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ESSAYS IN THE COMPARATIVE STATICS OF PORTFOLIO ADJUSTMENT

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
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* * * * *

The Ohio State University
1975

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Edward J. Kane
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To Erik, Tigran, Kerstin, and my Parents. Without their help and constant sacrifice this dissertation would never have been written.
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Chapter 1.

Introduction.

This dissertation consists of a set of interrelated essays dealing, except for Chapter 8, with various aspects of the comparative statics of portfolio selection in the Mean-Variance framework. Chapter 8 extends the analysis and applies it to aspects of the theory of consumer choice, as well as asset choice in general.

While a number of studies exist on the comparative statics of portfolio choice, I show that none provides a satisfactory analysis. This study attempts to reformulate and improve the previous analysis and also provide useful empirical propositions. The following is a brief summary of each chapter.

Chapter 2 consists of a discussion of the work of Levy (19), who attempts to derive demand as well as Slutsky equations for assets under risk using the Separation Theorem. I show that Levy misapplies the Separation Theorem and that his equations are wrong. In Chapter 3 I provide an analysis which remedies Levy’s mistakes and provides correct demand and Slutsky equations for risky assets, using the Separation Theorem.
Chapter 4 is a review of the rest of the literature on the comparative-statics optimal portfolio adjustment. The literature consists of the works of Allingham and Morishima (2), Bierwag and Grove (5), Royama and Hamada (27), and Morishima (23). I show that these studies misidentify or improperly identify the comparative-static effects underlying optimal portfolio adjustment. This misidentification is partly due to the fact that these studies treat assets in a manner analogous to commodities in traditional consumer theory, i.e., as ultimate objects of utility. Thus Royama and Hamada incorrectly identify, in the case of a change in the expected-return of an asset, or its variance-covariance, wealth and substitution effects which they claim are exactly analogous to the commodity-wealth and substitution effects of traditional consumer theory. The other authors identify "Veblen" or "Want-Pattern" effects for such changes, meaning that the utility function is shifting. The reason for the shift is that utility is expressed as a direct function of assets, so that changes in the qualities of assets, such as their expected return, or variance-covariance, directly affect the utility function or the preference-ordering over assets. In Chapter 6 I argue that in the Mean-Variance framework an asset is demanded indirectly because of its contribution to portfolio expected return and variance and not for other intrinsic properties.
Thus the demand for an asset is analogous to the demand for a factor of production.

The treatment of an asset as a factor of production places the problem of asset-demand within Lancaster's reformulation of demand theory. The Lancaster approach to demand incorporates into the analysis information on the objective properties or characteristics of goods. Under this approach, the demand for a good becomes a derived demand for the characteristics of the good where goods are inputs producing these characteristics. One important advantage of this approach, as we show, is that it makes possible the derivation of useful empirical propositions. The usefulness of the Lancaster approach is best illustrated by Lancaster's following remarks on the disadvantages of the traditional approach to demand:

"Once we are given the preference map, we have already incorporated all the information concerning goods as such. The physical properties of goods relevant to the consumer have been presumed to have been taken into account by the consumer in deciding whether he prefers one collection or another. If the goods had been different, the preference map would have been different; that is all we can say.

"Since the traditional analysis starts with this preference diagram, the properties of goods have been swallowed up in the preferences before the analysis even commences, and there is no possibility of using information concerning these properties anywhere in a later stage. With no theory of how the properties of goods affect the preferences at the beginning, traditional analysis can provide no predictions as to how demand would be affected by a specified change in one or more properties of a good, or how a "new" good would fit into the preference pattern over existing goods. Any change in any property of any good
implies that we have a new preference pattern for every individual: we must throw away any information derived from observing behavior in the previous situation and begin again from scratch."

The new approach incorporates into the analysis factors which are traditionally considered exogenous to the consumer's preference function. This underlies our reformulation, for example, of the literature on demand with "variable" preferences, in Chapter 8.

In Chapter 6, I derive demand and comparative-static equations for assets in a Lancaster-type framework. I identify changes in expected return, or variance-covariance, of an asset as equivalent to changes in the productivity of the asset in producing ultimately-desired characteristics. I call such effects productivity effects. We can break down the productivity effect into wealth and substitution effects. An alternative and empirically more useful breakdown of the productivity effect is also provided. Ultimately I generate some very useful empirical propositions with respect to changes in the various characteristics of assets. The framework of Chapter 6 can be generalized to analyze demand equations using utility functions which include in addition to E and V, other characteristics (or moments) of the distribution. Also, a more general measure of risk, instead of variance, may be used.

Chapter 5 provides an analysis of the mapping from Mean-Variance space to asset space which is useful to the
analysis in Chapter 6. In this context I show that Tsiang's (34) recent contention, that closed and circular indifference curves in asset space cannot exist if one assumes that the slopes of E-S indifference curves are always less than unity, is invalid.

Chapter 7 applies the framework of Chapter 6 to an analysis of taxation and the demand for a risky asset. Various types of taxes are discussed.

In Chapter 8 we generalize the framework developed in Chapter 6 to portfolio choice with more than two characteristics and also apply it to traditional consumer demand theory. Implications of the approach to the literature on demand under "variable" consumer preferences are examined. We find that such an approach to consumer theory provides a more useful model of analysis than the traditional one, with testable empirical implications.
Notes to Chapter One

1 See Lancaster (18). Note that this approach is not new. It has been suggested also by Hicks (13), and Morishima (22). See also Roberts (25) on the parallels between the Lancaster framework and portfolio analysis.

2 See Lancaster (18) page 4.

3 For the literature on demand under "variable" preferences see Ichimura (15), Tintner (31), Basmann (4).
Chapter 2

The Demand for Assets Under Conditions of Risk--Comment on Levy

In a recent article in the Journal of Finance,¹ H. Levy presents a model of the demand for assets under risk. Levy derives a set of equations which he claims are analogous to the Slutsky equation in consumer theory. I show, that his model is wrong, and that his equations do not correspond to the Slutsky equation in consumer theory.

Levy's analysis is in terms of the Mean-Variance framework. Investors are assumed to be risk-averse, regarding variance (an indicator of risk) as a bad and expected return as a good. The analysis is carried out in terms of n risky assets and one riskless asset. The Separation Theorem² is used, but its implications are largely neglected. This theorem holds that in the presence of a riskless asset, the investor can separate the choice of optimal composition of the risky portfolio from the choice of optimal total portfolio composition, between risky assets as a whole and the riskless asset. This procedure involves two separate optimization processes. Levy carries out only one of these, thus determining the optimal mix of risky
securities. He neglects the second optimization which determines optimal total portfolio composition. This neglect leads to a crucial mistake in the derivation of demand for a risky asset. It also implies, wrongly, that the demand for a risky asset is independent of the investor's preference function.

Levy minimizes the variance of the investor's wealth next period, subject to the expected value of his wealth next period, to obtain the locus of efficient portfolios. He minimizes a function of the form,

\[ C = \sum_{i=1}^{n} x_i^2 \sigma_i^2 + 2 \sum_{i < j}^{n} x_i x_j \rho_{ij} \sigma_i \sigma_j + 2 \rho \left[ u - \sum_{i=1}^{n} x_i u_i - \left( W_0 - \sum_{i=1}^{n} x_i \right) R_F \right] \]  

where, \( u = E_{w_1} - w_0 \),

and, \( E_{w_1} \) = the expected value of the investor's wealth next period,

\( \sigma_{w_1}^2 \) = the variance of the investor's wealth,

\( x_i \) = the amount invested in asset \( i \),

\( u_i \) = expected rate of return on asset \( i \),

\( R_F \) = rate of return on safe asset,

\( W_0 \) = initial wealth of the investor,
\[ \sigma_i^2 = \text{variance of asset } i, \]

\[ Q_{ij} = \text{the correlation coefficient between } i \text{ and } j, \]

\[ \rho = \text{is a Lagrange multiplier}. \]

He differentiates \( C \) with respect to \( X_i \), and gets \( n \) first order conditions. 3 He solves these to get

\[ X_i = \rho \left[ \frac{u_i - R_f}{\sigma_i^2} - \sum_{j=1}^{n} \frac{u_j - R_f}{\sigma_i \sigma_j} Q_{ij} \right], \quad (2) \]

where, \( \sigma_i^2 = \sigma_i^2 \frac{M}{M_{ii}} \), \( Q_{ij} = \frac{M_{ij}}{M_{ii} M_{jj}} \),

and, \( M \) is the correlation matrix, with unity on the diagonal and correlation coefficients on the off-diagonal elements.

\( \sigma_i^2 \) is called the residual variance of \( i \), the variance left after removing the effects of other variables on variance of \( i \). \( Q_{ij} \) is the partial correlation coefficient between \( i \) and \( j \).

Levy treats equation (2) as the demand function for asset \( i \). He rewrites (2) as

\[ X_i = \rho \cdot Y_i, \quad (3) \]

where

\[ Y_i = \left[ \frac{u_i - R_f}{\sigma_i^2} - \sum_{j=1}^{n} \frac{u_j - R_f}{\sigma_i \sigma_j} Q_{ij} \right], \quad i \]
Figure 1 gives the geometrical interpretation of the above optimization procedure. CBDA in the figure represents feasible risky portfolio combinations, in the absence of a riskless asset. The assumption of risk-aversion makes BDA the locus of efficient portfolio combinations, along which $\sigma_{W_1}^2$ is minimum for every $E_{W_1}$.

The introduction of a riskless asset expands the efficient set. Given a riskless rate $R_F$, at which both borrowing and lending are allowed, the new efficiency locus becomes the straight line aDa, which is tangent to the old efficiency set at D. The investor now confines his investment to different combinations, along aDa, of the riskless asset and the optimal composition of risky assets corresponding to D. This composite of risky assets has been called the Hicksian security.\(^4\)

The optimization procedure is equivalent to maximizing the slope of the line aDa subject to the locus BDA. Note that BDA has to be convex from above.

All $X_i$'s are assumed by Levy to be greater or equal to zero throughout, i.e., $0 \leq X_i \leq 0.5$. Levy also shows that $\rho$ is reciprocal of the slope of the line aDa. That,

$$\rho = \frac{\sigma_{W_1}^2}{E_{W_1} - W_0(1+R_F)}$$  \hspace{1cm} (4)
This means that the slope of aDa "represents the minimum premium that the investor requires when his portfolio risk $\sigma(W_1)$ rises by one unit." 6

Levy analyzes the comparative statics of effects of changes in various variables on $X_i$ in terms of equation (3).

He differentiates (3) with respect to $u_i$, holding the other independent variables constant, to get

$$\frac{\partial X_i}{\partial u_i} = \frac{\partial \rho}{\partial u_i} \chi + \rho \frac{\partial \gamma_i}{\partial u_i}; \tag{5}$$

He rewrites (5) as

$$\frac{\partial X_i}{\partial u_i} = \frac{\partial(E_{W_1})}{\partial u_i} \frac{\partial \rho}{\partial(E_{W_1})} \cdot \chi + \rho \frac{1}{\sigma_i^2}. \tag{6}$$

Levy claims that equation (6) is analogous to the Slutsky equation in consumer demand theory. Quoting, 7 "The change in $\frac{\partial X_i}{\partial u_1}$ includes two parts: the 'substitution effect,' i.e., the change in $X_i$ as a result of the change in $u_1$ when we treat expected wealth as a constant, and the 'wealth effect,' or 'income effect' i.e., the change in $X_i$ due to a change in the expected wealth." The first term on the right side of equation (6) is said to be the "income effect" and the second term the "substitution effect."
Notice Levy's concept of "real income." He claims that keeping expected wealth constant is analogous to keeping utility, or real income, constant in consumer theory. This is the way he defines his substitution effect. This procedure is wrong. Expected wealth may be held constant, but risk may be raised or lowered, yielding a lower or higher level of utility, or real wealth. That is, the investor's real wealth can be changed by a change in the asset's risk also. Thus Levy's substitution term does not hold utility, or real wealth, constant.

According to Levy, the substitution effect in equation (6) is always positive, since $\frac{1}{\sigma_i^2} > 0$. The income effect has ambiguous sign since $\frac{\partial (EW_1)}{\partial u_1} \geq 0$. Quoting Levy, "If the investor ends up with a portfolio with higher expected wealth, $\frac{\partial (EW_1)}{\partial u_1} > 0$. But since the investor might decide to move to portfolio with less risk, and less expected wealth, $\frac{\partial (EW_1)}{\partial u_1}$ might be negative."

Implicit in this argument, is that the income effect is determined by the investor's utility-of-wealth function. According to Levy $\frac{\partial f}{(EW_1)} > 0$, which contradicts equation (4). In equation (4) $\frac{\partial f}{(EW_1)} < 0$.

Levy shows the two effects graphically, as in Figures 2 (in the case of a change in $u_1$) and 3 (in the
Figure 2

Expected Wealth

Standard Deviation

$W_0(1+R_F)$
Figure 3
case of a change in $\sigma_i$). The curves $I_1$, $I_2$, $I_3$ are indifference curves in $E-\sigma$ space. He claims that his substitution and income effects in equation (6) correspond to the movements from $M$ to $M''$, and $M''$ to $M''$, respectively. Curiously, Levy's substitution term in equation (6), $\rho \frac{1}{\sigma_i}$, does not include parameters of the utility function drawn in the figure. Yet, the sign and magnitude of the substitution term do depend on parameters of the utility function $U(E, \sigma)$, in analogous fashion to the substitution effect in consumer theory. Similar remarks apply to Levy's income effect, which however is not explicitly formulated.

In the case of a change in $\sigma_i$, Levy's income and substitution effects become,

$$\frac{\partial X_i}{\partial \sigma_i} = \frac{\partial \rho}{\partial \sigma_i} \gamma_i + \rho \frac{\partial \gamma_i}{\partial \sigma_i}, \quad (7)$$

$$= \frac{\partial \sigma_{W_i}}{\partial \sigma_i} \frac{\partial \rho}{\partial \sigma_i} \gamma_i - \frac{\rho [u_i - R_F]}{\sigma_i^{\frac{3}{2}}} \sum_{j=1}^{n} \frac{u_j - R_F \sigma_{i_j}}{\sigma_j} - \frac{\rho}{\sigma_i} \frac{u_i - R_F}{\sigma_i^{\frac{3}{2}}}, \quad (8)$$

where, the second and third terms in (8) are said to represent the substitution effect, and the first term the income effect. The substitution effect is negative,
while the sign of the income effect is ambiguous. This ambiguity arises, according to Levy, because
\[ \frac{\partial \sigma_{i}}{\partial \sigma_{1}} \geq 0. \]

The criticisms made of Levy's comparative statics underlying equation (6) apply in analogous fashion to equation (8).

The optimization procedure carried out by Levy determined the efficiency locus aDa shown in Figure 1. According to Levy, Equation (3) depicts the optimal quantity of asset i demanded on the efficiency locus. But it does not. The optimal point on aDa is determined by maximizing the investor's preference function along it. Levy does not carry out this maximization procedure. Equation (3) merely states that the quantity of asset i demanded is proportional to its share in the Hicksian security.\(^{11}\) Equation (3) implies that the relative composition of the Hicksian security is invariant along the line aDa. That is,

\[ \frac{\gamma_i}{\gamma_j} = \frac{\rho \gamma_i}{\rho \gamma_j} = \frac{\gamma_i}{\gamma_j}, \quad (9) \]

for assets i and j.

While the relative composition of the Hicksian security is constant along the efficiency locus,
the absolute quantity of the Hicksian asset (and therefore the absolute quantity of asset i) varies along it. This variation, or substitution between the Hicksian asset and riskless asset, depends upon the investor’s preference function between risk and return. Figure 4 shows the equilibrium proportion \( Z \) (where \( Z \geq 0 \)) of the Hicksian security held in the total portfolio, for two alternative convex indifference curves \( I_1 \) and \( I_2 \). Note that for any point along aDa in the figure,

\[
E_{W_i} = W_0 (1 - Z) (1 + R_F) + Z E_P ,
\]

\[
\sigma_{W_i} = Z \sigma_P ,
\]

or

\[
Z = \frac{\sigma_{W_i}}{\sigma_P} ,
\]

where \( E_P \) and \( \sigma_P \) are expected return and standard deviation of the Hicksian security at D. The bottom portion of the figure shows equation (12). To each value of \( \sigma_{W_1} \) on the opportunity locus aDa corresponds a different value of \( Z \), and therefore, a different value of \( X_i \).

Thus, \( X_i \) in equation (3) does not depict the amount of asset i demanded by the investor at an equilibrium point such as M in Figure 2.

The income and substitution effects, correctly formulated, each correspond to movements from one initial
Figure 4
equilibrium point of utility maximum to another; yet Levy's income and substitution effects are not based on any conditions governing equilibria at these points. Correctly interpreted, Levy's substitution term \( \frac{1}{\sigma_i} \), merely shows the change in the amount of asset i composing the Hicksian security, as \( u_1 \) changes.

In Section IV in his article Levy states, \(^{12}\) "We shall prove in this section that in the mean-variance framework one cannot determine if security and expected wealth are normal goods." Security refers to the riskless asset.

Levy's proof consists of showing, using equation (6), that since for all assets \( i (i = 1, \ldots, n) \) the sign of \( \frac{\partial X_i}{\partial u_1} \) is ambiguous \( \left( \frac{\partial E_{W_1}}{\partial u_1} \right|_i \neq 0 \), one cannot determine whether \( \sum X_i \) increases or decreases as a result of a change in \( u_1 \). Hence, one cannot determine whether the amount of the riskless asset held, \( \sum X_i \), increases or decreases as a result of the change in \( u_1 \). Therefore, one cannot determine whether security is normal or inferior. He also reasons that one cannot determine whether expected wealth is normal, by arguing, \(^{13}\)

"Moreover, even if the investor decides to decrease his investment in each risky asset (i.e., security \( i \) is a normal good), one cannot determine if an expected wealth is a normal good. A decrease in each \( X_i \) and increase in the investment amount in bonds \( (W_o \sum X_i) \) induces a decrease in the expected wealth..."
of the new optimal portfolio. On the other hand, the rise of \( u_1 \) increases the expected wealth of the portfolio, and consequently, based on the above analysis, one cannot determine if expected wealth is a normal or inferior good."

Notice first that Levy's definition of normality is wrong. The correct definition concerns the change in the quantity of asset \( i \) held as wealth, not \( u_1 \), increases. Second, Levy's proof is circular. He uses the fact that expected wealth may be normal or inferior to establish that security and expected wealth are normal or inferior. More important, however, is that Levy's statement, and the proof, are misdirected simply because one cannot determine whether any good or asset is normal or inferior, in mean-variance or any other framework, without reference to the investor's utility function. Levy neglects the fact that one cannot derive demand functions without specifying utility functions.

In the same context, Levy criticises Tobin, "Although Tobin does not claim that both security and expected wealth are normal goods, in his analysis he gives an example which treats the two variables as normal goods." Tobin's example, however, is in terms of a specific pattern of indifference curves which lead to normality of expected wealth and security. Tobin, in fact, also mentions that a different pattern of indifference curves might lead to inferiority."
In section V, Levy derives substitution-complimentarity relationship between risky assets, such as $\frac{\partial X_i}{\partial u_j}$. His results are wrong, because the underlying demand relationship (3) is wrong.

To summarize, I have shown that the demand for a risky asset is determined, in the framework of the Separation Theorem, by two separate optimization processes. Levy's basic mistake lies in his failing to maximize a utility function subject to the efficiency locus. This produces a demand function for any risky asset that is independent of the investor's preference function. Levy does not derive a demand function, nor do his subsequent equations correspond to Slutsky equations.
Notes to Chapter Two

1. See Levy (19).

2. See Tobin (32), Lintner (20), Sharpe (29).

3. Second-order conditions for a minimum (which Levy does not give) require that the bordered Hessian determinant be positive definite, i.e.,

\[
H = \begin{pmatrix}
\frac{\partial^2 C}{\partial x_1^2} & \cdots & \frac{\partial^2 C}{\partial x_1 \partial x_j} & \cdots & \frac{\partial^2 C}{\partial x_1 \partial x_n} & \frac{\partial^2 C}{\partial \rho \partial x_1} \\
\vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
\frac{\partial^2 C}{\partial x_j \partial x_1} & \cdots & \frac{\partial^2 C}{\partial x_j^2} & \cdots & \frac{\partial^2 C}{\partial x_j \partial x_n} & \frac{\partial^2 C}{\partial \rho \partial x_j} \\
\vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
\frac{\partial^2 C}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 C}{\partial x_n \partial x_j} & \cdots & \frac{\partial^2 C}{\partial x_n^2} & \frac{\partial^2 C}{\partial \rho \partial x_n} \\
\frac{\partial^2 C}{\partial \rho \partial x_1} & \cdots & \cdots & \cdots & \frac{\partial^2 C}{\partial \rho \partial x_n} & 0
\end{pmatrix} < 0
\]

4. This name derives from the analogy of the concept with that of composite goods discussed by Hicks in (14); see his Mathematical Appendix, part 10. The term is due to Ed Kane (17).

5. Although Levy just assumes that all \(x_i\) are greater than zero, i.e., no short sales, the second order conditions given above in footnote (3), in fact guarantee that \(Y_i\)
in Equation (3) is greater than or equal to zero, if all $u_i$'s are greater than zero.

6 See Levy (19) page 81.

7 See Levy (19) page 85.

8 See Levy (19) page 86.

9 See Mathematical Appendix in Hicks (14).

10 Note that $\frac{\partial \sigma_i}{\partial \sigma_i} > 0$, so that the signs are not altered by differentiating with respect to $\sigma_i$ or $\sigma_i$.

11 See Tobin (32).

12 See Levy (19) page 91.

13 See Levy (19) page 92.

14 See Levy (19) page 91.

15 See Tobin (32) page 51.
Chapter 3
Comparative-Statics of Portfolio Choice using the Separation Theorem

In recent years, a number of authors have derived demand equations, as well as so-called Slutsky equations, for risky assets in the Mean-Variance framework. However, to my knowledge, no one has successfully incorporated the Separation Theorem into such analysis. In a recent article, H. Levy attempts to do just that, but I have shown elsewhere that Levy does not derive a demand equation and that his comparative-static results are wrong.

In this paper, I make use of the Separation Theorem (which holds that in the presence of a riskless asset, the investor can separate the choice of the optimal composition of the risky portfolio from the choice of optimal total portfolio composition, between risky assets as a whole and the riskless asset) to develop a comparative-static analysis of adjustments in the optimal composition of a portfolio containing n risky assets and one riskless asset.

I show that the demand for a risky asset is derived, in the framework of the Separation Theorem, by two separate optimization processes. In analogous fashion to some
previous results, the comparative-static results that I derive have ambiguous signs, which cannot be restricted without specifying both the investor's preference function between risk and return, and the values of the moments of the distribution of returns for assets. The analysis has important implications for, among others, Liquidity Preference Theory, as well as the Theory of Taxation and Risk-Taking.

**Derivation of Demand for a Risky Asset**

In the framework of the Separation Theorem, the locus of efficient asset combinations is obtained by minimizing the variance of the investor's wealth, subject to the expected value of the investor's wealth. Minimize a function $C$,

$$C = \sum_{i=1}^{n} \left[ X_i \sigma_i^2 + 2 \sum_{j > i}^{n} X_i X_j \rho_{ij} \sigma_i \sigma_j \right]$$

$$+ 2 \lambda \left[ E_W - \sum_{i=1}^{n} X_i (1 + u_i) - (W_0 - \sum_{i=1}^{n} X_i) (1 + R_F) \right],$$

where,

$E_W$ = the expected value of the investor's wealth,

$\sigma_W^2$ = the variance of the investor's wealth,
$W_0$ = initial wealth of the investor,
$X_i$ = the amount invested in asset $i$,
u$_i$ = the expected rate of return on asset $i$,
$R_F$ = rate of return on safe asset,
$\sigma_i^2$ = variance of asset $i$,
$Q_{ij}$ = the correlation coefficient between $i$ and $j$,
$\lambda$ = a Lagrange multiplier.

Differentiating $C$ with respect to $X_i$, we get $n$ first-order conditions for a minimum, which can be written in the matrix form

\begin{equation}
\nabla X = \lambda \left[ u - R_F 1 \right]
\end{equation}

where, $V$ is the variance-covariance matrix.

$$u' = \left[ u_1, u_2, \ldots, u_n \right],$$

$$X' = \left[ X_1, \ldots, X_n \right].$$

Second order conditions for a minimum require that the bordered Hessian determinant be positive definite, i.e.,
\[
\begin{bmatrix}
\frac{\partial^2 c}{\partial x_1^2} & \cdots & \frac{\partial^2 c}{\partial x_1 \partial x_j} & \cdots & \frac{\partial^2 c}{\partial x_1 \partial x_n} & \frac{\partial^2 c}{\partial x_1 \partial \lambda} \\
\vdots & & \ddots & & \vdots & \vdots \\
\frac{\partial^2 c}{\partial x_j \partial x_1} & \cdots & \frac{\partial^2 c}{\partial x_j^2} & \cdots & \frac{\partial^2 c}{\partial x_j \partial x_n} & \frac{\partial^2 c}{\partial x_j \partial \lambda} \\
\vdots & & \ddots & & \vdots & \vdots \\
\frac{\partial^2 c}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 c}{\partial x_n \partial x_j} & \cdots & \frac{\partial^2 c}{\partial x_n^2} & \frac{\partial^2 c}{\partial x_n \partial \lambda} \\
\frac{\partial^2 c}{\partial \lambda \partial x_1} & \cdots & \cdots & \cdots & \frac{\partial^2 c}{\partial \lambda \partial x_n} & 0
\end{bmatrix} < 0
\]

Solving the first-order conditions we get

\[
X^*_i = \lambda \left[ \frac{u_i - R_F}{\sigma_i^2} - \sum_{\substack{j=1 \atop j \neq i}}^n \frac{u_j - R_F}{\sigma_i \sigma_j} Q_{ij} \right], \quad (4)
\]

where, \(Q_{ij}\) is the partial correlation between \(i\) and \(j\), after removal of any part of the correlation due to the influence of other assets. It is defined as

\[
Q_{ij} = \frac{-M_{ij}}{\sqrt{M_{ii} M_{jj}}}
\]
$\Sigma$ is the correlation matrix with unity on the diagonal and correlation coefficients on the off-diagonal elements. $M_{ij}$ is the cofactor of the $(i,j)^{th}$ element of $\Sigma$.

Note also that $\sigma_i^2$ is the residual variance of $i$, the variance left after removing the effects of other variables on variance of $i$. It is defined as $\sigma_i^2 = \sigma_i^2 \frac{M_{ii}}{M}$.

In the two asset case, equation (4) becomes

$$X_i = \lambda \left[ \frac{u_i - R_F}{\sigma_i^2} - \frac{u_2 - R_F}{\sigma_2 \sigma_i} \right]$$

where $\sigma_i^2 = \sigma_1^2 (1-r^2)$, and $Q_{12} = r$. We can rewrite equation (4) in the form,

$$X_i = \lambda Y_i$$

where

$$Y_i = \left[ \frac{u_i - R_F}{\sigma_i^2} - \sum_{j=1, j \neq i}^{n} \frac{u_j - R_F}{\sigma_i \sigma_j} Q_{ij} \right]$$

The second order conditions (3) guarantee that $Y_i$, and therefore $X_i$, is greater than zero ($x_i \geq 0$) for any risky asset $i$ (that is, short-sales will not occur), when all $u_i$ are greater than $R_F$.

Equation (6) implies that the relative composition of risky assets is constant along the efficiency locus. That is,

$$\frac{X_i}{X_j} = \frac{\lambda Y_i}{\lambda Y_j} = \frac{Y_i}{Y_j}$$
Figure 5 provides a geometrical interpretation of the above optimization procedure, in the case of two risky assets $X_i$ and $X_j$. The lines $R_oR_o, R_1R_1$, etc., are constant-return loci. Each shows the different combinations of $X_i$ and $X_j$ yielding a given value of $E_w$. The lines $C_oC_o, C_1C_1$, etc., are constant-risk loci. Each shows the different combinations of $X_i$ and $X_j$ yielding a given value of $\sigma_w^2$. The optimization procedure in equation (1) yielded dominant combinations of $X_i$ and $X_j$ which minimize $\sigma_w^2$ for any $E_w$. The locus of such dominant combinations is the straight line OPP', along which the relative proportion of risky assets is constant, but the absolute amount varies.

H. Levy falsely treats equation (6) as the demand function for asset $i$. However, the optimum point along OPP', and thus the optimum quantity of asset $i$ demanded, is determined by the further step of maximizing the investor's utility function of $E_w, \sigma_w$ along it. Levy's approach implies wrongly, that the demand for a risky asset is independent of the investor's utility function.

The utility maximization procedure can best be shown in terms of Figure 6. CBDA, in the top quadrant of the figure, represents feasible risky portfolio combinations, in the absence of a riskless asset. The assumption of risk-aversion makes BDA the locus of efficient portfolio combinations, along which $\sigma_w^2$ is minimum for every $E_w$. 
\( \sigma_i > \sigma_o \)
\( \epsilon_i > \epsilon_o \)
The introduction of a riskless asset expands the efficient set. Given a riskless rate $R_p$, at which both borrowing and lending are allowed, the new efficiency locus (yielded by the optimization in equation (1)) becomes the straight line $aD_a$, which is tangent to the old efficiency set at $D$. The investor now confines his investment to different combinations, along $aD_a$, of the riskless asset and the optimal composition of risky assets corresponding to $D$. This composite of risky assets has been called the Hicksian security. Note again that the relative composition of the Hicksian security is invariant along the line $aD_a$.

While the relative composition of the Hicksian security is constant along the efficiency locus, the absolute quantity of the Hicksian asset (and therefore the absolute quantity of asset $i$) varies along it. This variation, or substitution between the Hicksian asset and riskless asset, depends upon the investor's preference function between risk and return. Figure 6 shows the equilibrium proportion $Z$ (where $Z \geq 0$) of the Hicksian security held in the total portfolio, for two alternative convex indifference curves $I_1$ and $I_2$.

Note that for any point along $aD_a$ in the figure,

$$E_w = W_o(1-Z)(1+R_F) + ZE_p ,$$  \hspace{1cm} (7)
Figure 6
\[ \sigma_w = \sqrt{\sigma_p}, \quad (8) \]

or

\[ Z = \frac{\sigma_w}{\sigma_p}, \quad (9) \]

where \( E_p \) and \( \sigma_p \) are expected return and standard deviation of the Hicksian security at \( D \). The bottom portion of the figure shows equation (9). To each value of \( \sigma_w \) on the opportunity locus \( dD_a \) corresponds a different value of \( Z \), and therefore, a different value of \( X_p \).

More formally, the individual maximizes a utility function \( U(E_w, \sigma_w) \) subject to the efficiency locus \( dD_a \). This yields the first-order condition

\[
\frac{-U_{E_w}}{U_{\sigma_w}} = \frac{E_p - \sigma_w(1+R_F)}{\sigma_p} , \quad (10)
\]

that the slope of the indifference curve equals the slope of the efficiency locus at the point of utility maximum.

Note that \( U(E_w, \sigma_w) \) is assumed to be continuous and at least twice differentiable, with \( U_{E_w} > 0 \) and \( U_{\sigma_w} < 0 \). This implies that the indifference curves are positive sloping, that is

\[
\left. \frac{dE_w}{d\sigma_w} \right|_{U = u_0} = -\frac{U_{\sigma_w}}{U_{E_w}} > 0 , \quad (11)
\]

I also assume that the determinant
which implies that the investor has indifference curves
convex from below, as shown in Figure 6. This guaran-
tees that we have a maximum.

The first order condition yields equilibrium values
for $E_w, \sigma_w$, and substituting these values into (9) we get
the equilibrium value for $Z$, as is shown in Figure 6.

