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EMBEDDING GRAPHS IN THE PROJECTIVE PLANE

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Chin San Wang, B.S., M.S.

** ** *

The Ohio State University

1975

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INTRODUCTION

The four color conjecture is well known. In 1890 P. J. Heawood [5] found the mistake in Kempe's "proof" of this conjecture and proved the corresponding result for five colors. In addition he formulated a generalization of the four color conjecture to general surfaces. He conjectured that the chromatic number \( n \), the number of colors needed to color the closed orientable surface of genus \( g \), \( M_g \) is the integer part

\[
n(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor
\]

and for the closed non-orientable surface of genus \( g \), \( \tilde{M}_g \) is

\[
\tilde{n}(g) = \left\lfloor \frac{7 + \sqrt{1 + 24g}}{2} \right\rfloor, \ g \neq 2.
\]

The Heawood conjecture for non-orientable surfaces was solved by Ringel [9] in 1952 and by Ringel and Youngs [10] for orientable surfaces in 1968. They used Heawood's "contiguity Lemma" which allowed them to replace this problem with an embedding problem. Namely they showed that the orientable genus, the smallest genus of any orientable surface containing \( K_n \) is the integer hull.
\[ g(n) = \left\lceil \frac{(n - 4)(n - 3)}{12} \right\rceil \]

and the non-orientable genus of \( K_n \) is

\[ \hat{g}(n) = \left\lceil \frac{(n - 4)(n - 3)}{6} \right\rceil . \]

The idea used in this replacement is an amalgamation technique similar to that in the proof of the five color theorem. Using this technique we may construct an example of a map on \( M_g \) (\( \tilde{M}_g \)) with exactly \( n(g) \) (\( \tilde{n}(g) \)) countries each two of which are contiguous.

By considering the dual to the graph of boundaries of this map (the graph of railroads between the capitals of these countries) we then get \( K_n \subset M_g \) (\( K_n \subset \tilde{M}_g \)) and the dual embedding problem (see Ringel [8]).

Using the solution of the Heawood conjecture as motivation we next ask whether we can find a method to compute the orientable (respectively non-orientable) genus of an arbitrary graph. That is, given arbitrary graph \( K \) find the integer \( g \) such that \( M_g \) contains \( K \) but \( M_{g-1} \) does not contain \( K \) (respectively \( \tilde{M}_g \) contains \( K \) but \( \tilde{M}_{g-1} \) does not contain \( K \)).

An answer to a special case of this question is the theorem of Kuratowski [7] which says that a graph is planar (\( K \subset \mathbb{R}^2 \)) if and only if \( K \) does not contain \( K_5 \) or \( K_{3,3} \). Equivalently, \( g(K) = 0 \) if and only if \( K \) does not contain \( K_5 \) or \( K_{3,3} \).
A first generalization of Kuratowski's theorem would be a similar characterization of graphs that embed in the projective plane $P$.

In this paper we prove part of this characterization. In particular we prove that the set of irreducible graphs $I(P)$ for $P$ (analogous to the set $I(R^2)$ consisting of $K_5$ and $K_{3,3}$) contains 103 graphs (Theorem 5.1).

Further we strongly conjecture that our list of 103 graphs gives the complete set $I(P)$. (See section 9 for a more complete discussion of this conjecture.)

The technique that we use to construct our list of 103 graphs in $I(P)$ is as follows: Given a surface $M$, Paul Kainen [6] introduced a partial ordering on $I(M)$, the set of irreducible graphs for $M$. Huneke and Glover [2] have introduced a study of the isotopy classes of embeddings of $K \subset M$ which makes it possible to study this partially ordered set. In particular they strongly conjecture that $I(P)$ has exactly five maximal elements. One of these five graphs was found by Neil Robertson. The 103 graphs we have found are obtained by using this partial order to construct graphs below these five. In this paper we do not show that these are all graphs $K$ in $I(P)$ below these five, but we do prove that each of the 103 graphs given are distinct and are in $I(P)$. The proof that these five graphs are the maximal elements will appear in [3]. The proof that these 103 graphs are all the graphs in $I(P)$ below these five will appear in [11]. Altogether
this will show that \( I(P) \) is precisely the set of these 103 graphs.

The plan of the paper follows: In section 1 we give basic definitions and prove that \( I(P) \) is a partially ordered set with respect to the order that we introduce (Proposition 1.2). In section 2 we discuss certain special elements in \( I(P) \) which are sinks and sources of a partially ordered subset of \( I(P) \). Many of these sources (minimal minors) were found independently by Robertson. We also discuss symmetries in graphs. In section 3 we give miscellaneous facts about embedding graphs in the projective plane \( P \). In section 4 we prove that all 103 graphs we list are distinct (not homeomorphic). In section 5 we use results in sections 6-8 to prove our main result: Theorem 5.1 - \( I(P) \) contains 103 graphs. In section 6 we prove that sources do not embed in \( P \). In section 7 we prove that sinks are irreducible. In section 8 we complete the proof of Theorem 5.1 by locating each graph between a source and a sink. In section 9 we discuss conjectures and unsolved problems related to our results. An index is included giving the pages on which each of the 103 graphs are discussed.
1. THE PARTIALLY ORDERED SET $I(M)$

By a surface $M$ we mean a 2-manifold, that is a locally Euclidean Hausdorff space of dimension 2.

By a graph $K$ we mean a finite 1-dimensional $CW$ complex. By an embedding $\zeta : K \rightarrow M$ we mean an injective map (continuous function). When we write $K \subset M$ or $M \supset K$ we mean that there exists an embedding $\zeta : K \rightarrow M$.

Given a surface $M$ we say that a graph $K$ is irreducible for $M$ if $K \not\subset M$ but for every proper subspace $L \not\subset K$, $L \subset M$.

Given surface $M$ let $I(M)$ denote the set of all irreducible graphs for $M$. As an example we note that Kuratowski's theorem says that

$$I(\mathbb{R}^2) = \{K_5, K_{3,3}\}$$

Given surface $M$ we next introduce a relation on $I(M)$. As motivation we observe that $K_{3,3}$ may be obtained from $K_5$ by splitting any vertex and deleting two edges.

![Diagram](image_url)
Specifically, given graph $K$ we may form a new graph $\tilde{K}$ constructed by replacing a vertex $v \in K$ by two vertices $v_1$ and $v_2$ connected by an edge $e$ and then connecting some of the edges incident to $v$ in $K$ so that they become incident to $v_1 \in \tilde{K}$ and the remaining edges in $K$ incident to $v$ so that they become incident to $v_2 \in \tilde{K}$.

To describe the operation above, let $\text{adj}(v)$ denote the set of all vertices in $K$ adjacent to $v$. Suppose $\text{adj}(v) = \{u_1, u_2, \ldots, u_n\}$. If $\tilde{K}$ is the graph resulting from splitting $v$ into $v_1, v_2$ with $\text{adj}(v_1) = \{u_1, u_2, \ldots, u_j\}$ in $\tilde{K}$, then we write $\tilde{K} = 3^v: (u_1, \ldots, u_j)K$. When referring to an arbitrary splitting of $v$, we shall use the notation $3^v(K)$. Furthermore, when the underlying graph $K$ is obvious, $v: (u_1, \ldots, u_j)$ is used instead of $3^v: (u_1, \ldots, u_j)K$.

$K - e$ will denote the graph given by deleting an edge, more particularly the interior of an edge. $K/e$ will denote the graph given by contracting $e$ (collapsing $e$ into a point). $K - \text{st}(v)$ will denote the operation of deleting the star of a vertex, i.e. $v$ together with the interiors of all incident edges. These operations may involve a set of edges or vertices instead of a single edge or vertex.

**Lemma 1.1** If $K \not\subset M$, then $3^vK \not\subset M$

**Proof:** Assume $3^vK \subset M$, i.e. there exists embedding $f: 3^vK \rightarrow M$. Note that there exists a commutative diagram
where \( \overline{f} \) and \( \overline{f} \) are induced by \( f \) and are hence embeddings.

Let \( K \in I(M) \), since \( S_v(K) \notin M \), there exists at least one graph \( L \subseteq S_v(K) \) such that \( L \in I(M) \). We define each such \( L \) to be less than \( K \) and define a relation \( < \) on \( I(M) \) to be the transitive relation generated by the above.

This means that if \( L < K \) and \( J < L \) then \( J < K \).

