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RECONSTRUCTION PROBLEMS OF
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DISSERTATION

Presented in Partial Fulfillment of the Requirement for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By
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The Ohio State University
1975

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In memory of my parents
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## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Graphical Terms</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>Star Geometry</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>Exceptional Graph for ((v,k,\lambda)=(4,3,2))</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>Illustration for Proposition 2.11</td>
<td>40</td>
</tr>
<tr>
<td>5</td>
<td>Transversals of 2-claws in (L(\Gamma))</td>
<td>41</td>
</tr>
<tr>
<td>6</td>
<td>Definitions of (d(x)) and (p(x))</td>
<td>53</td>
</tr>
<tr>
<td>7</td>
<td>(&lt;p,z&gt; = &lt;p,k&gt;)</td>
<td>54</td>
</tr>
<tr>
<td>8</td>
<td>(\mathcal{M}(A,B,C,x))</td>
<td>64</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>DEDICATION</td>
<td>ii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>iii</td>
</tr>
<tr>
<td>VITA</td>
<td>iv</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>v</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>Chapter</td>
<td></td>
</tr>
<tr>
<td>I. PRELIMINARIES</td>
<td>4</td>
</tr>
<tr>
<td>Conventions</td>
<td></td>
</tr>
<tr>
<td>Incidence Structures</td>
<td></td>
</tr>
<tr>
<td>Graphs</td>
<td></td>
</tr>
<tr>
<td>Partial Geometries</td>
<td></td>
</tr>
<tr>
<td>Affine Spaces</td>
<td></td>
</tr>
<tr>
<td>Projective Spaces</td>
<td></td>
</tr>
<tr>
<td>Designs</td>
<td></td>
</tr>
<tr>
<td>Projective Linear Group</td>
<td></td>
</tr>
<tr>
<td>Möbius Plane</td>
<td></td>
</tr>
<tr>
<td>II. RECONSTRUCTION PROBLEMS</td>
<td>23</td>
</tr>
<tr>
<td>Reconstructions Based on Eigenvalues of Graphs</td>
<td></td>
</tr>
<tr>
<td>Geometrization Theorem</td>
<td></td>
</tr>
<tr>
<td>Characterization of Graphs with the 4-Vertex</td>
<td></td>
</tr>
<tr>
<td>Condition</td>
<td></td>
</tr>
<tr>
<td>Linegraph of an Affine Space</td>
<td></td>
</tr>
<tr>
<td>Block-Residual Designs</td>
<td></td>
</tr>
<tr>
<td>Point-Residual Designs</td>
<td></td>
</tr>
<tr>
<td>III. CHARACTERIZATION OF LINEGRAPH OF AN AFFINE</td>
<td>49</td>
</tr>
<tr>
<td>SPACE</td>
<td></td>
</tr>
<tr>
<td>Statement of Theorem</td>
<td></td>
</tr>
<tr>
<td>Buekenhout's Theorem</td>
<td></td>
</tr>
<tr>
<td>Transversals</td>
<td></td>
</tr>
</tbody>
</table>

vi
Parallelism
Affine Planes
Proof of Theorem 3.1

IV. EMBEDDING OF A PSEUDO-BLOCK-RESIDUAL DESIGN
INTO A MOBIUS PLANE. .......................... 85

Motivation and Summary of the Proof of
Theorem
Fundamental Lemma
The Three Classes of Blocks
Tangents
Parallel Classes
Proof of Theorem 4.1 for q≥5
Proof of Theorem 4.1

V. EMBEDDING OF A PSEUDO POINT-RESIDUAL-DESIGN
INTO A MOBIUS PLANE. ............................. 126

REFERENCES. ........................................ 132
INTRODUCTION

One of the most widely studied areas in modern combinatorial theory is the determination of uniqueness of those existing incidence structures which satisfy certain properties. Various results have been obtained on the characterization of graphs, some of which are stated in the first three sections of Chapter II. We are particularly concerned with the characterization of the linegraph of an affine space.

Let \( \Gamma \) be an affine space\(^1\) over \( \text{GF}(q) \). Let \( G \) be a graph whose vertex set is the lines of \( \Gamma \), two vertices being adjacent if and only if the corresponding lines in \( \Gamma \) intersect. The linegraph \( G \) thus obtained is a \((q,k,q)\)-strongly regular graph. Furthermore, we show in Chapter II that the graph \( G \) possesses certain regularities on its triangles and 2-claws. Chapter III is devoted to the proof of the converse, which is stated in

**Theorem 3.1.** Let \( G \) be a \((q,k,q)\)-strongly regular graph with \( q \geq 4 \). Suppose \( G \) satisfies the following:

\[
(1) \quad k > \frac{1}{2}(q(q-1) + q(q+1)(q^2 - 2q + 2)).
\]

\(^1\) All definitions that we take for granted or are briefly introduced, are stated in detail in the text.
(2) For every triangle \((A,B,C)\) in \(G\), the triangle degree of \((A,B,C)\) either equals \(q(q-2)\) or is at least \(k-3\).

(3) For every 2-claw \((A;B,C)\) in \(G\), the 2-claw degree of \((A;B,C)\) equals either \(q(q-1)\) or \(2(q-1)\). Then \(G\) is isomorphic to the linegraph of an affine space \(AF(q,n)\). Furthermore, \(q\) is a prime power and \(k = (q^n-1)/(q-1)\).

We also discuss the cases where \(q \leq 3\).

Besides characterizing graphs, many have investigated problems of reconstructing residual designs. Bose, Singhi and Shrikhande \([5]\) recently proved the embedding theorem for a block-residual design of a symmetric pairwise balanced design. We consider a similar problem on block-residual designs of a Möbius plane.

Let \(M\) be a Möbius plane, an \(S(3,q+1,q^2+1)\). Let \(A\) be a fixed block in \(M\). If \(A\) is deleted from \(M\), then we obtain a block-residual design \(M'\) with parameters

\[ \lambda_0 = b = q^3 + q - 1, \quad \lambda_1 = r = q^2 + q, \quad \lambda_2 = q + 1, \quad \lambda_3 = 1, \]

\[ K = \{q+1, q, q-1\}, \quad v = q^2 - q. \]

We define a design to be a pseudo-block-residual design of order \(q\) (abbreviated by \(PBRD(q)\)) if it has these parameters. We also define a block \(B\) in \(M'\) to be \(r\)-tangent to another block \(B'\) at a point \(x\) if and only if \(B\) and \(B'\) are tangent
to each other at $x$ and there exists a $(q+1)$-valent block $C$ such that $C$ is tangent to $B$ at $x$ and secant to $B'$. We say that a PBRD($q$) satisfies the $r$-tangency condition if for any $q$-valent block $B$ and two points $x, y \notin B$, there exists at most one block which is $r$-tangent to $B$ and contains $x$ and $y$. Clearly, $M'$ satisfies the $r$-tangency condition. In Chapter IV, we derive the following:

**Theorem 4.1.** Let $q \notin 4$. If $D$ is a PBRD($q$) such that $D$ satisfies the $r$-tangency condition, then $D$ is uniquely embeddable into a Möbius plane.

We also look at the dual of this problem in Chapter V. Let $\infty$ be a point in $M$. If $\infty$ is deleted from $M$, then the point-residual design $M^*$ thus obtained is an $S_q(2, q+1, q^2)$. Any $S_q(2, q+1, q^2)$ is called a pseudo-point-residual design of order $q$. An $S_q(2, q+1, q^2)$ is said to have the tangency property if for any block $A$ and points $x$ and $y$, $x \in A$, $y \notin A$, there exists at most one block containing $y$ and tangent to $A$ at $x$. Clearly the point-residual design $M^*$ satisfies the tangency property. We prove

**Theorem 5.1.** Let $D$ be an $S_q(2, q+1, q^2)$. If $D$ satisfies the tangency condition, then $D$ is uniquely embeddable into a Möbius plane.
CHAPTER I
PRELIMINARIES

1.1 Conventions. Throughout this thesis we shall use the following notations. \( \mathbb{N} \) will denote the set of natural numbers \( \{1, 2, 3, \ldots \} \), \( \mathbb{N}_0 \) will denote the set \( \{0, 1, 2, \ldots \} \). The set of integers will be denoted by \( \mathbb{Z} \).

We shall be concerned with finite sets only. If \( X \) is a set, \(|X|\) denotes the cardinality of \( X \). If \(|X| = n\), then \( X \) is called an \( n \)-set. Any subset of \( X \) containing \( t \) elements is called a \( t \)-subset of \( X \). We shall often be concerned with subsets of a given set, and we adopt the convention that capital german letters \( A, B, \ldots \) denote collections of subsets of a given set. \( \emptyset \) is used to denote the empty set, often referred to as the null set.

If \( X \) and \( Y \) are two sets, \( X - Y = \{x \mid x \in X \text{ and } x \notin Y\} \). If \( Y \) is a singleton \( \{x\} \), then we often write \( X - x \) instead of \( X - \{x\} \).

Let \( f \) be a function and \( X \) be the domain of \( f \). For \( x \in X \), the image of \( f \) is given by \( f(x) \), and for a subset \( Y \) of \( X \), \( f(Y) = \{f(x) \mid x \in Y\} \). If \( f \) is a real-valued function over the domain \( X \), the average value of \( f \) over the whole domain is denoted by \( \text{ave } f(.) \). If the average is taken over
a subdomain \( Y \) of \( X \), the average is denoted by \( \text{ave} f(.) \).

A subset \( R \subseteq X \times X \) is said to be a relation on \( X \). If \((x,y) \in R\), then \( x \) and \( y \) are said to be \( R \)-related. If \( R_1, \ldots, R_j \) are \( j \) relations defined on a set \( X \), we use the notation \( \Delta_{i_1,i_2,\ldots,i_m}(x_1,\ldots,x_m) \) to denote the set of elements in \( X \) which are \( R_{i_t} \)-related to \( x_t \), for all \( t, 1 \leq t \leq m \), and \( \delta_{i_1,i_2,\ldots,i_m}(x_1,\ldots,x_m) \) to denote the cardinality of \( \Delta_{i_1,i_2,\ldots,i_m}(x_1,\ldots,x_m) \). In cases where \( i_1 = i_2 = \ldots = i_m \), we simply use \( \Delta_{i_1}(x_1,\ldots,x_m) \) to denote the set of elements \( R_{i_1} \)-related to the \( x_i \)'s, \( i=1,\ldots,m \), and correspondingly \( \delta_{i_1}(x_1,\ldots,x_m) \). Often, there is only one relation defined on the set \( X \). Then we simply write \( \Delta(x_1,\ldots,x_m) \) and \( \delta(x_1,\ldots,x_m) \) correspondingly.

§1.2 Incidence Structures. The study of combinatorics is often concerned with classes of entities which are related under certain rules. Here we are concerned with certain types of incidence structures.

Definition. An incidence structure is a triple \((X,B,I)\) where \( X \) and \( B \) are disjoint sets and \( I \subseteq X \times B \). Members of \( X \) are called points, or vertices, and elements of \( B \) are called blocks, or lines.
If \((x,b) \in I\), then we say that \(x\) and \(b\) are incident with each other. We also say that \(x\) is in \(b\) or \(b\) contains \(x\); usually we denote it by \(x \in b\). In cases where members of \(B\) are subsets of \(X\) and \(I\) is the inclusion relation, we simply write \((X,B)\) for \((X,B,I)\). Using the convention given above, if \(b\) is a block, then \(\Delta(b)\) denotes the set of points which are incident with \(b\), and \(\delta(b) = |\Delta(b)|\). Similarly, if \(x\) is a point in \(X\), then \(\Delta(x)\) denotes the set of blocks which contain \(x\), and \(\delta(x) = |\Delta(x)|\). An incidence structure is called a finite incidence structure if both \(X\) and \(B\) are finite sets. Two blocks \(b\) and \(b'\) are said to intersect if and only if there exists a point \(x\) which is incident with both \(b\) and \(b'\). Let \(\mathcal{A}\) be a collection of blocks in \(B\). A block \(b\) is a transversal of \(\mathcal{A}\) if and only if \(b\) intersects every block in \(\mathcal{A}\) in exactly one point.

From a given incidence structure \((X,B,I)\), new incidence structures can be obtained. One of the most useful ones is the dual incidence structure \((B,X,I^*)\), where \(I^* = \{(b,x) | (x,b) \in I\}\). One can easily check that the dual of the dual of an incidence structure is the incidence structure itself. If \(S\) is an incidence structure, then we write \(S^*\) for its dual. Let \(P_1\) and \(P_2\) be two properties satisfied by an incidence structure \(S\). \(P_1\) and \(P_2\) are said to be dual properties if and only if whenever \(P_1\) is satisfied by \(S\), \(P_2\) is satisfied by \(S^*\), and vice versa.

Other new incidence structures can be obtained by
deleting either a block or a point in $B$. Let $b_0$ be a fixed block in $B$. A **block-residual** substructure with respect to the given block $b_0$ is an incidence structure $(X',B',I')$ where $X'=X-\Delta(b_0)$, $B'=B-b_0$ and $I'=\cap(X'\times B')$.

Dual to the construction of a block-residual substructure is the construction of a point-residual incidence structure. Let $x_0$ be a point in $X$. A **point-residual** incidence structure with respect to the given point $x_0$ is an incidence structure $(X'',B'',I'')$ where $X''=X-x_0$, $B''=B-\Delta(x_0)$ and $I''=\cap(X''\times B'')$.

Let $D=(X,B,I)$ and $\overline{D}=(\overline{X},\overline{B},\overline{I})$ be two incidence structures. $D$ and $\overline{D}$ are said to be **isomorphic** if there exist bijections $\sigma:X\rightarrow\overline{X}$ and $\sigma':B\rightarrow\overline{B}$ such that $(x,b)\in I$ if and only if $(\sigma(x),\sigma'(b))\in\overline{I}$.

There are various kinds of incidence structures which are of great interest to mathematicians. Here we will discuss graphs, partial geometries, and designs in the following sections.

§1.3 **Graphs.** In this thesis, we shall be using only simple graphs and as such, we will drop the adjective simple. A **simple graph** is an incidence structure $(V,E,I)$ where for all $e\in E$, $e$ is incident with exactly two elements of $V$. Members of $V$ are called **vertices**, and a block in $E$ is called an **edge**. In order to specify the graph $G$, we often write $V(G)$ as the vertex set and $E(G)$ as the edge set. An edge $e$ will be denoted by $(x,y)$ where $x$ and $y$ are the
vertices incident with $e$.

Let $G$ be a graph. Two distinct vertices $x$ and $y$ of $G$ are said to be adjacent if and only if $(x,y) \in E(G)$. By convention, $\delta(x)$ denotes the number of vertices adjacent to a vertex $x$, and is called the vertex degree of $x$. If $(x,y) \notin E(G)$, then $\delta(x,y)$ is called the nonedge degree of $(x,y)$. A graph $G$ is said to be regular if and only if every vertex has the same vertex degree. It is called edge-regular if and only if every edge of $G$ has the same edge degree. The degree of a regular graph is the common degree of all its vertices.

Three distinct vertices are said to form a triangle if and only if they are pairwise adjacent. If $x$, $y$ and $z$ form a triangle in $G$, then $\delta(x,y,z)$ is called the triangle degree of $(x,y,z)$. Let $\{x,y_1,\ldots,y_n\}$ be a set of distinct vertices in $G$. If $x$ is adjacent to every $y_i$, $1 \leq i \leq n$, and the $n$ $y_i$'s are pairwise nonadjacent, then $\{x,y_1,\ldots,y_n\}$ is called a claw and is denoted by $(x;y_1,\ldots,y_n)$. Since there are $n$ $y_i$'s which are pairwise nonadjacent, it is called an $n$-claw. If $(x;y,z)$ is a 2-claw, then $\delta(x,y,z)$ is called the 2-claw degree of $(x;y,z)$. A claw is said to be maximal if and only if it is not properly contained in any other claw.

If $S=\{x_1,\ldots,x_n\}$ is a subset of $V(G)$ such that the vertices in $S$ are pairwise adjacent, then $S$ is called a clique. Since $S$ has $n$ vertices, $S$ is called an $n$-clique. A clique in $G$ is maximal if it is not a proper subset of any
other clique in $G$. If $G$ itself is a clique, then $G$ is called a **complete graph**.

![Graphical Terms](image)

- a vertex
- an edge
- a triangle

**Figure 1. Graphical Terms**

A $(v,n,\lambda,\mu)$ **strongly regular graph** is a graph $G$ satisfying the following:

(i) $V(G) = v$.

(ii) $G$ is regular of degree $n$.

(iii) $G$ is edge regular with edge degree $\lambda$.

(iv) $G$ is nonedge regular with nonedge degree $\mu$.

Corresponding to every incidence structure $(X,B,I)$, there is a graph $G$ whose vertex set $V(G)=X$, and two vertices are adjacent if and only if there exists a block $b$ containing them. The incidence graph is often a useful tool in the study of incidence structures, especially geometries. We will often say that two points in an incidence structure are **adjacent** if and only if they are incident with a block $b$. 
in B.

The linegraph \(L(D)\) of an incidence structure \(D=(X,B,I)\) is the incidence graph of the dual of \((X,B,I)\). That is, the vertex set of the linegraph is the collection of blocks in \(B\), and two vertices are adjacent if and only if their corresponding blocks are incident with a common point in \(D\). In Chapter III, we will give a characterization of the linegraph of an affine space. First, we need to develop the notions of partial geometries.

\[1.4 \textbf{Partial Geometries.} \quad \text{An \((r,k,t)\)-partial geometry is an incidence structure \((P,L,I)\) satisfying the following axioms:}\]

(A1) Every point in \(P\) is incident with exactly \(r\) lines in \(L\).

(A2) Every line in \(L\) is incident with exactly \(k\) points in \(P\).

(A3) A pair of distinct points in \(P\) are incident with at most one common line.

(A4) Given a line \(L\) and a point \(p\) not in \(L\), there exists exactly \(t\) lines containing \(p\) and intersecting \(L\).

If \(k=t\), then every pair of distinct points are contained in a unique line. A simple example of an \((r,k,t)\)-partial geometry can be seen from the star geometry is Figure 2. The point set of the partial geometry is the set of vertices in the graph, and the lines are the lines drawn in Figure 2. It can be easily checked that \(r=t=2\) and \(k=4\).
It is a well-known fact that the dual of an \((r,k,t)\)-partial geometry is a \((k,r,t)\)-partial geometry. Corresponding to every \((r,k,t)\)-partial geometry, there is a strongly regular graph, namely the incidence graph of the geometry, with parameters

\[
\begin{align*}
    v &= \frac{1}{t}(r-1)(k-1)(k-t) + r(k-1) + 1 \\
    n &= r(k-1) \\
    \lambda &= (t-1)(r-1) + k - 2 \\
    \mu &= rt.
\end{align*}
\]  

An \((r,k,t)\)-strongly regular graph \(G\) is a strongly regular graph with parameters as those given in (1.1); it is often called pseudo-geometric. The graph \(G\) is called geometrizable if and only if there exists an \((r,k,t)\)-partial geometry having its incidence graph isomorphic to \(G\).

Bose [2] proved a basic theorem regarding the geometrizability of strongly regular graphs:

**Theorem 2.4.** A pseudo-geometric \((r,k,t)\)-strongly regular graph is geometrizable if

\[k > \frac{1}{2}(r(r-1)+t(r+1)(r^2-2r+2)).\]
We shall give a proof of the theorem in Chapter II.

§1.5 Affine Spaces. Two of the most widely studied incidence structures are the affine and projective spaces. In this section, we shall discuss affine spaces and their associated linegraphs. Projective spaces will be discussed in the next section.

Let \( \Gamma \) be a triple \((P, \mathcal{L}, \mathcal{P})\) where \( P \) is a set of points, \( \mathcal{L} \) and \( \mathcal{P} \) are classes of subsets of \( P \) called lines and planes respectively. Two points \( p \) and \( q \) are said to be collinear if they are contained in some line \( L \) in \( \mathcal{L} \). We also say that the points \( p \) and \( p' \) lie on the line \( L \). A set \( S \) of points are called coplanar if they are contained in some plane \( \pi \) in \( \mathcal{P} \). We will also say that \( S \) lies on \( \pi \). Two lines are said to be parallel if they are equal, or if they are disjoint and coplanar. If \( A \) and \( B \) are parallel, we write \( A//B \).

A linear subspace of \( \Gamma \) is a subset \( P' \) of \( P \) satisfying the following:

(1) For every line \( L \) in \( \mathcal{L} \), if \( P' \) contains at least two points of \( L \), then \( P' \) contains all points of \( L \).

(2) If \( P' \) contains any three noncollinear points in \( \Gamma \), then all planes containing the three points are contained in \( P' \).

Clearly \( P \) itself is a linear subspace. It is easily seen that intersections of linear subspaces are also linear subspaces. For any subset \( X \) of \( P \), the subspace generated by \( X \) is the smallest subspace which contains \( X \). A 3-space is a
subspace generated by four noncoplanar points.

**Definition.** A triple \( \Gamma = (P, L, \Pi) \) is an **affine space** if \( \Gamma \) satisfies the following:

(B1) Each line contains at least 2 points and every two distinct points are contained in a unique line.

(B2) Each plane contains at least 3 noncollinear points, and every 3 noncollinear points are contained in a unique plane.

(B3) For every line \( L \) and any point \( p \), there exists a unique line containing \( p \) and parallel to \( L \).

(B4) If two planes contained in a 3-space have a point in common, then they have at least two points in common.

(B5) There are three noncollinear points.

(B6) There is a finite set of points which generate \( P \).

It is known that parallelism in an affine space is an equivalence relation. Further, an affine space \( \Gamma \) is related to a vector space \( V \) as given below. First, we define the **dimension** of an affine space to be 1 less than the smallest number of points which generate the space. A 2-dimensional affine space is often called an **affine plane**.

**Proposition 1.1.** If \( \Gamma \) is an affine space of dimension greater than or equal to 3, then there exists a vector
space $V$ over a division ring $R$ such that there is a bijection $\sigma : R \to V$ satisfying the following properties:

1. $\sigma$ is a bijection between points of $P$ and vectors of $V$.

2. $\sigma$ is a bijection between lines of $L$ and translates of 1-dimensional subspaces of $V$.

3. $\sigma$ is also a bijection between planes of $\Pi$ and translates of 2-dimensional subspaces of $V$.

4. For all subspaces $S, S'$ of $P$, $\sigma(S) \subseteq \sigma(S')$ if and only if $S \subseteq S'$.

Moreover, if each line of $L$ contains a finite number $q$ of points, then the division ring is the Galois field $GF(q)$ and $q$ is a prime power.

We often use $AF(q,n)$ to denote the $n$-dimensional affine space over $GF(q)$. The linegraph of an affine space $\Gamma$ is the linegraph defined on the incidence structure $(P,L)$. We denote the linegraph by $L(\Gamma)$.

**Proposition 1.2.** Let $\Gamma$ be an $n$-dimensional finite affine space with $n \geq 3$. $L(\Gamma)$ is a $(q,k,q)$-strongly regular graph where $q$ is the cardinality of each line in $L$ and $k = (q^n-1)/(q-1)$.

**Proof.** Clearly $L(\Gamma)$ is the incidence graph of the dual of $(P,L)$. Thus, it suffices to show that $(P,L)$ is a $(k,q,q)$-partial geometry and that every point in $P$ is incident with $(q^n-1)/(q-1)$ lines.
It is obvious that \((P, \mathcal{L})\) satisfies (A1), (A2) and (A3) in the definition of a partial geometry. To show (A4), let \(L\) be a line in \(\mathcal{L}\) and \(p\) a point not in \(L\). Consider a point \(x\) in \(L\), \(x \neq p\). By (B1) in the definition of an affine space, there exists a unique line \(L_x\) containing \(x\) and \(p\). Since \(L\) contains \(q\) points, there exist \(q\) lines containing \(p\) and intersecting \(L\). Therefore \((P, \mathcal{L})\) is an \((k,q,q)\)-partial geometry.

Let \(p\) be a point in \(P\). The number of lines containing \(p\) is the same as the number of 1-dimensional subspaces contained in an \(n\)-dimensional vector space \(V\) over \(GF(q)\). Hence, \(k = (q^n-1)/(q-1)/\)

\[1.6 \text{ Protective Spaces.} \] Closely related to affine spaces are projective spaces. Let \((P, \mathcal{L})\) be an incidence structure. A linear subspace of \(P\) is defined as in Section 1.5. A linear subspace will sometimes be called a subspace. \((P, \mathcal{L})\) is called a projective space if and only if it satisfies the following axioms:

(C1) Every pair of distinct points are contained in a unique line.

(C2) Every line contains at least 3 distinct points.

(C3) PASCH AXIOM: If two distinct lines \(L\) and \(M\) intersect at a point \(p\), and \(N\) and \(N'\) are distinct lines intersecting both \(L\) and \(M\) at points distinct from \(p\), then \(N\) and \(N'\) intersect each other.

(C4) There are three noncollinear points.
(C5) There is a finite set of points which generate P.

The projective dimension of a subspace of \((P,\mathcal{L})\) is 1 less than the cardinality of a minimal generating set of the subspace. Thus the projective dimension of \((P,\mathcal{L})\) is 1 less than the cardinality of a minimal generating set of P. If \((P,\mathcal{L})\) is of projective dimension 2, we usually call \((P,\mathcal{L})\) a projective plane.

If a projective space \((P,\mathcal{L})\) has projective dimension greater than 2, then there exists a division ring \(F\), a vector space \(V\) over \(F\), and a bijection \(\sigma\) between subspaces of \(P\) and subspaces of \(V\) such that for all subspaces \(S, S'\) of \(P\),

(1) \(\sigma(S)\subseteq\sigma(S')\) if and only if \(S\subseteq S'\), and

(2) dimension of \(\sigma(S) = 1 + \text{projective dimension of } S\).

§1.7 Designs. Very often we are interested in incidence structures whose blocks are closely related to \(t\)-subsets of the point sets for a given positive integer \(t\).

Let \(v, t \in \mathbb{N}_0\) such that \(v > t \geq 0\), and for every \(i\), \(0 \leq i \leq t\), let \(\lambda_i \in \mathbb{N}_0\). Let \(K\) be a set of nonnegative integers such that every member of \(K\) is smaller than or equal to \(v\). An incidence structure \(D = (X, \mathcal{A})\) is called a \((\lambda_0, \lambda_1, \ldots, \lambda_t; K,v)\) \(t\)-design (denoted by \(S(\lambda_0, \lambda_1, \ldots, \lambda_t; K,v)\)) if and only if

(D1) every \(i\)-subsets of \(X\) is contained in exactly \(\lambda_i\) blocks, for \(0 \leq i \leq t\),

(D2) for every block \(A \in \mathcal{A}\), \(|A| \in K\), and

(D3) \(|X| = v\).
If \(|A|=k\), then \(A\) is called a \(k\)-valent block. Obviously \(\lambda_0^2 \lambda_1^2 \cdots \lambda_t^2\). In cases where Axiom (D1) is only known to be satisfied by \(i=t\), the design \(D\) will be denoted by \(S_\lambda(t,K,v)\) where \(\lambda=\lambda_t\). Further, if \(\lambda=1\), we only use \(S(t,K,v)\). For simplicity, if \(K\) consists of a singleton \(k\), we write \(D\) as a \(S_\lambda(t,k,v)\) instead of \(S_\lambda(t,\{k\},v)\).

