \) generating in two ways a positive effect upon \( \bar{q} \); via directly lowering the price of land, and by one consuming more land and less public good.
Finally, as in Alonso's analysis, \( \bar{q} \) is a concave upward function of distance which is required for market equilibrium so that there will be maintained an invariant utility surface. Land, behaving as a normal good, suffers from diminishing marginal utility requiring the quantity of land consumed at increasing distances from the city center to increase at an increasing rate to balance increases in transportation expense, both time and price, and decreases in the public and composite good (Alonso, 1964; p. 66; Figure 24). See Figure 5.
FIGURE 5

QUANTITY OF LAND VERSUS DISTANCE
CHAPTER VI

SPATIAL EQUILIBRIUM QUANTITY OF PUBLIC GOOD, $\vec{g}$

Introduction

In the previous chapter the variable $\vec{r}$ was substituted into a structural relation and implications were drawn from the resulting relation; similarly, $\vec{r}$ will replace $r$ in the structural relation defining $\vec{g}$ in this chapter. The spatial equilibrium quantity of public good will be derived and analysed with respect to $\vec{r}$, $t$, and $s$. Drawing upon these results, a "second-best" spatial equilibrium solution will be illustrated to rationalize the birth of new municipalities about a central city.

The condition that the results contained within this monograph are optimal is contingent upon the lack of interdependent utility functions; otherwise, the solutions may be pessimal, that is, they may depict the worst of all possible worlds.\(^1\)

---

\(^1\)The theory of the second best has relevancy to geography:
- a) If spillovers are not capitalized into the site values of land, or if there are no institutional mechanisms for the pricing of externalities, then the Pareto-Optimal solution may represent the pessimal solution.
- b) If there are interdependent utility functions, Jones' consumption of space or location does affect me, then the Pareto-Optimal analysis
Drawing upon the derivations in equations 4.37 and 4.54, the spatial equilibrium quantity of public good may be formulated

\[ 6.01 \quad \bar{g} = \frac{\alpha \bar{r}}{\rho t} \]

\( \bar{g} \) is shown to be positively related, in a linear fashion, to \( \bar{r} \).

\[ 6.02 \quad \frac{\partial \bar{g}}{\partial \bar{r}} = \frac{\alpha t}{\rho} \frac{\partial \bar{r}}{\partial \bar{r}} = \frac{\alpha t}{\rho} > 0 \]

and

\[ 6.03 \quad \frac{\partial^2 \bar{g}}{\partial \bar{r}^2} = 0 \]

See Figure 6. Higher spatial equilibrium market land values will prompt the public to demand greater provision of public goods.

---

may not be used.

(c) Non-differentiation of products is required in a perfectly competitive market. Space is differentiable due to the uniqueness of location, beautiful views, etc. If this is the case, then Pareto-Optimal analysis should incorporate a second-best solution. The analysis of the second-best differs from the standard Pareto-Optimal analysis by recognizing the additional "non-market" or imperfect constraints. (Thrall, 1974)
FIGURE 6

QUANTITY OF PUBLIC GOOD VERSUS RENT
The change in the spatial equilibrium quantity of public good with respect to the charge per unit for the public good is

\[
\frac{\partial g}{\partial t} = - (\alpha \rho) \frac{\partial^2}{\partial t^2} (\alpha \rho + 1) < 0
\]

and the second order conditions are

\[
\frac{\partial^2 g}{\partial t^2} = K \left( \frac{\partial^2}{\partial t^2} \right) > 0
\]

where

\[
K = a^2 (\rho^2 \alpha + \rho)
\]

As the charge per unit of public good increases, the spatial equilibrium quantity of public good decreases and asymptotically approaches a minimum. \( g \) behaves as a normal good whose quantity demanded decreases as its price increases. See Figure 7.

The spatial relation of the optimal quantity of public good to distance is an interesting concave upward function.

\[
\frac{\partial g}{\partial s} = \left( \frac{\partial t}{\rho} \right) \frac{\partial^2}{\partial s^2} < 0
\]
FIGURE 7

QUANTITY OF PUBLIC GOOD VERSUS PUBLIC GOOD CHARGE
which may be interpreted by equation 4.58

\[ \frac{\partial \bar{g}}{\partial s} = - \left( \frac{\alpha t}{\rho} \right) \text{Re}^{-\Omega s} \left[ V \Omega + \frac{w \rho k}{V} \right] < 0 \]

The second derivative yields

\[ \frac{\partial^2 \bar{g}}{\partial s^2} = \left( \frac{\alpha t}{\rho} \right) \frac{\partial^2 \bar{r}}{\partial s^2} > 0 \]

The spatial equilibrium quantity of public good provided per unit of land has been shown to decline with increasing distance from the city center in Figure 8. The present model structurally requires this result since the spatial equilibrium quantities of land and public good are substitutes; this was proven in equations 5.04 through 5.16. The greater the consumption of land, *ceteris paribus*, the less public good will be demanded; Figure 5 portrays the increase in \( \bar{q} \) as distance from the city center increases. \( \bar{g} \) and \( \bar{r} \) are positively related; \( \bar{r} \) declines with increases in \( s \) thereby structurally decreasing \( \bar{g} \).

The present spatial distribution of public goods within urban areas of North America may conform to the present analysis. Public good provision is often more costly in low population density environments than it is in relatively high population density environments; for example, sewer provision is a centrally provided public good in high population density areas and is provided in the
FIGURE 8

QUANTITY OF PUBLIC GOOD VERSUS DISTANCE
form of septic tanks privately by the consumer population in low density areas. As persons become relatively more spatially remote from the city center, public provision of goods may be replaced by their private substitutes.

It will be shown in Chapter VII that if greater quantities of public good are provided than is the spatial equilibrium optimum, the land market value will be depressed. Inner urban area residents demand more public goods than urban fringe residents. A pure local public good by definition is where the quantity supplied at every distance from the city center is the same; hence, it is spatially invariant. Providing the quantity of public good which maximizes the utility of the urban area residents would decrease the utility of the fringe area residents. Given a continuous population distribution at every distance $s$ from the city center, only residents at one distance $s$, say $s^*$, would receive a spatial equilibrium quantity of public good; residents at all other $s$ would receive either too little or too great a quantity of public good. See Figure 9, where $\bar{g}_0$ represents the pure local public good which is spatially invariant by definition. Hence, the question as to which quantity of the spatially invariant pure local public good should be supplied has no single optimal geographic solution.