**Proposition 1.2** The relation \( < \) on \( I(M) \) is a partial ordering.

**Proof:** We need only check anti-symmetry. Observe that the lexicographical ordering of the valency sequences (each written as a non-increasing sequence) of graphs \( K \in I(M) \) gives a partial ordering on \( I(M) \) which refines \( < \) on \( I(M) \). Hence the result.
2. SOURCES, SINKS AND SYMMETRIES IN $\mathbf{I}(\mathbf{M})$

We first discuss sources and sinks.

$\mathbf{K} \in \mathbf{I}(\mathbf{M})$ is called a source if $\mathbf{K}/e \subseteq \mathbf{M}$ for all edges $e$ in $\mathbf{K}$. $\mathbf{K} \in \mathbf{I}(\mathbf{M})$ is called a sink if $\mathbf{S}_v \mathbf{K}$ is reducible for every vertex $v$ of $\mathbf{K}$.

**Lemma 2.1** Let $L = \mathbf{S}_v \mathbf{K}$, $\mathbf{K} \nsubseteq \mathbf{M}$, $L \in \mathbf{I}(\mathbf{M})$. Then $\mathbf{K} \in \mathbf{I}(\mathbf{M})$.

**Proof:** $\mathbf{K} = L/e$ for some edge $e \in L$. Thus

$\mathbf{K} - e' = L - e'/e$. Since $L - e' \subseteq \mathbf{M}$, so does $L - e'/e$.

**Lemma 2.2** Let $\mathbf{K}^i = \mathbf{S}_v \mathbf{K}^{i-1}$, $i = 1, 2, \ldots, n$. If $\mathbf{K}^0 \nsubseteq \mathbf{M}$ and $\mathbf{K}^n \in \mathbf{I}(\mathbf{M})$, then $\mathbf{K}^i \in \mathbf{I}(\mathbf{M})$ for all $i = 1, 2, \ldots, n$.

The proof of Lemma 2.2 follows immediately from 2.1.

We next discuss symmetry in graphs. If $\mathbf{K}$ is a simplicial graph without vertices of valency 2, then every homeomorphism of $\mathbf{K}$ induces a permutation of the vertices of $\mathbf{K}$. Note that every such permutation is a product of (simple) cycles. A homeomorphism which induces a permutation on vertices is called a symmetry on $\mathbf{K}$. The notation

$$s: [(v_1^l, v_2^l, \ldots, v_k^l), \ldots (v_1^r, v_2^r, \ldots, v_p^r)]$$
will represent a homeomorphism $\zeta$ which induces the permutation given by the product of the given cycles. Also we write $v_i \sim v_j$ if $v_i$, $v_j$ are in the same cycle of $S$. We also write $e \sim e'$ if $e = (v_1, v_2)$, $e' = (v_1^1, v_2^1)$ and $S(v_1) = v_1^1$, $S(v_2) = v_2^1$. The $S$ will be deleted and we will write $e \sim e'$, $v \sim v'$ when $S$ is clear from the context.
3. THE TOPOLOGY OF THE PROJECTIVE PLANE

Recall that the real projective plane $\mathbb{P}$ is defined to be the orbit space of the antipodal involution on the 2-sphere $S^2$. Note that $\mathbb{P}$ is homeomorphic to the unit disc in $\mathbb{R}^2$ with antipodal points in the boundary identified. A simple cycle $C$ in $\mathbb{P}$ is called essential if $\mathbb{P} - C$ is connected; null if $\mathbb{P} - C$ is not connected.

For a graph $K$, $(v_1, v_2)$ denotes an edge with incident vertices $v_1, v_2$. Further, $(v_1, v_2, \ldots, v_{n+1})$ will denote a path from $v_1$ to $v_{n+1}$. If $v_1 = v_{n+1}$ then we say $(v_1, \ldots, v_n)$ is a $n$-cycle.

A graph $K$ is called projective if there exists an embedding $\zeta: K \to \mathbb{P}$. $K$ is non-projective otherwise. A cycle $C$ in $K$ is said to be essential with respect to an embedding $\zeta$ if $\zeta(C)$ is essential in $\mathbb{P}$. A cycle $C$ in $K$ is said to be essential for $\mathbb{P}$ if for every embedding of $\zeta: K \to \mathbb{P}$, $C$ is essential with respect to $\zeta$.

A subgraph $L$ of a graph $K$ is called a $k$-graph if there exists a graph $M$ such that $L \subseteq M \subseteq K$ and either

1. $L$ is homeomorphic to $K_{2,3}$, and $M$ is homeomorphic to $K_{3,3}$ with one of the cubic vertices of $M$ not in $L$
(ii) $L$ is homeomorphic to $K_4$, and $M$ is homeomorphic to $K_5$, or $S_v(K_5)$ with the cubic vertices of $M$ not in $L$.

We use the notation \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) to represent $K = \begin{array}{ccc} a & b \\ c & d \end{array}$
and \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) represent $K$.

In case a $k$-graph $L$ in $K$ contains vertices of $K$ which are valency 2 in $L$, the notation will be expanded to include those vertices. For example

\[
\begin{pmatrix} a & b & 1 \\ x & y & z \end{pmatrix}
\] represents $\begin{array}{ccc} a & b & 1 \\ x & y & z \end{array}$

For most of the 103 graphs, each contains a subgraph homomorphic to $K_{3,3}$. To study the (non) embedding of these graphs we often embed this $K_{3,3}$ into $P$. As an example the graph below represents a graph $K$ in which $(2, b), (3, a), (c, 1)$ are edges in $K$ (through points of identification in $P$).

![Figure 3.1](image-url)

We close this section by giving the following lemmas which will be used in later sections.
Lemma 3.1 If a graph $K$ with $E$ edges and $V$ vertices embeds in $P$ with $F$ faces (components of the complement of $K$ in $P$), each of which is homeomorphic in $\mathbb{R}^2$, then $V - E + F = 1$.

Proof: The Euler characteristic of $P$ is 1. The given vertices, edges, and faces give a CW decomposition of $P$. Hence the result.

Lemma 3.2 Any two essential cycles $C_1$ and $C_2$ in $P$ intersect each other.

Proof: We use that, for $C_1$ essential, $P - C_1$ is homeomorphic to $\mathbb{R}^2$. This is a well known fact of topology. Assume $C_2$ is disjoint from $C_1$. By the Jordan curve theorem $\mathbb{R}^2 - C_2$ has 2 components. Hence $P - C_1 - C_2$ has two components and $C_1$ is in the boundary of only one of these components. Hence $P - C_2$ has 2 components. Hence $C_2$ is not essential.

Lemma 3.3 Let $K$ be a graph and let $\zeta: K \to P$ be an embedding.

Suppose $C_i$ are cycles in $K$, $i = 1, \ldots, n$ such that

(i) $\bigcup_{i=1}^{j-1} C_j \cap \bigcup_{i=1}^{j-1} C_i$ is nonempty and connected for each $j = 2, \ldots, n$, and

(ii) $C_i$ is null with respect to $\zeta$ for each $i = 1, \ldots, n$.

Then $\zeta \left( \bigcup_{i=1}^{n} C_i \right)$ is null.
Proof: If \( \bigcup_{i=1}^{n} C_i \) contains a cycle \( C \) which is essential with respect to \( \zeta \), then \( P - \zeta C \cong \mathbb{R}^2 \). Since \( \bigcup_{i=1}^{n} C_i \) is connected, each component of \( P - \zeta(\bigcup_{i=1}^{n} C_i) \) is homeomorphic to \( \mathbb{R}^2 \) and \( P - \bigcup_{i=1}^{n} C_i \) has \( 1 - V + E \) components by Lemma 3.1. However, \( n \), the number of independent cycles in \( \bigcup_{i=1}^{n} C_i \), is \( 1 - V + E \), the Betti number of \( \bigcup_{i=1}^{n} C_i \). It remains to prove by induction on \( k \) that \( P - \zeta(\bigcup_{i=1}^{k} C_i) \) has at least \( k + 1 \) components and hence \( F \geq n + 1 \), a contradiction. For \( k = 1 \), \( \zeta C_1 \) is null so that \( P - \zeta(C_1) \) has two components. For each \( k \), \( P - \zeta(C_k) \) has two components, each of which intersect one of the components of \( P - \zeta(\bigcup_{i=1}^{k-1} C_i) \). Hence, assuming inductively that \( P - \zeta(\bigcup_{i=1}^{k-1} C_i) \) has at least \( k \) components, \( P - \zeta(\bigcup_{i=1}^{k} C_i) \) has at least \( k + 1 \) components. Hence the result.