**Proposition 1.3.** An \(S_\lambda(t,k,v)\) is a \((\lambda_0,\lambda_1,\ldots,\lambda_t; k,v)\) \(t\)-design where

\[
\lambda_i = \frac{\lambda(t-1)}{(k-1)_{t-i}} \quad \text{exist.}
\]

Proof. Let \(P\) be any \(i\)-subset of \(X\). We will count the number of ordered pairs \(\{(y_{i+1},\ldots,y_t,A)\}\) where \(y_j \not\in P\) for all \(j, i+1 \leq j \leq t\), and \(A\) is a block containing \(P\cup\{y_{i+1},\ldots,y_t\}\).

Fixing a \((t-1)\)-subset in \(X-P\), there exist exactly \(\lambda\) choices of \(A\) and there are \(\binom{v-i}{t-1}\) choices of \((t-1)\)-subsets. On the other hand, fixing a block \(A\) containing \(P\), we have \(\binom{k-i}{t-1}\) choices of \((t-1)\)-subsets of \(X-P\). Hence

\[
\lambda_i \binom{k-i}{t-1} = \lambda(t-1) \binom{v-i}{t-1},
\]

and we have

\[
\lambda_i = \frac{\lambda(t-1)}{(k-1)_{t-i}}. \quad (1.2)
\]

For \(i=0\), \(\lambda_0\) is the number of blocks in \(D\) and is usually written as \(b\). For \(i=1\), \(\lambda_1\) is the number of blocks containing a given point and is usually written as \(r\).
For designs that have various block sizes, the proposition is not valid. A counterexample of such can be found for \( t=3, \, v=6, \, \lambda=1 \) and \( K=\{3,4\} \). Let \( X = \{1,2,3,4,5,6\} \), and the blocks are listed below,

\[
\begin{align*}
1 & 2 & 3 & 4 & 1 & 3 & 5 & 2 & 3 & 5 \\
1 & 2 & 5 & 6 & 1 & 3 & 6 & 2 & 3 & 6 \\
3 & 4 & 5 & 6 & 1 & 4 & 5 & 2 & 4 & 5 \\
\end{align*}
\]

It can be easily checked that this design is an \( S(3,\{3,4\},6) \), but it is not an \( S(A,\{3,4\},6) \) for any positive integer \( \lambda \).

Let \( D \) be a design. A block-residual design of \( D \) is a block-residual substructure \( D' \) of \( D \). If \( D \) is a \( (v,k,\lambda) \) \( t \)-design, then \( D' \) is a \( (v-k,K',\lambda) \) \( t \)-design where \( k \) is the size of the deleted block and \( K' \) is a subset of \( \mathbb{N}_0 \) such that every member of \( K' \) is not larger than the maximal member in \( K \). A design with parameters as those of a block-residual design is called a pseudo-block-residual design.

Dually, let us consider the point-residual substructure \( \bar{D} \) of \( D \). \( \bar{D} \) may not be a design. In the "six-points" example given above, if we delete the point "1" from the design, the resulting substructure is as follows:

\[
\begin{align*}
3 & 4 & 5 & 6 & 2 & 3 & 5 & 2 & 4 & 5 \\
2 & 3 & 6 & 2 & 4 & 6.
\end{align*}
\]

Clearly, it is not a design. However, if \( K \) is a singleton set, then the new structure \( \bar{D} \) is a design.
Proposition 1.4. Let $D$ be a $(v,k,\lambda)$ t-design. The point-residual structure $\overline{D}$ of $D$ is an $S_{\lambda',(t-1,k,v-1)}$ with $\lambda'=\lambda_{t-1}-\lambda$.

Proof. Let $P$ be any $(t-1)$-subset of $D$, and $x$ be the point deleted from $D$. $P\cup x$ is a $t$-set; hence it is contained in exactly $\lambda$ blocks in $D$. By Proposition 1.3, $P$ is contained in exactly $\lambda_{t-1}$ blocks in $D$ of which $\lambda$ blocks contain the deleted point $x$. But these $\lambda$ blocks are deleted in $\overline{D}$. Thus $P$ is contained in exactly $\lambda_{t-1}-\lambda$ blocks in $\overline{D}$, that is, $\lambda' = \lambda_{t-1} - \lambda$.

$\overline{D}$ is called a point-residual design. Any design with parameters as those of $\overline{D}$ is called a pseudo-point-residual design.

§1.8 Projective Linear Group. Let $V=V(n,p)$ be a vector space of dimension $n$ over a field $F=GF(p)$. A linear automorphism of $V$ is a permutation $\sigma$ of $V$ satisfying

$$\sigma(u+v) = \sigma(u) + \sigma(v) \quad \text{for all } u, v \in V,$$

and

$$\sigma(\alpha v) = \alpha \sigma(v) \quad \text{for all } \alpha \in F \text{ and } v \in V.$$

The group of all linear automorphisms of $V$ is called the General Linear Group of $V$, and is denoted by $GL(V)$ (or $GL(n,p)$). A linear automorphism $\sigma$ of $V$ is called a scalar transformation if there exists a nonzero element $\alpha \in F$ such that $\sigma(v)=\alpha v$ for all $v \in V$.

Let $G$ be a group. A homomorphism $\sigma$ of $G$ is a map
σ : G → G such that σ(g·g') = σ(g)·σ(g'). The kernel of σ is the set of elements in G whose images under σ are equal to the identity of G. A subgroup of G is called a normal subgroup if it is the kernel of some homomorphism of G. If H is a normal subgroup, then for every g ∈ G, a coset gH is the set of elements gh, where h ∈ H. The group defined on the set of cosets \{gH | g ∈ G\} by gH·g'H = g·g'H is called the quotient group \( G/H \).

The centre of a group G, denoted by \( Z(G) \), is the set of elements in G which commute with every other element in G. It is known that \( Z(GL(V)) \) is a normal subgroup of \( GL(V) \). Hence, the quotient group \( GL(V)/Z(GL(V)) \) exists. The quotient group denoted by \( PGL(V) \) (or \( PGL(n,p) \)) is called the Projective Linear Group of V.

Let us now compute the order of \( PGL(n,p) \).

**Proposition 1.5.** \(|PGL(n,p)| = p^{\frac{n}{2}n(n-1)} \prod_{i=2}^{n} (p^i-1)\).

**Proof.** Since \( PGL(n,p) = GL(n,p)/Z(GL(n,p)) \),

\[ |PGL(n,p)| = |GL(n,p)|/|Z(GL(n,p))|. \quad (1.3) \]

The centre of the general linear group is known to be a group of scalar transformations; hence \(|Z(GL(n,p))| = p-1.\)

Now, for every two distinct bases \( \{e_1, \ldots, e_n\} \) and \( \{v_1, \ldots, v_n\} \) of V, there exists a unique linear automorphism σ in \( GL(n,p) \) such that \( σ(e_1) = v_1. \) Thus, the order of \( GL(n,p) \) is equal to the number of distinct bases in V. Hence
\[ |\text{GL}(n,p)\| = (p^n-1)(p^n-p)\ldots(p^n-p^{n-1}) \]
\[ = p^{\frac{1}{2}n(n-1)} \prod_{i=1}^{n} (p^i-1). \quad (1.4) \]

From Equations (1.3) and (1.4), we obtain the result.

The proof of the proposition follows mainly the proof found in [1].

\[ \textbf{1.9 Möbius Plane}^2. \text{ Let } M \text{ be any } S(3,q+1,q^2+1). \]

We shall call M a Möbius plane. Finite models of Möbius planes are known and an example of such is described below.

Let \( X \) be the set of elements in \( \text{GF}(q^2) \cup \{\infty\} \) and \( A = \text{GF}(q) \cup \{\infty\} \). Consider the images of \( A \) under the projective linear group \( \text{PGL}(2,q^2) \)

\[ x \rightarrow \frac{ax+b}{cx+d} \quad \text{ad} \neq bc \text{ and } a,b,c,d \in \text{GF}(q^2). \quad (1.5) \]

The image of the element \( \infty \) under the mapping (1.5) is \( \frac{a}{c} \).

Also, for any element \( a \in \text{GF}(q^2) \), \( \frac{a}{0} \) is formally set to be equal to \( \infty \). Let \( \mathcal{A} \) be the set of distinct images of \( A \) and \( M=(X,\mathcal{A}) \). Then \( M \) is a Möbius plane. Clearly, \( |X|=q^2+1 \) and every block has \( q+1 \) elements. Let \( (x_1, x_2, x_3) \) and \( (y_1, y_2, y_3) \) be two distinct 3-subsets of \( X \). It can be easily seen that there exists a linear automorphism \( \sigma \) in \( \text{PGL}(2,q^2) \) such that \( \sigma(y_i)=x_i \) for \( i=1,2,3 \). Thus, every three points are contained in at least one block. It remains to show that they are contained in a unique block.

\[ ^2 \text{It is also known as the Inversive Plane.} \]
Consider two linear automorphism $\sigma$ and $\sigma'$ in $\text{PGL}(2,q^2)$ such that $\sigma(\text{PGL}(2,q)) = \sigma'(\text{PGL}(2,q))$, that is, there exists a $\rho \in \text{PGL}(2,q)$ with $\rho \sigma = \sigma'$. Since $\rho(\text{GF}(q) \cup \{\infty\}) = \text{GF}(q) \cup \{\infty\}$, we have $\sigma(\text{GF}(q) \cup \{\infty\}) = \sigma'(\text{GF}(q) \cup \{\infty\})$. Therefore, the images of $A$ under $\sigma$ and $\sigma'$ are identical. Hence, the number of distinct blocks in $M$ is at most $|\text{PGL}(2,q^2)| / |\text{PGL}(2,q)|$.

From Proposition 1.5, we obtain that $b$ is at most $q(q^2+1)$. Knowing $b$ and using the proof of Proposition 1.3, we compute that the average number of blocks containing 3 distinct points in $M$ is at most 1. Hence, every 3 distinct points are contained in a unique block, and $M$ is a $(q^2+1,q+1,1)$ 3-design. Since $M$ is a $(q^2+1,q+1,1)$ 3-design, by Proposition 1.3, we have

**Proposition 1.6.** $M$ is a $(b,r,\lambda_2,\lambda_3; k,\nu)$ 3-design with $b = q^3 + q$, $r = q^2 + q$, $\lambda_2 = q+1$, $\lambda_3 = 1$, $k = q+1$ and $\nu = q^2 + 1$. 

CHAPTER II
RECONSTRUCTION PROBLEMS

In this chapter we shall give a brief motivation of the work we have done in this thesis. The first three sections describe briefly the development of the reconstruction problems of graphs and the results obtained thus far. The last three sections are devoted to the discussion of the problems we are considering in this thesis. Theorem (3.1), Theorem (4.1) and Theorem (5.1) are the main results we have obtained.

§2.1 Reconstructions Based on Eigenvalues of Graphs. We have seen that from an arbitrary incidence structure, new incidence structures can always be derived. Hence, if we are given an incidence structure, we are interested in determining whether it is a derived structure, and if so, can we reconstruct the original structure. These reconstruction problems have been studied by many in recent years. The geometric properties of the various incidence structures are very often the means to the study of the subject.

Both A. J. Hoffman and D. K. Ray-Chaudhuri had done some important work in the development of the subject.
based on the eigenvalues of the graphs. Let $G$ be a simple graph with $v$ vertices. The \textit{adjacency matrix} of $G$ is defined to be a $(v \times v)$-matrix $A = (a_{ij})$, where $a_{ij} = 1$ or $0$ depending on whether the $i$-th vertex and the $j$-th vertex are adjacent or not, for all $i, j, i, j = 1, \ldots, v$. The eigenvalues of the adjacency matrix $A$ are called the \textit{eigenvalues} of the graph $G$.

If $D = (P, Q, I)$ is an incidence structure, then for every pair $(p, A) \in I$, $(p, A)$ is called a \textit{flag} of $D$. The \textit{flag-graph} $F(D)$ of the incidence structure $D$ is a graph whose vertex set is the set of flags in $D$, and two distinct vertices $(p, A)$ and $(p', A')$ are adjacent if and only if either $p = p'$ and $A \neq A'$, or $p \neq p'$ and $A = A'$. Hoffman and Ray-Chaudhuri gave a characterization of the flag-graph of an affine plane.

\textbf{Theorem 2.1.} (A.J. Hoffman and D.K. Ray-Chaudhuri [17])
Let $n$ be a positive integer. If $G$ is a regular, connected simple graph on $n^2(n+1)$ vertices with distinct eigenvalues $2n-1$, $-2$, $\frac{1}{2}(2n+3+\sqrt{4n+1})$, $\frac{1}{2}(2n+3-\sqrt{4n+1})$ and $n-2$, then there exists an affine plane $\pi$ such that $G$ is isomorphic to the flag-graph of $\pi$.

The same two authors also considered the flag-graph of a symmetric balanced incomplete block design, and showed that the flag-graph can be reconstructed from its distinct eigenvalues and connectedness. A balanced incomplete block design, abbreviated by BIBD, is a $S_\lambda(2,k,v)$ where $v > k > \lambda > 0$ and $v, k, \lambda$ are integers. A design is \textit{symmetric} if
simple graph with minimum eigenvalue equal to -2 such that the minimum degree of \( H \) is at least 46, and for any two distinct adjacent vertices \( x \) and \( y \) in \( H \) there exist at least two distinct vertices \( z \) and \( z' \) adjacent to \( x \), but not adjacent to \( y \), then \( H \) is isomorphic to \( L(G) \) for some simple graph \( G \).

Eigenvalues of graphs are important features in the study of the reconstructions of many graphs. If one studies the proofs of the above theorems, one notices that the notions of "claws" and "cliques" are heavily used. In fact, the geometries of claws and cliques form the bases for the reconstructions of many graphs, as we shall see in the following sections.

\[2.2 \text{ Geometrization Theorem.} \quad \text{One of the most important classes of graphs is the class of strongly regular graphs, which were introduced by Bose [2] in 1963. The characterization problems of various kinds of strongly regular graphs have been widely studied, and the most fundamental theorem in the subject is probably the Geometrization Theorem given below.} \]

Theorem 2.4. Geometrization Theorem. (Bose [2]) Let \( k > r > t > 1 \) be integers. A pseudo-geometric \((r,k,t)\)-strongly regular graph is \((r,k,t)\)-geometrizable if \( k > \frac{1}{2}(r(r-1)+t(r+1)(r^2-2r+2)) \).
and only if \( v = b \), where \( b \) is the number of blocks in the design.

**Theorem 2.2.** (A.J. Hoffman and D.K. Ray-Chaudhuri [16])

Let \( v > k > \lambda > 0 \) be integers and \((v,k,\lambda) \neq (4,3,2)\). If \( G \) is a regular, connected, simple graph with distinct eigenvalues \( 2k-2, -2, k-2+\sqrt{k-\lambda} \) and \( k-2-\sqrt{k-\lambda} \), then there exists a symmetric BIBD \( \pi \) such that \( G \) is isomorphic to the linegraph of \( \pi \). Further, if \((v,k,\lambda) = (4,3,2)\), then there exists exactly one exceptional graph (given in Figure 3).

**Figure 3.** Exceptional graph for \((v,k,\lambda) = (4,3,2)\)

Besides flag-graphs, the linegraphs of various incidence structures are of great interests to many. In particular, Ray-Chaudhuri proved an important theorem on the characterization of linegraphs.

**Theorem 2.3.** (Ray-Chaudhuri [19]) Let \( G \) be a finite simple graph. If the number of edges of \( G \) is greater than the number of vertices in \( G \), then the minimum eigenvalue of \( L(G) \) equals -2. Conversely, if \( H \) is a
simple graph with minimum eigenvalue equal to -2 such that the minimum degree of H is at least 46, and for any two distinct adjacent vertices x and y in H there exist at least two distinct vertices z and z' adjacent to x, but not adjacent to y, then H is isomorphic to L(G) for some simple graph G.

Eigenvalues of graphs are important features in the study of the reconstructions of many graphs. If one studies the proofs of the above theorems, one notices that the notions of "claws" and "cliques" are heavily used. In fact, the geometries of claws and cliques form the bases for the reconstructions of many graphs, as we shall see in the following sections.

§2.2 Geometrization Theorem. One of the most important classes of graphs is the class of strongly regular graphs, which were introduced by Bose [2] in 1963. The characterization problems of various kinds of strongly regular graphs have been widely studied, and the most fundamental theorem in the subject is probably the Geometrization Theorem given below.

Theorem 2.4. Geometrization Theorem. (Bose [2]) Let $k > r > t > 1$ be integers. A pseudo-geometric $(r,k,t)$-strongly regular graph is $(r,k,t)$ geometrizable if $k > \frac{1}{2}(r(r-1)+t(r+t)(r^2-2r+2))$. 
We simplify the proof of Bose to a certain extent.
First, let us consider the incidence graph G of an \((r,k,t)\)-partial geometry \(\Pi\). Every line \(L\) in \(\Pi\) gives rise to a clique \(L'\) in G. If \(\Sigma\) denotes the set of cliques in G which are derived from the lines \(L\) in \(\Pi\), then it is easy to see that \(\Sigma\) satisfies the following:

(E1) Any two adjacent vertices are contained in a unique clique in \(\Sigma\).

(E2) Each vertex is contained in \(r\) cliques of \(\Sigma\).

(E3) Each clique in \(\Sigma\) contains \(k\) vertices.

(E4) If \(L' \in \Sigma\) and \(p\) is a vertex not contained in \(L'\), then there are exactly \(t\) vertices \(q_1, \ldots, q_t\) in \(K\) such that \(p\) is adjacent to \(q_i\), for \(i = 1, \ldots, t\).

Lemma 2.5. Let \(G\) be an \((r,k,t)\)-strongly regular graph such that \(k > r > t\). If \(\Sigma\) is a set of cliques in \(G\) satisfying axioms (E1) and (E1), then \(\Sigma\) satisfies axioms (E3) and (E4).

Proof. Let \(L\) be a clique in \(\Sigma\) such that \(L\) contains \(k\) vertices and let \(\overline{L}\) denote the set of vertices not contained in \(L\). For every nonnegative integer \(x\), we define

\[\overline{L}_x = \{q \mid q \in \overline{L} \text{ and } q \text{ is adjacent to exactly } x \text{ vertices in } L\}.\]

and \(g(x) = |\overline{L}_x|\).

If \(v\) denotes the number of vertices in \(G\), then clearly,
If we count the number of pairs \((p, q)\) such that \(p \in L\), \(q \in \overline{L}\) and \(p\) and \(q\) are adjacent, then for every vertex \(p\) in \(L\), \(p\) is adjacent to every other vertex in \(L\); hence \(p\) is adjacent to only \(n-k+1\) vertices in \(\overline{L}\) where \(n\) is the vertex degree of \(G\). Thus, there are \(k(n-k+1)\) such pairs \((p, q)\). On the other hand, if \(q \in \overline{L}_x\), then \(q\) is adjacent to exactly \(x\) vertices in \(L\); hence the total number of pairs \((p, q)\) is

\[
\sum_{x=0}^{k} x g(x) = k(n-k+1). \quad (2.2)
\]

Next, we count the number of ordered triples \((p, p', q)\) where \(p \neq p'\), both \(p, p' \in L\), \(q \in \overline{L}\) and \(p, p'\) are both adjacent to \(q\). For every pair \((p, p')\) in \(L\) there exist \(\lambda-k+2\) choices of \(q\), where \(\lambda\) is the edge-degree of \(G\). Hence, there are \(k(k-1)(\lambda-k+2)\) such triples. On the other hand, if \(q \in \overline{L}_x\), then there are \(x(x-1)\) choices of \((p, p')\). Hence

\[
\sum_{x=0}^{k} x(x-1) g(x) = k(k-1)(\lambda-k+2). \quad (2.3)
\]

Using simple algebraic manipulations on Equations (2.1), (2.2) and (2.3), and the parameters \(v, n, \lambda\) given in (1.1), we obtain the following,

\[
\sum_{x=0}^{k} (x-t)^2 g(x) = 0. \quad (2.4)
\]

Therefore, \(g(x)=0\) for all \(x \neq t\), and \(g(t)=v-k\). In other words,
every vertex not contained in \( L \) is adjacent to exactly \( t \) vertices in \( L \). Thus, we have shown that every clique that contains \( k \) vertices satisfies (E4). We are left to prove that every clique in \( \Sigma \) contains \( k \) vertices.

Let \( L' \in \Sigma \). Suppose \(|L'| > k\) and let \( L \) be a clique contained in \( L' \) such that \(|L'| = k\). Let \( p \in L' - L \), \( p \) is adjacent to every vertex in \( L \); hence \( t = k \) and this contradicts the hypothesis \( k > t \). Now, it is sufficient to show that every clique in \( \Sigma \) contains at least \( k \) vertices. Suppose \( L \) is a clique in \( \Sigma \) such that \(|L| < k\). Let \( p \in L \) and let \( L_1, L_2, \ldots, L_r \) be the cliques in \( \Sigma \) containing \( p \). Since for every vertex \( q \) which is adjacent to \( p \), \( q \) is contained in exactly one of the cliques \( L, L_2, \ldots, L_r \), and \( p \) is adjacent to \( r(k-1) \) vertices, we have

\[
r(k-1) = (|L| - 1) + \sum_{i=2}^{r} (|L_i| - 1). \tag{2.5}
\]

But \(|L| < k\) and \(|L_i| < k\) for \( i = 2, \ldots, r \); hence \( r(k-1) < r(k-1) \) and this is impossible. Thus, every clique in \( \Sigma \) contains at least \( k \) vertices and the proof is complete.

It is obvious that any \((r,k,t)\)-strongly regular graph \( G \) which satisfies the hypothesis of the lemma is \((r,k,t)\)-geometrizable. Further, the point set of the corresponding partial geometry consists of the vertices of \( G \), and the line set of the geometry is \( \Sigma \).

Herewith, we shall use the notation \((p; S)\) to denote an \( s \)-claw in a graph \( G \), where \( S \) is an \( s \)-set of pairwise
nonadjacent vertices. Let $p(r,t) = \frac{1}{2}(r(r-1)+t(r+1)(r^2-2r+2))$.

Lemma 2.6. Let $G$ be an $(r,k,t)$-strongly regular graph with $k > r \cdot t > 1$ and $k > p(r,t)$. If $(p;S)$ is an $s$-claw in $G$, then the following hold:

1. $s < r$.
2. If $s < r$, then $(p;S)$ is not a maximal claw.
3. If $s = r-1$, then there are at least $k-1-(r-1)(t-1)$ vertices $u$ such that $(p;S \cup u)$ is an $r$-claw in $G$.
4. If $s = r$, then $f(1) \geq r(k-2)-r(r-l)(t-1)$ where $f(1)$ is the number of vertices $u$ in $\Delta(p)$ such that $u \notin S$ and $u$ is adjacent to exactly one vertex in $S$.

Proof. Let $T = \{u|u$ is adjacent to $p$ and $u \notin S\}$.

$$T_x = \{u|u \in T$ and $u$ is adjacent to exactly $x$ vertices in $S\}.$

and $$f(x) = |T_x|.$$

Since the sets $T_x$, $0 \leq x \leq s$, partition the set $T$, we have

$$\sum_{x=0}^{s} f(x) = n - s. \quad (2.6)$$

We then compute the number of ordered pairs $(u,u')$ such that $u \in S$, $u' \in T$ and $u$ is adjacent to $u'$. For every vertex in $S$, there are $s \lambda$ vertices $u'$ which are adjacent to both $p$ and $u$; thus there are $s \lambda$ such pairs $(u,u')$. On the other hand, if $u' \in T_x$, then there are $x$ choices of $u$. Hence

$$\sum_{x=0}^{s} xf(x) = s \lambda. \quad (2.7)$$
Next we count the number of triples \((q,q',u)\) where \(q \neq q'\), \(q,q' \in S\), \(u \in T\) and \(u\) is adjacent to both \(q\) and \(q'\). For every ordered pair \((q,q')\) in \(S\), \((q,q')\) is a nonedge; hence there exist at most \((\mu-1)\) choices of \(u\) which is adjacent to both \(q\) and \(q'\) where \(\mu\) is the nonedge degree of \(G\). Thus, the total number of triples \((q,q',u)\) is at most \(s(s-1)(\mu-1)\). On the other hand, if \(u \in T_x\), then there are \(x(x-1)\) choices of \((q,q')\). Therefore,

\[
\sum_{x=0}^{s} x(x-1)f(x) \leq s(s-1)(\mu-1) . \tag{2.8}
\]

To show (1), let us assume \(s=r+1\). From Equations (2.6), (2.7) and (2.8), we obtain

\[
f(0)+\frac{1}{2} \sum_{x=0}^{r+1} (x-1)(x-2)f(x) \leq -k+p(r,t) < 0 . \tag{2.9}
\]

But the left hand side of the inequality is nonnegative, thus \(s \leq r\). Now, Equations (2.6) and (2.7) together imply that

\[
f(0) > f(0) - \sum_{x=1}^{s} (x-1)f(x)
= (r-s)(k-1) - s(t-1)(r-1) . \tag{2.10}
\]

If \((p;S)\) is a maximal claw with \(s \leq r\), then \(f(0)=0\). Substituting \(f(0)=0\) into Equation (2.10), we find that

\[k \leq 1+(r-1)^2(t-1) < p(r,t)\]

and this contradicts our hypothesis. Thus, \((p;S)\) is not a maximal claw.
To show (3), let us substitute $s=r-1$ into Equation (2.10) and we obtain

$$f(0) \geq k-1-(r-1)^2(t-1). \quad (2.11)$$

Since every vertex $u$ in $T_0$ can be adjoined to $S$ to form an $r$-claw $(p;Swu)$ and $T_0$ has $f(0)$ vertices, (3) is clearly satisfied.

Finally, let us consider $s=r$. By (1), the claw $(p;S)$ is clearly maximal, hence $f(0)=0$. Thus, Equation (2.6) becomes

$$\sum_{x=1}^{r} f(x) = n-r = r(k-2). \quad (2.12)$$

Combining Equations (2.7) and (2.12), we have

$$r(k-2) - f(1) = \sum_{x=2}^{r} xf(x) \leq \sum_{x=2}^{r} (x-1)f(x) = r(r-1)(t-1). \quad (2.13)$$

This implies that $f(1) \geq r(k-2)-r(r-1)(t-1)$, and (4) is proved.

Next we shall construct the system of cliques in $\Sigma$ in a $(r,k,t)$-strongly regular graph $G$. A grand clique $L$ in $G$ is a maximal clique such that $|L| \geq k-(r-1)^2(t-1)$.

Lemma 2.7. Let $G$ be an $(r,k,t)$-strongly regular graph with $k>p(r,t)$. If $p$ and $q$ are two adjacent vertices in $G$, then $p$ and $q$ are contained in a unique grand clique of $G$. 
Proof. Let \((p; S)\) be an \((r-1)\)-claw in \(G\) that contains \(q\). Let \(T_q\) be the set as defined in Lemma (2.6). By (3) of Lemma (2.6), the vertices in \(T_q\) are pairwise adjacent and they are adjacent to \(p\) and \(q\). Hence there exists at least one grand clique containing \(p\) and \(q\). Suppose \(L\) and \(L'\) are two distinct grand cliques in \(G\) that contain \(p\) and \(q\). Since \(L\) and \(L'\) are maximal cliques, there exist two points \(x\) and \(y\) such that \(x \in L - L'\) and \(y \in L' - L\). Clearly, \(L \cap L' \subseteq (x, y)\), and

\[ |L \cap L'| \leq \mu = rt. \] (2.14)

On the other hand, \(L \cup L' \subseteq (p, q) \cup \{p, q\}\). Hence,

\[ |L \cup L'| \leq \lambda + 2 = k + (r-1)(t-1). \] (2.15)

Using Equations (2.14) and (2.15), we obtain

\[ |L| + |L'| = |L \cup L'| + |L \cap L'| \leq k + (r-1)(t-1) + rt. \] (2.16)

But both \(L\) and \(L'\) are grand cliques. Hence

\[ |L| + |L'| \geq 2(k - (r-1)^2(t-1)). \] (2.17)

From the last two equations, we observe that \(k \geq p(r, t)\), which contradicts the hypothesis of the lemma. Therefore every two adjacent vertices are contained in a unique grand clique./

In view of the previous lemma, it is only natural for us to define \(\Sigma\) to be the set of grand cliques in \(G\). It will
be sufficient to show that $\Sigma$ satisfies axioms (E1) and (E2). Since the lemma above shows that $\Sigma$ satisfies (E1), we are left to show that every vertex is contained in exactly $r$ grand cliques.