The total demand for asset $i$, $\bar{X}_i$, can now be written
as

$$\bar{X}_i = \frac{\sigma^2}{\sum_j Y_j} \left( \frac{Y_i}{\sum_j Y_j} \right), \quad j = 1, \ldots, i, \ldots, n; \quad Z > 0. \quad (13)$$

Note that $\frac{Y_i}{\sum_j Y_j}$ is the proportion of the risky asset $i$
in the Hicksian security. The Separation Theorem states
that this proportion is invariant of the preference func-
tion.

I now consider the comparative-statics of the effects
on $\bar{X}_i$ of changes in various variables.

(A) The Effect of a Change in Expected Yield

Considering the effect on $\bar{X}_i$ of a change in $u_i$, hold-
ing constant $\sigma_i, \sigma_j, u_{ij}, Q_{ij}$.
\[ \frac{\partial X_i}{\partial u_i} = W_0 \frac{\partial Z}{\partial u_i} \left[ \frac{Y_i}{\sum_j Y_j} \right] + W_0 Z \left[ \frac{\partial}{\partial u_i} \left( \frac{Y_i}{\sum_j Y_j} \right) \right], \quad (14) \]

The first term in equation (14) represents the change in the amount of asset \( i \) held, due to the change in the amount held of the Hicksian security. The composition of the Hicksian security is kept constant. This term contains both an income effect and a substitution effect.

The second term in Equation (14) represents the change in the amount of asset \( i \) held due to the change in the composition of the Hicksian security, as a result of the change in \( u_i \).

I first examine the sign of the second term in (14).

\[ \frac{\partial}{\partial u_i} \left[ \frac{Y_i}{\sum_j Y_j} \right] = \frac{\partial Y_i}{\partial u_i} \left( \sum_j Y_j \right) - Y_i \frac{\partial}{\partial u_i} \left( \sum_j Y_j \right), \quad (15) \]

\[ = \frac{1}{\sum_j Y_j} \left[ (\sum_j Y_j) - Y_i \right] + \sum_j \frac{1}{\sum_j Y_j} Q_{ij}. \quad (16) \]

If \( Q_{ij} \) is strictly nonnegative, the numerator in (16) is unambiguously greater than zero. Therefore,

\[ W_0 Z \left[ \frac{\partial}{\partial u_i} \left( \frac{Y_i}{\sum_j Y_j} \right) \right] > 0, \quad \text{if} \quad Q_{ij} > 0 \]
The sign of the first term in equation (14) is however ambiguous. The reason is that

From equation (9) we can solve for \( \frac{\partial Z}{\partial u_i} \),

\[
\frac{\partial Z}{\partial u_i} = \frac{[\frac{\partial \sigma_v}{\partial u_i}] \sigma_p - \sigma_v [\frac{\partial \sigma_p}{\partial u_i}]}{[\sigma_p]^2},
\]

Both \( \frac{\partial \sigma_v}{\partial u_i} \) and \( \frac{\partial \sigma_p}{\partial u_i} \) have ambiguous signs, so that \( \frac{\partial Z}{\partial u_i} \) is ambiguous. The sign and magnitude of \( \frac{\partial \sigma_w}{\partial u_i} \) depends upon the investor's preference function. The sign of \( \frac{\partial \sigma_p}{\partial u_i} \) is ambiguous, it depends upon the values of the parameters determining the Hicksian security, as shown in Appendix II.

Figure 7 demonstrates the factors underlying the sign and magnitude of \( \frac{\partial Z}{\partial u_i} \). The line aBa is the original opportunity locus and the line aDa is the opportunity locus after the increase in \( u_i \). The point \( C_0 \) represents the initial portfolio equilibrium point before the increase in \( u_i \). The line OLL' is the locus of points of equal \( Z \), such as \( C_0 \) and \( C_0' \), since

\[
Z_0 = \frac{\sigma_{w_0}}{\sigma_{p_0}} = \frac{NL}{NB} = \frac{ML'}{MB} = \frac{\sigma_{w_0}'}{\sigma_{p_0}'}. 
\]

Points to the right of \( C_0 \) or \( C_0' \) (along aBa or aDa) correspond to values of \( Z \) higher than that initially prevailing at \( C_0 \).

The movement from the initial equilibrium point \( C_0 \) to the final equilibrium point \( C_2 \) can be separated into
an income and a substitution effect. The sum total of these two effects determines the sign and magnitude of \( \frac{\partial g_w}{\partial u_i} \). We can write,

\[
\frac{\partial g_w}{\partial u_i} = (\frac{\partial g_w}{\partial u_i})_{u=U_c} + [W_o Z_o (\frac{Y_i}{\sum_j Y_j})] (\frac{\partial g_w}{\partial W}).
\]  \hspace{1cm} (18)

The first term represents the substitution effect, and corresponds to the movement from \( C_0 \) to \( C_1 \), as a result of the increase in \( u_i \), holding utility (or real wealth) constant at \( U_o \). The second term is the wealth effect, which corresponds to the movement from \( C_1 \) to \( C_2 \), as \( a'a' \) shifts parallel upward to \( aa'' \). The wealth effect arises, because when \( u_i \) increases, the same expected yield can be obtained with a lower amount of wealth invested in the Hicksian security. This wealth effect is proportional to the amount of asset \( i \) originally held in the Hicksian security, i.e., \( W_o Z_o (\frac{Y_i}{\sum_j Y_j}) \). Note that \( \frac{\partial g_w}{\partial W} \geq 0 \), depending upon the investor's preference function.

The sign of the substitution term is unambiguously positive (guaranteed by the convexity condition in (12)). The magnitude of the substitution effect, and the sign and magnitude of the wealth effect, depend upon parameters of the utility function, in analogous fashion to the substitution and income effects of consumer theory. If \( \frac{\partial g_w}{\partial W} < 0 \), then the preference function has increasing
absolute risk-aversion, while if $\frac{\delta \sigma}{\delta w} > 0$, then it has decreasing absolute risk-aversion. A preference function characterized by decreasing or constant absolute risk-aversion guarantees that $\frac{\delta \sigma}{\delta u_i} > 0$ in equation (18).

In Figure 3, the point of final equilibrium $C_2$ is to the right of $C_o$ along $aa''$, implying that $\frac{\delta z}{\delta u_i} > 0$ in this particular case; that is the numerator of (17) is greater than zero.

An important point to notice here is that if the risky portfolio consisted of only one asset, then $\frac{\delta \sigma}{\delta u_i} = 0$, and the sign of $\frac{\delta z}{\delta u_i}$ in (17) would depend solely on the sign of $\frac{\delta \sigma}{\delta u_i}$. This would guarantee, (when $u_i$ increases), a rise in the proportion of the risky asset in the portfolio, in the case of utility of wealth functions having constant or decreasing absolute risk-aversion. This has interesting implications for the discussion of demand for indexed and insured money. For example, in the context of the choice between holding money (the riskless asset) versus a risky asset, this type of utility function would provide a negative relationship between money holdings and expected yield on the risky asset. However, this result does not unambiguously extend to the case of multiple risky asset alternatives to money, since in that situation $\frac{\delta \sigma}{\delta u_i} \approx 0$, and therefore, $\frac{\delta \sigma}{\delta u_i} > 0$ does not guarantee $\frac{\delta z}{\delta u_i}$ (in equation (17)) to be always positive.
Thus Tobin\textsuperscript{17} derives a negative relationship between money holding and expected return on the risky asset, in the case of one risky asset and cash (the riskless asset), using a quadratic utility function. He concludes (as I do in the case of one risky asset), that this relationship depends solely upon the investor's attitude towards risk, i.e., the investor's utility-of-wealth function. That a negative relationship exists if the utility function exhibits increasing (or constant) absolute risk-aversion. Considering the situation involving multiple risky alternatives to cash, Tobin states that:

"The argument is not essentially changed, however, if $A_2$ is taken to be the aggregate share invested in a variety of non-cash assets, e.g., bonds and other debt instruments differing in maturity, debtor, and other features."\textsuperscript{16}

Tobin is correct because he defines $A_2$ as an aggregate of risky assets. However, an increase in the expected yield on one risky asset in a situation of many assets may lead to an increase in the amount held of that asset, as well as of money (the safe asset), while holdings of other assets decline. This may occur even if the utility function exhibits increasing absolute risk-aversion. In general the outcome is ambiguous.

I have established that while the second term in (14) is positive, when $q_{ij} > 0$, the first term has ambiguous sign. Therefore, the sign of $\frac{\partial x^*_1}{\partial u_1}$ is ambiguous.
(B) **The Effect of a Change in Risk**

We can in similar fashion as above consider the effect of a change in $\sigma_i$, on $\bar{X}_i$, holding constant $\sigma_j (j \neq i)$. $u_j, Q_{ij}$.

\[
\frac{\partial \bar{X}_i}{\partial \sigma_i} = u_i \frac{\partial Z}{\partial \sigma_i} \left[ \frac{Y_i}{\sum_j Y_j} \right] + W_o Z \left[ \frac{\partial}{\partial \sigma_i} \left( \frac{Y_i}{\sum_j Y_j} \right) \right],
\]

(19)

The first term represents the change in the amount of asset $i$ held due to the change in the amount held of the Hicksian security. The composition of the Hicksian security is kept constant.

The second term in (19) represents the change in the amount of asset $i$ held due to the change in the composition of the Hicksian asset, as a result of the change in $\sigma_i$.

I first analyze the sign of the second term in (19).

\[
\frac{\partial}{\partial \sigma_i} \left[ \frac{Y_i}{\sum_j Y_j} \right] = \frac{\partial Y_i}{\partial \sigma_i} \left( \frac{\sum_j Y_j}{\sum_j Y_j} \right) - Y_i \left( \frac{\partial}{\partial \sigma_i} \sum_j Y_j \right) \left[ \frac{\sum_j Y_j}{\sum_j Y_j} \right]^2,
\]

(20)

\[
= \frac{-2(u_i - R_F) \left[ \sum_j Y_j - Y_i \right] - \sum_j \frac{u_i - R_F}{\sigma_i^2 \sigma_j} Q_{ij} \left[ \sum_j Y_j \right] Y_i \left( \sum_j Y_j \right)^2}{\left[ \sum_j Y_j \right]^2}
\]

(21)
Figure 8
If \( Q_{ij} \) is strictly nonnegative, the numerator in (21) is unambiguously less than zero. Therefore, 

\[
W_0 \sum \frac{d}{d \sigma_i} \left[ \frac{Y_i}{\sum_j Y_j} \right] < 0 , \text{ if } Q_{ij} > 0 .
\]

The sign of the first term in equation (19) is however ambiguous, \( \frac{dZ}{d\sigma_i} \neq 0 \).

From equation (9) earlier,

\[
\frac{dZ}{d\sigma_i} = \frac{\frac{d\sigma_W}{d\sigma_i} \sigma_P - \sigma_W \frac{d\sigma_P}{d\sigma_i}}{\sigma_P^2} .
\] (22)

Both \( \frac{d\sigma_W}{d\sigma_i} \) and \( \frac{d\sigma_P}{d\sigma_i} \) have ambiguous sign, so that \( \frac{dZ}{d\sigma_i} \) is ambiguous. The sign of \( \frac{d\sigma_W}{d\sigma_i} \) depends upon the investor's preference function. The sign of \( \frac{d\sigma_P}{d\sigma_i} \) depends upon the values of the parameters involved in determining the Hicksian security, as shown in Appendix II.

Figure 8 demonstrates the factors underlying the sign and magnitude of \( \frac{dZ}{d\sigma_i} \). The approach is analogous to the case of a change in \( u_1 \). We start at an initial value of \( Z_0 \) corresponding to \( C_0 \), the point of initial portfolio equilibrium, along the opportunity locus aBa.

If the final portfolio equilibrium point, after a decrease
in \( \sigma_i \), is above \( C_0 \) (as in the case at \( C_2 \) in the figure) along \( aD_a'' \), the new opportunity locus, then \( \frac{\partial Z}{\partial \sigma_i} > 0 \); if it is below \( C_0 \) then \( \frac{\partial Z}{\partial \sigma_i} < 0 \). This follows from the fact, that \( \text{OND} \) and \( \text{OMB} \) are similar triangles, Thus,

\[
Z_o = \frac{\sigma_{w_0}}{\sigma_{p_0}} = \frac{ML'}{MB} = \frac{NL}{ND} = \frac{\sigma_{w_0}'}{\sigma_{p_0}'}.
\]

Note that \( \frac{\partial \sigma_w}{\partial \sigma_i} \) consists of an income and substitution effect, analogous to the case of a change in \( u_1 \).

\[
\frac{\partial \sigma_w}{\partial \sigma_i} = \left( \frac{\partial \sigma_w}{\partial \sigma_i} \right)_{U_0} + W_o Z_o \left( \frac{\gamma_i}{\sum_j \gamma_j} \right) \frac{\partial \sigma_w}{\partial W}; \quad (23)
\]

The first term represents the substitution effect and corresponds to the movement from \( C_0 \) to \( C_1 \), as a result of the change (decrease) in \( \sigma_i \); holding utility or real wealth constant at \( U_0 \). The second term is the wealth effect and corresponds to the move from \( C_1 \) to \( C_2 \) as \( a'a' \) shifts parallel upward to \( aD_a'' \). The substitution term has negative sign (guaranteed by the convexity condition (12)). The magnitude of the substitution effect, and the sign and magnitude of the wealth effect, depend upon the parameters of the utility function.

The sign of \( \frac{\partial \sigma_w}{\partial \sigma_i} \) completely determines the sign of \( \frac{\partial Z}{\partial \sigma_i} \) only when \( \frac{\partial \sigma_p}{\partial \sigma_i} = 0 \), as in the case of only one risky
asset. Otherwise, as before, the properties of the utility of wealth function do not provide sufficient information about the sign of $\frac{\partial z}{\partial c_i}$.

I have established that the second term in equation (13) has negative sign, when $Q_{ij} \geq 0$, while the first term has ambiguous sign. Therefore, the sign of $\frac{\partial x_i}{\partial c_i}$ is ambiguous.

(C) Substitution-Complementarity Relationships

In this section, I consider substitution-complementarity relationships among risky assets.

Change in $u_i$

Differentiating $x_i$ with respect to $u_j$ (the expected rate of return on another risky asset), we get (holding constant $u_i \neq j, c_j, Q_{ij}$)

$$\frac{\partial x_i}{\partial u_j} = w_o \frac{\partial z}{\partial u_j} \left[ \frac{Y_i}{\sum_j Y_j} \right] + w_o z \left[ \frac{\partial x_i}{\partial u_j} \left( \frac{Y_i}{\sum_j Y_j} \right) \right]. \quad (24)$$

As before, the first term in the equation represents the change in the amount of asset $i$ held, due to the change in the amount held of the Hicksian security as a whole. The second term represents the change in the amount of asset $i$ due to the change in the composition of the Hicksian security.
The sign of the second term in (24) depends upon the sign of $Q_{ij}$, the partial correlation coefficient between $i$ and $j$.

$$\frac{\partial}{\partial u_j} \left( \frac{y_i}{y_j} \right) = \frac{y_i \left( \sum_j y_j \right) - y_i \frac{\partial}{\partial u_j} \left( \sum_j y_j \right)}{\left[ \sum_j y_j \right]^2}$$

(25)

$$\sum_{j=1}^{n} \frac{-Q_{ij}}{\sigma_i \sigma_j} \left( \sum_j y_j \right)^{-1} y_i \left( \frac{1}{\sigma_j^2} - \sum_{i=1}^{n} \frac{1}{\sigma_i \sigma_j} \right) Q_{ij}$$

(26)

$$= \frac{n}{\left[ \sum_j y_j \right]^2}$$

That is,

$$\frac{\partial}{\partial u_j} \left( \frac{y_i}{y_j} \right) < 0 , \text{ if } Q_{ij} > 0 ,$$

$$> 0 , \text{ if } Q_{ij} < 0 .$$

This says, that the proportion of asset $i$ held in the Hicksian security varies inversely with the expected yield of asset $j$, if the partial correlation between $i$ and $j$ is positive or zero, and positively if the partial correlation is negative.

The sign of the first term in (24) is ambiguous, for reasons similar to those already discussed earlier in the context of changes in $u_i$ or $\sigma_i$. Briefly, the sign of $\frac{\partial y}{\partial u_j}$ depends upon the signs and magnitudes of $\frac{\partial \sigma_i}{\partial u_j}$ and $\frac{\partial \sigma_j}{\partial u_j}$. These depend upon parameters of the investor's preference.
function as well as upon parameters determining the Hick-
sian security.

Thus, \[ \frac{\partial \bar{X}_i}{\partial u_j} \neq 0. \]

Similarly, \[ \frac{\partial \bar{X}_i}{\partial u_i} \neq 0. \]

It can also be easily shown that \[ \frac{\partial \bar{X}_i}{\partial u_j} \neq \frac{\partial \bar{X}_i}{\partial u_i} \] in general.

Change in \( \sigma_j \). (holding constant \( \sigma_{ij}, \sigma_{i} \)).

Differentiating \( \bar{X}_i \) with respect to \( \sigma_j \), we get,

\[ \frac{\partial \bar{X}_i}{\partial \sigma_j} = \omega \frac{\partial X}{\partial \sigma_j} \left[ \frac{Y_i}{\sum Y_j} \right] + \omega \frac{\partial X}{\partial \sigma_j} \left[ \frac{\partial \left( \frac{Y_i}{\sum Y_j} \right)}{\partial \sigma_j} \right] \]

(27)

The sign of the second term in (27) depends upon the

sign of \( Q_{ij} \).

\[ \frac{\partial}{\partial \sigma_j} \left( \frac{Y_i}{\sum Y_j} \right) = \frac{\partial Y_i}{\partial \sigma_j} \left( \frac{\Sigma Y_j - Y_i \frac{\partial Y_i}{\partial \sigma_j} \Sigma Y_j}{\Sigma Y_j} \right) \]

(28)

\[ \left( \sum Y_j \right) \left[ \sum_{j=1}^{n} \left( \frac{U_j - R_i}{\sigma_{ij} \Sigma Y_j} \right) \right] + \frac{Y_i \Sigma Y_j}{\sigma_{ij}} \]

(29)

The expression in (29) is greater or less than zero

depending on whether \( Q_{ij} \) is greater (or equal) or less

than zero, respectively.
That is,
\[ \frac{\partial}{\partial \sigma_j} \left( \frac{\gamma_i}{\sum \gamma_j} \right) > 0, \quad \text{if} \quad \sigma_{ij} \geq 0, \]
\[ < 0, \quad \text{if} \quad \sigma_{ij} < 0. \]

Thus, the proportion of asset i held in the Hicksian security varies directly with the residual variance of asset j if the partial correlation between i and j is positive or zero, and inversely if the partial correlation is negative.

The sign of the first term in (27) is ambiguous because the signs and magnitudes of \( \frac{\partial \sigma_w}{\partial \sigma_j} \) and \( \frac{\partial \sigma_p}{\partial \sigma_j} \) are ambiguous. Again, these signs depend upon parameters of the investor's preference function and parameters determining the Hicksian security.

Therefore, \( \frac{\partial X_i}{\partial \sigma_j} \neq 0 \).

It can also be easily shown that \( \frac{\partial X_i}{\partial \sigma_j} \neq \frac{\partial X_i}{\partial \sigma_i} \) in general.

In chapter six, without using the Separation Theorem, I derive more general and more useful comparative-static equations with strong empirical implications.
APPENDIX I

I consider two risky assets and a riskless asset. We can write:

\[ E_w = X_1 (1 + u_1) + X_2 (1 + u_2) + (W_0 - X_1 - X_2)(1 + R_F), \]  

(1)

\[ \sigma_w^2 = \sigma_1^2 X_1^2 + \sigma_2^2 X_2^2 + 2 X_1 X_2 \rho \sigma_1 \sigma_2 r, \]

(2)

where,

- \( E_w \) = the expected value of the investor's wealth,
- \( \sigma_w^2 \) = the variance of the investor's wealth,
- \( X_1 \) = the amount invested in asset 1,
- \( X_2 \) = the amount invested in asset 2,
- \( W_0 \) = initial wealth of the investor,
- \( u_1 \) = expected rate of return on asset 1,
- \( u_2 \) = expected rate of return on asset 2,
- \( R_F \) = expected rate of return on safe asset,
- \( \sigma_1^2 \) = variance of asset 1,
- \( \sigma_2^2 \) = variance of asset 2,
- \( r \) = correlation coefficient between assets 1 and 2.

Note, \( u_1 > u_2 > R_F \), \( \sigma_1 > \sigma_2 \), \(-1 \leq r \leq 1\).
The locus of efficient portfolios, or asset combinations, can be obtained by maximizing $E_w$ subject to $\sigma_w^2$ or minimizing $\sigma_w^2$ subject to $E_w$. I do the latter.

Minimize a function $C$

$$C = \left[ x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2 + 2x_1 x_2 \sigma_1 \sigma_2 \right]$$

$$+ \lambda \left[ E_w - X_1(1+U) - X_2(1+U) - (W_0 - X_1 - X_2)(1+R_F) \right] ,$$

where $\lambda$ is a Lagrange multiplier. Differentiating (3) with respect to $X_1$, $X_2$, $\lambda$ respectively, we get the first order conditions for a minimum

$$\frac{dc}{dx_1} = \sigma_w^2 + 2\lambda (u_1 - R_F) = 0$$

$$= 2x_1 \sigma_1^2 + 2x_2 \sigma_2^2 - 2\lambda (u_1 - R_F) = 0 \ ,$$

$$\frac{dc}{dx_2} = \sigma_w^2 + 2\lambda (u_2 - R_F) = 0$$

$$= 2x_2 \sigma_2^2 + 2x_1 \sigma_1 \sigma_2 - 2\lambda (u_2 - R_F) = 0 \ ,$$

$$\frac{dc}{d\lambda} = E_w - X_1(1+U) - X_2(1+U)$$

$$- (W_0 - X_1 - X_2)(1+R_F) = 0 \ .$$
Second order conditions for a minimum require that
the bordered Hessian determinant be positive definite, that is

\[
H = \begin{vmatrix}
\frac{\partial^2 c}{\partial x_1^2} & \frac{\partial^2 c}{\partial x_1 \partial x_2} & \frac{\partial^2 c}{\partial x_1 \partial x_3} \\
\frac{\partial^2 c}{\partial x_2 \partial x_1} & \frac{\partial^2 c}{\partial x_2^2} & \frac{\partial^2 c}{\partial x_2 \partial x_3} \\
\frac{\partial^2 c}{\partial x_3 \partial x_1} & \frac{\partial^2 c}{\partial x_3 \partial x_2} & 0 \\
\end{vmatrix}
\]

which requires

\[
\sigma_1 \sigma_2 r (u_2 - R_F) (u_1 - R_F) < \sigma_2^2 (u_1 - R_F)^2,
\]

\[
\sigma_1 \sigma_2 r (u_2 - R_F) (u_1 - R_F) < \sigma_2^2 (u_2 - R_F)^2.
\]

From (4) we get

\[
X_1 = \frac{\lambda (u_1 - R_F) - X_2 \sigma_1 \sigma_2 r}{\sigma_1^2},
\]

Substituting (7) into (5) we get

\[
X_2 = \lambda \left[ \frac{(u_2 - R_F) \sigma_1 - (u_1 - R_F) \sigma_2 r}{\sigma_1 \sigma_2^2 (1 - r^2)} \right],
\]
by symmetry,

\[ X_1 = \lambda \left[ \frac{(u_1 - R_F)\sigma_2 - (u_2 - R_F)\sigma_1 r}{\sigma_2 \sigma_1^2 (1 - r^2)} \right]. \] (9)

The second-order conditions given above guarantee that the terms in the brackets in equations (8) and (9) are greater than zero, and therefore \( X_1, X_2 > 0 \), i.e., no short sales occur (given that \( u_1, u_2 > R_F \)).
APPENDIX II

We can show that \( \frac{\partial \sigma_p}{\partial u_i} \geq 0 \); holding constant \( u_j \neq i \), 

\[ \sigma_j \cdot Q_{ij}. \]

\[ \sigma_p^2 = \sum_{i=1}^{n} \left[ X_i^0 \cdot \sigma_i^2 + 2 \sum_{j=1 \neq i}^{n} X_i^0 X_j^0 Q_{ij} \sigma_i \sigma_j \right], \tag{1} \]

where \( X_i^0, X_j^0 \) correspond to the optimal amounts of \( X_i \) and \( X_j \) in the Hicksian security, when all the portfolio is invested in the Hicksian security, i.e., \( Z = 1 \).

Thus, 

\[ X_i^0 = W_0 \left( \frac{\gamma_i}{\sum_{j} \gamma_j} \right), \tag{2} \]

\[ X_j^0 = W_0 \left( \frac{\gamma_j}{\sum_{j} \gamma_j} \right). \tag{3} \]

From equation (1) we get,

\[ \frac{\partial \sigma_p}{\partial u_i} = \frac{1}{\sigma_p} \left[ \sum_{j=1}^{n} X_i^0 \frac{\partial X_j^0}{\partial u_i} \sigma_i^2 + \sum_{j=1 \neq i}^{n} \left( X_i^0 \frac{\partial X_j^0}{\partial u_i} Q_{ij} \sigma_i \sigma_j \right) \right], \tag{4} \]
We have shown in equations (16) and (26) in the main text of the paper that
\[
\frac{\partial \left( \frac{Y_i}{Z_j} \right)}{\partial u_i} \geq 0, \quad \text{if} \quad Q_{ij} \geq 0,
\]
\[
\frac{\partial \left( \frac{Y_i}{Z_j} \right)}{\partial u_i} \geq 0, \quad \text{if} \quad Q_{ij} \leq 0.
\]

It follows, by substitution into equations (2) and (3), that
\[
\frac{\partial x_i^0}{\partial u_i} \geq 0, \quad \text{if} \quad Q_{ij} \geq 0,
\]
and
\[
\frac{\partial x_i^0}{\partial u_i} \geq 0, \quad \text{if} \quad Q_{ij} \leq 0.
\]

Substituting \( \frac{\partial x_i^0}{\partial u_i} \) and \( \frac{\partial x_i^0}{\partial u_i} \) into equation (4), we see that \( \frac{\partial \sigma_p}{\partial u_i} \geq 0 \). Note that this ambiguity does not depend solely on the sign of \( Q_{ij} \). The sign of \( \frac{\partial \sigma_p}{\partial u_i} \) depends on the values of all the parameters that determine \( \sigma_p \).

We can also show that \( \frac{\partial \sigma_p}{\partial \sigma_i} \geq 0 \), holding constant \( \sigma_{j \neq i}, u_j, Q_{ij} \).

Differentiating equation (1) with respect to \( \sigma_i, 2^0 \).
We have shown in equations (20) and (29) that
\[
\frac{\partial \sigma_p}{\partial \sigma_i} = \frac{1}{2\sigma_p} \left[ \frac{x_i^0}{\sigma_i} \sigma_i + \sum_{j=1}^{n} \frac{x_j^0}{\sigma_j} \frac{\partial x_i^0}{\partial \sigma_i} \sigma_j^2 + \sum_{j=1, j \neq i}^{n} \frac{x_j^0}{\sigma_j} \frac{\partial x_i^0}{\partial \sigma_i} \sigma_j \delta_{ij} \right] + x_i^0 \frac{\partial x_i^0}{\partial \sigma_i} \delta_{ij} \sigma_i^2 \delta_{ij}.
\]

We have shown in equations (20) and (29) that
\[
\frac{\partial (\frac{y_i}{\delta y_j})}{\partial \sigma_i} \leq 0, \text{ if } Q_{ij} \geq 0,
\]
and
\[
\frac{\partial (\frac{y_j}{\delta y_j})}{\partial \sigma_i} \geq 0, \text{ if } Q_{ij} \geq 0.
\]

Hence, \( \frac{\partial x_i^0}{\partial \sigma_i} \leq 0, \text{ if } Q_{ij} \geq 0 \),

and \( \frac{\partial x_j^0}{\partial \sigma_j} \geq 0, \text{ if } Q_{ij} \geq 0 \).

Substituting \( \frac{\partial x_i^0}{\partial \sigma_i} \) and \( \frac{\partial x_j^0}{\partial \sigma_j} \) into equation (5), we see that \( \frac{\partial \sigma_p}{\partial \sigma_i} \leq 0 \). Again note that the sign of \( \frac{\partial \sigma_p}{\partial \sigma_i} \) depends upon values of all parameters that determine \( \sigma_p \).
Notes to Chapter 3

1 See G. Bierwag and M. Grove (5), M. Allingham and M. Morishima (2), M. Morishima (23), S. Royama and K. Hamada (27).

2 See J. Tobin (32), J. Lintner (20), W. Sharpe (29).

3 See H. Levy (19).

4 V. Aivazian (1) is a much shortened form of Chapter 2.

5 I compare my results with results from previous studies in the last section of the paper.

6 The following procedure for deriving the efficiency locus and equation (4) is similar to that of Levy (19). This procedure is not due to Levy, however, but goes back to Tobin (32).

7 For the two risky asset case, see Appendix I.

8 Equation (4) is obtained by solving equation (2) for \( X \), i.e., \( X = V^{-1}(u - R_\text{p,1}) \) and then expressing \( X_1 \) in terms of residual variances and covariances. See also Appendix I. For a slightly different procedure see Levy (19).

9 See Tobin (32) for use of such a diagram.

10 See H. Levy (19).

11 See V. Aivazian (1).

12 See Tobin (32), page 24, for the proof of this linearity.

13 The term Hicksian security is due to Ed Kane (17). The name derives from the analogy of the concept to that of
composite goods discussed by Hicks in (14) see Hicks' Mathematical Appendix, part 10.

14 With certain restrictions on the distribution of returns, we can in general assume that E(U) = U(E_w, \sigma_w). These restrictions are in addition to that the joint distribution of returns be normal. See K. Borch (9), M. Feldstein (12), J. Chipman (10). Chipman shows extensively some of the restrictions to be imposed on the function U(E_w, \sigma_w) in order that it may be an expected utility function. For other evaluations of the Mean-Variance approach see Samuelson (28), and the Borch, Bierwag, Levy, Tsiang exchange (9).

15 A similar diagram has been used by Stiglitz (30).

16 An increase in initial wealth W_0, holding other variables constant, increases E_p, \sigma_p, in proportion to their original magnitudes, so that the new Hicksian security with coordinates (E'_p, \sigma'_p) lies along the ray OD, which joins the origin to the initial Hicksian security with coordinates (E_p, \sigma_p). See figure on next page. Moreover, since the gross return on the riskless asset has increased in proportion to its initial value, the efficiency locus must shift parallel upward. Algebraically, we can write:

\[ E_p = W_0 (1 + e_p), \]
\[ \sigma_p = W_0 s_p, \]

where, \( e_p \) is the rate of return on the Hicksian security, and \( s_p \) is the standard deviation per dollar of the Hicksian security.

The slope of the efficiency locus is:

\[ \frac{W_0 (1 + e_p) - W_0 (1 + R_F)}{W_0 s_p} \]

so that if \( W_0 \) increases to \( W_1 \) this slope does not change.

(I would like to thank R. Lang and D. Resler for discussion on this point).
17 See Arrow (3), Stiglitz (30).

18 See Tobin (32).

19 See Tobin (32) page 21.

20 Note that \( \frac{\partial \sigma_i}{\partial \sigma_i} > 0 \), (since \( \frac{|M|}{M_{ii}} > 0 \)), so that the signs are not altered by a differentiating with respect to \( \sigma_i \) or \( \sigma_i \).
Chapter 4

Review and Criticism of the Literature Dealing with Slutsky Equations for Risky Assets in an E-V Framework

In this chapter I review and criticize the literature on Slutsky equations for risky assets in the Mean-Variance framework. These studies consist of Royama and Hamada (27) (henceforth R & H), Levy (19) (henceforth L), Bierwag and Grove (5) (henceforth B & G), Allingham and Morishima (2) (henceforth A & M), and Morishima (23) (henceforth M). There are basic differences in approach among these studies. L, whose work was discussed and criticized in Chapter 2, is framed in the Separation Theorem framework. R & H make the assumption that the utility of wealth function is quadratic. A & M and B & G do not constrain the form of their preference function. M uses a Revealed-Preference approach (the idea of an E-V preference function is implicit though) to derive results which are very similar to those of A & M. All the above models are single-period ones. They are also partial equilibrium models in the sense that they are concerned with the decision-making of a single investor.

