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By
Ronald Dee Baker, B. S.

* * * * *

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Reading Committee:

T. A. Dowling
G. N. Robertson
D. K. Ray-Chaudhuri
R. M. Wilson

Approved by

Richard M. Wilson
Adviser
Department of Mathematics
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VITA

October 9, 1947 ............... Born - Edmond, Oklahoma

1968 .......................... B. S., Central State University, Edmond, Oklahoma

1968-1975 ..................... Danforth Fellow

1968 .......................... University Fellow, Department of Mathematics, The Ohio State University, Columbus, Ohio


1971 .......................... University Fellow, Department of Mathematics, The Ohio State University, Columbus, Ohio

1971-1973 ..................... Teaching Associate, Department of Mathematics, The Ohio State University, Columbus, Ohio

1973-1974 ..................... University Fellow, Department of Mathematics, The Ohio State University, Columbus, Ohio

1974-1975 ..................... Teaching Associate, Department of Mathematics, The Ohio State University, Columbus, Ohio

FIELDS OF STUDY

Major Field: Mathematics


Studies in Algebra. Professors Harold D. Brown and Joseph C. Ferrar
TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>VITA</td>
<td>iv</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>vii</td>
</tr>
<tr>
<td>INTRODUCTION AND SUMMARY</td>
<td>1</td>
</tr>
<tr>
<td>Chapter</td>
<td></td>
</tr>
<tr>
<td>I. PRELIMINARY GRAPH THEORY</td>
<td>12</td>
</tr>
<tr>
<td>Edge Colored graphs</td>
<td></td>
</tr>
<tr>
<td>Decompositions</td>
<td></td>
</tr>
<tr>
<td>Factorizations</td>
<td></td>
</tr>
<tr>
<td>II. Γ-FACTORIZATION PROBLEMS</td>
<td>20</td>
</tr>
<tr>
<td>Families of graphs G</td>
<td></td>
</tr>
<tr>
<td>Factor types Γ</td>
<td></td>
</tr>
<tr>
<td>Some examples</td>
<td></td>
</tr>
<tr>
<td>III. Γ-INDUCED CONSTRAINTS ON G</td>
<td>26</td>
</tr>
<tr>
<td>Admissible</td>
<td></td>
</tr>
<tr>
<td>Near-Admissible</td>
<td></td>
</tr>
<tr>
<td>IV. TECHNIQUES FOR CONSTRUCTION</td>
<td>33</td>
</tr>
<tr>
<td>Direct</td>
<td></td>
</tr>
<tr>
<td>Product theorems</td>
<td></td>
</tr>
<tr>
<td>Mixing types and near-factorizations</td>
<td></td>
</tr>
<tr>
<td>Subfactorizations</td>
<td></td>
</tr>
<tr>
<td>V. APPLICATIONS OF TECHNIQUES</td>
<td>46</td>
</tr>
<tr>
<td>Whist, Directed Whist, Triple Whist</td>
<td></td>
</tr>
<tr>
<td>Balanced incomplete block designs</td>
<td></td>
</tr>
<tr>
<td>Spouse avoiding mixed doubles round robin</td>
<td></td>
</tr>
<tr>
<td>tournaments</td>
<td></td>
</tr>
<tr>
<td>Nearly Kirkman systems</td>
<td></td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
</tr>
<tr>
<td>-------------------</td>
<td>------</td>
</tr>
<tr>
<td>APPENDIX A</td>
<td>75</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>81</td>
</tr>
</tbody>
</table>

LIST OF FIGURES

1. $K_2$-factorization of $K_6$ ................................................. 3
2. $C_7$-factorization of $K_7$ ................................................. 3
3. Nearly Kirkman triple system on 18 points ......................... 5
4. Triple Whist tournament for 8 players ............................... 6
5. Types of Whist tables .................................................. 47
INTRODUCTION AND SUMMARY

The survey book of F. Harary [20] on graph theory devotes an entire chapter to what he calls "Factorization". In this chapter he states that a factor of a graph is a spanning subgraph and a factorization is a collection of edge disjoint subgraphs containing all edges of the graph. These definitions are in complete harmony with this dissertation in which these ideas are extended from simple graphs to edge-colored mixed graphs. This extension of the ideas produces a unified theory which encompasses not only problems traditionally viewed as graph theoretic, but also problems from the area of combinatorial designs. A 1-factor is a factor which is regular of degree 1, and similar definitions are given for k-factor and k-factorization. A Γ-factor is a factor which is the disjoint union of subgraphs all isomorphic to the graph Γ. Hence a 1-factor is a K₂-factor. Some examples will serve to illustrate. Harary states the following theorems:

Theorem: The complete graph K₂n admits a K₂-factorization.

Theorem: Every regular bipartite graph admits a K₂-factorization

* All definitions sketched or briefly introduced here are given in full detail in the text.
Theorem: The graph $K_{2n+1}$ admits a $C_{2n+1}$-factorization, i.e. a factorization in which each factor is a spanning cycle.

Theorem: The complete graph $K_{2n}$ is the edge disjoin union of a $K_2$-factor and a graph which admits a $C_{2n}$-factorization.

The first of these theorems is illustrated in Figure 1 and the third in Figure 2. It should be noted that a $C_v$-factor of a graph $G$ with $v$ vertices is known as a Hamilton cycle after Sir William Hamilton who popularized searching for them.

The next examples are from the area of combinatorial designs. A set of $v$ objects and a collection of $k$ element subsets such that any pair of objects is contained in precisely $\lambda$ of the subsets is known as a balanced incomplete block design or $(v,k,\lambda)$-configuration. When $\lambda = 1$ these are known as Steiner systems and for $k = 3$ as Steiner triple systems. If additionally the $k$ element subsets are arranged into classes (called parallel classes) which partition the $v$ objects, then the design is called resolvable. A resolvable Steiner triple system is known as a Kirkman triple system. Suppose a Kirkman triple system is given, and its set of $v$ objects are identified with the vertices of the complete graph $K_v$. Each parallel class defines a $K_3$-factor which has as components the triangles whose vertex sets are the triples of the parallel class. In this way the Kirkman triple system is a $K_3$-factorization of $K_v$. 
Fig. 1: $K_2$-factorization of $K_6$.

Fig. 2: $G_7$-factorization of $K_7$. 
Recognizing this connection between designs and graphs, A. Kotzig and A. Rosa [25] proposed the following generalization of Kirkman triple systems. A nearly Kirkman triple system is a $K_3$-factorization of the complement of a $K_2$-factor in the complete graph on $v$ vertices, and is denoted by $NKTS[v]$ . In this paper Kotzig and Rosa found infinitely many such systems, including a $NKTS[18]$ given below in Figure 3 (the vertex set is $\{1,2,\ldots,9,1',\ldots,9'\}$). In Chapter V constructions are given which produce $NKTS[v]$ for all but three possible values of $v$.

Thus resolvable designs may be viewed as factorizations of graphs, and clearly in the case when $\lambda$ is not one they are factorizations of graphs with multiple edges. But these examples are not the most general. The next example is due to E. H. Moore [32] who was asked to devise schedules for conducting Whist Tournaments. Whist is a card game played by four people who are partitioned into two partnerships, and thus the graph $K_4$ does not adequately represent the distinction between partners and opponents. In order to make this distinction the idea of an edge-colored graph is introduced. Figure 4 illustrates a triple Whist tournament for eight players, denoted $TWh[8]$ . The graph involved has eight vertices and three edges between any pair of vertices. The edges are of three colors, represented by solid, dashed, and zigzag segments, which may be interpreted as partners, first-opponents, and second-opponents, respectively. The factor type
Fig. 3: NKTS[18].
Fig. 4: TWh[8].
is a graph $\Gamma$ which is $K_4$ with two non-adjacent edges of each color, and thus the $TWh[8]$ is a $\Gamma$-factorization. A Whist tournament is a factorization involving only two colors, one for partners and one for opponents. Such a factorization may be obtained by considering any one of the three colors of a triple Whist tournament as representing partners (and identifying the other two colors as an opponent color), explaining the use of the word triple in the name. Since the appearance of the paper by Moore in 1896 numerous other authors have considered problems similar to the Whist tournament problem, generally dropping the condition that the tables of card play (usually bridge rather than Whist) be partitioned into rounds (see [7, 8, 10, 26, 34, 40, 43, 44]).

Aside from the $TWh[24]$ found in [34] no progress seems to have been made beyond Moore's infinite families in constructing such factorizations. There also seem to be no examples in the literature of the next type of arrangement which is naturally suggested by the order of play in these card games. The third related problem is directed Whist tournaments for which the basic graph has a simple edge and directed edges, in either direction, between any two vertices. The factor type $\Gamma$ of a directed Whist tournament is a directed quadrilateral with its (undirected) diagonals, the former corresponding to an orientation of the table and the latter the partnerships. Examples of directed Whist tournaments provide a counter-example to an "almost conjecture" found in [33] concerning the existence of a certain type of quasigroup (see Appendix A).
Another problem similar to the Whist problems has recently been introduced by R. K. Brayton, D. Coppersmith and A. J. Hoffman [9] for use in arranging tennis tournaments. The graph involved has $2n$ vertices which represent $n$ couples. Any two people other than the couples are joined by an edge with a color corresponding to opponents and any two people of opposite sex, other than the couples, are joined by an edge with another color corresponding to partnership. The factor type is the same as the Whist tournament, and the factorization is known as a resolvable spouse avoiding mixed doubles round robin tournament (SAMDRR). In considering this problem, without a resolvable condition, the aforementioned authors observed that such SAMDRR were equivalent to self-orthogonal Latin squares (SOLS). Resolvable SAMDRR are found for all but finitely many orders $n$ such that $n \not\equiv 2 \pmod{4}$ in Chapter V, and the connection with Latin squares is examined in Appendix A.

An important kind of problem closely related to $\Gamma$-factorizations is $\Gamma$-near-factorizations. A $\Gamma$-near-factor is a spanning subgraph which is the disjoint union of subgraphs isomorphic to $\Gamma$ and an edgeless subgraph which has no more vertices than $\Gamma$. The vertex set of the edgeless subgraph is said to index the $\Gamma$-near-factor. A $\Gamma$-near-factorization is a collection of edge disjoint $\Gamma$-near-factors whose union is the entire graph and whose indexing sets form an equicardinal partition of the vertex set of the graph.
SAMDRR provide an example to motivate the importance of near-
factorizations. If the number of couples, \( n \), is odd, then at
most \( \frac{1}{2}(n-1) \) tennis matches could be played simultaneously
(there are at most \( \frac{1}{2}(n-1) \) pairwise disjoint subgraphs isomorphic
to \( \Gamma \)). Thus one attempts to arrange sets of \( \frac{1}{2}(n-1) \) matches
to be played simultaneously while a couple sits out the round. These
\( \Gamma \)-near-factorizations are also known as resolvable SAMDRR. In the
same manner (triple, directed) Whist tournaments are defined for
a number of players congruent to one modulo four, and also near-
resolvable balanced incomplete block designs.

The first four chapters develop this unified theory of \( \Gamma \)-
factorizations and \( \Gamma \)-near-factorization. Chapter I presents necessary
background information from graph theory. Chapter II is devoted
to explaining and illustrating what a \( \Gamma \)-factorization problem is.
Briefly, a \( \Gamma \)-factorization problem is the determination of which
graphs from a particular family of graphs admit a \( \Gamma \)-factorization.
Chapter III derives certain constraints on a graph which must be
satisfied if it is to admit a \( \Gamma \)-factorization. Any graph which
satisfies these constraints is called \( \Gamma \)-admissible. Chapter IV
presents several techniques for construction of \( \Gamma \)-factorizations.
Because the several problems which are selected for illustration
have an heritage in the area of combinatorial designs or recreational
mathematics, the current mode of presentation may seem like needless
obfuscation. For this the author apologizes, and suggests that the
disturbed reader skip the first four chapters and read Chapter V.
Chapter V contains the results of applying the general techniques to a half dozen special problems. In the first section the three types of Whist problems are considered. The results are:
Whist tournaments exist for all numbers $v$ of players such that $v \equiv 0 \pmod{4}$, except possibly $v=152$, and such that $v \equiv 1 \pmod{4}$, except possibly $v=57$ or 129. Triple Whist tournaments exist for $v=4n$ players for all sufficiently large $n$, $n \not\equiv 2 \pmod{4}$, and for $v \equiv 1 \pmod{4}$ and sufficiently large. There are infinitely many directed Whist tournaments for $v \equiv 0 \pmod{4}$ and directed Whist tournaments exist for all $v \equiv 1 \pmod{4}$ except possibly $v=57$, 93 or 129. In the second section it is shown, as a corollary to the result on Whist tournaments, that resolvable balanced incomplete block designs for $k=4$ and $\lambda=3$ exist for all $v \equiv 0 \pmod{4}$ except possibly $v=152$. In the third section resolvable SAMDRR of order $n$ (for $n$ couples) are shown to exist for all sufficiently large $n$, $n \not\equiv 2 \pmod{4}$). In the fourth section nearly Kirkman triple systems are shown to exist for $v$ points for all $v \equiv 0 \pmod{6}$, $v > 12$, except possibly $v=84$, 102 or 174.

Appendix A is a discussion of the relationship between Latin squares and resolvable SAMDRR, triple Whist tournaments and directed Whist tournaments. In particular it is shown that there exist four mutually orthogonal Latin squares of order 33, this being one more than was previously known (see van Lint [41]).
Whist tournaments for $v=57$ and 152 have recently been found by H. Hanani, and for $v=129$ by R. Wilson, thus settling all admissible orders of $v$ for this problem.
CHAPTER I: PRELIMINARY GRAPH THEORY

A graph $G$ is a triple $(V, E, \tau)$ where $V$ is a set of vertices (points or nodes), $E$ is a set of edges (links or arcs), and $\tau$ is a map from $E$ to the set of two element subsets of $V$. The vertices $x$ and $y$ are called the ends of $a$ if $\tau(a) = \{x, y\}$. The sets $V$ and $E$ are also denoted $V(G)$ and $E(G)$, especially when several graphs are being discussed simultaneously. An edge $a$ is said to connect or join its ends. Two vertices $x$ and $y$ are adjacent if there exists some edge $a \in E$ such that $\tau(a) = \{x, y\}$. A vertex $x$ is incident with an edge $a$ if $x \in \tau(a)$. Two edges $a$ and $b$ are adjacent if $\tau(a)$ and $\tau(b)$ contain a common vertex, and are parallel if $\tau(a) = \tau(b)$. A graph is simple if $\tau(a) = \tau(b)$ implies that $a = b$, and $E$ will be identified with its image under $\tau$. A graph $H = (V(H), E(H), \tau)$ is a subgraph of $G$ if $V(H)$ is a subset of $V(G)$ (denoted $V(H) \subseteq V(G)$), $E(H) \subseteq E(G)$ and $\tau(a) = \tau_1(a)$ for all $a \in E(H)$. A complete graph on $v$ vertices, $K_v$, is a simple graph on any set $V$ of cardinality $v$ (denoted $|V| = v$) for which $E(K_v)$ is the set of all two element subsets of $V$.

The valence of a vertex $x$, denoted $\text{val}_G(x)$ (or $\text{val}(x)$ if $G$ is understood), is the number of edges incident with it.
The degree of a vertex \( x \), \( \text{deg}(x) \), is the number of vertices adjacent to it. The following theorem is sometimes called the first theorem of graph theory.

**Theorem:** For any graph \( G \), \( \sum_{x \in V(G)} \text{val}(x) = 2|E(G)| \).

**Proof:** Define an incidence function for \( G \) by

\[
I(x,a) = \begin{cases} 
0 & \text{if } x \notin \eta(a) \\
1 & \text{if } x \in \eta(a) 
\end{cases}
\]

Evidently then \( I \) is a function from \( V(G) \times E(G) \) to \{0,1\} such that \( \sum_{a \in E(G)} I(x,a) = \text{val}(x) \). Hence it follows that

\[
\sum_{x \in V(G)} \text{val}(x) = \sum_{x \in V(G)} \sum_{a \in E(G)} I(x,a) = \sum_{a \in E(G)} \sum_{x \in V(G)} I(x,a) = \sum_{a \in E(G)} 2 = 2|E(G)| ,
\]

which completes the proof.