A series of second best solutions are available (see Lipsey and Lancaster, 1956, for a discussion of the theory of the second best). The "mean quantity" may be found so that half the population is under, and half over supplied. The mean quantity weighted by income
FIGURE 9

QUANTITY OF PUBLIC GOOD VERSUS DISTANCE
GIVEN A SPATIALLY INVARIANT SUPPLY OF PUBLIC GOOD
distribution may be found achieving, say, an "ability to pay" solution. Residents may pay an equal proportional share of their income less transportation cost as another solution. Alternatively, the charge per unit of public good per unit of land may be made a function of $s$, declining with increasing distance from the city center. Allow $t$ to vary so that $\overline{g}_0$ may become the spatially optimal quantity of public good at every distance $s$; this, however, is unconstitutional.

A solution which may be observed within the United States is where the politically unbounded space, assumed in this model, has been divided into many independent municipalities. In this manner, holding agglomeration economies constant and assuming independent utility functions with zero externalities, a second best solution has been to maximize the number of $s^*$ thereby minimizing the deviation about $s^*$ resulting in the highest possible societal welfare by being as close as possible to the $\overline{g}(s)$ function. See Figure 10 where each $s^*_i$ ($i = 1, \ldots, n$) is halfway from the innermost and outermost portion of the municipality from the central city center. The upward bound upon the number of $s^*$ and the actual city size, cities providing large quantities of public goods being of greater population or tract than those offering less, may be constrained by economies of agglomeration. Contracting out may be thought of in this context as an attempt to increase such upward bounds (see Bish, 1971, for a discussion of contracting out). Figure 10 has been drawn by moving equal distances along the $\overline{g}(s)$ function; Figure 11 has been drawn by restricting the change in $\overline{g}(s)$, measured on the "y" axis, to be constant. The solid
FIGURE 10

QUANTITY OF PUBLIC GOOD VERSUS DISTANCE
GIVEN SPATIALLY INVARIANT SUPPLIES OF PUBLIC GOODS
EQUAL INCREMENTS ON THE PUBLIC GOOD FUNCTION
QUANTITY OF PUBLIC GOOD VERSUS DISTANCE
GIVEN SPATIALLY INVARIANT SUPPLIES OF PUBLIC GOODS
EQUAL INCREMENTS ALONG THE PUBLIC GOOD AXIS

FIGURE 11
rings on the "x" axis have been drawn by dividing the distance between each s*, or s', in half; the broken rings restrict s* to be in the center of the municipality. Different criterion yield different spatial structures of the urban environment. Figure 10 allows for an infinite number of municipalities while the inner municipalities are relatively larger than those in Figure 11, which has a finite number of municipalities though the farthest out municipality is infinite in size and the centroid is of indeterminant location.

One would not expect, empirically, ring cities as in Figure 11 and 12. Historical modes of transportation during the urban area's development along with political inertia maintaining historical governmental boundaries, among just a few boundary determinants, are most likely of greater importance in determining the actual boundaries of municipalities. However, one should find greater similarities of public good provision among cities located within a ring than would be expected between the rings. At the least, the "ring theory" may aid in explaining the proliferation of the number of municipalities in the United States' urban environment and an alternative to Tiebout's model (Tiebout, 1956).

Theoretically, the model does provide an inner and outer bound for governmental jurisdictions. To be complete, the radaii bounds may be a reflection of communication and transportation costs which often have "in-out" bias with respect to the central city, along with theories of agglomeration and compactness. (See Christaller on theorems of the administrative principle, K_1 = 7^4; Christaller, 1966.)
CHAPTER VII

A SKETCH OF $\bar{g}_t$

Introduction

The change in the spatial equilibrium land value, $\bar{r}$, may not be calculated with respect to a change in the optimal total public good expenditure per unit of land, $\bar{g}_t$, by the usual calculus techniques since $\bar{g}$ has been determined endogenously. A sketch of the relation between $\bar{g}_t$ and $\bar{r}$ may be shown in heuristic fashion.

$\bar{g}_t$

$\bar{g}$ has been shown to be the optimal quantity of public good per unit of land; $t$ is the charge per unit for this public good. $t\bar{g}$ is then the total expenditure per unit of land which maximizes individual welfare given a homogeneous society in spatial equilibrium.

Cities within the United States must maintain either a balanced budget, or revenue deficits when they do occur are financed through city borrowing and are limited to a proportion of the city's taxable wealth. Therefore, in real dollars, increases in millage rates will likely be accompanied by increases in public goods.
From equations 4.43 and 4.37, restructure the spatial equilibrium land value surface to include the variable \( \bar{g}t \).

\[
7.01 \quad \bar{F} = \phi(\bar{g}t)(\lambda \alpha)
\]

where

\[
7.02 \quad \phi = \left( w^{-\gamma} (1 + \frac{\alpha}{\rho}) - (\rho + \alpha) \pi (\rho + \alpha)^{\rho + \alpha} t^{-\alpha} e^{-\Omega s} p^{-\gamma} (y - ks)^{\gamma} \right)^{\lambda} \]

\[
7.03 \quad \lambda = \frac{\rho}{1 + \alpha \rho}
\]

As \( \bar{g}t \) changes, due to say a change in taste, measured by a relative change in the parameter \( \alpha \), and given a constant exogeneous price per unit of public good, \( t \), then \( \bar{F} \) will change in the manner of a parabola as in Figure 12. The charge per unit of public good, which has been shown to be similar to the nominal tax rate, and the specific optimal land tax, say the gross tax per unit of land, have important and dramatically different behavior; this may be easily visualized by comparing Figures 12 with 2, and Figure 13 with 4.