**Proof of Theorem 2.4.** Let $(p; s_1, \ldots, s_r)$ be an $r$-claw in $G$. By Lemma 2.7 every pair of adjacent vertices $(p, s_i)$, $i=1, \ldots, r$, is contained in a unique grand clique $L_i$; hence there are at least $r$ grand cliques containing the point $p$. Suppose $L'$ is another grand clique containing $p$. For every $j$, $j=1, \ldots, r$, let $H_j=\{u|u$ is adjacent to both $p$ and $s_j$, but $u$ is not adjacent to $s_i$, for $i \neq j$ and $1 \leq i \leq r\}$. Since $(p; s_1, \ldots, s_r)$ is a maximal claw, the vertices in $H_j$ are pairwise adjacent for $j=1, \ldots, r$. Thus, $H_j \cup \{p, q_j\} \subseteq L_j$ and the number of vertices adjacent to $p$ is

$$n = r(k-1) \geq \sum_{j=1}^{r} |L_j-p| + |L'-p|$$

$$\geq \sum_{j=1}^{r} |H_j| + r + |L'-p|$$

$$= f(1)+r+|L'-p|$$

$$= r(k-2)-r(r-1)(t-1)+r+k-(r-1)^2(t-1).$$

This implies that $k \leq p(r,t)$ and contradicts the hypothesis of the theorem. Thus the proof is complete. /

It should be noted that under the conditions of Theorem (2.4) on $(r,k,t)$-strongly regular graph, $G$ is in fact uniquely geometrizable; that is, if $\pi$ and $\pi'$ are two $(r,k,t)$-partial geometries whose incidence graphs are both
isomorphic to G, then π is isomorphic to π'.

For the characterization of strongly regular graphs, the Geometrization Theorem plays an important role. Bose and Laskar [4] proved a very general theorem which is very helpful for characterizing certain classes of graphs. Let $r, k, e, b \in \mathbb{N}_0$ such that $r \geq 1$ and $k \geq 2$. Let $G$ be a graph satisfying the following:

(F1) $G$ is regular of degree $r(k-1)$.

(F2) $G$ is edge-regular with edge-degree $k-2+e$.

(F3) For every nonedge $(x,y)$ in $G$, $\delta(x,y) \leq 1+b$.

(F4) $k > \max \{p(r,e,b), q(r,e,b)\}$ where

$$p(r,e,b) = \frac{1}{2}(r+1)(rb-2e) \quad \text{and} \quad q(r,e,b) = 1+b+(2r-1)e.$$  

We define a clique in $G$ to be a **grand clique** if it is both maximal and has size at least $k-(r-1)e$.

**Theorem 2.8.** (Bose-Laskar [4]). If $G$ is a graph satisfying (F1)-(F4), then any two adjacent vertices in $G$ are contained in exactly one grand clique, and each vertex of $G$ is contained in exactly $r$ grand cliques.

Based on this theorem, Bose and Laskar characterized tetrahedral graphs [4], and T. A. Dowling characterized cubic lattice graphs [8] and $T_m$ graphs [9]. Recently, Bose, Singhi and Shrikhande [5] generalized the notion of a partial geometry to a partial geometric design and proved a geometrization theorem regarding edge-regular multigraphs.
§2.3 Characterization of Graphs with the 4-Vertex Condition. In characterizing graphs of various nature, one often realizes that the fact the graph is an \((r, \ldots, t)\)-strongly regular graph does not usually lead to the result; other geometric properties are necessary. Higman [14] and Sims [24] studied a class of strongly regular graphs and observed that all the graphs under study satisfy the 4-vertex condition. Let \(x\) and \(y\) be two distinct vertices in \(G\), and let \(d(x,y)\) denote the number of edges of \(G\) whose endpoints are in \(\Delta(x,y)\). The 4-vertex condition requires that there exist integers \(a\) and \(b\) such that \(d(x,y) = a\) for any edge \((x,y)\) in \(G\) and \(d(x,y) = b\) for any nonedge \((x,y)\) in \(G\).

Let us first observe that if \(K_v\) denotes a complete graph on \(v\) vertices, then \(K_v\) is a strongly regular graph. Disjoint unions of \(K_v\) and the complements of these graphs are also strongly regular. We shall call this class of strongly regular graphs trivial strongly regular graphs. The following theorem gives a characterization of a large class of strongly regular graphs.

**Theorem 2.9.** (C.C. Sims [24]) Let \(m\) be a positive integer greater than 1. There exists a finite class of graphs \(\mathcal{A}\) such that if \(G\) is a nontrivial strongly regular graph with minimum eigenvalue equal to \(-m\) and satisfying the 4-vertex condition, then either \(G \not\in \mathcal{A}\) or \(G\) belongs to one of the 4 classes of graphs defined below:
(1) Linegraph of a complete graph on \( v \) vertices. 
(This is a \((2,v-1,2)\)-strongly regular graph.)

(2) Linegraph of a complete bipartite graph on \( v+v \) vertices. (This is a \((2,n,1)\)-strongly regular graph.)

(3) Linegraph of a projective space \( \text{PG}(d-1,q) \), where \( q \) is a prime power and \( d \) is a positive integer not less than 4. (This is a \((q+1,k,q+1)\)-strongly regular graph.)

(4) Graphs with vertex set \( V \times V \) where \( V \) is a \( d \)-dimensional vector space over \( \text{GF}(q) \), \( d \geq 2 \) and \( q \) is a prime power such that two distinct vertices \((x,y)\) and \((z,w)\) are adjacent if and only if \( x-z \) and \( y-w \) span a 1-dimensional subspace of \( V \). (This is a \((q+1,k,q)\)-strongly regular graph.)

Recently D.K. Ray-Chaudhuri and A.P. Sprague proved some interesting theorems similar to that of Sims'. Let \( V \) be a \( d \)-dimensional vector space over \( \text{GF}(q) \) where \( q \geq 2 \) and \( q \) is a prime power. Let \( W_i \) denote the set of \( i \)-dimensional subspaces of \( V \), \( 1 \leq i \leq d \). If \( s \) is a positive integer, \( 1 \leq s \leq d \), then \( (W_{s-1}, W_s) \) is an incidence structure. Any incidence structure \( \pi \) which is isomorphic to \((W_{s-1}, W_s)\) is called an \((s,q,d)\)-projective incidence structure, denoted by \( P(s,q,d) \). It is easily seen that the graphs in class (3) above are incidence graphs of \( P(2,q,d) \). In the case \( q=1 \), \( P(s,1,d) \) is defined as an incidence structure \((X_{s-1}, X_s)\) where \( X_i \) is the
set of i-subsets of \( X \), \( i=s-1,s \).

Let \( \pi=(P,\mathcal{L}) \) be a finite incidence structure. Let
\[
r(\pi) = \min \{ \delta(p) | p \in P \}
\]
and
\[
k(\pi) = \min \{ \delta(L) | L \in \mathcal{L} \}.
\]
If \( q=1 \), then let \( s(\pi,q) = k(\pi) \); otherwise, let \( s(\pi,q) \) denote the unique
real number \( s \) that satisfies \( q^{s-1} = k(\pi)(q-1) \). Let \( d(\pi,q) \) be
the unique real number \( d \) satisfying \( q^{d-s(\pi,q)+1}-1 = (q-1)r(\pi) \),
for \( q \geq 2 \). If \( q=1 \), then \( d(\pi,q) = r(\pi) + s(\pi,1) - 1 \).

**Theorem 2.10.** (D.K. Ray-Chaudhuri and A.P. Sprague [21]). Let \( q \geq 1 \) be an integer and \( \pi \) be a finite
incidence structure which satisfy the following:

1. \( s \leq s(\pi,q) < d(\pi,q)-1 \).
2. There exists at most one line joining two distinct
   points.
3. Let \( p \) be a point and \( L \) be a line. If \( p \not\in L \) and
   there exists a point \( p' \) in \( L \) such that \( p \) and \( p' \)
   are collinear, then there exist exactly \( q+1 \) lines
   which contain \( p \) and intersect \( L \).
4. Let \( p \) and \( p' \) be two distinct noncollinear points.
   If there exists a point \( p'' \) such that \( p,p'' \) and \( p', p'' \)
   are both collinear pairs, then there exist
   exactly \( (q+1) \) lines \( L \) such that \( L \) contains \( p \) but
   not \( p' \), and \( p' \) is collinear with some point \( \overline{p} \) in
   \( L \).
5. The incidence graph of \( \pi \) is connected.

Then \( s = s(\pi,q), \ d = d(\pi,q) \) are both integers, and \( q=1 \) or
q is a prime power; further, π is a P(s,q,d). Conversely, for 3 ≤ s ≤ d-1, any P(s,q,d) satisfies (1)-(5).

The 4-vertex condition plays an important role in the characterization of many graphs, in particular, graphs connected with projective spaces. However, for many other graphs, the 4-vertex condition is not found, and other geometric configurations have to be explored. E.E. Shult [23] had considered triangles in a graph and the vertices adjacent to a triangle. In this thesis, we also consider triangles and 2-claws for the characterization of the linegraph of an affine space.

2.4 Linegraph of an Affine Space. We have seen in Section 1.5 that the linegraph of an affine space is a (q,k,q)-strongly regular graph. Conversely, given a (q,k,q) strongly regular graph we would like to know if it is the linegraph of an affine space. From Sims' Theorem, we observe that both the linegraphs of complete graphs and the linegraphs of projective spaces are (q,k,q)-strongly regular graphs, with q=2 for the former case, and q=q'+1 for the latter, where q' is the order of the field. Hence, it is obvious that we need to find some other geometric properties in order to characterize the linegraph of affine spaces.

Proposition 2.11. Let Γ be an affine space and L(Γ) be the linegraph of Γ. If (L,M,N) is a triangle in L(Γ), then δ(L,M,N) either equals q(q-2) or is at
least $k-3$.

Proof. A triangle $(L,M,N)$ in $L(\Gamma)$ corresponds to three intersecting lines in $\Gamma$. Either the three lines all meet at a point $p$, or they meet pairwise at 3 different points $x$, $y$ and $z$. In the former case, every line that contains $p$ intersects $L$, $M$ and $N$ at $p$. Since there are $k-3$ other lines containing $p$, the triangle degree of $(L,M,N)$ is at least $k-3$.

In the latter case, $L$, $M$ and $N$ are coplanar. Let us assume that $x$ and $y$ are the points of intersection of $L$, $M$ and $L$, $N$ respectively. Consider a point $w$ in $L$, distinct from $x$ and $y$; there are $q-1$ lines other than $L$ containing $w$ and intersecting $N$. But one of these lines is parallel to $M$. Hence there exist only $q-2$ lines containing $w$ and intersecting both $M$ and $N$. If $w=x$, then any line containing $x$ and intersecting $N$ intersects $L$, $M$ and $N$, and there are $q-2$ such lines. Similarly for $w=y$, there are $q-2$ lines containing $y$ and intersecting $L$, $M$ and $N$. Thus there are $q(q-2)$ lines that intersect $L$, $M$ and $N$; so $\delta(L,M,N)=q(q-2)$.

Figure 4. Illustration for Proposition 2.11

**Proposition 2.12.** Let $\Gamma$ be an affine space and $L(\Gamma)$ be the linegraph of $\Gamma$. If $(L,M,N)$ is a 2-claw in $L(\Gamma)$,
then $\delta(L,M,N)$ is either $q(q-1)$ or $2(q-1)$.

Proof: To compute $\delta(L,M,N)$ is to compute the number of transversals of the three lines in $\Gamma$. If $M$ is parallel to $N$, then $L$, $M$ and $N$ are coplanar. Every line that intersects $M$ and is not parallel to $L$ is a transversal of $L$, $M$ and $N$. Since there are $q(q-1)$ such transversals $\delta(L,M,N)=q(q-1)$.

If $M$ is not parallel to $N$, then $L$, $M$ and $N$ are non-coplanar. Let $x$ and $y$ denote the points of intersection of $L$ and $M$, and $L$ and $N$ respectively. Any line that contains $x$ and intersects $N$, intersects $L$, $M$ and $N$. There are $q-1$ such lines. Similarly there are $q-1$ transversals of $L$, $M$ and $N$ that contain the point $y$. Hence $\delta(L,M,N) \geq 2(q-1)$. If there exists another transversal $T$, then $(T,M,L)$ and $(T,N,L)$ are triangles in $L(\Gamma)$, that is, they are coplanar. But this contradicts the fact that $M$ is not parallel to $N$. Hence $\delta(L,M,N)=2(q-1)$.

From these two propositions, we see that the linegraph of an affine space is a $(q,k,q)$-strongly regular graph with certain "regularities" concerning the triangles and the
2-claws. Based on these properties we are able to show the following:

**Theorem 3.1.** Let G be a \((q,k,q)\)-strongly regular graph with \(q \geq 4\). Suppose G satisfies the following:

1. \(k > \frac{1}{2}(q(q-1)+q(q+1)(q^2-2q+2))\).
2. For every triangle \((A,B,C)\) in G, \(\delta(A,B,C)\) either equals \(q(q-2)\) or is at least \(k-3\).
3. For every 2-claw \((A,B,C)\) in G, \(\delta(A,B,C)\) equals either \(q(q-1)\) or \(2(q-1)\).

Then G is isomorphic to the linegraph of an affine space \(AF(q,n)\). Furthermore, \(q\) is a prime power and \(k=(q^n-1)/(q-1)\).

### 2.5 Block-Residual Designs

Besides characterizing graphs, another reconstruction problem of great interest is the characterization of pseudo-residual designs. We have seen in the previous chapter that residual designs can be obtained from a given design by deleting either a block or a point. Usually one considers residual designs of a highly symmetric design. A design whose parameters are the same as those of a residual design is called a pseudo-residual design. There had been many investigations in which one studies the question whether a pseudo-residual design is isomorphic to a residual design. Equivalently one finds conditions under which a pseudo-residual design can be embedded into the parent symmetric design.
Definition. Let $D'$ be a pseudo-residual design. $D'$ is said to be embeddable if and only if there exists a design $D$ such that the residual design $D^*$ obtained from $D$ is isomorphic to $D'$.

Let us recall that a symmetric design is a $S(2,k,v)$ with $v=b$. If $\lambda=1$ then a symmetric design is a projective plane; the residual design of a projective plane can be easily checked to be an affine plane. It is well-known that an affine plane can be embedded into a projective plane. W.S. Connor and M. Hall proved the embedding theorem for $\lambda=2$.

Theorem 2.13. (W.S. Connor and M. Hall [13]). Let $v'$, $k'$, $\lambda'$ be positive integers satisfying $k'(k'-1) = \lambda'(v'-1)$. If $D$ is an $S(b,r,\lambda,2,k,v)$ with $b=v'-1$, $r=k'$, $k=k'-\lambda'$, $v=v'-k'$ and $\lambda'=1$ or 2, then $D$ can be embedded as a block-residual design in a symmetric design with parameters $v', k', \lambda'$.

Recently, Bose, Singhi and Shrikhande [5] extended the result to $\lambda \geq 3$. They showed that if $D$ is an $S(2,k,v)$ with $k$ greater than a certain function of $\lambda$, then $D$ is uniquely embeddable into a symmetric design.

In this thesis we are interested in embedding the pseudo-residual designs into a Möbius plane. First, we have to compute the parameters of these residual designs. We shall do so for the block-residual designs in this section.
followed by the point-residual designs in the next section.

Let \( M \) be a Möbius plane, that is, an \( S(3,q+1,q^2+1) \). Blocks of \( M \) will be called circles also. Let \( M' \) be the block-residual design obtained from \( M \) by deleting a block \( A \). Clearly, \( M' \) is a 3-design with \( \lambda_2=1 \). We now study some of the properties of the residual design \( M' \).

**Proposition 2.15.** \( M' \) is a 2-design with \( \lambda_2=q+1 \).

Proof: It suffices to show that \( M \) is a 2-design with \( \lambda_2=q+1 \). From Proposition 1.3, it is clear that \( \lambda_2=q+1 \).

**Proposition 2.16.** \( M' \) is an \( S'\), \( r, \lambda_2, \lambda_3; K, v \) with \( b=q^3+q-1, r=q^2+q, \lambda_2=q+1, \lambda_3=1, K=\{q+1, q, q-1\} \) and \( v=q^2-q \).

Proof. From Proposition 1.3, \( M \) is a \( (q^3+q, q^2+q, q+1, 1; q+1, q^2+1) \) 3-design. Since \( M' \) is obtained from \( M \) by deleting a block \( A \), it is clear that \( b=q^3+q-1, r=q^2+q \) and \( v=q^2-q \). Now \( M \) is a 3-design with \( \lambda_3=1 \), any two distinct blocks in \( M \) cannot have more than 2 points in common. Thus for any block \( B \) in \( M \), \( |B \cap A| \) takes values of 0, 1 and 2. Hence the residual block sizes take values of \( q+1, q \) and \( q-1 \).

Besides being a 3-design, a Möbius plane \( M \) possesses interesting geometric properties concerning the blocks. Let \( B \) be a block in \( M \). A block \( B' \) is said to be tangent to \( B \) at a point \( x \) if and only if \( \Delta(B,B')=\{x\} \). \( B' \) is said to be secant to \( B \) if and only if \( \delta(B,B')=2 \). The set of circles
tangent to a given circle $B$ at a point $x$ satisfies the following:

**Proposition 2.17.** Let $B$ be a block in $M$ and $x$ be a point incident with $B$. If $C$ and $C'$ are two distinct blocks tangent to $B$ at $x$, then $C$ and $C'$ are mutually tangent at $x$.

**Proof.** We shall first show that given any point $z$ not in $B$, there exists a unique block $C$ incident with $z$ and tangent to $B$ at $x$. Consider a point $y$ in $B$, distinct from $x$, the three points $x$, $y$, and $z$ determine a unique block. Since there are $q$ such points $y$ in $B$, there are $q$ blocks incident with $x$ and $z$, and secant to $B$. But $\lambda_2=q+1$ hence there exists a unique block incident with both $x$ and $z$ and tangent to $B$ at $x$.

Since each block is incident with $q+1$ points, and there are $q^2+1$ points in $M$, it is obvious that there are $q-1$ blocks tangent to $B$ at $x$ and they are mutually tangent.

We shall call a maximal set of mutually tangent circles at a point $x$ to be a **pencil** with **carrier** $x$. From the proposition, we get

**Proposition 2.18.** Each pencil in $M$ consists of $q$ circles.

**Proof.** Let $A$ be a fixed circle in a pencil with carrier $x$. We count the number of ordered pairs $(y,B)$ such that $y\neq x$, $y \in B$ and $B=A$ or $B$ is tangent to $A$ at $x$. For every point $y$
Definition. Let $D'$ be a pseudo-residual design. $D'$ is said to be embeddable if and only if there exists a design $D$ such that the residual design $D^*$ obtained from $D$ is isomorphic to $D'$.

Let us recall that a symmetric design is a $S_{\lambda}(2,k,v)$ with $v=b$. If $\lambda=1$ then a symmetric design is a projective plane; the residual design of a projective plane can be easily checked to be an affine plane. It is well-known that an affine plane can be embedded into a projective plane. W.S. Connor and M. Hall proved the embedding theorem for $\lambda=2$.

Theorem 2.13 (W.S. Connor and M. Hall [13]). Let $v'$, $k'$, $\lambda'$ be positive integers satisfying $k'(k'-1) = \lambda'(v'-1)$. If $D$ is an $S(b,r,\lambda_2,k,v)$ with $b=v'-1$, $r=k'$, $k=k'-\lambda'$, $v=v'-k'$ and $\lambda_2=1$ or 2, then $D$ can be embedded as a block-residual design in a symmetric design with parameters $v',k',\lambda'$.

Recently, Bose, Singhi and Shrikhande [5] extended the result to $\lambda \geq 3$. They showed that if $D$ is an $S_{\lambda}(2,k,v)$ with $k$ greater than a certain function of $\lambda$, then $D$ is uniquely embeddable into a symmetric design.

In this thesis we are interested in embedding the pseudo-residual designs into a M"obius plane. First, we have to compute the parameters of these residual designs. We shall do so for the block-residual designs in this section.
tangent to a given circle B at a point x satisfies the following:

**Proposition 2.17.** Let B be a block in M and x be a point incident with B. If C and C' are two distinct blocks tangent to B at x, then C and C' are mutually tangent at x.

Proof. We shall first show that given any point z not in B, there exists a unique block C incident with z and tangent to B at x. Consider a point y in B, distinct from x, the three points x, y and z determine a unique block. Since there are q such points y in B, there are q blocks incident with x and z, and secant to B. But \( \lambda_2 = q + 1 \) hence there exists a unique block incident with both x and z and tangent to B at x.

Since each block is incident with q+1 points, and there are \( q^2 + 1 \) points in M, it is obvious that there are q-1 blocks tangent to B at x and they are mutually tangent./

We shall call a maximal set of mutually tangent circles at a point x to be a pencil with carrier x. From the proposition, we get

**Proposition 2.18.** Each pencil in M consists of q circles.

Proof. Let A be a fixed circle in a pencil with carrier x. We count the number of ordered pairs \((y,B)\) such that \( y \neq x, y \in B \) and \( B = A \) or B is tangent to A at x. For every point y
there exists a unique choice of \( B \) and there are \( q^{2} + 1 - 1 \) points in \( M \) distinct from \( x \). Hence the number of ordered pairs is \( q^{2} \).

On the other hand, for every block \( B \) in the pencil, there are \( q \) choices of \( y \). Therefore

\[
(Number \ of \ blocks \ in \ a \ pencil) \cdot q = q^{2}.
\]

Equivalently, there are \( q \) blocks in a pencil.

It is obvious that if two blocks \( B \) and \( B' \) in \( M \) are tangent to each other at a point \( x \) and \( x \) is not a point in the deleted block \( A \), then \( B \) and \( B' \) are tangent to each other in \( M' \). However, if \( B \) and \( B' \) are secant to each other in \( M \) and if exactly one of their points of intersection is contained in the deleted block \( A \), then \( B \) and \( B' \) are also tangent to each other in \( M' \). We shall call this type of tangency to be \( r \)-tangency.

**Proposition 2.19.** Let \( B \) be a \( q \)-valent block in \( M' \) and let \( x \) and \( y \) be two distinct points not in \( B \). There exists at most one block \( C \) such that \( C \) is \( r \)-tangent to \( B \) and \( C \) contains both \( x \) and \( y \) (The \( r \)-Tangency Condition).

Proof. If \( C \) is \( r \)-tangent to \( B \), then \( C \) must contain the deleted point \( x' \) in \( B \). Since \( x \), \( y \) and \( x' \) are three distinct points, there exists a unique block containing them. Hence there exists at most one block \( C \) such that \( C \) is \( r \)-tangent to
Let us consider two blocks $B$ and $C$ in $M'$ such that $B$ is $r$-tangent to $C$ at a point $x$. One can easily check that there exists a $(q+1)$-valent block $A$ such that $A$ is tangent to $B$ at $x$ and is secant to $C$. Based on this, we generalize the definition of $r$-tangency to an arbitrary design.

**Definition.** Let $D$ be an $S\lambda(t,K,v)$ with $K=\{k+1,k,k-1\}$. A block $B$ in $D$ is said to be $r$-tangent to another block $B'$ at a point $x$ if and only if there exists a $(k+1)$-valent block $A$ such that $A$ is tangent to $B$ at $x$ and is secant to $B'$.

In Chapter IV we show the following:

**Theorem 4.1.** Let $q^2$ be a positive integer. If $M''$ is an $S(b,r,A_2,A_3;K,v)$ with $b=q^3+q-1$, $r=q^2+q$, $A_2=q+1$, $A_3=q$, $K=\{q+1,q,q-1\}$ and $v=q^2-q$ such that $M''$ satisfies the $r$-tangency condition, then $M''$ can be uniquely embedded into a Möbius plane. Conversely, if $M$ is an $S(3,q+1,q^2+1)$, then a block-residual design $M'$ of $M$ satisfies the $r$-tangency condition.

§2.6 **Point-Residual Designs.** Dual to the embedding problems of block-residual designs are the embedding problems of point-residual designs.

Let $M$ be a Möbius plane with $v=q^2+1$, $k=q+1$, $A=1$. Let $M^*$ be a point-residual design obtained from $M$ by deleting
a point $x$. It can be easily seen that $M^*$ is not a 3-design. However, it is a 2-design.

**Proposition 2.20.** $M^*$ is an $S_q(2,q+1,q^2)$.

**Proof.** Since $M$ is an $S(3,q+1,q^2+1)$, by Proposition 1.4 we immediately get the result.

**Proposition 2.21.** Let $A$ be a block in $M^*$. If $x$ and $y$ are two distinct points such that $x \in A$ and $y \notin A$, then there exists at most one block containing $y$ and tangent to $A$ at $x$ (*The Tangency Condition*).

**Proof.** By Proposition 2.17, we see that the Möbius plane $M$ satisfies the tangency condition; hence $M^*$ satisfies the tangency condition also.

In Chapter V, we prove the following,

**Theorem 5.1.** Let $D$ be an $S_q(2,q+1,q^2)$. $D$ is uniquely embeddable into a Möbius plane if and only if $D$ satisfies the tangency condition.
CHAPTER III
CHARACTERIZATION OF LINEGRAPH
OF AN AFFINE SPACE

§3.1 Statement of Theorem. We have seen in Chapter II that the linegraph of an affine space is a \((q,k,q)\)-strongly regular graph. Further, we know that an arbitrary \((q,k,q)\)-strongly regular graph is not necessarily derived from the lines of an affine space. Hence we study the triangle-degrees and the 2-claw degrees of the linegraph of an affine space and arrive at the results as stated in Propositions 2.11 and 2.12. Again we ask the question whether a \((q,k,q)\)-strongly regular graph having triangle-degrees and 2-claw degrees as those given in Proposition 2.11 and 2.12 is isomorphic to the linegraph of an affine space. In this chapter, we devote our attention to this characterization question.

For the case where \(q=1\), a \((1,k,1)\)-strongly regular graph consists of a singleton point, and is a trivial graph. For \(q=2\), the dual of a \((2,k,2)\)-strongly regular graph is simply a complete graph on \(k\) vertices. If a \((2,k,2)\)-strongly regular graph is isomorphic to the linegraph of an affine space \(AF(2,n)\), then we have seen in Chapter I that
k=2\(^n\)-1. Since complete graphs of arbitrary sizes \(k\) exist, not every \((2,k,2)\)-strongly regular graph is isomorphic to the linegraph of an affine space \(AF(2,n)\). For \(q=3\), the Hall matroid on 81 points, denoted by \(H(81)\), constructed by Hall [11], has the properties that the linegraph of \(H(81)\) is a \((3,k,3)\)-strongly regular graph, and each plane is isomorphic to an affine plane \(AF(3,2)\), but \(H(81)\) is not isomorphic to an affine space \(AF(3,3)\). Thus we see that for \(q=2,3\), the answer to our characterization question is negative. However, for \(q\geq 4\), we have the following result.