A graph \( G \) is called **bipartite** if there exists a partition of the vertex set into two parts, \( V(G) = X_1 \cup X_2 \), such that \( \eta(a) \notin X_1 \) and \( \eta(a) \notin X_2 \) for each edge \( a \in E(G) \). Similarly, \( G \) is called **multipartite**, or **multipartite**, if there is a partition of the vertex set into \( m \) parts such that no part contains both ends of an edge. A **complete multipartite graph** of type \( (n_1,n_2,\ldots,n_m) \) is a simple graph on \( n_1 + n_2 + \cdots + n_m \) vertices \( V(G) \) which admits a partition \( X_1 \cup X_2 \cup \cdots \cup X_m \) such that \( |X_i| = n_i \).
and if \( x, y \) are in different parts of the partition then \((x, y)\) \( \in E(G) \). Such a graph is denoted by \( K_{n_1, n_2, \ldots, n_m} \).

Given \( X \subseteq V(G) \) and \( S \subseteq E(G) \), there is a unique subgraph \( H \) with \( V(H) = X \) and \( E(H) = S \) if and only if for each \( a \in S \), \( \eta(a) \subseteq X \). Given \( H \) a subgraph of \( G \), also denoted \( H \subseteq G \), \( H \) is a spanning subgraph of \( G \) if and only if \( V(H) = V(G) \). For \( S \subseteq E(G) \), the spanning subgraph of \( G \) with edge set \( S \) is denoted by \( G; S \). Thus \( V(G; S) = V(G) \) and \( E(G; S) = S \). For \( X \subseteq V(G) \), the subgraph of \( G \) induced by \( X \), \( G[X] \), is the subgraph with vertex set \( V(G[X]) = X \) and edge set \( E(G[X]) \) consisting of all edges of \( G \) with both ends in \( X \). \( G; S \) denotes the subgraph obtained from \( G; S \) by deleting its isolated vertices, i.e. \( G; S = (G; S)[X] \) where \( X = \{ x \in V(G): \text{val}_{G; S}(x) > 0 \} \).

A path \( P \) is a string \( x_1, a_1, x_2, a_2, \ldots, a_k, x_{k+1} \) of vertices and edges such that \( \eta(a_i) = \{ x_i, x_{i+1} \} \) and \( a_i = a_j \) implies \( i = j \). The vertices \( x_1 \) and \( x_{k+1} \) are called the ends of \( P \), and \( P \) is said to connect \( x_1 \) to \( x_{k+1} \). If \( x_1 = x_{k+1} \), then \( P \) is closed. The path \( P \) is said to have length \( k \) and, if not closed, size \( k+1 \). The path \( P \) is simple if all vertices are distinct, save possibly \( x_1 \) and \( x_{k+1} \). A simple closed path is a polygon or circuit. In a simple graph the path \( P \) may also be denoted \( x_1, x_2, \ldots, x_{k+1} \) since there is no ambiguity. A single vertex \( x \) is allowed as a path of length zero, or size one. Clearly if \( x_1, a_1, x_2, a_2, \ldots, a_k, x_{k+1} \) is a path, then so is \( x_{k+1}, a_k, \ldots, a_2, x_2, a_1, x_1 \).
Also if \(x_1, a_1, \ldots, a_k, x_{k+1}\) and \(y_1, b_1, \ldots, b_{t}, y_{t+1}\) are paths with \(x_{k+1} = y_{l}\), then \(x_1, a_1, \ldots, a_k, x_{k+1}, b_1, \ldots, b_{t}, y_{t+1}\) is a path if all the edges are distinct, or otherwise some \(x_1, a_1, \ldots, a_s, x_s, y_t, b_{t+1}, \ldots, b_{t}, y_{t+1}\) is a path. Hence the relation on \(V(G)\) of being connected to is an equivalence relation, and thus partitions \(V(G)\) into equivalence classes. If \(X\) is such an equivalence class, then the subgraph \(G[X]\) is a connected component of \(G\).

In a simple graph, \(\text{val}(x) = \text{deg}(x)\). In particular if \(\text{val}(x) = r\) for all \(x \in V(G)\), \(G\) is called regular of valency (or degree) \(r\). There is a natural correspondence of a simple graph \(G\) to a graph \(\overline{G}\) given by \(V(\overline{G}) = V(G)\) and \(E(\overline{G}) = \overline{E(G)}\).

A graph \(G\) is called regular of valency \(r\), degree \(d\), if \(G\) is regular of degree \(d\) and for each edge \(\{x, y\}\) of \(G\) there are exactly \(r/d\) edges \(a \in E(G)\) such that \(\overline{a} = \{x, y\}\). A graph \(G\) is pseudo-regular if \(G\) is regular.

A directed graph, or digraph, \(D\) is a triple \((V, E, \delta)\) where \(V\) is a set of vertices (points or nodes), \(E\) is a set of edges (links, or arcs), and \(\delta\) is a map from \(E\) to \(V \times V - \{(x, x) : x \in V\}\), the set of ordered pairs of distinct vertices. If \(\delta(a) = (x, y)\) then the vertex \(x\) is called the tail (initial vertex or source) of \(a\) and \(y\) is called the head (terminal vertex or sink) of \(a\). The tail of \(a\) will also be denoted by \(\delta^- (a)\) or \(a^-\), and the head by \(\delta^+ (a)\) or \(a^+\). To a digraph \(D\) there naturally corres-
ponds a graph \( G(D) \) given by \( V(G(D)) = V(D) \), \( E(G(D)) = E(D) \), and with ends map \( \eta(a) = (\delta^+(a), \delta^-(a)) \). Most terminology for graphs is applied to digraphs via this correspondence. A digraph is \textit{simple} if \( \delta(a) = \delta(b) \) implies \( a = b \), and its edge set is identified with its image under \( \delta \). A digraph \( D' = (V', E', \delta') \) is a \textit{subdigraph} of \( D \) if \( V' \subseteq V \), \( E' \subseteq E \) and \( \delta'(a) = \delta(a) \) for all \( a \in E' \). A \textit{complete digraph} on \( v \) vertices \( D_v \) is a simple digraph on any vertex set \( V \) such that \( |V| = v \) and \( E = \{ (x, y) \in V \times V : x \neq y \} \).

In addition to the connected components of \( D \) defined by \( G(D) \), there is also the notion of a \textit{strongly connected component}. A \textit{directed path} \( P \) is a string \( x_1, a_1, x_2, \ldots, a_k, x_{k+1} \) such that the edges are distinct and \( \delta(a_i) = (x_i, x_{i+1}) \). \( P \) is said to strongly connect \( x_1 \) to \( x_{k+1} \). If \( X \subseteq V(D) \) has the property that any ordered pair of its vertices are strongly connected and is maximal (under containment) with respect to this property, then \( D[X] \) is a strongly connected component.

The \textit{in valence} of a vertex \( x \), denoted \( \text{val}_1(x) \), is the number of edges with head \( x \) and the \textit{out valence}, \( \text{val}_0(x) \), the number of edges with tail \( x \). A vertex \( y \) is a predecessor of \( x \) if there is some edge \( a \) such that \( \delta(a) = (y, x) \) and a successor if there is some edge \( a \) such that \( \delta(a) = (x, y) \). The \textit{in degree} of \( x \), \( \text{deg}_1(x) \), is the number of predecessors and the \textit{out degree}, \( \text{deg}_0(x) \), is the number of successors.
Subgraphs $H_1, H_2 \subseteq G$ are edge disjoint if and only if $E(H_1) \cap E(H_2) = \emptyset$ (the empty set), and disjoint if and only if $V(H_1) \cap V(H_2) = \emptyset$.

There are several important operations on graphs, i.e. ways to assign a unique third graph to an ordered pair of graphs. The following will be used in this dissertation. For a simple graph $G$, $\overline{G}$, will denote the graph with $V(\overline{G}) = V(G)$ and as edges all pairs $(x,y) \notin E(G)$. $\overline{G}$ is called the complement of $G$.

For graphs $G_1, G_2$ their union $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, where any $a \in E(G_1) \cap E(G_2)$ must have $\tau_1(a) = \tau_2(a)$. When $V(G_1) = V(G_2)$ the edge disjoint union of $G_1$ and $G_2$, denoted $G_1 \cup G_2$, is defined on the vertex set $V(G_1)$ to have edge set the formal disjoint union of $E(G_1)$ and $E(G_2)$. In other words $E(G_1 \cup G_2) = X_1 \cup X_2$, where $X_1 \cap X_2 = \emptyset$, and there exist bijections $\psi_1$ from $X_1$ to $E(G_1)$ such that for $a \in X_1$, $\tau(a) = \tau_1(\psi_1(a))$. Unless there is some chance of confusion $a$ will be identified with $\psi(a)$. The disjoint union of graphs $G_1$ and $G_2$, denoted $G_1 \sqcup G_2$, is defined as the union where both the vertex sets and the edge sets are formally disjoint.

A mixed graph $M$ is the union of a graph $G$ and a digraph $D$ on the same vertex set, i.e. $V(M) = V(G) = V(D)$, $E(M) = E(G) \cup E(D)$ and $M$ has an associated ends map $\mu$ which agrees with $\tau$ on $E(G)$ and $\delta$ on $E(D)$. Two mixed graphs $M_1$ and $M_2$ are iso-
morphic if there exists a bijection (i.e. a one-to-one correspondence or matching) \( \psi \) from \( V(M_1) \cup E(M_1) \) to \( V(M_2) \cup E(M_2) \) such that if \( \mu_1(a) = (x,y) \) or \((x,y)\) then \( \mu_2(\psi(a)) = (\psi(x),\psi(y)) \) or \((\psi(x),\psi(y))\) respectively and \( \psi(V(M_1)) = V(M_2) \) (hence \( \psi(E(M_1)) = E(M_2) \)). Since a graph or a digraph is a mixed graph also, this defines isomorphism for graphs and for digraphs. Henceforth, the term graph will be used with reference to mixed graphs unless there is a danger of ambiguity.

An (edge-) colored graph \( G \) is a graph \( G \) together with a map \( \Pi : E(G) \to C \), for \( C \) some set of colors. For any color \( c \in C \), there is a graph \( G_c \) given by \( V(G_c) = V(G) \) and \( E(G_c) = \Pi^{-1}(c) = \{ a : \Pi(a) = c \} \). Any graph \( G \) may be viewed as a colored graph \( G^{(c)} \) by assuming a constant color map \( \Pi : E(G) \to \{ c \} \). Two colored graphs \( G_1 \) and \( G_2 \), with color maps \( \Pi_1 \) and \( \Pi_2 \) respectively, are isomorphic provided there exists an isomorphism \( \psi \) from \( G_1 \) to \( G_2 \) as graphs with the property that \( \Pi_1(a) = \Pi_2(\psi(a)) \) for all \( a \in E(G_1) \).

Suppose \( \Gamma \) and \( G \) are colored graphs. A decomposition of \( G \) is a family of edge disjoint (colored) subgraphs \( G_1, G_2, \ldots, G_b \) such that \( G = G_1 \cup G_2 \cup \ldots \cup G_b \). A \( \Gamma \)-decomposition is a decomposition such that each \( G_i \) is isomorphic to \( \Gamma \). A factor of \( G \) is a spanning subgraph \( G_i \). A \( \Gamma \)-factor is a spanning subgraph \( G_i \) which is the disjoint union of subgraphs isomorphic to \( \Gamma \). In particular if \( \Gamma \) is connected these subgraphs are
the components of $G_i$. A factorization of $G$ is a family of edge disjoint factors $G_1, G_2, \ldots, G_r$ such that $G = G_1 + G_2 + \ldots + G_r$.

A $\Gamma$-factorization is a factorization in which each $G_i$ is a $\Gamma$-factor.

Now suppose $V(G)$ admits a partition $X_1 \cup X_2 \cup \ldots \cup X_r$ such that each subgraph $G[X_i]$ is edgeless and each set $X_i$ contains $m$ vertices for some $m \leq |V(G)|$. A $\Gamma$-near-factor of $G$ (with respect to the given partition of $V(G)$) is a spanning subgraph $G_i$ which is the disjoint union of subgraphs isomorphic to $\Gamma$ and the (edgeless) subgraph $G[X_i]$ ($G_i$ is said to be indexed by $X_i$).

A $\Gamma$-near-factorization of $G$ is a family of edge disjoint $\Gamma$-near-factors $G_1, G_2, \ldots, G_r$ indexed by the $X_i$ such that $G = G_1 + G_2 + \ldots + G_r$. The trivial partition of $V(G)$ into singleton sets certainly satisfies the condition on induced subgraphs.

Throughout the remainder of the text the term graph will be used for colored mixed graphs, since context will alert the reader to whether edges are colored and whether edges are directed or not.

The notation $I_n$ is used to denote the set $\{1,2,\ldots,n\}$. 
CHAPTER II: FACTORIZATION PROBLEMS

A factorization question is "given $\Gamma$ and $G$, does $G$ admit a $\Gamma$-factorization?". Since there exists a great diversity of choices for $\Gamma$ and for $G$ in formulating such a question, it is proper to ask what is a "good" factorization question? There is no absolute answer to such a question since it involves value judgements. Three criteria are suggested for measuring the "goodness" of a factorization question. The first is whether the question arises naturally in the investigation of another problem. The second criterion is whether the graphs $\Gamma$ and $G$ are appealing in the sense that they possess symmetry and simplicity. The third criterion is whether an affirmative answer to the question will produce useful, although possibly surprising, byproduct results. Indeed, all three criteria may be simultaneously satisfied. Rather than attempting to further outline how to ascertain the "goodness" of a question, the concept of a factorization problem is introduced and then many "good" examples are given to illustrate the concepts.

What is a factorization problem? A "good" $\Gamma$ is selected and fixed. Then a family of "good" $G$ are selected, a family of $G$ which might reasonably be expected to admit $\Gamma$-factorizations and which are easily distinguished by a parameter or a small set of parameters. The factorization problem is then to determine for
which parameters (or which sets of parameters) the corresponding graphs admit a $\Gamma$-factorization. A factorization problem is then specified by giving $\Gamma$ and the family of graphs $G$.

Consider the family of graphs parameterized by a natural number $v$ (denoted $v \in \mathbb{N}$) and some fixed natural number $\lambda$, the graph indexed by $v$ being the edge disjoint union of $\lambda$ distinct $K_v$ on the same set of $v$ vertices, $\lambda K_v$. For this family the following $\Gamma$ define "good" factorization problems. Suppose $\Gamma = K_k^v$ for some natural number $k$, then a $K_k^v$-factorization is also known as a resolvable balanced incomplete block design with block size $k$, on $v$ points or varieties, and with index $\lambda$. In particular for $k = 3$ and $\lambda = 1$, these factorizations are known as Kirkman triple systems [35]. Suppose $\Gamma$ is the graph of a polygon or circuit of length $k$, $C_k$, then a $\Gamma$-factorization is also known as a resolvable balanced circuit design [36]. For $k = 3$ this is again a Kirkman triple system since $K_3$ is a triangle. Suppose $\Gamma$ is a bipartite graph, then a $\Gamma$-factorization is also known as a resolvable balanced bipartite design [22,23]. For $\Gamma$ a multipartite graph a $\Gamma$-factorization might be called a resolvable balanced multipartite design. The most interesting cases of $\Gamma$ multipartite are when each part of the vertex partition is of the same size, or in the bipartite case when one part is a single vertex. Finally for this family of graphs $G$ consider the case when $\Gamma$ is the graph of a path. Such a $\Gamma$-factorization has been called a resolvable
balanced path design, or a resolvable handcuffed design [21,27,28].