The policy implications of Figure 12 are significant and manifold. Call the parabola \( \bar{F} \) in Figure 12, \( \bar{F}^* \) since it is a function of \( \bar{g}t \); the government may maximize public welfare by remaining on the "pseudo expansion path" defined by \( \bar{F}^* \). Quantities of the public good provided greater than the optimum will be treated as a cost, analogous to \( t \),
$RENT$

FIGURE 12

RENT VERSUS THE OPTIMAL TAX
depressing the spatial equilibrium land market value. If this quantity of public good is preferred by society, *ceteris paribus*, the land market value will increase with increases in the public good.

Figure 13 portrays, again descriptively, the relation between the optimal quantity of land consumed and increases in optimal tax as brought about by change in the parameter $\alpha$. Equations 5.09 through 5.16 imply that land and the public good are substitutes; if society chooses to increase the optimal tax, the quantity of land consumed by each family unit will decline. The optimal quantity of land is positively related, in Figure 4, to the charge per unit of public good as expected of normal good substitutes. $t$ and $\delta t(\alpha)$ affect the optimal consumption of land in opposing fashion. See Figures 14 and 15 for the relations within a system of optimal gross taxes and within a system of charges per unit of public good.

**Conclusion to the Chapter**

Within this chapter a rough sketch of $\bar{t}$ has been drawn depicting in heuristic fashion the relation between this optimal gross tax and the spatial equilibrium land value surface. If societies taste preferences change to desire more units of public good at a given $t$, analogous to a rightward shift in a demand schedule, we would expect less land to be consumed as a consequence.

$\bar{t}$ is an underidentified parameter for investigation in a concise manner; at the least a production function for public goods must be assumed for further analysis which is warranted.
FIGURE 13

QUANTITY OF LAND VERSUS THE OPTIMAL TAX
FIGURE 14

INTERRELATIONS WITH $\ddot{g}_t$
FIGURE 15

INTERRELATIONS WITH $t$
CHAPTER VIII

SPATIAL RELATIONS

Introduction

A variety of relations which reflect the spatial structure of the ideal urban environment may be extracted from derivations in the previous chapters. This chapter will analyze the effect upon the ideal urban environment by t and $\bar{\mathbf{g}}t$. No attempt will be made to provide explicit functions since to do so would require further structural assumptions; only the sign of the relation, say the direction of the relative change, will be evaluated along with the methodology for solving a function if structural relations were given. Population density, $\bar{P}(s)$, total urban land value, $\bar{U}(s)$, and the spatial equilibrium radius of the urban environment will be presented. These concepts will be utilized for a discussion of urban sprawl.

Population Density as a Function of t

Increases in t will decrease the spatial equilibrium land value surface and increase the spatial equilibrium quantity of land consumed per household; furthermore, increases in t will decrease the demand for public goods and increase the demand for its substitute, land,
thereby lowering the spatial equilibrium population density.

\[ D(s) = \frac{m}{q(s, r)} \]

where \( D(s) \) is the spatial equilibrium population density at distance \( s \) from the city center, and \( m \) represents household size. The first and second derivatives of \( D(s) \) with respect to \( t \) can be shown to be negative and positive respectively defining a reversed J shaped function suggesting that population density may be treated analogous to a normal good; increases in the charge per unit of public good will increase land consumption, the substitute, and thereby decrease population density.

The equilibrium quantity of population within radius \( s \) is found by integrating over the population density

\[ P(s) = 2\pi n \int_0^s D(s) s \, ds \]

\( n \) is a constant representing the proportion of urbanized land to total land. The change in total population within radius \( s \) with respect to the public good may be shown to be negative (see Appendix C). This result is expected since equation 8.01 requires the family unit to consume more land as the charge per unit of public good increases; there will be an outward expansion of the urban population requiring

\[ \frac{\partial P}{\partial t} > 0 \]
The aggregate population within every radius $s$ from the city center will decline for $s < \bar{s}$; the population within the total urban environment is, by Casetti's constant $\bar{u}$, structurally required to remain the same.

\[ \bar{s}, \text{ The Spatial Equilibrium Radius} \]

The spatial equilibrium limit of the urban environment may be solved for by first integrating over the population density to $\bar{s}$

\[ 8.04 \quad \bar{P}(\bar{s}) = 2\pi n \int_0^\bar{s} s \bar{D}(s) \, ds \]

As in the preceding section, total urban population must remain constant before and after a change in $t$, consequently

\[ 8.05 \quad \frac{\partial \bar{P}(\bar{s})}{\partial t} = 0 \]

Equation 8.05 implicitly defines the spatial equilibrium radius of the urban environment. Various methods may be used to solve 8.05 for $\bar{s}$; one method is presented in Appendix D.

\[ \bar{U}(s), \text{ Spatial Equilibrium Aggregate Land Value} \]

Let $\bar{U}(s)$ represent the urban land value within radius $s$ from the central city. $\bar{U}(s)$ may be derived by integrating over the land value surface
\[ 8.06 \quad \overline{U}(s) = 2\pi n \int_0^s \frac{r}{s} s \, ds \]

It is shown in Appendix E that

\[ 8.07 \quad \frac{\partial \overline{U}(s)}{\partial t} = -(\alpha)R \overline{C}_4 t \]

where \( R \) and \( C_4 \) are constants greater than zero. Hence, as the charge per unit of public good increases, the aggregate urban land value, for \( s < \overline{s} \), decreases.