**Theorem 3.1.** Let \(q\geq 4\). Suppose \(G\) is a \((q,k,q)\)-strongly regular graph satisfying the following:

\((G1)\) \(k > \frac{1}{2}(q(q-1)+q(q+1)(q^2-2q+2))\).

\((G2)\) For every triangle \((A,B,C)\) in \(G\), \(\delta(A,B,C)\) either equals \(q(q-2)\) or is at least \(k-3\).

\((G3)\) For every 2-claw \((A;B,C)\) in \(G\), \(\delta(A,B,C)\) equals either \(2(q-1)\) or \(q(q-1)\).

Then \(G\) is isomorphic to the linegraph of an affine space \(AF(q,n)\). Furthermore, \(q\) is a prime power and \(k=(q^n-1)/(q-1)\).

We shall prove the theorem by first defining parallel vertices in \(G\), followed by constructing affine planes on its dual. Finally, using Buekenhout's Theorem [6], we obtain the result.
§3.2 Buekenhout's Theorem. In this section we shall show the characterization of affine spaces based on lines, which is given by Buekenhout [6]. The proof we give here differs slightly from that given by Buekenhout.

Theorem 3.2. (Buekenhout [6]). Let $P$ be a linear space with points, lines and planes satisfying the following:

(H1) Every 3 noncollinear points determine a unique plane.

(H2) Every plane of $P$ is an affine plane.

(H3) There exists three noncollinear points.

(H4) Every line contains at least four points.

Then $P$ is an affine space.

With these hypotheses, it is clear that $P$ satisfies all the axioms given in the definition of an affine space except the one axiom concerning a 3-space. Thus to show that $P$ is an affine space is to show that any two intersecting planes in a 3-space intersect each other at more than two points. To do this, we compute the number of points in a 3-space.

Lemma 3.3. Let $\pi$ be an affine plane such that every line in $\pi$ contains $q$ points. If $x$ is a point in $\pi$, then $x$ is contained in exactly $q+1$ lines in $\pi$.

Proof. Let $L$ be a line in $\pi$ that contains $x$ and let $L'/L$. Since $L$ is the unique line containing $x$ that is parallel to $L'$, every other line containing $x$ intersects $L'$. L'
contains q points; hence x is contained in q+1 lines.

Lemma 3.4. Let \( \pi \) be an affine plane of order q. If \( L \) is a line in \( \pi \) then \( L \) is parallel to exactly q lines in \( \pi \).

Proof. Let \( L \) and \( L' \) be two intersecting lines in \( \pi \). For every point \( x \) in \( L' \), there exists a unique line \( L_x \) containing \( x \) which is parallel to \( L \). Hence there are at least q lines parallel to \( L \). Since every line that is parallel to \( L \) intersects \( L' \), these are the only q lines parallel to \( L \).

Lemma 3.5. Let \( \pi \) be an affine plane. If every line in \( \pi \) contains q points, then \( \pi \) contains \( q^2 \) points, i.e. \( |\pi| = q^2 \).

Proof. Let \( L \) be a fixed line in \( \pi \). We count the number of ordered pairs \((x, L')\) where \( x \) is a point not contained in \( L \), \( L' \) is a line containing \( x \), and \( L' \parallel L \). For every point \( x \) not in \( L \), there exists a unique line \( L' \) containing \( x \) which is parallel to \( L \). Hence there are \(|\pi|-q \) such pairs. On the other hand, if \( L' \) is a line parallel to \( L \) and is distinct from \( L \), then there are \( q \) choices of \( x \). From the previous lemma, there are \((q-1)\) lines distinct from \( L \) that are parallel to \( L \). Hence,

\[ |\pi|-q = q(q-1). \]

This implies that \(|\pi|=q^2\).
Let us now consider a 3-space \( V \). \( V \) is generated by four noncoplanar points, \( a, b, c \) and \( d \). Let \( \pi \) denote the affine plane generated by \( a, b \) and \( c \) and let \( L \) be the line containing \( a \) and \( d \); clearly \( L \) does not lie in \( \pi \). By Lemma 3.3, \( a \) is contained in \( q+1 \) lines, \( L_1, \ldots, L_{q+1} \), that lie in \( \pi \). Let \( \pi_i = \langle L_i, L \rangle \), \( i=1, \ldots, q+1 \), be \( q+1 \) distinct affine planes that contain \( L \). We shall consider the set of points in \( V' = \bigcup_{i=1}^{q+1} \pi_i \), and show that \( V' \) is the set of all points of \( V \).

**Lemma 3.6.** \( V' \) is a linear space.

Proof. To show \( V' \) is a linear space is to show that if \( x \) and \( y \) are two distinct points in \( V' \), then the line containing \( x \) and \( y \) is contained in \( V' \). Before we proceed with the proof, we shall introduce some notations for easier reference.

Let \( x \in V' \). Then \( x \) is contained in \( \pi_i \) for some \( i, 1 \leq i \leq q+1 \). If \( x \notin L \), there exists a unique line in \( \pi_i \) which is parallel to \( L_i \) and intersects \( L \) at a point \( d(x) \). Let \( p \) be a point in \( L \). If \( x \notin L \), then the line \( \langle x, p \rangle \) lies in \( \pi_i \); furthermore if \( p \neq d(x) \), then \( \langle x, p \rangle \) intersects \( L_1 \) at a unique point \( p(x) \). Clearly, \( p(x) \in \pi \), (see Figure 6). Also, if \( p(x) \neq p(y) \), then \( \langle p(x), p(y) \rangle \notin \pi \).

![Figure 6. Definitions of \( d(x) \) and \( p(x) \)](image)
Let $x$ and $y$ be two distinct points in $V'$. We have to show that $\langle x, y \rangle \subseteq V'$. If $x, y$ and $L$ are coplanar, then $x, y$ and $L$ are contained in $\pi_i$ for some $i$, $1 \leq i \leq q + 1$; hence $\langle x, y \rangle \subseteq \pi_i \subseteq V'$ and we are done. Henceforth, we shall assume that $x, y$ and $L$ are noncoplanar, that is, $x$ and $y$ are contained in 2 distinct $\pi_i$'s, say $\pi_1$ and $\pi_2$. It is noteworthy that for any point $p$ in $L$ which is distinct from $a$, $d(x)$ and $d(y)$, $p(x)$ and $p(y)$ are distinct points in $L_1$ and $L_2$ respectively. Furthermore, the points $a$, $p(x)$ and $p(y)$ are non-collinear.

Let $z$ be a point in $\langle x, y \rangle$. If $\langle p, z \rangle$ intersects $\langle p(x), p(y) \rangle$ at a point $k$ for some $p$ in $L$ such that $p$ is distinct from $a$, $d(x)$ and $d(y)$, then $k \in \pi_i$; hence, $k \in L_i$ for some $i$, $1 \leq i \leq q + 1$. Since both points $k$ and $p$ are contained in $\pi_i$, the line $\langle p, k \rangle$ lies in $\pi_i$. But the point $k$ is in $\langle p, z \rangle$, so $\langle p, z \rangle = \langle p, k \rangle$ and we have that $z \in \pi_i$ and $z \in V'$. (See Figure 7)

![Figure 7. $\langle p, z \rangle = \langle p, k \rangle$](image)

From the remark above, we see that if $\langle x, y \rangle \parallel \langle p(x), p(y) \rangle$ for some $p$ in $L$ that is distinct from $a$, $d(x)$ and $d(y)$, then for every point $z$ in $\langle x, y \rangle$, $\langle p, z \rangle$ naturally intersects $\langle p(x), p(y) \rangle$; hence $\langle x, y \rangle \subseteq V'$. Thus we may assume that $\langle x, y \rangle$
is not parallel to \( <p(x),p(y)> \) for all points \( p \) in \( L \) that are distinct from \( a, d(x) \) and \( d(y) \). Since \( <x,y> \) is not parallel to any line \( <p(x),p(y)> \) and \( <x,y> \) is not contained in \( \pi \), the line \( <x,y> \) intersects \( \pi \) at a unique point; let \( w \) denote this unique point of intersection. Let us also note that if \( <x,y> \) is not parallel to \( <p(x),p(y)> \) for all \( p \) in \( L \), \( p \neq a,d(x),d(y) \), then there exists at most one point \( u \) in \( <x,y> \) such that \( u \notin V' \), namely the point \( u \) in \( <x,y> \) such that \( <p,u> \parallel <p(x),p(y)> \). We will show that either \( u \) lies in \( V' \) or we will arrive at a contradictory statement.

Case 1. \( d(x)=d(y) \): Consider the plane \( \pi^* \) which is generated by \( x, y \) and \( d(x) \). Since \( w \in <x,y> \) and \( w \in \pi \), \( w \in \pi \cap \pi^* \). Suppose there exists a point \( z \) in \( \pi \cap \pi^* \) such that \( z \neq w \). The line \( <z,w> \) lies in \( \pi \) and is therefore disjoint from \( <x,d(x)> \); hence \( <z,w> \) is parallel to \( <x,d(x)> \) in \( \pi^* \). Similarly, \( <z,w> \) is parallel to \( <y,d(x)> \) in \( \pi^* \), but this contradicts the fact that \( <x,d(x)> \) is not parallel to \( <y,d(x)> \). Thus, \( w \) is the unique point of intersection of \( \pi \) and \( \pi^* \).

Let \( x' \) be a point in \( <x,d(x)> \) such that \( x' \neq x \) and \( x' \neq d(x) \). The line \( <x',y> \) is distinct from \( <x,y> \); hence \( <x',y> \) does not contain the point \( w \) in \( <x,y> \). But \( <x',y> \) lies in \( \pi^* \) which intersects \( \pi \) at \( w \) only. So, \( <x',y> \not\parallel \emptyset \).

Since \( <x',y> \) and \( \pi \) are disjoint, \( <x',y> \) is parallel to \( <p(x),p(y)> \) for some \( p \) in \( L \), \( p \neq a,d(x'),d(y) \). By previous remark, \( <x',y> \) lies in \( V' \). Thus the only points in \( \pi^* \) that may not be contained in \( V' \) are those contained in the lines
\( \langle x, y \rangle \) and \( \langle d(x), y \rangle \). However, \( u \) is the only such point.

Consider any point \( z \) in \( \langle u, d(x) \rangle \) such that \( z \neq u \) and \( z \neq d(x) \). Since \( z \in \pi^* \), \( z \) is contained in \( V' \), thus the plane \( \pi' \) generated by \( z \) and \( L \) is also contained in \( V' \). But the points \( z \) and \( d(x) \) are contained in \( \pi' \). It follows that the line \( \langle z, d(x) \rangle \) which is the same line as \( \langle u, d(x) \rangle \), is contained in \( \pi' \). Hence \( u \) is contained in \( \pi' \), and is therefore in \( V' \).

Case 2. \( d(x) \neq d(y) \): Let \( p \) be any point in \( L \) that is distinct from \( a \). Clearly there exists at least a point \( z \) in \( \langle x, y \rangle \) such that \( d(z) \neq p \). Let us consider the plane \( \pi' \), generated by the 3 noncollinear points \( x, y \) and \( p \). The plane \( \pi' \) contains \( x \) and \( y \), so it also contains \( w \) and \( z \); hence \( p(z) \) is a point contained in both \( \pi' \) and \( \pi \). But \( w \) is also contained in \( \pi \), so we have that \( \langle w, p(z) \rangle \in \pi \cap \pi' \); in fact, \( \pi \cap \pi' = \langle w, p(z) \rangle \).

For every 2 distinct points \( z \) and \( z' \) in \( \langle x, y \rangle \) such that \( z \) and \( z' \) are distinct from \( w \) and \( u \), the points \( p(z) \) and \( p(z') \) are both contained in \( \pi \cap \pi' \); hence \( \langle w, p(z) \rangle = \langle w, p(z') \rangle \). Thus, we have observed that the line \( \langle w, p(z) \rangle \) is independent of the choice of \( z \) and we use \( L(p) \) to denote the line \( \langle w, p(z) \rangle \).

Let \( \langle p, p' \rangle \) be the unique line that contains \( p \) and is parallel to \( \langle x, y \rangle \), and let \( p' \) be the point of intersection of \( \langle p, p' \rangle \) and \( L(p) \). Consider any line \( L_i \) in \( \pi \) such that \( L_i \) intersects \( L(p) \) at a point \( v \), distinct from \( p' \). Since \( \langle p, p' \rangle \) is the unique line containing \( p \) which is parallel to \( \langle x, y \rangle \), the line \( \langle p, v \rangle \) intersects \( \langle x, y \rangle \); so does the plane
\[\pi_1\] which contains \(<p,v>\). Therefore there exists at most two lines \(L_1\) and \(L_j\) that contain \(a\) and whose corresponding planes \(\pi_1\) and \(\pi_j\) are disjoint from \(<x,y>\); namely the lines \(<a,p'>\) and the unique line \(M_1(p)\) which is parallel to \(L(p)\) and contains \(a\). We shall denote the line \(<a,p'>\) by \(M_2(p)\).

If \(u\) is not contained in \(V'\), then \(<x,y>\) has exactly \(q-1\) points contained in \(V'\). But there are \(q+1\) planes \(\pi_1, \ldots, \pi_{q+1}\), each intersects \(<x,y>\) at at most one point. Hence there are exactly two planes \(\pi_{i_0}\) and \(\pi_{j_0}\) such that \(\pi_{i_0}\) and \(\pi_{j_0}\) are disjoint from \(<x,y>\). Thus both \(M_1(p)\) and \(M_2(p)\) are lines that contain \(a\) and whose corresponding planes \(\pi_{i_0}\) and \(\pi_{j_0}\) are disjoint from \(<x,y>\).

Let \(p_1, p_2\) and \(p_3\) be three distinct points in \(L\) that are distinct from \(a\). For each \(j, j=1,2,3\), both \(M_1(p_j)\) and \(M_2(p_j)\) exist. Since there are only two such lines, we have

\[M_1(p_i)=M_2(p_j)\quad \text{and} \quad M_2(p_i)=M_1(p_j)\quad i\neq j\quad 1\leq i,j\leq 3 \quad (3.1)\]

or

\[M_1(p_i)=M_1(p_j)\quad \text{and} \quad M_2(p_i)=M_2(p_j)\quad i\neq j\quad 1\leq i,j\leq 3 \quad (3.2)\]

If \(M_1(p_i)=M_1(p_j)\) for some \(i,j\), \(i\neq j\) and \(1\leq i,j\leq 3\) as that given in Equation (3.2), then \(L(p_i)\) and \(L(p_j)\) are both parallel to \(M_1(p_i)\) (By definition of \(M_1(p_i)\)); that is, \(<w,p_i(x)>=<w,p_j(x)>\). But this implies that \(w, p_i, p_j\) and \(x\) are coplanar and are contained in \(\pi_k\) for some \(k, 1\leq k\leq q+1\), contradicting the fact that \(<x,y>\) is not contained in \(V'\).
Thus, $M_1(p_i) \neq M_1(p_j)$ for all $i, j$ such that $i \neq j$ and $1 \leq i, j \leq 3$.

In the case $M_1(p_i) = M_2(p_j)$ as that given in Equation (3.1), we have $M_1(p_1) = M_2(p_2) = M_1(p_3)$. But in the paragraph above, we have shown that $M_1(p_1) \neq M_1(p_3)$. Hence, $M_1(p_1) \neq M_2(p_j)$ for distinct pairs $i, j, 1 \leq i, j \leq 3$.

As a conclusion, we see that $u$ must be contained in $V'$. Therefore, $V'$ is a linear space.

**Theorem 3.7.** Let $P$ be a linear space with points, lines and planes that satisfy axioms $(H1)-(H4)$. If every line in $P$ contains $q$ points and $V$ is a 3-space in $P$, then $|V| = q^3$.

Proof. Let $V$ be generated by four noncoplanar points $a, b, c$ and $d$ and let $V'$ be defined as in Lemma 3.6. Since $V'$ is a linear space containing $a, b, c, d$ and is contained in $V$, by the minimality property of $V$ generated by $a, b, c, d$, $V = V'$.

Thus to find $|V|$ is to compute $|V'|$.

Consider the $q+1$ affine planes $\pi_1, \ldots, \pi_{q+1}$ contained in $V'$. For each pair $i, j$, $i \neq j$ and $1 \leq i, j \leq q+1$, $\pi_i$ intersects $\pi_j$ at a unique line $L$, that is, $|\pi_i \cap \pi_j| = q$. By Lemma 3.5, each affine plane $\pi_i$ contains $q^2$ points, we obtain

$$|V'| = \sum_{i=1}^{q+1} |\pi_i - L| + |L| = (q^2 - q)(q+1) + q$$

$$= q^3.$$  

Knowing the number of points contained in a 3-space,
we can derive the following:

**Corollary 3.8.** Let $V$ be a 3-space in $P$, where $P$ is the same linear space defined in the theorem. If $L$ is any given line in $V$, then the number of affine planes containing $L$ is $q+1$.

Proof. Let us count the number of ordered pairs $(x, \pi)$, where $\pi$ is an affine plane in $V$ containing $L$ and $x$ is a point in $\pi$ but is not contained in $L$. For every point $x$ that is not contained in $L$, there exists a unique affine plane containing both $x$ and $L$, namely the plane generated by $x$ and $L$. Hence, there are $q^3-q$ such pairs. On the other hand, for every affine plane $\pi$ containing $L$, there are $q^2-q$ points which are contained in $\pi$ but are not contained in $L$. Thus,

$$q^3-q = \text{(number of affine planes containing } L) \cdot (q^2-q),$$

and the number of affine planes containing $L$ is $q+1$.

Using the lemmas and the theorem we have developed thus far, we can now proceed to prove Theorem 3.2.

**Proof of Theorem 3.2.** Let $\Gamma_1$ and $\Gamma_2$ be two distinct affine planes in a 3-space $V$ such that $\Gamma_1$ intersects $\Gamma_2$. Let $a$, $b$ and $c$ be any three noncollinear points in $\Gamma_1$, and let $d$ be a point that is in $\Gamma_2$ but is not contained in $\Gamma_1$. Recall the definitions of $\pi$ and $L$ in the proof of Lemma 3.6. They are the affine plane generated by $a$, $b$ and $c$ and the line
generated by a and d respectively. Then \( \pi_1, \pi_2, \ldots, \pi_{q+1} \) are the only \( q+1 \) planes in \( V \) containing \( L \). Since both points a and d are contained in \( \Gamma_2 \) and \( \Gamma_2 \) is contained in \( V \), \( \Gamma_2 = \pi_i \) for some \( i, 1 \leq i \leq q+1 \). Similarly, since \( \Gamma_1 \) is generated by a, b and c, \( \Gamma_1 = \pi \). But \( \pi_1 \cap \pi \) at a line \( L_1 \), so \( \Gamma_1 \cap \Gamma_2 \) at a line \( L_1 \), and we have \( |\Gamma_1 \cap \Gamma_2| \geq 2 \).

Hence the linear space \( P \) satisfies axiom (B4) in the definition of an affine space. Obviously \( P \) satisfies all the other axioms, so \( P \) is an affine space and the proof is complete.

Based on Buekenhout's Theorem, we see that if we can define linear space \( P \) with points of \( P \) as the grand cliques of a geometrizable \((q,k,q)\)-strongly regular graph \( G \), the lines of \( P \) as the vertices of \( G \), and the planes of \( P \) to be such that each plane is an affine plane, then \( P \) is an affine space; hence \( G \) is isomorphic to the linegraph of the affine space \( P \). Thus, in order to prove the characterization theorem on the linegraph of an affine space, we need to establish the planes from the graph \( G \), and to show that each plane is an affine plane. The next few sections shall be devoted to this purpose.

\section{3.3 Transversals.}

Let \( G \) be a geometrizable \((q,k,q)\)-strongly regular graph. From the proof of Bose's Geometrization Theorem, we have seen that every two adjacent vertices \( A \) and \( B \) in \( G \) are contained in exactly one grand
clique. We shall use \( C(A,B) \) to denote the unique grand clique containing both \( A \) and \( B \). Since each grand clique in \( G \) corresponds to a line in the corresponding \((q,k,q)\)-partial geometry \( \pi(G) \), we sometimes call a grand clique a line. Hence if \( A \) and \( B \) are contained in a line, \( A \) and \( B \) are called collinear. In a \((q,k,q)\)-strongly regular graph \( G \), the set of lines in \( G \) has the following property:

**Proposition 3.9.** If \( G \) is a geometrizable \((q,k,q)\)-strongly regular graph, then any two distinct grand cliques in \( G \) intersect each other at a unique vertex.

**Proof.** Suppose there exist two nonintersecting grand cliques in \( G \), say \( x \) and \( y \). Let \( A \) be any vertex in \( x \). \( A \) is a vertex not in \( y \); hence \( A \) is adjacent to exactly \( q \) vertices \( B_1, \ldots, B_q \) in \( y \). For each \( i, 1 \leq i \leq q \), \( B_i \) and \( A \) determine a unique clique \( C(A,B_i) \) containing \( A \); thus, there are \( q \) distinct grand cliques containing \( A \). Now \( x \) is also a grand clique containing the vertex \( A \), but is distinct from the \( q \) grand cliques \( C(A,B_i), 1 \leq i \leq q \). This contradicts the fact that there are exactly \( q \) grand cliques containing \( A \). Hence any two distinct grand cliques intersect at a unique point. /

By virtue of this proposition, we are able to adopt the notation \( x \wedge y \) for the unique point of intersection of the two grand cliques \( x \) and \( y \). It is also clear that if we define the dual of the \((q,k,q)\)-partial geometry \( \pi(G) \) to be a linear space \( P \), then every two distinct points in \( P \)
determine a unique line and every two distinct lines intersect at at most one point. Next, we have to establish parallel lines and obtain the affine planes. To this end, we study the geometries of the set of vertices which are adjacent to two other vertices in G. Throughout the rest of the chapter, we shall assume that G satisfies axioms (G1)-(G3) in Theorem 3.1, and $q \geq 4$.

**Definition.** If $A$ and $B$ are two nonadjacent vertices in $G$, then a transversal of $A$ and $B$ is an element in $\Delta(A,B)$. If $A$ and $B$ are two distinct adjacent vertices in $G$, then a transversal of $A$ and $B$ is a vertex in $\Delta(A,B)$ which is not contained in $C(A,B)$.

The set of transversals of $A$ and $B$ is denoted by $T(A,B)$.

We shall first study transversals of two nonadjacent vertices. Let $A$ and $B$ be two nonadjacent vertices. For every vertex $C \in T(A,B)$, the triple $(C,A,B)$ is a 2-claw in $G$. It is clear that every grand clique containing either $A$ or $B$ contains exactly $q$ transversals of $A$ and $B$. Hence if $C \in T(A,B)$, then there are at least $2(q-1)$ other transversals of $A$ and $B$ that are adjacent to $C$, namely the transversals contained in $C(A,C)$ and $C(B,C)$. Thus, for any 2-claw $(C,A,B)$, $\delta(C,A,B) \geq 2(q-1)$.

**Lemma 3.10.** Let $A$ and $B$ be any two nonadjacent vertices in $G$. If $C$ and $D$ are two distinct
transversals of A and B, such that C is adjacent to D and \((A,C,D)\) and \((B,C,D)\) are noncollinear triples, then 
\[ \delta(A,B,D) = \delta(A,B,C) = q(q-1). \]

Proof. Since \((A,C,D)\) and \((B,C,D)\) are noncollinear triples, \(D \not\in C(A,C)\) and \(D \not\in C(B,C)\). From the remark above, there are 
\(2(q-1)\) transversals of A and B that are adjacent to C and are contained in either \(C(A,C)\) or \(C(B,C)\). Now, D is a transversal of A and B which is adjacent to C and is distinct from these \(2(q-1)\) transversals; hence 
\[ \delta(A,B,C) > 2(q-1). \] By Axiom (G2), \(\delta(A,B,C) = q(q-1)\). Similarly, \(\delta(A,B,D) = q(q-1).\]  

We shall now partition the transversals of two nonadjacent vertices A and B in accordance to the different cliques containing either A or B. Let us first introduce the following notations. If A is a vertex in G, then \(\mathcal{L}(A)\) denotes the set of grand cliques containing A. Let A and B be two distinct vertices in G, and \(x \in T(A,B)\). If \(x \in \mathcal{L}(A) \cup \mathcal{L}(B)\), but \(x \not\in C(A,B)\), then 
\[ M(A,B,C,x) = \{ D \in T(A,B) \mid D \text{ is adjacent to } C \text{ and is contained in } x \}. \]
and \(m(A,B,C,x) = |M(A,B,C,x)|. \) (See Figure 8)

\textbf{Lemma 3.11.} Let A and B be two nonadjacent vertices and \(C \in T(A,B)\). If \(x \in \mathcal{L}(A) \cup \mathcal{L}(B)\), then \(q-1 \geq m(A,B,C,x) \geq 1.\)
Proof. Without loss of generality we may assume that B is contained in x. Since C(A,C) and x are two distinct grand cliques, they intersect each other at a unique point D. If D≠C, then clearly D∈M(A,B,C,x) and m(A,B,C,x)≥1. If D=C, then C is contained in x; A is adjacent to exactly q-1 vertices of x other than C and we have m(A,B,C,x)=q-1.

To show q-1≥m(A,B,C,x), we only have to consider C as a vertex not contained in x. Since C is not in x, C is adjacent to exactly q vertices in x and one of which is B; hence m(A,B,C,x)≤q-1.

Lemma 3.12. Let (C;A,B) be a 2-claw. If there exists a grand clique x₀ in $\mathcal{K}(A)\cup\mathcal{K}(B)$ such that $x₀∉C(B,C)$, $x₀∉C(A,C)$ and $m(A,B,C,x₀)>1$, then

\[ \text{ave } m(A,B,C,.) = q-1 \]

where the average runs over all grand cliques in $\mathcal{K}(A)\cup\mathcal{K}(B)$.

Proof. Without loss of generality, let B be contained in $x₀$. Since $m(A,B,C,x₀)>1$, there exists a transversal D of
A and B contained in \( x_0 \) which is adjacent to C. But both \((B,C,D)\) and \((A,C,D)\) are noncollinear triples; furthermore, 
\( x_0 \not\in C(B,C) \) and \( x_0 \not\in C(A,C) \); hence by Lemma 3.10, 
\( \delta(A,B,C) = q(q-1) \).

Next, we count the number of ordered pairs \((x,E)\) where \( x \in \mathcal{L}(A) \cup \mathcal{L}(B) \) and \( E \in T(A,B) \) such that \( E \) is contained in \( x \) and is adjacent to C. Fixing a grand clique \( x \), \( x \in \mathcal{L}(A) \cup \mathcal{L}(B) \), there are \( m(A,B,C,x) \) choices of \( E \). On the other hand, if we fix a transversal \( E \) of \( A \) and \( B \) such that \( E \) is adjacent to \( C \), then there are exactly 2 choices of \( x \), namely \( C(A,E) \) and \( C(B,E) \). Since there are \( \delta(A,B,C) \) choices of \( E \), we have

\[
\sum_{x} m(A,B,C,x) = 2q(q-1).
\]

The vertices A and B are nonadjacent, so the cliques containing either A or B are all distinct, and we have 
\( |\mathcal{L}(A) \cup \mathcal{L}(B)| = 2q \). Thus

\[
\text{ave } m(A,B,C,\ldots) = q-1. /
\]

From these two lemmas, we obtain

**Proposition 3.13.** Let A and B be two nonadjacent vertices in G. If \( C \in T(A,B) \) and \( x \in \mathcal{L}(A) \cup \mathcal{L}(B) \), then \( m(A,B,C,x) \in \{1, q-1\} \).