Now consider the family of graphs $G$ which are all complete multipartite graphs. When $\Gamma$ is $K_k$ these are known as resolvable partially balanced incomplete block designs based on group divisible association schemes, or simply as resolvable group divisible designs. Two restricted versions of this problem, namely to the families of complete $m$-partite graphs or to complete multipartite graphs for which each part of its partition is of size $n$, are especially interesting. These may be referred to as resolvable group divisible designs on a fixed number $m$ of groups or of constant group size $n$, respectively. In the latter case if $n = 1$, then the family is simply the family of complete graphs and has already been discussed. For values of $n, 1 \leq n < k$, resolvable group divisible designs of constant group size are also known as nearly Kirkman systems [25]. In particular for $n = 2$ and $k = 3$, these are called nearly Kirkman triple systems [25]. A $K_k$-factorization of $K_{n_1,n_2,\ldots,n_k}$ for $n_1 = n_2 = \ldots = n_k$ is known as a resolvable transversal design.

Now consider the family of graphs $G = \lambda_1 K_v^{(c_1)} + \lambda_2 K_v^{(c_2)} + \ldots + \lambda_m K_v^{(c_m)}$. Of special interest are the cases $m = 2$ or $3$, as well as the obvious $m = 1$ already discussed. For $m = 2$, consider the graph $\Gamma = (K_{s_1} \uplus K_{t_1}) + K_{s,t}^{(c_2)}$ where the partition for the bipartite $\Gamma_{c_2}$ is given by the components of the $\Gamma_{c_1}$. 
A $\Gamma$-factorization is called a resolvable calibration design, or when $s = t$ a resolvable balanced weighing design [7,8]. In particular for $\lambda_1 = 1, \lambda_2 = 2$ and $s = t = 2$, a $\Gamma$-factorization is called a Whist tournament. For a Whist tournament the vertices of $G$ are called players, $\Gamma$ and the components of $\Gamma$-factors Whist-tables, $\Gamma$-factors Whist-rounds, $c_1$ colored edges partners and $c_2$ colored edges opponents [32]. When $m = 3, \lambda_1 = \lambda_2 = \lambda_3 = 1$, and $\Gamma$ is $K_4$ with color map $\Pi$ such that $E(\Gamma_c_1), E(\Gamma_{c_2})$ and $E(\Gamma_{c_3})$ each contain two non-adjacent edges, then a $\Gamma$-factorization is called a triple Whist tournament [32]. For a triple Whist tournament the vertices of $G$ are called players, $\Gamma$ and the components of $\Gamma$-factors triple-Whist-tables, $\Gamma$-factors triple-Whist-rounds, $c_1$ colored edges partners, $c_2$ colored edges first opponents and $c_3$ colored edges second opponents.

Another family are the graphs $G$ indexed by $n$ with $V(G) = I_n \times I_2$ and color map $\Pi: E(G) \rightarrow \{c_1, c_2\}$ such that $\Pi^{-1}(c_2)$ contains all edges of $K_{2n}$ except those in $S = \{(x,1),(x,2): x \in I_n\}$ and $\Pi^{-1}(c_2)$ contains all edges of $K_{n,n}$ (with partition $V(G) = (I_n \times \{1\}) \cup (I_n \times \{2\})$) except those in $S$. The elements of $S$ are known as spouse pairs. The graph $\Gamma$ is the same as the graph described above for Whist tournaments. These $\Gamma$-factorizations are known as resolvable spouse avoiding mixed doubles round robin tournaments [9], and are related to arranging tennis tournaments.
Still another family of graphs are those directed graphs on \( v \) vertices which contain \( \lambda \) edges in each set \( S^{-1}((x,y)) \) for every ordered pair of distinct vertices \( (x,y) \), known as the \( \lambda \)-fold complete directed graphs \( \lambda D_v \). The graph \( \Gamma \) is taken as a directed circuit of length \( k \), \( C_k \), and \( \Gamma \)-factorizations are called resolvable directed circuit designs. In particular, for \( k = 3 \) these are called directed Kirkman designs \([24,29]\).

The last family considered here are the graphs \( \lambda_1 K_v + \lambda_2 D_v \) on \( v \) vertices. The graph \( \Gamma \) is a mixed graph such that \( G(\Gamma) = K_k \). Special attention is given to the case \( G(\Gamma) = K_4 \) and \( \Gamma = H + D \) such that \( E(H) \) is a pair of non-adjacent edges and \( D \) is a directed quadrilateral. Such \( \Gamma \)-factorizations for the family with \( \lambda_1 = \lambda_2 = 1 \) are called directed Whist tournaments. Edges of \( K_v \) are called partners and edges of \( D_v \) are called opponents. Both \( \Gamma \) and the components of the \( \Gamma \)-factors are called directed-Whist-tables. The \( \Gamma \)-factors are called directed-Whist-rounds, and vertices of \( G \) are called players.

Certain of the problems listed above have already received much attention in the literature. In particular, resolvable balanced incomplete block designs have been found for all admissible values of \( v \) when \( k = 3 \) by D. K. Ray-Chaudhuri and R. M. Wilson \([35]\) and when \( k = 4 \) by H. Hanani, Ray-Chaudhuri and Wilson \([19]\). Here admissible values of \( v \) are those of the form \( 6t + 3 \) for \( k = 3 \) and are those of the form \( 12t + 4 \) for \( k = 4 \). In
Chapter III the idea of admissible parameters is introduced. Several problems are selected for application of the techniques given in Chapter IV, and presented in Chapter V. These problems are Whist tournaments, triple Whist tournaments, directed Whist tournaments, resolvable balanced incomplete designs with $k = 4$, $\lambda = 3$, resolvable spouse avoiding mixed doubles round robin tournaments, and nearly Kirkman triple systems.
CHAPTER III: Γ-INDUCED CONSTRAINTS ON G

Suppose one assumes that G admits a Γ-factorization, then what properties is G forced to possess as a result of properties of Γ? If c is a color of Γ, then Gc admits a Γc-factorization. Hence attention may be restricted to a single color of edges. Moreover, since directed edges are distinguishable from undirected edges, these may also be separated. For clarity assume that G and Γ are graphs with only undirected edges (which may have arisen as Gc and Γc). Let \( V(Γ) = \{x_1, x_2, \ldots, x_k\} \) and \( \text{val}_Γ(x_i) = d_i \). Fix a vertex \( x \in V(G) \), then for each Γ-factor \( G_1, G_2, \ldots, G_r \) there is a subgraph \( G_i, x \) which contains the vertex \( x \) and admits a isomorphism \( ψ_i \) from \( G_i, x \) to Γ, with say \( ψ_i(x) = x_{j_i} \). Since the \( G_i \) partition the edge set of G it follows that

\[
\text{val}_G(x) = \sum_{i=1}^{r} \text{val}_{G_i}(x) = \sum_{i=1}^{r} \text{val}_Γ(ψ_i(x)) = \sum_{i=1}^{r} \text{val}_Γ(x_{j_i}) = \sum_{i=1}^{r} d_{j_i}.
\]

In other words \( \text{val}_G(x) \) is an non-negative integral linear combination of the numbers \( d_1, d_2, \ldots, d_k \). In particular if Γ is regular of valence \( d \), then \( \text{val}_G(x) = rd \). Thus G must be regular of valence \( rd \).
In the case that $\Gamma$ is a directed graph the above observation takes the form \[ \text{val}_I(x) = \sum_{i=1}^{r} e_i \] and \[ \text{val}_O(x) = \sum_{i=1}^{r} f_i \] where $e_i$ and $f_i$ are in and out valences of $x_i$.

The second observation is that since each $\Gamma$-factor of $G$ is the disjoint union of subgraphs isomorphic to $\Gamma$, there being $p$ such subgraphs, then it follows that $|V(G)| = p|V(\Gamma)|$, or $p = |V(G)| / |V(\Gamma)|$.

The third observation arises from counting the edges of $G$. Since $|E(G_i)| = p|E(\Gamma)|$, it follows that $|E(G)| = pr|E(\Gamma)|$. This equation may also be written as $pr = |E(G)| / |E(\Gamma)|$.

It is now possible to determine whether a given graph $G$ might admit a $\Gamma$-factorization. This is summarized in a definition and a proposition.

**Definition:** A graph $G$ is $\Gamma$-admissible provided that:

1. If $\Gamma_c$ is undirected, then each $\text{val}_{\Gamma_c}(x)$ is divisible by the greatest common divisor of the valencies of the vertices in $\Gamma_c$. If $\Gamma_c$ is directed, then the same condition applies to both in- and out-valency.

2. The number $p$ computed from $p = \frac{|V(G_c)|}{|V(\Gamma_c)|}$ is an integer.

3. The number $pr$ computed from $pr = \frac{|E(G_c)|}{|E(\Gamma_c)|}$ is an integer independent of $c$. Moreover for a factorization problem a parameter, or parameter set, is called $\Gamma$-admissible if the graph it indexes is $\Gamma$-admissible.
**Proposition:** If $G$ admits a $\Gamma$-factorization, then $G$ is $\Gamma$-admissible.

**Proof:** This follows from the discussion above.

Below are computations determining which parameters are admissible for the examples of the previous chapter (notice the "$\Gamma$" is dropped if no confusion will arise).

Consider $G = \lambda K_v$, indexed by $v$, and $\Gamma = K_k$. The valence of a vertex of $G$ is $\lambda(v-1)$ and of a vertex of $\Gamma$ is $k-1$.

The first theorem of graph theory yields that $|E(G)| = \frac{1}{2} \lambda v(v-1)$ and $|E(\Gamma)| = \frac{1}{2} k(k-1)$. Thus the conditions are that $k-1$ divides $\lambda(v-1)$, $v/k$ is an integer and $\frac{1}{2} \frac{\lambda v(v-1)}{k(k-1)}$ is an integer. Thus $G$ is $\Gamma$-admissible if both $k$ divides $v$ and $k-1$ divides $\lambda(v-1)$. Special cases are: $k = 3$, $\lambda = 1$ where then $v \equiv 3 \pmod{6}$ are the admissible values of $v$; $k = 4$, $\lambda = 1$ where then $v \equiv 4 \pmod{12}$ are the admissible values of $v$; and $k = 4$, $\lambda = 3$ where then $v \equiv 0 \pmod{4}$ are admissible. This example shows that the conditions are not necessarily independent.

For the remaining cases only a minimal set of conditions will be noted.

For $G = \lambda K_v$ and $\Gamma = C_k$ the conditions are that $\frac{1}{2} \lambda(v-1)$ and $v/k$ are integers. Those conditions may be simplified according to the parity of $\lambda$ and of $k$. If $\lambda$ is even, then the condition is that $k$ divide $v$. If $\lambda$ is odd, then $v$ must
be odd and hence \( k \) cannot be even. Thus for \( \lambda \) odd and \( k \) even no \( v \) are admissible and for \( \lambda \) odd and \( k \) odd then \( v = k \mod 2k \) are the admissible values.

For \( G = \lambda K_v \) and \( \Gamma = K_{n,m} \) the conditions are that \( \lambda(v-1) \) is divisible by the g.c.d. of \( n \) and \( m \), \( v/n+m \) is an integer, and that \( \frac{\lambda(v-1)(n+m)}{2mn} \) is an integer. As a special case consider \( n = m \), which reduces to \( v/2n \) and \( \frac{\lambda(v-1)}{2n} \) being integers.

Another special case is \( m = 1 \), which reduces to \( v/n+1 \) and \( \frac{\lambda(v-1)(n+1)}{2n} \) being integers.

For \( G \) the complete multipartite graph with partition type \((n_1, n_2, \ldots, n_m)\) and \( \Gamma = K_k \) the conditions are that \( v-n_i \) is divisible by \( k-l \) for each \( i \) \((v = n_1 + n_2 + \ldots + n_m)\) and
\[
\sum_{i=1}^{m} \frac{n_i(v-n_i)}{k(k-1)} \quad \text{and} \quad \frac{v}{k}, \text{are integers. In particular if} \quad n=n_1=n_2=\ldots=n_m \\
\text{these reduce to the conditions that the expressions} \quad \frac{n(m-l)}{k-l}, \quad \frac{nm n(m-l)}{k(k-1)} \quad \text{and} \quad \frac{nm}{k} \text{are integers (any two of which imply the third).} \]

Clearly \( m=k \) provide admissible graphs corresponding to resolvable transversal designs.

For \( G = \lambda_1 K_v + \lambda_2 K_v \) and \( \Gamma = K_{2n} \) with color map \( \Pi(a) = c_1 \) if \( a \in K_n \) and \( \Pi(a) = c_2 \) otherwise, there are two sets of conditions corresponding to the two colors. These conditions are that \( \frac{\lambda_1(v-1)}{n-1} = \frac{\lambda_2(v-1)}{n} \) and \( \frac{v}{2n} \) are integers. When \( \lambda_1 = 1 \),
\( \lambda_2 = 2 \) and \( n = 2 \) such \( \Gamma \)-factorizations are called Whist tournaments, and all \( v \equiv 0 \pmod{4} \) are admissible. It should be noted that 
\[ \lambda_1 n = \lambda_2 (n-1), \text{ (for } n = 2 \text{ this is } 2\lambda_1 = \lambda_2) \] hence the choices of \( \lambda_1 = 1 \) and \( \lambda_2 = 2 \) are minimal values for \( n = 2 \).

A triple Whist tournament is a \( \Gamma \)-factorization of \( G = K_v^{(c_1)} + K_v^{(c_2)} + K_v^{(c_3)} \) for \( \Gamma = K_4 \) and \( \Gamma_c = K_2 \perp K_2 \) for \( i = 1, 2, 3 \). The conditions are that \( v/4 \) and \( v-1 \) are integers, i.e. \( v \equiv 0 \pmod{4} \).

The condition for a resolvable spouse avoiding mixed doubles round robin tournament with \( V(G) = I_n \times I_2 \) is that \( n \) is even.

The condition for resolvable directed circuit designs for \( G = \lambda D_v \) is that \( v \equiv 0 \pmod{k} \), and is interesting in that \( \lambda \) is not involved.

A directed Whist tournament has \( G = K_v + D_v \) and \( \Gamma = (K_2 \perp K_2) + C_4 \) so that \( G(\Gamma) = K_4 \). The admissible \( v \) are all \( v \equiv 0 \pmod{4} \).

The remainder of the chapter is devoted to a parallel discussion to the one above for near-factorizations.

**Definition:** A graph \( G \) is \( \Gamma \)-near-admissible provided that:

(i) If \( \Gamma_c \) is undirected, then each \( \text{val}_{\Gamma_c}(x) \) is divisible by the greatest common divisor of the valencies of each vertex in \( \Gamma_c \). If \( \Gamma_c \) is directed, then the same condition applies to both in- and out-valency.

(ii) \( V(G) \) admits a partition \( X_1 \cup X_2 \cup \ldots \cup X_r \) such that \( |X_i| = m \leq |V(\Gamma)| \) and \( G[X_i] \) is edgeless.
(iii) The number $p = \frac{|V(G_c)|}{|E(G_c)|}$ is an integer.

(iv) The ratio $\frac{|E(G_c)|}{|E(\Gamma_c)|}$ is $pr$ (p and r given in (iii) and (ii) above) for all colors $c$.

**Proposition:** If $G$ admits a $\Gamma$-near-factorization, then $G$ is $\Gamma$-near admissible.

**Proof:** Condition (ii) follows from the definition of a near-factorization. The remaining conditions follow from arguments analogous to the factorization case.

Consider $G = \lambda K_v$, indexed by $v$, and $\Gamma = K_k$. The partition is the discrete partition into singletons so that the conditions are

(i) $\frac{\lambda(v-1)}{k-l}$ is an integer, (ii) $r = v$, (iii) $p = \frac{v-1}{k}$ is an integer, and (iv) $\frac{1}{2} \lambda v(v-1) = v \frac{(v-1)}{k}$. Thus $G$ is near-admissible when $\lambda = k-1$ and $v \equiv 1 \pmod{k}$. The case $k = 3$ and $\lambda = 2$ has been solved by H. Hanani [18], and $k = 4$, $\lambda = 3$ is considered in Chapter V.

If $G$ is any of the three types of Whist graphs, then the condition is $v \equiv 1 \pmod{4}$ with the discrete partition into singletons.