\( \overline{Q}(s) \), Aggregate Quantity of Land

Within radius \( s \) from the city center, the quantity of urbanized land is found by integrating over the urbanized land surface. If we assume that the ideal city is a circular city, one need only solve for the volume of a circle out to \( s \), or \( \overline{s} \). A star shaped, or other shaped city, would require the additional complication of integrating over a discontinuous land surface. The spatial equilibrium quantity of land within radius \( s \) from the city center is

\[ 8.08 \quad \overline{Q}(s) = \pi n s^2 \]

\( n \) has been defined as the proportion of land in urban use; land in nonurban use may include public and military land, flood plains, agricultural, and speculative uses for example. If all land that
potentially may be urbanized through the market process, within radius s, has been urbanized, then

\[ 8.09 \quad \frac{\partial Q(s)}{\partial t} = 0 \]

If \( n \) is a function of speculation or agricultural productivity, say \( n^* \), then

\[ 8.10 \quad \frac{\partial Q(s)}{\partial t} = \pi s^2 \frac{\partial n^*}{\partial t} \quad \wedge 0 \]

if

\[ 8.11 \quad \frac{\partial n^*}{\partial t} \quad \wedge 0 \]

The change in \( n^* \) with respect to \( t \) will be positive if the tax makes the land no longer profitable with respect to the land's opportunity cost. Equation 8.10 will be negative if land is speculatively held from immediate urbanization; this may be the case if land is taxed differentially with respect to types of land use as may be the case when agricultural land receives preferential treatment over urban land by assessors or through lower millage rates.
The radius of the optimal city has been derived in Appendix D. The change in the optimal city's land mass may be derived

\[
\frac{\partial Q(s)}{\partial t} = \pi n z s - \frac{\partial s}{\partial t} > 0
\]

given no land speculation, and

\[
\frac{\partial Q(s)}{\partial t} = \pi n z s + \pi n s^2 \frac{\partial n^*}{\partial t}
\]

when the land market allows speculation. The second order conditions are indeterminant for both cases unless explicit relations for \(f(s, t, n^*)\) are given. Equation 8.13 is indeterminant without explicit values for the parameters; however, a sketch may be drawn of its implications. This will be done later in this chapter under the heading, "Sprawl".

The Spatial System and \(\bar{g}_t\)

Since \(\bar{g}\) is endogeneous, only a descriptive analysis may be presented for the spatial implications of \(\bar{g}_t\).

\(t\), was shown to decrease population density within the ideal city since \(\bar{q}\), and \(t\), are positively related while population density, \(\bar{D}(s)\), is inversely related to \(\bar{q}\). It has been observed heuristically that \(\bar{q}(s)\) is inversely related to \(\bar{g}_t\), Figure 13, implying that \(\bar{D}(s)\) and
gt(α) are positively related. Casetti has shown that utility declines with increases in population density (Casetti, 1971). This analysis has suggested that population densities may be treated analogous to normal goods. Increases in the population density will not decrease utility if brought about by society choosing a greater consumption of the substitute, public good. Zoning higher population densities, as in the Casetti case, will decrease personal utility, holding interdependence of utility functions and agglomeration economies constant.

gt and t yield contrasting results in other spatial formulations. The land value surface will decrease as the public good charge increases. An increase in the optimal gross tax, gt(α), would imply an increase in \( \bar{r} \); therefore \( \bar{U}(s) \) would likely increase with increases in \( gt(α) \). Land would be expected to absorb a greater proportion of personal income if society chooses higher optimal taxes. By the same reasoning, \( \bar{gt} \) is expected to decrease the size of the urban environment. The change in \( \bar{s} \) as \( \bar{gt} \) increases is expected to be negative. Clearly note that \( \bar{gt} \) may only change endogeneously through a change in household taste preferences for public goods, analogous to a shift upward and to the right of a demand curve.

**Sprawl**

Increasing the public good charge, t, will encourage low density development connotating suburban sprawl. Conversely, relatively higher population densities are consistent with high optimal gross taxes.

Confusion about whether taxes hinder or harvest sprawl persist not only due to the opposing effects of t, and \( \bar{gt} \), but, from structural
assumptions when modelling. Inferences about the inverse relation of suburban sprawl and the public good charge, t, derive from abstractions that agricultural rents are, relative to urban rents, both elastic and transformable to a zero rent intercept without mutation of the model's interpretation as has been the strategy within this paper. To illustrate, if has been discussed here that the public good charge, t, decreases the spatial equilibrium land value surface causing sprawl.

Contrary to the model thus far developed, assume a steep agricultural rent gradient, say \( \bar{a}(s) \); relate the previously derived spatial equilibrium urban rent schedule to \( \bar{a}(s) \) as in Figure 16. Levying the public good charge, t, on urban and not on agricultural property will decrease the total available urban land because of the downward shift in \( \bar{r}_1 \) to \( \bar{r}_2 \). Differential land taxation will, therefore, alter a model's spatial implications as will assuming greater than one significant rent schedule. The structure of the spatial equilibrium urban environment may be influenced by differential land taxation.

Differential land taxation has salient implications for today's land market within the United States and Canada. Regions with rapid population growth hear pleas from urban periphery agrarians that it is unfair to tax agricultural land as urban since agrarians are not consuming the urban social good and urban taxes will force agrarians to sell their land since urban fringe farmers suffer relatively higher costs, t, and yet yield similar revenues per acre as nonfringe farmers; the periphery is therefore thought to be at a comparative
disadvantage for farming.

Ohio has recently succumbed to this with passage of the 1973 proposition, "Save Open Spaces", which will tax and assess land at its current use, say agricultural, rather than its best alternative use, say urban. Current use taxes are also used in California.

Let the spatial equilibrium land rent be initially $\bar{r}_2$ in Figure 16. Let a new rent gradient be $\bar{r}_1$ in Figure 16 due to an increase in either population or income. Holders of land near $\bar{s}_2$ have the incentive to maintain agricultural land use until public good externalities, such as utility and sewer service, are capitalized. Land owners near $\bar{s}_1$ will develop their property earlier than $\bar{s}_2$ since capitalization of externalities for them has a longer time horizon. The cost of speculation within the zone $\bar{s}_2$ to $\bar{s}_1$ is the opportunity cost if land is taxed at its current use.