I discuss the above works in the order given, except for L which was discussed in Chapter 2.
Royama and Hamada's Work

R & H use a utility function of the quadratic form,

\[ U(W) = W - \frac{1}{2} \alpha W^2 , \quad \alpha > 0 , \tag{1} \]

where \( W \) is the investor's end-of-period wealth. \( \alpha > 0 \) implies diminishing marginal utility of wealth. To keep utility nonnegative, \( W \) is restricted to values of \( W < \frac{2}{\alpha} . \)

There are \( n \) risky assets, with \( X_i \) the value of the \( i \)th asset at the beginning of the period, and \( W_0 \) is the beginning of period wealth. \( q_i \) is the rate of return on asset \( i \), a random variable. By definition

\[ \sum_{i=1}^{n} X_i = W_0 \tag{2} \]
\[ W = \sum_{i=1}^{n} (1+q_i) X_i \tag{3} \]

The investor's problem is to maximize expected utility of wealth \( E(U(W)) \) subject to initial wealth \( W_0 \). Thus the problem becomes, using (1), the maximization of the function \( L \) with respect to \( X_i \), where

\[ L = \left[ \sum_{i=1}^{n} \mu_i X_i - \frac{1}{2} \alpha \sum_{i,j} (\sigma_{ij} + \mu_i \mu_j) X_i X_j \right] \]
\[ + \lambda \left[ \sum_i X_i - W_0 \right] , \tag{4} \]

and where
\[ \mu_i = E(1+q_i) , \]
\[ \sigma_{ij} = \text{cov}(q_i,q_j) , \]
\[ \lambda \text{ is a Lagrange multiplier.} \]

The first-order conditions, from (4), are
Second order conditions for a maximum require that the principal minors of the following matrix $H$ alternate in sign, where

$$H = \begin{bmatrix}
m_{11} & \cdots & m_{1n}
\vdots & \ddots & \vdots
m_{n1} & \cdots & m_{nn}
\end{bmatrix}
$$

and where $m_{ij} \equiv -a(\sigma_{ij} + \mu_i/\mu_j)$.

This guarantees that $\frac{D_{jj}}{D} < 0$, where $D$ is the determinant of $H$ and $D_{ij}$ is the cofactor of the $(i,j)$th element of $D$.

To derive the response of demand to changes in expected returns, they totally differentiate (5) and (6), holding all $\sigma_{ij}$ constant, to get, in matrix form

$$\begin{bmatrix}
dX_1 \\
\vdots \\
dX_n
d\lambda
\end{bmatrix}
= \begin{bmatrix}
-d\mu_i + a \sum_{j} (\sigma_{ij} \mu_j + \mu_i \mu_j) X_j \\
\vdots \\
-d\mu_n + a \sum_{j} (\mu_j \mu_n + \mu_n \mu_j) X_j \\
dW_0
\end{bmatrix}, \quad (7)$$
which is rewritten as

\[
\begin{bmatrix}
\frac{dx_1}{d\lambda} \\
\vdots \\
\frac{dx_n}{d\lambda}
\end{bmatrix}
= H^{-1} \begin{bmatrix}
-d\mu_j + a \sum_j (\mu_j d\mu_i + \mu_i d\mu_j) X_j \\
\vdots \\
-d\mu_n + a \sum_j (\mu_j d\mu_n + \mu_n d\mu_j) X_j \\
-dW_0
\end{bmatrix}
\] (8)

From (8), R & H derive

\[
\left( \frac{\partial x_j}{\partial W_0} \right)_{\text{const.}} = \frac{D_{n+1,j}}{|D|} 
\] (9)

and

\[
\left( \frac{\partial x_j}{\partial W_0} \right)_{W_0 \text{ const.}} = -\left(1-a \sum_{k=1}^{n} \mu_k X_k \right) \frac{D_{ij}}{|D|} + a X_i \sum_{k=1}^{n} \mu_k \frac{D_{kj}}{|D|} 
\] (10)

They then define \( S_{ij} \) as
\[ S_{ij} = -\left( 1 - a \sum_{k=1}^{n} \mu_k X_k \right) \frac{D_{ij}}{|D|}, \quad (11) \]

So that rewriting (11) they get

\[ S_{ij} = \left( \frac{\partial X_j}{\partial \mu_i} \right)_{W_0 \text{ const.}} - a X_i \sum_{k=1}^{n} \mu_k \frac{D_{ij}}{|D|}; \quad (12) \]

They claim that

\[ a \sum_{k} \mu_k \frac{D_{kj}}{|D|} = \frac{\partial X_j}{\partial E(W)} \mu_k \text{ const.} \quad (13) \]

R & H do not point out that, to be meaningful, the term \( \frac{\partial X_j}{\partial E(W)} \mu_k \text{ const.} \) should be interpreted as involving a parametric change in \( E(W) \), such as a change in \( T \) in equation (14) below. This is clarified below.

From (13), R & H write

\[ S_{ij} = \left( \frac{\partial X_j}{\partial \mu_i} \right)_{W_0 \text{ const.}} - X_i \left[ \frac{\partial X_j}{\partial E(W)} \right]_{\mu_k \text{ const.}} \quad (14) \]
R & H call $S_{ij}$ the substitution effect of a change in $\mu_i$ on $X_j$, and (13) the Slutsky equation of such a change, claiming complete analogy to consumer theory. Thus in their words:

"Suppose all $X_k$'s remain unchanged in spite of a small increase in the expected return on the $i$th asset $d\mu_i$. Then the investor automatically enjoys an increase in the value of his future expected wealth by $X_j d\mu_i$ without increasing risk. Also suppose this amount is taken away from the investor to keep him at the same combination of risk and return. Then the effect of a deduction from expected wealth will change the demand for $X_j$ by

$$-X_j a \sum_k \mu_k \left( \frac{D_{kj}}{D} \right) d\mu_i.$$  

I show below, however, that taxing the investor's expected wealth by $X_j d\mu_i$ does not keep him at the original combination of risk and return. The term

$$X_j a \sum_k \mu_k \left( \frac{D_{kj}}{D} \right)$$

is called by R & H the expected wealth effect and is interpreted as being completely analogous to the wealth effect of consumer theory. $S_{ij}$ is defined as the "effect of the change in $\mu_i$ on $X_j$ provided that the investor is compensated for the change in $\mu_i$ so as to enable him to enjoy the same expected wealth with the same risk." I argue below that these interpretations of the terms composing (10) are not valid.

A central point of criticism concerns the nature of the term in (13). R & H write

$$E(W) = \sum_i \mu_i X_i - T$$  

(15)
and correctly show that

$$V(W) = \sum_i \sum_j X_i X_j \sigma_{ij}$$

(16)

and

$$\frac{\partial x_{ij}}{\partial T} = -a \sum_k \mu_k \frac{D_{ij}}{D_t}$$

(17)

where $T$ stands for a lump-sum tax on future wealth.

Thus the terms in (13) equal $-\frac{\partial x_{ij}}{\partial T}$.

From (15) and (16) R & H argue that if, after the original change in $x_i$, a lump-sum tax of $x_i d\mu_i$ is imposed on the investor, then $E$ would decrease by the amount of this tax and $V$ would stay unchanged, bringing the investor back to the original combination of $E$ and $V$.

Hence, they contend, the second term in (12) is the effect on $x_j$ of an expected-wealth compensation which keeps the investor at the original levels of $E$ & $V$.

However, a lump-sum tax of $x_i d\mu_i$ does not keep the investor at his original $E, V$ combination.

Implicit in R & H's above argument is that

$$\frac{\partial E(W)}{\partial T} = -1$$

(18)

and

$$\frac{\partial V(W)}{\partial T} = 0$$

(19)
However, neither equation is valid. Correctly formulated,

\[
\frac{\partial E(W)}{\partial T} = \sum_i \mu_i \frac{\partial X_i}{\partial T} - 1 \tag{20}
\]

\[
\frac{\partial V(W)}{\partial T} = \sum_i \sum_j \sigma_{ij} \left( X_i \frac{\partial X_i}{\partial T} + X_j \frac{\partial X_j}{\partial T} \right) j \tag{21}
\]

The implications of (20) and (21) are that a lump-sum tax of \( X_i \, d\mu_i \), i.e., an original parametric decrease in \( E \) by \( X_i \, d\mu_i \), leads in turn to adjustments in \( E \) and \( V \) to new optimal values. Since these are, along with the \( X_i \)'s, choice variables and not parameters of the system. Unlike the parameter \( T \) (or \( \mu_i, \sigma_{ij}, W_0 \)), the variables \( E, V, X_i \) are determined by the investor's choice or utility-maximizing mechanism. A parametric change in \( E(W) \) (due to the change in \( T \)) shifts the \( E(W) \) function (15) without in general bringing the investor back ultimately to the original optimal values of \( E \) and \( V \).

R & H's failure to distinguish between choice variables and parameters leads to mistaken interpretations of the terms in (10). We conclude that, contrary to R & H, placing a lump-sum tax of \( X_i \, d\mu_i \) on the investor's expected wealth does not in general keep the investor at the original \( E, V \) combination, or at a
point on the original utility frontier; hence \( S_{ij} \) is not the substitution effect of \( \mu_i \) on \( X_j \) holding the investor at the original combination of \( E \) and \( V \).

The term \( \frac{\partial X_j}{\partial E(W)} \) is misleading. It implies that \( X_j \) responds in some unique manner to changes in \( E(W) \); while parametric changes or shifts in the \( E(W) \) function do affect \( X_j \) in some manner, they also affect \( E(W) \) and \( V(W) \). All these choice variables are jointly determined. The treatment of \( \frac{\partial X_j}{\partial E(W)} \) as the wealth effect is criticized below.

R & H write, "By analogy to consumer demand theory, let us call an asset \( X_j \) a normal asset if \( \frac{\partial X_j}{\partial E(W)} > 0 \), an inferior asset if \( \frac{\partial X_j}{\partial E(W)} < 0 \). Naturally all assets cannot be inferior or normal." In a mean-variance framework, expected wealth alone is not a measure of real wealth or utility, since variance of wealth also enters the utility function. Expected wealth may stay constant, but variance raised or lowered yielding lower or higher levels of utility and real wealth. Hence R & H's definition of the wealth effect as based solely on changes in expected wealth is inappropriate. The second part of the above statement by R & H is not valid either. In consumer theory, from the income constraint, one derives that all goods cannot be normal or inferior. The analogue to that in the R & H system is first-order
condition (6) (derived from the initial wealth constraint) which yields, after differentiating by \( W_0 \), that

\[
\sum_i \frac{\partial x_i}{\partial W_0} = 1
\]

where the constraint is on \( \frac{\partial x_i}{\partial W_0} \) and not on \( \frac{\partial x_i}{\partial E(W)} \). There is nothing in the system which imposes a similar constraint on \( \frac{\partial x_i}{\partial E(W)} \).

Morishima (23) also criticizes R & H's interpretation of equation (10) as a Slutsky equation. He argues that unlike the traditional theory of demand underlying the Slutsky equation where prices appear in the budget equation, in the theory of portfolio choice expected rates of return (or variance-covariance) are not in the budget equation, but in the preference function affecting the rankings of portfolios. Morishima identifies effects of changes in these variables as "Want Pattern" effects, distinguishing them from the wealth and substitution effects of the usual Slutsky equation. Morishima is correct in pointing out that expected rates of return (or variance-covariance) do not enter into the budget equation, but I show below that for changes in these variables one can still identify a wealth effect and a utility-constant substitution effect. I discuss the "Want Pattern" effect later in this chapter and more critically in chapters five and six.

We next separate in the manner of traditional consumer theory wealth and substitution effects underlying equation (10).
Equation (23) gives the effect on \( X_j \) of a change in \( \mu_i \), holding constant other \( \mu_j \) and \( \sigma_{ij} \), and \( W_0 \). To isolate the pure substitution effect of this change we set \( \frac{\partial E(U)}{\partial \mu_i} \) equal to zero.

\[
\frac{\partial E(U)}{\partial \mu_i} = X_i + \sum_j \mu_j \frac{\partial X_j}{\partial \mu_i} - a \sum_j X_j \mu_j \\
- a \sum_j X_j \frac{\partial X_j}{\partial \mu_i} \sum_i (\sigma_{ij} + \mu_i \mu_j) = 0
\]  

(23) reduces to

\[
\sum_j \frac{\partial X_j}{\partial \mu_i} \left[ \mu_j - a X_j \sum_i (\sigma_{ij} + \mu_i \mu_j) \right] = X_i \left[ a \sum_j X_j \mu_j - 1 \right]
\]  

From the first-order condition (5), the term in the brackets on the left-hand side of (24) equals \( \lambda \), so (24) can be rewritten as

\[
\sum_j \frac{\partial X_j}{\partial \mu_i} = \frac{1}{\lambda} X_i \left[ a \sum_j X_j \mu_j - 1 \right]
\]  

(25)
From first-order condition (6) follows that the left-hand side of (25) is the change in wealth \( W_0 \) due to the change in \( \mu_1 \). Thus (25) implies that the change in wealth required to keep \( dU = 0 \) is equal to

\[-\frac{1}{\lambda} a \sum_j X_j \left[ \mu_j - 1 \right] \].

Hence the wealth effect on \( X_j \), associated with the change in \( \mu_1 \), is

\[-\frac{1}{\lambda} a \sum_j X_j \left[ \mu_j - 1 \right] \frac{D_{n+1, j}}{1/D} \].

This is an implicit wealth effect arising because of the increase in the "productivity" of \( X_j \) in producing expected wealth, as we explain in Chapter 6. Note the difference between this wealth effect term and R & H's so-called wealth effect term.

The wealth-compensated substitution effect of a change in \( \mu_1 \) is, using (10),

\[
\frac{\partial X_j}{\partial \mu_i} \bigg|_{U=U_0} = - \left( 1 - \sum_{k=1}^{n} \mu_k X_k \right) \frac{D_{ij}}{1/D} + \frac{a X_i \sum_{k=1}^{n} \mu_k D_{ij}}{1/D} - \frac{1}{\lambda} a \sum_j X_j \left[ \mu_j - 1 \right] \frac{D_{n+1, j}}{1/D} ;
\]

R & H also establish substitutes and complements relationships using properties of the term \( S_{ij} \), claiming certain analogies to the substitution effect of consumer
theory. None of their inferences is valid, however, since $S_{ij}$ is not the substitution effect. Finally, their analysis of changes in variance of an asset, and correlation coefficients between assets, suffer from similar conceptual mistakes as the above.

II The Analyses of B & G and A & M.

B & G and A & M identify "Veblen" or "Want Pattern" effects in their analysis of the effect of a change in the expected return (or variance) of one asset on the demand for any asset. By these they mean that changes in expected return (or variance) of an asset affect the investor's utility function. They distinguish these Veblen effects from wealth and substitution effects associated with a change in an asset's price. The theory of asset-choice, when there are changes in expected return (or variance), is treated in a manner analogous to the theory of consumer demand with variable preferences. 6

We see in Chapter 5, in the context of our discussion of the problem of mapping from characteristic or E-V space to asset space, that in E-V space, changes in expected return (or variance) of an asset simply shift the efficiency locus and do not affect the preference function. One can readily identify wealth and substitution effects in E-V space. The Veblen effects in asset space become merely components of wealth and substitution effects in characteristic or E-V space. Changes in expected return
(or variance) are equivalent to changes in the productivity of assets in characteristic space, which motivates my renaming these Veblen effects as productivity effects, in Chapter 6.

We see in chapter six that an approach framed in E-V or characteristic space yields Slutsky equations of asset demand. We are also able to generate useful empirical generalizations.

(i) **Bierwag and Grove's analysis**

B & G's analysis is exactly analogous to the Ichimura (15), Tintner (31), and Basmann (4) approach to the problem of demand with variable consumer preferences. This approach is presented and criticized, for lack of theoretical and empirical content, in Chapter 8. In that chapter, I develop an alternative treatment of the problem of demand under variable preferences in terms of the general approach to consumer theory suggested by the works of Hicks (13), Lancaster (18), and Morishima (22). The B & G analysis suffers from the shortcomings of the Ichimura-Tintner-Basman approach.

B & G maximize a preference function \( U(\mathcal{E}, V) \) subject to the investor's initial wealth \( N = \sum_i P_i X_i^0 \), where

\[
E = \sum_i \epsilon_i X_i, \quad i = 1, \ldots, n,
\]

\[
V = \sum_i \sum_j X_i X_j \sigma_{ij}, \quad j = 1, \ldots, n,
\]

and
\( x_i \) = the investor's planned holdings of the \( i \)th asset,
\( x_i^0 \) = the investor's initial holdings of the \( i \)th asset,
\( p_i \) = the price of the \( i \)th asset,
\( e_i \) = the expected return on the \( i \)th asset,
\( \sigma_{ij} \) = the covariance between returns on the \( i \)th and \( j \)th assets;
\( p_i^e \) = expected price of the \( i \)th asset.

First-order conditions for utility-maximum are

\[
U_i = U_E E_i + U_V V_i = \lambda P_i,
\]
\[
\sum_i p_i (x_i - x_i^0) = 0
\]
where \( U_i \) = the marginal utility of asset \( i \),
\( U_E \) = the marginal utility of \( E \),
\( U_V \) = the marginal utility of \( V \);

\( E_i, V_i \) are partial derivatives of \( E \) and \( V \) with respect to \( i \); \( \lambda \) is a Lagrange multiplier.

Second-order conditions require that the principal minors of \( D \) alternate in sign, where

\[
D = \begin{bmatrix}
U_{11} & \cdots & U_{1n} & U_{11} \\
\vdots & \ddots & \vdots & \vdots \\
U_{n1} & \cdots & U_{nn} & U_{n1} \\
U_{1} & \cdots & U_{n1} & 0
\end{bmatrix}
\]
Differentiating the first-order conditions with respect to a general parameter \( \alpha \), and solving for \( \frac{dX_i}{d\alpha} \), they obtain

\[
\frac{dX_j}{d\alpha} = \sum_i \left( \frac{dP_i}{d\alpha} \right) \frac{D_{ij}}{|D|} - \sum_i U_{i\alpha} \frac{D_{ij}}{|D|} + \lambda \left[ \sum_i P_i \frac{dX_i^0}{d\alpha} + \sum_i (X_i^0 - X_i) \frac{dP_i}{d\alpha} \right] \frac{D_{n+1,j}}{|D|}
\] (30)

where \( D_{ij} \) is the cofactor of the \((i,j)\)th element of \( D \).

\( U_{i\alpha} \) is the partial derivative of the marginal utility \( U_i \) with respect to \( \alpha \).

Next they derive equations for changes in various variables by equating \( \alpha \) to a particular variable.

Thus, from (30), B & G obtain for a change in \( N \), initial wealth,

\[
\frac{\partial X_j}{\partial N} = \lambda \frac{D_{n+1,j}}{|D|}
\] (31)

which is the wealth effect on \( X_j \) in analogous fashion to that of consumer theory.

A Change in the Expected Return of Asset K.

For the effect a change in \( \varepsilon_k \), holding the other independent variables constant, B & G derive, from (30),
\[
\frac{\partial X_i}{\partial e_k} = -\sum_{i=1}^{n} \frac{U_i e_k}{\lambda} S_{ij},
\]

(32)

where \( S_{ij} = \lambda \frac{D_{ij}}{D} \), the Hicks-Slutsky substitution term, and where \( U_i e_k \) is the partial derivative of the marginal utility of asset \( i \), \( U_i \), with respect to \( e_k \). According to B & G equation (6) "is the Slutsky equation for a revision in \( e_k \). It states that the effect on optimal asset holdings due to a revision in a predicted price is given by a linear combination of Slutsky-Hicks substitution terms whose weights are the changes in the marginal utilities of the assets induced by the revision in the parameter." Equation (32) has the same form as the Tintner-Ichimura relation of Chapter 8. Equation (32) is not a Slutsky equation, however, since it cannot be broken down into substitution and wealth effects. This is because the utility function, or the investor's taste-pattern over assets, itself is changing (because of changes in \( U_i \)) making comparisons of before and after utility levels meaningless, and thus impossible to identify wealth effects and utility-constant substitution effects. In another article, Bierwag and Grove (4) characterize the effect of a change in \( e_k \) on asset demand as a Veblen effect, because \( e_k \) affects the investor's utility function.
However one characterizes equation (32), it is devoid of much empirical content. The sign of $U_{i_1 e_k}$ is unclear since no information is incorporated into the analysis on the nature of $U_i$. The sign of $\frac{\partial X_i}{\partial e_k}$ is thus also ambiguous with no possible restrictions on it. Neither can (6) be broken down into components separating in general empirically significant effects from empirically insignificant ones, such as the substitution and wealth effects of consumer theory. The only restriction on the sign of $\frac{\partial X_i}{\partial e_k}$, yielded by the above approach, follows from the budget constraint, that

$$\sum_j P_j \frac{\partial X_i}{\partial e_k} = 0,$$

or

$$\sum_j P_j x_j \eta(x_j, e_k) = 0,$$  \hspace{1cm} (33)

where $\eta$ stands for elasticity.

That there is at least one asset for which $\eta(x_j, e_k) < 0$, and at least another for which $\eta(x_j, e_k) > 0$, unless all elasticities are zero.

Further in their article, by making certain assumptions about the initial price, expected return, and variance-covariance, of a numeraire asset $X_1$, B & G derive another equation from (32). They assume that the initial price $P_1$, and expected return $e_1$, of the numeraire asset are unity, and its covariances with all assets,
\( a_{11} \), are zero (thus assuming the existence of a riskless asset). They then rewrite the first-order conditions \((27)\) as
\[
\begin{align*}
\rho_i &= e_i + \rho V_i = P_i, \quad i = 2, \ldots, n. \\
\end{align*}
\] (34)
where \( \rho_i = \frac{u_i}{\bar{u}_i} \), the marginal rate of substitution of \( X_i \) for \( X_1 \).

The term \( \rho \) is the marginal rate of substitution of \( E \) for \( V \), a measure of the investor's risk-aversion. Equation \((34)\) states \(^9\) "that the investor adjusts his holdings of \( X_i \) at the margin so that in equilibrium the current price of \( X_i \) in terms of the numeraire equals its predicted price plus an allowance for risk."

Employing the above assumptions, they rewrite \((32)\) as
\[
\frac{\partial x_i}{\partial e_k} = - \sum_{i=2}^{n} r_i e_k S_{i,j},
\] (35)
\[
= - S_{kj} - \sum_{i=2}^{n} \left( \frac{\partial \beta_i}{\partial e_k} \right) S_{i,j}, \quad k \neq 1, \quad j
\] (36)
where \( \beta_i = \rho V_i \) is interpreted \(^{10} \) "as a discount for risk whose components are the investor's tastes for and estimate of the incremental risk of units of \( X_i \) in his portfolio."
Of the terms in (36)

"The first is the Slutsky-Hicks substitution term which is negative for \( k = j \) so that the effect of an upward revision in an non-predicted price is to increase the optimal holdings of the asset, ceteris paribus. The second term is a linear combination of the Slutsky-Hicks substitution terms whose weights are the revisions in the investor's risk allowances induced by his revision in the predicted price. Since the elements under summation cannot be restricted a priori, the sign of the second term is ambiguous."

In an Appendix to their paper, and after extensive algebraic manipulation of (36), B & G derive the following empirical approximation

\[
\eta(x_j, e_k) = -\frac{e_k}{p_k} \eta(x_j, P_k),
\]

subject to the condition that \( \frac{e_k x_k}{E} \) is very small; they interpret (37) to mean

"In words, if the anticipated value of the holdings of \( X_k \) is small relative to the anticipated value of the total portfolio, the predicted price elasticity is the same as the net worth compensated price elasticity modified by the factor \( \left( -\frac{e_k}{P_k} \right) \).

One might regard this result as a logical extension of the familiar Marshallian proposition of traditional demand theory."

Note that \( \eta(x_j, P_k) \) is the wealth-compensated price elasticity, defined here as \( S_{kj} \left( \frac{x_j}{P_k} \right) \).

In chapter six I show that one can derive the empirical approximation (37) in a simpler manner and directly from the first-order conditions by utilizing fully information these contain on the characteristics of the utility function. My approach does not make
the unnecessary assumptions underlying (36). The reason B & G's expressions (35) or (36) are more useful than (32) is that the former incorporate into the analysis, although indirectly through $\beta_i$, information on the nature of the preference function in terms of its $E,V$ characteristics. It is this that makes the derivation of equation (36) possible.

The advantages to asset-choice theory, or consumer theory, of an approach which fully incorporates information on the utility function in terms of its important characteristics (i.e., $E,V$ above), are fully discussed in chapters six and eight.

The Effect of a Change in initial price

For a change in $P_k$, the initial price of the $k$th asset, they obtain from (30),

$$\frac{\partial x_j}{\partial p_k} = S_{kj} + (x_k^0 - x_k) \frac{\partial x_j}{\partial x_k}$$

which they claim is the Slutsky equation of a change in the price of asset $k$ on the amount demanded of asset $j$, holding other prices and initial wealth constant. They identify the first term on the right, $S_{kj}$, as the wealth-compensated substitution effect, and the second term as the wealth effect. According to B & G,13 "The Slutsky equation for a change in a current price may, thus, be regarded as being composed of a substitution
The term which is negative for \( k = j \), and a networth term which cannot in general be restricted in sign. However, equation (38) is incomplete. A change in \( P_k \) affects \( e_k \) since by definition \( e_k = P^e_k - P_k \). Thus \( U_{ie_k} \) in (30) is not zero. The preceding argument does not violate the B & G assumption underlying (38) that changes in \( P_k \) do not affect \( P^e_k \) (i.e. the elasticity of expectations are zero); the latter is another route by which \( P_k \) might affect \( e_k \). By taking the effect of \( P_k \) on \( e_k \) into account, \( \frac{\partial X_j}{\partial P_k} \) becomes,

\[
\frac{\partial X_j}{\partial P_k} = S_{k,j} + \sum_i U_{ie_k} \frac{D_{ij}}{P_i} + (X^e_k - X_k) \frac{\partial X_j}{\partial N}; \quad (39)
\]

Equation (39) contradicts also B & G's contention that the demand for \( X_j \) is homogeneous of degree zero in initial asset prices.

(ii) The Works of Allingham and Morishima (A & M) and Morishima (M).

The papers of A & M and M place similar interpretations on the comparative statics of optimal portfolio adjustment. The basic difference between them is that while A & M makes use of the utility function explicitly, M is in the Revealed-Preference framework. Because of the similarity of the two papers my discussion will
center more on the A & M work, but will comment on some aspects of the M work.

In similar manner to B & G, A & M differentiate between the usual income-substitution effects and a Want-Pattern (or Veblen) effect, in their analysis of optimal portfolio adjustment.

A & M maximize a utility function of \( E \) and \( V \),

\[
E \left( \sum_i y_i \mu_i, \sum_i \sum_j y_i y_j \sigma_{ij} \right), \quad i, j = 1, \ldots, m ,
\]

subject to

\[
\sum_i q_i y_i = W ,
\]

where \( y_i \) = quantity of asset \( i \),
\( \mu_i \) = expected return on asset \( i \),
\( \sigma_{ij} \) = covariance between \( i \) and \( j \),
\( q_i \) = initial price of asset \( i \)
\( W \) = initial wealth of investor.

The first-order conditions for a maximum are

\[
f_{\mu} \mu_i + 2f_{\sigma} \sum_j y_j \sigma_{ij} = \lambda q_i ,
\]

\[
\sum_i q_i y_i = W ,
\]

where, \( f_{\mu} \) is the marginal utility of expected wealth,
\( f_{\sigma} \) is the marginal utility of \( \sigma \),
\( \lambda \) is a Lagrange multiplier.
For the effect on asset \( k \) of a change in the initial price of asset \( r \), they derive, by differentiating the first-order conditions,

\[
\frac{\partial q_k}{\partial q_r} = -\gamma_k \gamma_k + \lambda Y_{rk} + \sum_j (R \delta_{jr} + \frac{\partial R}{\partial \mu} \delta_{jr} \gamma_j \gamma_j) \frac{\partial \mu}{\partial q_r} \gamma_{jk} \quad (43)
\]

where \( R = \frac{\sigma}{\mu} \), the marginal rate of substitution between \( \mu \) and \( \sigma \);

\[
\delta_{jr} = 1 \text{ for } j = r,
\]

\[
= 0 \text{ for } j \neq r,
\]

\( Y_k \), \( Y_{rk} \) are the ratios of the cofactors of \( q_k \) and \( g_{rk} \) to the value of the determinant, respectively in the determinant

\[
\begin{vmatrix}
0 & q_1 & \cdots & q_m \\
q_1 & q_{ii} & \cdots & q_{im} \\
\vdots & \vdots & \ddots & \vdots \\
q_m & q_{m1} & \cdots & q_{mm}
\end{vmatrix}
\]
According to A & M, on the right side of expression (43), "the first term represents the proper income effect, the second the proper substitution effect, and the third the proper want pattern effect." This last effect is defined by A & M as "the change in demand arising through a change in tastes (or in preference orderings over commodities) brought about by the change in prices." This is the same as B & G’s Veblen effect. I show in Chapter 6 that the A & M identification of the terms in (43) is inappropriate; that the terms in (43) are components solely of substitution and wealth effects, and that no taste changes are involved. The so-called Veblen or Want-Pattern effects simply reflect changes in the productivity of assets in terms of their objective $E,V$ characteristics, i.e., changes in $\mu_i, \sigma_{ij}$.

In the case where the change is in expected return alone, rather than in the initial price of an asset, A & M derive

$$ \frac{\partial y_k}{\partial \mu_r} = f_\sigma \rho Y_{rk} + f_\sigma \frac{\partial \rho}{\partial \mu_r} y_r \sum_j \mu_j Y_{jk} \quad , $$

where

$$ g_{ij} = f_{\mu} \mu_i \mu_j + 2 f_{\sigma} \mu_j \sum_k \sigma_{ik} y_k + \sigma_{ij} \sum_k \sigma_{ik} y_k $$

$$ + 2 f_{\sigma} \sigma_{ij} + 4 f_{\rho} (\sum_k \sigma_{ik} y_k) (\sum_k \sigma_{jk} y_j) \quad ; $$
which is their Want-Pattern effect alone.

A & M correctly point out that equation (44) is the same as R & H's equation (10) earlier, except that the latter is explicitly framed in terms of the quadratic utility function. However, unlike R & H, who identify the first term on the right as the substitution effect and the second term as the wealth effect, A & M identify these as the Relative Want-Pattern Effect and Absolute Want-Pattern Effect, respectively.

A & M, like R & H, claim that the second term on the right in (44) is \( y \left[ \frac{\partial y_k}{\partial E(N)} \right] \). Such an interpretation is meaningful if the change in \( E(W) \) is a parametric change caused by a change in \( T \), the lump-sum tax, as we noted in the context of our discussion of the R & H paper.

Regardless of what they call these terms, the description of the effects by both sets of authors are very similar. A & M describe their effects in the following manner, where the equations (7) and (9) are our (42.) and (44):

"A change in \( \mu_r \) affects the individual's want pattern via two channels: through the change in the relativity of \( \mu_r \) to other \( \mu_i \)'s, and through the change in the absolute level of \( \mu = \sum_{i=1}^{N} \mu_i y_i \). In order to obtain the pure effect on \( y_k \) of the former, we have to introduce an imaginary change in \( \mu \), so as to leave the individual on the same level of \( \mu \) as before. Such a compensated purely relative change in \( \mu_r \) gives rise to a change in the \( y_i \)'s,
since, as (7) shows, the marginal rates of substitution between the \( y_i \)'s depends on their relative expected values. This effect, which we call the relative want pattern effect, is represented by the first term in (9).”