If $G$ is the graph of a spouse avoiding mixed doubles round robin tournament with $V(G) = I_n \times I_2$, then the conditions reduce to $n$ being odd.

If $G$ is the complement of a $K_2$-factor of $K_{2n}$, $\Gamma = K_3$, and the partition is given by the vertex sets of the subgraphs in the
$X_2$-factor, then the condition is that $n \equiv 1 \pmod{3}$. These near-factorizations always exist as is shown in Chapter V.
CHAPTER IV: TECHNIQUES FOR CONSTRUCTION

In this chapter several important methods for obtaining the existence of a $\Gamma$-factorization of a graph $G$ are discussed. These methods are arranged according to the order they are likely to be used for a particular problem. Thus the first type are the direct constructions. For a direct construction no reference is made to any previously obtained $\Gamma$-factorization. The second method is the use of product theorems, i.e. the assertion of the existence of a $\Gamma$-factorization for a graph indexed by $vw$ from the existence of $\Gamma$-factorizations for graphs indexed by $v$ and by $w$. The third method involves utilization of near-factorizations and factorizations of a different type, i.e. for a different $\Gamma$ or a different family of graphs $G$, or both. The last method demonstrates the role of subfactorizations in generalizing the above techniques.

The first and most naïve approach to finding a $\Gamma$-factorization of an admissible graph $G$ is to systematically enumerate the spanning subgraphs of $G$ checking to see if they are $\Gamma$-factors, and from among these searching for a collection which partition the edge set of $G$. This technique is known as brute force, and may actually be practical if both $G$ and $\Gamma$ are small. The brute force method can be refined in some cases by utilizing symmetry of the graph $G$. An example of such a refinement is the method of differences employed.
by R. C. Bose [5, 16]. Let $R$ be an additive group of automorphisms of $G$, i.e. the elements of $R$ are isomorphisms of $G$ to itself (as a colored mixed graph). For each edge $a \in E(G)$ there is an orbit $\Theta = \{b \in E(G) : b = \rho(a) \text{ for some } \rho \in R\}$, and these orbits partition the edge set of $G$. Suppose moreover that $R$ is semi-regular on the edges of $G$, i.e. if $\rho(a) = a$ then $\rho = 0$ (the identity of $R$). Then it suffices to find one $\Gamma$-factor $G_0$ with the property that $\Theta \cap E(G_0)$ is a single edge for each orbit $\Theta$. The remaining $\Gamma$-factors are given by $G_{\rho} = \rho(G_0)$ for $\rho \in R$, $\rho \neq 0$. Thus the following theorem has been proved.

**Theorem A:** Suppose $R$ is an additive group of automorphisms of $G$ which is semi-regular on the edges of $G$ and $G_0$ is a $\Gamma$-factor which meets each (edge) orbit in a single edge, then $G$ admits a $\Gamma$-factorization. (Moreover this may be generalized to a family of initial factors $G_0$ such that exactly one factor meets each orbit in one edge.)

In practice the method of differences is the most powerful of the direct construction techniques, since there exist families of additive groups possessing similar properties, and hence often many factorizations can be described via one construction. This is again illustrated in the next chapter as it has been in the work of R. C. Bose [5] and many other authors.

The next general method is that of the product theorem. The first form of a product theorem is based on the following principle:
Suppose $H_1$ is a graph with $v$ vertices and $H_2$ is a graph with $w$ vertices each of which admits a $\Gamma$-factorization, and $G$ is a graph with $vw$ vertices which can be written as $G_1 + G_2$ where $G_1$ admits a $H_1$-factorization, and $G_2$ admits a $H_2$-factorization, then $G$ admits a $\Gamma$-factorization. The union of the first $\Gamma$-factor of each subgraph isomorphic to $H_1$ in the first $H_1$-factor form the first $\Gamma$-factor of $G$, and the remaining $\Gamma$-factors are obtained analogously. For a family of graphs indexed by the number of vertices, a strong product theorem is a method of expressing the graph $G$ with $vw$ vertices as the edge disjoint union of graphs $G_i$ admitting $H_i$-factorizations ($i = 1, 2$), where $H_1$ and $H_2$ are the graphs with $v$ and $w$ vertices respectively. Such a product theorem is proven for the family of graphs independent of $\Gamma$, and is thus both quite useful and difficult to obtain. A qualified strong product theorem might stipulate that such an expression for the graph on $vw$ vertices exists provided $v$ and $w$ satisfy some additional condition, for example that $v$ is a prime power larger than $w$. While such conditions might be tantamount to assuming the method of expressing $G$, they may nevertheless be easier to check in practice than actually displaying the way of expressing $G$ and consequently are viewed as bona fide product theorems. This is summarized in the following theorem.

**Theorem B:** For a family of graphs indexed by the number of vertices, let $G$ have $vw$ vertices, $H_1 v$ vertices and $H_2 w$ vertices. If
G = G₁ + G₂ such that G₁ admits an H₁-factorization and G₂ admits an H₂-factorization, then Γ-factorizations of H₁ and H₂ imply a Γ-factorization of G.

The second form of a product theorem depends on Γ as well as the family of graphs. As above let G, H₁, and H₂ be graphs of the family on v,w, v and w vertices respectively. The idea is to write G as G₁ + G₂ where G₁ is a H₁-factor and G₂ admits a Γ-factorization because H₂ does. More precisely G₂ should admit a factorization whose type is a graph on kv vertices using the Γ-factorization of H₂, and then the factor type on kv vertices should admit a Γ-factorization via a qualified strong product theorem. As an illustration suppose the family of graphs is the family of complete graphs. Then Kvw can be written as the edge disjoint union of a Kv-factor and the w-partite graph K_v,v,...,v. Now if Γ is K_k it follows easily that K_w admitting a Γ-factorization implies the w-partite graph K_v,v,...,v admits a M-factorization where M is the k-partite graph K_v,v,...,v. Hence to complete the product theorem for resolvable balanced incomplete block designs of index unity with block size k it suffices to know the existence of resolvable transversal designs of index unity with group size v. Thus in the second form a method of expressing G is used which requires a H₁-factor and a graph admitting an M-factorization (using H₂), and then requires a Γ-factorization of M. Since the parameters of M are determined by H₁ and Γ, a
qualified version of this product theorem might restrict \( v \) but allow \( w \) to be arbitrary (subject to a \( \Gamma \)-factorization of \( H_2 \)). This contrasts with the qualified strong product theorem which usually specifies some relationship between \( v \) and \( w \) (such as the existence of a resolvable transversal design on \( w \) groups of size \( v \)). This is summarized in the next theorem.

**Theorem C:** For a family of graphs indexed by the number of vertices, let \( G, H_1 \) and \( H_2 \) be the graphs with \( vw, v \) and \( w \) vertices respectively. If \( G = G_1 + G_2 \) such that \( G_1 \) is an \( H_1 \)-factor and \( G_2 \) admits a \( \Gamma \)-factorization if \( H_2 \) does, then \( \Gamma \)-factorization of \( H_1 \) and \( H_2 \) imply a \( \Gamma \)-factorization of \( G \).

Methods involving utilization of factorizations of a different type have already been introduced since these actually include product theorems. The strong product theorem is a form of the general substitution principle. Namely if \( G \) can be written as an edge disjoint union \( G_1 + G_2 + \ldots + G_m \) such that each \( G_i \) admits a \( H_i \)-factorization and each \( H_i \) admits a \( \Gamma \)-factorization, then \( G \) admits a \( \Gamma \)-factorization. The general substitution principle includes the transitive property that if \( G \) admits a \( H \)-factorization and \( H \) admits a \( \Gamma \)-factorization, then \( G \) admits a \( \Gamma \)-factorization. Thus the existence of resolvable balanced incomplete block designs with \( k = 4 \) [19] provides triple Whist tournaments for one-third of all admissible parameters, clearly demonstrating the power of this idea. Again this is summarized in a theorem.
Theorem D: If \( G \) is the edge disjoint union of graphs \( G_1 \) which admit \( H_1 \)-factorizations, then \( \Gamma \)-factorizations of each \( H_1 \) imply a \( \Gamma \)-factorization of \( G \).

The use of near-factorizations is possibly the most subtle of all the techniques. The first method of using near-factorizations involves writing \( G \) as \( G_1 + G_2 \). The graph \( G_1 \) is the union

\[ H_1 \cup H_2 \cup \ldots \cup H_n \]

such that each \( H_i \) admits a \( \Gamma \)-factorization with \( r_i \) \( \Gamma \)-factors and for any distinct \( i \) and \( j \), \( 1 \leq i, j \leq n \),

\[ V(H_i) \cap V(H_j) = X_\emptyset \]. Moreover, there are partitions of \( V(H_i) - X_\emptyset = X_1 = X_1 \cup X_2 \cup \ldots \cup X_{r_i} \) such that the graphs \( G[X_\emptyset] \) and

\[ G_2[X_{i,j}] \]

are edgeless and all \( |X_{i,j}| = m, 1 \leq i \leq n, 1 \leq j \leq r_i \).

Finally assume that there is a collection of edge disjoint subgraphs \( M \) of \( G_2 \) whose union is \( G_2[V(G_2) - X_\emptyset] \) and which possess the properties: (i) for any \( i, 1 \leq i \leq n \), and \( M \) there exists a \( j, 1 \leq j \leq r_i \), such that \( V(M) \cap V(H_i) = X_{i,j} \), (ii) for any \( X_{i,j} \) and \( X_{s,t} \) with \( i \neq s \) there exists a unique \( M \) with \( X_{i,j} \cup X_{s,t} \subseteq V(M) \), and (iii) each \( M \) admits a \( \Gamma \)-near-factorization with respect to the partition given by the \( X_{i,j}, 1 \leq i \leq n, 1 \leq j \leq r_i \). Then \( G \) admits a \( \Gamma \)-factorization. The \( \Gamma \)-factors of \( G \) are indexed by the sets \( X_{i,j} \). Corresponding to \( X_{i,j} \) a \( \Gamma \)-factor of \( G \) is the union of the \( j \)-th \( \Gamma \)-factor of \( H_i \) with the \( \Gamma \)-near-factors indexed by \( X_{i,j} \) for each subgraph \( M \) with \( V(M) \cap V(H_i) = X_{i,j} \).

It should be noted that \( X_\emptyset \) is not restricted in size and indeed \( X_\emptyset = \emptyset \) is a possibility. This construction is summarized below.
Theorem E: If \( G = G_1 + G_2 \) where \( G_1 \) is the union of \( \Gamma \)-factorable graphs and \( G_2[V(G_2) - X_0] \) is the union of edge disjoint subgraphs satisfying the properties (i), (ii), (iii) above, then \( G \) admits a \( \Gamma \)-factorization.

The second method of using near-factorizations is of the product theorem type. Assume \( G \) is \( G_1 + G_2 \) where \( G_1 \) is a \( H_1 \)-factor with subgraphs \( L_1, L_2, \ldots, L_s \) and \( G_2 \) admits a \( H_2 \)-factor-ization \( M_1 + M_2 + \ldots + M_t \) where \( M_1 \) has subgraphs \( N_1, N_2, \ldots, N_r \). Moreover, assume that the intersections \( V(L_i) \cap V(N_j) \) are all sets of a given uniform partition \( X_1 \cup X_2 \cup \ldots \cup X_r = V(L_i) \) with respect to which each \( L_i \) admits a \( \Gamma \)-near-factorization, and additionally that \( H_2 \) admits a \( \Gamma \)-factorization. Then \( G \) admits a \( \Gamma \)-factorization. For each \( H_2 \)-factor \( M_1 \) form \( \Gamma \)-factors of \( G \) as the union of \( \Gamma \)-factors of the subgraph of \( M_1 \), except for one \( \Gamma \)-factor in each subgraph \( N_j \) of \( M_1 \). Now for each subgraph \( N_j \) of \( M_1 \) form a \( \Gamma \)-factor of \( G \) as the union of the reserved \( \Gamma \)-factor of \( N_j \) and \( \Gamma \)-near-factorizations of each \( L \) indexed by \( V(L_i) \cap V(N_j) \). Thus this method might be employed to form a product theorem for a \( \Gamma \)-factorization indexed by \( v \) (\( v = |V(H_2)| \)) and a \( \Gamma \)-near-factorization indexed by \( w \) (\( w = |V(H_1)| \)), producing a \( \Gamma \)-factorization indexed by \( vw \). Thus the following theorem has been proved.

Theorem F: If \( G = G_1 + G_2 \) where \( G_1 \) is an \( H_1 \)-factor and \( G_2 \) admits an \( H_2 \)-factorization subject to the condition outlined above,
then $H_1$ having a $\Gamma$-near-factorization and $H_2$ a $\Gamma$-factorization imply $G$ admits a $\Gamma$-factorization.

Finally the idea of a subfactorization is introduced and used for a generalization of the first method using near-factorization. Suppose a graph $G$ admits a $\Gamma$-factorization, $X \subseteq V(G)$ and set $H = G[X]$. The $\Gamma$-factorization of $G$ is said to contain a subfactorization with respect to $H$ (or $X$) if it contains $\Gamma$-factors $G_1, G_2, \ldots, G_n$ such that $H_i = G_i[X]$ are the $\Gamma$-factors of a $\Gamma$-factorization of $H$. Now modify the hypothesis of the first method using near-factorizations as follows: $G$ is written as $G_1 + G_2$ where $G_1$ is the union $H_1 \cup H_2 \cup \ldots \cup H_n$ such that each $H_i$ admits a $\Gamma$-factorization with $r_i + s$ $\Gamma$-factors and for any distinct $i$ and $j$, $V(H_i) \cap V(H_j) = X_0$. Again assume that there are partitions of $V(H_i) - X_0 = X_i = X_{i,1} \cup X_{i,2} \cup \ldots \cup X_{i,r_i^1}$ such that the graphs $G_i[X_{i,j}]$ are edgeless and $|X_{i,j}| = m$. But now assume each $H_i$ contains a subfactorization with respect to $X_0$, with $s$ $\Gamma$-factors, and indeed these subfactorizations are identical. The other assumptions are retained and the construction is also duplicated to form $r_1 + r_2 + \ldots + r_n$ of the $\Gamma$-factors using those of each $H_i$ not related to the subfactorization of $X_0$. The final $s$ $\Gamma$-factors are the union of the $s$ $\Gamma$-factors related to $X_0$ over all the $H_i$. This form of the method actually includes the earlier form if the convention that any edgeless graph admits a $\Gamma$-factorization (for any $\Gamma$) is adopted. Hence the theorem listed below is proved.
Theorem G: If \( G = G_1 + G_2 \) where \( G_1 \) is the union of \( \Gamma \)-factorable graphs and \( G_2 \{V(G_2) - X_0\} \) is the union of edge disjoint subgraphs satisfying the properties above, then \( G \) admits a \( \Gamma \)-factorization.

Theorems A, B, C, D, E, F and G are concerned with the existence of factorizations. Notice that among these Theorems E, F, G mention near-factorizations in their hypothesis. Thus it is useful to have some theorems concerned with the existence of near-factorization, and hence some follow. The first observation is that Theorem A is valid for near-factorizations also. The last three theorems of this chapter are uniquely about near-factorizations.

Suppose a graph \( G \) and a partition \( X_1 \cup X_2 \cup \ldots \cup X_r \) of its vertex set are given (with \( |X_1| = m \)). Suppose \( G \) can be written as \( G_1 + G_2 + \ldots + G_m \) so that each \( G_i \) admits an \( H_i \)-decomposition where the vertex sets of the subgraphs of the decomposition are unions of some of the \( X_j \). Suppose moreover that if \( M_1, M_2, \ldots, M_t \) are all the subgraphs of the decomposition such that \( X_j \subseteq V(M_s) \), \( s = 1,2,\ldots,t \), then the \( V(M_s) - X_j \) partition \( V(G) - X_j \). Then each \( H_i \) admitting a \( \Gamma \)-near-factorization (relative to the induced partition of the \( X_j \)) implies \( G \) admits a \( \Gamma \)-near-factorization. The union of the \( \Gamma \)-near-factors of the subgraphs indexed by \( X_j \) provides the \( \Gamma \)-near-factor of \( G \) indexed by \( X_j \). In summary:

Theorem H: If \( G \) can be written as the edge disjoint union of \( G_i \) so that each \( G_i \) admits an \( H_i \)-decomposition and the subgraphs \( M \) of the decomposition on each \( X_j \) (i.e. \( X_j \subseteq V(M) \)) only meet in
Then $X_j$ and cover $V(G)$, then $\Gamma$-near-factorizations of the $H_i$ imply a $\Gamma$-near-factorization of $G$.