**Proof.** Case 1. \( x = C(A,C) \) or \( x = C(B,C) \): From the proof of Lemma 3.11, we have \( m(A,B,C,x) = q-1 \).

Case 2. \( x \not\in C(A,C) \) and \( x \not\in C(B,C) \): If \( m(A,B,C,x) > 1 \), then from
the previous lemma, ave m(A,B,C,.)=q-1 where the average runs over all grand cliques in \( \mathcal{L}(A) \cup \mathcal{L}(B) \). By Lemma 3.11, 
\[ m(A,B,C,x) \leq q-1 \]; hence \( m(A,B,C,x) = q-1 \).

Similar to nonadjacent vertices, we would like to study the transversals of two adjacent vertices \( A \) and \( B \) which are also adjacent to a fixed transversal \( C \) of \( A \) and \( B \).

**Lemma 3.14.** Let \( A \) and \( B \) be two adjacent vertices. If \( C \in T(A,B) \), then \( C \) is adjacent to exactly \((q-1)(q-2)\) other transversals of \( A \) and \( B \).

Proof. Since \( G \) is a \((q,k,q)\)-strongly regular graph and \( A \) and \( B \) are two adjacent vertices, \( \delta(A,B) = k-2+(q-1)^2 \). But \( C(A,B) \) is a grand clique containing \( k-2 \) vertices other than \( A \) and \( B \), which are not transversals of \( A \) and \( B \); hence \( |T(A,B)| = (q-1)^2 \).

If \( C \) is a transversal of \( A \) and \( B \), then \( C \notin C(A,B) \). Therefore, \( C \) is adjacent to exactly \( q-2 \) vertices in \( C(A,B) \) other than \( A \) and \( B \). Hence, for sufficiently large \( k \)

\[ \delta(A,B,C) \leq |T(A,B)|-1+q-2 \]
\[ = (q-1)^2+q-3 \]
\[ < k-3. \]

By Axiom \((G3)\) in Theorem 3.1, \( \delta(A,B,C) = (q-2)q \). However, we have seen that the set \( \Delta(A,B,C)\cap C(A,B) \) consists of \( q-2 \) vertices which are not transversals of \( A \) and \( B \). Hence \( C \) is adjacent to exactly \((q-2)(q-1)\) transversals of \( A \) and \( B \).
A direct consequence of the above lemma is,

**Lemma 3.15.** Let A and B be two adjacent vertices. If 
C ∈ T(A, B), then C is nonadjacent to exactly q-2 transversals of A and B.

Again, we would like to know how these \((q-1)(q-2)\) transversals of A and B which are adjacent to C are partitioned in accordance to the various cliques containing A or B.

**Lemma 3.16.** Let A and B be two adjacent vertices in G. If 
C ∈ T(A, B), then \(\text{ave } m(A, B, C, \ldots) = q-2\) where the average runs over all grand cliques in \(\mathcal{L}(A) \cup \mathcal{L}(B) - C(A, B)\).

**Proof.** We shall count the number of ordered pairs \((x, D)\) where \(x\) is a grand clique in \(\mathcal{L}(A) \cup \mathcal{L}(B) - C(A, B)\) and 
\(D \in M(A, B, C, x)\). Fixing \(x\), there are \(m(A, B, C, x)\) choices of \(D\).

On the other hand, if we fix a vertex \(D\) such that \(D \in \Delta(A, B, C)\) and \(D \in T(A, B)\), then there are exactly two choices of \(x\), namely \(C(A, D)\) and \(C(B, D)\). From Lemma 3.14, there are \((q-1)(q-2)\) choices of \(D\); hence

\[
\sum_{x} m(A, B, C, x) = q(q-1)(q-2).
\]

Since \(|\mathcal{L}(A) \cup \mathcal{L}(B) - C(A, B)| = 2(q-1)|, we have

\(\text{ave } m(A, B, C, \ldots) = q-2.\)

**Lemma 3.17.** Let A and B be two adjacent vertices and 
C ∈ T(A, B). If \(x\) is a grand clique in \(\mathcal{L}(A) \cup \mathcal{L}(B) - C(A, B)\),
then $m(A, B, C, x) \geq q - 2$.

Proof. Without loss of generality, we may assume $x$ to be a clique containing $A$.

Case 1. $x = C(A, C)$: Every vertex in $x$ different from $A$ and $C$, which is adjacent to $B$, is contained in $M(A, B, C, x)$; hence $m(A, B, C, x) = q - 2$.

Case 2. $x \neq C(A, C)$: Since $B$ is not contained in $x$, $B$ is adjacent to $q$ vertices in $x$, namely $A, D_2, \ldots, D_q$. If $C$ is adjacent to each $D_i$, $2 \leq i \leq q$, then clearly, $m(A, B, C, x) \geq q - 2$.

Suppose $C$ is nonadjacent to $D_2$; let us consider the 2-claw $(B, C, D_2)$. The grand clique $x$ contains $D_2$; hence by Proposition 3.13, $m(C, D_2, B, x)$ equals either 1 or $q - 1$.

Since $A$ and $C(B, C) \cap x$ are two distinct vertices in $M(C, D_2, B, x)$, we have $m(C, D_2, B, x) = q - 1$. This means that for every $i$, $3 \leq i \leq q$, $D_i$ is adjacent to $C$. Consequently, $m(A, B, C, x) \geq q - 2$.

From the results of Lemmas 3.16 and 3.17, we arrive at the following:

**Proposition 3.18.** Let $A$ and $B$ be two adjacent vertices and $C \in T(A, B)$. If $x$ is a grand clique in $\mathcal{L}(A) \cup \mathcal{L}(B) - C(A, B)$, then $C$ is adjacent to exactly $q - 2$ transversals of $A$ and $B$ which are contained in $x$.

In the next proposition, we shall see how the transversals of two adjacent vertices $A$ and $B$ can be partitioned.
in accordance to the various grand cliques in G.

**Proposition 3.19.** Let A and B be two adjacent vertices. If x is a grand clique in G that contains neither A nor B, then there exists either none or exactly q-2 transversals of A and B contained in x.

Proof. Let C be a transversal of A and B which is contained in x. Since A, B and C are noncollinear, B is a transversal of A and C. Hence, by the previous proposition, we have m(A,B,C,x)=q-2; that is, there are exactly q-1 vertices in x including C which are adjacent to both A and B. But one of these q-1 vertices is the vertex x\(\mu\)C(A,B), which is not a transversal of A and B. Thus there are exactly q-2 transversals of A and B which are contained in x./

Using the results concerning transversals developed in this section, we shall differentiate between two types of nonadjacencies in G and define the relation 'parallelism' accordingly.

**§3.4 Parallelism.** In this section we shall establish two types of nonadjacencies in G. Let A and B be two nonadjacent vertices in G. If C is a transversal of A and B and x is a grand clique in \(\mathcal{L}(A)\cup\mathcal{L}(B)\), then by Proposition 3.13, m(A,B,C,x) equals either 1 or q-1. Furthermore, for every transversal C of A and B, if x\(\notin\)C(A,C), x\(\notin\)C(B,C) and m(A,B,C,x)=q-1, then m(A,B,C,y)=q-1 for all y in \(\mathcal{L}(A)\cup\mathcal{L}(B)\).
Thus, in differentiating the two types of nonadjacencies, we may assume that the function \( m(A, B, C, x) \) depends only on the 2-claw \((C; A, B)\).

**Definition.** Let \( A \) and \( B \) be two nonadjacent vertices. \( A \) is **parallel** to \( B \) if and only if either \( A= B \) or there exists a transversal \( C \) of \( A \) and \( B \) such that \( m(A, B, C, x)= q-1 \), where \( x \in \mathcal{L}(A) \cup \mathcal{L}(B) \) and \( x \notin C(A, C), x \notin C(B, C) \). We shall denote the parallelism by \( A//B \).

First, we have to show that the relation of parallelism is independent of the choice of the transversal \( C \) of \( A \) and \( B \).

**Lemma 3.20.** Let \( A \) and \( B \) be two nonadjacent vertices and let \( C \in T(A, B) \). If \( m(A, B, C, x)= q-1 \) for some grand clique \( x \in \mathcal{L}(A) \cup \mathcal{L}(B) \) and \( x \notin C(A, C), x \notin C(B, C) \), then \( B \) is adjacent to every vertex in \( x \), which is distinct from \( A \) and is adjacent to \( C \).

**Proof.** The vertex \( C \) is not contained in \( x \), so \( C \) is adjacent to exactly \( q \) vertices in \( x \), namely \( A, D_2, \ldots, D_q \). Since \( m(A, B, C, x)= q-1 \), \( B \) must be adjacent to each \( D_i, 2 \leq i \leq q \).

**Proposition 3.21.** If \( m(A, B, C, x)= q-1 \) for some grand clique \( x \in \mathcal{L}(A) \cup \mathcal{L}(B), x \notin C(A, C) \) and \( x \notin C(B, C) \), then for any transversal \( C' \) of \( A \) and \( B \), \( m(A, B, C', x)= q-1 \).

**Proof.** Without loss of generality, we may assume that \( x \notin \mathcal{L}(A) \). For clarity, we shall denote \( C(B, C) \) by \( y \). Let us
consider the vertex $C'$. If $C'$ is contained in $x$, then clearly $m(A,B,C',x)=q-1$. Henceforth, we shall assume that $C'$ is not contained in $x$.

Case 1. $C'$ is contained in $y$. Since both vertices $C$ and $C'$ are contained in $y$ and $A$ is not in $y$, $C$ is not contained in the clique determined by $A$ and $C'$, so $C$ is a transversal of $A$ and $C'$. By Proposition 3.18, $C$ is adjacent to exactly $q-2$ transversals of $A$ and $C'$ which are contained in $x$. But by the previous lemma, the vertex $B$ is also adjacent to all of these $q-2$ transversals of $A$ and $C'$. Hence

$$m(A,B,C',x) > q-2 > 1.$$ 

This implies that $m(A,B,C',x)=q-1$.

Case 2. $C'$ is not contained in $y$.

Subcase 2.1. $C'$ is adjacent to $C$. Let $z$ denote the grand clique containing $C$ and $C'$, and let $D$ be the point $x \in z$. The vertex $D$ is contained in $x$ and is adjacent to $C$, by Lemma 3.20, $B$ is adjacent to $D$. Furthermore, $x \in C(B,C')$ is another point contained in $x$ and is adjacent to $A$, $B$ and $C'$. Thus, $m(A,B,C',x) > 2$ and hence $m(A,B,C',x)=q-1$.

Subcase 2.2. $C'$ is non-adjacent to $C$. Let $z$ denote the grand clique containing $B$ and $C'$. If we can show that there exists a transversal $C''$ of $A$ and $B$ contained in $z$ such that $C''$ is adjacent to $C$, then $C''$ is not contained in either $x$ or $y$, and by the previous subcase, $m(A,B,C'',x)=q-1$. But $C'$ is a vertex contained in $z$ and not in $x$, by case 1,
we have \( m(A,B,C',x) = q-1 \). Hence, we are left to show the existence of C'.

Since \( m(A,B,C,x) = q-1 \) and \( z \in \mathcal{L}(B) \), \( m(A,B,C,z) = q-1 \); this means that there are \( q-1 \) transversals of A and B which are contained in \( z \) and are adjacent to C. Thus, C' exists and the proof is complete.

It is clear from this proposition that 'parallelism' is a well-defined relation on pairs of nonadjacent vertices. We still have to construct the affine planes in the linear space \( P \) defined in the previous section. Before that, let us establish the next two theorems concerning parallelism.

**Theorem 3.22.** Let \( x \) be a grand clique in \( G \). If \( B \) is a vertex in \( G \), not contained in \( x \), then there exists a unique vertex A contained in \( x \) such that A is parallel to B.

**Proof.** Let \( n(x,B) \) denote the number of vertices contained in \( x \) which are parallel to B. We shall count the number of triples \( (A,C,D) \) where A is a vertex in \( x \) nonadjacent to B, C is a transversal of A and B which is not contained in \( x \), and D is a vertex in \( x \) adjacent to both B and C. Fixing a vertex A in \( x \) nonadjacent to B, there are \( q^2 - q \) transversals C of A and B which are not contained in \( x \). If A is parallel to B, then for every transversal C of A and B, \( m(A,B,C,x) = q-1 \), that is, there are \( q-1 \) choices of D. If A is not parallel to B, then \( m(A,B,C,x) = 1 \) and there exists a unique
choice of $D$. Since there are $n(x,B)$ vertices in $x$ which are parallel to $B$, there are $k-q-n(x,B)$ vertices in $x$ which are neither adjacent nor parallel to $B$. Hence, the number of triples $(A,C,D)$

$$= n(x,B)(q^2-q)(q-1) + (k-q-n(x,B))(q^2-q).1. (3.3)$$

Next, we consider a vertex $C$ which is adjacent to $B$ but is not contained in $x$, and count the number of vertices $D$ in $x$ which is adjacent to both $B$ and $C$. If $D \notin C(B,C)x$, then $D$ is a transversal of $B$ and $C$. But by Proposition 3.19, there exists either one or exactly $q-1$ choices of $D$. In the former case, any vertex $A$ which is contained in $x$, distinct from $D$ and adjacent to $C$, is nonadjacent to $B$ and there are exactly $q-1$ such vertices $A$. In the latter case, since there are $q-1$ vertices $D$ in $x$ which are adjacent to both $B$ and $C$, there exists a unique vertex $A$ in $x$, which is adjacent to $C$ but is nonadjacent to $B$. Thus, we see that in both cases there are $q-1$ choices of the pairs $(A,D)$. Since there are $q(k-1)-q$ vertices $C$ which are adjacent to $B$ and not contained in $x$, we obtain that the number of triples $(A,C,D)$

$$= (q(k-1)-q)(q-1) = q(k-2)(q-1). \quad (3.4)$$

Equating Equations (3.3) and (3.4), we have $n(x,B)=1$. Consequently, there exists a unique vertex in $x$ which is parallel to the vertex $B$. /
Theorem 3.23. If $A$ and $B$ are two distinct vertices which are parallel to $C$, then $A$ is nonadjacent to $B$.

Proof. Suppose $A$ is adjacent to $B$. Since $C$ is parallel to both $A$ and $B$, $C$ is a vertex not in the grand clique $C(A,B)$. By the previous theorem, there exists a unique vertex in $C(A,B)$ which is parallel to $C$, but this contradicts the fact that both $A$ and $B$ are parallel to $C$. Hence, $A$ is nonadjacent to $B$.

Corollary 3.24. Let $x$ be a grand clique in $G$. If $A$ is a vertex contained in $x$, then $A$ itself is the unique vertex contained in $x$, which is parallel to $A$.

Proof. By definition, $A$ is parallel to itself. If there exists another vertex $B$ contained in $x$ which is parallel to $A$, then $A$ and $B$ are nonadjacent; this contradicts that $A$ and $B$ are both contained in $x$.

§3.5 Affine Planes. In this section we shall define affine planes on $P$, based on the notion of parallel lines developed in the previous section. Henceforth, two parallel vertices will be called $p$-related, two adjacent vertices will be called $a$-related. For every distinct pair of vertices $A$ and $B$, $\Delta_{pa}(A,B)$ will denote the set of vertices that are parallel to $A$ and are adjacent to $B$. Clearly, if $A$ and $B$ are parallel, then $\Delta_{pa}(A,B)$ is empty. Hence, we are only interested in two adjacent vertices $A$ and $B$. First,
we shall compute the number of vertices in \( \Delta_{pa}(A,B) \).

**Lemma 3.25.** If \( A \) and \( C \) are two adjacent vertices, then 
\[ |\Delta_{pa}(A,C)| = q. \]

Proof. By Theorem 3.22 and Corollary 3.24, for every grand clique \( x \) in \( \mathcal{L}(C) \), there exists a unique vertex \( D \) in \( x \) which is parallel to \( A \). Since there are \( q \) grand cliques in \( \mathcal{L}(C) \), 
\[ |\Delta_{pa}(A,C)| = q. \]

Let us now define a linear subspace \( \mathcal{P}(A,B,C) \) in \( P \) based on a 2-claw \( (C;A,B) \) in \( G \) such that \( A \) is parallel to \( B \). Let \( \Theta \) be a set of vertices in \( G \), we define

\[ \mathcal{L}(\Theta) = \{ x \mid x \text{ is a grand clique containing a vertex } D \text{ in } \Theta \}. \]

Let \( \mathcal{L}(A,B,C) = T(A,B) \cup \Delta_{pa}(A,C) \),
and \( \mathcal{P}(A,B,C) = \mathcal{L}(\mathcal{L}(A,B,C)) \).

We shall show that \( \mathcal{P}(A,B,C) \) is an affine plane with \( \mathcal{L}(A,B,C) \) as the set of lines of the plane and \( \mathcal{P}(A,B,C) \) as the set of points with the obvious incidence relation. First, we show that \( \mathcal{L}(A,B,C) \) contains the correct number of lines in an affine plane.

**Proposition 3.26.** \[ |\mathcal{L}(A,B,C)| = q^2 + q. \]

Proof. Since every transversal of \( A \) and \( B \) cannot be parallel to \( A \), the two sets \( T(A,B) \) and \( \Delta_{pa}(A,C) \) are
disjoint, thus

$$|\mathcal{L}(A,B,C)| = |T(A,B)| + |\Delta_{pa}(A,C)|$$

$$= q^2 + q. /$$

Next, we shall show that $\Phi(A,B,C)$ has the correct number of points in an affine plane.

**Lemma 3.27.** Let $A$ and $B$ be two adjacent vertices and let $C \in T(A,B)$. If $x$ is a grand clique containing either $A$ or $B$ but not both, and $x$ does not contain $C$, then there exists a unique vertex $D$ in $x$ such that $D \in T(A,B)$ and $D$ is nonadjacent to $C$. In fact, $D$ is the unique vertex parallel to $C$.

**Proof.** Without loss of generality, we may assume $x \in \mathcal{L}(A)$. By Proposition 3.18, $m(A,B,C,x) = q-2$, but there exist $q-1$ transversals of $A$ and $B$ in $x$. Hence there exists a unique transversal $D$ in $x$ which is not adjacent to $C$. Consider the 2-claw $(B;C,D)$. Since $x$ is a grand clique containing $D$ but not $B$ and there are at least $q-2$ vertices in $x$ which are adjacent to both $B$ and $C$, $m(C,D,B,x) \geq q-2$. Thus, $m(C,D,B,x) = q-1$ and $C$ is therefore parallel to $D$. /

**Lemma 3.28.** Let $A$ be parallel to $B$. If $C \in T(A,B)$, then $\Delta_{pa}(A,C) = \Delta_{pa}(B,C)$.

**Proof.** It suffices to show that for any $D \in \Delta_{pa}(A,C)$, $D$ is
parallel to B. Let x be a grand clique in \( \Delta(A) \) such that x does not contain C. Let D be a transversal of A and B in x such that D is adjacent to C. Furthermore, let D be such that D is not contained in \( C(B,C) \). (Note that at least two such D's exist, because \( m(A,B,C,x)=q-1 \) and \( q^4 \)). Since neither one of the triples \( (A,C,D) \) and \( (B,C,D) \) is collinear, both A and B are transversals of the adjacent vertices C and D.

Using the previous lemma, we see that for every grand clique containing C which does not contain A or D (that is, \( y \neq C(A,C) \) and \( y \neq C(D,C) \)), there exists a unique vertex \( A_y \) such that \( A_y \parallel A \) and \( A_y \in T(C,D) \). If \( y \) contains the vertex B, then clearly \( A_y = B \) and \( A_y \) is parallel to B. If \( y \) does not contain the vertex B and both \( A_y \) and B are parallel to A, then by Theorem 3.23, \( A_y \) and B are nonadjacent. But by the previous lemma the unique transversal \( A_y \) of C, D which is nonadjacent to B is parallel to B. Thus, for all vertices E in \( \Delta_{pa}(A,C) \) such that \( C(E,C) \neq C(A,C) \) and \( C(E,C) \neq C(D,C) \), E is parallel to B. Hence, \( E \in \Delta_{pa}(B,C) \).

If \( E \in \Delta_{pa}(A,C) \) and \( C(E,C) = C(A,C) \), then E is the unique vertex parallel to A and is contained in \( C(A,C) \); hence \( E = A \). Obviously, \( A \parallel B \); so \( E \parallel B \) and \( E \in \Delta_{pa}(A,C) \).

It remains to show that if \( E' \in \Delta_{pa}(A,C) \) such that \( C(E',C) = C(D,C) \), then \( E' \in \Delta_{pa}(B,C) \). At the beginning of the proof we have observed that there exists another vertex \( D' \) which has the same properties as those of D. But
C(D',C) ≠ C(D,C); hence C(E',C) ≠ C(D',C) and using the same arguments as above, E' ∈ Δ_{pa}(B,C).

From this lemma, we observe that Φ(A,B,C) = Φ(B,A,C).
So A and B are 'equivalent' in the sense that they can be interchanged without affecting the definition of Φ(A,B,C).

Lemma 3.29. Let E and E' be two vertices in Φ(A,B,C).
If E'//A and E'//A, then E'//E'.

Proof. Since E'//A and E'∈Φ(A,B,C), E is adjacent to C.
Thus, C∈T(A,E). By Lemma 3.28, Δ_{pa}(A,C) = Δ_{pa}(E,C). But E'//A and E'∈Φ(A,B,C), E'∈Δ_{pa}(A,C). This implies that E'//E'.

Theorem 3.30. |Φ(A,B,C)| = q^2.

Proof. Let us first compute the number of grand cliques contained in Λ(Δ_{pa}(A,C)). Since the vertices in Δ_{pa}(A,C) are pairwise nonadjacent, we have

|Λ(Δ_{pa}(A,C))| = ∑_{E∈Δ_{pa}(A,C)} |Λ(E)| = q^2.

Thus, if we can show that Λ(Δ_{pa}(A,C)) = Φ(A,B,C), then we are done. But Δ_{pa}(A,C)⊆Λ(A,B,C); hence Λ(Δ_{pa}(A,C))⊆Φ(A,B,C).
So we only have to show that Φ(A,B,C)⊆Λ(Δ_{pa}(A,C)).

Let x∈Φ(A,B,C). If x∈Λ(Δ_{pa}(A,C)), then we have nothing else to prove. If x∉Λ(T(A,B)), then there exists a transversal D of A and B such that D is contained in x. If x contains either A or B, then clearly x∈Δ_{pa}(A,C). Henceforth
we shall assume that x does not contain A or B.

Case 1. D is adjacent to C. Since A and B are nonadjacent and both are adjacent to C and D, at least one of the two vertices A and B is a transversal of C and D. By the remark that A and B are 'equivalent', without loss of generality, we may assume that A∈T(C,D). By Lemma 3.27, there exists a unique transversal E of C and D in x such that E//A; that is E∈Δpa(A,C) and E is contained in x. Thus, x∈L(Δpa(A,C)).

Case 2. D is nonadjacent to C. Let E denote the vertex x∈C(3,C). Clearly, E is adjacent to both 3 and C. If E is adjacent to A, then E∈T(A,3) and E is contained in x; hence using Case 1 of the proof, x∈L(Δpa(A,C)). So we only need to show that E is adjacent to A.

Both the vertices D and E are contained in x, so D and E are adjacent. But B is not contained in x; hence B∈T(D,E). Let z denote the clique C(A,D). Clearly z does not contain B. Thus, there exists a unique transversal T of D and E in z such that T//B. Since A is the unique vertex in z which is parallel to B, A=T. Therefore, T is adjacent to E and the proof is complete.

Next, we shall show that φ(A,B,C) is an affine plane.

Theorem 3.31. φ(A,B,C) is an affine plane.

Proof. We shall first show that every two distinct points in φ(A,B,C) determine a unique line in L(A,B,C). Let us count the number of triples (x,y,L) where x and y are
distinct points in $\mathcal{P}(A,B,C)$ and $L$ is a line in $\mathcal{L}(A,B,C)$ that contains both $x$ and $y$. For every line $L$ in $\mathcal{L}(A,B,C)$, $L$ contains $q$ points; hence there are $q(q-1)$ choices of $(x,y)$. Since $|\mathcal{L}(A,B,C)|=q(q+1)$, we have

$$\text{Number of triples } (x,y,L) = q(q+1)q(q-1). \quad (3.5)$$

On the other hand, if we fix a pair $(x,y)$ in $\mathcal{P}(A,B,C)$ and let $f(x,y)$ denote the number of lines in $\mathcal{L}(A,B,C)$ that contain $x$ and $y$, then there are $\sum f(x,y)$ such triples $(x,y,L)$ where the sum runs over all ordered pairs of distinct points in $\mathcal{P}(A,B,C)$. Thus,

$$\text{Number of triples } (x,y,L) = q^2(q^2-1)\text{ave } f(...) \quad (3.6)$$

Equating Equations (3.5) and (3.6), we obtain $\text{ave } f(...) = 1$.

Since every two distinct grand cliques $x$ and $y$ in $G$ intersect at a unique point in $G$, $f(x,y)=1$. Hence, $f(x,y)=1$, that is, every two distinct points in $\mathcal{P}(A,B,C)$ determine a unique line in $\mathcal{L}(A,B,C)$.

Next, we will show that for every line in $\mathcal{L}(A,B,C)$ and a point $x$ not in $L$, there exists a unique line $L_x$ in $\mathcal{L}(A,B,C)$ such that $L_x$ contains $x$ and is parallel to $L$. Let us consider $\Delta_{pa}(A,C)$. The set of parallel lines in $\Delta_{pa}(A,C)$ partitions the points in $\mathcal{P}(A,B,C)$. Hence, for every line $L$ in $\Delta_{pa}(A,C)$ and every point $x$ not in $L$, there exists a unique line $L_x$ containing $x$ which is parallel to $A$. By Lemma 3.29, $L_x$ is parallel to $L$ and $L_x \in \mathcal{L}(A,B,C)$. 
Consider a line $L \in T(A,B)$ and a point $x$ in $\mathcal{P}(A,B,C)$ such that $x$ is not in $L$. If $x$ is a point in $A$, then let $C$ be the line determined by $x$ and $C(B,L)$; otherwise, let $C$ be the line containing $x$ and the point $C(A,L)$. Since $A$ and $B$ are 'equivalent', without loss of generality, we may assume $C$ to be the latter. Consider the pair $(B,C)$. $L$ is a transversal of $B$ and $C$, and $x$ is a grand clique containing $C$ but neither $B$ nor $L$. By Lemma 3.27, there exists a unique transversal $L_x$ of $B$ and $C$ such that $L_x$ is contained in $x$ and $L_x \parallel L$. If $L_x$ is adjacent to $A$, then $L_x \in T(A,B)$ and $L_x$ is contained in $\mathcal{L}(A,B,C)$. So we only need to show that $L_x$ is adjacent to $A$. Since $A \parallel B$, $m(A,B,C,x)=q-1$, that is, every vertex in $x$ which is adjacent to $B$ is adjacent to $A$. Thus, $L_x$ is adjacent to $A$.

Since $\mathcal{P}(A,B,C)$ possesses the two properties we have derived above, and clearly $\mathcal{P}(A,B,C)$ contains 3 noncollinear points, $\mathcal{P}(A,B,C)$ is an affine plane and the proof is complete.

Thus far we have defined affine planes $\mathcal{P}(A,B,C)$ on $P$ based on a 2-claw $(C;A,B)$. But in order to show that these affine planes are well-defined, we have to show that they are independent of the choice of the transversal $C$ of $A$ and $B$, and are also independent of the choice of the pair of parallel lines $A$ and $B$ in the plane.