Let land be taxed at its current use, $\bar{s}_1$ is developed first, and that transportation facilities to link $\bar{s}_1$ to the central city are developed. $\bar{s}_2$ will capitalize these externalities in their land values at zero cost. If further shifts upward and to the right occur in $\bar{r}$, for example due to increasing the accessibility as urbanized outer rings are connected to the city center, the end product will be a leap-frog suburban expansion with inner areas like $\bar{s}_2$ developed last. Less intensive development will also follow from the process of connecting outer rings to inner areas by a transport network; such processes lower the rent gradient near the inner city. Thus, differential taxation and the receipt of $\bar{s}$ as a free good by some
FIGURE 16

THE SPATIAL LIMIT OF THE URBAN AREA
concentric zones about the inner city will result in variations in the spatial theory presented earlier. Hence, this model has powerful planning and policy implications.

Conclusion to the Chapter

Within this chapter a variety of relations reflecting an ideal city's spatial structure have been analyzed with respect to the exogeneous parameter $t$ and the endogeneous optimal gross tax $\tilde{g}_t$. Increases in $t$ will decrease population density; populations within regions of relatively high $t$ are expected to inhabit urban environments of relatively great areal extent. $\tilde{g}_t$ was found to behave in an opposite manner; however, $\tilde{g}_t$ may only change endogeneously via new taste preferences by society. A generalized method for solving for the spatial equilibrium radius of the urban environment was also provided.
TABLE 1
SIGNS OF THE PARTIAL DERIVATIVES OF \( \bar{r}, \bar{q}, \bar{D}(s)^*, \bar{U}(s)^*, \bar{Q}(\bar{s}), \bar{P}(s)^* \)
WITH RESPECT TO \( \bar{u}, p, y, k, t, s \)

<table>
<thead>
<tr>
<th></th>
<th>( \bar{u} )</th>
<th>( s )</th>
<th>( y )</th>
<th>( k )</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( \bar{r} )</td>
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<tr>
<td>( \bar{Q} )</td>
<td>-**</td>
<td>+</td>
<td>+**</td>
<td>-**</td>
<td>+**</td>
</tr>
<tr>
<td>( \bar{P} )</td>
<td>*</td>
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<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( \bar{g} )</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
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</tr>
</tbody>
</table>

* for \( s < \bar{s} \)
** for \( s = \bar{s} \)
CHAPTER IX

THE BURDEN

Introduction

The model which has been developed within this dissertation provides a basis for investigating many factors in the spatial public economy. This chapter will investigate the question: upon whom does the burden of the specific land tax fall given that the actors are either landlords or new purchasers of land after the tax increase.

The burden of an increase in the charge per unit of public good will be calculated by finding the ratio of the change in land rent resulting from the increase in t to the change in total revenue. If the land price decreases by the same amount as the increase in the public good charge, the ratio will equal -1.00. Land values decrease due to increases in the public good charge, as shown in this model, requiring the ratio to be negative; the burden will fall upon the landlord since he must pay the tax while in possession of the land and furthermore suffer a loss in real property value. The owner may not regain the value of the public good charge by selling the parcel. New purchasers of land after the public good charge increase will be affected less the greater is the absolute value of the ratio. The
ratio will be +1.00 if the land prices increase by the same amount of
the tax increase; the owner may "pass forward" to renters or new
purchasers of land all of the increase in the public good charge; this
may be the case if society has a change in taste preferences, increasing
\( \bar{g} \) for every \( t \), thereby increasing \( \bar{g}t \). In this latter case, if the ratio
is positive and lay between 0 and 1, then the landlord may pass forward
a portion of the present increase to future new purchasers of land.
If the value is greater than +1.00, then more than 100% of the new
charge is passed forward. (This has been shown to be the case for
nonland oligopolies and monopolies. Cragg, Harberger, and Mieszkowski,
1962; Harberger, 1962; Gordon, 1967; Meiszowski, 1969; Oakland, 1969,
1972.)

This index for the burden suggests that increases in the charge
per unit of public good, \( t \), is "passed backward" onto the landlord;
at most it is shared by the landlord and new purchasers. The burden
will fall upon new purchasers only if there is a structural change in
\( \bar{g}t \), say due to a change in societies taste.

The Burden, \( \beta \)

The burden, \( \beta \), may be calculated by the following relation

\[
\beta = \frac{c_1 \int_0^{\bar{s}} (\bar{r}_1 - \bar{r}_0) \text{ ns ds}}{c_2 \int_0^{\bar{s}} (t_1 - t_0) \text{ ns ds}}
\]
where \( r^* \) is the spatial equilibrium land value surface after a change, say an increase, in the charge per unit of public good in time period \( i; \) \( r^0 \) is the spatial equilibrium land value surface in the first time period. \( t_0 \) represents the initial charge per unit of public good and \( t_i \) is the charge in the \( i \)th time period for public goods. \( c_1 \) and \( c_2 \) are both the integral constant and \( 2\pi \). Land values have been shown to be an inverse function of the public good charge; thus,

\[
9.02 \quad \frac{r^*}{r_i} [t^*] < \frac{r^*}{r_i} [t_i] < \frac{r^*}{r_o}
\]

for

\[
9.03 \quad t^* > t_i > t_o
\]

as shown in equations 4.60 and 4.61. Since the numerator is negative and the denominator positive in equation 9.01

\[
9.04 \quad \beta < 0
\]

The second order conditions for maximum utility, see equations 4.29, 4.30 and Figure 2, restrict \( t_i \) to be less than \( t^* \). \( t^* \) may be found when \((\rho + \alpha) = 1\). From equation 6.01

\[
9.05 \quad t^* = \frac{\hat{r}}{\Lambda} (\alpha/\rho) \quad (\rho + \alpha) = 1
\]
The numerator in equation 9.01 is bounded by

\[ 0 < c_1 \int_0^s (\bar{r} - \bar{r}_0) \text{ ns } ds \]
\[ < \int_0^s (\bar{r}_1[t^*] - \bar{r}_0) \text{ ns } ds \]

\( s \) is the spatial equilibrium radius of the urban environment. The smallest value that inequality 9.06 may assume is

\[ 9.07 \quad - \int_0^s \bar{r}_0 \text{ ns } ds \]

\( \beta \) is consequently constrained by

\[ 9.08 \quad 0 \leq \beta \leq c_3 \int_0^s (\bar{r}_1[t^*] - \bar{r}_0) (t_i^* - t_i)^{-1} \text{ ds} \]

where

\[ 9.09 \quad c_3 = c_1/c_2 \]

The solution of equality 9.01 requires explicit values for the parameters; however, a general statement may be presented about the burden. \( \beta \) is negative implying the burden is on the landowner; however, there is no guarantee that \( \beta \) will equal -1.00. \( \beta \) may be less than -1.00 placing a greater burden on the landowner than the initial change in the charge for public goods. This may be empirically
analogous to the converse of the microeconomic case.