The next theorem is the basis of a product theorem for near-factorizations. Suppose $G = G_1 + G_2$ such that $G_1$ is an $H_1$-factor and relative to the partition $X_1 \cup X_2 \cup \ldots \cup X_v$ of $V(G)$, where each $X_i$ is the vertex set of a subgraph of $G_1$, $G_2$ admits an $H_2$-factorization. Suppose each $X_i$ admits a partition $X_{i,1} \cup X_{i,2} \cup \ldots \cup X_{i,r}$, with $|X_{i,j}| = m$, relative to which the subgraph isomorphic to $H_1$ admits a $\Gamma$-near-factorization and $H_2$ admits a $\Gamma$-factorization with $r \Gamma$-factors. Then $G$ admits a $\Gamma$-near-factorization relative to the partition of the $X_{i,j}$. The $\Gamma$-near-factor indexed by $X_{i,j}$ is formed as follows: For the $H_2$-near-factor indexed by $X_i$, label the $\Gamma$-factors of each subgraph isomorphic to $H_2$ by the sets $X_{i,1}, X_{i,2}, \ldots, X_{i,r}$. Such a labeling, called the external indexing, is possible since $H_2$ has $r \Gamma$-factors. The $\Gamma$-near-factor indexed by $X_{i,j}$ is then the union of the $\Gamma$-near factor of the subgraph isomorphic to $H_1$ on $X_i$ and indexed by $X_{i,j}$, with the $\Gamma$-factors having external index $X_{i,j}$. As an example suppose $G$ is $K_{vw}$. Then $G_1$ would be a $K_{w}$-factor and $G_2$ the complete multipartite graph with $v$ parts of size $w$. If $\Gamma$ is $K_k$, then $H_2$ would be the complete multipartite graph with $k$ parts of size $w$ and a product theorem is valid if $H_2$ admits a $\Gamma$-factorization, i.e. if there exists a resolvable transversal design. Thus the following theorem is proven.
Theorem I: If $G = G_1 + G_2$ where $G_1$ is an $H_1$-factor, $G_2$ admits an $H_2$-near-factorization indexed by the vertex sets of the subgraphs of $G_1$, $H_1$ admits a $\Gamma$-near-factorization with $r$ $\Gamma$-near-factors, and $H_2$ admits a $\Gamma$-factorization with $r$ $\Gamma$-factors, then $G$ admits a $\Gamma$-near-factorization.

The final theorem of this chapter is a method of obtaining a near-factorization from a factorization known as the point deletion technique. Suppose $G$ admits a $\Gamma$-factorization such that if $M_i$ is the subgraph of the $i$-th $\Gamma$-factor $G_i$ with $x \in V(M_i)$, then $G[V(M_i)]$ is $M_i$. Suppose moreover that the sets $X_i = V(M_i) - \{x\}$ partition $X = V(G) - \{x\}$. Then $G_o$ admits a $\Gamma$-near-factorization relative to the partition given by the $X_i$, where the $\Gamma$-near-factor indexed by $X_i$ has edge set $E(G[X - X_i])$ and $G_o$ is the graph with vertex set $X$ and edge set $E(G)$ minus the edges of all the $M_i$. The main example of this type is $G = K_v$ and $\Gamma = K_k$. In this case all conditions are satisfied, $G_o$ is the complete $r$-partite graph on $r$ sets of size $k-1$. This construction is summarized below.

Theorem J: If $G$ admits a $\Gamma$-factorization and $x$ is a vertex of $G$ which satisfies the properties above, then $G_o$, the $x$-deletion of $G$, admits a $\Gamma$-near-factorization.

In most of the above theorems the hypothesis includes the expression of the graph $G$ as the edge disjoint union of two or more subgraphs which satisfy certain constraints. These hypotheses are generalizations
of constructions based on certain well known types of designs, which are discussed below.

A {	extit{pairwise balanced design}} on $v$ points with block sizes $K$ (here $K \subset \mathbb{N}$ and is used since the notation is standard), denoted $\text{PBD}_\lambda(K,v)$, is a decomposition of $\lambda K_v$ for which each subgraph is isomorphic to $K_k$ for some $k \in K$. The hypothesis of Theorem H may be satisfied by a $\text{PBD}_\lambda(K,v)$. A {	extit{uniformly resolvable}} $\text{PBD}_\lambda(K,v)$ is a factorization of $\lambda K_v$ such that each factor is a $K_k$-factor for some $k \in K$. The special cases of $K = \{k\}$ are known as balanced incomplete block designs (BIBD), denoted $B_\lambda(k,v)$, and resolvable BIBD's. Whenever $\lambda = 1$ it is dropped from the notations above. The hypothesis of Theorem D may be satisfied by a uniformly resolvable $\text{PBD}_\lambda(K,v)$ or by a resolvable $B_\lambda(k,v)$.

A {	extit{group divisible design}} on $v$ points with block sizes $K$ and group sizes $G$, denoted $\text{GDD}(K,G,v)$, is a factorization of $K_v$ as $G_1 + G_2$ such that $G_1$ is the disjoint union of graphs each isomorphic to $K_g$ for some $g \in G$, and a decomposition of $G_2$ for which each subgraph is isomorphic to $K_k$ for some $k \in K$. The subgraphs of $G_1$ are groups and of $G_2$ are blocks. A $\text{PBD}(K',v+1)$ may be constructed from a $\text{GDD}(K,G,v)$, where $K' = K \cup \{g + 1 : g \in G\}$, as follows: Let $K_v \subseteq K_{v+1}$ with $x \in V(K_{v+1})$ but $x \notin V(K_v)$, and take as subgraphs the blocks together with the subgraphs $K_{v+1}[V(H) \cup \{x\}]$ where $H$ is a group of the GDD. The hypothesis of Theorem E may be satisfied in this manner. A uniformly resolvable
GDD\([K,G,v]\) is one in which the graph \(G_2\) admits a factorization such that each factor is a \(K_k\)-factor for some \(k \in K\). The special cases of \(K = \{k\}\), \(G = \{g\}\), and \(v = kg\) are known as transversal designs, denoted \(TD[k,g]\), and resolvable transversal designs. A resolvable \(T[k,g]\) may satisfy the hypothesis of Theorem B.
CHAPTER V: APPLICATIONS OF TECHNIQUES

In this chapter the previously highlighted factorization problems are solved or partially solved using the techniques outlined in Chapter IV. These factorization problems, with indicated section numbers, are as follows: 1. Whist tournaments, directed Whist tournaments, and triple Whist tournaments; 2. resolvable balanced incomplete block designs; 3. resolvable spouse avoiding mixed doubles round robin tournaments; and 4. nearly Kirkman triple systems.

The approach to the main theorem of each section is based on Theorem E. The hypothesis of Theorem E is satisfied by a group divisible design. In order to reduce the problem to finitely many values of \( v \), group divisible designs are constructed inductively using finite sets of block and group sizes. This is illustrated by Proposition 1.9 and Proposition 1.17.

1. **Whist.** The three types of factorizations which have Whist in their name are so similar that they are discussed together below. In the interest of economy of notation a Whist tournament for \( v \) players is denoted \( Wh[v] \), while directed Whist tournaments and triple Whist tournaments are denoted \( DWh[v] \) and \( TWh[v] \). Moreover the same notations are used for corresponding \( T \)-near-factorizations when \( v \equiv 1 \pmod{4} \), where the partition of the graphs \( G \) is into
singletons. The next proposition lists results, whose proofs are in the indicated references.

**Proposition 1.1:**

(i) (E.H. Moore) If \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) is the factorization of \( n > 1 \) into powers of distinct primes, then there exists a resolvable transversal design with \( m \) groups of size \( n \) for \( m \leq \min_{1 \leq i \leq r} \alpha_i \) [32].

(ii) There exists a resolvable transversal design with 4 groups of size \( n \) for all \( n \equiv 0, 1 \) (mod 4) [41].

In order to facilitate the exposition the notation \((x_1, x_2; x_3, x_4)\) is adopted for Whist-tables, directed-Whist-tables and triple-Whist-tables as illustrated by Figure 5.

**Fig. 5:** Types of Whist tables

**Proposition 1.2:**

(i) If there exists a DWh[v], then there exists a Wh[v].

(ii) If there exists a TWh[v], then there exists a Wh[v].

**Proof:** (i) Replacing \( G = K_v + D_v \) with \( K_v^{(c_1)} + 2K_v^{(c_2)} \) produces the Wh[v].
\( (ii) \) Replacing \( G = K_v^{(c_1)} + K_v^{(c_2)} + K_v^{(c_3)} \) with \( K_v^{(c_1)} + 2K_v^{(c_2)} \) produces the \( \text{Wh}[v] \).

**Proposition 1.3:** If \( v = 4n + 1 \) is a prime power, then there exists a \( \text{DWh}[v] \).

**Proof:** This proposition is an illustration of Theorem A, the difference method. The group is the additive group of the finite field with \( v \) elements, \( \text{GF}(v) \). Let \( x \) be a primitive element of \( \text{GF}(v) \), i.e. a generator of the cyclic multiplicative group of non-zero elements. \( G_0 \) has subgraphs \( T = (r, rx^{2n}; rx^n, rx^{3n}) \) as \( r \) ranges over \( 1, x, x^2, \ldots, x^{n-1} \).

**Proposition 1.4:** Cyclic \( \text{Wh}[v] \) exist for \( v = 4, 8, 12, 16, 20, 28, 32, 36 \) and \( 40 \), and in fact cyclic \( \text{TWh}[v] \) exist for \( v = 4, 8, 16 \) and \( 24 \).

**Proof:** See [32] and [34] for constructions and definition of cyclic.

**Proposition 1.5:** If there exists a uniformly resolvable \( \text{PBD}[K,v] \) such that for each \( k \in K \) there exists a \( \text{Wh}[v] \) (\( \text{TWh}[k] \) or \( \text{DWh}[k] \)), and \( k \equiv 0 \pmod{4} \), then there exists a \( \text{Wh}[v] \) (resp. \( \text{TWh}[v] \) or \( \text{DWh}[v] \)).

**Proof:** A uniformly resolvable \( \text{PBD}[K,v] \) is a method of writing \( K_v = G_1 + G_2 + \ldots + G_t \) such that each \( G_i \) admits a \( K_k \)-factorization for some \( k \in K \). The proposition is then an application of Theorem D.
Corollary 1.6: If there exists a resolvable $B[k,v]$ such that there exists a $Wh[k]$, $k \equiv 0 \mod 4$, then there exists a $Wh[v]$.
Similarly for $TWh[v]$ and $DWh[v]$.

Proof: A resolvable $B[k,v]$ is a $K_k$-factorization of $K_v$.

Proposition 1.7: If there exists a uniformly resolvable $GDD[K,(g),v]$ such that (i) $g \equiv 1 \mod 4$ and there exists a $Wh[g]$, (ii) $k \equiv 0 \mod 4$ and there exists a $Wh[k]$ for each $k \in K$, (iii) there is some parallel class of the resolution which is transversal, i.e. has blocks of size $k_o$ where $v = gk_o$, then there exists a $Wh[v]$.
Similarly for $TWh[v]$ and $DWh[v]$.

Proof: This is an application of Theorem F, where the groups are a $K_g$-factor of $K_v$. The first condition is that $K_g$ admits an appropriate near-factorization, while the third condition is the restriction on one of the factors in Theorem F.

Proposition 1.8: If there exists a $PBD[K,v]$ such that for each $k \in K$ there exists a $Wh[k]$, $k \equiv 1 \mod 4$, then there exists a $Wh[v]$. Similarly for $TWh[v]$ and $DWh[v]$.

Proof: This is an application of Theorem H. A $PBD[K,v]$ is a presentation of $K_v$ as $G_1 + \ldots + G_m$ such that each $G_i$ admits a $K_k$-decomposition for some $k \in K$.

The next proposition is the basis for the reduction of the $Wh[v]$ problem to a finite number of values for $v$. 
Proposition 1.9: If there exists a GDD[K, G, v-1] satisfying
(i) for each \( g \in G \) there exists a Wh[g+1], \( g \equiv 3 \pmod{4} \) and
(ii) for each \( k \in K \) there exists a Wh[k], \( k \equiv 1 \pmod{4} \), then
there exists a Wh[v]. Similarly for TWh[v] and DWh[v].

Proof: This is an application of Theorem E. The GDD gives a
presentation \( K_v = G_1 + G_2 \) where \( G_1 = H_1 \cup H_2 \cup ... \cup H_n \) such that
\( V(H_1) \cap V(H_j) = \{x\} \), a single vertex. Moreover, \( G_2[V(K_v) - \{x\}] = \)
\( M_1 \cup M_2 \cup ... \cup M_b \) where each \( M_i \) is isomorphic to \( K_k \) for some
\( k \in K \).

The next two propositions relate resolvable spouse avoiding
mixed doubles round robin tournaments (SAMDRR) to Whist and triple
Whist tournaments. Recall that a resolvable SAMDRR of order \( n \) is
a factorization or a near-factorization according to whether \( n \) is
even or odd.

Let \( H_i = (K_n \sqcup K_n)^{(c_1)} + (K_{n,n} - M)^{(c_j)} + (K_{n,n} - M)^{(c_s)} \) where
the \( K_{n,n} \) are with respect to the partition of the vertex set given
by the \( K_n \), \( M \) is a \( K_2 \)-factor of the \( K_{n,n} \), and \( (i, j, s) = \)
\( [1, 2, 3] \) with \( i = 1, 2, \) or \( 3 \). Let \( H_4 \) be the graph \( (L \sqcup L)^{(c_1)} + \)
\( (L \sqcup L)^{(c_2)} + (L \sqcup L)^{(c_3)} \), with the underlying graph \( K_{n,n,n,n} \) minus
a \( K_4 \) factor, where \( L \) is \( K_{n,n} \) minus a \( K_2 \)-factor.

Lemma 1.10: There exists a resolvable SAMDRR of order \( n \), \( n \) even,
if and only if there exists a \( \Gamma \)-factorization of \( H_i \), \( i = 1, 2 \) or \( 3 \),
where \( \Gamma = (K_2 \sqcup K_2)^{(c_1)} + (K_2 \sqcup K_2)^{(c_2)} + (K_2 \sqcup K_2)^{(c_3)} \). There
exists a resolvable SAMDRR of order $n$, $n$ odd, if and only if there exists a $\Gamma$-near-factorization of $H_i$, $i = 1, 2$ or $3$. Moreover, if there exists a resolvable SAMDRR of order $n$, then there exists a resolvable transversal design on 4 groups of size $n$, and hence a $\Gamma$-factorization of $H_4$. Finally, for $n$ odd, the existence of a resolvable SAMDRR of order $n$ implies that $H_4$ admits a $\Gamma$-near-factorization (relative to the partition given by the subgraphs of the $K_4$-factor removed from $K_{n,n,n,n}$).

Proof: Since $\Gamma_c$ is independent of $c$ it suffices to show the first part of the lemma for $i = 3$. The case $n$ even is considered first. Suppose there exists a resolvable SAMDRR, i.e. a factorization of the graph $G = (K_n \sqcup K_n)^2 + (K_{n,n} - M)^1 + (K_{n,n} - M)^2$.

By recoloring the edges of $G_{c_2}$, not in $G_{c_1}$, to have color $c_3$, $G$ is transformed into $H_3$. Moreover, relative to this transformation each subgraph of a factor in the SAMDRR becomes isomorphic to $\Gamma$ since all triangles of $H_3$ with exactly one edge of color $c_1$ contain exactly one edge each of colors $c_2$ and $c_3$. To produce the converse, recolor all $c_3$ edges of $H_3$ in color $c_2$. The proof for $n$ odd is identical, except that one reads near-factor for factor in each instance.