Lemma 3.4: Let \( K \) be non-projective, let \( (v_1, v_2, v_3) \) be a 3-cycle in \( K \) with \( v_1 \) cubic (valency 3) then \( K - (v_2, v_3) \) is non-projective.
Proof: Suppose there is an embedding \( \zeta : K \to (v_1, v_2) \to P \).
Use a neighborhood of the edges \((v_1, v_2)\) and \((v_1, v_3)\) to
give an embedding of \( K \). This contradiction proves the
result.

Lemma 3.5 If \( L \) is a \( k \)-graph in \( K \) and \( \zeta : K \to P \) is an
embedding, then there is a cycle \( C \subseteq L \) such that \( \zeta(C) \) is
essential in \( P \).

Proof: It is sufficient to assume that \( K = K_5 \) or \( K_{3,3} \).
Suppose \( \zeta(C) \) is null for all \( C \subseteq L \). Then there
exists \( C_0 \subseteq P \) disjoint from \( \zeta(L) \) which is essential.
As a result \( L \subseteq P - C_0 \cong \mathbb{R}^2 \) such that \( L \cap (K - L) \)
is contained in the closure of \( L \) region. Since \( K \) has
only one vertex not in \( L \), \( K \subseteq \mathbb{R}^2 \) contradicting
Kuratowski's theorem. Hence the result.
4. DISTINCTNESS OF GRAPHS IN \( I(P) \)

We shall prove that the 103 graphs in the appendix are pairwise distinct.

We name our graphs according to their first Betti number
\[ \beta = 1 - V + E \]
where \( V, E \) denote the number of vertices and edges of the graph respectively. More specifically each \( A_i \) is a graph with Betti number \( \beta = 12 \), \( B_i \) with \( \beta = 11 \), \( C_i \) with \( \beta = 10 \), \( D_i \) with \( \beta = 9 \), \( E_i \) with \( \beta = 8 \), \( F_i \) with \( \beta = 7 \), and \( G \) is the only graph with \( \beta = 6 \). Let us call all \( A_i \)'s the \( A \)-graphs, and similarly for \( B_i \)'s, \( C_i \)'s, etc. Since the Betti number \( \beta \) is clearly invariant under graph isomorphism, we see all \( A \)-graphs are distinct from \( B \)-graphs, and vice versa. Similarly for \( C \) (or \( D, E, F, G \))-graphs.

The valency sequences provided under each graph in the appendix are in the general form \( s_0 \ (s_1, s_2, \ldots, s_n) \) where \( s_0 = n \) denotes the number of vertices of the graphs and \( s_i \ (i > 0) \) is the valency of each of the vertices arranged in descending order.

If \( L, M \) are graphs of the same class (e.g., both are \( A \)-graphs) with sequences \( s_0 \ (s_1, s_2, \ldots, s_n) \), and \( t_0 \ (t_1, t_2, \ldots, t_n) \) then \( L, M \) are isomorphic implies that \( s_i = t_i \) for all \( i = 0, 1, \ldots, n \). Denote \[ [s_0 \ (s_1, s_2, \ldots, s_n)] = [G] \] such that
the valency sequence of $G$ is $s_0(s_1, s_2, \ldots, s_n)$. We need only to check that the graphs in $[s_0(s_1, s_2, \ldots, s_n)]$ are distinct from each other.

Looking into the appendix, we see for A-graphs, C-graphs and G-graphs each $[s_0(s_1, \ldots, s_n)]$ is a singleton set, hence each is distinct from each other. Thus all we have left to check are the classes $[s_0(s_1, \ldots, s_n)]$ in B-graphs, D-graphs, E-graphs and F-graphs.

**Theorem 4.1** The 103 graphs in the appendix are distinct.

**Proof:** The format of our proof is as follows:

For each class we first list $[s_0(s_1, \ldots, s_n)]$ by lexicographic order. Then for each subclass we exhibit a binary tree which will distinguish each graph in the same class. Each edge in the binary tree is distinguished by a property $P_i$ or its logical complement $\overline{P_i}$. The property $P_i$ will be stated precisely following the binary tree. As a hypothetical example consider

$$[s_0(s_1, s_2, \ldots, s_n)] = \{A_{n_1}, \ldots, A_{n_4}\}$$

![Figure 4-1](image_url)
$P_1$: $K$ is a disconnected graph.

$P_2$: The 3 vertices of valency 4 are in a 3-cycle in $K$.

$P_3$: $K$ contains no five cycle.

In the discussion that follows $P_1$ will be omitted as it is obvious from the binary tree when $P_1$ is specified.

Let $K$ be a graph, $U \subseteq V(K)$, recall the definition from section 1 that $K - U$ denotes the graph resulting by deleting each vertex in $U$ and its incident edges. The following notations will be used frequently.

$$V(s_1, s_2, \ldots, s_p) = \{ v \in V(K) \mid v \text{ is a vertex of valency } s_i \text{ for some } i = 1, 2, \ldots, p \}$$

$$\overline{V}(s_1, s_2, \ldots, s_p) = \{ v \in V(K) \mid v \in V(s_1, \ldots, s_p) \text{ or there exists } u \in V(s_1, \ldots, s_p) \text{ such that } v \text{ is adjacent to } u \}$$

Note that $\overline{V}$ is defined in similar fashion from $\overline{V}$.

**4-1 B-graphs**

The following are those non-singleton classes

(1) $[3(6, 6, 4, 4, 4, 4)] = \{ B_2, B_6, B_7 \}$
$P_1$: $K$ has vertex-connectivity 2.

$P_2$: Each vertices of valency less than 6 in $K$ is adjacent to a vertex of valency 6.

(2) \[9(6, 4, 4, 4, 4, 4, 4, 4)] = \{B_3, B_{10}\}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure4-3}
\caption{Figure 4-3}
\end{figure}

$P$: $K$ has vertex-connectivity 2.

(3) \[10(4, 4, 4, 4, 4, 4, 4, 4)] = \{B_1, B_4, B_{11}\}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure4-4}
\caption{Figure 4-4}
\end{figure}
$P_1$: $K$ is disconnected.

$P_2$: $K$ has vertex-connectivity 2.

4-2 $D$-graphs

The following are non-singleton class on valency sequences.

(1) $[9(5, 5, 4, 4, 4, 3, 3, 3)] = \{D_2, D_3, D_4, D_5, D_{12}\}$

Figure 4-5

$P_1$: The two vertices of valency 5 are adjacent in $K$.

$P_2$: The vertices of valency 4 form a 3-cycle in $K$.

$P_3$: $K = V(5, 4)$ is a 4-cycle.

(2) $[10(5, 4, 4, 4, 4, 3, 3, 3, 3)] = \{D_6, D_8, D_{13}\}$

Figure 4-6
$P_1$: $K - V(5,3)$ is a $K_4$.

$P_2$: $K - V(5,3)$ contains a 3-cycle.

(3) $[10(5, 5, 4, 4, 3, 3, 3, 3, 3, 3)] = \{D_7, D_9, D_{20}\}$

$P_1$: $K$ contains no 3-cycle.

$P_2$: $K - \overline{V(4)}$ contains an isolated vertex.

(4) $[11(4, 4, 4, 4, 3, 3, 3, 3, 3, 3)] = \{D_{10}, D_{14}, D_{18}\}$

$P_1$: $K$ is disconnected

$P_2$: $K$ has vertex-connectivity 2

4-3 E-graphs

Only the following are non-singleton classes.
(1) \([11(6, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)] = \{E_1, E_26\}\)

P: K is point-separated

(2) \([9(5, 5, 4, 3, 3, 3, 3, 3)] = \{E_{11}, E_{27}\}\)

P: \(V(5, 4)\) are mutually non-adjacent in K.

(3) \([10(5, 5, 3, 3, 3, 3, 3, 3, 3)] = \{E_4, E_{12}, E_{28}\}\)

P_1: K - V(5) is a single vertex.