**Lemma 3.32.** Let $A$ and $B$ be two distinct parallel
vertices in $G$. If $C$ and $D$ are both in $T(A,B)$, then

$$\Delta_{pa}(A,C) = \Delta_{pa}(A,D).$$

Proof. Let $E \in \Delta_{pa}(A,C)$. If $E$ is adjacent to $D$, then clearly $E \in \Delta_{pa}(A,D)$. If $E$ is not adjacent to $D$, then $E$ and $D$ are parallel lines in $\mathcal{P}(A,B,C)$. Since $E$ is also parallel to $A$, $D$ is parallel to $A$. But this contradicts the fact that $D \in T(A,B)$. Hence, $E \in \Delta_{pa}(A,D)$ and $\Delta_{pa}(A,C) \subseteq \Delta_{pa}(A,D)$. By symmetry, $\Delta_{pa}(A,D) \subseteq \Delta_{pa}(A,C)$ and the proof is complete.

By virtue of the above lemma, we may simplify the notation $\mathcal{P}(A,B,C)$ to $\mathcal{P}(A,B)$ where $A$ and $B$ are two distinct parallel lines.

**Lemma 3.33.** Let $A$ and $B$ be two distinct parallel lines in $P$. If $M$ and $N$ are two distinct parallel lines in $\mathcal{L}(A,B)$, then $\mathcal{P}(A,B) = \mathcal{P}(M,N)$.

Proof. Since $\mathcal{P}(A,B)$ and $\mathcal{P}(M,N)$ are both determined by the lines in $\mathcal{L}(A,B)$ and $\mathcal{L}(M,N)$ respectively, we shall show that $\mathcal{L}(A,B) = \mathcal{L}(M,N)$.

Case 1. $M \parallel A$. By Lemma 3.29, any line that is parallel to $A$ must be parallel to $M$. Similarly, any transversal of $A$ and $B$ must be a transversal of $M$ and $N$. Thus, $T(A,B) \subseteq T(M,N)$. Let $C \in T(A,B)$, then $C \in T(M,N)$. If $E \in \Delta_{pa}(A,C)$, then clearly $E \in \Delta_{pa}(M,C)$. Therefore, we have $\mathcal{L}(A,B) \subseteq \mathcal{L}(M,N)$. Since both $A$ and $B$ are in $\mathcal{L}(M,N)$, by similar arguments, we have $\mathcal{L}(M,N) \subseteq \mathcal{L}(A,B)$. Hence, $\mathcal{L}(A,B) = \mathcal{L}(M,N)$. 
Case 2. M is adjacent to A. Clearly \( M \in T(A,B) \) and \( A \in T(M,N) \). Let \( E \in T(A,B) \). If \( E \) is adjacent to \( M \), then \( E \) is adjacent to \( N \); hence \( E \in T(M,N) \) and \( E \) is also contained in \( \mathcal{L}(M,N) \). If \( E \) is parallel to \( M \), then \( E \in \Delta_{pa}(M,A) \) and \( E \) is also contained in \( \mathcal{L}(M,N) \). Thus, \( T(A,B) \subseteq \mathcal{L}(M,N) \).

If \( E \in \Delta_{pa}(A,N) \), then \( E \) is adjacent to \( M \). Hence \( E \) is adjacent to \( N \). Thus, \( E \in T(M,N) \) and \( T(M,N) \subseteq \mathcal{L}(M,N) \). So \( \Delta_{pa}(A,N) \subseteq \mathcal{L}(M,N) \).

Consequently, we have \( \mathcal{L}(A,B) \subseteq \mathcal{L}(M,N) \). By Proposition 3.26, \( |\mathcal{L}(A,B)| = |\mathcal{L}(M,N)| \). Hence, \( \mathcal{L}(A,B) = \mathcal{L}(M,N) \).

Based on these two lemmas, one easily sees that the affine planes \( \mathcal{G}(A,B) \) defined on a pair of distinct parallel lines \( A \) and \( B \) are well-defined planes.

§3.6 Proof of Theorem 3.1. Let \( G \) be a geometrizable \((q,k,q)\)-strongly regular graph and let \( \pi(G) \) denote its corresponding \((q,k,q)\)-partial geometry. If \( \pi^*(G) \) is the dual geometry of \( \pi(G) \), then \( \pi^*(G) \) is a \((k,q,q)\)-partial geometry. Let \( P \) be a linear space with its point set as the set of points in \( \pi^*(G) \) and the line set as the set of lines in \( \pi^*(G) \), then every two distinct points determine a unique line and every two intersecting lines intersect at a unique point.

If, in addition, \( q \geq 4 \) and \( G \) has its triangle degrees equal to either \( q(q-2) \) or to be at least \( k-3 \), and its 2-claw degrees equal to \( q(q-1) \) or \( 2(q-1) \), then affine planes \( \mathcal{G}(A,B) \)
can be defined on $P$ based on any two distinct parallel lines $A$ and $B$ in $P$. Thus, $P$ is a linear space with points, lines and planes such that each plane is an affine plane.

Let $x$, $y$ and $z$ be 3 noncollinear points in $P$. There exists a unique line $L$ containing $z$ and parallel to the line $\langle x,y \rangle$. Since $z \not\in L$ and $z \not\in \langle x,y \rangle$, $L \not\subset \langle x,y \rangle$; hence, $L$ and $\langle x,y \rangle$ determine a unique affine plane which contains $x$, $y$ and $z$. So every 3 noncollinear points in $P$ determine a unique plane.

From the above discussion, we see that $P$ is a linear space that satisfies axioms $(H1)-(H4)$. By Buekenhout's Theorem, $P$ is an affine space. Clearly the linegraph of $P$ is isomorphic to the $(q,k,q)$-strongly regular graph $G$. Therefore, we have characterized the linegraph of an affine space.
CHAPTER IV
EMBEDDING OF A PSEUDO-BLOCK-RESIDUAL DESIGN INTO A MOBIUS PLANE

§4.1 Motivation and Summary of the Proof of the Main Theorem. Let us recall that a Möbius plane \( M \) is an \( S(3,q+1,q^2+1) \), and \( M \) has the property that if \( B \) and \( B' \) are two distinct circles that are tangent to a circle \( A \) at \( x \), then \( B \) and \( B' \) are mutually tangent at \( x \). A maximal set of circles which are mutually tangent at a point \( x \) is called a pencil with carrier \( x \). By Proposition 2.18, every pencil in \( M \) consists of \( q \) circles. If the point \( x \) is deleted from \( M \), then the circles in a pencil with carrier \( x \) are pairwise disjoint. A set of pairwise disjoint circles that partition the set of points in an incidence structure is called a parallel class of circles. Clearly, a pencil in \( M \) with carrier \( x \) is a parallel class of circles in \( M-x \).

Let \( A_\infty \) be a block in \( M \) and let \( M' \) be the block-residual design obtained from \( M \) by deleting \( A_\infty \). We have seen that \( M' \) is an \( S(\lambda_0,\lambda_1,\lambda_2,\lambda_3;K,v) \) with

\[
\lambda_0=q^3+q-1, \quad \lambda_1=q^2+q, \quad \lambda_2=q+1, \quad \lambda_3=1, \quad K=\{q+1,q,q-1\}, \quad v=q^2-q.
\]  

(4.1)
Any 3-design with parameters as those given in (4.1) is called a pseudo-block-residual design of order \( q \), abbreviated by PBRD\( (q) \). Our purpose is to embed the design \( D \) which is a PBRD\( (q) \) into a Möbius plane, and to show that it is uniquely embeddable. Before we proceed to solve the embedding problem, it will be helpful to study the Möbius plane \( M \) and its block-residual design \( M' \) in greater detail.

Let \( M, M' \) and \( A_\infty \) be defined as before. For every deleted point \( x \) in \( A \), there exists a pencil \( \Omega' \) in \( M \) with \( x \) as the carrier which contains \( A_\infty \). Clearly, \( \Omega' - A_\infty \) forms a parallel class of circles in \( M' \); moreover, each circle in \( \Omega' - A_\infty \) is \( q \)-valent. Conversely, given any parallel class of \( q \)-valent circles in \( M' \), there corresponds a unique point deleted from \( A_\infty \). Thus, in order to embed a PBRD\( (q) \) into a Möbius plane \( M \), first of all, we have to establish the \( q+1 \) parallel classes of \( q \)-valent blocks in \( D \).

When the parallel classes are established in \( D \), we still have to find means to 'complete' the \( (q-1) \)-valent circles to \( (q+1) \)-valent blocks. Again, we are motivated by examining the \( (q-1) \)-valent blocks in \( M' \). Let \( B \) be any \( (q-1) \)-valent block in \( M' \) and let \( x \) and \( y \) be the corresponding deleted points in \( A_\infty \). Consider the pencil \( \Omega' \) in \( M \) with carrier \( x \); every circle in \( \Omega' \) intersects \( B \cup \{x,y\} \) at two points. Hence, every circle in the corresponding parallel class is \( r \)-tangent to \( B \) at a point. We shall define a circle \( B \) to be an \textit{\( r \)-transversal} of a parallel
class $\mathfrak{A}$ if and only if $B$ is $r$-tangent to every circle in $\mathfrak{A}$. It is obvious then, that any $(q-1)$-valent block in $M'$ is an $r$-transversal of exactly two parallel classes. Conversely, given any two parallel classes of $q$-valent blocks in $M'$, there are $q$ common $r$-transversals, namely the circles in $M$ that contain the two corresponding deleted points. Hence, if we can show that every $(q-1)$-valent circle in a pseudo-block-residual design $D$ is an $r$-transversal of exactly two parallel classes, then we can 'complete' the $(q-1)$-valent blocks by adjoining their corresponding parallel classes. Our Fundamental Lemma stated in the next section shows how these parallel classes and $r$-transversals lead to the embedding of an $S(3,\{k+1,k,k-1\},v)$ into an $S(3,k+1,v+k+1)$.

From the above discussions, we see that we have to set up the parallel classes in PBRD$(q)$. To do so we consider the set of blocks that are $r$-tangent to a given $q$-valent block. We define a maximal set of circles which are mutually $r$-tangent at $x$ and contains at least 4 circles to be an $r$-pencil with carrier $x$. We show that each $r$-pencil consists of one $q$-valent block and $q$ $(q-1)$-valent blocks. Using the $r$-tangency condition, we prove that the $r$-tangents of a $q$-valent block $B$ which contain a common point $x$ are mutually $r$-tangent. Furthermore, these $r$-tangents determine a unique $r$-pencil whose $q$-valent block is either $B$ itself or disjoint from $B$. From this, we obtain the $q+1$ parallel classes of $q$-valent blocks. We also show that every
(q-1)-valent block that is r-tangent to a q-valent block B is an r-transversal of the parallel class containing B. Thus, we are able to establish the main theorem as stated below:

**Theorem 4.1.** Let q≠4. If D is a pseudo-block-residual design that has parameters as those given in (4.1) and satisfies the r-tangency condition, then D is uniquely embeddable into a Möbius plane of order q. Conversely, the block-residual design of a Möbius plane M satisfies the r-tangency condition.

**§ 4.2 Fundamental Lemma.** In this section we shall reconstruct a 3-design from a pseudo-block-residual design by means of parallel classes and r-transversals. Let us first state the result as follows:

**Lemma 4.2.** (The Fundamental Lemma). Let D be an S(3,K,v) where K={k+1,k,k-1}. Suppose D satisfies the following conditions:

(I1) The collection of k-valent blocks can be partitioned into k+1 parallel classes.

(I2) Every (k-1)-valent block is the r-transversal of exactly two parallel classes and every two parallel classes have exactly k common (k-1)-valent r-transversals which are pairwise disjoint.

(I3) Given any two distinct points x and y and a
parallel class \( \mathcal{A} \), either there exists exactly one block in \( \mathcal{A} \) that contains \( x \) and \( y \), or there exists exactly one \((k-1)\)-valent block that contains \( x \) and \( y \) and is an \( r \)-transversal of \( \mathcal{A} \).

Then \( D \) can be uniquely embedded into an \( S(3,k+1,v+k+1) \).

In proving the Fundamental Lemma, we reconstruct the \( k+1 \) 'missing' points and adjoin them to the \( k \)-valent and \((k-1)\)-valent blocks in \( D \). Finally, we show that it is a 3-design. Before we proceed, let us establish two simple lemmas.

Lemma 4.3. Let \( D \) be as defined in the Fundamental Lemma. Then, \( v = k^2 - k \).

Proof. Since every \((k-1)\)-valent block in \( D \) is an \( r \)-transversal of a parallel class \( \mathcal{A} \), \( \mathcal{A} \) consists of \( k-1 \) \( k \)-valent blocks; hence, \( v = k(k-1) \).

Lemma 4.4. If \( \mathcal{A} \) and \( \mathcal{A}' \) are two distinct parallel classes in \( D \) and \( x \) is a point, then there exists a unique \((k-1)\)-valent block that contains \( x \) and is a common \( r \)-transversal of \( \mathcal{A} \) and \( \mathcal{A}' \).

Proof. Since the \( k \) common \( r \)-transversals of \( \mathcal{A} \) and \( \mathcal{A}' \) are pairwise disjoint, they partition the \( k^2 - k \) points in \( D \). Hence, given any point \( x \) in \( D \), there exists a unique common \( r \)-transversal of \( \mathcal{A} \) and \( \mathcal{A}' \) that contains \( x \).
Now we can proceed to prove the lemma.

Construction of 'New Points'. Let \( \alpha_1, \ldots, \alpha_{k+1} \) be the \( k+1 \) parallel classes of \( k \)-valent blocks in \( D \). Corresponding to every parallel class \( \alpha_i \), we define a 'new' point \( \alpha_i^* \), \( 1 \leq i \leq k+1 \). Let \( X \) be the set of points consisting of the points \( X \) in \( D \) and the \( k+1 \) new points \( \alpha_1^*, \ldots, \alpha_{k+1}^* \).

Construction of 'New Blocks'. Let \( A \) be a block in \( D \).

1. If \( A \) is a \( (k+1) \)-valent block, then we let \( A' \) to be \( A \).
2. If \( A \) is \( k \)-valent, then \( A \) is contained in a unique parallel class \( \alpha_i \) for some \( i, 1 \leq i \leq k+1 \). We define \( A' \) to be a block consisting of all points in \( A \) and the new point \( \alpha_i^* \).
3. If \( A \) is \( (k-1) \)-valent, then there exist exactly two parallel classes \( \alpha_i \) and \( \alpha_j \), \( i \neq j \), \( 1 \leq i, j \leq k+1 \), such that \( A \) is a common \( r \)-transversal of both classes. We extend \( A \) to a block \( A' \) consisting of all points in \( A \), together with the two new points \( \alpha_i^* \) and \( \alpha_j^* \).

Finally, we let \( A_\infty \) to be a block consisting of the \( k+1 \) new points and

\[
\overline{A} = \{A_\infty\} \cup \{A' | A \text{ is a block in } D\}.
\]

Construction of a \( 3 \)-design. Let \( S \) be the incidence structure \((X, \overline{A})\). We shall show that \( S \) is an \( S(3, k+1, v+k+1) \).

From the constructions of new points and new blocks, it is clear that \( S \) has \( v+k+1 \) points and each block in \( S \) has \( k+1 \) points. We only have to show that \( \lambda=1 \). Let \( x, y \) and \( z \)
be any three distinct points in \( X \).

Case 1. \( x, y, z \in X \). There exists a unique block \( A \) in \( D \) that contains \( x \), \( y \) and \( z \), then the extended block \( A' \) in \( \overline{A} \) is the unique block containing \( x \), \( y \) and \( z \).

Case 2. \( x, y \in X \) and \( z = \alpha^e_i \) for some \( i \), \( 1 \leq i \leq k+1 \). Consider the two points \( x \) and \( y \) and the parallel class \( \alpha^e_1 \) in \( D \). By Axiom (I3), either there exists a \( k \)-valent block \( A \) in \( \alpha^e_1 \) containing \( x \) and \( y \), or there exists a \( (k-1) \)-valent block \( A \) that contains \( x \) and \( y \) and is an \( r \)-transversal of \( \alpha^e_1 \). In either cases the extended block \( A' \) in \( \overline{A} \) is the unique block containing \( x \), \( y \) and \( z \).

Case 3. \( x \in X \), \( y = \alpha^e_i \) and \( z = \alpha^e_j \) with \( 1 \leq i < j \leq k+1 \). Let us consider the point \( x \) and the two parallel classes \( \alpha^e_1 \) and \( \alpha^e_j \) in \( D \). By Lemma 4.4, there exists a unique \( (k-1) \)-valent block \( A \) containing \( x \) which is a common \( r \)-transversal of \( \alpha^e_1 \) and \( \alpha^e_j \). Thus, \( A' \) is the unique block in \( S \) that contains \( x \), \( y \) and \( z \).

Case 4. \( x = \alpha^e_i \), \( y = \alpha^e_j \) and \( z = \alpha^e_m \) with \( 1 \leq i < j < m \leq k+1 \). The block \( \alpha^e_\infty \) is the unique block that contains \( x \), \( y \) and \( z \).

Proof of Lemma 4.2. It is clear that if we delete the block \( \alpha^e_\infty \) from \( S \), then the block-residual design \( S' \) thus obtained is isomorphic to \( D \). Moreover, since the \( k+1 \) new points and the extended blocks in \( S \) are uniquely determined, \( D \) is uniquely embeddable./

By virtue of this lemma, we see that if we can establish the parallel classes and the \( r \)-transversals that satisfy Axioms (I1)-(I3), then a PBRD(q) can be uniquely
4.3 The Three Classes of Blocks. Let $D=(X, \mathcal{B})$ be any PBRD$(q)$. For $k \in K = \{ q+1, q, q-1 \}$, we denote the set of $k$-valent blocks by $\mathcal{B}(k)$. Clearly, $\mathcal{B}$ is partitioned into the three classes, $\mathcal{B}(q+1)$, $\mathcal{B}(q)$ and $\mathcal{B}(q-1)$. Throughout this chapter, we shall use the alphabets $A$, $B$ and $C$ to denote members of $\mathcal{B}(q+1)$, $\mathcal{B}(q)$ and $\mathcal{B}(q-1)$ respectively; other alphabets will be used to denote blocks of various sizes. Let us first compute the order of $\mathcal{B}(k)$ for each $k$ in $K$.

Lemma 4.5. Let $D$ be a PBRD$(q)$. For $k \in K$, let $b(k)$ denote the number of $k$-valent blocks contained in $D$. Then

$$
\begin{align*}
b(q+1) &= \frac{1}{2} \cdot q(q-1)(q-2), \\
b(q) &= (q+1)(q-1), \\
b(q-1) &= \frac{1}{2} \cdot q^2(q+1).
\end{align*}
$$

(4.2)

Proof. Total number of blocks in $D$

$$
= q^3 + q - 1 = b(q+1) + b(q) + b(q-1). 
$$

(4.3)

Total number of triples $(x, y, z)$ where $x$, $y$ and $z$ are distinct points in $D$

$$
= \frac{q^2 - q}{3} = \left( \frac{q+1}{3} \right) b(q+1) + \left( \frac{q}{3} \right) b(q) + \left( \frac{q-1}{3} \right) b(q-1). 
$$

(4.4)

Next, let us count the number of ordered pairs $(x, E)$ where $x$ is a point incident with a block $E$ in $D$. Fixing a point $x$, there exists $\lambda_1$ choices of $E$ and there are $q^2 - q$ points
in D. Hence the total number of ordered pairs equals 
(q^2 + q)(q^2 - q). On the other hand, for a fixed k-valent block E, there are k choices of x. Hence,

\[(q^2 + q)(q^2 - q) = (q+1)b(q+1)+qb(q)+(q-1)b(q-1). \] (4.5)

Using Equations (4.3), (4.4) and (4.5), the parameters b(q+1), b(q) and b(q-1) are easily computed to be those given in (4.2).

**Lemma 4.6.** Let D be a PBRD(q). If for k ∈ K, r(k) denotes the number of k-valent blocks containing a given point in D, then for every point x in D,

\[r(q+1) = \frac{1}{2} (q+1)(q-2),\]
\[r(q) = q+1, \quad (4.6)\]
\[r(q-1) = \frac{1}{2} q(q+1).\]

**Proof.** Let x be any point in D. Since x is contained in \(q^2 + q\) blocks, we have

\[r(q+1)+r(q)+r(q-1) = q^2 + q. \] (4.7)

Next, let us count the number of triples (x, y, z) where y and z are points distinct from x and y ≠ z. Then,

\[\binom{q}{2} r(q+1)+\binom{q-1}{2} r(q)+\binom{q-2}{2} r(q-1) = \frac{q^2 - q - 1}{2}. \] (4.8)

Lastly, we count the number of ordered pairs (y, E) where y is a point distinct from x and both x and y are incident with the block E. For every point y distinct from x, there
are $q+1$ blocks containing both $x$ and $y$. Therefore,

$$qr(q+1)+(q-1)r(q)+(q-2)r(q-1) = (q^2-q-1)(q+1). \quad (4.9)$$

Combining Equations (4.7), (4.8) and (4.9), we get the parameters $r(q+1)$, $r(q)$ and $r(q-1)$ as those given in (4.6).

From these two lemmas, we observe that for each $k$ in $K$, $(X, \mathcal{O}(k))$ is an $S_{r(k)}(1,k,q^2-q)$. Even though none of the three designs is a 2-design, for each $k$ in $K$, $\lambda_2(k)$ takes only 2 values.

**Lemma 4.7.** Let $D$ be a PBDRD$(q)$ and for each $k \in K$, let $\lambda_2(k)$ denote the number of $k$-valent blocks containing two given points in $D$.

If $q \equiv 1 \pmod{2}$, then $\lambda_2(q)=0$ or 2 and $\lambda_2(q+1)=\lambda_2(q-1)$.

If $q \equiv 0 \pmod{2}$, then $\lambda_2(q)=1$ or $q+1$ and $\lambda_2(q+1)=\lambda_2(q-1)$.

Proof. Let $x$ and $y$ be two distinct points in $D$. Since $D$ is a 2-design,

$$\lambda_2(q+1)+\lambda_2(q)+\lambda_2(q-1) = \lambda_2 = q+1. \quad (4.10)$$

The blocks containing $x$ and $y$ partition the points of $D$ distinct from $x$ and $y$. Hence,

$$(q-1)\lambda_2(q+1)+(q-2)\lambda_2(q)+(q-3)\lambda_2(q-1) = q^2-q-2. \quad (4.11)$$

From Equations (4.10) and (4.11), we obtain
\[ \lambda_2(q+1) = \lambda_2(q-1). \] (4.12)

From Equations (4.10) and (4.12), we have

\[ \lambda_2(q+1) = \lambda_2(q-1). \]

Case 1. \( q \equiv 1 \pmod{2} \). Since \( q+1 \equiv 0 \pmod{2} \), from Equation (4.12), we see that \( \lambda_2(q) \equiv 0 \pmod{2} \).

Suppose \( \lambda_2(q) \not\equiv 0 \). Then there exists a block \( B \in \mathcal{O}(q) \) such that \( B \) contains \( x \) and \( y \). For each point \( y_i \) in \( B \), \( y_i \neq x \), the points \( x \) and \( y_i \) are contained in \( B \) and \( B \) is \( q \)-valent. Since \( \lambda_2(q) \equiv 0 \pmod{2} \), there exists at least one other \( q \)-valent block containing \( x \) and \( y_i \); let \( B_i \) be such a block. Clearly for \( i \neq j \), \( B_i \neq B_j \) and there are \( q-1 \) such \( B_i \)'s. But \( r(q) = q+1 \) implies that there exists another \( q \)-valent block \( B' \) containing \( x \). If \( B' \) contains \( x \) and \( y \), then \( \lambda_2(q) = 3 \) and contradicts that \( \lambda_2(q) \equiv 1 \pmod{2} \). Therefore, \( \lambda_2(q) = 2 \) and \( B' \) is not secant to \( B \). In fact, \( B' \) is the unique \( q \)-valent block that is tangent to \( B \) at \( x \).

Case 2. \( q \equiv 0 \pmod{2} \). Since \( q+1 \equiv 1 \pmod{2} \), from Equation (4.12), we see that \( \lambda_2(q) \equiv 1 \pmod{2} \).

Suppose \( \lambda_2(q) \not\equiv q+1 \). Then there exists a \((q+1)\)-valent block \( A \) containing \( x \) and \( y \). Let \( B_1, \ldots, B_{\lambda_2(q)} \) be the \( q \)-valent blocks containing \( x \) and \( y \), and \( B_1', \ldots, B_n' \) be the other \( q \)-valent blocks containing \( x \). For every point \( z \) in \( A \), distinct from \( x \) and \( y \), \( z \) is not contained in \( B_i' \) for all \( i \), \( 1 \leq i \leq \lambda_2(q) \). Since \( \lambda_2(q) \equiv 1 \pmod{2} \), \( z \) is contained in at
least one $B'_1, 1 \leq n$. But there are $q-1$ points in $A$ that are
distinct from $x$ and $y$. Hence

$$n \geq q-1. \quad (4.13)$$

Since $r(q)=q+1$, $\lambda_2(q)+n=q+1$, or equivalently,

$$n = q+1-\lambda_2(q). \quad (4.14)$$

From Equations (4.13) and (4.14), we have $\lambda_2(q) \leq 2$. But
$\lambda_2(q) \equiv 1 \pmod{2}$; hence $\lambda_2(q) \equiv 1.$/

**Corollary 4.8.** Let $B \in \mathcal{B}(q)$ and $x$ be a point in $B$.

If $q \equiv 1 \pmod{2}$, then there exists a unique $q$-valent
block tangent to $B$ at $x$.

If $q \equiv 0 \pmod{2}$, then there exists no $q$-valent block
tangent to $B$ at $x$.

Proof. Case 1. $q \equiv 1 \pmod{2}$. The result is clear from the
previous proof.

Case 2. $q \equiv 0 \pmod{2}$. If for every point $y$ in $B$, $y \neq x$, $B$ is
the only $q$-valent block containing $x$ and $y$, then every
other $q$-valent block containing $x$ is tangent to $B$; furthermore,
they are mutually tangent. This implies that there
are $q^2$ points in $D$ contradicting that $v=q^2-q$. Hence, there
exists a point $y$ in $B$ such that every $q$-valent block
containing $x$ contains $x$ and $y$. Hence, there exists no
$q$-valent block tangent to $B$ at $x.$/

We shall need these lemmas later. Meanwhile, let us
divert our attentions to blocks that are tangent to each other in D.

\section{4.4 Tangents} Let us recall that two blocks $E$ and $E'$ in $D$ are said to be tangent to each other if they intersect in exactly one point, and if they intersect in exactly two points, then they are secant to one another. Furthermore, if $E$ and $E'$ are tangent at $x$ and there exists a $(q+1)$-valent block $A$ which is tangent to $E$ at $x$ and secant to $E'$, then $E$ is said to be r-tangent to $E'$ at $x$. In this section, we shall establish the existence of r-tangents in $D$.

\textbf{Lemma 4.9.} Let $i \in \{0, 1, 2\}$ and $E$ be a fixed $(q+1-i)$-valent block in $D$. If $x$ is a point incident with $E$ and $y$ is a point not in $E$, then there exist exactly $i+1$ circles which are tangent to $E$ at $x$ and contain $y$.

\textbf{Proof.} Let $z$ be a point in $E$ which is distinct from $x$. The three points $x$, $y$ and $z$ determine a unique block $E'$ in $D$, and $E'$ is clearly secant to $E$. Since there are $q-i$ distinct points in $E$ that are different from $x$, there are $q-i$ blocks in $D$ which contain $x$ and $y$ and are secant to $E$. This implies that all other blocks containing $x$ and $y$ are tangent to $E$ at $x$. Since there are $q+1$ blocks containing $x$ and $y$ and $(q+1) - (q-1) = i+1$, the conclusion of the lemma follows.
Lemma 4.10. Let \( i \in \{0,1,2\} \) and \( E \) be a fixed \((q+1-i)\)-valent block in \( D \). If \( x \) is a point incident with \( E \), then there exist exactly \((i+1)q-1\) blocks which are tangent to \( E \) at \( x \).