The effect of the compensation, that leaves the individual on the same level of \( \mu \) as before, that A & M have in mind, is the negative of the second term in (44), namely

\[
-\frac{\partial y}{\partial \mu} \left( \frac{\partial y_k}{\partial E(W)} \right) \mu; \text{ const.}
\]

which also equals \( \frac{\partial y}{\partial \mu} \left[ \frac{\partial y_k}{\partial E(W)} \right] \).

We argued extensively earlier, in the context of our discussion of the R & H paper, that such a compensation does not leave the individual on the same level of \( \mu \) as before since \( \mu \) is a choice variable, rather than a parameter, and an original parametric decrease in \( \mu \) leads the investor to adjust the values of \( \mu \) and \( \sigma \) to new optimal levels. Thus the description of the first term in (44) as keeping the individual on the same level of \( \mu \) is inaccurate. A & M go on, \textsuperscript{17}

"On the other hand, the change in the absolute level of \( \mu \) induced by the change in \( \mu \) affects the marginal rate of substitution between \( \mu \) and \( \sigma \). If \( \frac{\partial \sigma}{\partial \mu} > 0 \), then the individual will be prepared to bear more risk than before when the absolute level of \( \mu \) is raised. Such a change in the individual's risk aversion brings about a change in the holding of assets, which is represented by the second term of (9) and may be called the Absolute Want Pattern Effect."

A & M's equation (9) is our (44).

Again, in accordance with our criticism above, this latter effect does not represent the movement from the
original to the final level of $\mu$, but rather the movement from some combination of $\mu$ and $\sigma$, in general other than the original, to the final combination of $\mu$ and $\sigma$. Although $\left[\frac{\partial y_k}{\partial E(W)}\right]_\mu$ is written as a partial derivative, it does not hold variance constant.

$M$ identifies similar effects as $A \& M$ in an analysis framed in the Revealed-Preference framework. Note that in the following quotation from $M$, $\mu$ stands for a vector of expected returns $\mu_1, \ldots, \mu_n$; $X$ stands for a vector of assets $x_1, \ldots, x_n$; $\bar{W}$ stands for initial wealth.

"The general effect of a change in $\mu$ on $X$ is split up into two components: one due to a change in the general level of $\mu_1, \ldots, \mu_n$ and the other due to a change in their relativity. In order to isolate them from each other, we introduce a third situation with $\mu^* \sim \mu^1$ but whose absolute level is adjusted such that $\mu^1 x^0 = \mu^0 x^0$. In terms of $\mu^2$, the levels of $\mu^0$ and $\mu^1$ may be defined as

$$\frac{\mu^0 x^0}{\mu^3 x^0} = 1 \text{ and } \frac{\mu^1 x^0}{\mu^2 x^0} = \lambda,$$

respectively.

"Let $X^2$ be an $X$ which is chosen at $\mu^2$ by the individual with given $\bar{W}$. Then the change in $X$ from $X^0$ to $X^1$ can be regarded as the 'resultant' of the change from $X^0$ to $X^2$ and the further change from $X^2$ to $X^1$, which we call the Relative and the Absolute Expected Return Effects, respectively. This analysis of the total effect is visualized by the formula

$$X^1 - X^0 = (X^1 - X^2) + (X^2 - X^0)\quad (1)$$

and can be given the following explanation."

"If the individual continued to choose the same portfolio $X^0$ as before in spite of an increase in the expected returns, he could expect a greater amount of wealth at the end of the period, without putting himself in a riskier position. Therefore, we could imagine the third circumstance where a
relative increase in some elements of $\mu$ happened in combination with such a general decrease in $\mu$ as would result in the expectation of the same amount of wealth if the individual stuck to $X^0$ despite the change. A change from the original to the third circumstance would give rise to a change in the portfolio selected, which is given by the Relative Expected Return Effect. And a general increase in $\mu$ which compensates the general decline in $\mu$ imagined in the third circumstance gives the Absolute Expected Return Effect.  

M writes the following equation to describe these changes,

$$\frac{\partial X_i}{\partial \mu_i} = X_i^0 \frac{\partial X_j}{\partial E} + X_j^0$$

(45)

where the first term is the Absolute Expected Return Effect and the second term is the Relative Expected Return Effect.

Criticizing R & H's characterization of their equation as a Slutsky equation, M writes about (45) (his (4)')

"As is at once noticed, this equation is very similar to the Slutsky equation which Hicks called the 'fundamental equation of value theory,' except that the latter has a minus sign before $X^0$ on the right-hand side. In analogy with the Slutsky equation, Royama and Hamada called the first and second terms of (4)' the wealth and the substitution effects respectively. However, (4)' is no more than the analysis of what I called the 'want pattern' effect in the previous chapter. It must be remembered that in the traditional theory of demand, prices appear in the budget equation but not in the utility function, while in the theory of portfolio choices, the expected rates of returns are not in the budget equation but affect the rankings of various portfolios."

As mentioned earlier, in the subsequent chapters we examine the nature of this effect.
My criticisms of the effects identified by \( \text{M} \) are similar to the ones given of \( A \& M \). The Relative Expected Return Effect does not hold expected wealth (or variance) constant, since although we have set \( \mu^0 x^0 = \mu^2 x^0 \), this does not imply that \( \mu^0 x^0 = \mu^2 x^2 \) (also in general \( \sigma(x_0) \neq \sigma(x_2) \)), where \( x^2 \) is the optimal portfolio selected with the vector \( \mu^2 \). In words, while the initial change in \( \mu \) is compensated to provide the individual with the same expected wealth were he to stay at \( x^0 \), the individual does not stay at \( x^0 \) and hence expected wealth and variance do not stay the same. The movement from \( x^2 \) to \( x^1 \), the Absolute Expected Return Effect, does not represent the effect of an increase in the general level of \( \mu \) from the initial level \( \mu^0 x^0 (= \mu^2 x^2) \) to the final level \( \mu^1 x^1 \), but rather it represents the movement from \( \mu^2 x^2 \) to \( \mu^1 x^1 \). Also this movement in general involves a change in variance, since \( \sigma(x_2) \neq \sigma(x_1) \). \( \sigma^2 \) in (45) does not hold variance constant.
Notes to Chapter Four

1. Throughout this chapter I use the symbols actually used by the authors of the articles.

2. See R & H (27), page 33.

3. See R & H (27), page 33.

4. See R & H (27), page 33.

5. See M (23), page 292-293.

6. See the literature on consumer demand with variable preferences: Ichimura (15), Tintner (31), Basmann (4).

7. See B & G (5), page 117.

8. For a more thorough discussion of the shortcomings of the Tintner-Ichimura relation see my Chapter 8.

9. See B & G (5), page 118.

10. See B & G (5), page 118.

11. See B & G (5), page 119.

12. See B & G (5), page 126.

13. See B & G (5), page 117.


15. See A & M (2), page 244.

17 See A & M (2), page 260.
18 See M (23), page 291–292.
19 See M (23), page 292–293.
Chapter 5
Transformation from Mean-Variance Space to Asset Space

In this chapter we discuss the nature of the mapping from mean-variance space to asset space. This will supplement and clarify our discussion in Chapter 6.

The mean-variance hypothesis of investment behavior assumes that the investor maximizes a utility function of expected end-of-period wealth, $E$, and variance of wealth, $V$, subject to a boundary condition defining attainable values of $E$ and $V$. The utility function $U(E,V)$ is assumed to have $U_E > 0$, $U_V < 0$; also the determinant

$$
\begin{vmatrix}
U_{EE} & U_{EV} & U_E \\
U_{EV} & U_{VV} & U_V \\
U_E & U_V & 0
\end{vmatrix}
$$

is assumed positive guaranteeing convexity of the indifference curves from below. These indifference curves are shown in Figure 9 by $U_0$, $U_1$, $U_2$, etc.

The boundary of attainable values of $E$ and $V$, the parabola $LL$, is obtained by minimizing $V$ for any given value of $E$. 

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We can map the optimization procedure from E-V space into asset space, assuming two assets only.

In asset space (Figure 9) the minimization of $V$ for any given value of $E$ yields the line AA which is the locus of points of tangency between "isomean lines" $E_0$, $E_1$, $E_2$, etc. and "isovariance ellipses" $V_0$, $V_2$, etc. Considering the indifference curve $U_0$ in Figure 10, it crosses LL at B and D with coordinates $(E_0, V_0)$ and $E_2, V_2$ respectively. B and D thus correspond to $B'$ and $D'$ in asset space on the line AA. The point C on $U_0$ has coordinates $E_1, V_1$ and is not an efficient point.
Figure 10

It is off AA in Figure 10. By mapping the points on $U_0$ into asset space we trace the circular indifference curve $U_0'$ in Figure 10. Doing the same for $U_1$ and $U_2$ in Figure 9 we trace the indifference curves $U_1$ and $U_2$ respectively in Figure 10.

Thus given values for expected rates of return, variance, and covariance, of $X_1$ and $X_2$, we derive a set of circular indifference curves in asset space. Changes in any of these variables or parameters lead to shifts in LL in Figure 9, and to shifts in the utility surfaces in Figure 10. It is highly important to note
that while changes in these variables shift only the efficiency locus or constraint in characteristics space, they lead to shifts in utility surfaces in asset space.

In Figure 11 we examine the optimization process in asset space, as well as the effects of changes in the variables. Initial prices for $X_1$ and $X_2$ as well as the initial wealth of the investor determine the position of budget constraint $KN$. As we show mathematically in Chapter 6, the utility maximum point is at $F$ where the budget constraint is tangent to $U'$ on its
convex portion. An increase in the expected rate of return of one of the assets shifts the efficiency locus upward in Figure 9. In terms of Figure 11, this leads to changes in the utility surfaces in asset space, with the asset combination at \( F \) now corresponding to a higher level of utility than \( U'_0 \). Similar arguments apply to changes in variance and covariance. Changes in all these variables lead to changes in the pattern of preferences in asset space, and the final outcome, concerning new equilibrium value of \( X_1 \) and \( X_2 \), is in general ambiguous since the direction and magnitude of these shifts depend on the actual values of the parameters of the system. Note, however, that no change in the pattern of preferences has occurred in \( E-V \) space.

A change in the price of an asset, such as a decrease in the price of \( X_2 \) (holding other variables constant) leads to the new budget constraint \( KN' \). At the same time, since a decrease in the price of \( X_2 \) affects the rate of return on \( X_2 \), the preference surfaces shift as well. Again, owing to these two types of shifts, it is in general unclear whether the new equilibrium values of \( X_1 \) and \( X_2 \) are greater or less than before. Notice that contrary to assertions by Bierwag and Grove (6) and B & G (5), changes in asset prices do cause shifts in the utility surfaces in Figure 11, even when we have zero elasticities of expectation, since changes
in current asset prices affect rates of return on assets.

B & G ([5]) and A & M ([2]) identify "Veblen" or "Want Pattern" effects in their analysis, meaning that changes in the expected return (or variance or covariance) of an asset affects the investor's utility function. They distinguish these Veblen effects from wealth and substitution effects associated with a change in an asset's price. Thus the theory of asset choice when there are changes in the former variables is treated in a manner analogous to the theory of consumer demand with variable preferences.

An important point made earlier was that in characteristics or E-V space, changes in expected return (or variance or covariance) simply shift the efficiency locus and do not affect the preference function. One can readily identify wealth and substitution effects in E-V space as I do in Chapter 6. The Veblen effects in asset space become merely components of wealth and substitution effects in characteristics space. Changes in expected return (or variance or covariance) are equivalent to changes in the productivity of assets in characteristic space. This motivates my renaming these Veblen effects as Productivity effects.

I show in Chapter 6 that only an approach framed in characteristics space provides the correct form for
Slutsky equations of asset demand, as well as meaningful definitions of substitutability and complementarity among assets.

Finally, we can discuss the effect of changes in initial wealth (holding other variables constant) on asset demand. An increase in initial wealth shifts KN parallel to the right to K'N' as shown in Figure 12.

The new equilibrium point is at G in Figure 12. In terms of characteristic mean-standard deviation space (E-S), the efficiency locus $L_0L_0$ shifts to $L_1L_1$ as
shown in Figure 13, with each point on $L_0O_0$ increasing in proportion to its original E-S magnitude.

Figure 13

No shifts in the utility function are associated with changes in wealth in asset space.

If we keep increasing wealth we trace the efficiency locus $LL$ which is the envelope of parabolas $L_0L_0$, $L_1L_1$, $L_2L_2$, etc. in Figure 14 each drawn for a given value of initial wealth. On $LL$ we determine the point $J$ which provides the highest level of utility $U_1$, for given values
of expected return, variance, covariance, and initial asset prices.

Point J above corresponds to point I in Figure 12 which Bierwag and Grove (6) identify as the "bliss point," the highest level of utility attainable. They argue that if wealth were to increase beyond the level corresponding to I in Figure 12 a phenomenon of "contamination by risk" arises, such as at point F', where utility is lower than at I. This arises, according to
these authors, because the investor must hold all his wealth in the form of these assets. However, as pointed out in footnote 3, and contrary to these authors' contention, point $F'$ is not a point of utility maximum because the second-order conditions of convexity of the indifference curve are not satisfied at this point. Put more simply, at points such as $F'$, at least one of the assets has become a bad (provides negative marginal utility). Utility-maximizing behavior thus requires that the investor use only part of his initial endowment of wealth to reach the highest level of utility at $I$. In other words, maximizing behavior requires the investor to discard part of his wealth rather than hold it in the form of an asset yielding negative marginal utility. Or, in a more complete model, rational behavior might imply that instead of discarding a portion of wealth the investor allocate more of it to consumption.

In a recent article, Tsiang criticizes the validity of closed indifference curves in asset space. Quoting Tsiang,

"For instance, C.O. Eierwag and Myron Grove by neglecting the constraint on the slopes of $E-S$ indifference curves, were led to the strange conclusion that the indifference curves for any two risky assets (the yield-risk ratios of which are unspecified) are closed curves, which are drawn in their Figure 3 as concentric circles, with the center at some bliss point, apparently not related to the concept of an ultimate bliss in certain
types of utility functions such as the negative exponential function. This would imply that a portfolio of, say 10 shares each of, say, GM and IBM might yield the same expected utility as a portfolio of, say, a million shares each of these two stocks; and that, in the latter situation, the owner could increase his expected utility by destroying some of both assets—a phenomenon which Bierwag and Grove call "contamination by risk," but which is obviously contrary to our common sense.

"Now if we realize that the E-S indifference curves would never have slopes greater than unity, then a ray from the origin in the positive E-S space with a slope greater than unity would never cut the same E-S indifference curve twice. Mapped into the two asset space, it means that so long as none of the two assets has a risk greater than its own expected terminal value (which can certainly be said for either GM or IBM stocks, even though they are not riskless), no rays from the origin representing proportionate variations of given portfolios of these two stocks should cut any asset indifference curves more than once. Thus the paradoxical conclusion of Bierwag and Grove can be shown to be incorrect, because they assumed that E-S indifference curves can slope up to be asymptotic to the vertical, so that any rays, with slope of 45° or more, can still cut each E-S indifference curve at two points. Actually the so-called phenomenon of contamination by risk cannot take place unless the risk of the portfolio involved is greater than its expected terminal value."

Tsiang's criticism is invalid. First, as I showed earlier, one can dispose of the "contamination by risk" argument by considering the second-order convexity conditions for utility maximum and showing that these do not hold at "contaminated" points such as F' in Figure 4. Thus the existence of circular indifference curves in asset space does not imply the existence of the "contamination" phenomenon, as Bierwag and Grove (6) and Tsiang
mistakenly claim. Second, constraining the slope of indifference curves in characteristics space to values less than or equal to unity does not in any way affect the existence of circular indifference curves in asset space, contrary to Tsiang's claim. Tsiang does not realize that indifference curves in asset space are derived from intersection points of the efficiency locus LL in Figure 9 with the indifference curves \( U_0, U_1, \) etc. The slope of the parabola LL depends upon the variance, covariance, and expected return of each asset. The slope of LL cannot be constrained to any value even if we constrain each asset to an expected return always greater than its variance. Thus LL and \( U_0, U_1, \) etc, can each have two intersections in spite of Tsiang's constraints, and circular indifference curves exist regardless.

Figure 15 shows, in geometric terms, a counterexample to Tsiang. We have two assets A and B with \( E_A > S_A \) and \( E_B > S_B, \) i.e. with both points lying above the 45° line in E-S space. If we do not constrain the correlation coefficient between these two assets to unity, we can derive the efficiency locus CS (there are no constraints on the slope of this line at any point). We draw linear indifference curves \( U_0, U_1, U_2, \) with slopes less than unity (less than 45°). As shown each indifference curve intersects the efficiency locus twice, at points such as
C, D or C', D' or C'', D'', respectively, generating circular indifference curves in asset space. Other contrary examples can be easily devised.
Notes to Chapter Five

1 The mechanics of mapping from E-V space to asset space is due to Bierwag and Grove (6). Following them, LL is drawn assuming no constraint on the investor's wealth. We show later that LL is the envelope of wealth-constant boundaries in E-V space.

2 See Markowitz (21) for the derivation of AA. The line AA in asset space (Figure 10) maps into LL in characteristic space (Figure 9).

3 Contrary to Bierwag and Grove (6), points such as F' in Figure 11 are not utility maximizing points because the condition of convexity of the indifference curve is not satisfied at F'. This has strong implications to the "contamination by risk" argument of Bierwag and Grove. More on this is given later. Note also that in general there are no restrictions to make the equilibrium values of both $X_1$ and $X_2$ positive. One of these may very well be negative.

4 For the theory of consumer demand with variable preferences see Basmann (4). B & G's analysis is exactly along the lines of Basmann in dealing with taste changes in consumer theory. See also Chapter 8.

5 I use mean-standard deviation E-S space for expositional purposes, to keep the effect of wealth linear in characteristic E-S space. This also makes our graphics comparable to Tsiang's (34) who works in E-S space and whose work I criticize later.

6 For this envelope property see also Bierwag and Grove (6) and Roy (26).

7 See Bierwag and Grove (6), p. 340.

8 See Bierwag and Grove (6), p. 341.

9 See Tsiang (34), footnote 12, page 363.
Chapter 5
The Comparative Statics of Portfolio Adjustment

In this chapter I reformulate and extend the previous studies of the comparative statics of optimal portfolio adjustment.

In a mean-variance framework an asset is demanded indirectly because of its contribution to portfolio expected return and variance, and not for other intrinsic properties. The demand for an asset is thus analogous to the demand for a factor of production.

The treatment of an asset as an input of production places the problem of asset-demand within Lancaster's reformulation of demand theory (18). Lancaster argues that many of the limitations of traditional consumer theory would be overcome by "breaking away from the traditional approach that goods are the direct objects of utility and, instead, supposing that it is the properties or characteristics of the goods from which utility is derived."¹ Thus the demand for a good becomes a derived demand for the characteristics of the good, where goods are inputs producing these characteristics.² In a recent note Roberts (25) points out the close parallel between modern portfolio theory and Lancaster's framework.³ In mean-variance

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portfolio theory assets have two characteristics, namely, portfolio risk and expected return. Unlike Lancaster's model, the transformation from asset or commodity space to characteristics space in mean-variance theory is neither linear nor additive.¹

I The demand for a risky asset i is derived by maximizing the investor's utility function of end-of-the period expected wealth and variance $U(E,V)$, subject to the investor's initial wealth constraint given by equation (49) below. Thus, maximize a function

$$ L = U(E,V) - \lambda \left( \sum_{i} p_{i}^{o} x_{i} - W_{0} \right), \quad (46) $$

where

$$ E = \sum_{i} x_{i} \mu_{i}, \quad (47) $$

$$ V = \sum_{i} \sum_{j} X_{i} X_{j} \sigma_{ij}, \quad (48) $$

$$ W_{0} = \sum_{i} p_{i}^{o} x_{i} \quad \quad (49) $$

The symbols are:

- $E$ = portfolio expected return,
- $V$ = portfolio variance,
- $x_{i}$ = amount of $i^{th}$ asset,
- $p_{i}^{o}$ = price of $i^{th}$ asset at beginning of period,
- $p_{i}^{e}$ = expected price of $i^{th}$ asset at end of period,
- $\mu_{i}$ = $p_{i}^{e} - p_{i}^{o}$, is the expected return of $i^{th}$ asset,
- $\sigma_{i}^{2}$ = is variance of $i^{th}$ asset,
\[ W_0 = \text{investor's initial wealth}, \]
\[ \lambda = \text{a Lagrange multiplier}, \]
\[ \sigma_{ij} = \text{covariance between returns of } i \text{ and } j. \]

From the maximization in (46) we get the following first-order conditions:

\[ U_E \mu_i + 2 U_V \sum_j \sigma_{ij} x_j = \lambda P_i^0, \quad (i = 1, \ldots, n), \quad (50) \]
\[ W_0 - \sum_i P_i^0 x_i = 0 \quad , \quad (51) \]

These first order conditions can be solved to derive the demand function for asset \( i \).

Second order conditions for a maximum require that the principal minors of the determinant \( D \), obtained by differentiating the equation system (50) and (51) with respect to the \( x_i \)'s, alternate in sign, where

\[ |D| = \begin{vmatrix} Z_{11} & \cdots & Z_{1n} & P_1^0 \\ \vdots & \ddots & \vdots & \vdots \\ Z_{n1} & \cdots & Z_{nn} & P_n^0 \\ P_1^0 & \cdots & P_n^0 & 0 \end{vmatrix}, \]

and where

\[ Z_{ij} = U_{EE} \mu_i \mu_j + 2 U_{EV} \mu_i \sum_r \sigma_{ir} X_r + 2 U_{VE} \mu_i \sum_r X_r \sigma_{ir} + 2 U_V \sigma_{ij} + 4 U_{VV} \left( \sum_r \sigma_{ir} X_r \right)^2. \]
In order to keep the exposition simple I initially assume that only two assets exist. I later generalize the results to the case of an indefinite number of assets.

First order conditions in the two-asset case are

\[ U_\varepsilon \mu_1 + 2 U_v (X_1 \sigma_1^2 + X_2 \sigma_{12}) = \lambda P_1^0, \quad (52) \]

\[ U_\varepsilon \mu_2 + 2 U_v (X_2 \sigma_2^2 + X_1 \sigma_{12}) = \lambda P_2^0, \quad (53) \]

\[ P_1^0 X_1 + P_2^0 X_2 = W_0; \quad (54) \]

II The Effect of a Change in Price

(i) I examine the effect of a change in \( P_1^0 \) on the quantity demanded of \( X_1 \), holding constant \( P_1^0, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_{12}, P_2^0, W_0 \). Differentiating (52), (53), (54) with respect to \( P_1^0 \), we get

\[ U_{\varepsilon \psi} \mu_1 \frac{\partial E}{\partial P_1^0} + U_{\varepsilon V} \mu_1 \frac{\partial V}{\partial P_1^0} + U_\varepsilon \frac{\partial \mu_1}{\partial P_1^0} + 2 U_{\psi V} (X_1 \sigma_1^2 + X_2 \sigma_{12}) \frac{\partial V}{\partial P_1^0} + 2 U_{\psi E} (X_1 \sigma_1^2 + X_2 \sigma_{12}) \frac{\partial E}{\partial P_1^0} \]

\[ + 2 U_{\psi V} (X_1 \sigma_1^2 + X_2 \sigma_{12}) \frac{\partial E}{\partial P_1^0} + 2 U_{\psi V} (X_1 \sigma_1^2 + X_2 \sigma_{12}) \frac{\partial V}{\partial P_1^0} = \lambda + P_1^0 \frac{\partial \psi}{\partial P_1^0}, \quad (55) \]

\[ U_{\varepsilon \psi} \mu_2 \frac{\partial E}{\partial P_1^0} + U_{\varepsilon V} \mu_2 \frac{\partial V}{\partial P_1^0} + 2 U_{\psi V} (X_2 \sigma_2^2 + X_1 \sigma_{12}) \frac{\partial V}{\partial P_1^0} + 2 U_{\psi E} (X_2 \sigma_2^2 + X_1 \sigma_{12}) \frac{\partial E}{\partial P_1^0} \]

\[ + 2 U_{\psi V} (X_2 \sigma_2^2 + X_1 \sigma_{12}) \frac{\partial E}{\partial P_1^0} + 2 U_{\psi V} (X_2 \sigma_2^2 + X_1 \sigma_{12}) \frac{\partial V}{\partial P_1^0} = P_2^0 \frac{\partial \psi}{\partial P_1^0}, \quad (56) \]

\[ P_1^0 \frac{\partial X_1}{\partial P_1^0} + P_2^0 \frac{\partial X_2}{\partial P_1^0} = -X_1; \quad (57) \]
Also,

$$\frac{\partial E}{\partial P_1^0} = \mu_1 \frac{\partial X_1}{\partial P_1^0} + X_1 \frac{\partial \mu_1}{\partial P_1^0} + \mu_2 \frac{\partial X_2}{\partial P_1^0},$$

$$\frac{\partial V}{\partial P_1^0} = 2 \sigma_1^2 X_1 \frac{\partial X_1}{\partial P_1^0} + 2 \sigma_2^2 X_2 \frac{\partial X_2}{\partial P_1^0} + 2 X_1 \sigma_1 \frac{\partial X_2}{\partial P_1^0} + 2 X_2 \sigma_1 \frac{\partial X_1}{\partial P_1^0},$$

Substituting these equations into (56), (57), (58), and writing the system in matrix form, we get

$$\begin{bmatrix} Z_{11} & Z_{12} & P_1^0 \\ Z_{21} & Z_{22} & P_2^0 \\ P_1^0 & P_2^0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial X_1}{\partial P_1^0} \\ \frac{\partial X_2}{\partial P_1^0} \end{bmatrix} = \begin{bmatrix} \lambda + R_1 \left( \frac{\partial \mu_1}{\partial P_1^0} \right) \\ R_2 \left( \frac{\partial \mu_1}{\partial P_1^0} \right) \\ -X_1 \end{bmatrix},$$

(58)

where

$$R_1 = - \left[ U_{EE} + U_{EE} \mu_1 X_1 + 2 U_{VE} X_1 (X_1 \sigma_1^2 + X_2 \sigma_1 \sigma_2) \right],$$

$$R_2 = - \left[ U_{EE} \mu_2 X_1 + 2 U_{VE} X_1 (X_2 \sigma_2^2 + X_1 \sigma_1 \sigma_2) \right].$$
By definition, \( \frac{\partial \mu_1}{\partial P^0_i} = -1 \).

Note that \( R_1 \) and \( R_2 \) denote respectively shifts in the marginal utility functions of \( X_1 \) and \( X_2 \) resulting from the change in the marginal productivity of \( X_1 \), namely \( \mu_1 \). The productivity change is the change in \( \mu_1 \) caused by the change in \( P^0_1 \).

Using Cramer's rule we solve (58) for \( \frac{\partial X_1}{\partial P^0_1} \):

\[
\frac{\partial X_1}{\partial P^0_1} = \frac{\begin{vmatrix}
\lambda + R_1 & Z_{12} & P^0_1 \\
R_2 & Z_{22} & P^0_2 \\
-X_1 & P^0_2 & 0
\end{vmatrix}}{|D|}
\]

or

\[
\frac{\partial X_1}{\partial P^0_1} = -X_1 \frac{D_{31}}{|D|} + \lambda \frac{D_{41}}{|D|} - \frac{D_{11} R_1 + D_{21} R_2}{|D|}, \tag{59}
\]

where \( D_{ij} \) is the cofactor of the \((i,j)\) the element of \( D \).

Equation (59) corresponds to the relationship obtained by A & M. They identify the first term on the right as the income effect, the second term as the substitution effect, and the third term as the "Want-Pattern" or "Veblen" effect (reflecting taste changes), of a change in \( P^0_1 \).

I show that the A & M identification of the terms in (59) is inappropriate; that the terms in (59) are components solely of substitution and wealth effects, and that
no taste changes are involved. The so-called "Want-Pattern" effects simply reflect changes in the productivity of assets in terms of their objective $E,V$ characteristics, i.e., changes in $\mu_1$, $\sigma_1^2$, $\sigma_{ij}$.

In analyzing the effect of a change in $P_1^0$, I distinguish between two channels through which effects operate: One that affects the productive characteristics of $X_1$ and all others. The productive characteristics of $X_1$ are affected because a change in $P_1^0$ affects $\mu_1$, since $\mu_1 = P_1^0 - P_1^e$. However, a change in $P_1^o$ affects $X_1$ via also the traditional income and substitution effects of consumer theory. We can separately identify these latter effects by constraining $\mu_1$ to stay constant.

$\mu_1$ can be kept constant, by assuming $P_1^e$ to vary proportionally to $P_1^o$, e.g. by assuming that a change in $P_1^o$ leads the investor to revise $P_1^e$ equally in same direction, so that $\frac{\partial \mu_1}{\partial P_1^e} = \frac{\partial P_1^e}{\partial P_1^o} - 1 = 0$. The assumption of $\frac{\partial \mu_1}{\partial P_1^o}$ leads to $R_1(\frac{\partial \mu_1}{\partial P_1^o}) = R_2(\frac{\partial \mu_1}{\partial P_1^o}) = 0$ on the right of equation (58). Thus the terms reflecting asset-productivity changes disappear. Equation (59) reduces to

$$\frac{\partial X_1}{\partial P_1^o} \bigg|_{d\mu_1 = 0} = -\chi_i \frac{D_{3i}}{(D)} + \lambda \frac{D_{ii}}{(D)} ; \quad (60)$$

We next show that the second term on the right side of (60) is the traditional substitution effect and the first term the traditional income effect of a price change.
To derive the substitution effect we set \( \frac{dU}{d\mu_1} \bigg|_{d\mu_1=0} = 0 \).

\[
dU \bigg|_{d\mu_1=0} = U_\varepsilon \left( \frac{\partial E}{\partial x_1} dx_1 + \frac{\partial E}{\partial x_2} dx_2 \right) + U_v \left( \frac{\partial V}{\partial x_1} dx_1 + \frac{\partial V}{\partial x_2} dx_2 \right) = 0 ,
\]

which reduces to

\[
\frac{dx_1}{dx_2} = - \frac{U_\varepsilon \mu_2 + U_v (x_2 \sigma_1^2 + x_1 \sigma_{12})}{U_\varepsilon \mu_1 + U_v (x_1 \sigma_1^2 + x_2 \sigma_{12})} ,
\]

From the first-order conditions (52), (53), the right side of (63) equals \( -\frac{P_2^0}{P_1^0} \), so that \( \frac{dU}{d\mu_1} \bigg|_{d\mu_1=0} = 0 \) can be written as

\[
P_1^0 dx_1 + P_2^0 dx_2 = 0 ,
\]

or

\[
P_1^0 \frac{\partial x_1}{\partial P_1^0} + P_2^0 \frac{\partial x_2}{\partial P_1^0} = 0 ;
\]

(64) implies that the rate of change of wealth with respect to \( P_1^0 \) is zero. Equation (60) thus reduces to

\[
\frac{\partial x_1}{\partial P_1^0} \bigg|_{u=U_0} = \lambda \frac{\partial U}{\partial P_1^0} \bigg|_{d\mu_1=0} ,
\]

(65)
which is the expression for the substitution effect (neglecting productivity changes) of a change in $P^0_1$. The sign of

$$\frac{\partial X_1}{\partial P^0_1} \bigg|_{\mu_0, \delta_1=0}$$

is negative since $|D| > 0$, and $D_1 < 0$.

Next we show that the first term in (60) represents the wealth effect of a change in $P^0_1$.