P_2: K - V(5) consists of 2 isolated vertices.
(4) \[10(5, 4, 4, 3, 3, 3, 3, 3, 3)] = \{E_{13}, E_{14}, E_{29}, E_{39}\}

- Figure 4-12
  - \(P_1\): \(V(4)\) in \(K\) are mutually non-adjacent.
  - \(P_2\): \(K - \overline{V}(5, 4)\) is a single vertex
  - \(P_3\): \(K - \overline{V}(4) = \emptyset\)

(5) \[11(5, 4, 3, 3, 3, 3, 3, 3, 3, 3)] = \{E_5, E_{17}, E_{32}\}

- Figure 4-13
  - \(P_1\): \(V(5, 4)\) are adjacent in \(K\)
  - \(P_2\): \(K\) is not 3-connected

(6) \[12(5, 3, 3, 3, 3, 3, 3, 3, 3, 3)] = \{E_2, E_{18}\}

- Figure 4-14
  - \(P\): \(K\) is point-separated
(7) \[9(4, 4, 4, 4, 3, 3, 3)] = \{E_7, E_8, E_{21}, E_{35}\}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4-15.png}
\caption{Figure 4-15}
\end{figure}

- \(P_1: K - V(4)\) is a 4-cycle.
- \(P_2: K - V(4) = \)
- \(P_3: K - V(4)\) is a graph of 4 isolated vertices.

(8) \[10(4, 4, 4, 3, 3, 3, 3, 3)] = \{E_9, E_{22}, E_{23}, E_{36}, E_{38}, E_{40}\}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4-16.png}
\caption{Figure 4-16}
\end{figure}

- \(P_1: K - V(3)\) is \(K_4\)
- \(P_2: K - V(3) = \)
- \(P_3: K - \overline{V}(4)\) is a single vertex.
P_4: K - V(3) is a graph of 4 isolated points.

P_5: K - V(3) is a 4-cycle.

(9) \[11(4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3)\] = \{(E_{15}, E_{16}, E_{24}, E_{30}, E_{31}, E_{37}, E_{41})\}

P_1: K - V(3) is a graph of 3 isolated vertices.

P_2: K - V(4) contains 2 isolated vertices.

P_3: K - \overline{V}(4) is a single isolated vertex.

P_4: K - \overline{V}(4) is a graph of two isolated vertices.

P_5: K - \overline{V}(4) are in a 4-cycle in K.

P_6: K - \overline{V}(4) is a null graph.

(10) \[12(4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3)\] = \{(E_6, E_{19}, E_{25}, E_{33})\}

Figure 4-17

Figure 4-18
$P_1$: $V(4)$ are adjacent in $K$.

$P_2$: $K - \overline{V(4)}$ is a graph of 3 isolated vertices.

$P_3$: $K - V(4)$ is disconnected.

(11) $[13(4,3,3,3,3,3,3,3,3,3,3,3,3,3,3)] = \{E_3, E_{34}\}$

\[ \text{Figure 4-19} \]

$P$: $K$ is point-separated.

4-4 $F$-graphs

The following are non-singleton classes.

(1) $[10(4,4,3,3,3,3,3,3,3,3)] = \{F_2, F_3, F_4, F_5, F_6\}$

\[ \text{Figure 4-20} \]

$P_1$: $V(4)$ are adjacent in $K$.

$P_2$: $K - V(4)$ is disconnected.
$P_3$: $K - \overline{V}(4) = \emptyset$

$P_4$: $K - \overline{V}(4) = 2$ isolated vertices with distance 2 (the length of the shortest path from one to the other).

(2) \[11(4,3,3,3,3,3,3,3,3,3)] = \{F_7, F_8, F_9, F_{10}\}

Figure 4-21

- $F_1$: $K - \overline{V}(4)$ is disconnected.
- $F_2$: $K - \overline{V}(4)$ is a null graph.
- $F_3$: $K$ is not 3-connected.

(3) \[12(3,3,3,3,3,3,3,3,3,3,3,3)] = \{F_{11}, F_{12}, F_{13}, F_{14}, F_{15}\}

These are five of the six distinct cubic irreducible graphs found by Glover and Huneke [4].
5. THE MAIN RESULT

We outline the results of sections 6, 7, 8 and from these prove our main result that each of the 103 graphs of the appendix are in \( I(P) \); these 103 graphs are distinct by section 4. Hence:

Theorem 5.1 \( I(P) \) contains 103 graphs.

Recall by Lemmas 1.1 and 2.1: if \( L = S_y K \), then \( K \) non-projective implies \( L \) non-projective. Furthermore \( L \) irreducible implies \( K \) irreducible. In particular if \( K^1 = S_y K^2 \) and 
\( K^2 = S_y K^3 \), and \( K^3 \) is non-projective then \( K^2, K^1 \) are non-projective. If furthermore, \( K^1 \) is irreducible then \( K^2, K^3 \) are irreducible, and hence \( K^1 \) is in \( I(P) \) for \( i = 1, 2, 3 \).

We will prove that each source does not embed in \( P \) in section 6 and that each sink is irreducible in section 7. In view of the Lemma 2.2, for graphs which are not both a source and a sink, we shall consider a directed graph structure in \( I(P) \) as follows: Each element in \( I(P) \) is a vertex and there is an edge from \( K \) to \( L \) if \( L = S_y K \). We include in section 8 a spanning sub-directed graph of the directed graph \( I(P) \) in which every graph in \( I(P) \) follows a source and is followed by a sink. More precisely, we observe the following:
(1) If $K$ is both a source and a sink then the non-projectivity and the irreducibility of $K$ are discussed in section 6 and section 7 respectively.

(2) If $K$ is a source but not a sink then non-projectivity of $K$ is discussed in section 6. To see irreducibility one finds a branch of the spanning sub-graph of $I(P)$ in section 8

$$K + K^1 + \ldots + K^n$$

with $K^n$ a sink. Since irreducibility of $K$ is implied by that of $K^n$ one needs only to check irreducibility of $K^n$ in section 7.

(3) Similarly, for a graph which is a sink but not a source its irreducibility is checked in 7 and non-projectivity follows from section 8 and section 6.

(4) If $K$ is neither a source nor a sink, then to see that $K$ is in $I(P)$ one finds a path in section 8

$$K^1 + \ldots + K + \ldots + K^n$$

such that $K^1$ is a source and $K^n$ is a sink. Hence $K$ is in $I(P)$ by Lemma 2.2

This discussion with the results of sections 6, 7 and 8 completes a proof of our main result.

Page references for the discussion of each graph is given in the Index.
6. SOURCES DO NOT EMBED

There are, among the 103 graphs, 35 sources. Each is proved to be non-projective. Each source will be considered as a separate case \( M_i, 1 \leq i \leq 35 \).

Lemma 6.1 If \( K \) is a spanning subgraph of the complete bipartite graph \( K_{n,m} \) with more than \( 2(n + m - 1) \) edges, then \( K \) is non-projective.

Proof: Suppose \( K \) embeds in \( P \). Then by Lemma 3.1,

\[
l = V - E + F < (n + m) - 2(n + m - 1) + F, \text{ so } F > n + m - 1.
\]

However, the boundary of each face contains a cycle and each cycle in \( K_{n,m} \), hence in \( K \), contains at least four edges. Also each edge is in the boundary of at most two faces, so

\[
2E \geq 4F, \text{ hence } n + m - 1 \geq F, \text{ a contradiction. Therefore,}
\]

\( K \) does not embed in \( P \).

Case \((M1)\), graph \( E_{10} \).

![Diagram of graph](image)

Figure 6.1

Observe that \( E_{10} = K_{3,5} \) and has 15 edges. Hence \( E_{10} \) is
non-projective by Lemma 6.1.

**Case (M2), graph $E_{20}$**.

![Figure 6.2](image)

Observe that $E_{20}$ is a subgraph of $K_{4,4}$ with 15 edges so it is non-projective by Lemma 6.1.

**Lemma 6.2** Let $K$ be a graph which embeds in $P$. Let $C$ be a cycle in $K$ disjoint from a $k$-graph of $K$. Then $C$ is null in $P$.

**Proof:** By Lemma 3.5 for any embedding of $K$ in $P$, there is a cycle in each $k$-graph which is essential. By Lemma 3.2 $P$ contains no disjoint essential cycles so $C$ is null.