Proof. For every point \( y \) in \( E \), different from \( x \), the points \( x \) and \( y \) are contained in \( q \) blocks other than \( E \). Hence, there are \( q(q-1) \) blocks which contain \( x \) and are secant to \( E \). But every block incident with \( x \) other than \( E \) is either a secant or a tangent of \( E \), and since there are \( q^2+q-1 \) blocks incident with \( x \) other than \( E \), the number of blocks tangent to \( E \) at \( x \) is \( q^2+q-1-q(q-1) \), or \( q(i+1)-1 \).

Lemma 4.11. If \( A \) is a \((q+1)\)-valent block and \( x \) is a point in \( A \), then the \( q-1 \) tangents of \( A \) at \( x \) are mutually tangent to each other at \( x \).

Proof. Let \( E \) and \( E' \) be two distinct tangents of \( A \) at \( x \), and suppose \( E \) and \( E' \) intersect at two distinct points \( x \) and \( y \). Since \( y \) is a point not in \( A \), by Lemma 4.9, there exists a unique block containing \( y \) which is tangent to \( A \) at \( x \). But both \( E \) and \( E' \) contain \( y \) and are tangent to \( A \) at \( x \). Hence, \( E \) and \( E' \) are mutually tangent at \( x \).

By virtue of this lemma, we see that the tangents of \( A \) at \( x \) together with \( A \) constitute a pencil in \( D \) with carrier \( x \).
Proposition 4.12 If $\mathcal{A}$ is a pencil with carrier $x$ such that $\mathcal{A}$ contains a $(q+1)$-valent block $A$, then $\mathcal{A}$ contains $q$ blocks and $\mathcal{A}$ partitions the points in $D$ that are distinct from $x$.

Proof. Let $A$ be a $(q+1)$-valent block in $\mathcal{A}$. For every point $y$ distinct from $x$, there exists a unique block $E$ in $\mathcal{A}$ such that $E$ contains $y$ and $E$ is tangent to $A$ at $x$. Hence, $\mathcal{A}$ partitions the points that are distinct from $x$. It is clear from the previous lemma that $\mathcal{A}$ contains $q$ blocks. 

From the above, we see that a $(q+1)$-valent block $A$ cannot be $r$-tangent to any other block $E$. The converse is also valid.

Lemma 4.13. Let $E$ be a block in $D$. If $A$ and $A'$ are two distinct $(q+1)$-valent blocks that are tangent to $E$ at a point $x$, then $A$ and $A'$ are mutually tangent at $x$.

Proof. Suppose $A$ is not tangent to $A'$ at $x$. Let $x$ and $y$ be the two points of intersection of $A$ and $A'$. For every point $z$ in $A'$ such that $z \neq x$ and $z \neq y$, there exists a unique block tangent to $A$ at $x$ which contains $z$. Hence, there are $q-1$ blocks that are tangent to $A$ at $x$ and are secant to $A'$. But there are only $q-1$ tangents of $A$ at $x$ of which $E$ is one. This contradicts that $E$ is tangent to $A'$ at $x$. Hence, $A$ and $A'$ are mutually tangent. 

From this, we observe that if $A$ is a $(q+1)$-valent
block and A is not r-tangent to E, then E is not r-tangent to A. We shall establish this property for every block E in D. That is, we want to show that the relation, 'r-tangency', is a symmetric relation.

**Lemma 4.14.** Let \( q \geq 4 \) and let E be a block in D such that E is not \((q+1)\)-valent. If \( x \) is a point in E, then there exists at least one \((q+1)\)-valent block A tangent to E at \( x \).

**Proof.** Let \( |E| = q+1-i \) where \( i \in \{1, 2\} \). For \( k \in K \), let \( t_i(k) \) denote the number of \( k\)-valent blocks tangent to E at \( x \). By Lemma 4.10, there are \((i+1)q-1\) blocks tangent to E at \( x \).

Hence,

\[
t_i(q+1)+t_i(q)+t_i(q-1) = (i+1)q-1. \tag{4.15}
\]

Next, we count the number of ordered pairs \((y, E')\) such that \( y \in E' \), \( y \notin E \) and \( E' \) is tangent to E at \( x \). For each point \( y \) not in E, there are \( i+1 \) blocks containing \( y \) and tangent to E at \( x \). Since there are \( v-(q+1-i) \) such points \( y \), we have

\[
qt_i(q+1)+(q-1)t_i(q)+(q-2)t_i(q-1) = (i+1)(q^2-2q-1+i). \tag{4.16}
\]

Combining Equations (4.15) and (4.16), we obtain

\[
2t_i(q+1)+t_i(q) = i^2+q-3. \tag{4.17}
\]

**Case 1.** \( |E| = q \) (i.e. \( i=1 \)). By Corollary 4.8 there exists
either none or exactly one \( q \)-valent block tangent to \( E \) at \( x \), depending on whether \( q \) is even or odd. Hence, \( t_1(q) \leq 1 \).

From Equation (4.16), we obtain the following inequality,

\[ 2t_1(q+1) \geq q-3. \]

For \( q \geq 4 \), \( t_1(q+1) > 0 \). Hence, there exists at least one \((q+1)\)-valent block tangent to \( E \) at \( x \).

Case 2. \( |E| = q-1 \) (i.e. \( i=2 \)). Suppose there exists no \((q+1)\)-valent block tangent to \( E \) at \( x \), then \( t_2(q+1) = 0 \) and \( t_2(q) = q+1 \). Hence, every \( q \)-valent block that contains \( x \) is tangent to \( E \). Consider a point \( y \) in \( E \) distinct from \( x \). The points \( x \) and \( y \) are not contained in any \( q \)-valent block.

If \( q \equiv 0 \pmod{2} \), then by Lemma 4.7, there exists at least one \( q \)-valent block containing \( x \) and \( y \). Hence, we arrive at a contradiction. Therefore, \( t_2(q+1) \neq 0 \), or there exists at least one \((q+1)\)-valent block tangent to \( E \) at \( x \).

If \( q \equiv 1 \pmod{2} \), then by Lemma 4.7, for each \( y \) in \( E \) distinct from \( x \) there exist \( \frac{1}{2}(q+1) \) \((q-1)\)-valent blocks containing \( x \) and \( y \). Thus, there are \( \frac{1}{2}(q+1)(q-2) \) \((q-1)\)-valent blocks containing \( x \) and secant to \( E \). Since there are \( \frac{1}{2}(q+1)q \) \((q-1)\)-valent blocks containing \( x \), there are \( q \) \((q-1)\)-valent blocks tangent to \( E \) at \( x \). But if \( t_2(q+1) = 0 \) and \( t_2(q) = q+1 \), then by Equation (4.17), \( t_2(q-1) = 2q-2 \). Hence, \( q = 2q-2 \) or \( q = 2 \). This contradicts that \( q \equiv 1 \pmod{2} \). Consequently, \( t_2(q+1) \neq 0 \), and there exists at least one \((q+1)\)-valent block tangent to \( E \) at \( x \).
Lemma 4.15. Let E and E' be two distinct blocks in D. If E is r-tangent to E' at x, then E' is r-tangent to E at x.

Proof. If E is r-tangent to E' at x, then there exists a (q+1)-valent block A that is tangent to E at x and is secant to E'. Suppose E' is not r-tangent to E at x; then for every (q+1)-valent block A' that is tangent to E' at x, A' is also tangent to E. Consider the two (q+1)-valent blocks A and A'. Since both A and A' are tangent to E at x, A is tangent to A' at x. But E' is also tangent to A' at x; hence by Lemma 4.11, E' is tangent to A at x. This contradicts that A is secant to E'. Therefore, E' is r-tangent to E at x.

Thus we see that r-tangency is a symmetric relation. Let us define two distinct blocks E and E' to be M-tangent at x if and only if E is tangent to E' but is not r-tangent to E' at x. A pencil with carrier x is called an M-pencil if the blocks in the pencil are mutually M-tangent.

Lemma 4.16. If E is M-tangent to E' at x and E' is M-tangent to E'' at x, then E is M-tangent to E'' at x.

Proof. Let A be a (q+1)-valent block in D such that A is tangent to E at x. Since E is not r-tangent to E', A is tangent to E' at x. But E' is M-tangent to E'' at x;
hence A is also tangent to $E''$ at x. Consequently, E is not r-tangent to $E''$ at x, or equivalently, E is $M$-tangent to $E''$ at x.

From this lemma, we observe that any pencil that contains a $(q+1)$-valent block is an $M$-pencil. Next, we shall study blocks that are tangent to a given $q$-valent block $B$.

**Lemma 4.17.** Let $B$ be a fixed $q$-valent block in D and $x$ be a point incident with $B$. If A is a $(q+1)$-valent block tangent to $B$ at x, then there exist exactly q blocks which are tangent to $B$ at x and are secant to A.

**Proof.** For $i=1,2$, let

$$T_i = \{ E \mid E \text{ is a block in } D \text{ tangent to } B \text{ at } x, E \neq A \text{ and } |E \cap A| = i \}.$$ 

The two sets $T_1$ and $T_2$ partition the set of tangents of $B$ other than A at x. By Lemma 4.10,

$$|T_1| + |T_2| = 2q - 2. \quad (4.18)$$

Next, we count the number of ordered pairs $(y,E)$ where E is tangent to $B$ at x, $E \neq A$ and $y \in E \cap A$. If y is distinct from x, then by Lemma 4.9, there exist two blocks containing y and tangent to $B$ at x, of which one is A. Hence, there exists a unique choice of E. If $y=x$, then by Lemma
4.10, there are $2q-2$ choices of $E$. Since there are $q$ points in $A$ that are distinct from $x$, the number of ordered pairs $(y,E)$ is $q+2q-2$. On the other hand, if $E$ is tangent to $B$ at $x$ and $|E \cap A|=i$, $i=1,2$, then there are $i$ choices of $y$. Thus,

$$|T_1|+2|T_2|=q+2q-2.$$  \tag{4.19}

Using Equations (4.18) and (4.19), we obtain $|T_2|=q$. Therefore, there are $q$ blocks tangent to $B$ at $x$ and secant to $A$.

It should be noted that $B$ is $r$-tangent to these $q$ blocks at $x$. Next, we show that they are mutually tangent at $x$.

**Lemma 4.18.** Let $B$ be a $q$-valent block and $A$ be a $(q+1)$-valent block tangent to $B$ at a point $x$. If $E$ and $E'$ are two distinct blocks tangent to $B$ at $x$ and secant to $A$, then $E$ and $E'$ are mutually tangent at $x$.

**Proof.** Suppose $E$ and $E'$ intersect each other at two points $x$ and $y$. If $ycA$, then $E$, $E'$ and $A$ are three blocks containing $y$ and tangent to $B$ at $x$. This contradicts the fact that there exist only two such tangents (by Lemma 4.9). Thus, the point $y$ is not contained in $A$.

Since $y \notin A$ and $A$ is $(q+1)$-valent, by Lemma 4.9, there exists a unique block $E''$ containing $y$ and tangent to $A$ at $x$. But both $E''$ and $B$ are tangent to $A$ at $x$, by the
previous lemma, $E''$ and $B$ are mutually tangent at $x$. Hence, $E$, $E'$ and $E''$ are three blocks tangent to $B$ at $x$ and containing $y$. This contradicts that there are only two such blocks.

Therefore, $E$ and $E'$ cannot intersect at two points and hence, they are mutually tangent at $x$.\/

**Lemma 4.19.** Let $A$ and $B$ be $(q+1)$-valent and $q$-valent blocks respectively such that $A$ is tangent to $B$ at $x$. If $C$ is tangent to $B$ at $x$ and secant to $A$, then $C$ is a $(q-1)$-valent block.

**Proof.** Let $C_1, \ldots, C_q$ be the $q$ blocks that are tangent to $B$ at $x$ and secant to $A$. Since for $1 \leq i < j \leq q$, $C_i$ and $C_j$ are mutually tangent at $x$ and each $C_i$ has at least $q-2$ points other than $x$, we have

$$q(q-2) \leq \sum_{i=1}^{q} (|C_i|-1) \leq q^2 - q - |B|.$$ 

But $B$ is $q$-valent, hence $\sum_{i=1}^{q} (|C_i|-1) = q(q-2)$. Consequently, each $C_i$ is a $(q-1)$-valent block.\/

From the proof, we also observe that the blocks $C_1, \ldots, C_q$ and $B$ partition the points distinct from $x$. In fact, we shall show that they form an $r$-pencil in $D$ with carrier $x$.

**Lemma 4.20.** Let $B, C_1, \ldots, C_q$ be as defined in the previous lemma and $x$ be their common point. Let
$T=\{B, C_1, \ldots, C_q\}$. If $E$ and $E'$ are two distinct blocks in $T$ and $A$ is a $(q+1)$-valent block tangent to $E$ at $x$, then $A$ is secant to $E'$.

Proof. Consider a point $y$ in $A$ distinct from $x$. Since the blocks in $T$ partition the points distinct from $x$, there exists a unique block $E_y$ in $T$ such that $E_y$ contains $x$ and $y$. Thus, there are $q$ blocks in $T$ that are secant to $A$.

But $T$ consists of $q+1$ blocks; hence there exists a unique block in $T$ that is tangent to $A$ at $x$, and $E$ is such a block. Thus, $|A\cap E'|=2$ and the conclusion of the lemma follows.

Proposition 4.21. Let $B$ be a $q$-valent block in $D$ and $x$ be a point in $B$. The pair $(x, B)$ determines a unique $r$-pencil in $D$ with carrier $x$, denoted by $\mathcal{P}(x, B)$. Furthermore, the $r$-pencil $\mathcal{P}(x, B)$ consists of $q+1$ blocks and they partition the points distinct from $x$. We shall call $B$ the carrier block of $\mathcal{P}(x, B)$.

Proof. Let $T=\{B, C_1, \ldots, C_q\}$ be defined as above. For every block in $T$, there exists a $(q+1)$-valent block tangent to $E$ at $x$. Hence, by the previous lemma, $E$ is $r$-tangent to every other block in $T$. Thus, they form an $r$-pencil at $x$.

Next, we count the number of points contained in $T$. Since the $q$ $(q-1)$-valent blocks and the $q$-valent block $B$ in $T$ are mutually tangent at $x$, we have $q(q-2)+q=q^2-q$. 
points contained in T. Thus, the blocks in T partition the points distinct from x.

Since for every (q+1)-valent block A that is tangent to B at x, A is secant to $C_i$ in T, 1 ≤ i ≤ q, and by Lemma 4.17, $C_1, \ldots, C_q$ are the only blocks that are r-tangent to B at x. Thus, $p(x, B)$ is the unique r-pencil with carrier x and carrier block B./

Corollary 4.22. Let B be a q-valent block in D. B is r-tangent to exactly $q^2$ (q-1)-valent blocks in D.

Corollary 4.23. Let B be a q-valent block in D. If x is a point not in B, then B is r-tangent to exactly q blocks in D that contain x. Furthermore, these q blocks are (q-1)-valent.

Proof. Let y be a point in B. $p(y, B)$ partition the points distinct from y; hence there exists a unique block $C_y$ containing x and is r-tangent to B at y. Since there are q points in B, there are q blocks containing x that are r-tangent to B. Clearly, these are the only r-tangents of B that contain x./

Eventually, we would like to show that these q (q-1)-valent blocks that are r-tangent to B and contain x determine an r-pencil in D with carrier x. Let us first conclude our discussions on q-valent blocks in the following theorem.
Theorem 4.24. Let $B$ be a $q$-valent block. If $x$ is a point in $B$, then there exists exactly one $r$-pencil and one $M$-pencil with carrier $x$ that contain $B$.

Proof. Since $B$ is $q$-valent, by Lemma 4.10, there are $2q-1$ blocks which are tangent to $B$ at $x$. By Proposition 4.21, $q$ of these tangents together with $B$ form an $r$-pencil $p(x,B)$ with carrier $x$. Among the remaining $q-1$ tangents of $B$, there is a $(q+1)$-valent block $A$. Since every tangent of $B$ at $x$ that is not $r$-tangent to $B$ is tangent to $A$ at $x$, these $q-2$ tangents of $B$ together with $A$ and $B$ form an $M$-pencil with carrier $x$.

Finally, we shall study the tangents of a $(q-1)$-valent block $C$ at a point $x$.

Lemma 4.25. Let $C$ be a $(q-1)$-valent block in $D$ and $x$ be a point in $C$. If $g(C,x)$ denotes the number of $q$-valent blocks $r$-tangent to $C$ at $x$, then $\text{ave } g(C,x)=2$.

Proof. We count the number of ordered pairs $(B,C)$ where $B$ and $C$ are $q$-valent and $(q-1)$-valent blocks respectively and $B$ is $r$-tangent to $C$ at $x$. For every $q$-valent block $B$ containing $x$, there are $q$ choices of $C$. Since there are $q+1$ $q$-valent blocks containing $x$, there are $q(q+1)$ ordered pairs $(B,C)$. On the other hand, if $C$ is a $(q-1)$-valent block containing $x$, then there are $g(C,x)$ choices of $B$. Hence $\sum g(C,x)=q(q+1)$. But there are $\frac{1}{2}q(q+1)$ $(q-1)$-valent
blocks containing \( x \), thus \( g(.,x)=2 \).

Proposition 4.26. Let \( C \) be a \((q-1)\)-valent block in \( D \). If \( x \) is a point in \( C \), then there exist exactly two \( q \)-valent blocks that are \( r \)-tangent to \( C \) at \( x \).

Proof. We shall show that there exist at most two \( q \)-valent blocks that are \( r \)-tangent to \( C \) at \( x \), then using the previous lemma, we obtain the conclusion of the proposition.

Let us recall that for \( k \in K \), \( t_2(k) \) denotes the number of \( k \)-valent blocks tangent to \( C \) at \( x \). From Equation (4.17) we have

\[
2t_2(q+1)+t_2(q) = q+1. \tag{4.20}
\]

For \( k \in K \), let \( t'(k) \) denote the number of \( k \)-valent blocks \( M \)-tangent to \( C \) at \( x \). Clearly, \( t'(q+1)=t_2(q+1) \). Let \( A \) be a \((q+1)\)-valent block \( M \)-tangent to \( C \) at \( x \), and let \( \mathcal{P} \) denote the \( M \)-pencil with carrier \( x \) that contains \( A \). \( C \) is a block in \( \mathcal{P} \). Since for every block \( E \) in \( \mathcal{P} \), \( E \neq C \), \( E \) is \( M \)-tangent to \( C \) at \( x \) and they are the only \( M \)-tangents of \( C \) at \( x \),

\[
t_2(q+1)+t'(q)+t'(q-1)+1 = |\mathcal{P}| = q. \tag{4.21}
\]

On the other hand, the blocks in \( \mathcal{P} \) partition the points that are distinct from \( x \), we have

\[
qt_2(q+1)+(q-1)t'(q)+(q-2)(t'(q)+1)+1 = q^2-q. \tag{4.22}
\]
Using Equations (4.21) and (4.22), we obtain

\[ 2t_2(q+1) + t'(q) = q-1. \]  

(4.23)

Combining Equations (4.20) and (4.23), we get

\[ t_2(q) - t'(q) = 2. \]

Thus, there exists at most 2 q-valent blocks r-tangent to C at x and the proof is complete.

**Proposition 4.27.** Let C be a (q-1)-valent block in D. If x is a point in C, then C is contained in exactly one M-pencil with carrier x. Furthermore, there are at least 2 r-pencils with carrier x that contain C.

**Proof.** From the proof of the previous lemma, if A is a (q+1)-valent block tangent to C at x, then the M-pencil with carrier x that contains A is the unique M-pencil with carrier x that contains C.

Since there are exactly 2 q-valent blocks r-tangent to C at x and each of these two blocks determines a unique r-pencil with carrier x, C is contained in at least two r-pencils.

We shall show that C is contained in exactly two r-pencils with carrier x.

**Lemma 4.28.** Let \( \mathcal{P}(x,B) \) and \( \mathcal{P}(x,B') \) be two distinct r-pencils with carrier x and carrier block B and B'
respectively. If \( c(B,B') \) denotes the number of common blocks in \( \mathcal{Q}(x,B) \) and \( \mathcal{Q}(x,B') \), then \( \text{ave } c(\ldots) = 1 \) where average runs over all pairs of distinct \( q \)-valent blocks containing \( x \).

Proof. Let us count the number of triples \( (\mathcal{Q}(x,B), \mathcal{Q}(x,B'), C) \) where \( \mathcal{Q}(x,B) \) and \( \mathcal{Q}(x,B') \) are distinct \( r \)-pencils with carrier \( x \) and \( C \) is a common block in \( \mathcal{Q}(x,B) \) and \( \mathcal{Q}(x,B') \). For every \( (q-1) \)-valent block \( C \) that contains \( x \), there are exactly two \( q \)-valent blocks \( B \) and \( B' \) that contain \( x \) and are \( r \)-tangent to \( C \). Hence, there are exactly two \( r \)-pencils with carrier \( x \) that contain a \( q \)-valent block and \( C \). Thus, there are two choices of the pair \( \mathcal{Q}(x,B) \), \( \mathcal{Q}(x,B') \)). But there are \( \frac{1}{2}(q+1)q \) choices of \( C \) that contains \( x \), so the total number of triples is \( (q+1)q \). On the other hand, for every distinct pair \( \mathcal{Q}(x,B), \mathcal{Q}(x,B') \)), there are \( c(B,B') \) choices of \( C \). Hence,

\[
(\sum_{\mathcal{Q}(x,B), \mathcal{Q}(x,B')} c(B,B') = q(q+1).
\]

But for every \( q \)-valent block \( B \) containing \( x \), there corresponds a unique \( r \)-pencil \( \mathcal{Q}(x,B) \); hence there are \( (q+1)q \) distinct pairs \( \mathcal{Q}(x,B), \mathcal{Q}(x,B') \)). Thus,

\[
(q+1)q \text{ ave } c(\ldots) = q(q+1),
\]

or equivalently, \( \text{ave } c(\ldots) = 1 \).
Lemma 4.29. If \( \mathcal{P}(x, B) \) and \( \mathcal{P}(y, B') \) are two distinct \( r \)-pencils with carriers \( x \) and \( y \), and carrier blocks \( B \) and \( B' \) respectively, then \( |\mathcal{P}(x, B) \cap \mathcal{P}(y, B')| \leq 1 \). In particular, if \( x = y \), then \( |\mathcal{P}(x, B) \cap \mathcal{P}(x, B')| = 1 \).

Proof. Case 1. \( x \neq y \) and \( B = B' \). Clearly \( \mathcal{P}(x, B) \cap \mathcal{P}(y, B') = B \).

Case 2. \( x \neq y \) and \( B \neq B' \). Suppose \( |\mathcal{P}(x, B) \cap \mathcal{P}(y, B')| \geq 2 \) and let \( C, C' \) be two common \( r \)-tangents of \( B \) and \( B' \) at \( x \) and \( y \) respectively. Since by Proposition 4.21, the blocks in \( \mathcal{P}(x, B) \) partition the points distinct from \( x \), there exists a unique block \( E \) in \( \mathcal{P}(x, B) \) that contains \( y \). But this contradicts that both \( C \) and \( C' \) contain \( y \). Hence, \( |\mathcal{P}(x, B) \cap \mathcal{P}(y, B')| \geq 1 \).

Case 3. \( x = y \) and \( B \neq B' \). For every point \( z \) in \( B' \), \( z \neq x \), there exists a unique block in \( \mathcal{P}(x, B) \) that contains \( z \). Since there are \( q-1 \) points in \( B' \) that are distinct from \( x \), there are \( q-1 \) blocks in \( \mathcal{P}(x, B) \) that are secant to \( B' \). But \( \mathcal{P}(x, B) \) consists of \( q+1 \) blocks; hence there are two blocks in \( \mathcal{P}(x, B) \) that are tangent to \( B' \) at \( x \). If both these blocks are \( M \)-tangent to \( B' \), then they are contained in the \( M \)-pencil with carrier \( x \) that contains \( B' \); hence they are mutually \( M \)-tangent to each other. This contradicts that they are both in \( \mathcal{P}(x, B) \). Thus, there exists at least one block in \( \mathcal{P}(x, B) \) that is \( r \)-tangent to \( B' \) at \( x \). By the previous lemma, we obtain that there exists exactly one, that is, \( |\mathcal{P}(x, B) \cap \mathcal{P}(x, B')| = 1 \).
Lemma 4.30. Let $C$ be a $(q-1)$-valent block and $x$ be a point in $C$. Let $\mathcal{C}(x,B)$ and $\mathcal{C}(x,B')$ be the two $r$-pencils with carrier $x$ and carrier blocks $B$ and $B'$ that contain $C$. If $C'$ is a $(q-1)$-valent block that is $r$-tangent to $C$ at $x$, then $C'$ is contained in exactly one of the two $r$-pencils $\mathcal{C}(x,B)$ and $\mathcal{C}(x,B')$.

Proof. Since each of the two $r$-pencils consists of $q+1$ blocks and $C$ is a common block in $\mathcal{C}(x,B)$ and $\mathcal{C}(x,B')$, there are $2q$ blocks that are $r$-tangent to $C$ at $x$ and are contained in $\mathcal{C}(x,B)$ and $\mathcal{C}(x,B')$. By Lemma 4.10, there are $3q-1$ tangents of $C$ at $x$ of which $q-1$ are contained in the unique $M$-pencil with carrier $x$ that contains $C$. Thus, there are only $2q$ blocks that are $r$-tangent to $C$ at $x$, and they are contained in either $\mathcal{C}(x,B)$ or $\mathcal{C}(x,B')$. Clearly, an $r$-tangent $C'$ of $C$ cannot be contained in both $\mathcal{C}(x,B)$ and $\mathcal{C}(x,B')$; otherwise $C$ and $C'$ are two common blocks of the $r$-pencils and this contradicts the previous lemma.

Lemma 4.31. Let $C$ and $C'$ be two distinct $(q-1)$-valent blocks. If $C$ is $r$-tangent to $C'$ at $x$, then there exists a unique block which is $r$-tangent to $C'$ and $M$-tangent to $C$ at $x$.

Proof. Let $\mathcal{C}$ denote the $M$-pencil with carrier $x$ that contains $C$. For every point $y$ in $C'$, $y \neq x$, there exists a unique block $E$ in $\mathcal{C}$ such that $E$ contains $x$ and $y$. Hence,
there are \( q-2 \) blocks in \( \mathcal{F} \) that are \( M \)-tangent to \( C \) but are secant to \( C' \). But \( \mathcal{F} \) consists of \( q \) blocks of which one is \( C \); hence there exists a unique block \( E \) \( M \)-tangent to \( C \) at \( x \) which is tangent to \( C' \). Clearly, \( E \) is not \( M \)-tangent to \( C' \) at \( x \); otherwise, \( C' \in \mathcal{F} \); this contradicts that \( C \) is \( r \)-tangent to \( C' \). Hence \( E \) is \( r \)-tangent to \( C' \) at \( x \). /

**Lemma 4.32.** Let \( C \) and \( C' \) be two distinct \((q-1)\)-valent blocks containing \( x \). If \( C \) is \( r \)-tangent to \( C' \) at \( x \), then there exist exactly \( q \) blocks \( r \)-tangent to both \( C \) and \( C' \) at \( x \).

Proof. Let \( \mathcal{P}(x,B) \) be an \( r \)-pencil with carrier \( x \) and carrier block \( B \) that contains both \( C \) and \( C' \). Since \( \mathcal{P}(x,B) \) contains \( q+1 \) blocks, there are at least \( q-1 \) blocks that are \( r \)-tangent to both \( C \) and \( C' \).