Differentiate equations (52), (53), and (54), with respect to $\omega$ holding $P^0_1, P^0_2, \mu_1^2, \sigma_1^2, \sigma_2^2, \sigma_{12}, P^0_1$ constant, to get

$$U_{EE} \mu_1 \frac{\partial E}{\partial \omega} + U_{EV} \mu_1 \frac{\partial V}{\partial \omega} + 2 U_{VV} \left( X_1 \sigma_1^2 + X_2 \sigma_{12} \right) \frac{\partial V}{\partial \omega}$$

$$+ 2 U_{VE} \left( X_1 \sigma_1^2 + X_2 \sigma_{12} \right) \frac{\partial E}{\partial \omega} + 2 U_{VV} \left( \sigma_1 \frac{\partial X_1}{\partial \omega} + \sigma_{12} \frac{\partial X_2}{\partial \omega} \right) = P^0_1 \frac{\partial \lambda}{\partial \omega}, \quad (66)$$

$$U_{EE} \mu_2 \frac{\partial E}{\partial \omega} + U_{EV} \mu_2 \frac{\partial V}{\partial \omega} + 2 U_{VV} \left( X_2 \sigma_2^2 + X_1 \sigma_{12} \right) \frac{\partial V}{\partial \omega}$$

$$+ 2 U_{VE} \left( X_2 \sigma_2^2 + X_1 \sigma_{12} \right) \frac{\partial E}{\partial \omega} + 2 U_{VV} \left( \sigma_2 \frac{\partial X_2}{\partial \omega} + \sigma_{12} \frac{\partial X_1}{\partial \omega} \right) = P^0_2 \frac{\partial \lambda}{\partial \omega}, \quad (67)$$

$$P^0_1 \frac{\partial X_1}{\partial \omega} + P^0_2 \frac{\partial X_2}{\partial \omega} = 1, \quad (68)$$
Also,

\[
\frac{\partial E}{\partial w_0} = \mu_1 \frac{\partial x_1}{\partial w_0} + \mu_2 \frac{\partial x_2}{\partial w_0},
\]

\[
\frac{\partial v}{\partial w_0} = 2\sigma_1 x_1 \frac{\partial x_1}{\partial w_0} + 2\sigma_2 x_2 \frac{\partial x_2}{\partial w_0} + 2x_1 \sigma_{12} \frac{\partial x_1}{\partial w_0} + 2x_2 \sigma_{12} \frac{\partial x_2}{\partial w_0};
\]

Substituting the latter two equations into (66), (67), (68), and writing the system in matrix form, we get

\[
\begin{bmatrix}
Z_{11} & Z_{12} & P_1^o \\
Z_{21} & Z_{22} & P_2^o \\
P_1^o & P_2^o & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x_1}{\partial w_0} \\
\frac{\partial x_2}{\partial w_0} \\
\frac{\partial x_1}{\partial w_0}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},
\]

(69)

We solve for \(\frac{\partial x_1}{\partial w_0}\) using Cramer's rule,

\[
\frac{\partial x_1}{\partial w_0} = \frac{Z_{12}P_2^o - P_1^o Z_{22}}{|D|} = \frac{D_{31}}{|D|};
\]

(70)
is the coefficient of \(-X_1\) in equation (60). The sign of \(\frac{D_{31}}{D}\) is ambiguous. Note that the change in real wealth caused by a small change in \(P_1^0\) is \(-X_1dP_1^0\), so that the rate of change in real wealth with respect to \(P_1^0\) is \(-X_1\). Thus the wealth effect of a change in \(P_1^0\) is \(-X_1\frac{D_{31}}{D}\), which is the first term of equation (60).

(ii) Allowing for Productivity Changes

We have analyzed the effect of a price change on \(X_1\) without allowing for changes in the productivity of \(X_1\). Next we complete the analysis by allowing for productivity changes.

Suppose now \(d\mu_1 \neq 0\). (In the context of our earlier assumption that \(d\mu_1 = dP_1^e - dP_1^0 = 0\), we now have \(P_1^e\) revert back to its old level while \(P_1^0\) stays constant at its new level; we thus isolate the change in \(\mu_1\) caused by the initial change in \(P_1^0\)). Differentiate (52), (53), (54), with respect to \(\mu_1\), holding constant \(P_1^0, \sigma_1^2, \sigma_2^2, \mu_2, \sigma_1, \sigma_2, P_2^0, U_0\).

\[
U_{EE} \mu_1 \frac{\partial E}{\partial \mu_1} + U_{EV} \mu_1 \frac{\partial V}{\partial \mu_1} + U_E + 2 U_{VE} (\sigma_1 \sigma_2 + \sigma_1 \sigma_{12}) \frac{\partial E}{\partial \mu_1} \\
+ 2 U_{VV} (\sigma_1 \sigma_2 + \sigma_2 \sigma_{12}) \frac{\partial V}{\partial \mu_1} + 2 U_V (\sigma_1 \frac{\partial X_1}{\partial \mu_1} + \sigma_2 \frac{\partial X_2}{\partial \mu_1})
\]

\[
= \frac{\partial \lambda}{\partial \mu_1}
\]
\[
U_{EE} \mu_2 \frac{\partial E}{\partial \mu_1} + U_{EV} \mu_2 \frac{\partial V}{\partial \mu_1} + 2 U_{VE} (X_2 \sigma_2^2 + X_1 \sigma_{12}) \frac{\partial E}{\partial \mu_1} \\
+ 2 U_{VV} (X_2 \sigma_2^2 + X_1 \sigma_{12}) + 2 U_{V} (\sigma_2^2 \frac{\partial X_2}{\partial \mu_1} + \sigma_{12} \frac{\partial X_1}{\partial \mu_1}) = P_2 \frac{\partial \lambda}{\partial \mu_1},
\]

\[
P_1 \frac{\partial X_1}{\partial \mu_1} + P_2 \frac{\partial X_2}{\partial \mu_1} = 0
\]

Also,

\[
\frac{\partial E}{\partial \mu_1} = \mu_1 \frac{\partial X_1}{\partial \mu_1} + X_1 + \mu_2 \frac{\partial X_2}{\partial \mu_1},
\]

\[
\frac{\partial V}{\partial \mu_1} = 2 \sigma_1^2 X_1 \frac{\partial X_1}{\partial \mu_1} + 2 \sigma_2^2 X_2 \frac{\partial X_2}{\partial \mu_1} + 2 X_1 \sigma_{12} \frac{\partial X_2}{\partial \mu_1}
\]

Substituting these into (71), (72), (73), we get

\[
\begin{bmatrix}
P_{11} & P_{12} & P_1 \\
P_{21} & P_{22} & P_2 \\
P_1 & P_2 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial X_1}{\partial \mu_1} \\
\frac{\partial X_2}{\partial \mu_1}
\end{bmatrix}
= \begin{bmatrix}
R_1 \\
R_2
\end{bmatrix}
\]

(74)
Using Cramer's rule we get

$$\frac{\partial x_1}{\partial \mu_1} = \frac{D_{11} R_1 + D_{12} R_2}{1D1}$$

(75)

which is the negative of the last term in (59). This is because \( \frac{\partial \mu_1}{\partial \mu_0} = -1 \). Expression (75) corresponds to the "Want Pattern" or "Veblen" effects of A & M and B & G. Its sign is ambiguous.

Expression (75) shows the change in \( x_1 \) resulting from a change in the productivity of \( x_1 \) in yielding \( E \), i.e. \( \frac{\partial E}{\partial x_1} \). It does not involve changes in the investor's preference function. It involves shifts in the E-S efficiency locus, and can be broken down into "productivity" wealth and "productivity" substitution effects in E-S space, as shown in Figure 16 and explained below. I call (75) the Productivity Effect.

An increase in \( \mu_1 \) (the yield on the low-risk asset) shifts the efficiency locus AB to A'B in Figure 16. The optimum E,S combination moves from N to N'. This movement can be broken down into substitution and wealth components. By decreasing the investor's initial wealth we shift A'B parallel down along OA'-OB until it is just tangent to the original utility surface \( U_0 \) at N''. The movement from N to N'' along \( U^0 \) is the wealth-compensated productivity-substitution effect of a change in \( \mu_1 \), and that from N'' to N' is the productivity-wealth effect of a change in \( \mu_1 \).
In terms of Figure 10 in Chapter 5, changes in $\mu_1$ result in shifts in the utility surfaces in asset space. However, one should keep in mind that these changes in preference orderings over assets are due to changes in the relative productivity of assets, not the investor's tastes.

The sign of the wealth effect in E-S space depends upon the investor's utility-of-wealth function, i.e. whether the investor has increasing, decreasing, or constant, absolute risk-aversion. The convexity of the utility function guarantees that the sign of the substitution effect in terms of $E$ and $S$ is unambiguous. However, it is ambiguous in asset space because, in addition to substitution among characteristics, substitution also occurs among assets because of changes in their relative productivities.

More formally, one can identify the wealth and substitution effects of a change in $\mu_1$, holding $\mu_2$, $\sigma_2^2$, $\sigma_1^2$, $\sigma_{12}$, $P_1^0$, $P_2^0$, $W_0$, constant. Note that the same analysis applies regardless of whether a change in $\mu_1$ is caused by a change in $P_1^0$ or $P_1^e$.

For the substitution effect, set $\frac{\partial u}{\partial \mu_1} = 0$, or

$$\frac{\partial u}{\partial \mu_1} = u_E \left( \frac{\partial E}{\partial \mu_1} \right) + u_V \frac{\partial V}{\partial \mu_1} = 0 , \quad (76)$$

$$u_E \left( \mu_1 \frac{\partial x_1}{\partial \mu_1} + x_1 + \mu_2 \frac{\partial x_2}{\partial \mu_1} \right) + u_V \left( \frac{\partial V}{\partial x_1} \frac{\partial x_1}{\partial \mu_1} + \frac{\partial V}{\partial x_2} \frac{\partial x_2}{\partial \mu_1} \right) = 0 ,$$
or
\[
\frac{\partial x_1}{\partial \mu_1} \left( u_{E,\mu_1} + u_V \frac{\partial V}{\partial x_1} \right) + \frac{\partial x_2}{\partial \mu_1} \left( u_{E,\mu_2} + u_V \frac{\partial V}{\partial x_2} \right) + u_E x_1 = 0 \tag{77}
\]

Substituting \( \lambda p_1^0 \) and \( \lambda p_2^0 \) from (52), (53), into (77), we get
\[
\frac{\partial x_1}{\partial \mu_1} (\lambda p_1^0) + \frac{\partial x_2}{\partial \mu_1} (\lambda p_2^0) + u_E x_1 = 0 \tag{78}
\]

or
\[
\frac{\partial x_1}{\partial \mu_1} p_1^0 + \frac{\partial x_2}{\partial \mu_1} p_2^0 = -\frac{u_E x_1}{\lambda} \tag{78}
\]

In view of equation (54), (78) implies that the change in wealth required to keep \( dU = 0 \) is equal to \( -\frac{u_E}{\lambda} x_1 \).

Hence the wealth effect associated with a change in \( \mu_1 \) is \( \frac{u_E}{\lambda} x_1 \) \( \left( \frac{D_{31}}{D} \right) \). This is an implicit wealth effect arising because of the productivity increase in \( x_1 \). It depends on the marginal utility of expected wealth, \( u_E \). Note that
\[
\frac{D_{31}}{D} = \frac{\partial x_1}{\partial w_0}.
\]

The wealth-compensated substitution effect of a change in \( \mu_1 \) is
\[
\frac{\partial X_1}{\partial \mu} \bigg|_{u_0} = \frac{D_{11} R_1 + D_{21} R_2}{|D|} \left( -X_1 \frac{U_E}{\lambda} \left( \frac{D_{31}}{|D|} \right) \right), \quad (79)
\]

which can be rewritten as (80) after writing \( R_1, R_2 \) in their explicit forms and rearranging,

\[
\frac{\partial X_1}{\partial \mu} \bigg|_{u_0} = -\frac{U_E}{\lambda} \left[ \frac{\lambda D_{11}}{|D|} + X_1 \frac{D_{31}}{|D|} \right] - U_{EE} X_1 \left[ \frac{D_{11}}{|D|} \mu_1 + \frac{D_{21}}{|D|} \mu_2 \right] - 2U_{EE} X_1 \left[ \frac{D_{11}}{|D|} (X_1 \sigma_1^2 + X_2 \sigma_{12}) + \frac{D_{21}}{|D|} (X_2 \sigma_2^2 + X_1 \sigma_{12}) \right] \right); \quad (80)
\]

The sign of the above expression is ambiguous. We have shown that the Productivity Effect (or A \& M's "Want Pattern" Effect) can be separated into productivity-wealth and productivity-substitution effects.

We now put together both effects associated with a change in \( P_1^0 \), to get for the total substitution effect

\[
\frac{\partial X_1}{\partial P_1^0} \bigg|_{u_0} = \frac{\lambda D_{11}}{|D|} + \frac{U_E}{\lambda} \left[ \frac{\lambda D_{11}}{|D|} + X_1 \frac{D_{31}}{|D|} \right] + U_{EE} X_1 \left[ \frac{D_{11}}{|D|} \mu_1 + \frac{D_{21}}{|D|} \mu_2 \right] + U_{EE} X_1 \left[ \frac{D_{11}}{|D|} (X_1 \sigma_1^2 + X_2 \sigma_{12}) + \frac{D_{21}}{|D|} (X_2 \sigma_2^2 + X_1 \sigma_{12}) \right] \right); \quad (81)
\]
rather than A & M's and B & G's expressions which contain only the first term on the right.

For the total wealth effect of a change in $P_1$, we have

$$- X_1 \frac{D^2_1}{d}\left( \frac{U_E}{\lambda} + 1 \right)$$

(82)

Both the substitution-effect and the wealth-effect terms in (81) and (82) respectively have ambiguous signs.

A more direct way of deriving the results in (80) or (79) in the case of a change in $P_1$ would be to differentiate (52), (53), (54), subject to the side condition that $dU = 0$. Similarly, we derive (81) in the case of a change in $P_1$ by differentiating (52), (53), (54), with respect to $P_1$, subject to $dU = 0$.

III The Effect of a Change in $\sigma_1^2$

We examine the effect of a change in $\sigma_1^2$ on $X_1$ holding constant $P_0^0$, $P_1^0$, $\mu_2$, $\sigma_2^2$, $\sigma_{12}$, $\nu_0$. Differentiate (52), (53), (54) with respect to $\sigma_1^2$,

$$\mu_1 U_{EE} \frac{dE}{d\sigma_1^2} + \mu_1 U_{EV} \frac{dV}{d\sigma_1^2} + 2(\chi_1 \sigma_1^2 + \chi_2 \sigma_{12}) \left( U_{VE} \frac{dE}{d\sigma_1^2} + U_{VV} \frac{dV}{d\sigma_1^2} \right)$$

$$+ 2U_{V} \left( \chi_1 + \sigma_1^2 \frac{d\chi_1}{d\sigma_1^2} + \sigma_{12} \frac{d\chi_2}{d\sigma_1^2} \right) = P_1^0 \frac{d\lambda}{d\sigma_1^2} ,$$

(83)
\[ \mu_2 U_{EE} \frac{\partial E}{\partial \sigma_1^2} + \mu_2 U_{EV} \frac{\partial V}{\partial \sigma_1^2} + 2 (x_2 \sigma_1^2 + x_1 \sigma_2) (U_{EE} \frac{\partial E}{\partial \sigma_1^2} + U_{V} \frac{\partial V}{\partial \sigma_1^2}) \]
\[ + 2 U_V (\sigma_2^2 \frac{\partial x_2}{\partial \sigma_1^2} + \sigma_2 \frac{\partial x_1}{\partial \sigma_1^2}) = p_1 \frac{\partial \lambda}{\partial \sigma_1^2} \quad \text{(84)} \]

\[ p_1 \frac{\partial x_1}{\partial \sigma_1^2} + p_2 \frac{\partial x_2}{\partial \sigma_1^2} = 0 \quad \text{(85)} \]

Also,
\[ \frac{\partial E}{\partial \sigma_1^2} = \mu_1 \frac{\partial x_1}{\partial \sigma_1^2} + \mu_2 \frac{\partial x_2}{\partial \sigma_1^2} \quad \text{(86)} \]

\[ \frac{\partial V}{\partial \sigma_1^2} = x_1^2 + 2 \sigma_2 \sigma_1 \frac{\partial x_1}{\partial \sigma_1^2} + 2 \sigma_2^2 \sigma_1 \frac{\partial x_2}{\partial \sigma_1^2} + 2 \sigma_1 \sigma_2 \frac{\partial x_2}{\partial \sigma_1^2} \]
\[ + 2 \sigma_2 \sigma_1 \frac{\partial x_1}{\partial \sigma_1^2} \quad \text{(87)} \]

Substituting these into (83), (84), (85), and writing the system in matrix form, we get

\[
\begin{bmatrix}
Z_{11} & Z_{12} & p_1^0 \\
Z_{21} & Z_{22} & p_2^0 \\
p_1^0 & p_2^0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x_1}{\partial \sigma_1^2} \\
\frac{\partial x_2}{\partial \sigma_1^2} \\
\frac{\partial \lambda}{\partial \sigma_1^2}
\end{bmatrix}
= 
\begin{bmatrix}
L_1 \\
L_2 \\
0
\end{bmatrix}
\quad \text{(86)}
\]
where
\[ L_1 = - \left[ x_1^e u_{x_1 v} + 2 x_1^v u_{x_1 v} (x_1^2 + x_2^2) + 2 u_v x_1 \right], \]
\[ L_2 = - \left[ x_2^e u_{x_1 v} + 2 x_2^v u_{x_1 v} (x_2^2 + x_1^2) \right]. \]

\( L_1 \) and \( L_2 \) denote respectively shifts in the marginal utility functions of \( x_1 \) and \( x_2 \) resulting from the change in the productivity of \( x_1 \) in producing \( v \), namely \( \sigma_1^2 \).

Using Cramer's rule we solve for \( \frac{\partial x_1}{\partial \sigma_1^2} \):
\[
\frac{\partial x_1}{\partial \sigma_1^2} = \frac{L_1 D_{11} + L_2 D_{21}}{|D|}, \tag{87}
\]

Expression (87) shows a Productivity Effect. It shows the change in \( x_1 \) resulting from a change in the marginal productivity of \( x_1 \) in producing variance, i.e. \( \frac{\partial v}{\partial x_1} \). The sign of expression (87) is ambiguous. We can again identify productivity—wealth and productivity—substitution effects.

For the substitution effect set \( \frac{\partial u}{\partial \sigma_1^2} = 0, \)
\[
\frac{\partial u}{\partial \sigma_1^2} = u_e \frac{\partial E}{\partial \sigma_1^2} + u_v \frac{\partial v}{\partial \sigma_1^2} = 0.
\]
or

$$U_e \left( \mu_1 \frac{\partial \pi_1}{\partial \sigma_1^2} + \mu_2 \frac{\partial \pi_2}{\partial \sigma_2^2} \right) + U_v \left( 2 \sigma_1 \sigma_1^2 \frac{\partial X}{\partial \sigma_1^2} + 2 \sigma_2 \sigma_2^2 \frac{\partial X}{\partial \sigma_2^2} + 2 \sigma_1 \sigma_2 \frac{\partial X}{\partial \sigma_1^2} \right) + U_v X_1^2 = 0 \quad , \quad (88)$$

which can be rewritten as

$$\left( \frac{\partial X_1}{\partial \sigma_1^2} \right) \left( U_e \mu_1 + U_v \frac{\partial \nu}{\partial X_1} \right) + \left( \frac{\partial X_2}{\partial \sigma_2^2} \right) \left( U_e \mu_2 + U_v \frac{\partial \nu}{\partial X_2} \right) + U_v X_1^2 = 0 \quad , \quad (89)$$

substituting $\lambda P_1^0$ and $\lambda P_2^0$ from (52), (53) into (89), we get

$$\frac{\partial X_1}{\partial \sigma_1^2} \left( \lambda P_1^0 \right) + \frac{\partial X_2}{\partial \sigma_2^2} \left( \lambda P_2^0 \right) + U_v X_1^2 = 0 \quad ,$$

or

$$\frac{\partial X_1}{\partial \sigma_1^2} P_1^0 + \frac{\partial X_2}{\partial \sigma_2^2} P_2^0 = - \frac{U_v X_1^2}{\lambda} \quad ; \quad (90)$$

The term on the right side of (90) is the required change in wealth to keep $dU = 0$. Hence the wealth effect associated with the productivity change is $\left( \frac{U_v X_1^2}{\lambda} \right) \left( \frac{D_3^1}{D_1} \right)$.
The wealth-compensated substitution effect of the change in $\sigma_1^2$ is

$$\frac{\partial x_1}{\partial \sigma_1^2} \bigg|_{U_0} = \frac{L_1 D_{11} + L_2 D_{21}}{D}$$

(91)

Writing $L_1$ and $L_2$ in their explicit forms we derive

$$\frac{\partial x_1}{\partial \sigma_1^2} \bigg|_{U_0} = -X_1 \frac{U_y}{\lambda} \left( 2 \lambda \frac{D_{11}}{D} + X_1 \frac{D_{31}}{D} \right) - X_1^2 \left( \frac{\partial U_y}{\partial \sigma_1^2} \left( \frac{D_{11}}{D} + \mu \frac{D_{31}}{D} \right) ight)$$

(92)

$$- 2 X_1^2 U_{yy} \left[ (X_1 \sigma_1^2 + X_2 \sigma_2^2) \frac{D_{11}}{D} + (X_2 \sigma_2^2 + X_1 \sigma_1^2) \frac{D_{31}}{D} \right]$$

Expressions (91) or (92) can be derived more directly by differentiating (52), (53), (54), with respect to $\sigma_1^2$ subject to $dU = 0$.

Geometrically, we can show the "Productivity" substitution and wealth effects in terms of Figure 17.

An increase in $\sigma_1^2$ (e.g., the variance of the low-risk asset in the Figure) shifts AB to A'B'. The optimum E-S combination moves from N to N'. This movement can be broken into substitution and wealth effect components. The movement from N to N'' along $U_0$ is the wealth-compensated productivity-substitution effect, and that from N'' to N' is the productivity-wealth effect.
Figure 17
IV. Generalizing to an Indefinite Number of Assets

We analyze the effect of a change in the expected return of an asset, in the case of an indefinite number of assets, holding other parameters constant. For a generalization of the effect of a change in initial price of an asset see Appendix II.

For a change in $\mu_r$, the expected return on asset $r$, we differentiate first-order conditions (50) and (51) with respect to $\mu_r$, to get

$$
\mu_i \left( \mu_E \frac{\partial E}{\partial \mu_r} + \mu_V \frac{\partial V}{\partial \mu_r} \right) + \delta_{ir} \mu_E^2 + 2 \sum_j \xi_j \sigma_{ij} \left( \mu_{VE} \frac{\partial \mu_E}{\partial \mu_r} + \mu_{VV} \frac{\partial \mu_V}{\partial \mu_r} \right)
$$

$$
+ 2 \mu_V \sum_j \sigma_{ij} \frac{\partial x_i}{\partial \mu_r} = P_i^0 \frac{\partial \lambda}{\partial \mu_r} \quad ;
$$

$$
\delta_{ir} = \begin{cases} 
1 & \text{if } i = r, \\
0 & \text{if } i \neq r \end{cases}
$$

$$
\sum_i P_i^0 \frac{\partial x_i}{\partial \mu_r} = 0 \quad ;
$$

Note that

$$
\frac{\partial E}{\partial \mu_r} = \sum_i \mu_i \frac{\partial x_i}{\partial \mu_r} + x_r
$$

$$
\frac{\partial V}{\partial \mu_r} = \sum_i \sum_j \sigma_{ij} \left( x_j \frac{\partial x_i}{\partial \mu_r} + x_i \frac{\partial x_i}{\partial \mu_r} \right) ;
$$

Substituting these into (93) and (94), and writing the equation system in matrix form, we have
where \( R_i = -(U_{E, E} S_{ir} + U_{E, E} X_r \mu_i + 2 U_{E, E} X_r \sum_j X_j c_{ij}) \)

\[
\begin{align*}
S_{ir} &= 1 \quad \text{for } i = r, \\
       &= 0 \quad \text{for } i \neq r.
\end{align*}
\]

\( R_i \) denotes the shift in the marginal utility function of \( X_i \) resulting from the change in the marginal productivity of \( X_r \) in producing \( E \), namely \( \mu_r \).

Using Cramer's rule we solve for \( \frac{\partial X_k}{\partial \mu_r} \),

\[
\frac{\partial X_k}{\partial \mu_r} = \sum_i R_i \frac{D_{ik}}{D} \]  

(96)

where \( D_{ik} \) is the cofactor of the \((i, k)\)th element of \( D \). Note that \( \frac{D_{ik}}{D} \) is the traditional wealth-compensated substitution term between \( X_i \) and \( X_k \). In the terminology of the previous section, it is the wealth-compensated substitution effect neglecting productivity changes. In the traditional Hicks- Slutsky sense of consumer theory, \( X_i \) and \( X_k \) are substitutes if \( \frac{D_{ik}}{D} > 0 \), and complements if \( \frac{D_{ik}}{D} \leq 0 \). All
other traditional results of consumer theory hold identically when we neglect productivity changes. We define assets as substitutes and complements in the exact manner of traditional consumer theory.

Writing (96) more explicitly,

\[
\frac{\partial X_k}{\partial \mu_r} = -U_E \frac{D_{rk}}{PD} - X_r \left[ U_{EE} \sum \mu_i \frac{D_{ik}}{PD} + 2 U_{VE} \sum \sum \mu_i \sigma_j \frac{D_{ij}}{PD} \right],
\]

(97)

The second term on the right-hand-side of (97) equals \(-X_r \frac{\partial X_k}{\partial T_E}\), where \(T_E\) is a lump-sum tax on expected wealth; this follows from our results in Chapter 7. We can interpret the terms in equation (97) in the following manner: If we keep all \(X_i\)'s unchanged, a change in \(\mu_r\) produces a change in \(E\) of \(X_r d \mu_r\) without changing \(V\). Lump-sum taxing away this change in \(E\) while keeping \(\mu_r\) at its new level, enables the investor to be at the original endowment of \(E\) and \(V\). The effect of this lump-sum tax on \(X_k\) is \(-X_r \frac{d X_k}{d T_E}\), or the negative of the second term in (97). The first term of (97) represents the effect on \(X_k\) of the change in \(\mu_r\) subsequent to the imposition of the lump-sum tax of \(X_r d \mu_r\) on portfolio expected wealth. The graph below makes the nature of these movements clearer.

Graphically, as shown by Figure 18, a change in \(\mu_r\) moves the individual from equilibrium point A on the
original efficiency locus LL to point B on the final efficiency locus LL'. This movement from A to B can be broken down into two components corresponding to the terms in (97). The lump-sum tax effect corresponds to the movement from point B to point C, as the efficiency locus LL' shifts parallel down by the amount of the tax \( X_r d\mu_r \) to MM. While this shift to MM enables the investor to be at original optimum point A, the investor does not stay at A but moves to point C which yields a higher level of utility. The movement between C and A, caused by the change in the efficiency locus from MM to LL due to the change in \( \mu_r \), (having already allowed for the lump-sum-tax effect), is given by the first term in (97).

We can also identify "productivity" substitution and wealth effects in equation (97) in the manner of Section II.

For the productivity wealth-effect we have,

\[
X_r \frac{D_{n+1,k}}{|D|} \left( \frac{U_E}{\lambda} \right).
\]

For the productivity substitution effect we have

\[
\frac{\partial X_k}{\partial \mu_r} \bigg|_{U_0} = - \frac{U_E}{\lambda} \left[ \frac{\lambda}{|D|} + X_r \frac{D_{n+1,k}}{|D|} \right] - U_{EE} X_r \sum_i \mu_i \frac{D_{ik}}{|D|} \
- U_{VE} X_r \sum_i \frac{D_{ik}}{|D|} \left( \sum_j X_{ij} \sigma_{ij} \right),
\]

(98)

The sign of either expression is again ambiguous.

To analyze the cross-effect relationship between

\[
\frac{\partial X_k}{\partial \mu_r} \quad \text{and} \quad \frac{\partial X_r}{\partial \mu_k}
\]

we now derive \( \frac{\partial X_r}{\partial \mu_k} \). By symmetry to the
previous results (see (96))
\[ \frac{\partial X_r}{\partial \mu_k} = \sum_i M_i Di_r , \]  \hspace{1cm} (99)

where
\[ M_i = - \left( U_{ESi} \mu_i + U_{EE} X_k \mu_i + 2 U_{VE} X_k \sum_j X_j \sigma_{ij} \right) \]
with
\[ S_{ik} = 1 \quad \text{for } i = k , \]
\[ = 0 \quad \text{for } i \neq k ; \]

\( M_i \) denotes the shift in the marginal utility function of \( X_i \) resulting from the change in the marginal productivity of \( X_k \) in producing \( E \), namely \( \mu_k \). Writing (99) more explicitly,
\[ \frac{\partial X_r}{\partial \mu_k} = - U_{E} \frac{D_{kr}}{|D|} - U_{EE} X_k \sum_i \mu_i \frac{D_{kr}}{|D|} \]
\[ - 2 U_{VE} X_r \sum_j \sum_i (X_j \sigma_{ij}) \frac{D_{ir}}{|D|} ; \]  \hspace{1cm} (100)

Comparing equations (97) and (100) we see that in general
\[ \frac{\partial X_r}{\partial \mu_k} \neq \frac{\partial X_k}{\partial \mu_r} . \]

V. The Effect of a Change in Variance-Covariance

We analyze the effect of a change in \( \sigma_{rf} \), the covariance of returns of assets \( X_r \) and \( X_f \) holding other parameters constant; we differentiate first-order conditions (50) and (51) with respect to \( \sigma_{rf} \), to get
\[ \mu_i \left( \frac{\partial E}{\partial \sigma_{rf}} + U_{Ev} \frac{\partial V}{\partial \sigma_{rf}} \right) + \sum_j \sum_d \sigma_{ij} \frac{\partial X_j}{\partial \sigma_{rf}} \left( \frac{\partial V}{\partial \sigma_{rf}} + U_{Ev} \frac{\partial E}{\partial \sigma_{rf}} \right) \]
\[ + 2 U_v \sum \sigma_{ij} \frac{\partial X_j}{\partial \sigma_{rf}} + 2 U_v \left( X_f \delta_{ir} + X_r \delta_{if} \right) = P_i \frac{\partial \lambda}{\partial \sigma_{rf}} \] (101)

\[ \sum_i P_i \frac{\partial X_i}{\partial \mu_r} = 0 \] (102)

Note that,
\[ \frac{\partial E}{\partial \sigma_{rf}} = \sum_i \mu_i \frac{\partial X_i}{\partial \sigma_{rf}} \]
\[ \frac{\partial V}{\partial \sigma_{rf}} = 2 \left( X_f X_i \delta_{ir} + X_r X_i \delta_{if} \right) + \sum_i \sum_j \sigma_{ij} \left( X_i \frac{\partial X_i}{\partial \sigma_{rf}} + X_j \frac{\partial X_j}{\partial \sigma_{rf}} \right) \]

Note also that \( \delta_{ir} = 1 \) for \( i = r \),
\( = 0 \) for \( i \neq r \),
\( \delta_{if} = 1 \) for \( i = f \),
\( = 0 \) for \( i \neq f \).

Substituting the expressions for \( \frac{\partial E}{\partial \sigma_{rf}} \) and \( \frac{\partial V}{\partial \sigma_{rf}} \) into (101) and (102) and writing the system in matrix form.
\[
\begin{bmatrix}
Z_{ii} & \ldots & Z_{in} & P_{i}^0 \\
\vdots & \ddots & \vdots & \vdots \\
Z_{ni} & \ldots & Z_{nn} & P_{n}^0 \\
\end{bmatrix}
\begin{bmatrix}
\frac{\partial X_i}{\partial \sigma_{rf}} \\
\vdots \\
\frac{\partial X_n}{\partial \sigma_{rf}} \\
\end{bmatrix}
= \begin{bmatrix}
S_i \\
\vdots \\
S_n \\
\end{bmatrix};
\]  

(103)

\(S_i\) denotes the shift in the marginal utility of asset \(i\) resulting from the change in \(\sigma_{rf}\), i.e., the change in the productivities of assets \(r\) and \(f\) in producing variance.