**Lemma 6.3** Let $K$ be a graph containing cycles $C_i$, $i = 1, \ldots, n$ such that:

1. $C_j \cap \bigcup_{i=1}^{j-1} C_i$ is nonempty and connected for each $j = 2, \ldots, n$, and
2. $C_i$ is disjoint from a $k$-graph in $K$ for each $i = 1, \ldots, n$, and
3. $\bigcup_{i=1}^{n} C_i$ contains a $k$-graph in $K$.

Then $K$ is non-projective.
Proof: Assume $K \subseteq P$. By Lemma 5.2 $C_i$ is null for each $i = 1, \ldots, n$. Hence by Lemma 3.3 $\bigcup_{i=1}^{n} C_i$ is null.

However, by Lemma 3.5 $\bigcup_{i=1}^{n} C_i$ must be essential, a contradiction. Hence the result.

Case (M3), graph $E_{35}$.

The cycles $C_1 = (1,a,3,b)$, $C_2 = (a,2,y,3)$, $C_3 = (3,y,x,b)$ satisfy the hypothesis of Lemma 6.3 as

\[
\begin{pmatrix} a & b \\ 1 & 3 & 2 \end{pmatrix} \leq \bigcup_{i=1}^{3} C_i
\]

is a $k$-graph in $E_{35}$. Hence $E_{35}$ is non-projective.

Case (M4), graph $D_1$.

The cycles $C_1 = (1,p,q)$, $C_2 = (2,p,q)$, $C_3 = (c,p,q)$ satisfy the hypothesis of Lemma 6.3 as

\[
\begin{pmatrix} p & q \\ 1 & 2 & 3 \end{pmatrix} \leq \bigcup_{i=1}^{3} C_i
\]
is a $k$-graph. Hence $D_1$ is non-projective.

**Case (M5), graph $F_1$.**

![Figure 6.5](image)

The cycles $C_1 = (1,p,q,x)$, $C_2 = (a,p,q,x)$ satisfy Lemma 6.3 as

$$
\begin{pmatrix}
  p & q \\
  1 & x & a
\end{pmatrix}
= C_1 \cup C_2
$$

is a $k$-graph in $F_1$. Hence $F_1$ is non-projective.

**Case (M6), graph $B_3$.**

![Figure 6.6](image)

The cycles $C_i = (i,p,q)$ for $i = 1, 2, 3$ satisfy Lemma 6.3 as

$$
\begin{pmatrix}
  p & q \\
  1 & 2 & 3
\end{pmatrix}
\subseteq \bigcup_{i=1}^{3} C_i
$$

is a $k$-graph in $B_3$. Hence $B_3$ is non-projective.

**Case (M7), graph $A_4$.**
Figure 6.7

The cycles $c_i = (i, i+1, i+2)$ for $i = 1, 2, 3$ and $c_4 = (1,3,5)$ satisfy Lemma 6.3 as

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \leq \bigcup_{i=1}^{4} c_i$$

is a $k$-graph in $A_4$. Hence $A_4$ is non-projective.

Lemma 6.4 If $K$ is non-projective, $\text{adj}(v) = \{v_1, v_2, \ldots, v_k\}$ $v, v_1, \ldots, v_k \in V(K)$. Suppose $v_1, v_2$ are adjacent, then

$\bar{S}(v_1, v_2)(K) - (v_1, v_2)$ is non-projective.

Proof: Let $v'$ be the new vertex of $\bar{S}(v_1, v_2)(K) - (v_1, v_2)$. $v'$ is cubic and cycle $(v', v_1, v_2)$ is a 3-cycle. Hence the result by Lemma 3.4.

Case (M8), graph $B_{12}$.

Using the notation of Case (M7), $B_{12} = \bar{S}_2(4, 6)A_4 - (4, 6)$ and $A_4$ is non-projective by Case (M7), so $B_{12}$ is non-
projective by Lemma 6.4.

**Case (M9), graph C_9.**

![Figure 6.9]

Using the notation of Case (M8), \( C_9 = S_{5:(1,3)}(B_{12}) - (1,3) \).
So by Lemma 6.4, \( C_9 \) is non-projective.

**Case (M10), graph C_8.**

![Figure 6.10]

Using the notation of Case (M8), \( C_8 = S_{2:(1,3)}(B_{12}) - (1,3) \).
So \( C_8 \) is non-projective by Lemma 6.4.

**Case (M11), graph D_{17}.**

![Figure 6.11]

Using the notation of Case (M10), \( D_{17} = S_{x:(1,p)}(C_8) - (1,p) \).
So \( D_{17} \) is non-projective by Lemma 6.4.
Case (M12), graph $E_{26}$.  

![Diagram of Figure 6.12](image)

Using the notation of Case (M11), $E_{26} = S_{y:(p,2)(D_1)} - (p,2)$. So $E_{26}$ is non-projective by Lemma 6.4.

Case (M13), graph $E_{27}$.  

![Diagram of Figure 6.13](image)

Using the notation of Case (M4), $E_{27} = S_{c:(p,q)(D_1)} - (p,q)$. So $E_{27}$ is non-projective by Lemma 6.4.

**Lemma 6.5** If $K$ contains two disjoint $k$-graphs, then $K$ is non-projective.

**Proof:** Let $L$ be a $k$-graph in $K$ which is disjoint from a $k$-graph $L'$ in $K$. By Lemma 3.5 $L$ and $L'$ each contain a cycle which is essential with respect to any embedding of $K$. However $L \cap L' = \emptyset$ and $P$ contains no disjoint essential cycles by Lemma 3.2. Hence the result.

In each case (M1), $14 \leq i \leq 19$, each graph either is disconnected, each component containing a $k$-graph, or contains a
cutpoint whose complement contains two components, each containing a k-graph of K. Hence the graphs in these cases are non-projective by Lemma 6.5.

Case (M14), graph $A_1 = K_5 \vee K_5$.

Case (M15), graph $B_1 = K_5 \perp K_5$.

Case (M16), graph $C_1 = K_5 \vee K_{3,3}$.

Case (M17), graph $D_{18} = K_5 \perp K_{3,3}$.

Case (M18), graph $E_1 = K_{3,3} \vee K_{3,3}$.

Case (M19), graph $F_{15} = K_{3,3} \perp K_{3,3}$.

In each of the remaining cases (M$i$) $20 \leq i \leq 35$, either the graph contains disjoint k-graphs, so it is non-projective by Lemma 6.5 (the disjoint k-graphs are listed in each of these cases) or the graph is non-projective by Lemma 6.4 (the splitting is given in each of these cases).

Case (M20), graph $C_{10}$.

Figure 6.14

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are disjoint k-graphs.
Case (M21), graph $D_{19}$.

Figure 6.15

$(1\ 2)$ and $(a\ b)$ are disjoint $k$-graphs.

Case (M22), graph $D_{20}$.

Figure 6.16

$(a\ b\ c\ d)$ and $(l\ 2\ 3\ 4\ 5\ 6)$ are disjoint $k$-graphs.

Case (M23), graph $D_{16}$.

Figure 6.17

$(a\ x), (b\ c), (p\ q), (1\ 2\ 3)$ are disjoint $k$-graphs.
**Case (M24)**, graph $E_{40}$.

$E_{40} = \mathcal{G}_{a:(p,q)(D_{16}) - (p,q)}$, see Case (M23) for notation.

**Case (M25)**, graph $E_{39}$.

$E_{39} = \mathcal{G}_{x:(p,q)(D_{16}) - (p,q)}$, see Case (M23) for notation.

**Case (M26)**, graph $D_{12}$.

$(a \ x) \cdot (b\ c)$ and $(p\ q) \cdot (1\ 2\ 3)$ are 2 disjoint $k$-graphs.

**Case (M27)**, graph $E_{4}$.
$E_4 = 3 \cdot (p, q) (D_{12}) - (p, q)$, see Case (M26) for notations.

**Case (M28)**, graph $B_2$.

(1 2), (a b) are disjoint k-graphs.

(3 4), (c d)

**Case (M29)**, graph $C_5$.

$C_5 = 3 \cdot b : (a, c)(B_2) = (a, c)$, see Case (M28) for notation.

**Case (M30)**, graph $D_{15}$.
Figure 6.24

\[ D_{15} = \frac{3}{r: (p, q)(c_5) = (p, q) }, \text{ see Case } \text{(M29) for notation.} \]

Case (M31), graph \( E_8 \).