Let \( \mathcal{P}(x,B') \) be the other \( r \)-pencil containing \( C \). If \( E \) is a block \( r \)-tangent to \( C \) at \( x \) and \( E \notin \mathcal{P}(x,B) \), then \( E \in \mathcal{P}(x,B') \). We shall show that there exists exactly one block \( E \) in \( \mathcal{P}(x,B') \) such that \( E \) is \( r \)-tangent to both \( C \) and \( C' \) at \( x \).

For every point \( y \) in \( C' \), \( y \neq x \), there exists a unique block \( E \) in \( \mathcal{P}(x,B') \) such that \( E \) contains \( x \) and \( y \). Hence, there are \( q-1 \) \( r \)-tangents of \( C \) at \( x \) that are secant to \( C' \). But \( \mathcal{P}(x,B') \) contains \( q+1 \) blocks, of which one is \( C \); hence there are two blocks in \( \mathcal{P}(x,B') \) that are tangent to \( C' \) at \( x \). By the previous lemma, one of these two blocks is \( M \)-tangent to \( C' \) at \( x \). Therefore, there exists a unique
block \( E \) in \( \mathcal{O}(x,B') \) such that \( E \) is \( r \)-tangent to \( C' \) at \( x \).

Consequently, there are exactly \( q \) blocks \( r \)-tangent to both \( C \) and \( C' \) at \( x \).

Proposition 4.33. Let \( C \) be a \((q-1)\)-valent block. If \( x \) is a point in \( C \), then \( C \) is contained in exactly two \( r \)-pencils with carrier \( x \). Moreover, every \( r \)-pencil in \( D \) contains a unique \( q \)-valent carrier block.

Proof. Suppose \( \mathcal{O} \) is another \( r \)-pencil containing \( C \) such that \( \mathcal{O} \) is distinct from \( \mathcal{O}(x,B) \) and \( \mathcal{O}(x,B') \). Let \( C' \) be a block in \( \mathcal{O} \) and \( C' \) be in \( \mathcal{O}(x,B) \). Since \( \mathcal{O} \neq \mathcal{O}(x,B) \), there exists a block \( C'' \) in \( \mathcal{O} \) such that \( C'' \notin \mathcal{O}(x,B') \). By the previous lemma, \( C'' \) is the only other block in \( \mathcal{O} \). Thus, \( |\mathcal{O}| = 3 \). But by the definition of an \( r \)-pencil, \( |\mathcal{O}| \leq 4 \); hence \( \mathcal{O}(x,B) \) and \( \mathcal{O}(x,B') \) are the only two \( r \)-pencils with carrier \( x \) that contain \( C \).

Proposition 4.34. Let \( E_1, E_2, E_3 \) be mutually tangent at a point \( x \) such that they are not contained in any \( M \)-pencil or any \( r \)-pencil with carrier \( x \). If \( \mathcal{O} \) is a maximal set of mutually tangent blocks containing \( x \) such that \( \mathcal{O} \) contains \( E_1, E_2 \) and \( E_3 \), then \( \mathcal{O} \) contains at most 4 blocks.

Proof. Since \( E_1, E_2 \) and \( E_3 \) are not contained in any \( M \)-pencil or any \( r \)-pencil, either they are mutually \( r \)-tangent and are contained in 2 distinct \( r \)-pencils at \( x \), or \( E_1 \) is
M-tangent to $E_2$.

Case 1. $E_1$, $E_2$ and $E_3$ are mutually $r$-tangent. From the previous proposition, there exists no other block that is $r$-tangent to all $E_i$'s at $x$. Let $E$ be a block in $\mathcal{P}$, $E \not\in E_i$, $i=1,2,3$. Without loss of generality, $E$ is $M$-tangent to $E_1$ at $x$, then $E$ is not $M$-tangent to $E_2$ and $E_3$. Otherwise, $E_1$ and $E_2$ are mutually $M$-tangent and this contradicts our assumption. Hence, $E_1$ is $r$-tangent to $E_2$ at $x$. By Lemma 4.31, $E$ is the unique block that is $M$-tangent to $E_1$ and $r$-tangent to $E_2$ at $x$. Thus, $|\mathcal{P}| \leq 4$.

Case 2. $E_1$ is $M$-tangent to $E_2$. Clearly, $E_3$ is $r$-tangent to both $E_1$ and $E_2$ at $x$. Moreover, $E_1$, $E_3$ and $E_2$, $E_3$ are contained in distinct $r$-pencils at $x$. Let $E \in \mathcal{P}$, $E \not\in E_i$, $i=1,2,3$.

Subcase 2.1. $E$ is $r$-tangent to $E_i$, $i=1,2,3$. Since the pairs $(E_1, E_3)$ and $(E_2, E_3)$ are in distinct $r$-pencils at $x$, either $E$, $E_1$, $E_3$ or $E$, $E_2$, $E_3$ are three mutually $r$-tangent blocks at $x$ that are contained in distinct $r$-pencils. By case 1, $|\mathcal{P}| \leq 4$.

Subcase 2.2. $E$ is $M$-tangent to $E_i$ for some $i$, $1 \leq i \leq 3$. Suppose $E$ is $M$-tangent to $E_1$ at $x$, then $E$ is not $M$-tangent to $E_3$; otherwise, $E_1$ and $E_3$ are mutually $M$-tangent at $x$. But then $E_2$ and $E$ are 2 blocks that are $M$-tangent to $E_1$ and $r$-tangent to $E_3$; this contradicts that there exists a unique such block. Hence, $E$ is $r$-tangent to $E_1$ at $x$. Similarly, $E$ is $r$-tangent to $E_2$ at $x$. Thus, $E$ is $M$-tangent
Suppose $E' \in \mathcal{E}$, $E' \neq E, E_1$, $i = 1, 2, 3$. Using the same arguments as above, $E'$ is $r$-tangent to both $E_1$ and $E_2$ at $x$. If $E'$ is also $r$-tangent to $E_3$, then by subcase 2.1, $|\mathcal{E}| \leq 4$. This contradicts that $E' \neq E$. Hence, $E'$ is $M$-tangent to $E_3$ at $x$. But then $E$ and $E'$ are two distinct blocks that are $M$-tangent to $E_3$ and $r$-tangent to $E_1$ at $x$. This contradicts that there exists only one such block. Hence $E'$ does not exist, and $|\mathcal{E}| \leq 4$.

§4.5 Parallel Classes. In this section, we shall establish the parallel classes of $q$-valent blocks by looking at the $r$-pencils in $D$. First let us state

The $r$-Tangency Condition. Let $B$ be a $q$-valent block. If $x$ and $y$ are two distinct points not in $B$, then there exists at most one block containing $x$ and $y$ which is $r$-tangent to $B$.

Let $D$ be a PBRD($q$) such that $D$ satisfies the $r$-tangency condition. Let $B$ be a $q$-valent block in $D$ and let $x$ be a point in $B$. By Corollary 4.23, there are $q$ ($q$-1)-valent blocks that contain $x$ and are $r$-tangent to $B$. We shall show that the $r$-tangency condition implies that these $q$ blocks are mutually $r$-tangent at $x$.

Lemma 4.35. Let $B$ be a $q$-valent block in $D$. If $x$ is a point not in $B$, then the $q$ blocks that contain $x$ and
are $r$-tangent to $B$ are mutually tangent to each other.

Proof. Suppose $C$ and $C'$ are two $r$-tangents of $B$ that contain $x$ and intersect at two points $x$ and $y$. Clearly, $x$ and $y$ are two distinct points not in $B$. But this contradicts the $r$-tangency condition. Hence, $C$ and $C'$ are mutually tangent.

Proposition 4.36. Let $q \geq 5$ and $B$ be a $q$-valent block in $D$. If $x$ is a point not in $B$, then the $q$ blocks containing $x$ and $r$-tangent to $B$ are mutually $r$-tangent to each other.

Proof. Let $C_1, \ldots, C_q$ be the $(q-1)$-valent blocks $r$-tangent to $B$ which contain $x$. If $C_1, \ldots, C_q$ are mutually $M$-tangent, then there exists a $(q+1)$-valent block $A$ containing $x$ which is tangent to $C_1, \ldots, C_q$. But this contradicts that every $M$-pencil contains only $q$ blocks. Suppose $C_1$ is $M$-tangent to $C$ and $C_1$ is $r$-tangent to $C_2$, then by Proposition 4.34, a maximal set $p$ of mutually tangent blocks that contain $C_1$, $C_2$ and $C_3$ contains at most $4$ blocks. But $q \geq 5$; hence, $C_1, \ldots, C_q$ are mutually $r$-tangent at $x$.

Corollary 4.37. The blocks $C_1, \ldots, C_q$ determine a unique $r$-pencil $\rho(x,B')$ with carrier $x$.

Proof. By Proposition 4.34, $C_1, \ldots, C_q$ are contained in the same $r$-pencil $\rho(x,B')$ with carrier $x$. 

Proposition 4.38. Let $q \geq 5$ and $B$ be a $q$-valent block in $D$. Let $x$ be a point not in $B$. If $\mathcal{P}(x, B')$ is the $r$-pencil with carrier $x$ such that each $(q-1)$-valent block in $\mathcal{P}(x, B')$ is $r$-tangent to $B$, then $B'$ is disjoint from $B$.

Proof. Since there are $q$ $(q-1)$-valent blocks in $\mathcal{P}(x, B')$, for each point $y$ in $B$, there exists a unique $(q-1)$-valent block $r$-tangent to $B$ at $y$. The blocks in $\mathcal{P}(x, B')$ partition the points distinct from $x$; hence, $B$ and $B'$ are disjoint.

Corollary 4.39. Every $r$-tangent of $B$ is an $r$-tangent of $B'$, and vice versa.

Proof. For every point $x$ in $B'$ there exist $q$ $(q-1)$-valent blocks $r$-tangent to $B$ which contain $x$. These $q$ blocks, together with $B'$, form an $r$-pencil $\mathcal{P}(x, B')$. Hence, they are also $r$-tangents of $B'$. Since there are $q$ points in $B'$, there are $q^2$ blocks that are $r$-tangents of both $B$ and $B'$. But by Corollary 4.22, $B$ has only $q^2$ $r$-tangents. Thus, every $r$-tangent of $B$ is an $r$-tangent of $B'$.

Next, we shall construct the parallel classes.

Definition. Let $B$ and $B'$ be two $q$-valent blocks in $D$. $B$ is said to be parallel to $B'$ if and only if either $B=B'$, or $B$ is disjoint from $B'$ and every $r$-tangent of $B$ is an $r$-tangent of $B'$ and vice versa. We shall denote them by $B//B'$. 
Proposition 4.40. If $B // B'$ and $B' // B''$, then $B // B''$.

Proof. Suppose $x \in B \cap B''$. Consider the $q$ $(q-1)$-valent blocks that contain $x$ and are $r$-tangent to $B'$, these $q$ blocks determine a unique $r$-pencil $\mathcal{P}(x, B)$ with carrier $x$. Hence, $B = B''$.

Suppose $B \nsubseteq B''$, then every $r$-tangent of $B$ is an $r$-tangent of $B'$, which, in turn, is an $r$-tangent of $B''$. Thus every $r$-tangent of $B$ is an $r$-tangent of $B''$ and $B \cap B'' = \emptyset$, so $B // B''$.

Proposition 4.41. Each $q$-valent block $B$ is contained in a parallel class $\mathcal{O}(B)$, and $\mathcal{O}(B)$ consists of $q-1$ blocks.

Proof. Let us count the number of ordered pairs $(x, B')$ such that $x \in B'$ and $B // B'$. For every point $x$ in $D$, there are $q$ $(q-1)$-valent blocks $r$-tangent to $B$ and containing $x$. They determine a unique $q$-valent block $B'$ parallel to $B$. Hence, there are $q^2 - q$ pairs. On the other hand, for every block parallel to $B$, there are $q$ choices of $x$, hence,

$$q \cdot \text{number of blocks parallel to } B = q^2 - q,$$

or $$\text{number of blocks parallel to } B = q - 1.$$

Since parallelism is a transitive relation, these $q-1$ blocks are mutually parallel to each other. Furthermore, they partition the points in $D$; hence, they form a parallel
Corollary 4.42. There are \( q+1 \) parallel classes in \( D \).

Proof. Since each parallel class contains \( q-1 \) blocks and there are \( q^2 - 1 \) \( q \)-valent blocks in \( D \), there are \( q+1 \) parallel classes in \( D \).

4.6 Proof of Theorem 4.1 for \( q \geq 5 \). From the previous section, we have found the \( q+1 \) parallel classes in \( D \). Next we have to establish the \( r \)-transversals of these parallel classes.

Lemma 4.43. Let \( C \) be a \( (q-1) \)-valent block in \( D \). \( C \) is an \( r \)-transversal of exactly two parallel classes in \( D \).

Proof. Let \( x \) be a fixed point in \( C \). There exist two \( q \)-valent blocks \( B \) and \( B' \) containing \( x \) and \( r \)-tangent to \( C \). Clearly, \( B \) and \( B' \) are in different parallel classes \( \mathcal{A}(B) \) and \( \mathcal{A}(B') \). Since \( C \) is an \( r \)-tangent of \( B \), \( C \) is an \( r \)-tangent of every block in \( \mathcal{A}(B) \), that is, \( C \) is an \( r \)-transversal of \( \mathcal{A}(B) \). Similarly, \( C \) is an \( r \)-transversal of \( \mathcal{A}(B') \). Clearly, \( \mathcal{A}(B) \) and \( \mathcal{A}(B') \) are the only two parallel classes of which \( C \) is an \( r \)-transversal.

Next, we show that there are \( q \) common \( r \)-transversals for every two distinct parallel classes.

Lemma 4.44. Every two distinct parallel classes have exactly \( q \) common \( r \)-transversals and they are disjoint.
Proof. Let $O(B)$ and $O(B')$ be two distinct parallel classes. We first show that any two common $r$-transversals of $O(B)$ and $O(B')$ are disjoint. Suppose $C$ and $C'$ are two common $r$-transversals such that $x \in C \cap C'$. Let $B$ and $B'$ be the $q$-valent blocks in $O(B)$ and $O(B')$ respectively such that $B$ and $B'$ contain $x$. Since $C$ and $C'$ are both $r$-tangents of $B$ and $B'$ at $x$, $|\rho(x,B) \cap \rho(x,B')| \geq 2$. This contradicts that there exists a unique block $r$-tangent to both $B$ and $B'$ at $x$. Thus, the common $r$-transversals of $O(B)$ and $O(B')$ are pairwise disjoint.

Let us count the number of triples $(C, O(B), O(B'))$ where $C$ is a common $r$-transversal of $O(B)$ and $O(B')$. For every $(q-1)$-valent block $C$ in $D$, there exist exactly two parallel classes of which $C$ is an $r$-transversal. Since there are $\frac{1}{2}q^2(q+1)$ $(q-1)$-valent blocks,

$$\sum \text{number of common } r\text{-transversals of } O(B) \text{ and } O(B') = \frac{1}{2} q^2(q+1) \cdot 2 \cdot 1 = q^2(q+1),$$

where the sum runs over all pairs $(O(B), O(B'))$. But there are $q+1$ distinct parallel classes; hence

average number of common $r$-transversals of 2 distinct parallel classes = $(q^2(q+1))/(q+1)q = q$.

Since the common $r$-transversals are pairwise disjoint and there are $q^2-q$ points in $D$, there are at most $q$ common $r$-transversals of $O(B)$ and $O(B')$. Thus, every two distinct
parallel classes have exactly q common r-transversals.

Thus far we see that D is a PBRD(q) that satisfies axioms (I1) and (I2) in the Fundamental Lemma. Next, we shall establish axiom (I3).

Lemma 4.45. Let x and y be two distinct points in D. If \( \mathcal{Q}(B) \) is a parallel class in D, then either there exists a q-valent block in \( \mathcal{Q}(B) \) containing x and y, or there exists exactly one r-transversal of \( \mathcal{Q}(B) \) containing x and y.

Proof. Let B be the block in \( \mathcal{Q}(B) \) containing x. If \( y \in B \), then we are done. If \( y \not\in B \), then there exists a unique block C containing y and r-tangent to B at x. Since C is an r-tangent of B, C is clearly an r-transversal of \( \mathcal{Q}(B) \). The proof is thus complete.

From the lemmas, we see that D satisfies axioms (I1)-(I3) in the Fundamental Lemma; hence D is embeddable into a Möbius plane. Thus we conclude,

Theorem 4.46. Let \( q \geq 5 \). If D is a PBRD(q) that satisfies the r-tangency condition, then D is uniquely embeddable into a Möbius plane of order q.

4.7 Proof of Theorem 4.1. The block-residual design of a Möbius plane obviously satisfies the tangency condition. Let D be a PBRD(q) that satisfies the r-tangency condition. If \( q \geq 5 \), then by Theorem 4.46, D is uniquely
embeddable. Next, we consider \( q = 1, 2 \) and 3.

For \( q = 1 \), the design \( \text{PBRD}(q) \) is a null design and is trivially embeddable.

For \( q = 2 \), \( \text{PBRD}(2) \) consists of 2 points and 9 blocks. Let the points be \( \{1, 2\} \). Since \( \text{PBRD}(2) \) is a 1-design and by Lemma 4.5, there are no 3-valent block, 3 2-valent blocks and 6 1-valent blocks. The blocks of \( \text{PBRD}(2) \) are,

\[
\begin{align*}
1 & 2, \\
1 & 2, \\
1 & 2,
\end{align*}
\]

To complete this design to a Möbius plane of order 2, we adjoin the new points \( \{3, 4, 5\} \) to the blocks and form

\[
\begin{align*}
1 & 2 & 3, \\
1 & 2 & 4, \\
1 & 2 & 5, \\
3 & 4 & 5.
\end{align*}
\]

Hence, \( \text{PBRD}(2) \) can be uniquely embedded into a Möbius plane.

For \( q = 3 \), there are 6 points in \( \text{PBRD}(3) \). Let them be \( \{1, 2, 3, 4, 5, 6\} \). Using Lemma 4.5 and the fact that it is a 2-design, one can check that the blocks of \( \text{PBRD}(3) \) are isomorphic to the following,
If we define

\[ \alpha_1 = \{1\ 3\ 5,\ 2\ 4\ 6\} \]
\[ \alpha_2 = \{1\ 3\ 6,\ 2\ 4\ 5\} \]
\[ \alpha_3 = \{1\ 4\ 5,\ 2\ 3\ 6\} \]
\[ \alpha_4 = \{1\ 4\ 6,\ 2\ 3\ 5\} , \]

then they are the four parallel classes of 3-valent blocks. It can be easily checked that every 2-valent block is an \( r \)-transversal of exactly two parallel classes, and they satisfy axioms (I1)-(I3) in the Fundamental Lemma. Hence, it is uniquely embeddable into the Möbius plane.
CHAPTER V
EMBEDDING OF PSEUDO-POINT-RESIDUAL DESIGNS INTO MOBIUS PLANE

In this chapter we shall study a dual problem of the embedding of a pseudo-block-residual design. Let M be a Möbius plane of order q and let oo be a point in M. If we delete the point oo and the incident blocks from M, then we obtain a 2-design with parameters

\[ v=q^2, \quad k=q+1, \quad \lambda=q. \]  (5.1)

Any 2-design with parameters as those given in (5.1) is called a pseudo-point-residual design of order q, abbreviated by PPRD(q). Our purpose is to reconstruct the Möbius plane from a given PPRD(q). Let D* be a PPRD(q). If D* is embeddable into a Möbius plane, then clearly no 3 distinct points can be contained in more than one block in D*.

Furthermore, if A is a block in D* containing a point x, and y is a point not in A, then there exists at most one block containing y which is tangent to A at x. We shall show that this is a sufficient condition for D* to be embeddable.
Definition. Let $A$ be a block in a design $D^*$ and $x$ be a point in $A$. For every point $y$ not in $A$, the Tangency Condition requires that there exists at most one block containing $y$ and tangent to $A$ at $x$.

Theorem 5.1. Let $D^*$ be a pseudo-point-residual design with parameters as those given in (5.1). If $D^*$ satisfies the Tangency Condition, then $D^*$ is uniquely embeddable in a Möbius plane $M$.

We shall prove the theorem in a series of lemmas. Let us first define that two blocks $A$ and $B$ are said to be tangent at a point $x$ if and only if $A \cap B = \{x\}$. $A$ and $B$ are said to be secant to each other if $|A \cap B| \geq 2$. A set of points are said to be concyclic if they are contained in a block.

Lemma 5.2. Let $D^*$ be an $S_q(2,q+1,q^2)$. Every point in $D^*$ is contained in $q^2-1$ blocks and there are $q^2(q-1)$ blocks in $D^*$.

Proof. Since $v=q^2$, $k=q+1$ and $\lambda=q$, by Proposition 1.3, $b=\lambda_0=q^2(q-1)$ and $r=\lambda_1=q^2-1$.

Lemma 5.3. Let $D^*$ be an $S_q(2,q+1,q^2)$. Let $A$ be a block in $D^*$. If $x$ is a point in $A$, then there exists at least $(q-2)$ blocks that are tangent to $A$ at $x$.

Proof. For every point $y$ in $A$, $y \neq x$, there are $q-1$ blocks, distinct from $A$, that contain $x$ and $y$. Hence, there are at
most $q(q-1)$ blocks containing $x$ and secant to $A$. But every point in $D^*$ is contained in $(q+1)(q-1)$ blocks; hence there are at least $q-2$ blocks tangent to $A$ at $x$. /

**Lemma 5.4.** Let $D^*$ be a PPRD($q$) such that $D^*$ satisfies the tangency condition. If $A$ is a block in $D^*$ and $x$ is a point in $A$, then there exists exactly $q-2$ blocks tangent to $A$ at $x$.

**Proof.** Since for every point $y$ not in $A$, there exists at most one block tangent to $A$ at $x$ which contains $y$. The tangents of $A$ at $x$ are mutually tangent. Hence, if $t$ denotes the number of tangents of $A$ at $x$, then

$$qt + q + 1 \leq q^2,$$

and $t \leq q - 1 - \frac{1}{q}$.

Since $t$ is an integer, $t \leq q-2$. From the previous lemma, there are exactly $q-2$ blocks tangent to $A$ at $x$. /

Henceforth, we shall assume that $D^*$ satisfies the tangency condition.

**Lemma 5.5.** Every 3 distinct points in $D^*$ are contained in at most one block.

**Proof.** Suppose $x$, $y$ and $z$ are 3 distinct points contained in 2 blocks $A$ and $A'$. Consider the blocks containing $x$ and secant to $A$. Since $A'$ is a secant of $A$ which intersects $A$
at both y and z, there are at most $q(q-1)-1$ secants of $A$ that contain $x$. This implies that there are at least $q-1$ blocks tangent to $A$ at $x$. But this contradicts the fact that there exist exactly $q-2$. Hence every 3 distinct points are contained in at most one block.

**Lemma 5.6.** For every 2 distinct points $x$ and $y$ in $D^*$, there are exactly $q-2$ points $z$ in $D^*$ such that $x$, $y$ and $z$ are not concyclic.

**Proof.** Let $A_1, \ldots, A_q$ be the blocks containing $x$ and $y$. Since no 3 points are contained in more than one block, $A_1, \ldots, A_q$ intersect each other at $x$ and $y$ only. Hence there are $2+q(q-1)$ points that are covered by $A_1, \ldots, A_q$. But $v=q^2$, this implies that there are exactly $q-2$ points $z$ such that $x$, $y$ and $z$ are not concyclic.

**Lemma 5.7.** If $(x,y,z)$ and $(x,y,z')$ are two nonconyclic triples, then $(x,z,z')$ is a nonconyclic triple.

**Proof.** Suppose $x$, $z$ and $z'$ are concyclic. Then there exists a unique block $A$ containing them. Consider the point $y$. $y$ is not in $A$. For every point $v$ in $A$, $v \not\in x,z,z'$, there exists at most one block containing $x$, $y$ and $v$. Since there are $q-2$ points in $A$ that are distinct from $x$, $z$ and $z'$, there are at most $q-2$ blocks containing $x$ and $y$ which are secant to $A$. But $x$ and $y$ are contained in $q$ blocks; hence there are at least two blocks containing $y$ which are tangent
to A at x. This contradicts the tangency condition. Therefore, \( x, z \) and \( z' \) are nonconcyclic.

For every pair of distinct points \( x \) and \( y \) in \( D^* \), let us define \( A(x,y) = \{ z | (x,y,z) \text{ is a nonconcyclic triple} \} \). From the above lemma, we observe

**Lemma 5.8.** If \( u \) and \( w \) are two distinct points in \( A(x,y) \), then \( A(x,y) \subseteq A(u,w) \).

Proof. Let \( z \in A(x,y) \). Since \( (x,y,z) \) and \( (x,y,u) \) are non-concyclic triples, \( (x,z,u) \) is nonconcyclic. Similarly, \( (x,z,w) \) is a nonconcyclic triple. But by the previous lemma \( z, u \) and \( w \) are nonconcyclic. Hence, \( z \in A(u,w) \). Thus, \( A(x,y) \subseteq A(u,w) \). By symmetry, \( A(u,w) \subseteq A(x,y) \) and the proof is complete.

Proof of Theorem 5.1. Let \( D^* = (X, \Omega) \) be an PPRD(q) such that \( D^* \) satisfies the tangent condition. We define

\[
\overline{X} = X \cup \{ \infty \},
\]

and \( \overline{\Omega} = \Omega \cup (\overline{A(x,y)} | \{ x,y \} \text{ is a 2-subset of } X) \)

where \( \overline{A(x,y)} = A(x,y) \cup \{ x,y, \infty \} \). We shall show that \( \overline{D} = (\overline{X}, \overline{\Omega}) \) is a Möbius plane of order \( q \). Clearly, \( |\overline{X}| = q^2 + 1 \) and every block in \( \overline{\Omega} \) contains \( q+1 \) points. We are left to show that every 3 distinct points in \( \overline{X} \) are contained in a unique block.

Let \( x, y \) and \( z \) be three distinct points in \( \overline{X} \). We
first show that there exists at least a block containing them. If \( z = \infty \), then the block \( \overline{A}(x,y) \) contains \( x, y \) and \( z \). If \( x, y \) and \( z \) are in \( X \) and \( (x,y,z) \) is a concyclic triple in \( X \), then \( \overline{A}(x,y) \) contains \( x, y \) and \( z \). If \( (x,y,z) \) is a non-concyclic triple, then \( z \in \overline{A}(x,y) \) and the block \( \overline{A}(x,y) \) contains \( x, y \) and \( z \). Hence every three distinct points are contained in at least one block.

Next we shall count the number of blocks in \( \overline{A} \). Let us compute the number of triples \((x,y,\overline{A}(x,y))\) such that \( x \neq y \). For every distinct pair \((x,y)\), there exists a unique \( \overline{A}(x,y) \). Hence, there are \( q^2(q^2-1) \) such triples. On the other hand, if \( \overline{A}(x,y) \) is a block in \( \overline{A} \), then there are \( q(q-1) \) choices of \((x,y)\). Therefore, number of blocks \( \overline{A}(x,y) \) is \( q(q+1) \). But \( |\overline{A}|=q^2(q-1) \); hence \( \overline{a}=q^3+q \).

Using Proposition 1.3, we can compute the average number of blocks containing a subset of 3 points which turns out to be 1. Hence, every 3 distinct points are contained in a unique block.

Thus, \((X,\overline{A})\) is an \( S(3,q+1,q^2+1) \), or a Möbius plane.
LIST OF REFERENCES