The Burden, \( \bar{\beta} \)

As in the derivation of \( \beta \), let \( \bar{\beta} \) represent the burden due to a change in the gross optimal tax, \( \bar{\text{gt}} \), say due to an increase in tastes for public goods by society analogous to a shift upward and to the right in the demand schedule. A more complete analysis would also view the supply side of public goods by assuming a social good production function (see Thrall, Casetti, 1975). A general sketch of \( \beta \) may be drawn even though the function may not be specifically evaluated. Let

\[
9.10 \quad \bar{\beta} = \frac{c_4 \int_0^{\bar{s}} (r_j - r_o) \, ns \, ds}{c_5 \int_0^{\bar{s}} \left[ \bar{\text{gt}}_j - \bar{\text{gt}}_o \right] \, ns \, ds}
\]

where \( \bar{r}_j \) is the new spatial equilibrium land value surface after the change in \( \bar{\text{gt}} \) during time period \( j \). Let

\[
9.11 \quad \left[ \bar{\text{gt}}_j \right] > \left[ \bar{\text{gt}}_o \right]
\]

then

\[
9.12 \quad \bar{r}_j > \bar{r}_o
\]
which may be heuristically verified in Figure 12. Noting that the numerator and denominator in equation 9.10 have the same sign

\[ \beta > 0 \]

and is bounded by

\[ 0 < c_6 \int_0^\beta (r_j - r_o)(gt_j - gt_o)^{-1} ds \]

\( c_6 \) represents the constant \( c_4/c_5 \).

Equation 9.14 is not uniquely identifiable without explicitly given parameters and assumptions about the social good production function; however, a general statement may be given about the burden \( \beta \). \( \beta \) is positive implying that the burden of the optimal gross tax per unit of land falls entirely upon new purchasers of land. \( \beta \) is not restricted to be less than +1.00, allowing more than the initial value of the optimal gross tax to be passed forward, given that housing market conditions would allow for this. This is feasible and consistent with market requirements since municipalities may be thought of as differentiated products; see Figures 9, 10, and 11. This case, furthermore, is analogous to the microeconomic monopolist case.

**Conclusion**

It has been shown that the optimal gross tax, \( \tilde{g}t \), and the charge per unit of public good, \( t \), behave in diametrically opposite fashion;
this is not surprising since the two "taxes" represent two different concepts. \( \hat{g}t \)'s burden falls upon new land purchasers. Conversely, the burden of \( t \) falls upon the land owner.

As a planning guide, public expenditures should be "visible" as well as should the charges. Increases in the charge, \( t \), erodes equity stored by a household in land. Increases in \( \hat{g}t \), where consistent with societies preferences, may increase social wealth at a point in space; however, it may represent a windfall gain to old landlords. An example of the first case, \( t \), is where public employees receive a pay increase in real dollars when there is no increase in productivity; an example of the latter case, \( \hat{g}t \), is where a landlord has property tangential to a newly created park or green-belt.

An evaluation of the burden of the tax between landlords and renters requires further specification of the relation between land values and rent over time. Given the above as a basis, such an analysis would be a feasible extension and would be of value to planners interested in the spatial equity of local land taxes.
CHAPTER X

CONCLUSION

Within this dissertation a specific land tax has been explored in an urban spatial equilibrium setting. It has been assumed that a city rests upon an isotropic surface; the behavior of the inhabitants of this city is as if they have both perfect knowledge and are utility maximizers. All of the households are uniform in composition, income and tastes.

There are two classes of land use, urban and agricultural; however, for the most part, the agricultural rent gradient conforms to the abscissa. Further simplifying assumptions in the definitions of the parameters were given for brevity.

The spatial equilibrium land value surface was first derived utilizing Casetti's operationalization of Alonso's model with the addition of a structural inclusion of a simplified public economy. The endogeneous land value surface is substituted into the structural formulations defining household land consumption, the composite good, and for the first time, the public good. The spatial equilibrium quantity of public good is consequently derived. Multiplying the charge per unit of public good by the spatial equilibrium quantity of
public good, the spatial equilibrium optimum gross tax per unit of land is created. When this latter value is multiplied by the spatial equilibrium quantity of land, the optimal proportion of income devoted to supporting the public sector is achieved.

Except for the analysis of the public economy, the implications of the model developed within this dissertation is similar to that by Alonso and later by Casetti since the basic formulation is identical in all three. A basis for analyzing a spatial public economy was introduced by this dissertation. It was shown that the spatial equilibrium quantity of public good will decline with respect to distance from the central city; if the good is to be consumed it may be provided privately at relatively large distances from the city center. The gross tax rate, given that the cost of providing the public good is everywhere the same, should consequently decline in a negative exponential fashion from the city center. The further one is from the central city, the lower will be the proportion of income devoted to supporting the public economy. Consistent with normal goods, increases in the charge for public goods will increase the demand for substitutes, land in this model, and decrease the demand for public goods at every distance from the central city. Where the cost of providing public goods is high, say in rural areas, one would expect to observe fewer public goods and greater land consumption by the household. This result, though quite intuitive, was summarized in Figures 10 and 11 with interesting implications; these figures provide a pure geographic rationale for the Tiebout hypothesis for municipality
formations. In addition, the Tiebout hypothesis requires very strong assumptions for household mobility, population composition, perfect knowledge by both the household and municipality, and assumptions about the production function of the public good including externalities. Providing real world assumptions within the model developed within this dissertation would not require deviation from the equilibrium condition requiring a multicenter setting.