The second part of the lemma is shown in two stages, namely the existence of the transversal design first and then the factorization of $H_4$ from it. In the first case suppose $n$ is even and $V(H_3) = I_n \times I_2$. Let $V(G) = I_n \times I_4$ for $G = K_{n,n,n,n}$. Now
for each subgraph $B$ isomorphic to $\Gamma$ in the factorization of $H_3$ with $V(B) = \{(i,1),(j,1),(s,2),(t,2)\}$ and the edge $\{(i,1),(s,2)\}$ colored $c_1$, construct two graphs isomorphic to $K_4$ with vertex sets $\{(i,1),(j,2),(s,3),(t,4)\}$ and $\{(i,2),(j,1),(s,4),(t,3)\}$.

This produces a $K_4$-factor of $G$ for each $\Gamma$-factor of $H_3$, and the remaining $K_4$-factor is the one removed from $G$ to form the underlying graph of $H_3$. The case of $n$ odd is virtually the same.

Here each $\Gamma$-near-factor of $H_3$ produces $n-1$ subgraphs of $G$ isomorphic to $K_4$. Now if $\{(i,1),(j,2)\}$ is the set indexing the near-factor, then a $K_4$ on $\{(i,1),(j,2),(i,3),(j,4)\}$ is the $n$-th subgraph used to form the $K_4$-factor of $G$. Thus the existence of a SAMDRR of order $n$ implies the existence of a resolvable transversal design on $4$ groups of size $n$.

Now assume there exists a resolvable transversal design for $G = K_n + K_n + K_n + K_n$. Remove one $K_4$-factor from $G$ to form the underlying graph of $H_4$. The lemma is proved by introduction of the color map $\Pi$ of $H_4$. The final part of the lemma was already proved above when it was observed that each $\Gamma$-near-factor of $H_3$ produced $n-1$ subgraphs of $G$, i.e., a $K_4$-near-factor of $G$. Hence the result follows from the introduction of the color map $\Pi$ of $H_4$.

**Proposition 1.1:** If there exists a resolvable SAMDRR of order $n$, then there exists a $\text{TWh}[4n]$.

**Proof:** This proposition is essentially an application of Theorem D. The graph is $G = K_{4n} + K_{4n} + K_{4n}$ and the block
type \( \Gamma \) is the triple-table, as given in Lemma 1.10. \( G \) can be written as \( G_1 + G_2 + G_3 + G_4 + H_4 \), where \( G_1 \) is an \( H_1 \)-factor, \( G_2 \) is an \( H_2 \)-factor, \( G_3 \) is an \( H_3 \)-factor, and \( G_4 \) is a \( K_4 \)-factor. This decomposition is accomplished by letting \( V(G) = I_n \times I_4 \).

Then \( G_4 \) is the disjoint union of \( K_4 \)'s with vertex sets \( \{t\} \times I_4 \) for \( t = 1, 2, \ldots, n \). For each \( G_i \) there are two subgraphs isomorphic to \( H_1 \), their vertex sets being \( I_n \times \{s, t\} \) where \( \{s, t\} \) is \( \{1, 2\} \) and \( \{3, 4\} \) for \( i = 1 \), \( \{1, 3\} \) and \( \{2, 4\} \) for \( i = 2 \), and \( \{1, 4\} \) and \( \{2, 3\} \) for \( i = 3 \). The remaining edges clearly form the \( H_4 \).

Now if \( n \) is even, the proposition is an immediate consequence of Theorem D in view of Lemma 1.10, since there exists a TWh[4] by Proposition 1.14. If \( n \) is odd, some modification is necessary.

Since a TWh[4] has three rounds, \( G_4 \) produces three \( \Gamma \)-factors, say \( F_1 \), \( F_2 \), and \( F_3 \). Now consider \( G_1 + F_1 \), \( G_2 + F_2 \) and \( G_3 + F_3 \). Each \( G_i \) admits a \( \Gamma \)-near-factorization with near factors indexed by the sets \( \{t\} \times I_4 \). But each \( F_i \) contains exactly one subgraph isomorphic to \( \Gamma \) on the vertex set \( \{t\} \times I_4 \), and these subgraphs exhaust the \( F_i \). Thus \( G_i + F_i \) admits a \( \Gamma \)-factorization, thus proving the proposition via Theorem D.

**Proposition 1.12:** If there exists a resolvable SAMDRR of order \( n \), \( n \) odd, then there exists a Wh[4n+1].

**Proof:** Let \( G \) be defined on the vertex set \( (I_n \times I_4) \cup \{\omega\} \). Then \( G \) can be written as \( G_1 + G_2 + G_3 + G_4 + G_5 + H_4 \) as in Proposition 1.11, except that \( G_5 \) is the union of graphs \( I_t \) on the vertex
sets \((\{t\} \times I_i) \cup \{\omega\}\), each isomorphic to \(K_2\). By Propositions 1.2 and 1.3 there exists a \(WH[5]\), which gives five subgraphs of \(L_t\) isomorphic to \(\Gamma\). Denote by \(L_{t,i}\) the subgraph of \(L_t\) which is indexed by \((t,i)\) for \(i = 1, 2, 3, 4\) and by \(L_{t,\omega}\) the subgraph indexed by \(\omega\). The rounds of the \(WH[n+1]\) are as follows: For \(\omega\), the graph \(\bigvee_{1,\omega} \cup L_{2,\omega} \cup \ldots \cup L_{n,\omega}\). For \((t,i)\), \(i = 1, 2, 3\), the union of the \(\Gamma\)-near-factor of \(G_i\) indexed by \((t) \times I_i\) and \(L_{t,i}\). For \((t,4)\), the union of the \(\Gamma\)-near-factor of \(H_4\) indexed by \((t) \times I_4\) and \(L_{t,4}\).

The next proposition is a weak product theorem, a mixed product theorem, and a near-factorization product theorem for the three forms of Whist.

**Proposition 1.13:** If there exists \(WH[v]\) and \(WH[w]\), then there exists a \(WH[vw]\). Similarly for \(TWH\) and \(DWH\).

**Proof:** There are three cases depending on the parity of \(v\) and \(w\).

**Case 1:** \(v \equiv w \equiv 0 \pmod{4}\). This is an application of Theorem C. Let \(G\) have vertex set \(I_v \times I_w\). Then \(G = G_1 + G_2\) where \(G_1\) is the disjoint union of graphs with vertex sets \((i) \times I_w\) (and all possible edges of all three colors) and \(G_2\) is the graph with the remaining edges. Clearly \(G_1\) is a \(K_{w}^{(c_1)} + K_{w}^{(c_2)} + K_{w}^{(c_3)}\)-factor, and thus admits a \(\Gamma\)-factorization by Theorem D. For each table of a \(WH[v]\) on \(I_v\), say \((i,j; s,t)\), consider a graph \(H_4\) on vertex set \((i,j,s,t) \times I_w\) similar to the \(H_4\) in Lemma 1.10, where the
colors correspond to those of the table according to first coordinates of vertices. Such a correspondence produces an $H_4$-factor of $G_2$ for each round of the $Wh[v]$, and taken together they form an $H_4$-factorization of $G_2$. By Proposition 1.1 there exists a resolvable transversal design on four groups of size $w$, and thus by Lemma 1.10 a $T$-factorization of $H_4$. Thus the proposition follows by Theorem D.

Case 2: $v \equiv 0 \pmod{4}$, $w \equiv 1 \pmod{4}$. This is an application of Theorem F. The decomposition of $G$ into $G_1$ and $G_2$ is exactly as in case 1, thus satisfying the hypothesis of Theorem F.

Case 3: $v \equiv w \equiv 1 \pmod{4}$. This is an application of Theorem I. The decomposition of $G$ into $G_1$ and $G_2$ is as in case 1, and since each $H_4$-factor of $G_2$ has $w$ $T$-factors the external indexing is possible, thus completing the proof for Whist.

The cases above involved a variation of the $H_4$ in Lemma 1.10 to two colors. For $TWh$ the graph "$H_4$" is just as in Lemma 1.10, while for $DWh$ the variation is to a mixed graph in the obvious manner.

Before the above propositions can be utilized to prove the main theorem it is necessary to prove a series of lemmas designed to provide the necessary input. In particular Proposition 1.9 will be exploited for Whist-tournaments after a lemma on the existence of GDD's. The result on triple-Whist-tournaments follows from the
main theorem of paragraph 3 below and Proposition 1.11. The result on directed-Whist-tournaments follows from Proposition 1.8 and the work of R. M. Wilson.

**Lemma 1.14:** If there exists a $K_k$-decomposition of the complete $k$-partite graph $G = \overline{K_{k-1}} \cup K_{k-1} \cup \ldots \cup K_{k-1}$, then there exists a $K_k$-decomposition of the graph $P = \overline{K_k} \cup K_1 \cup K_1 \cup \ldots \cup K_1$ where $|V(P)| = k^2 - k + 1$.

**Proof:** Observe that $G$ is a subgraph of $P$ and let $(x) = V(P) - V(G)$. Then the remaining $K_k$ type subgraphs are obtained as $P[X_1 \cup (x)]$ where the partition of $V(G)$ is $X_1 \cup X_2 \cup \ldots \cup X_k$ and $i = 2, 3, \ldots, k$.

**Lemma 1.15:** If there exists a $K_k$-decomposition of the complete $k$-partite graph $H = \overline{K_k} \cup K_k \cup \ldots \cup K_k$, then there exists a $K_k$-decomposition of the graph $A = \overline{K_k} \cup K_k \cup K_k \cup \ldots \cup K_k$ with $|V(A)| = k^2$.

**Proof:** Observe that $H$ is a subgraph of $A$ and that each $A[X_i]$ is a $K_k$ for the sets $X_i$, $i = 3, 4, \ldots, k$ of the partition of $V(H)$.

**Lemma 1.16:** If $k$ is a prime power, then there exists a $K_{k+1}$-decomposition of the $(k+1)$-partite graph $G = \overline{K_k} \cup K_k \cup \ldots \cup K_k$ and a $K_k$-decomposition of the $k$-partite graph $H = \overline{K_k} \cup K_k \cup \ldots \cup K_k$.

**Proof:** Observe that $H$ is a subgraph of $G$ and in fact the induced subgraph on $V(H)$. Moreover, the decomposition of $H$ arises from
that of \( G \) via replacing all subgraphs \( L \) by the subgraphs \( L[V(H)] \).

The decomposition of \( G \) is a corollary of Proposition 1.1(i) and is proven in [16].

The next proposition illustrates the main technique for constructing group divisible designs.

**Proposition 1.17:** If \( k \) and \( k+1 \) are prime powers and there exists a \( TD[k^{2}+1, g_{o}] \), then there exists a \( GDD[K, G_{o}, v] \) where
\[
K = \{k^{2} + 1, k + 1\}, \quad G_{o} = \{g_{o}, g_{o} + km_{1}, g_{o} + km_{2}\}
\]
whenever
\[
o \leq m_{1}, \quad m_{2} \leq g_{o}
\]
and \( v = (k^{2} + 1) g_{o} + km_{1} + km_{2} \). Moreover, if there exists \( GDD[K, G, g] \) for all \( g \in G_{o} \), then there exists a \( GDD[K, G, v] \).

**Proof:** A \( TD[k^{2}+1, g_{o}] \) is a \( K_{k^{2}+1} \)-decomposition of the complete \((k^{2}+1)\)-partite graph \( G = \bigcup_{g} K_{g_{o}} \). First a set valued map \( B \) from \( V(G) \) is defined. Suppose \( X_{1} \cup X_{2} \cup \ldots \cup X_{k^{2}+1} \) is the partition of \( V(G) \) with \( X_{i} = \{x_{i,1}, \ldots, x_{i,g_{o}}\} \), then

\[
B(x_{i,j}) = \begin{cases} 
\{y_{i,j,1}, \ldots, y_{i,j,k+1}\} & \text{if } i=1 \text{ or } 2 \text{ and } 1 \leq j \leq m_{i} \\
\{x_{i,j}\} & \text{otherwise}
\end{cases}
\]

where \( B(x_{i,j}) \cap B(x_{s,t}) = \emptyset \) for \((i,j) \neq (s,t)\). For any set \( S \subset V(G) \) let \( B(S) \) denote the union of all \( B(x) \) where \( x \in S \). Now there is a graph \( G' \) with \( V(G') = \bigcup(V(G)) \) which is a complete multipartite graph relative to the partition given by the \( B(X_{i}) \).
The proposition requires that $G'$ admit a decomposition into subgraphs isomorphic to $K_{k^2+1}$ or $K_{k+1}$. For any subgraph $H$ of $G$ in its $K_{k^2+1}$-decomposition let $H'$ denote the graph $G[B(H)]$. There are precisely three possibilities, namely $H'$ is a $K_{k^2+1}$, or is one of the $(k^2+1)$-partite graphs $K_{k+1} \cup K_1 \cup \ldots \cup K_1$ or $K_{k+1} \cup K_{k+1} \cup K_1 \cup \ldots \cup K_1$. In the first case $H'$ is retained for the decomposition whereas in the latter two cases there exist subgraphs of $H'$ isomorphic to $K_{k+1}$, by Lemmas 1.14, 1.15 and 1.16. Hence $G'$ can be decomposed into $K_{k^2+1}$ and $K_{k+1}$ subgraphs.

To show the latter half of the proposition requires observing that the relevant graph is the union of the graphs for the $GDD[K, G_o, v]$ and the several for the $GDD[K, G, g]$, which are edge disjoint.

**Lemma 1.18:** For all $v \geq 32^4$, $v \equiv 0 \pmod{4}$, there exists a $GDD[K, G, v-1]$ where $K = \{5, 17\}$ and $G = \{19, 23, 27, \ldots, 315, 319\} - \{151\}$.

**Proof:** It suffices to enumerate a list of values $g_o$ subject to the constraints that $g_o \equiv 3 \pmod{4}$, there exists a $TD[17, g_o]$ and such that the intervals $17 g_o$ to $25 g_o$ cover all $v-1 \geq 323$.

If such a list is given then setting $k = 4$ and avoiding $g_o$, $g_o + 4m_1$ or $g_o + 4m_2$ being $151$ proves the lemma from Proposition 1.17. Such a list of values for $g_o$ is $19, 27, 31, 47, 67, 83, 107, 139, 199, 243, 323, 443, 619, 863, 1207, 1679$, and then $3^8, 3^{s-3}, 31,
Lemma 1.19: For all $v-1 \in \{19, 23, 27, \ldots, 319\} - \{151\}$ there exists a $Wh[v]$.

Proof: Corollary 1.6 shows that there exist $TWh[v]$ for all $v \equiv 4 \pmod{12}$ since there exist resolvable $B[4,v]$ by Proposition 2.3 below, and hence $Wh[v]$ by Proposition 1.2 (ii). For $v = 4n$ where $n = 6s + 1$ or $n = 3^t$, $t \geq 2$, Proposition 1.11 applies to the resolvable SAMDRR of Proposition 3.2. Since there exist $TWh[4]$ and $TWh[8]$ (Proposition 1.4) it follows from Proposition 1.13 that there exist $TWh[2^r]$ for $r \geq 2$. For the values 48, 60, 72, 84, 96, 104, 120, 132, 156, 168, 192, 200, 204, 224, 240, 264, 276, 296, and 300 the construction follows from Proposition 1.13. For 56, 216 and 248 Proposition 1.9 is applied to $TD[k,n]$ for $(k,n) = (5,11),(5,43)$ and $(13,19)$. Finally 228 and 312 follow from Proposition 1.11 and resolvable SAMDRR of Remark 3.5.

Theorem 1.20: For all $v \equiv 0 \pmod{4}$, except possibly $v = 152$, there exists a $Wh[v]$.

Proof: By Lemma 1.19 the theorem is true for all $v \leq 320$, and hence by Lemma 1.18 and Proposition 1.9 the result holds for all $v$.