Solving (103) for \(\frac{\partial X_k}{\sigma_{rf}}\),

\[
\frac{\partial X_k}{\partial \sigma_{rf}} = \sum_i S_i \frac{D_{ik}}{|D|};
\]

(104)

Writing (104) more explicitly,

\[
\frac{\partial X_k}{\partial \sigma_{rf}} = -2U_v (X_f \frac{D_{vk}}{|D|} + X_r \frac{D_{vk}}{|D|}) - 2X_rX_f \left[U_e v \sum_i N_i \frac{D_{ik}}{|D|}
\right.
\]

\[
+ 2U_{vv} \sum_j \sum_i (X_i \sigma_{ij}) \frac{D_{ik}}{|D|}\];
\]

(105)
In the case where \( r = f \), i.e., involving a change in the variance of asset \( r \), the above becomes

\[
\frac{\partial X_k}{\partial \sigma_r^2} = -4U \left( X_r \frac{D_{rk}}{D_r} \right) - X_r^2 \left[ 2U_{Ev} \sum \mu_i \frac{D_{ik}}{D_r} \right. \\
+ 2U_{Vv} \sum \left( \frac{X_j \sigma_{ij}}{D_r} \right) \frac{D_{ik}}{D_r} \left. \right]
\]

(106)

The signs of expressions (105) and (106) are ambiguous in general.

The second term on the right-hand-side of (105) equals

\[-X_rX_f \frac{\partial X_K}{\partial T_V} \]

where \( T_V \) is a lump-sum tax on total portfolio variance or risk; this follows from our results in Chapter 7. Similarly, the second term in (106) is equal to

\[-X_r^2 \frac{\partial X_K}{\partial T_V} \]

We can interpret the terms in (105) in the following manner: If we keep all \( X_i \)'s unchanged, a change in \( \sigma_{rf} \) produces a change in \( V \) of \( 2X_rX_f \sigma_{rf} \) without changing \( E \). Lump-sum-taxing away this change in \( V \) while keeping \( \sigma_{rf} \) at its new level, enables the investor to be at the original endowment of \( E \) and \( V \). The effect of this lump-sum tax on \( X_k \) is \( X_rX_f \frac{\partial X_k}{\partial T_V} \), or the negative of the second term in (105). The first term in (105) represents the effect on \( X_k \) of the change in \( \sigma_{rf} \) subsequent to the imposition
of the lump-sum tax on portfolio variance. The graph below makes the nature of these movements more clear. Note that an analogous interpretation to the above applies in the case of a change in \( \sigma_r^2 \).

Graphically, in Figure 19, suppose a change in \( \sigma_{rf} \) moves the individual from equilibrium point A on the original efficiency locus LAL to point B on the final efficiency locus LB. This movement from A to B can be broken down into two components corresponding to the terms in (105). The lump-sum tax effect corresponds to the movement from B to C, as the efficiency locus shifts parallel leftward by the amount of the lump-sum-tax to MM. Notice that while MM enables the investor to be at the original optimum point A, the investor does not stay at A but moves to C which yields a higher level of utility. The movement between C and A, caused by the change in the efficiency locus from MM to LAL due to the change in \( \sigma_{rf} \) (having adjusted for the lump-sum tax effect), is given by the first term in (105).

We can also identify productivity-substitution and wealth effects in the foregoing. Thus in the case of a change in variance, \( \sigma_r^2 \), we have, generalizing from equation (92) of section III, for the productivity-wealth effect,

\[ -X_r^2 \left( \frac{U_v}{\lambda} \frac{D_{n+1,k}}{|D|} \right) j \]  

(107)
and for the productivity-substitution effect,

\[ \frac{\partial X_k}{\partial \sigma_{k}^2} \bigg|_{\mu_0} = -X_r \frac{U_y}{\lambda} \left( 2 \lambda \frac{D_{ik}}{|D|} + X_r \frac{D_{n+1,k}}{|D|} \right) 
- 2X_r \left[ U_{EV} \sum_i N_i \frac{D_{ik}}{|D|} + U_{VV} \sum_j (X_j \sigma_{ij}) \frac{D_{ik}}{|D|} \right] ; \quad (108) \]

The sign of either expression is ambiguous.

To analyze cross-effect relationships, we now derive

\[ \frac{\partial X_r}{\partial \sigma_{k}^2} \]. By symmetry to previous results

\[ \frac{\partial X_r}{\partial \sigma_{k}^2} = \sum_i N_i \frac{D_{ik}}{|D|} ; \quad (109) \]

where

\[ N_i = -2U_v (X_k \delta_{ik} + X_f \delta_{if}) - 2U_{EV} \mu_i X_k X_f 
- 2U_{VV} X_r X_f \sum_j X_j \sigma_{ij} \quad j \]

where \( \delta_{ik} = 1 \) for \( i = k \),
\( = 0 \) for \( i \neq k \);
and \( \delta_{if} = 1 \) for \( i = f \),
\( = 0 \) for \( i \neq f \).

\( N_i \) denotes the shift in the marginal utility of asset \( i \) resulting from the change in \( \sigma_{k}^2 \), i.e., the change in the productivities of assets \( k \) and \( f \) in producing variance.
Writing (109) more explicitly, we have

\[
\frac{\partial x_r}{\partial \sigma_{kf}} = -2uv \left( x_f \frac{D_{rk}}{\mid D \mid} + x_k \frac{D_{fr}}{\mid D \mid} \right) - 2x_kx_f \left( u_{ev} \sum_{i} \mu_i \frac{D_{ir}}{\mid D \mid} \right) \\
+ 2u_{vv} \sum_{i} \sum_{j} (x_j \sigma_{ij}) \frac{D_{ir}}{\mid D \mid} \right) ; \tag{110}
\]

In the case where \( k = f \),

\[
\frac{\partial x_r}{\partial \sigma_k^2} = -4uv \left( x_k \frac{D_{rk}}{\mid D \mid} \right) - 2x_k^2 \left[ u_{ev} \sum_{i} \mu_i \frac{D_{ir}}{\mid D \mid} \right] \\
+ u_{vv} \sum_{i} \sum_{j} (x_j \sigma_{ij}) \frac{D_{ir}}{\mid D \mid} \right] \] ; \tag{111}

Comparing these expressions with those of \( \frac{\partial x_k}{\partial \sigma_{rf}} \) and \( \frac{\partial x_k}{\partial \sigma_r^2} \), we see that in general

\[
\frac{\partial x_k}{\partial \sigma_{rf}} \neq \frac{\partial x_r}{\partial \sigma_{kf}} ,
\]

and \( \frac{\partial x_k}{\partial \sigma_r^2} \neq \frac{\partial x_r}{\partial \sigma_k^2} \).

VI Empirical Generalizations

In this section we derive empirical generalizations from the previous results.
(i) In the case of a change in $\mu_r$, we can write equation (97) in elasticity form,

$$\mathcal{E}(X_k, \mu_r) = -U_E \frac{\mu_r}{\lambda P_r} \mathcal{E}(X_k, P_r^o) \frac{d\mu_r}{d\mu_r = 0}$$

$$2 \frac{\mu_r}{E} \mathcal{E}(U_E, E) U_E \sum_i \frac{\mu_i}{\lambda P_i} \mathcal{E}(X_k, P_i^o) \frac{d\mu_i}{d\mu_i = 0}$$

$$2 \frac{\mu_r}{E} \mathcal{E}(U_V, E) U_V \sum_i \frac{1}{\lambda P_i} \left( \sum_j \sigma_{ij} \mathcal{E}(X_k, P_i^o) \frac{d\mu_i}{d\mu_i = 0} \right)$$

where $\mathcal{E}(X_k, P_r^o) \frac{d\mu_r}{d\mu_r = 0}$ is the compensated cross-elasticity between $X_k$ and $X_r$. $\mathcal{E}(X_k, P_r^o) \frac{d\mu_r}{d\mu_r = 0} = \frac{\lambda D_{r\sigma} P_r}{(d\sigma)} X_k$; As noted before, $\frac{\lambda D_{r\sigma}}{(d\sigma)} X_k$ is the wealth-compensated substitution effect net of productivity changes, (i.e., $d\mu_t = 0$).

It follows from results of traditional consumer theory that

$$\mathcal{E}(X_k, P_r^o) \frac{d\mu_r}{d\mu_r = 0} < 0 \text{ for } r = k,$$

$$\mathcal{E}(X_k, P_r^o) \frac{d\mu_r}{d\mu_r = 0} \geq 0 \text{ for } r \neq k.$$

If the latter inequality is less than zero then $r$ and $k$ are called complements, and if greater than zero they are called substitutes. Also follows from traditional results that all assets cannot be complements. Thus in the case of only two assets they must be substitutes.
\( \varepsilon(U^2, E) \) and \( \varepsilon(U, V, E) \) stand respectively for the elasticity of the marginal utility of expected wealth with respect to expected wealth, and the elasticity of the marginal utility of variance with respect to expected wealth. The signs of these elasticities depend upon the nature of the \( U(E, V) \) function.

In general the sign of expression (112) is ambiguous. However, an empirical generalization about \( \varepsilon(X_k, \mu_r) \) can be derived from equation (94); the latter equation reduces to

\[
\sum_k P^0_k X_k \varepsilon(X_k, \mu_r) = 0, \quad \text{for} \quad k = 1, \ldots, n. \quad (113)
\]

Expression (113) implies that there is at least one asset \( k \) for which \( \varepsilon(X_k, \mu_r) < 0 \), and at least another for which \( \varepsilon(X_k, \mu_r) > 0 \), unless all \( \varepsilon(X_k, \mu_r) \) are zero. \( \varepsilon(X_k, \mu_r) > 0 \), unless all \( \varepsilon(X_k, \mu_r) \) are zero.

Further empirical generalizations about \( \varepsilon(X_k, \mu_r) \) that the contribution of asset \( r \) to portfolio expected return is very small, i.e., \( \frac{X_r \mu_r}{E} = 0 \); then the last two terms in (112) can be ignored and it reduces to

\[
\varepsilon(X_k, \mu_r) = -U_E \frac{\mu_r}{\lambda P^0_r} \varepsilon(X_k, P^0_r) u_0 \quad _{d \mu_r = 0} \quad (114)
\]
The sign of (114) depends upon the sign of $E(x_k, \mu_r) u_o$, given that $u_o > 0$.

The following empirical generalizations hold from (114):

(A) $E(x_k, \mu_r) > 0$ for $k = r$, since $E(x_k, \mu_r) u_o < 0$ in this case:

(B) $E(x_k, \mu_r) > 0$, for $k \neq r$, if $E(x_k, \mu_r) u_o < 0$, i.e., when the goods are complements.

(C) $E(x_k, \mu_r) < 0$, for $k \neq r$, if $E(x_k, \mu_r) u_o > 0$, i.e., when the goods are substitutes.

To conclude, if an asset's contribution to portfolio expected return is insignificant relative to total portfolio expected return, then an increase (decrease) in the asset's expected return will increase (decrease) the quantity held of the asset, and increase (decrease) the quantity held of assets which are complements to it, and decrease (increase) the quantity held of assets which are substitutes to it.

If, instead of assuming that the relative contribution of every asset to $E$ is small, we assume that the utility function is such that $E(U_E, E)$ and $E(U_V, E)$ are both zero, then the last two terms of (112) again disappear and the above empirical generalizations again hold. We show in Chapter 7 that a sufficient condition for the last two terms in (112) to disappear is that the utility function
be such that \( \frac{\partial (U_e/U_V)}{\partial E} = 0 \).

To establish empirical generalizations for cross-effects, we express (100) in elasticity form, to get

\[
\mathcal{E}(x_r, \mu_k) = - \frac{\mu_k}{\lambda P_r^0} \mathcal{E}(x_r, P_r^0) u_o \tag{115}
\]

\[
2 \frac{\mu_k x_k}{E} \mathcal{E}(u, E) u_e \sum \frac{\mu_i}{\lambda P_i^0} \mathcal{E}(x_r, P_i^0) u_o \tag{116}
\]

\[
2 \frac{x_k + \mu_k}{E} \mathcal{E}(u, E) u_v \sum \frac{1}{\lambda P_i^0} \left( \frac{\partial X_i}{\partial P_i^0} \right) \mathcal{E}(x_k, P_i^0) u_o \tag{117}
\]

If we assume that \( \frac{X_k X_k}{E} = 0 \), or that the utility function \( U(E, V) \) is such that the last two terms in (115) disappear, then we can write,

\[
\mathcal{E}(x_r, \mu_k) = - \frac{\mu_k}{\lambda P_r^0} \mathcal{E}(x_r, P_r^0) u_o \tag{115}
\]

Comparing (114) and (116), we see that \( \mathcal{E}(x_r, \mu_r) \) and \( \mathcal{E}(x_k, \mu_k) \) have the same signs since \( \mathcal{E}(x_r, P_r^0) u_o \) and \( \mathcal{E}(x_k, P_r^0) u_o \) have the same signs. Thus the cross elasticities have the same signs. Moreover, if we write \( \mathcal{E}(x_r, \mu_k) \) and \( \mathcal{E}(x_k, \mu_r) \) in the above equations more explicitly, we get

\[
\frac{\partial X_k}{\partial \mu_r} \frac{\mu_r}{X_k} = - \frac{\mu_r}{\lambda P_r^0} \left( \frac{\partial X_k}{\partial P_r^0} \right) u_o \left( \frac{P_r^0}{X_k} \right) \tag{117}
\]
and
\[ \frac{\partial X_r}{\partial \mu_k} \frac{\mu_k}{X_r} = -U_e \frac{\mu_k}{P_k^0} \left( \frac{\partial X_r}{\partial P_k^0} \right) \frac{u_r}{\frac{dP_k^0}{dH_k} \frac{P_k^0}{X_r}}. \]

(118)

Since
\[ \left( \frac{\partial X_k}{\partial P_r^0} \right)_{u_0} \frac{u_0}{\frac{dP_r^0}{dH_k} \frac{P_k^0}{X_r}} = \left( \frac{\partial X_r}{\partial P_k^0} \right)_{u_0} \frac{u_0}{\frac{dP_k^0}{dH_k} \frac{P_k^0}{X_r}}. \]

The above equations lead to
\[ \frac{\partial X_k}{\partial \mu_r} = \frac{\partial X_r}{\partial \mu_k}. \]

Thus, cross partials with respect to expected returns not only have the same sign but are equal in magnitude; this result is subject to the above empirical approximation and/or the assumption about the \( U(E,V) \) function.

(ii) For the case of a change in \( \sigma_{r_f} \), we write equation (105) in elasticity form,
\[ \varepsilon(x_k, \sigma_r) = - \frac{2u_v}{\lambda} \left[ x_f \frac{\sigma_f}{p_r^o} \varepsilon(x_k, p_r^o) \frac{u_o}{d\mu_r = 0} + x_r \frac{\sigma_r^2}{p_r^o} \varepsilon(x_k, p_r^o) \frac{u_o}{d\mu_r = 0} \right] \]

\[ - \frac{2x_r \sigma_r}{\sqrt{v}} \varepsilon(u_v, v) \frac{u_v}{\lambda} \left[ \sum_i \frac{1}{p_i^o} \varepsilon(x_k, p_i^o) \frac{u_o}{d\mu_i = 0} \right] \]

In the case where \( r = f \), (120) becomes

\[ \varepsilon(x_k, \sigma_r^2) = - \frac{2u_v}{\lambda} \left[ \frac{2x_r \sigma_r}{p_r^o} \varepsilon(x_k, p_r^o) \frac{u_o}{d\mu_r = 0} \right] \]

\[ - \frac{2x_r \sigma_r^2}{\sqrt{v}} \varepsilon(u_v, v) \frac{u_v}{\lambda} \left[ \sum_i \frac{1}{p_i^o} \varepsilon(x_k, p_i^o) \frac{u_o}{d\mu_i = 0} \right] \]

\[ - \frac{2x_r \sigma_r^2}{\sqrt{v}} \varepsilon(u_v, v) \frac{u_v}{\lambda} \left[ \sum_i \frac{1}{p_i^o} \varepsilon(x_k, p_i^o) \frac{u_o}{d\mu_i = 0} \left( \sum_j X_j \sigma_{ij} \right) \right] \]

(121)

\( \varepsilon(u_v, v) \) and \( \varepsilon(u_v, v) \) are respectively the elasticities of marginal utility of expected wealth with respect to variance, and marginal utility of variance with respect to variance. The signs of these elasticities depend upon the nature of the \( U(E, V) \) function.

In general the sign of either (120) or (121) is ambiguous. However, an empirical generalization about \( \varepsilon(x_k, \sigma_r) \) can be derived from equation (102); the latter equation reduces to
\[ \sum_{k} P_k \sigma_k \mathcal{E}(X_k, \sigma_{fr}) = 0, \quad \text{for } k = 1, \ldots, n, \quad (122) \]

Expression (122) implies that there is at least one asset \( k \) for which \( \mathcal{E}(X_k, \sigma_{fr}) > 0 \), and at least another for which \( \mathcal{E}(X_k, \sigma_{fr}) < 0 \), unless all \( \mathcal{E}(X_k, \sigma_{fr}) \) are zero.

If we make the assumption that an asset's contribution to portfolio variance is very small, i.e., in the case of asset \( r \),
\[ \frac{X_r \sigma_r^2 + X_r X_f \sigma_{rf}}{\sqrt{V}} = 0, \]
then expressions (120) and (121) reduce to, respectively,
\[
\mathcal{E}(X_k, \sigma_{rf}) = -2 \frac{U_Y}{\lambda} \left[ \frac{X_f \sigma_{rf}}{P_r^0} \mathcal{E}(X_k, P_r^0) \bigg|_{\mu = 0} + \right.
\left. X_r \left[ \frac{\sigma_{rf}}{P_r^0} \mathcal{E}(X_k, P_r^0) \bigg|_{\mu = 0} \right] \right];
\]
\[ (123) \]
\[
\mathcal{E}(X_k, \sigma_r^2) = -2 \frac{U_Y}{\lambda} \left[ 2 X_r \sigma_r^2 \mathcal{E}(X_k, P_r^0) \bigg|_{\mu = 0} \right];
\]
\[ (124) \]

If, instead of assuming that the relative contribution of every asset to \( V \) is small, we assume that the utility function is such that \( \mathcal{E}(U_E, V) \) and \( \mathcal{E}(U_Y, V) \) are both zero, then the last two terms of (120) again disappear and above equations (123) and (124) hold. We show in Chapter 7 that a sufficient condition for the last two terms in (123) or
(124) to disappear is that the utility function be such that 

$$\frac{\partial [u_e/u_v]}{\partial v} = 0.$$ 

From equation (123) we derive the following generalizations on the sign of $\varepsilon(x_k, \sigma_{rf})$, noting that $u_v < 0$. Also note that the terms substitute and complement are again used in their traditional senses referring to signs of compensated price elasticities.

(A) $\varepsilon(x_k, \sigma_{rf}) > 0$, if $\varepsilon(x_k, p_r^0) u_0 > 0$ and 
$\varepsilon(x_k, p_r^0) u_0 > 0$, and $\sigma_{rf} > 0$; i.e., an increase (decrease) in $\sigma_{rf}$ increases (decreases) holdings of asset $k$, if $x_k$ and $x_r$ are substitutes, $x_k$ and $x_r$ are substitutes, and $\sigma_{rf}$ is greater than zero.

(B) $\varepsilon(x_k, \sigma_{rf}) < 0$, if $\varepsilon(x_k, p_r^0) u_0 > 0$ and 
$\varepsilon(x_k, p_r^0) u_0 > 0$, and $\sigma_{rf} < 0$; i.e., an increase (decrease) in $\sigma_{rf}$ decreases (increases) holdings of asset $k$, if $x_k$ and $x_r$ are substitutes, $x_k$ and $x_r$ are substitutes, and $\sigma_{rf}$ is less than zero.

(C) $\varepsilon(x_k, \sigma_{rf}) < 0$, if $\varepsilon(x_k, p_r^0) u_0 < 0$ and 
$\varepsilon(x_k, p_r^0) u_0 < 0$, and $\sigma_{rf} > 0$; i.e., an increase (decrease) in $\sigma_{rf}$ decreases (increases) holdings of asset $k$, if $x_k$ and $x_r$ are complements, $x_k$ and $x_r$ are complements, and $\sigma_{rf}$ is greater than zero.
(D) \( \mathcal{E}(x_k, \sigma_{rf}) > 0 \) if \( \mathcal{E}(x_k, P_f^{o}) u_0 < 0 \) and \( \mathcal{E}(x_k, P_f^{o}) u_0 < 0 \), and \( \sigma_{rf} > 0 \); i.e., an increase (decrease) in \( \sigma_{rf} \) increases (decreases) holdings of asset \( k \), if \( x_k \) and \( x_r \) are complements, \( x_x \) and \( x_f \) are complements, and \( \sigma_{rf} \) is less than zero.

(E) When \( k = r \), \( \mathcal{E}(x_k, P_f^{o}) u_0 < 0 \), and we derive the following generalizations:

\[
\mathcal{E}(x_k, \sigma_{rf}) > 0 \quad \text{if} \quad \mathcal{E}(x_k, P_f^{o}) u_0 < 0 \quad \text{and} \quad \sigma_{rf} > 0; \quad \frac{d\sigma_{rf}}{d\mu_f} = 0
\]

\[
\mathcal{E}(x_k, \sigma_{rf}) < 0 \quad \text{if} \quad \mathcal{E}(x_k, P_f^{o}) u_0 < 0 \quad \text{and} \quad \sigma_{rf} < 0; \quad \frac{d\sigma_{rf}}{d\mu_f} = 0
\]

When \( \mathcal{E}(x_k, P_f^{o}) u_0 > 0 \), the sign of \( \mathcal{E}(x_k, \sigma_{rf}) \) depends on the relative magnitudes of \( \mathcal{E}(x_k, P_f^{o}) u_0 \) and \( \mathcal{E}(x_k, P_f^{o}) u_0 \), as well as the sign of \( \sigma_{rf} \).

Next, we derive from equation (124) the following generalizations on the sign of \( \mathcal{E}(x_k, \sigma_r^2) \):

(F) \( \mathcal{E}(x_k, \sigma_r^2) < 0 \) for \( k = r \), since in this case \( \mathcal{E}(x_k, P_f^{o}) u_0 < 0 \).

(G) \( \mathcal{E}(x_k, \sigma_r^2) < 0 \) if \( \mathcal{E}(x_k, P_f^{o}) u_0 < 0 \),

\( \sigma_r^2 > 0 \) if \( (x_k, P_f^{o}) u_0 > 0; \quad \frac{d\sigma_r^2}{d\mu_f} = 0 \)
(F) and (G) imply that an increase (decrease) in the variance of an asset decreases (increases) the quantity held of that asset, also decreases (increases) the quantity of assets which are complements to it, and increases (decreases) the quantity of assets which are substitutes to it.

We derive cross-effect relationships, again making the assumption that any asset's contribution to total portfolio variance is very small relative to the size of the portfolio; we get

\[
\mathcal{E}(X_r, \sigma_{fk}) = -\frac{2}{\lambda} u \left[ X_f \frac{\sigma_{fk}} {\sigma_{k0}} \mathcal{E}(X_r, \sigma_{k0}) u_o \right] + \frac{X_k \sigma_{fk} \mathcal{E}(X_r, \sigma_{k0}) u_o}{\lambda},
\]

or when \( f = k \)

\[
\mathcal{E}(X_r, \sigma_{k0}) = -\frac{2}{\lambda} u \left[ 2X_k \frac{\sigma_{k0}} {\sigma_{k0}} \mathcal{E}(X_r, \sigma_{k0}) u_o \right] ;
\]

Comparing (125) and (123) we notice that these terms are in general not equal; a sufficient condition for the signs of \( \mathcal{E}(X_r, \sigma_{fk}) \) and \( \mathcal{E}(X_k, \sigma_{rf}) \) to conform is that the compensated substitution terms in both equations, as well as \( \sigma_{rf} \) and \( \sigma_{fk} \), have the same signs.
Comparing (124) and (126), we see that $\mathcal{E}(x_r, \sigma_k^2)$ and $\mathcal{E}(x_k, \sigma_r^2)$ have the same sign since $\mathcal{E}(x_r, p_k^0)_{\mu^k=0}$ and $\mathcal{E}(x_k, p_r^0)_{\mu^r=0}$ have the same sign.

**Conclusion**

In this chapter we provided a complete analysis of the portfolio adjustment mechanism as well as useful empirical implications. Our approach can be generalized to analyze demand equations using utility functions, which include, in addition to $E$ and $V$, other characteristics (or moments) of the distribution. Also, a more general measure of risk, instead of variance, may be used.

In Appendix I we derive restrictions on the signs of productivity–substitution effects, for changes in $\mu_r$ and $\sigma_r^2$, in the case of an indefinite number of assets. In Appendix II we examine the effects of an initial price change in the case of an indefinite number of assets.
We examine restrictions on the signs of the productivity-substitution effects. In the case of a change in $\mu_r$, writing (98) in elasticity form,

$$
\varepsilon(X_i, \mu_{\text{re}})_U = - \frac{U_E \mu_r}{\lambda P_i^0} \varepsilon(X_i, P_i^0)_U \left|_{\mu_r = 0} \right. - \frac{U_E \mu_r}{P_i^0} \frac{P_i^0 \varepsilon(X_i, W)}{W_o} \left|_{\mu_r = 0}
$$

$$
= \frac{2 \mu_r X_r}{E} \varepsilon(U_E, E) \left|_{\mu_r = 0} \right. \sum_i \frac{M_i}{\lambda P_i^0} \varepsilon(X_i, P_i^0)U_0 \left|_{\mu_r = 0}
$$

$$
- \frac{2 \mu_r X_r}{E} \varepsilon(U_V, E) \left|_{\mu_r = 0} \right. \sum_i \frac{1}{\lambda P_i^0} \varepsilon(X_i, P_i^0)U_0 \left( \sum_j x_j \sigma_{ij} \right) \left|_{\mu_r = 0}
$$

(127)

(127) is the same as (112) except for the second term on the right-hand-side which is the wealth-effect term. If we make the assumption that investment in any asset forms a very small part of the investor's initial wealth, i.e.,

$$
\frac{P_i^0 X_i}{W_o} \approx 0,
$$

then together with the assumption that

$$
\frac{X_r \mu_r}{E} \approx 0,
$$

(127) reduces to

-153-
\[ \mathcal{E}(x_k, \mu_r) u_0 \approx -UE \frac{\mu_r}{\lambda P_r^0} \mathcal{E}(x_k, P_r^0) u_0 \quad (128) \]

which is the same expression as (114). We discussed restrictions on the sign of this expression.

For the productivity-substitution effect of a change in variance, we express (108) in elasticity form

\[ \mathcal{E}(x_k, \sigma_r^2) \bigg|_{u_0} = -2 \frac{U_V}{\lambda} \left[ 2 X_r \frac{\sigma_r^2}{P_r^0} \mathcal{E}(x_k, P_r^0) u_0 \right] \quad (129) \]

\[ + X_r \frac{U_V}{\lambda} \frac{X_r P_i^0}{W_0} \mathcal{E}(x_k, W_0) \frac{\sigma_r^2}{P_r^0} - 2X_r^2 \frac{\sigma_r^2}{P_r^0} \mathcal{E}(U_e, V) \frac{U_e}{V} \left[ \sum_{i=1}^{P_r} \frac{E(X_k, P_i^0)}{u_0} \right] \]

\[ - \frac{2X_r^2 \sigma_r^2}{V} \mathcal{E}(U_V, V) \frac{U_V}{\lambda} \left[ \sum_{j=1}^{P_r} \frac{1}{P_r} \mathcal{E}(X_k, P_i^0) u_0 \right]_{u_0} \quad (129) \]

(129) is the same as (121) except for the second term on the right-hand-side which is the wealth effect term.

Making the assumption that \( \frac{X_r P_i^0}{W} \approx 0 \), and that

\[ \frac{X_r^2 \sigma_r^2}{V} \approx 0 \], (129) reduces to
\[ \mathcal{E}(\chi, \sigma^2)_{\mu_0} = -2 \frac{u_r}{\lambda} \left[ \frac{2\chi_r}{\lambda \sigma^2} \mathcal{E}(\chi, P_{\lambda})_{\mu_0} \right] \]

which is the same as expression (124). We discussed restrictions on the sign of this expression.
APPENDIX II

(i) The Effect of a Change in Initial Price of an Asset

Generalizing from (59), we derive, in the case of \( n \) assets, for the total effect on \( X_k \) of a change in \( P_r^0 \) holding the other variables constant,

\[
\frac{\partial X_k}{\partial P_r^0} = -X_r \frac{D_{n+1,k}}{|D|} + \lambda \frac{D_{rk}}{|D|} - \sum_i R_i \frac{D_{ik}}{|D|}, \tag{131}
\]

The first term in (131) is the Wealth-Effect net of productivity changes, the second term is the Substitution-Effect net of productivity changes, the third term is the Productivity-Effect.

Note that \( \frac{\partial X_k}{\partial W_0} = \frac{D_{n+1,k}}{|D|} \).

Separating (131) into its total wealth and substitution components, where the total includes productivity and other effects of a change in initial price, we have for the total wealth effect of a change in \( P_r^0 \) on \( X_k \), holding the other variables constant, (from (82)).

\[
-X_r \left( \frac{D_{n+1,k}}{|D|} \right) \left( \frac{U \xi}{\lambda} + 1 \right) \tag{132}
\]
The total substitution effect (from (81))

\[
\frac{\partial X_k}{\partial P_r^0} \bigg|_{U_0} = \lambda \frac{D_{rk}}{|D|} + \frac{U_E}{\lambda} \left[ \lambda \frac{D_{rk}}{|D|} + X_r \frac{D_{n+1,k}}{|D|} \right] \\
+ U_{EE} X_r \sum_i \frac{D_{ik}}{|D|} + 2 U_{VE} X_r \sum_i \frac{D_{ik}}{|D|} \left( \sum_j \sigma_i^j \right) ;
\]

(133)

The signs of the expressions in both (132) and (133) are in general ambiguous.

For cross effects we have for the effect on \( X_r \) of a change in \( P_k^0 \),

\[
\frac{\partial X_r}{\partial P_k^0} = -X_r \frac{D_{n+1,r}}{|D|} + \lambda \frac{D_{rk}}{|D|} - \sum_i M_i \frac{D_{ir}}{|D|} ;
\]

(134)

Comparing (131) and (134), the second terms in both expressions, which is the Substitution Effect net of productivity changes, are equal, while the first and last terms are in general unequal. Thus in general

\[
\frac{\partial X_k}{\partial P_r^0} \neq \frac{\partial X_r}{\partial P_k^0} ;
\]

Separating (134) into its total wealth and substitution components, we have for the total wealth effect
\[ X_k \frac{Dn+1, r}{D1} \left( \frac{U_e}{\lambda} + 1 \right) \]

(135)

The total substitution effect is

\[
\frac{\partial X_r}{\partial P_k} \bigg|_{U_0} = \lambda \frac{Dr_k}{D1} + \frac{U_e}{\lambda} \left[ \lambda \frac{Dr_k}{D1} + X_k \frac{Dn+1, r}{D1} \right]
\]

\[ + U_{ee} X_k \sum_i \mu_i \frac{Dr_i}{D1} + 2 U_{ee} X_k \sum_i \frac{Dr_i}{D1} (\sum_j X_j \delta_{ij}) \]

(136)

Comparing (137) to (135) we see that in general the total wealth effect of a change in \( P_k^0 \) on \( X_r \) is not equal to that of a change in \( P_r^0 \) on \( X_r \). Comparing (133) to (136) we also see that in general

\[
\frac{\partial X_r}{\partial P_k} \bigg|_{U_0} \neq \frac{\partial X_k}{\partial P_r} \bigg|_{U_0}
\]

In view of the results of Chapter 6 and Appendix I, if we write (133) in elasticity form, and make the assumptions that asset \( r \) forms a small proportion of the investor's initial wealth, and also that \( \frac{\mu_r X_r}{E} \approx 0 \), then (133) reduces to
\begin{equation}
\mathcal{E}(X_K, P_r^0)_{U_0} = \mathcal{E}(X_K, P_r^0)_{U_0} - U_E \frac{\mu_r}{\lambda P_r^0} \mathcal{E}(X_K, P_r^0)_{U_0} \tag{137}
\end{equation}

where \( \mathcal{E}(X_K, P_r^0)_{U_0} \) is the traditional compensated-substitution effect net of productivity changes, while \( \mathcal{E}(X_K, P_r^0)_{U_0} \) is the productivity-substitution effect of a change in initial price \( P_r^0 \).