It was found that increases in the public good charge would decrease population densities thereby encouraging suburban sprawl. The cause for leap-frog sprawl was explored. It is expected that societies which have relatively high taste preferences for public goods, given an invariant charge for public goods, would most likely have relatively great population density gradients; their cities would be expected to have relatively less areal development. Planners aware of these theoretical spatial effects may be able to encourage the development of ideal urban centers and manipulate land uses other than by zoning controls.

There has been disagreement about the effect property taxes have upon market land values; consequently, this disagreement extends to the burden and, given an income distribution, the incidence. These concepts may not be seriously explored unless the spatial environment is considered. In this dissertation, given the assumption of two actors, landlords and future purchasers of land, the burden of an increase in the charge per unit of public good will fall upon the landlord. If society changes taste preferences towards greater
consumption of public good, landlords will be able to pass the cost forward to new purchasers of land; in this case, landlords may actually receive windfall gains.
APPENDIX A

So that a comparative static analysis of the cross effects between the public good and land may be succinctly presented, let \( p = z = 0 \), then the bordered determinant 4.17 becomes

\[
\begin{vmatrix}
0 & -(r+tg) & -tq \\
-(r+tg) & z_{qq} & z_{qg} \\
-tq & z_{gq} & z_{gg}
\end{vmatrix}
\]

where

\[2.A \begin{vmatrix} 0 & -(r+tg) \end{vmatrix} = -(r+tg)^2 < 0\]

and

\[3.A \begin{vmatrix} 0 & -(r+tg) \end{vmatrix} = 2(r + tg)(tq)z_{gq} - (r + tg)^2z_{gg} - (tq)^2z_{qq} > 0\]

Equation 3.A is positive since

\[z_{gg}, z_{qq} < 0\]
Equations 2.A and 3.A guarantee a maximum in this example.

Totally differentiate equations 4.06, 4.07 and 4.08 with respect to income, $y$

\[
\begin{vmatrix}
\partial \lambda / \partial y \\
\partial q / \partial y \\
\partial g / \partial y
\end{vmatrix}
\begin{vmatrix}
qr_y - 1 \\
-ry \\
0
\end{vmatrix}
\]

and with respect to the charge per unit of public good, $t$

\[
\begin{vmatrix}
\partial \lambda / \partial t \\
\partial q / \partial t \\
\partial g / \partial t
\end{vmatrix}
\begin{vmatrix}
\tau_t \\
-\lambda g \\
-\lambda q
\end{vmatrix}
\]

Solving 4.A and 5.A by Cramer's rule for the change in $q$ with respect to income and public good charge

\[
6.A \quad \frac{\partial q}{\partial y} = \frac{1}{H_3'}((r + tg)(qr_y - 1)Z_{gg} - tq(qr_y - 1) + (tq)^2 r_y(\lambda))
\]

\[
7.A \quad \frac{\partial q}{\partial t} = \frac{1}{H_3'}(qr_y Z_{gq} - tq^2(r + tg)(\lambda) - (tq)^2 g(\lambda) - r_y q(r + tg)Z_{gg})
\]

Because $\tau_t$ and $r_y$ are not equal to zero, the income and substitution
effects may not be identified; however,

\[ 8.A \quad \frac{\partial q}{\partial t} \leq 0 \]

if

\[ 9.A \quad t q r_t z_{qq} + t q^2 (\lambda) (r + t g) + (t q)^2 (\lambda) g \leq r_t (r + t g) z_{gg} \]

and

\[ 10.A \quad \frac{\partial q}{\partial y} \leq 0 \]

if

\[ 11.A \quad (t q)^2 (\lambda) r_y + t q (q r_y - 1) z_{qg} \leq (q r_y - 1)(r + t g) z_{gg} \quad \text{for} \quad (q r_y > 1) \]

Hence, in this geographic setting, the Slutsky income and substitution effects may not be clearly derived nor can a comparative static analysis as in equations 8.A through 11.A be presented without explicit values for the parameters. It is suspected that within this Slutsky formulation is hidden important spatial implications which warrant intensive investigation in the future.
APPENDIX B

In production theory, assuming a Cobb-Douglas production function, restricting the parameters to sum to less than 1 is necessary for a perfectly competitive market; an analogy to consumer theory would be weak except for the fact that the elements in the utility function must be independent. Solve for $\alpha$, $\pi$, and $\rho$.

1. $\alpha = \frac{\lambda t q g}{u}$

2. $\pi = \frac{\lambda p z}{u}$

3. $(\rho + \alpha) = \frac{\lambda (r + t g) q}{u}$

Substituting 3.01.B into 3.03.B

4. $\rho = \frac{\lambda q r}{u}$

Equation 4.34 requires that
$5. B \quad 0 < \frac{\lambda t q g}{u} + \frac{\lambda p z}{u} + \frac{\lambda q r}{u} < 1$

5. B simplified is

6. B \quad 0 < y - ks < u/\lambda

$\lambda$ is interpreted in consumer theory as the marginal utility. The point elasticity is defined as the ratio of the marginal function to the average function:

$7. B \quad \frac{du/dy}{u/y}$

Divide equation 6. B by $u/\lambda$

8. B \quad 0 < \lambda \frac{y - ks}{u} < 1

or

9. B \quad 0 < \eta_{y',u} < 1

where $y'$ represents the income not spend on transportation. Hence, equation 4.34 requires that the income elasticity of utility be less
than one; that is, it must be relatively inelastic. A relatively small change in income not spent on transportation may bring about a relatively large change in utility. A small change in transportation cost, due to either an increase in k or s, will result in a large change in utility.
APPENDIX C

The equilibrium quantity of population within radius $s$ is found by integrating over the population density

$$\bar{p}(s) = t^{-\alpha} L \int_0^s (y - ks)^a e^{-bs} s \, ds$$

where

$$L = (6.28)n m v(\rho + \alpha)^{-1}(\bar{u} - 1) p^{-\pi} \rho$$

$$b = \rho \Omega$$

$$a = wp - 1$$

The change in total population within radius $s$ with respect to the charge per unit of public good is found by first evaluating the integral in 1.C by Maclaurin's expansion technique and then evaluating the derivative

$$\frac{\partial \bar{p}(s)}{\partial t} = -\alpha t^{-(\alpha+1)} y^a s L(1 + \frac{s}{2}) < 0$$
Hence, an increase in the charge per unit of public good will decrease equilibrium population density within this static model since the second derivative is positive, this change occurs at a decreasing rate.
APPENDIX D

The spatial equilibrium limit of the urban environment may be solved for given the rate of change of $\bar{s}$ with respect to $t$. Alternatively, if the rate of change of $\bar{s}$ is determined by exogeneous institutional constraints, the spatial equilibrium limit of the urban environment may be presented.