Figure 6.25

\[
\begin{pmatrix}
x & y \\
a & p
\end{pmatrix}, \begin{pmatrix}
b & c \\
1 & 2 & 3
\end{pmatrix}
\]

are disjoint k-graphs.

Case (M32), graph \( E_7 \).

Figure 6.26

\[
\begin{pmatrix}
x & y \\
a & p
\end{pmatrix}, \begin{pmatrix}
b & c \\
1 & 2 & 3
\end{pmatrix}
\]

are 2 disjoint k-graphs.

Case (M33), graph \( F_5 \).
Figure 6.27

\[(a \ p), (b \ c)\] are 2 disjoint k-graphs.

Case (M34), graph $F_6$.

Figure 6.28

\[(q \ y), (1 \ 3)\] are 2 disjoint k-graphs.

\[(x \ p \ 2), (a \ b \ c)\] are 2 disjoint k-graphs.

Case (M35), graph $G$.

Figure 6.29

\[(a \ p), (b \ c)\] are disjoint k-graphs.

\[(x \ y \ z), (1 \ 2 \ 3)\] are disjoint k-graphs.
7. SINKS ARE IRREDUCIBLE

There are among the 103 graphs, 38 sinks. To see that each sink \( K \) is non-projective observe that either it is also a source, or there exist graphs \( K^1 \) such that \( K^1 \rightarrow \cdots \rightarrow K \) in the directed set \( I(P) \) (see section 8) with \( K^1 \) a source; hence since each source is non-projective by section 6, repetitive application of Lemma 2.1 proves that each sink is non-projective.

We now show that each sink \( K \) is irreducible by proving that \( K - e \subseteq P \) for each \( e \in E(K) \), \( E(K) \) the set of all edges of \( K \).

For terminology, \( K \) non-projective, \( e \in E(K) \), \( e \) is called irreducible if \( K - e \) is projective. Observe that \( K \) irreducible if and only if \( e \) is irreducible for all \( e \in E(K) \).

Different methods are employed to check irreducibility of these sinks. Accordingly, we group the 38 sinks into four classes. Class 1 consists of nine graphs, denoted by Case N1 to N9, and is handled by Lemma 7.1 and Lemma 7.3. Class 2 consists of eight graphs, denoted by Case N10 to N17. We describe explicit embedding for \( K - e \) for each graph \( K \) and each edge \( e \) of \( K \) in class 2. Class 3 consists of six cubic graphs denoted by Case N18 to N23, irreducibility of which was proved by Glover and Huneke [4]. Class 4 consists of 15 graphs denoted by
Case $N^2$ to $N^3$. We use the irreducibility of graphs discussed in classes 1, 2, 3 and Lemma 7.7 and Lemma 7.8 to assist in showing each graph in class 4 is irreducible.

When describing an embedding of $K-e$ explicitly the following lemma is useful.

**Lemma 7.1** Let $K=(v_1,v_2)$ be embedded in $P$ (hence $P$ is divided into regions). Let $\{R_1, \ldots, R_m\}$ be regions containing $v_1$, $\{S_1, \ldots, S_n\}$ be regions containing $v_2$. If $e$ is an edge common to the regions $R_i$ and $S_j$ for some $i,j$, then $K-e \subseteq P$.

**Proof:** Using the embedding of $K=(v_1,v_2)$, $R_i \cup S_j \cup e$ is a single region for $K=(v_1,v_2)-e \subseteq P$. Since $v_1, v_2$ are both in the same region $R_j \cup S_j \cup e$, we can embed the edge $(v_1,v_2)$ in the region $R_i \cup S_j \cup e$ and this gives an embedding for $K-e$.

We shall often represent a graph in $P$ in such a way that there are exactly two edges crossing each other. These two edges will be called the crossing edges with respect to the representation. Clearly if $(v_1,v_2)$ is either one of the crossing edges, then $K-(v_1,v_2) \subseteq P$ and by Lemma 7.1 each edge that is common to any two regions containing $v_1, v_2$ respectively is also irreducible.

**Lemma 7.2** Let $K=L_1 \cup L_2$ where either

1) $L_1, L_2$ disjoint, or
ii) $L_1 \cap L_2$ is a vertex.

If $L_1$ is projective and $L_2$ is planar, then $K$ is projective.

For proof see Battle et al [1].

**Lemma 7.3** Let $K = L_1 \cup L_2$ where either

1) $L_1$, $L_2$ disjoint or

2) $L_1 \cap L_2$ is a vertex. If $L_1, L_2 \in I(R^2)$, then $K$ is irreducible.

**Proof:** $L \in I(R^2) \Rightarrow L$ is projective and $L - e$ is planar.

The conclusion follows immediately from Lemma 7.2.

**Case N1-N5**

The graphs $A_3 = K_5 \cup K_5$, $B_1 = K_5 \cup K_5$, $C_4 = K_3 \cup K_3 \cup K_3 \cup K_3$, $D_{18} = K_3 \cup K_5$, $E_3 = K_3 \cup K_3$ are irreducible by Lemma 7.3.

**Lemma 7.4** Let $K = L_1 \cup L_2$ where $L_1 \cap L_2$ contains exactly two vertices, $v_1, v_2$. If $L_1 \cup (v_1, v_2)$ is projective and $L_2 \cup (v_1, v_2)$ is planar then $K$ is projective.

**Proof:** Since $L_2 \cup (v_1, v_2)$ is planar it embeds into an open disc. An embedding of $L_1 \cup (v_1, v_2)$ in $P$ extends to an embedding $L_1 \cup (v_1, v_2) \cup L_2$ in $P$.

**Lemma 7.5** Let $K = L_1 \cup L_2$ such that $L_1 \cap L_2 = \{v_1, v_2\}$ where $v_1$ is not valency 2 in $L_j$, $i = 1, 2$ and $j = 1, 2$. Suppose

1) $L_1$ or $L_1 \cup (v_1, v_2) \in I(R^2)$
ii) $L_2$ or $L_2 \cup (v_1, v_2) \in I(R^2)$

iii) $(v_1, v_2) \notin K$.

Then $K - e$ is projective for each edge $e$ of $K$.

**Proof:** Let $e \in E(K)$, without loss of generality suppose $e \in L_1$. Observe that $L_2$ or $L_2 \cup (v_1, v_2) \in I(R^2)$ implies that $L_2 \cup (v_1, v_2)$ is projective. Hence by Lemma 7.4 it is sufficient to show that $(L_1 - e) \cup (v_1, v_2)$ is planar.

If $L_1 \cup (v_1, v_2) \in I(R^2)$, then $L_1 \cup (v_1, v_2) - e$ is planar, hence the result. Hence, assume $L_1 \in I(R^2)$.

If $L_1 \approx K_3$ then $v_1, v_2$ are vertices of $K_3$ (valency of $v_i$ not 2), thus $(v_1, v_2) \in L \subset K$ a contradiction to (iii).

If $L_1 = K_{3,3}$ then $v_1, v_2$ are non-adjacent vertices of $K_{3,3}$, so $(L_1 \cup (v_1, v_2)) - e$ has only five vertices (excluding vertex of degree 2) and has a trivalent vertex, so is homeomorphic to a proper subgraph of $K_3$. Hence it is planar.

**Cases N6 - N9**

Lemma 7.5 proves the following graphs are irreducible.

$B_4$:

![Figure 7.1](image_url)
We state the following lemma which will be used throughout the discussion of the following eight graphs.

**Lemma 7.6** Let $K$ be represented in $P$ with exactly two crossing edges then $K - (v_1, v_2) \subseteq P$ for each of the crossing edges $(v_1, v_2)$. Suppose $e$ is a common edge of regions $R$ and $S$.
where \( v_1 \in R \), \( v_2 \in S \), then \( e \) is irreducible in \( K \).

**Proof:** It follows immediately from Lemma 7.1.

**Case N.10**

\[ E_{19}: \]

\[ s_1: [(1,2), (r,s)] \]

\[ s_2: [(p,q), (x,y), (b,c)] \]

**NOTE:**

1. The six edges \((p,s), (p,y), (q,r), (q,x), (x,l), (s,2)\) are irreducible (Figure 7.5 and Lemma 7.6).
2. The five edges \((b,1), (b,2), (c,1), (c,x), (y,3)\) are irreducible (Figure 7.6 and Lemma 7.6).
3. The two edges \((a,2), (a,3)\) are irreducible (consider the representation form by crossing \((b,1), (p,y)\) in Figure 7.6, then apply Lemma 7.6).
4. The three edges \((p,r), (q,s), (a,1)\) are irreducible (by (1), (3), and \(s_1\)).
5. The three edges \((b,y), (c,2), (x,3)\) are irreducible (by (2) and \(s_2\)).
Thus all 19 edges are irreducible in $E_{19}$. So $E_{19}$ is irreducible.