The sign of (137) depends upon whether \( X_K \) and \( X_r \) are substitutes or complements.

For \( \mathcal{E}(X_r, P_k^0)_{U_0} \), we have

\begin{equation}
\mathcal{E}(X_r, P_k^0)_{U_0} = \mathcal{E}(X_r, P_k^0)_{U_0} - U_E \frac{\mu_k}{\lambda P_k^0} \mathcal{E}(X_r, P_k^0)_{U_0} \tag{138}
\end{equation}

We know from traditional results that

\begin{equation}
\mathcal{E}(X_r, P_k^0)_{U_0} = \mathcal{E}(X_r, P_k^0)_{U_0} \tag{139}
\end{equation}

The relationship between the signs \( \mathcal{E}(X_r, P_k^0)_{U_0} \) and \( \mathcal{E}(X_K, P_r^0)_{U_0} \) depends upon whether these two assets are substitutes or complements.
Notes to Chapter Six

1 See Lancaster (18), page 133.

2 This idea is not very new. Others who have proposed it include Hicks (13), and Korishima (22).

3 See also Kane (16).

4 Thus variance is not linearly homogeneous in assets.

5 See A & M (2), pages 258-259.
Chapter 7

Taxation and Demand for a Risky Asset

The problem of the effect of taxation on risk-taking in an E-V framework has been discussed by several authors such as (7), (11), (24). These authors show that the effect of taxation on total risk-taking depends upon the investor's preference function of risk and expected return. However, I am aware of no analysis in the literature dealing with the comparative-statics of the effects of taxation on the demand for an individual risky asset, and providing restrictions on the sign of this effect. In this chapter I attempt such an analysis within the framework already developed in Chapter 6.

The paper is divided into two parts. In part I we analyze the effect on the demand for a risky asset of a lump-sum tax on future wealth (which is the same as a lump-sum tax on expected wealth), and also the case of a lump-sum tax on portfolio variance or risk. Part II examines the effect of a proportional tax on future returns of a single asset (as well as a proportional tax on future wealth in general) on the demand for any asset in a portfolio. Notice that we do not consider the case of a lump-sum tax on end-of-period return of a specific asset, since the comparative-static analysis...
there is identically equivalent to the case of a change
in the expected yield of a particular asset, already
discussed in Chapter 6.

\( \text{I} \) (i) With only a lump-sum tax \( T_E \) on future wealth or
expected wealth, we have

\[
E = \sum_{i=1}^{n} x_i / \mu_i - T_E, \quad (1)
\]

\[
V = \sum_{i,j} x_i x_j \sigma_{ij}, \quad i, j = 1, \ldots, n \quad (2)
\]

Maximizing a utility function \( U(E, V) \) subject to
a wealth constraint, we derive first-order conditions
which are the same as equations (50) and (51) in Chap­
ter 6. Second-order conditions again require that the
determinant \( D \) in Chapter 6 be negative definite.

To derive the effect on \( x_i \) of a change in \( T_E \), we
differentiate the first-order conditions with respect
to \( T_E \), to get

\[
\mu_i (U_{EE} \frac{\partial E}{\partial T_E} + U_{EV} \frac{\partial V}{\partial T_E}) + 2 \sum_j x_j \sigma_{ij} (U_{EE} \frac{\partial E}{\partial T_E} + U_{VV} \frac{\partial V}{\partial T_E}) \]

\[
+ 2 U_V \sum_j \sigma_{ij} \frac{\partial x_j}{\partial T_E} = P_\lambda \frac{\partial \lambda}{\partial T_E}, \quad (3)
\]

\[
\sum_{i} p_\lambda \frac{\partial x_i}{\partial T_E} = 0 \quad (4)
\]
Note that

\[ \frac{\partial E}{\partial T_E} = \sum_i \mu_i \frac{\partial x_i}{\partial T_E} - 1 \quad (5) \]

\[ \frac{\partial V}{\partial T_E} = \sum_i \sum_j \sigma_{ij} \left( x_i \frac{\partial x_j}{\partial T_E} + x_j \frac{\partial x_i}{\partial T_E} \right) \quad (6) \]

Substituting (5) and (6) into (3) and (4), and writing the system in matrix form, we have

\[
\begin{bmatrix}
Z_{11} & \cdots & Z_{1n} & P_1 \\
\vdots & \ddots & \vdots & \vdots \\
Z_{n1} & \cdots & Z_{nn} & P_n \\
P_0 & \cdots & P_0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x_1}{\partial T_E} \\
\vdots \\
\frac{\partial x_n}{\partial T_E}
\end{bmatrix}
= 
\begin{bmatrix}
J_1 \\
\vdots \\
J_n
\end{bmatrix}
\]  

(7)

where \( Z_{ij} \) is as defined in Chapter 6, and

\[ J_i = \mu_i U_{EE} + 2U_{VE} \sum_j x_j \sigma_{ij} \]

\( J_i \) denotes the shift in the marginal utility of asset \( i \) due to the change in \( T_E \) which affects the average productivity of every asset in producing \( E \).

Solving (7) for \( \frac{\partial x_k}{\partial T_E} \),

\[ \frac{\partial x_k}{\partial T_E} = \sum_i \frac{J_i D_{ik}}{|D|} \quad (8) \]
or, \[ \frac{\partial X_k}{\partial T_E} = U_{EE} \sum_i \mu_i \frac{D_{ik}}{D} + 2U_{VE} \sum_i \sum_j X_j \sigma_{ij} \frac{D_{ik}}{D} ; \quad (9) \]

In elasticity form,

\[ \mathcal{E}(X_k, T_E) = \frac{T}{E} \left[ \mathcal{E}(U_E, E) \frac{U_E}{X_k} \sum_i \mu_i \frac{D_{ik}}{D} ight. \]

\[ \left. + 2 \mathcal{E}(U_V, E) \frac{U_V}{X_k} \sum_i \sum_j X_j \sigma_{ij} \frac{D_{ik}}{D} \right] ; \quad (10) \]

The signs of (9) and (10) are in general ambiguous.

Equation (9) can also be written in an alternative form. Substituting \[ \Sigma x_j \sigma_{ij} \] from the first-order conditions (i.e. \( \Sigma x_j \sigma_{ij} = \lambda \beta \frac{p_i}{U} - U_E \mu_i \)), and noting that

\[ \Sigma p_i \frac{D_{ik}}{D} = 0 \] (expansion of a determinant by alien cofactors), the second term on the right-hand-side in (9) becomes 

\[ -U_{VE} \left( \frac{U_E}{U_V} \right) \sum_i \mu_i \frac{D_{ik}}{D} . \]

(9) can then be rewritten as

\[ \frac{\partial X_k}{\partial T_E} = \left[ U_{EE} - U_{VE} \left( \frac{U_E}{U_V} \right) \right] \sum_i \mu_i \frac{D_{ik}}{D} ; \quad (11) \]

or

\[ \frac{\partial X_k}{\partial T_E} = U_V \left[ \frac{\partial \left( \frac{U_E}{U_V} \right)}{\partial E} \right] \sum_i \mu_i \frac{D_{ik}}{D} ; \quad (12) \]

(12) implies that if the utility function \( U(E, V) \) has the property that the marginal rate of substitution between \( E \) and \( V \) is invariant to changes in \( E \) (i.e., \( \frac{\partial \left( \frac{U_E}{U_V} \right)}{\partial E} = 0 \))

then \( \frac{\partial X_k}{\partial T_E} = 0 \).
Finally, it follows from equation (4) that

$$\sum_{i} x_i P_i^o E(x_i; T_E) = 0$$

(13)

which implies that there is at least one asset i for which \( E(x_i; T_E) < 0 \), and at least another for which \( E(x_i; T_E) > 0 \), unless all \( E(x_i; T_E) \) are zero.

(ii) With a lump-sum tax \( T_V \) on portfolio variance or risk, we have

$$E = \sum_i \mu_i x_i$$

(14)

$$V = \sum_i \sum_j x_i x_j \sigma_{ij} - T_V$$

(15)

\( \lambda, j = 1, \ldots, n \).

First and second-order conditions are the same as in section (i). To derive the effect on \( x_1 \) of a change in \( T_V \) above, we differentiate the first-order conditions with respect to \( T_V \), to get

$$\mu_i (U_{EE} \frac{\partial E}{\partial T_V} + U_{EV} \frac{\partial V}{\partial T_V}) + 2 \sum_j x_j \sigma_{ij} (U_{VE} \frac{\partial E}{\partial T_V} + U_{VV} \frac{\partial V}{\partial T_V})$$

$$+ 2 U_{V} \sum_j \sigma_{ij} \frac{\partial x_i}{\partial T_V} = P_i^o \frac{\partial \lambda}{\partial T_V} \right) \right)$$

(16)

$$\sum_{i} P_i^o \frac{\partial x_i}{\partial T_V} = 0 \right) \right)$$

(17)
Note that

\[
\frac{\partial E}{\partial T_V} = \sum_i \mu_i \frac{\partial X_i}{\partial T_V} \tag{18}
\]

\[
\frac{\partial V}{\partial T_V} = \sum_i \sum_j \sigma_{ij} \left( X_i \frac{\partial X_i}{\partial T_V} + X_j \frac{\partial X_j}{\partial T_V} \right) - 1 \tag{19}
\]

Substituting (18) and (19) into (16) and (17) we get

\[
\begin{bmatrix}
Z_m & \cdots & Z_m & P_o \\
\vdots & \ddots & \vdots & \vdots \\
Z_m & \cdots & Z_m & P_o \\
P_o & \cdots & P_o & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial X_i}{\partial T_V} \\
\vdots \\
\frac{\partial X_i}{\partial T_V} \\
\frac{\partial X_j}{\partial T_V}
\end{bmatrix}
= \begin{bmatrix}
Q_i \\
\vdots \\
Q_i \\
0
\end{bmatrix} \tag{20}
\]

where \( Q_i = \mu_i U_{EV} + 2 \left( \sum_j \sigma_{ij} \right) U_{VV} \).

\( Q_i \) denotes the shift in the marginal utility of asset \( i \) due to the change in \( T_V \) which here affects the average productivity of every asset in producing \( V \).

Solving (20) for \( \frac{\partial X_k}{\partial T_V} \), we have

\[
\frac{\partial X_k}{\partial T_V} = \sum_i Q_i \frac{D_{ik}}{1D_i} \tag{21}
\]

or

\[
\frac{\partial X_k}{\partial T_V} = U_{EV} \sum_i \mu_i \frac{D_{ik}}{1D_i} + 2 U_{VV} \sum_j \sum_i X_j \sigma_{ij} \frac{D_{ik}}{1D_i} \tag{22}
\]
In elasticity form, (21) becomes

\[ \varepsilon(X_k, T_V) = \frac{T_V}{V} \left[ \frac{U_E}{X_k} \varepsilon(U_E, V) \sum_i \mu_i \frac{D_{ik}}{|D_1|} \right] + 2 \varepsilon(U_V, V) \frac{U_V}{X_k} \sum_j \sum_i X_i \delta_{ij} \frac{D_{ik}}{|D_1|} \right]; \tag{23} \]

The signs of (22) or (23) are in general ambiguous.

Equation (22) can be written in an alternative form. Again substituting for \( \sum x_{ij} \delta_{ij} \) from the first-order conditions and simplifying, the second term in (22) becomes

\[ -U_{VV} \left( \frac{U_E}{U_V} \right) \sum_i \mu_i \frac{D_{ik}}{|D_1|} \]

so that (22) can be rewritten as,

\[ \frac{\partial X_k}{\partial T_V} = \left[ U_{EV} - U_{VV} \left( \frac{U_E}{U_V} \right) \right] \sum_i \mu_i \frac{D_{ik}}{|D_1|}, \tag{24} \]

or

\[ \frac{\partial X_k}{\partial T_V} = U_V \left[ \frac{\partial (U_E/U_V)}{\partial V} \right] \sum_i \mu_i \frac{D_{ik}}{|D_1|}; \tag{25} \]

Equation (25) implies that if the utility function has the property that the marginal rate of substitution between \( E \) and \( V \) is invariant to changes in \( V \) (i.e.,

\[ \frac{\partial (U_E/U_V)}{\partial V} = 0 \]) then \( \frac{\partial X_k}{\partial T_V} = 0 \).
Again, it follows from equation (17) that

$$\sum_{i} p_i x_i \mathbb{E}(X_i, T_Y) = 0$$

which implies that there is at least one asset i for which $E(x_i, T_Y) < 0$, and at least another for which $E(x_i, T_Y) > 0$, unless all $E(x_i, T_Y)$ are zero.

II With proportional taxes on end-of-period returns on assets, the investor's expected end-of-period wealth and variance of wealth become, respectively,

$$E = \sum_{i}(1-t_i) x_i \mu_i, \quad i = 1, \ldots, n, $$

$$V = \sum_{i=1}^{n} (1-t_i)(1-t_j) x_i x_j \sigma_{ij}, \quad j = 1, \ldots, n, $$

where $t_i, t_j$ are the proportional taxes on end-of-period returns of assets i and j respectively. There are n assets with different tax rates.

Note that the marginal product of asset i in producing $E$ and $V$, respectively, is

$$\frac{\partial E}{\partial X_i} = (1-t_i) \mu_i,$$

$$\frac{\partial V}{\partial X_i} = \sum_{j} (1-t_i)(1-t_j) x_j \sigma_{ij}.$$ 

In a mean-variance framework, as before, the investor's optimization problem becomes the maximization of the utility function subject to the investor's wealth constraint, or the maximization of $L$, where
\[ L = U(E, V) + \lambda \left( W_0 - \sum_i P_i^0 X_i \right) \quad ; \]  

First-order conditions for a maximum are

\[ U_E (1-t_i) \mu_i + U_V (1-t_i) \sum_j (1-t_j) \sigma_{ij} = \lambda P_i^0 , \]  

\[ \sum_i P_i^0 X_i = W_0 \quad j \]  

Second-order conditions for a maximum require that the principal minors of the determinant \(|G|\) alternate in sign, where

\[ G = \begin{bmatrix} g_{11} & \cdots & g_{1n} & P_1^0 \\ \vdots & \ddots & \vdots & \vdots \\ g_{n1} & \cdots & g_{nn} & P_n^0 \\ P_1^0 & \cdots & P_n^0 & 0 \end{bmatrix} \]

and

\[ g_{ir} = U_E E (1-t_i)(1-t_r) \mu_i \mu_r + U_E V (1-t_i) \mu_i \sum_j (1-t_j) \sigma_{ij} \sigma_{jr} \]

\[ + U_V (1-t_i)(1-t_r) \sigma_{ir} + U_V E (1-t_i) \sum_j (1-t_j)(1-t_r) \mu_r X_j \sigma_{ij} \]

\[ + U_V V (1-t_r)(1-t_i) \left[ \sum_j (1-t_j) X_j \sigma_{ij} \right] \left[ \sum_j (1-t_j) X_j \sigma_{jr} \right] , \]

\[ i, r = 1, \ldots, n ; j \]
The effect of variation in a tax parameter is obtained by differentiating the first-order conditions with respect to that parameter. In the case of a change in $t_r$, the tax on end-of-period return on asset $r$, we have, differentiating the first-order conditions with respect to $t_r$, holding all other variables constant,

$$\frac{dE}{dt_r} = \mu_r + (1-t_i) \mu_i \left[ \frac{dE}{dt_r} + \frac{dV}{dt_r} \right] -$$

$$2uv \delta_{ir} \sum_j (1-t_j) x_j \sigma_{ij} - 2uv (1-t_i) x_r \sigma_{ir} +$$

$$2uv (1-t_i) \sum_j (1-t_j) \sigma_{ij} \frac{dX_i}{dt_r} + 2(1-t_i) \sum_j x_j \sigma_{ij} \left[ \frac{dE}{dt_r} \right]$$

$$+ uv \frac{dV}{dt_r} = P_i \frac{dX_i}{dt_r}, \quad (16)$$

$$\sum_i P_i \frac{dX_i}{dt_r} = 0 \quad , \quad (17)$$

$$\delta_{ir} = 1 \quad \text{for } i = r,$$

$$= 0 \quad \text{for } i \neq r;$$

Note that,

$$\frac{dE}{dt_r} = -\mu_r x_r + \sum_j (1-t_j) \mu_j \frac{dX_i}{dt_r},$$

$$\frac{dV}{dt_r} = -2 \sum_j (1-t_j) x_r x_j \sigma_{ij} +$$

$$+ \sum_i \sum_j (1-t_i)(1-t_j) \sigma_{ij} \frac{dX_i}{dt_r} + \sum_i \sum_j (1-t_i)(1-t_j) x_j \sigma_{ij} \frac{dX_i}{dt_r};$$
Substituting these into (16), (17) and writing the system in matrix-form, we get

\[
\begin{bmatrix}
g_{11} & \cdots & g_{1n} & P_{10} \\
\vdots & & \vdots & \vdots \\
g_{n1} & \cdots & g_{nn} & P_{n0} \\
P_{10} & \cdots & P_{n0} & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x_i}{\partial t_r} \\
\vdots \\
\frac{\partial x_n}{\partial t_r} \\
\frac{\partial \lambda}{\partial t_r}
\end{bmatrix}
= \begin{bmatrix}
I_1 \\
\vdots \\
I_n \\
0
\end{bmatrix},
\tag{18}
\]

where

\[I_i = U_E \Sigma \mu_r + (1-t_i) \mu_i U_E \mu_r X_r +
2 (1-t_i) \mu_i U_E \Sigma_j (l-t_j) X_j \sigma_{ij} + 2 U_v \Sigma \mu_r \Sigma_j (l-t_j) X_j \sigma_{ij}
+ 2 U_v (1-t_i) X_r \sigma_{ir} + 2 (1-t_i) U_v \mu_r X_r \Sigma_j (l-t_j) X_j \sigma_{ij}
+ 4 (1-t_i) U_v \left( \Sigma_j (l-t_j) X_j \sigma_{ij} \right) \left( \Sigma_j (l-t_j) X_r X_j \sigma_{rj} \right) \]

\[I_i \] denotes the shift (increase or decrease) in the marginal utility function of asset \( i \) resulting from the change in the marginal productivity of the asset \( r \) in producing \( E \) and \( V \), since the tax affects both characteristics.

Using Cramer's rule we solve the system (18) for

\[\frac{\partial x_k}{\partial t_r} , \text{ to get}\]
$$\frac{\partial x_k}{\partial t_r} = \sum_i \frac{G_{ik}}{|G|}$$

(19)

where $G_{ik}$ is the cofactor of the $i,j$th element of $G$.

We can write expression (19) more explicitly and in elasticity form, as

$$\mathcal{E}(x_k, t_r) = \frac{u_r t_r}{\lambda} \mu_r \mathcal{E}(x_k, P_i^0) u_0 + t_r \frac{u_r}{\lambda} \mathcal{E}(u_r, E) \sum_i (1-t_i) \frac{\mu_i}{P_i^0} \mathcal{E}(x_k, P_i^0) u_0$$

$$+ 2 \text{tr} \left[ \sum_j (1-t_j) x_j g_{ij} \right] \frac{u_r}{\lambda} \mathcal{E}(u_r, V) \sum_i (1-t_i) \frac{\mu_i}{P_i^0} \mathcal{E}(x_k, P_i^0) u_0$$

$$+ 2 \text{tr} \left[ \sum_j (1-t_j) x_j g_{ij} \right] \frac{u_r}{\lambda} \mathcal{E}(u_r, E) \sum_i (1-t_i) \frac{\mu_i}{P_i^0} \mathcal{E}(x_k, P_i^0) u_0$$

$$+ 4 \text{tr} \left[ \sum_j (1-t_j) x_j g_{ij} \right] \frac{u_r}{\lambda} \mathcal{E}(u_r, V) \sum_i (1-t_i) \frac{\mu_i}{P_i^0} \mathcal{E}(x_k, P_i^0) u_0$$

(20)

Note from Chapter 6 that

$$\left( \frac{\partial x_k}{\partial P_i^0} \right)_{u_0} = \lambda \frac{\text{Det}}{|D|}$$

The sign of expression (20) is ambiguous, depending upon the relative magnitudes of the parameters of the utility function as well as the distribution parameters.

From equation (17) we derive

$$\sum_i P_i^0 x_i \mathcal{E}(x_k, t_r) = 0$$

(21)

from which follows that there is at least one $i$ for which $\mathcal{E}(x_k, t_r) < 0$, and at least another for which $\mathcal{E}(x_k, t_r) > 0$, barring the circumstance when all $\mathcal{E}(x_k, t_r)$ are zero.
If we make the assumption, as we did in Chapter 6, that the relative contribution of asset \( r \) to total portfolio expected return and variance is very small, namely that \( \frac{\mu_r}{E} \approx 0 \) and \( \frac{\Sigma x_r x_j \sigma_{rj}}{\nu} \approx 0 \), then (20) becomes

\[
\mathcal{E}(x_k, tr) = \frac{\mu_E}{\sigma} \frac{tr}{\sigma_r} \mathcal{E}(x_k, \sigma_r) u_0 \frac{d\mu_r}{0} \\
+ \frac{\mu_v}{\sigma} \frac{tr}{\sigma_r} \mathcal{E}(x_k, \sigma_r) u_0 \frac{d\sigma_r}{0} \mathcal{E}(x_k, \sigma_r) u_0 \\
- \frac{\mu_r}{\sigma} \sum_i (1-t_i) \sigma_{ir} \mathcal{E}(x_k, \sigma_i) u_0 \frac{d\mu_r}{0} 
\]

(22)

The sign of (22) is still ambiguous; it depends upon the parameters of the utility function and the distribution function of returns. The reason for ambiguity is that a change in the proportional tax on an asset, unlike changes in each of expected return or variance alone, affects more than one characteristic; it affects \( E \) and \( V \). In the case of a change in expected return on an asset, for example, further empirical generalizations were possible because only one characteristic, namely \( E \), was affected.

Finally, for the case where all \( t_i \) are equal to \( t \), equivalent to a proportional tax on future wealth of \( t \), (22) becomes
\[ E(x_k, t) = \frac{u_e}{\lambda} \frac{t}{P_r^o} \mu_r E(x_k, P_r) u_0 \bigg|_{d\mu_r = 0} + \frac{2t}{P_r^o} \frac{U_v}{\lambda} E(x_k, P_r^o) u_0 \]

\[ + \frac{2t}{P_r^o} \frac{U_v}{\lambda} E(x_k, P_r^o) u_0 \bigg|_{d\mu_r = 0} (1-t) \sum_j \sigma_{ij} + \]

\[ \frac{2t}{\lambda} U_v (1-t) \sum_i \sigma_{ir} \frac{E(x_k, P_i^o) u_0}{P_i^o} \bigg|_{d\mu_i = 0} \]

(23)

The sign of (23) is ambiguous for the same reasons noted above.
Chapter 8
The Case of Indefinitely Many Characteristics--Implications to Demand with "Variable" Consumer Preferences

The analysis of Chapter six is now extended to the case of an indefinite number of characteristics. Implications of such an approach to consumer theory, as well as to portfolio theory with assets of more than two characteristics, are examined. The literature on demand under variable preferences is also reformulated in terms of this approach.

The advantages to consumer theory of assuming that it is the characteristics of commodities, rather than the commodities themselves, that are the direct objects of utility, have been discussed by Hicks (13), Morishima (22), and Lancaster (18). Morishima alone formally develops Slutsky equations for a commodity price change in this new framework, making the assumption that each characteristic is a linearly homogeneous function of the commodities. None of the above authors examines the implications of the new approach to the literature on demand with "variable" consumer preferences. This latter literature is within the usual Slutsky-Hicks approach to consumer theory and is discussed later.

The shortcomings of the traditional approach to consumer theory are well discussed by Lancaster:

"Once we are given the preference map, we have already incorporated all the information concerning
goods as such. The physical properties of goods relevant to the consumer have been presumed to have been taken into account by the consumer in deciding whether he prefers one collection or another. If the goods had been different, the preference map would have been different; that is all we can say.

"Since the traditional analysis starts with this preference diagram, the properties of goods have been swallowed up in the preferences before the analysis even commences, and there is no possibility of using information concerning these properties anywhere in a later stage. With no theory of how the properties of goods affect the preferences at the beginning, traditional analysis can provide no predictions as to how demand would be affected by a specified change in one or more properties of a good, or how a "new" good would fit into the preference pattern over existing goods. Any change in any property of any good implies that we have a new preference pattern for every individual: we must throw away any information derived from observing behavior in the previous situation and begin again from scratch."

The new approach to consumer theory incorporates information on the different objective properties or characteristics of goods into the analysis. Underlying this approach are two relationships. First, a technical relationship between quantities of goods and any given characteristic. Second, a preference function in terms of characteristics, since it is the characteristics of goods in which consumers are ultimately interested. Under the new approach, the demand for a commodity becomes a derived demand for the characteristics of the commodity, where commodities are inputs producing these characteristics.
In considering implications to the traditional literature on demand with "variable" consumer preferences, the new approach treats many types of variations in preference-orderings over commodities simply as variations in productivity of commodities in producing ultimately-desired characteristics. Without denying the possibility of taste-changes in characteristics space, we show that such an approach, which endogenizes many factors normally considered exogenous to the consumer's preference function, provides a more useful model than the traditional one with testable empirical implications. Thus factors such as advertising, quality-improvements of commodities, introduction of a new commodity, etc., all traditionally depicted as shifting the preference function, are now depicted as affecting the objective properties of commodities in producing ultimately--desired characteristics, without shifting the preference function. The only assumption made in this paper about the mapping from characteristic to commodity space, i.e., the nature of the technical relationship between commodities and characteristics, is that the relation is homogeneous. Slutsky equations for a price change are first developed and then the implications of the model to demand with "variable" consumer preferences are considered.
I  The Model

Assume that there are m commodities and n characteristics. The utility function is

\[ U = U(C_1, \ldots, C_n) \quad (1) \]

where first and second derivatives exist. The characteristics function is

\[ C_i = C_i(\alpha_{i1}^1 X_1, \ldots, \alpha_{i1}^j X_j, \ldots, \alpha_{im}^i X_m) \quad (2) \]

i = 1, ..., n) stands for characteristics, j = 1, ..., m) stands for commodities. \( \alpha_{ij}^i \) is a shift-parameter representing shifts in the productivity schedule of commodity j in producing characteristic i. Initially we set all \( \alpha_{ij}^i \) at unity. The function \( C_i \) is assumed to be homogeneous (not necessarily linear) in \((X_1, \ldots, X_m)\), implying simply that if all commodities were to increase by the same proportion, \( C_i \) would increase by some proportion. We also assume \( \frac{\partial C_i}{\partial X_j} > 0 \).

The characteristic function \( C_i \) can be regarded as either a purely technical relationship constrained solely by the degree of information available in society, or alternately as a subjective relationship expressing the consumer's conception of the productive characteristics of commodities. Either interpretation is consistent with our approach.

The utility maximization procedure subject to the individual's budget constraint leads to the maximization of function \( L \) with respect to \( X_j \), or
Max \[ L = U(c_1, \ldots, c_n) + \lambda (P_{i_1}x_i + \cdots + P_{m}x_m - y_0) \], \tag{3}

w.r.t. \( x_j \)

where \( P_j \) is price of commodity \( x_j \); we assume that \( P_j \) does not affect the value of any characteristic. \( y_0 \) is the individual's income. First-order conditions for utility maximum are,

\[ \frac{\partial U}{\partial c_i} \frac{\partial c_i}{\partial x_j} + \cdots + \frac{\partial^n U}{\partial c\partial x_j} = \lambda P_j, \quad (j = 1, \ldots, m) \tag{4} \]

\[ P_{i_1}x_1 + \cdots + P_{m}x_m = y_0 \tag{5} \]

Second-order conditions for a maximum require that determinant \( D \) (which consists of parameters of both functions (1) and (2)) is negative definite,

\[ D = \begin{bmatrix} D_{u} & \ldots & D_{u m} & P_1 \\ \vdots & \ddots & \vdots & \vdots \\ D_{m u} & \ldots & D_{m m} & P_m \\ P_1 & \ldots & P_m & 0 \end{bmatrix} \]

where \( D_{jk} = \frac{\partial^2 U}{\partial c_i \partial c_j} \left[ \frac{\partial^2 U}{\partial c_i^2} \frac{\partial c_i}{\partial x_k} + \cdots + \frac{\partial^2 U}{\partial c_i \partial c_n} \frac{\partial c_n}{\partial x_k} \right] \]

\[ + \frac{\partial^2 U}{\partial c_i \partial x_j} \left[ \frac{\partial^2 U}{\partial c_i \partial c_j} \frac{\partial c_i}{\partial x_k} + \cdots + \frac{\partial^2 U}{\partial c_n \partial x_j} \frac{\partial c_n}{\partial x_k} \right] \]

\[ + \frac{\partial^2 U}{\partial x_j \partial x_j} \frac{\partial^2 U}{\partial c_i \partial x_j} \frac{\partial c_i}{\partial x_k} \]

\[ + \frac{\partial^2 U}{\partial c_n \partial x_j} \frac{\partial^2 U}{\partial x_j \partial x_j} \frac{\partial c_n}{\partial x_k} \]
Morishima (6) makes the restrictive assumption that the characteristic function which he calls the means-objectives relationship, is linearly homogeneous and is able to establish a Separation property between choices in objective (or characteristic) space and means (or commodity) space. This Separation property implies that the consumer's choice of the best combination of means or commodity-inputs can be separated from the choice of best combination of objectives or characteristics. The second-order conditions in this case merely require that, the bordered Hessians of functions (1) and (2) be negative definite respectively.

II  **Effect of a Change in Price**

We now analyze the comparative-statics of a price change. Differentiating first-order conditions with respect to \( p_x \), holding other prices and income constant, we get
\[ \frac{\partial^j C_i}{\partial x_j} \left[ \frac{\partial^u C_i}{\partial C_i \partial x_j} + \ldots + \frac{\partial^u C_i}{\partial C_i \partial x_n \partial x_j} \right] + \frac{\partial^j C_i}{\partial x_j} \frac{\partial^u C_i}{\partial x_j \partial x_j \partial x_j} + \ldots + \frac{\partial^n C_i}{\partial x_j \partial x_j \partial x_j} \]

where \( S_{jr} = 1 \) for \( j = r \),
\( = 0 \) for \( j \neq r \).

Also, \( \sum_j P_j \frac{\partial x_j}{\partial x_j} + \lambda_j S_{j} = 0 \quad j \)

Note that,
\[ \frac{\partial C_i}{\partial x_j} = \sum_j \frac{\partial^j C_i}{\partial x_j} \frac{\partial x_j}{\partial x_j} \]
\[ \frac{\partial^2 C_i}{\partial x_j \partial x_k} = \sum_k \frac{\partial^2 C_i}{\partial x_j \partial x_k} \frac{\partial x_k}{\partial x_k} \quad (k = 1, \ldots, m) \]

Substituting these above, we get for (6) and (7)
Writing (8) and (7) in matrix form, we get the following system

\[
\begin{bmatrix}
\frac{\partial C_1}{\partial x_{11}} & \cdots & \frac{\partial C_1}{\partial x_{1m}} & P_1 \\
\vdots & & \vdots & \vdots \\
\frac{\partial C_n}{\partial x_{m1}} & \cdots & \frac{\partial C_n}{\partial x_{mm}} & P_m \\
P_1 & \cdots & P_m & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x_1}{\partial P_r} \\
\vdots \\
\frac{\partial x_m}{\partial P_r} \\
\frac{\partial x_{n+1}}{\partial P_r}
\end{bmatrix}
= \begin{bmatrix}
\lambda \delta_{r1} \\
\vdots \\
\lambda \delta_{rm} \\
-X_r
\end{bmatrix},
\]

(9)

Using Cramer's rule, we get the Slutsky equation

\[
\frac{\partial x_j}{\partial P_r} = \lambda \frac{Z_{ij}}{|D|} - X_r \frac{Z_{n+1,j}}{|D|},
\]

(10)

where \( Z_{ij} \) is the cofactor of the \((i,j)\) element of \( D \).