From equation 1.C

$$\bar{p}(\bar{s}) = \bar{s}^{\gamma} t^{-\alpha} L(1 + \frac{\bar{s}}{t})$$

However, Casetti's constant $\bar{u}$ constrains $\bar{p}(\bar{s})$ to be invariant with respect to $t$. Hence

$$\frac{\partial \bar{p}(\bar{s})}{\partial t} = y^{\gamma} L t^{-\alpha} \left[ \frac{\partial \bar{s}}{\partial t} (1 + \frac{\bar{s}}{t} \right] \right] - \frac{\alpha}{2} \left(1 + \frac{\bar{s}}{t}\right)^{2}$$

First solve for the rate of change of the spatial equilibrium radius given an exogeneous change in $t$.
4. D \[ \frac{\partial s^t}{\partial t} = (-)^t s \]

Given that \( \frac{\partial s^t}{\partial t} \) has been empirically estimated to be some rate, say \( \theta[s] \), then

5. D \[ \frac{\partial s^t}{\partial t} = (-)^t \theta[s] \]
APPENDIX E

Let \( \bar{U}(s) \) represent the urban land value within radius \( s \) from the central city. \( \bar{U}(s) \) may be derived by integrating over the land value surface

\[
1.E \quad \bar{U}(s) = 2\pi n \int_0^s \frac{r}{s} s \, ds
\]

Simplifying equation 1.E

\[
2.E \quad U(s) = RT \int_0^s s(y - ks)^{\omega_p} e^{-\Omega s} \, ds
\]

\[
3.E \quad R = (\bar{u}^{-1} p^{-\pi} t^{-\alpha} H)^{\rho}
\]

\[
4.E \quad T = 2\pi n
\]

Expand the function in 2.E using MaClaurin's expansion method so that the integral may be more easily solved.

\[
5.E \quad f(s) = s(y - ks)^{\omega_p} e^{-\Omega s}
\]

\[
6.E \quad f(0) = 0
\]
7.E \[ f'(0)s = sy^{wp} = sC_1 > 0 \]

8.E \[ f''(0) \frac{s^2}{2} = -\frac{s^2}{2} [y^{wp-1} + y^{wp} \Omega] = -s^2C_2 < 0 \]

9.E \[ f'''(0) \frac{s^3}{6} = \frac{s^3}{6} [\Omega y^{wp} (y^{-1} + \Omega) + \Omega y^{wp-2}k + ky^{wp-2}] \]


10.E \[ \bar{U}(s) = R T \int_0^s (sC_1 - s^2C_2 + s^3C_3)ds \]

Equation 10.E is easily solved, with the restriction that \( \bar{U}(s) \) be positive

11.E \[ \bar{U}(s) = R T [\frac{s^2C_1}{2} - \frac{s^3C_2}{3} + \frac{s^4C_3}{4}] > 0 \]

The restriction that 11.E be positive requires that

12.E \[ \frac{s^2C_1}{2} + \frac{s^4C_3}{4} > \frac{s^3C_2}{3} \]

The change in \( \bar{U}(s) \) with respect to \( t \) may be derived
\[ 13.E \quad \frac{\partial \tilde{U}(s)}{\partial t} = -(\alpha)RC_4 t \]

where

\[ 14.E \quad C_4 = \frac{s^2 C_1}{2} - \frac{s^3 C_2}{3} + \frac{s^4 C_3}{4} > 0 \]

The second derivative of 10.E is

\[ (\alpha)(\alpha + 1)RC_4 t - \alpha RC_4 \]

\[ 15.E \quad = \alpha RC_4 (at + t - 1) \leftarrow 0 \]

if

\[ 16.E \quad t(\alpha + 1) \leftarrow 1 \]

As the charge per unit of public good increases, say the millage rate increases, the aggregate urban land value, for \( s < \bar{s} \) decreases.
APPENDIX F

Classify public goods with respect to two spatial subdivisions, spatially variant and spatially invariant public goods. A spatially invariant public good conforms to the traditional economic definition of a pure public good with the additional restriction that there occurs nonrival consumption across space and that the marginal cost of providing the public good an additional unit distance from the source is zero; hence, households or regions may not, or should not, be restricted from consumption of the public good. Many nationally distributed public goods conform to this classification; for example, the space program, military expenditure, foreign policy, and funding for scientific research. Perhaps a community's "good will" is the singular item which conforms to this restrictive criterion on the local level.

Spatially invariant public goods decline in either a discrete or continuous fashion, so that the marginal cost of providing the public good an additional unit distance from the source is greater than zero. Restrictions on consuming the good to specified regions should be considered. Most local public goods conform to this second sub-classification; each public good is expected to have a different cost-distance or access-cost gradient depending if the public good is
distributed to the household, say water or electricity, or if the household must travel to the source to consume the public good, say recreation land or educational facilities. The slope of the gradient may be due to the traditional geographic distance decay phenomenon, for example, agglomeration economies, transportation cost, information, and economies of scale. See Figures 17 and 18 for examples of these hypothetical relationships.
$\text{COST}$

Cost To Additional Household For Linking To Sewer or Water

Cost Of Laying Sewer Or Water Pipe Per Foot

Police Cost Per Mile Cruised

FIGURE 17

HYPOTHETICAL PUBLIC GOODS' COST DISTANCE RELATIONS
FIGURE 18

HYPOTHETICAL PUBLIC GOODS' ACCESS COST DISTANCE RELATIONS
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