**Case N11**

$E_{18}$:

![Figure 7.7](image)

![Figure 7.8](image)

$S_1$: $[(a,c), (x,y), (p,q)]$

$S_2$: $[(2,z), (x,x_1,y), (a,c,q,p)]$

$S_3$: $[(x,1), (x,y), (a,q), (c,p)]$

**NOTE:**

1. The five edges $(b,3)$, $(p,x)$, $(p,3)$, $(q,x)$, $(r,z)$ are irreducible (Lemma 7.6 apply to Figure 7.7).
2. The three edges $(p,x)$, $(q,y)$, $(q,3)$ are irreducible (apply $S_1$ on (1)).
3. The four edges $(a,1)$, $(c,y)$, $(c,3)$, $(z,1)$ are irreducible (apply $S_3$ on (1)).
4. The three edges $(a,x)$, $(a,3)$, $(c,1)$ are irreducible (apply $S_1$ on (3)).
5. The two edges $(y,2)$, hence $(x,2)$ by $S_1$, are irreducible (by $S_2$ on $(z,1)$).
(6) The two edges (b,z), hence (b,2) by $S_2$, are irreducible (by Figure 7.8).

Thus all 19 edges in $E_{18}$ are irreducible.

**Case N12**

$E_{16}$:

![Figure 7.9 and Figure 7.10]

$s_1$: [(a,b), (2,3), (x,y)]

$s_2$: [(1,2), (q,a), (x,x)]

$s_3$: [(b,q), (1,3), (x,y)]

**NOTE:**

(1) The four edges (p,x), (p,y), (q,r), (r,l) are irreducible (Figure 7.9 and Lemma 7.6).

(2) The three edges (a,x), (p,x), (x,2) are irreducible ($s_2$ on (1)).

(3) The two edges (b,y), (y,3) are irreducible ($s_3$ on (1)).

(4) The three edges (b,1), (c,2), (q,5) are irreducible (Lemma 7.6 on Figure 7.10).

(5) The three edges (a,3), (b,2), (c,1) are irreducible ($s_2$ on (4)).
(6) The three edges \((a, l), (c, 3), (q, 2)\) are irreducible
\((S_{\perp} \text{ on } (4)).\)

Thus all 18 edges are irreducible in \(E_{16}\), hence \(E_{16}\) is irreducible.

Case N13

\(E_{25}\):

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure7_11}
\caption{Figure 7.11}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure7_12}
\caption{Figure 7.12}
\end{figure}

\(S_1: [(p, l), (q, b), (y, a, z, c), (3, w, 2, x)]\)

\(S_2: [(a, c), (y, z), (w, x), (2, 3)]\)

**NOTE:**

(1) The four edges \((b, 3), (p, y), (q, x), (q, w)\) are irreducible (Lemma 7.6 on Figure 7.11).

(2) The four edges \((a, l), (b, 2), (c, l), (p, z)\) are irreducible \((S_1, S_2 \text{ on } (1)).\)

(3) The three edges \((a, x), (p, l), (x, y)\) are irreducible (Lemma 7.6 and Figure 7.12).

(4) The four edges \((a, 3), (w, c), (w, z), (z, 3)\) are irreducible (by \(S_1, S_2 \text{ on } (3)).\)

(5) The two edges \((c, 2), (y, 2)\) are irreducible (by \(S_2 \text{ on } (4)).\)
The two edges \((b,1), (p,q)\) are irreducible (for \((p,q)\), embed \(\begin{array}{c} q \\ \hline \hline \end{array} x \hline \hline \end{array}\) through region denoted by \(R\) in Figure 7.12, \((b,1)\) follows by \(S_1\).

So we have 19 edges are irreducible.

**Case N14**

\[D_{11}\]:

**NOTE:**

1. The six edges \((b,3), (c,z), (c,2), (p,x), (p,z), (q,1)\) are irreducible (Lemma 7.6 on Figure 7.13).
2. The eight edges \((a,x), (a,3), (b,2), (p,1), (q,c), (q,y), (x,y), (y,2)\) are irreducible (Lemma 7.6 on Figure 7.14).
3. The three edges \((a,1), (b,1), (z,3)\) are irreducible (Lemma 7.6 on Figure 7.15).
4. The two edges \((c,1), (p,q)\) are irreducible (by \(S\) on (2)).

So all 19 edges are irreducible.
**Case N15**

$E_{28}$:

![Figure 7.16](image)

$s_1$: [(p,q)]

$s_2$: [(1,3)]

$s_3$: [(a,b), (c,r), (x,2), (p,3,q,l)]

**NOTE:**

1. The eight edges (a,q), (b,p), (b,q), (b,3), (c,3), (p,r), (x,r), (x,2) are irreducible by Figure 7.16.
2. The four edges (a,p), (b,l), (c,l), (q,r) are irreducible (by $s_1$, $s_2$ on (1)).
3. The three edges (a,l), (a,3), (c,2) are irreducible (by $s_3$ on (1)).
4. The two edges (a,x), (b,2) are irreducible (represent $E_{28}$ by crossing (p,b) with (a,q) from Figure 7.16 and apply Lemma 7.6 to (p,b), we get that (a,x) is irreducible, (b,2) follows by applying $s_3$ on (a,x)).

Thus all 17 edges are irreducible.
Case N.16

\[ E_{34} : \]

\begin{align*}
\text{Figure 7.17} & \quad \approx \\
\text{Figure 7.18} & \\
\text{Figure 7.19} & \quad \approx \\
\text{Figure 7.20} & \\
\end{align*}

\[ S = [(q,b), (p,3), (a,r)] \]

**NOTE:**

1. \((b,w), (b,3), (q,z), (w,1)\) are irreducible (Figure 7.17).
2. \((b,z), (p,q), (q,w)\) are irreducible (by \( S \) on (1)).
3. \((a,y), (p,r), (x,r), (x,1)\) are irreducible (Figure 7.18).
4. \((a,x), (a,3), (r,y)\) are irreducible (by \( S \) on (3)).
5. \((c,p), (y,2), (z,2)\) are irreducible (Figure 7.19).
6. \((c,3)\) is irreducible (\( S \) on (5)).
7. \((c,1)\) is irreducible (Figure 7.20).
8. \((c,2)\) is irreducible (by placing \( a \) in region \( \mathcal{R} \) in Figure 7.18, cross \((a,3)\) with \((c,2)\)).
Thus all 20 edges are irreducible.

**Case N17**

\[ D_{20} \]

\[ \begin{array}{c}
\text{Figure 7.21} \\
\text{Figure 7.22}
\end{array} \]

\[ S_1: [(a,d), (b,c), (2,4), (3,5), (1,6)] \]
\[ S_2: [(2,3)] \]
\[ S_3: [(4,5)] \]

**NOTE:**

1. (c,6), (d,4), (b,4), (a,6) are irreducible by Figure 7.22 and Lemma 7.6.
2. (d,5), (b,5) are irreducible (by \( S_3 \) and (1)).
3. (d,6) is irreducible (by placing 6 on \( R_2 \)).
4. (c,4) is irreducible (by placing 4 on \( R_2 \)).
5. (c,5) is irreducible (by \( S_3 \) and (4)).
6. All other edges are irreducible by \( S_1 \) and (1), (2), (3), (4), (5).

Let \( L, K \) be graphs such that \( L = S_y K \). Then there is a natural correspondence between edges and vertices of \( L \) and \( K \).
In the discussion that follows, no distinction will be made unless it is necessary for clarity.

We have been calling a vertex cubic if it is of valency 3. Recall an edge \( e \) is called cubic of \((v_1,v_2,v_3)\) if \( e \) is in a 3 cycle \((v_1,v_2,v_3)\) and if \( e = (v_1,v_2) \) say, and \( v_3 \) is cubic.

**Lemma 7.7** Let \( K \) be non-projective, \( L = S_v(K) - e \) where \( e \) is cubic of \((v_1,v_2,v_3)\) in \( S_v(K) \). If \( L \) is irreducible, then all edges of \( K \), except possibly the 3 edges on \((v_1,v_2,v_3)\) are irreducible.