The first term in (10) is the substitution effect and the second term is the income effect of a change in \( P_r \). From second-order conditions it follows that \( \frac{Z_{jj}}{|D|} < 0 \), which means that the own-substitution term is negative.

Also, all other traditional results of consumer theory
hold, as for example, that

(i) Compensated cross-substitution terms are equal (since \( Z_{ij} = Z_{ji} \))

(ii) Not all commodities can be complements (where in the traditional way of consumer theory commodities are defined as substitutes if \( \lambda \frac{Z_{ij}}{|D|} < 0 \), and complements if \( \lambda \frac{Z_{ij}}{|D|} > 0 \))

since \( \sum_j P_j \frac{Z_{ij}}{|D|} = 0 \).

From which also follows that in the case of only two commodities, they must be substitutes.

As mentioned earlier Morishima (22), by assuming \( C_i \) to be linearly homogeneous, is able to establish a Separation property between choices in characteristic (or objective) space and commodity (or means) space. This enables Morishima to break down the compensated substitution effect in his Slutsky equation for a change in the price of a commodity, into an "objectives-substitution" term and a "means-substitution" term. In his words: 

"... a reduction in the price \( p_r \) reduces the cost of the objective \( j \) for which good \( r \) is a means, and accordingly sets up a tendency for the objective \( j \) to be substituted for others. The meaning of the second term (the means-substitution term)... is that a reduction in \( p_r \) reduces the cost of \( r \)-intensive methods of attaining \( y_j \) relative to other
methods, and accordingly sets up a tendency for good r
to be substituted for others."

This breakdown of the compensated-substitution term
enables Morishima to provide empirically more meaningful
definitions of substitutability and complementarity.

III The Traditional Literature on Demand with "Variable"
Consumer Preferences.

The need for incorporating variations in the pre­
ference function into consumer theory is well pointed­
out in the following quotation from Basmann 7 (4):

"Consumer demand theory, however, not having
taken variable preferences explicitly into ac­
count until very recently, has not asserted any
hypothetical laws governing relations among
the shifts in demand functions which a change
in preference orderings should cause. It is
desirable that such hypothetical laws be de­
rived, and that econometricians bring them
under empirical test. The logical consequences
of the assumption of fixed preferences differs
markedly from experience in the modern economy,
for in the latter it is commonly observed that
the consumption behavior of real individuals
and households is changed more or less syste­
 matically by advertising and other forms of
selling effort, and by changes in social and
technological factors exogenous with respect
to the consumer economy. In econometric demand
analysis, the introduction of time as an inde­
pendent trend variable to "explain" the effects
of changes in taste is at best an expedient it
would be better to avoid, if possible, since
trend parameters are not capable of causal or
legal interpretation."
The classic articles in this area are the two by Ichimura (15) and Tintner (31). These authors derived, for shifts in demand for a commodity as a result of a change in consumer's preferences, an expression which is a linear combination of the usual Hicks-Slutsky compensated substitution terms in consumer theory. Their approach has been adopted by Basmann (4) in the analysis of advertising, and by Bierwag and Grove (5) in their attempt to derive Slutsky equations for risky assets. I summarize the Ichimura-Tintner approach.

They start from the usual first and second-order conditions of consumer utility maximization:

\[ U_j = \lambda P_j, \quad j = 1, \ldots, m \]
\[ \sum_{j} P_j X_j = Y_0 \]

where \( U = U(x_1, \ldots, x_m) \) is the utility function here;

and that

\[ U = \begin{bmatrix} U_{11} & \cdots & U_{1m} & P_1 \\ \vdots & \ddots & \vdots & \vdots \\ U_{m1} & \cdots & U_{mm} & P_m \\ P_1 & \cdots & P_m & 0 \end{bmatrix}, \]

is negative definite.

They differentiate the first-order conditions with respect to a shift parameter \( \gamma \) which denotes a shift in the original preference function. This gives

\[ U_j, \frac{\partial X_1}{\partial \gamma} + U_jx, \frac{\partial X_2}{\partial \gamma} + \cdots + U_jm \frac{\partial X_m}{\partial \gamma} + U_j, = P_j \frac{\partial \lambda}{\partial \gamma}, \quad (11) \]
In matrix form, we have

\[
\begin{bmatrix}
U_{11} & \cdots & U_{1m} & p_1 \\
\vdots & \ddots & \vdots & \vdots \\
U_{m1} & \cdots & U_{mm} & p_m \\
p_1 & \cdots & p_m & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial X_1}{\partial \gamma} \\
\vdots \\
\frac{\partial X_m}{\partial \gamma} \\
\frac{\partial X_m}{\partial \gamma}
\end{bmatrix}
= 
\begin{bmatrix}
U_{1y} \\
\vdots \\
U_{my} \\
o
\end{bmatrix},
\]

which gives

\[
\frac{\partial X_j}{\partial \gamma} = \sum_{i=1}^{m} \frac{H_{jk} U_{ij}}{|H|},
\]

where \( H_{jk} \) are the cofactors of the \( j,k \)th element of the matrix \( U \).

Expression (14) is known as the Tintner-Ichimura relation; it is a linear combination of the Hicks-Slutsky substitution terms \( \frac{H_{jk}}{|H|} \), the weights being \( U_{ij} \).

As a theoretical hypothesis with empirical implications equation (14) is devoid of content. The sign and meaning of \( U_{ij} \) is unclear. The sign of \( \frac{\partial X_j}{\partial \gamma} \) ambiguous, except for the implication of (12) above which is, that \( \frac{\partial X_j}{\partial \gamma} \) must be positive for at least one commodity \( j \), and negative for
at least another, to preserve the overall sum equal to zero. This is true except obviously in the case where all \( \frac{dx_i}{dx} \) are equal to zero.

Neither can expression (14) be broken down into components which delineate empirically important effects from empirically unimportant ones, such as substitution and income effects as in the Slutsky equation of consumer theory. In fact the concept of a Slutsky equation does not apply here because the preference function is changing, making comparisons of before and after utility levels meaningless. In the same vein, all prior information concerning the nature of the preference function has been discarded. We may as well talk about a different individual. We have absolutely no information on how tastes are affected by the properties of commodities initially and thus we can make no predictions in this model about how demand would be affected by a change in these tastes.

Thus the incorporation of the Tintner-Ichimura relations into traditional demand analysis adds little to our ability to empirically explain the effects of various factors, other than relative prices and income, such as improved information technology, advertising, etc., on consumption behavior.

These shortcomings can be overcome by framing the problem along lines suggested earlier in this chapter. In the next section we approach the problem of "variations"
in the consumer's tastes as one of changes in the productivity of commodities in producing characteristics. Note that our analysis does not deny the possibility of taste changes in characteristic space and the total validity of the above Tintner-Ichimura equations for such changes. Our approach merely asserts that one should separate taste-changes from changes in the objective characteristics of goods (which the Tintner-Ichimura analysis does not) in order to provide greater generality to the analysis.

Our model in the next section provides this separation.

IV The Effect of a Change in the Productivity of a Commodity

Continuing with the framework developed in Sections I and II, we differentiate first-order conditions with respect to $\alpha_u^d$, the productivity of commodity $d$ in producing characteristic $u$. 
\[ a_j ' \frac{\partial c_i}{\partial x_j} \left[ \frac{\partial^2 u}{\partial c_i \partial d_u} + \cdots + \frac{\partial^2 u}{\partial c_i \partial c_n \partial d_u} \right] + \]

\[ a_j ' \frac{\partial u}{\partial c_i} \left[ \frac{\partial c_i}{\partial x_j \partial x} + \cdots + \frac{\partial c_i}{\partial x_j \partial c_n \partial d_u} \right] + \frac{\partial u}{\partial c_i} \frac{\partial c_i}{\partial x_j} \delta_{jd}^{iu} \]

\[ + d_j \frac{\partial c_n}{\partial x_j} \left[ \frac{\partial^2 u}{\partial c_n \partial d_u} + \cdots + \frac{\partial^2 u}{\partial c_n \partial c_n \partial d_u} \right] + \]

\[ a_j n \frac{\partial u}{\partial c_n} \left[ \frac{\partial c_n}{\partial x_j \partial x} + \cdots + \frac{\partial c_n}{\partial x_j \partial c_n \partial d_u} \right] + \]

\[ \frac{\partial u}{\partial c_n} \frac{\partial c_n}{\partial x_j} \delta_{jd}^{iu} = P_j \frac{\partial x_i}{\partial d_u} , \quad (j = 1, \ldots, m), \quad (15) \]

\[ \sum_j P_j \frac{\partial x_i}{\partial d_u} = 0 ; \quad (16) \]

where \( \delta_{jd}^{iu} = 1 \) for \( i = u \) and \( j = d \),

\[ = 0 \quad \text{otherwise}. \]
Note that
\[
\frac{\partial C_i}{\partial \alpha_d} = \sum_j a_j \frac{\partial C_i}{\partial \alpha_j} \left( \frac{\partial X_j}{\partial \alpha_d} + X_j \delta_{j_d} \right)
\]

Substituting this into (15), we get
\[
\alpha_j \frac{\partial C_i}{\partial \alpha_j} \left[ \frac{\partial^2 U}{\partial C_i \partial C_i} \left( \sum_j \alpha_j \frac{\partial C_j}{\partial \alpha_j} \frac{\partial X_j}{\partial \alpha_d} + \frac{\partial C_i}{\partial \alpha_j} X_j \delta_{j_d} \right) \right] + \alpha_j \frac{\partial U}{\partial C_i} \left( \frac{\partial C_i}{\partial \alpha_j} \frac{\partial X_j}{\partial \alpha_d} \right) + \ldots
\]
\[
\frac{\partial^2 U}{\partial C_i \partial C_i} \left( \sum_j \alpha_j \frac{\partial C_j}{\partial \alpha_j} \frac{\partial X_j}{\partial \alpha_d} + \frac{\partial C_i}{\partial \alpha_j} X_j \delta_{j_d} \right) + \frac{\partial U}{\partial C_j} \frac{\partial C_i}{\partial \alpha_j} X_j \delta_{j_d} + \ldots
\]
\[
\frac{\partial^2 U}{\partial C_i \partial C_i} \left( \sum_j \alpha_j \frac{\partial C_j}{\partial \alpha_j} \frac{\partial X_j}{\partial \alpha_d} + \frac{\partial C_i}{\partial \alpha_j} X_j \delta_{j_d} \right) + \frac{\partial U}{\partial C_i} \frac{\partial C_i}{\partial \alpha_j} X_j \delta_{j_d} + \ldots
\]
\[
\frac{\partial U}{\partial C_i} \frac{\partial C_i}{\partial \alpha_j} X_j \delta_{j_d} + \ldots
\]
\[
\frac{\partial U}{\partial C_i} \frac{\partial C_i}{\partial \alpha_j} X_j \delta_{j_d} + \ldots
\]
\[
= P_j \frac{\partial \lambda}{\partial \alpha_d} \quad (i = 1, \ldots, n), \quad (17)
\]

Rearranging (17) and (16) and writing the system in matrix form, we have
where

\[-R_j = \sum_i \alpha_i^j \frac{\partial C_i}{\partial X_j} \left[ \frac{\partial^2 U}{\partial C_i^2} \left( \sum \frac{\partial C_i}{\partial X_j} X_j s_{ij}^u \right) \right] + \ldots \]

\[+ \frac{\partial^2 U}{\partial C_i \partial C_n} \left( \sum \frac{\partial C_n}{\partial X_j} X_j s_{ij}^u \right) + \sum \frac{\partial U}{\partial C_i} \frac{\partial C_i}{\partial X_j} s_{ij}^u.\]

$R_j$ can be interpreted as the shift in the marginal utility of $X_j$ due to the change in the productivity of $X_d$ in producing characteristic $C_u$.

Using Cramer's rule we solve for $\frac{\partial X_k}{\partial d_u}$,

\[\frac{\partial X_k}{\partial d_u} = \sum_j R_j \frac{Z_{jk}}{|D|} \quad j, \]

which is the same as the Tintner-Ichimura expression except that we now have an explicit expression, $R_j$, for the shift in the marginal utility of $X_j$. 

\[\text{Eq. 18}\]

\[\begin{bmatrix}
D_m & \cdots & D_m & P_1 \\
\vdots & \ddots & \vdots & \vdots \\
D_m & \cdots & D_m & P_m \\
P_1 & \cdots & P_m & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial X_i}{\partial d_u} \\
\vdots \\
\frac{\partial X_m}{\partial d_u} \\
\frac{\partial \lambda}{\partial d_u}
\end{bmatrix}
= \begin{bmatrix}
R_1 \\
\vdots \\
R_m \\
0
\end{bmatrix} \]
Writing (19) more explicitly we have,

\[
\frac{\partial X_k}{\partial \alpha_d} = - \sum_j \sum_i \left[ \frac{\partial C_i}{\partial \alpha_d} \left( \frac{\partial^2 u}{\partial C_i \partial \alpha_d} \frac{\partial C_u}{\partial \alpha_d} X_d \frac{\partial C_u}{\partial \alpha_d} \right) \right] \frac{Z_{jk}}{|D|} - \frac{\partial u}{\partial C_u} \frac{\partial C_u}{\partial \alpha_d} \frac{Z_{ak}}{|D|} \right]
\]

Writing the above in elasticity form, and noting that

\[
\frac{Z_{jk}}{|D|} = \frac{1}{\lambda} \left( \frac{\partial X_j}{\partial P_k} \right) u_0,
\]

which is the substitution effect term of a pure price change derived earlier, we have

\[
\hat{E}(X_{k^*}, \alpha_d^u) = - \left( \frac{\partial C_u}{\partial x_d} \right) \alpha_d^u \sum_j \frac{\partial}{\partial x_d} \hat{E}(U_{c_i}, C_{u_i}) \alpha_i^u \frac{\partial C_i}{\partial x_j} \frac{\hat{E}(X_k, P_{j^*})}{\partial P_j} u_0
\]

\[
- \frac{\partial u}{\partial C_u} \frac{\partial C_u}{\partial x_d} \frac{1}{\lambda} \hat{E}(X_{k^*}, P_{d^*}) u_0,
\]

\[
i = 1, \ldots, n,
\]

\[
j = 1, \ldots, m.
\]

The sign of expression (21) is ambiguous depending upon the signs of \(E(U_{c_i}, C_{u_i})\) and \(E(X_k, P_{j^*})u_0\). However, it follows from (16) that \(\sum \pi_k X_{k^*} \hat{E}(X_{k^*}, \alpha_d^u) = 0\) for \(k = 1, \ldots, j, \ldots, m\) so that (barring the circumstance when all \(E(X_k, \alpha_d^u)\) are zero) there is at least one \(k\) for which \(E(X_k, \alpha_d^u) < 0\), and at least another one for which \(E(X_k, \alpha_d^u) > 0\).

Further empirical generalizations can be derived if we assume, as an approximation that the contribution of
commodity \( d \) to characteristic \( u \) is very small \( (x_d \frac{\partial C_u}{\partial x_d} \ll C_u) \)
then the first term in (21) can be ignored and we have

\[
\varepsilon(x_k, a_d^u) = -\frac{1}{\lambda} \left( \frac{\partial U}{\partial C_u} \right) \left( \frac{\partial C_u}{\partial x_d} \right) \varepsilon(x_k, p_d) u_0, \quad (22)
\]

The sign of the above expression depends upon the signs of \( \frac{\partial U}{\partial C_u} \) and \( \varepsilon(x_k, p_d) u_0 \). We know that

\[
\varepsilon(x_k, p_d) u_0 < 0 \quad \text{for} \quad k = d, \\
\geq 0 \quad \text{for} \quad k \neq d;
\]

(In the case of only two commodities, \( \varepsilon(x_k, p_d) u_0 > 0 \) for \( k \neq d \)).

Now, if \( \frac{\partial U}{\partial C_u} > 0 \), then:

(i) \( \varepsilon(x_k, a_d^u) > 0 \) if \( k = d \),

i.e. an increase in the productivity of commodity \( X_d \) in \( C_u \) increases the demand for \( X_d \);

(ii) \( \varepsilon(x_k, a_d^u) < 0 \) if \( \varepsilon(x_k, p_d) u_0 > 0 \), for \( k \neq d \).

(i.e. when the goods are substitutes).

(iii) \( \varepsilon(x_k, a_d^u) > 0 \) if \( \varepsilon(x_k, p_d) u_0 < 0 \), for \( k \neq d \).

(i.e. when the goods are complements).
Verbally, if a commodity's contribution to a characteristic is insignificant relative to the total size of that characteristic, then, if the marginal utility of the characteristic is positive, an increase in the productivity of the commodity in producing that characteristic increases the amount held of that commodity, decreasing the amounts held of substitute commodities and increasing amounts held of complement commodities to that commodity. If the marginal utility of the characteristic is negative (\(\frac{\partial Y}{\partial C_\alpha} < 0\)) then symmetrically opposite results to the above hold. Also, if, instead of assuming that the relative contribution of the commodity to the characteristic is very small, we assume that the utility function is such that the marginal utility of every characteristic is constant (so that \(E(U_{C_\alpha}, C_\alpha) = 0\)), then the same results as above hold since again the first term in (21) disappears.

Simple empirical generalizations are easy to establish within the present framework. That is the main advantage of this model over the Tintner-Ichimura-Basmann one.

V The Effect of a Lump-Sum Tax on Characteristic \(r\)

We show in this section that the first term of equation (20) is the negative of the effect of a lump-sum tax on characteristic \(u\) of magnitude \(X_\alpha \frac{\partial C_u}{\partial X_\alpha}\).

With a lump-sum tax \(T_u\) on characteristic \(u\), the \(C_u\) function becomes,
First and second order conditions for a maximum are the same as before. Differentiating first-order conditions (4) and (5) with respect to $T_u$, we get

$$
\begin{align*}
\frac{\partial}{\partial x_j} \left( \frac{\partial C_i}{\partial x_j} \right) + \frac{\partial^2 C_i}{\partial x_j \partial x_i} + \cdots + \frac{\partial^2 C_i}{\partial x_j \partial x_h} 
+ \frac{\partial}{\partial x_i} \left( \frac{\partial C_i}{\partial x_i} \right) + \cdots + \frac{\partial^2 C_i}{\partial x_i \partial x_h} 
+ \frac{\partial}{\partial x_h} \left( \frac{\partial C_i}{\partial x_h} \right) + \cdots + \frac{\partial^2 C_i}{\partial x_h \partial x_i} 
\end{align*}
$$

Note that

$$
\frac{\partial C_i}{\partial T_u} = \sum_j \frac{\partial C_i}{\partial x_j} \frac{\partial x_j}{\partial T_u} - 1
$$

Substituting this into (24) and writing the system in matrix form, we get

$$
\begin{bmatrix}
D_{11} & \cdots & D_{1m} & P_1 \\
\vdots & \ddots & \vdots & \vdots \\
D_{m1} & \cdots & D_{mm} & P_m \\
P_1 & \cdots & P_m & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x_1}{\partial T_u} \\
\vdots \\
\frac{\partial x_m}{\partial T_u} \\
\frac{\partial \lambda}{\partial T_u}
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
\vdots \\
F_m \\
0
\end{bmatrix}
$$
where
\[ F_j = \sum_j a_j \frac{\partial C_i}{\partial x_j} \left[ \frac{\partial^2 u}{\partial c_i \partial c_u} \right] ; \quad (i = 1, \ldots, n) \] (27)

\( F_j \) represents the shift in the marginal utility of commodity \( X_j \) due to the change in \( T_u \); note that changes in \( T_u \) affect the average productivity of every commodity in producing characteristic \( u \).

Solving (26) for \( \frac{\partial X_k}{\partial T_u} \), we get

\[ \frac{\partial X_k}{\partial T_u} = \sum_j F_j \frac{Z_{jk}}{D} \] (28)

or

\[ \frac{\partial X_k}{\partial T_u} = \sum_j \sum_i a_{ij} \frac{\partial C_i}{\partial x_j} \left[ \frac{\partial^2 u}{\partial c_i \partial c_u} \right] \frac{Z_{jk}}{D} \] (29)

Notice that the first term of (20) equals

\[ -X_d \frac{\partial C_u}{\partial x_d} \left( \frac{\partial X_k}{\partial T_u} \right) \] (30)

Equation (30) represents the effect on \( X_k \) of a lump-sum tax equal to \( X_d \frac{\partial C_u}{\partial x_d} \). Notice that if we keep all \( X_j \)'s unchanged, a change in \( \frac{\partial C_u}{\partial x_d} \) produces a change in the value of characteristic \( u \) of \( X_d \frac{\partial C_u}{\partial x_d} \) keeping other characteristics at same values. Taxing away this productivity change, to enable the individual
to consume his original endowment of characteristics, affects $x_k$ according to the negative of the first term in (20).

Graphically, in the case of only two characteristics $c_1$, $c_2$, as shown in Figure 20, an increase in $\alpha_d$ moves the individual from the original optimal point $A$ on the original efficiency locus $LL$ to the final optimal point $B$ on the new efficiency locus $LL'$. This movement can be broken down into two components corresponding to the terms in (20). The lump-sum-tax effect which is the negative of the first term in (20), corresponds to the movement from optimal point $B$ to optimal point $C$ as the efficiency locus $LL'$ shifts parallel leftward, by the amount of the lump-sum tax, to $MM$. While $MM$ enables the individual to consume the original endowment of characteristics at $A$, the individual does not stay at $A$, but moves to a higher level of utility at $C$. The movement between $A$ and $C$ caused by the change in the efficiency locus from $LL$ to $MM$, corresponds to the second term in (20).

VI. Substitution and Income Effects

Alternatively, one can delineate income and substitution effects within the present framework; this was not possible in the Tintner-Ichimura-Basmann framework.

Differentiate the utility function with respect to $\alpha_d$, and set it equal to zero to get the substitution term,
\[ \sum_i \frac{\partial u}{\partial c_i} \frac{\partial c_i}{\partial x_d} = 0 \]  

(31)

or, writing \( \frac{\partial c_i}{\partial x_d} \) explicitly, we have

\[ \sum_i \sum_j \frac{\partial u}{\partial c_i} \alpha^i_j \frac{\partial c_i}{\partial x_j} \left( \frac{\partial x_j}{\partial x_d} + x_j s_{jd} \right) = 0 \]  

(32)

or

\[ \frac{\partial x_1}{\partial x_d} \left( \sum_i \frac{\partial u}{\partial c_i} \alpha^i_1 \frac{\partial c_i}{\partial x_1} \right) + \cdots + \frac{\partial x_m}{\partial x_d} \left( \sum_i \frac{\partial u}{\partial c_i} \alpha^i_m \frac{\partial c_i}{\partial x_m} \right) \\
+ \frac{\partial u}{\partial c_u} x_d \frac{\partial c_u}{\partial x_d} x_d = 0 \]  

(33)

From first-order conditions (4),

\[ \sum_i \frac{\partial u}{\partial c_i} \alpha^i_j \frac{\partial c_i}{\partial x_j} = \lambda^j P_j \]  

Substituting this into (33), we get

\[ \frac{\partial x_1}{\partial x_d} (\lambda P_1) + \cdots + \frac{\partial x_m}{\partial x_d} (\lambda P_m) + \frac{\partial u}{\partial c_u} x_d \frac{\partial c_u}{\partial x_d} = 0 \]  

(34)
or

$$\sum_j \frac{\partial x_j}{\partial x_d} p_j = -x_d (a_d^u \frac{\partial y / \partial x_d}{\lambda} \frac{\partial c_u}{\partial x_d}) \quad (35)$$

In view of equation (16), equation (35) implies that the change in income required to keep \( \Delta u = 0 \) is equal to

$$-x_d \left( a_d^u \frac{\partial y / \partial x_d}{\lambda} \frac{\partial c_u}{\partial x_d} \right).$$

The income effect on \( x_j \) associated with a change in \( a_d^u \)

is

$$-x_d \left( a_d^u \frac{\partial y / \partial x_d}{\lambda} \frac{\partial c_u}{\partial x_d} \right) \left( \frac{Z_{n+1,j}}{1D} \right).$$

This is an implicit income effect arising because of the productivity increase in \( x_d \). Note that \( \frac{Z_{n+1,j}}{1D} = \frac{\partial x_j}{\partial y_0} \).

The income-compensated substitution effect of a change in \( a_d^u \) is, from (20),

$$\frac{\partial x_j}{\partial a_d^u } \bigg|_{u_0} = -\sum_i \sum_j a_{ij} \frac{\partial c_i}{\partial x_j} \left[ \frac{\partial y / \partial c_i \partial x_d}{\partial c_i \partial x_d} \right] \left( \frac{Z_{jk}}{1D} \right)$$

$$\Rightarrow \frac{\partial y / \partial c_u \partial x_d}{\partial x_d} \left( \frac{Z_{jk}}{1D} \right) - x_d \left( \frac{\partial y / \partial x_d}{\lambda} \frac{\partial c_u}{\partial x_d} \right) \left( \frac{Z_{n+1,j}}{1D} \right)$$

(36)
Thus we can identify substitution and income effects within the present framework, each with ambiguous signs.

If we express (36) in elasticity form, the third term on the right becomes

\[
\frac{X_a P_d}{W_o} \left( \frac{\partial U/\partial C}{\lambda P_d} \frac{\partial C}{\partial X_a} \right) \mathcal{E}(X_j, W_o),
\]

which can be ignored if we assume \( \frac{X_a P_d}{W_o} \) is very small. Thus as an empirical approximation the sign of \( \frac{\partial X_j}{\partial \lambda_d} \) is the same as that of \( \frac{\partial X_j}{\partial \lambda_d} \).

Graphically, as shown in Figure 21, the substitution effect is the movement from point A on the original efficiency locus LL, to point D on the wealth-compensated efficiency locus NN. The wealth effect is the movement from point D to B on LL', the final efficiency locus. Note that the shift from NN to LL' is a parallel shift away from the origin, where every point on LL moves away proportionally along a vector joining it to the origin.

In Chapter 6, I called the above effects productivity-substitution and productivity-wealth effects.
Figure 21
Notes to Chapter Eight

1 See articles by Basmann (4), Bierwag and Grove (5), Ichimura (15), Tintner (31).

2 See Lancaster (18), page 4.

3 It is a restrictive assumption, for example, since it excludes the E-V framework where variance is not linearly homogeneous in assets or commodities.

4 See Hicks (13).

5 See Morishima (22), page 157.

6 See Morishima (22), pages 157-158.

7 See Basmann (4), page 48.

8 See Basmann (4), page 50.

9 We can show, for example (in a manner similar to that in Chapter 6), that if the contribution of any commodity to a characteristic is small, then cross-effects have the same sign, i.e., \( \xi(x_k, d_u^i) \) and \( \xi(x_d, d_k^u) \) have the same sign.
Chapter 9

Summary

I have presented a set of interrelated essays dealing with various aspects of the comparative statics of portfolio selection. I show that previous studies fail to provide satisfactory analyses of the comparative statics of portfolio selection. I reformulate the analysis and provide the correct form and interpretation of the comparative-static effects as well as useful empirical generalizations.

Chapter 2 critically discusses Levy's attempt to derive Slutsky equations for assets under risk using the Separation Theorem. I show that Levy misapplies the Separation Theorem and that his equations are wrong. Chapter 3 provides correct demand and Slutsky equations for risky assets using the Separation Theorem.

Chapter 4 presents a review of the rest of the literature on the comparative statics of portfolio adjustment. It shows that these studies misidentify or improperly identify the comparative-static effects. The misidentification is due to the treatment of assets, in these studies, in a manner analogous to commodities of
traditional consumer theory, i.e., as ultimate objects of utility. Thus Royama and Hamada incorrectly identify, in the case of a change in the expected-return of an asset, or its variance-covariance, wealth and substitution effects which they claim are exactly analogous to the commodity-wealth and substitution effects of traditional consumer theory. The other authors identify "Veblen" or "Want-Pattern" effects for such changes, meaning that the utility function is shifting. The reason for the shift is that utility is expressed as a direct function of assets, so that changes in the qualities of assets, such as their expected return, or variance-covariance, directly affect the utility function or the preference-ordering over assets. In Chapter 6 I argue that in the Mean-Variance framework an asset is demanded indirectly because of its contribution to portfolio expected return and variance and not for other intrinsic properties. Thus the demand for an asset is analogous to the demand for a factor of production. By placing the analysis into such a framework we show in Chapter 6 that no "Veblen" or "Want-Pattern" effects underlie the comparative statics of portfolio adjustment.

The treatment of an asset as a factor of production places the problem of asset-demand within Lancaster's reformulation of demand theory. The Lancaster approach to demand incorporates into the analysis information on
the objective properties or characteristics of goods. Under this approach, the demand for a good becomes a derived demand for the characteristics of the good where goods are inputs producing these characteristics. One important advantage of this approach, as we show, is that it makes possible the derivation of useful empirical propositions.

In Chapter 6, I derive demand and comparative-static equations for assets in a Lancaster-type framework. I identify changes in expected return, or variance-covariance, of an asset as equivalent to changes in the productivity of the asset in producing ultimately-desired characteristics. I call such effects productivity effects. We can break down the productivity effect into wealth and substitution effects. An alternative and empirically more useful breakdown of the productivity effect is also provided. Ultimately I generate some very useful empirical propositions with respect to changes in the various characteristics of assets. For example, I show that if an asset's contribution to portfolio expected return is insignificant relative to total portfolio expected return, then an increase (decrease) in the asset's expected return will increase (decrease) the quantity held of assets which are complements to it, and decrease (increase) the quantity held of assets which are substitutes to it.
The framework of Chapter 6 can be generalized to analyze demand equations using utility functions which include in addition to $E$ and $V$, other characteristics (or moments) of the distribution. Also, a more general measure of risk, instead of variance, may be used.

Chapter 5 provides an analysis of the mapping from Mean-Variance space to asset space which is useful to the analysis in Chapter 6. In this context I show that Tsiang's (34) recent contention, that closed and circular indifference curves in asset space cannot exist if one assumes that the slopes of $E-S$ indifference curves are always less than unity, is invalid.

Chapter 7 applies the framework of Chapter 6 to an analysis of taxation and the demand for a risky asset. Various types of taxes are discussed.

In Chapter 8 we generalize the framework developed in Chapter 6 to portfolio choice with more than two characteristics and also apply it to traditional consumer demand theory. Implications of the approach to the literature on demand under "variable" consumer preferences are examined. The advantages to consumer theory of assuming that it is the characteristics of commodities, rather than the commodities themselves, that are the direct objects of utility, have been discussed by Hicks, Morishima, and Lancaster.
The new approach to consumer theory incorporates information on the different objective properties or characteristics of goods into the analysis. Underlying this approach are two relationships. First, a technical relationship between quantities of goods and any given characteristic. Second, a preference function in terms of characteristics, since it is the characteristics of goods in which consumers are ultimately interested.

Under the new approach, the demand for a commodity becomes a derived demand for the characteristics of the commodity, where commodities are inputs producing these characteristics. We find that such an approach to consumer theory provides a more useful model of analysis than the traditional one, with testable empirical implications. Thus the Tintner-Ichimura relation in the literature on demand under "variable" consumer preferences is reformulated in terms of the new approach to yield more useful comparative-static results which have, unlike the traditional results, unambiguous signs.
REFERENCES


(13) Hicks, J., A Revision of Demand Theory, 1956.


REFERENCES


REFERENCES