Theorem 1.21: For all $v \equiv 1 \pmod{4}$, except possibly 57 or 129, there exists a $Wh[v]$.
Proof: Those values other than 93 are given in Theorem 1.27 below because of Proposition 1.2 (i). The existence of a Wh[93] follows from Proposition 1.12 and the existence of a resolvable SAMDRR of order 23 (see Proposition 3.2).

Theorem 1.22: For all large \( v = 4n \) such that \( n \not\equiv 2 \pmod{4} \) there exists a TWh[\( v \)].

Proof: By Proposition 1.11 the result holds provided there exist the resolvable SAMDRR of order \( n \), and the latter is established in section 3.

Theorem 1.23: For all large \( v \), \( v \equiv 1 \pmod{4} \), there exists a TWh[\( v \)].

Proof: In [42] R. M. Wilson shows that for all sufficiently large \( v \), \( v \equiv 1 \pmod{4} \), there exists a PBD\([K,v]\) where \( K = \{29, 37\} \). Hence, the result follows from Proposition 1.8 and the existence of TWh[29] and TWh[37] given below (by difference methods):

- TWh[29]: Symbols are the elements of GF(29) and the tournament is cyclic with initial round:
  
  \((1,2; 9,27), (16,3; 28,26), (24,19; 13,10), (7,14; 5,15), (25,21; 22,8), (23,17; 4,12)\) and \((20,11; 6,18)\).

- TWh[37]: Symbols are the elements of GF(37) and the tournament is cyclic with initial round:
  
  \((1,2; 17,4), (16,32; 13,27), (34,31; 23,25), (26,15; 35,30),\)
Theorem 1.24: There exists a $\text{DWh}[40]$, and hence infinitely many $\text{DWh}[v]$ for $v \equiv 0 \pmod{4}$.

Proof: Players are $(\mathbb{Z}_7 \cup \{\infty\}) \times \mathbb{Z}_5$. From each of the following five pairs of tables an initial round is obtained by developing modulo 5 in the second coordinate, and then seven rounds are obtained from each initial round by developing modulo 7 in the first coordinate (with $\infty$ fixed). The pairs of tables are:

- $((0,0), (2,1); (1,2), (4,2)) \& ((0,0), (6,0); (3,1), (5,0))$
- $((0,0), (4,0); (2,3), (1,4)) \& ((0,0), (5,3); (6,2), (3,4))$
- $((0,0), (1,3); (4,1), (2,3)) \& ((0,0), (3,2); (5,3), (6,4))$
- $((0,0), (2,2); (1,4), (4,0)) \& ((0,0), (6,5); (3,0), (5,1))$
- $((0,0), (5,4); (6,0), (3,1)) \& ((0,0), (1,3); (4,0), (2,0))$

Now on each set $\{x\} \times \mathbb{Z}_5$, for $x \in \mathbb{Z}_7 \cup \{\infty\}$, the tables of a $\text{DWh}[5]$ are formed. One round of the above 35 is removed and each of its pairs of tables is coupled with the tables from the $\text{DWh}[5]$'s omitting the appropriate player just as in the proof of Proposition 1.13. This completes the construction of the $\text{DWh}[40]$.

To obtain infinitely many $\text{DWh}[v]$ for $v \equiv 0 \pmod{4}$ it suffices to apply Proposition 1.13.

Lemma 1.25: For all $n \equiv 1 \pmod{4}$, $n \geq 289$, there exists a $\text{GDD}[K,G,n]$ with $K = \{5, 17\}$ and $G = \{17, 21, 25, \ldots, 281\} - \{57, 93, 129\}$.
Proof: Apply Proposition 1.17 with $k = 4$ and $g_0$ selected as $17, 25, 29, 37, 49, 61, 81, 121, 173, 241, 353, 519$, and then $3^s, 3^{s-2}, 13, 3^{s-2}, 17, 3^{s-2}, 25, 3^{s-2}, 29, 3^{s-2}, 41, 3^{s-2}, 57, 3^{s+2}$ for $s = 6, 8, 10, \ldots$.

Lemma 1.26: For all $v \in \{5, 9, 13, 17, \ldots, 285\} - \{57, 93, 129\}$ there exists a $\text{DWh}[v]$.

Proof: All those $v$ which are prime powers follow from Proposition 1.3. For $v = 33$ the players are the residues modulo 33 and the rounds are developed from the initial round: $(25, 14; 27, 32), (26, 10; 6, 29), (23, 4; 21, 13), (16, 7; 12, 15), (3, 24; 2, 31), (5, 20; 17, 11), (30, 28; 22, 9)$ and $(18, 19; 8, 1)$. For the remaining values of $v$ there exists a $\text{PEB}[K,v]$ with block sizes $K$ prime power congruent to one modulo four or 33, hence a $\text{DWh}[v]$ by Proposition 1.8 (see [42]).

Theorem 1.27: For all $v \equiv 1 \pmod{4}$, except possibly 57, 93 or 129, there exists a $\text{DWh}[v]$.

Proof: By Lemma 1.26 the result holds for $v \leq 289$, and hence by Lemma 1.25 and Proposition 1.8 the theorem is true.

* The $\text{DWh}[33]$ was found with the aid of the IBM 370/165 of the Instruction and Research Computer Center of The Ohio State University.
2. **Resolvable BIBD's.** In this section one new theorem (2.5) is proven concerning the existence of resolvable balanced incomplete block designs (BIBD's). Recall the notation "resolvable $B_\lambda^\alpha[k,v]" for a $K_k$-factorization of $\lambda K_v$.

**Theorem 2.1:** (Ray-Chaudhuri and Wilson): There exists a resolvable $B[3,v]$ for all $v = 3 \pmod{6}$.

**Proof:** See [35].

The next lemma is a generalization of a construction of E. H. Moore [32] which is useful in proving the theorem following it.

**Lemma 2.2:** Suppose $V(\Gamma) = \{y,z,v,w\}$ and $E(\Gamma) = \{(y,z),(v,w),(y,v),(y,w),(z,v),(z,w)\}$, in other words $\Gamma$ is a mixed graph whose simple edges form a $K_2 \perp K_2$ and directed edges form a bond. If $G = K_n + D_n$ admits a $\Gamma$-near-factorization, then there exists a resolvable $B[4,v]$ for $v = 3n + 1$. In particular if $n$ is a prime power congruent to one modulo four, such a $\Gamma$-near-factorization exists.

**Proof:** Let the graph $K_{3n+1}$ have as vertex set $(V(G) \times I_3) \cup \{\infty\}$. Suppose $G_u$ is the $\Gamma$-near-factor indexed by $u$, then form a $K_4$-factor of $K_{3n+1}$ as follows: One $K_4$ has vertex set $\{(u) \times I_3\} \cup \{\infty\}$. For a subgraph isomorphic to $\Gamma$ with vertex set $\{y,z,v,w\}$ as above form three $K_4$ subgraphs with vertex sets $\{(y,1),(z,1),(v,j),(w,j)\}$ for $(i,j) = (1,2), (2,3)$ or $(3,1)$.
The remark for \( n = 4m+1 \) prime power follows by selecting 
\[ y = r, \quad z = rx^{2m}, \quad v = rx^m, \quad w = rx^{3m} \]
where \( x \) is a primitive element of \( \text{GF}(n) \) and \( r = 1, x, x^2, \ldots, x^{m-1} \) to form an initial \( \Gamma \)-near-factor.

**Theorem 2.3** (Hanani, Ray-Chaudhuri and Wilson): There exists a resolvable \( B[4,v] \) for all \( v \equiv 4 \) (mod 12).

**Proof:** See [19].

**Theorem 2.4** (Hanani): There exists a resolvable \( B_2[3,v] \) for all \( v \equiv 0 \) (mod 3), \( v \neq 6 \).

**Proof:** See [18].

**Theorem 2.5:** There exist a resolvable \( B_3[4,v] \) for all \( v \equiv 0 \) (mod 4) except possibly \( v = 152 \).

**Proof:** There exist \( W_h[v] \) for all these values and hence the result follows from ignoring the color map.

3. **Resolvable SAMDRR.** This paragraph is organized in the same manner as paragraph 1, namely there are the constructive propositions, the lemmas, and the main theorems.

**Proposition 3.1:**

(i) If there exists a \( TWh[v] \), then there exists a resolvable SAMDRR of order \( v \).

(ii) If there exists a \( DWh[v] \), then there exists a resolvable SAMDRR of order \( v \).
Proof: Both halves are proven simultaneously. Suppose the graph $G$ for the resolvable SAMDRR has vertex set $X \times I_2$ where $X$ is the vertex set of the graph of the $\text{TWh}[v]$ or $\text{DWh}[v]$. From each table $(x_1, x_2; x_3, x_4)$ form two subgraphs having vertex sets $\{(x_1, 1), (x_2, 1), (x_3, 2), (x_4, 2)\}$ and $\{(x_1, 2), (x_2, 2), (x_3, 1), (x_4, 1)\}$ in which the edges corresponding to $\{x_1, x_3\}$ and $\{x_2, x_4\}$ are colored $c_1$. In this manner a $\Gamma$-factor (or $\Gamma$-near-factor) is produced from each round, thus proving the proposition.

Proposition 3.2: (i) If $n = 6s + 1$, then there exists a resolvable SAMDRR of order $n$.
(ii) If $n = 3^t$, $t \geq 2$, then there exists a resolvable SAMDRR of order $n$.

Proof: (i) Let $V(G) = \mathbb{Z}_n \times I_2$ and construct the subgraphs $((i, 1), (j, 1); (2i-j, 2), (2j-i, 2))$ where the representation is the same as for Whist tournaments. For each $c \in \mathbb{Z}_n$, a $\Gamma$-near-factor $G_c$ is all such subgraphs for which $\frac{1}{2}i + \frac{1}{2}j = c$.
(ii) Let $V(G) = \text{GF}(3^t) \times I_2$ and $u \in \text{GF}(3^t) - \text{GF}(3)$. The subgraphs are all $((i, 1), (j, 2); (ui + (1-u)j, 2), (uj + (1-u)i, 2))$, with near factors $G_c$ containing all such subgraphs for which $-i-j = c$ for various $c \in \text{GF}(3^t)$.

Proposition 3.3: If there exists a $\text{GDD}[K, G, v-1]$ such that
(i) there exists a resolvable SAMDRR of order $k$ for each $k \in K$, 

all $k$ odd, and (ii) there exists a resolvable SAMDRR of order $g + 1$ for each $g \in G$, all $g$ odd, then there exists a resolvable SAMDRR of order $v$.

**Proof:** This is an application of Theorem E. The graph $G$ has vertex set $I_v \times I_2$ and the GDD may be taken on the vertex set $I_{v-1}$. It follows that $G$ is written as desired by replacing all subgraphs $H$ from the GDD by $G[V(H) \times I_2]$.

**Proposition 3.4:** If there exists a PBD[$K,v$] such that all $k \in K$ are odd and there exist resolvable SAMDRR of order $k$, then there exists a resolvable SAMDRR of order $v$.

**Proof:** This is an application of Theorem H. The graph $G$ has vertex set $I_v \times I_2$ and the PBD may be taken on the vertex set $I_v$. It follows that $G$ is written as desired by replacing all subgraphs $H$ from the PBD by $G[V(H) \times I_2]$.

**Remark 3.5:** The orders 50 and 78 follow from Proposition 3.3 and TD[7,7] and GDD[7,11], {7}, 77] respectively. The GDD arises from a TD[7,11] by removal of a $K_i$-factor and adjoining the groups as a $K_{11}$-factor. Order 57 follows from Proposition 3.4 and a TD[7,8].

**Proposition 3.6:** If there exist resolvable SAMDRR of orders $n$ and $m$, then there exists a resolvable SAMDRR of order $nm$. 
Proof: This is an application of Theorem C. The graph $G$ on vertex set $I_{nm} \times I_2$ can be written as $G_1 + G_2$ where $G_1$ is an $H$-factor for $H$ the graph of a resolvable SAMDR of order $n$ and such that $G_2$ admits an $H_4$-factorization where $H_4$ has underlying graph $K_{n,n,n,n}$ and admits a $\Gamma$-factorization as in the proof of Lemma 1.10.

Lemma 3.7: For all $n \equiv 0 \pmod{4}$, $324 \leq n \leq 880$ or $1136 \leq n$, there exists a GDD[$K$, $G$, $n-1$] where $K = \{5, 17\}$ and $G = G_1 \cup G_2$ for $G_1 = \{19, 23, 27, \ldots, 319\} - \{47, 55, 59, 71, 103, 131, 151, 167, 227, 239, 263, 295\}$ and $G_2 = \{879, 883, \ldots, 1135\} - \{887, 911, 919, 923, 947, 967, 971, 1019, 1031, 1063, 1067, 1091, 1103, 1111\}$.

Proof: Comparing with Lemma 1.18 it is noted that fewer group sizes are in $G_1$ than in this previous lemma, in particular $g_o = 47$ was used in Lemma 1.18. Thus values of $n$ from 880 to 1136 are not covered when the remaining initial values $g_o$ are used. Values formerly dependent on group sizes in this range now utilize group sizes from $G_2$. With these observations the lemma follows from Proposition 1.17.

Lemma 3.8: For all $n \equiv 3 \pmod{4}$, $n \geq 323$, there exists a GDD[$K$, $G$, $n$] with $K = \{5, 17\}$ and $G = \{19, 23, 27, \ldots, 319\} - \{39, 51, 75, \ldots, 303\}$.

Proof: Apply Proposition 1.17 with the same values $g_o$ as in Lemma 1.18.
Theorem 3.9: For all large \( n \equiv 0 \pmod{4} \) there exists a resolvable SAMDRR of order \( n \).

Proof: By Lemma 3.7 and Proposition 3.3 it suffices to consider the orders \( g + 1 \) such that \( g \) is one of the group sizes of Lemma 3.7. Propositions 3.1 and 3.2 cover all \( n \equiv 4 \pmod{12} \) or \( n = 4m \) where \( m \) is any of the values of Proposition 3.2. Additional values follow from Proposition 3.6. Applying Proposition 3.4 to \( m = 21 \) produces via Proposition 3.6 \( n = 84 \). For 120, 156, 204, 216, 248 and 300 Proposition 3.3 may be applied to TD\([k,m]\), or a GDD\([k,m], [k], km\) derived from such as in Remark 3.5. These arise from \((k,m) = (5, 31), (5, 43), (13, 19)\) or \((13, 23)\) and \((k,m) = (7, 17)\) or \((7, 29)\) respectively. Finally 900, 936, 996, 1016, 1080 and 1128 follow from Proposition 3.3 also with \((k,m) = (29, 31), (5, 187), (5, 199), (5, 203), (15, 83)\) and \((23, 49)\).

Theorem 3.10: For all large odd \( n \) there exists a resolvable SAMDRR of order \( n \).

Proof: If \( n \equiv 1 \pmod{4} \), then the result follows from Theorem 1.27 and Proposition 3.1(ii).

If \( n \equiv 3 \pmod{4} \), then by Proposition 3.4 and Lemma 3.8 it suffices to show the existence for orders \( g \) of Lemma 3.8. Proposition 3.2 covers most of these values. For the orders 147, 231 and 291 Proposition 3.4 is applied to PBD\([5,7], 147\), PBD\([5, 47], 231\) and PBD\([5, 59], 291\).
4. Nearly Kirkman Triple Systems. Recall the definition of a nearly Kirkman triple system as a $K_3$-factorization of $K_v$ minus a $K_2$-factor. Such a system will be denoted by $NKTS[v]$. All $v \equiv 0 \pmod{6}$ are admissible. In this section it will be shown that this condition is sufficient except for $v = 6, 12, 84, 102$ or $174$ (among these $v = 6$ and $12$ do not admit $NKTS$ while the remaining values are not settled). For brevity a resolvable $B[3, u]$ is denoted $KTS[u]$.

Lemma 4.1: There exists a $K_3$-near-factorization of $K_v$ minus a $K_2$-factor for all $v \equiv 2 \pmod{6}$, $v > 2$.

Proof: This is an application of Theorem J to Theorem 2.1 ($v = 2$ is omitted since $K_2$ minus a $K_2$-factor is trivial).

Lemma 4.2: There exists a $K_3$-factorization of $K_{v, v, v}$ for all $v \neq 2$ or $6$.

Proof: This is the same as a resolvable $TD[3, v]$ and are known to exist (see [16]).

Proposition 4.3 (Kötzig-Rosa): If there exists a $KTS[u]$ and an $NKTS[v]$, then there exists an $NKTS[uv]$.

Proof: This is an application of Theorem C, noting that the underlying graphs actually come from different families. Let $G$ be the graph for the $NKTS[uv]$, then $G = G_1 + G_2$ where $G_1$ is an $H_1$-factor ($H_1$ the graph of an $NKTS[v]$) and $G_2$ is a $H_2$-factor ($H_2$ is $K_{v, v, v}$). The result then follows from Lemma 4.2.
**Proposition 4.4** (Kötzig-Rosa): If there exist KTS[u] and KTS[v], \( v > 3 \), then there exists an NKTS[u(v-1)].

**Proof:** By Lemma 4.1 there exists a non-trivial \( K_3 \)-near-factorization of \( K_v \) minus a \( K_2 \)-factor. Theorem F is then applied to this \( K_3 \)-near-factorization and the \( K_3 \)-factorization of \( K_u \).

**Proposition 4.5:** If there exists a KTS[u], an NKTS[v] and a \( TD[k,n] \) for \( k = \frac{1}{2}(u-1) \) and \( n = \frac{1}{2}(v-2) \), then there exists an NKTS[2kn + 2].

**Proof:** This is an application of Theorem E. From the TD with graph \( G_0 \), a new graph \( G \) is obtained on vertex set \((V(G_0) \cup \{w\}) \times I_2\) having decomposition given by the subgraphs \( G[V(H) \times I_2] \) for each subgraph \( H \) of \( G_0 \).

**Proposition 4.6:** If there exists a GDD[\( K, G, n \)] such that

(i) for each \( g \in G \) there exists an NKTS[2g + 2] and

(ii) for each \( k \in K \) there exists a KTS[2k + 1], then there exists an NKTS[2n + 2].

**Proof:** This is another application of Theorem E and is proven in the same manner as Proposition 4.5.

**Corollary 4.7:** If there exists an NKTS[v], then there exists an NKTS[4v - 6].

**Proof:** Let \( u = 9 \) in Proposition 4.5 since there exists a \( TD[4, \frac{1}{2}(v-2)] \) for all such \( v \).
Proposition 4.8: If there exists a KTS[u], an NKTS[v] and a resolvable TD[k,n] for $k = \frac{1}{2}v$ and $n = \frac{1}{2}(u-1)$, then there exists an NKTS[2kn].

Proof: This is an application of Theorem B which doubles the vertex set of the TD as in the propositions above.

Proposition 4.9: If there exists a uniformly resolvable PBD[K,n] and for each $k \in K$ there exists an NKTS[2k], then there exists an NKTS[2n].

Proof: This is an application of Theorem D which doubles the vertex set of the PBD as above.

Lemma 4.10: For all $n \equiv 2 \pmod{3}$, $n \geq 110$, there exists a GDD[K, G, n] where $K = [4, 10]$ and $G = \{11, 14, 17, \ldots, 107\} - \{41, 50, 86\}$.

Proof: Proposition 1.17 is applied with $k = 3$ and the initial values $g_0$ selected as 11, 17, 23, 29, 32, 47, 53, 71, 83, 89, 101, $11^s \cdot 13$, $11^s \cdot 19$, $11^s \cdot 25$, $11^s \cdot 31$, $11^s \cdot 49$, $11^s \cdot 73$, $11^s \cdot 103$, $11^{s+2}$ for $s = 1, 3, 5, \ldots$.

Lemma 4.11: There exists an NKTS[v] for all $v \equiv 0 \pmod{6}$ such that $\frac{1}{2}(v-2) \in G$, $G$ as in Lemma 4.10.

Proof: Assuming 18, 30, 36, 48 and 192 are known, all the remaining values follow from Propositions 4.3 to 4.9. An NKTS[18] is given in the introduction.
Below are direct constructions (difference method) for the remaining three values.

NKTS[30]: Points are \((\mathbb{Z}_7 \times \{1,2,3,4\}) \cup \{^1_1, ^2_2\}\). Each column is an initial parallel class which is then developed mod 7 to give the remaining classes.

\[
\begin{array}{ccccccc}
(4,1) & (5,2) & (0,3) & (0,1) & (2,2) & (6,3) \\
(6,1) & (3,2) & (4,3) & (6,1) & (5,2) & (1,4) \\
(1,1) & (6,2) & (5,4) & (0,2) & (5,3) & (4,4) \\
(5,1) & (1,2) & (6,4) & (1,2) & (0,3) & (3,4) \\
(2,1) & (6,3) & (0,4) & (1,1) & (2,1) & (4,1) \\
(0,1) & (1,3) & (3,4) & (3,2) & (4,2) & (6,1) \\
(3,1) & (5,3) & (2,4) & (1,3) & (2,3) & (4,3) \\
(0,2) & (3,3) & (1,4) & (6,4) & (0,4) & (2,4) \\
(2,2) & (2,3) & ^1_1 & (5,1) & (5,4) & ^1_1 \\
(4,2) & (4,4) & ^2_2 & (3,1) & (3,3) & ^2_2 \\
\end{array}
\]

NKTS[36]: Points are \((\mathbb{Z}_{17} \times \{1,2\}) \cup \{^1_1, ^2_2\}\). An initial parallel class is given, the remaining classes being obtained by developing mod 17.

\[
\begin{array}{ccccccc}
(2,1) & (3,2) & (4,2) & (12,1) & (6,2) & (9,2) & (3,1) & (8,2) & (13,2) \\
(11,1)(10,2)(0,2) & (7,1) & (2,2) & (11,2) & (0,1) & (9,1) & (7,2) \\
(13,1)(8,1) & (16,2) & (10,1) & (14,1)(4,1) & (15,1)(16,1)(1,1) \\
(12,2)(14,2)(1,2) & ^1_1 & (5,1) & (5,2) & (6,1) & (15,2) & ^2_2 \\
\end{array}
\]
NKTS[48]: Points are \((Z_{23} \cup \{\infty\}) \times Z_2\). Half an initial parallel class is given, the other obtained by developing mod 2. The remaining classes are developed mod 23.

| (0,0) | (14,0) | (22,1) | (6,0) | (15,0) | (13,1) |
| (1,0) | (2,0)  | (4,0)  | (8,0) | (19,0) | (18,1) |
| (5,0) | (10,0) | (20,0) | (9,0) | (16,0) | (12,1) |
| (3,0) | (7,0)  | (21,1) | (11,0) | (17,0) | (0,1) |

Finally a NKTS[192] must be constructed. Since \(K_{192} = (K_4 \sqcup K_4 \sqcup K_4 \sqcup K_4) \cup K_{48,48,48,48}\) it follows that \(G = (H \sqcup H \sqcup H \sqcup H) + H_0\) where \(G\) is a graph of an NKTS[192], each \(H\) is a graph of an NKTS[48], and \(H_0\) is \(K_{48,48,48,48}\). Unions of factors of the four \(H\) provide the first 23 factors of \(G\), and it only remains to show that \(H_0\) admits a \(K_3\)-factorization (providing the remaining 72 \(K_3\)-factors of \(G\)). To show \(H_0\) admits a \(K_3\)-factorization it is first shown to admit a \(K_{6,6,6,6}\)-factorization, which in turn admits a \(K_3\)-factorization (Theorem D). Let \(V = X \times Y \times W\) be the vertex set of \(H_0\), where \(|X| = 4\), \(|Y| = 8\), and \(|W| = 6\), and the sets \((X) \times Y \times W\) form the partition. Let \(F\) be a \(K_8,8,8,8\) on the vertex set \(X \times Y\) with partition given by the sets \((X) \times Y\). By Proposition 1.1 \(F\) admits a \(K_4\)-factorization with factors \(F_1, F_2, \ldots, F_8\). Form a \(K_{6,6,6,6}\)-factor \(H_1\) of \(H_0\) having as subgraphs all \(H_0[V_O \times W]\) where \(V_O\) is the vertex set of a \(K_4\) subgraph of \(F_1\). Thus the construction is completed by showing that each of the \(H_1\) admits a \(K_3\)-factorization. Such a
Construction is given in [42] utilizing the embedding of the projective plane of order 3 in the projective plane of order 9 (see [13] for a discussion of finite projective planes).

Theorem 4.12: There exist an NKTS[v] for all \( v = 0 \pmod{6} \), \( v > 12 \), except possibly \( v = 84, 102, \) or 174.

Proof: By Lemma 4.11 the result is true for \( v \leq 222 \), and hence the theorem follows from Lemma 4.10 and Proposition 4.6.
APPENDIX A

The purpose of this appendix is to demonstrate the relationship between Latin squares and three factorization problems, namely resolvable SAMDRR, triple Whist tournaments and directed Whist tournaments.

Definition A.1: A quasigroup $Q$ is a set $S$ together with a binary operation which satisfies the condition that any two elements of the equation $xy = z$ uniquely determine the third.

Definition A.2: A Latin square of order $n$ on the symbol set $S$ ($|S| = n$), the row set $R$ ($|R| = n$) and the column set $C$ ($|C| = n$) is a mapping $L$ from $R \times C$ to $S$ such that the equation $L(r,c) = s$ has a unique solution for any one of $r$, $c$ or $s$ if the other two are fixed.

Proposition A.3: The class of Latin squares with $S = R = C$ are coexistent with the class of quasigroups on $S$.

Proof: Write $xy = z$ for $L(x,y) = z$ and conversely.

In light of the above proposition the language of quasigroups will be used throughout the discussion whenever it is convenient. Clearly the structure of a quasigroup of order $n$ may be transferred to any other set of $n$ elements by an arbitrary bijection between...
the sets, hence two quasigroups of the same order may be assumed to have the same set.

**Definition A.4:** Quasigroups $Q$ and $Q'$ on the set $S$, with operations written as $*$ and $\cdot$ respectively, are orthogonal if for each $(s,t) \in S \times S$ there exist a unique ordered pair $(x,y) \in S \times S$ such that $x * y = s$ and $x \cdot y = t$. $Q'$ is called a mate of $Q$. A set of mutually orthogonal quasigroups is a set of quasigroups such that any two are orthogonal. $Q$ is called idempotent if $x * x = x$ for all $x \in S$, unipotent if $x * x = y * y$ for all $x,y \in S$, and symmetric (or commutative) if $x * y = y * x$ for all $x,y \in S$. The $(1,2)$-conjugate of $Q$ is the quasigroup of $S$ with operation $\circ$ given by $x \circ y = y * x$. An idempotent $Q$ is called self-orthogonal if $Q$ is orthogonal to its $(1,2)$-conjugate. $Q'$ is a mate to a self-orthogonal $Q$ if it is orthogonal to both $Q$ and its $(1,2)$-conjugate.

**Definition A.5:** Latin squares $L$ and $L'$ are orthogonal if their associated quasigroups are, and $L'$ is called a mate of $L$. A set of mutually orthogonal Latin squares (MOLS) is a set of Latin squares any two of which are orthogonal. $L$ is idempotent, unipotent or symmetric if its associated quasigroup $Q$ is. The Latin square associated with the $(1,2)$-conjugate of $Q$ is called the transpose of $L$. $L$ is self-orthogonal if it is orthogonal to its transpose.
L' is a mate of a self-orthogonal L if L' is orthogonal to both L and its transpose.

The above correspondence between Latin squares and quasigroups may be found in [14], which also contains proofs of the following well known result.

**Proposition A.6:** The following are equivalent:

(i) A set of k-2 MOIS (or mutually orthogonal quasigroups) of order n.

(ii) A TD[k,n].

(iii) A resolvable TD[k-1,n].

The following propositions illustrate the relationships previously alluded to.

**Proposition A.7:** There exists a resolvable SAMDRR of order n if and only if there exists a self-orthogonal Latin square of order n with a symmetric mate which is idempotent if n is odd or unipotent if n is even.

**Proof:** This is a refinement of a theorem in [9] asserting the equivalence of a SAMDRR and a self-orthogonal Latin square. Suppose the graph G of the resolvable SAMDRR has vertex set \( I_n \times I_2 \) and the Latin squares L and L' have symbol (row and column) set \( I_n \).

Then \{ (s, 1), (t, 1), (u, 2), (v, 2) \} is the vertex set of a subgraph with edges \{ (s, 1), (u, 2) \} and \{ (t, 1), (v, 2) \} colored c₁ in
factor \( G_i \) if and only if \( L(s, t) = u, L(t, s) = v \), and \( L'(s, t) = i \). \( L(i, i) = i \) for \( i \in I_n \), and \( L'(i, i) = i \) if \( n \) is odd or \( L'(i, i) = n \) if \( n \) is even. The verification that this correspondence is as asserted is straightforward.

**Corollary A.8:** If there exists a TWh[\( v \)] or DWh[\( v \)], then there exists a self-orthogonal Latin square of order \( v \) with symmetric mates.

**Proof:** By Proposition 3.1 the existence of a TWh[\( v \)] or DWh[\( v \)] implies the existence of a resolvable SAMDR of order \( v \).

It is possible to recognise those self-orthogonal Latin squares with symmetric mates which arise from TWh[\( v \)] or DWh[\( v \)] via the construction of Cor. A.8. The next proposition illustrates using the quasigroup language for convenience.

**Proposition A.9:** Suppose there exists a self-orthogonal quasigroup \( Q \) of order \( v \) with commutative mate \( Q' \), the operations denoted by juxtaposition and \( * \) respectively. Also assume that \( x * y = xy * yx \). If \( Q \) satisfies Schröder's second law, namely \( (xy)(yx) = x \), then there exists a TWh[\( v \)]. If \( Q \) satisfies Stein's third law, namely \( (xy)(yx) = y \), then there exists a DWh[\( v \)].

**Proof:** Either condition implies that the correspondence of \( \{x, y\} \) to \( \{xy, yx\} \) is a matching, thus since the number of two element subsets is even it must be that \( v = 0, 1 \mod 4 \). Form one table

---

**Form one table**
(x, y; yx, xy) for each such pair of two element sets, and form round \( G_s \) from those tables for which \( x \cdot y = s \).

If the mate \( Q' \) is dropped from the hypothesis of Proposition A.9 then the conclusions are the existence of the appropriate decompositions rather than factorizations. This is discussed in [2].

It should be noted that Norton and Stein [33] were tempted to conjecture that quasigroups satisfying Schröder's second law only exist if \( v \equiv 0 \pmod{4} \) and those satisfying Stein's third law only if \( v \equiv 1 \pmod{4} \). The results of section 1 in Chapter V show that such a conjecture is false by the construction of Corollary A.8. The final observation is that Corollary A.8 can be extended for cyclic \( TWh[v] \) or \( DWh[v] \), \( v \equiv 1 \pmod{4} \), to assert the existence of another Latin square orthogonal to the first three.

**Proposition A.10:** If there exists a cyclic \( TWh[v] \) or \( DWh[v] \), \( v \equiv 1 \pmod{4} \), then there exist 4 MOLS of order \( v \).

**Proof:** Since \( v \equiv 1 \pmod{4} \) the \( TWh[v] \) or \( DWh[v] \) is a near-factorization. Suppose \( x \) is the player omitted in the initial round and define \( L''(y, y) = x \) for the new Latin square \( L'' \).

For \( y \neq z \), there exists a unique table \((y, z; s, t)\), which is identically the table \((y' + r, z' + r; s' + r, t' + r)\) for a table \((y', z'; s', t')\) of the initial round. Thus define \( L''(y, z) = y' \).

It may be verified directly that \( L'' \) is orthogonal to \( L' \), \( L \) and the transpose of \( L \) as given in Corollary A.8.
Corollary A.11: There exist 4 MOIS of order 33.

Proof: A cyclic DWh[33] is given in section 1 of Chapter V. This is a new world's record for MOIS of order 33 (see [41]).
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