**Proof:** Let \( e' \) not be in the 3-cycle \((v_1,v_2,v_3)\). Suppose \( K - e' \not\in P \). Then \( S_v(K) - e' \not\in P \) by Lemma 1.1. Hence \( S_v(K) - e' = e \not\in P \) by Lemma 3.4, as \( e \) is cubic. But then \( L - e' = (S_v(K) - e) - e' \not\in P \) which contradicts that \( L \) is irreducible, so \( e' \) is irreducible on \( K \).

**Lemma 7.8** Let \( K \) be non-projective. \( L = S_v(K) - (e_1,e') \) where \( e \) is cubic of \((v_1,v_2,v_3)\) in \( S_v(K) \) and \( e' \) is cubic of \((v'_1,v'_2,v'_3)\) in \( S_v(K) - e \). If \( L \) is irreducible then if \( \bar{e} \) is not an edge in the two 3-cycles \((v_1,v_2,v_3), (v'_1,v'_2,v'_3)\) then \( \bar{e} \) is irreducible.

**Proof:** Suppose \( \bar{e} \) is an edge of \( K \) not in \((v_1,v_2,v_3)\) or \((v'_1,v'_2,v'_3)\) such that

\[ K - \bar{e} \not\in P \]
Then $S_v(K) - e \not\subseteq P$ by Lemma 1.1. Hence $S_v(K) - e - e \not\subseteq P$ by Lemma 3.4, as $e$ is cubic in $S_v(K)$. Hence $S_v(K) - e - e - e' \not\subseteq P$ by Lemma 3.4 as $e$ is cubic in $S_v(K) - e$. Hence $L = S_v(K) - (e, e') - e \not\subseteq P$, which contradicts the irreducibility of $L$.

We shall now apply Lemma 7.7 and Lemma 7.8 to check irreducibility of the following graphs. In all cases either 3 or 6 edges remain to check. For completeness, we first list the following trivalent graphs for our reference.

**Cases N18-N23**

The graphs $F_{11}, F_{12}, F_{13}, F_{14}, F_{15}, G$ are irreducible (see Glover and Huneke [4]).

**Case N24**

$E_9:

![Figure 7.23](image)

$S: [(x,y), (l,2)]$

Since $S_{x:(p,a)}(E_9) - (p,a) = F_{10}$, since $F_{10} + F_{12}$ (see section 8) and $F_{12}$ is irreducible (see Case N19), $F_{10}$ is irreducible. Hence by Lemma 7.7 we only need to check the three edges
$\delta = \{(p,x), (p,a), (a,x)\}$. But $(p,a)$ is clearly irreducible.

$(a,x) \sim (a,y)$ by $S$ and $(a,y) \not\in \delta$ is irreducible, hence $(a,x)$ is irreducible.

Similarly $(p,x) \sim (p,y)$ by $S$ is irreducible.

**Case N.25**

$D_{19}$:

![Figure 7.24](image)

Note that $\delta_1: (2,4)(D_{19}) - (2,4) = E_7$ but $E_7 + E_9$ (see section 8). Now $E_9$ irreducible $\Rightarrow E_7$ irreducible. Hence by Lemma 7.7, we need only check the three edges $\{(1,2), (1,4), (2,4)\}$.

But by symmetry $S$ we have

$(1,2) \sim (a,b)$

$(1,4) \sim (a,d)$

$(2,4) \sim (b,d)$

so all edges are irreducible.
Case N 26

$D_{17}$:

![Figure 7.25]

$S_1$: $[(a,x), (p,z), (1,3)]$

$S_2$: $[(a,b), (y,1), (z,2)]$

Note that $S_z:(p,2)(D_{17}) = (p,2) = E_{26}$ and $E_{26} + E_{25}$ (see section 8) and $E_{25}$ irreducible (Case N13) $\Rightarrow E_{26}$ is irreducible. Now applying Lemma 7.7, we only have to check three edges $[(z,p), (2,z), (p,2)]$. Now $(p,2)$ is clearly irreducible by Figure 7.25 and $(p,z) \sim (p,2)$ (by $S_2$) and $(z,2) \sim (p,2)$ (by $S_1$). Therefore all edges are irreducible.

Case N 27

$D_{16}$:

![Figure 7.26]

$S$: $[(p,q), (b,c)]$
Note that $s_{a:p,q}(D_{16}) - (p,q) = E_{40}$ and $E_{40} + E_{15} + E_{19}$ (see section 8). Since $E_{19}$ is irreducible (Case N10), we have $E_{40}$ is irreducible. But then by Lemma 7.7 we need only to check irreducibility of edges $((a,p), (a,q), (p,a))$ but clearly $(a,q)$ is irreducible (Figure 7.26) and $(a,p) \sim (a,q)$ (by $S$) is also irreducible. To see $(p,q)$ is irreducible, we place $p$ in region $R$ in Figure 7.26 and crossing only $(p,q)$ and $(x,2)$.

Case N28

$D_{14}$:

![Figure 7.27]

S: $[(p,q)]$

Note that $s_{y:p,q}(D_{14}) - (p,q) = E_6$. $E_6$ is irreducible by Case N9. So by Lemma 7.7 only $(p,y), (q,y), (p,q)$ need to be checked. But $(p,y)$ is irreducible by Figure 7.27. $(q,y) = (p,y)$ because $p \sim q$. For $(p,q)$, we place $q$ in the region $R$ in Figure 7.27 and crossing only $(p,q)$ and $(y,z)$. 
Case N 29

\[ D_{10} : \]

![Figure 7.28](image)

\[
S: [(q, b), (x, 2), (z, l)]
\]

Note that \( S_{x: (p, q)}(D_{10}) = \{(p, q), (2, b)\} = F_{14} \). Since \( F_{14} \) is irreducible by Lemma 7.8 we need only check irreducibility of \( \delta = \{(b, 2), (b, p), (p, q), (p, x), (q, x), (p, 2)\} \). Now \( (q, x), (b, p), (p, 2) \) are irreducible by Lemma 7.6, Figure 7.28, but then \( (b, 2), (p, q), (p, x) \) are irreducible by \( S \).

Case N 30

\[ D_9 : \]

![Figure 7.29](image)

\[
S: [(p, q), (b, c), (x, y)]
\]

Note that \( S_{a: (p, q)}(D_9) = E_{32} \) and \( E_{32} \) is irreducible (Case N 16). Now \( E_{34} \) irreducible (Case N 16) = \( E_{32} \) irreducible.
Now applying Lemma 7.7, we only need to check the following edges \([(a,p), (a,q), (p,q)]\). But \((a,q)\) is irreducible (from Figure 7.29) and \((a,p) \sim (a,q)\), hence irreducible. To see \((p,q)\) is irreducible, place \(p\) in region \(\mathcal{R}\) in Figure 7.29 and cross only \((a,2)\) and \((p,q)\).

**Case N31**

\[D_2:\]

![Figure 7.30](image)

**Figure 7.30**

\(S: [(p,q)]\)

Note that \(S_{x:(p,q)}(D_2) - (p,q) = E_{28}\) and \(E_{28}\) is irreducible (by Case N15). So applying Lemma 7.7, we need only check the three edges \([(p,q), (p,x), (q,x)]\), but \((p,x)\) is irreducible (from Figure 7.30) and \((q,x) \sim (p,x)\) hence is irreducible. To see \((p,q)\) is irreducible, place \(p\) in the region \(\mathcal{R}\) crossing only \((p,q)\) and \((x,2)\).
Case N32

\[ C_{10} : \]

Figure 7.31

S: [(3,1), (1,1), (2,a), (4,c)]

Note that \( S: (2,4) \cdot (C_{10}) = D_2 \). \( D_2 \) is irreducible by Case N31. Applying Lemma 7.7, we need only check three edges \([(1,2), (1,4), (2,4)]\). But by symmetry S we have

\[ (1,4) \sim (b,c) \]
\[ (2,4) \sim (a,c) \]
\[ (1,2) \sim (a,b) \]

hence all are irreducible.

Case N33

\[ C_9 : \]

Figure 7.32

S: [(4,6), (1,3)]
Note that $S_{1}(x, y) (C_8) = (x, y) = D_8$ and $D_8 + D_{11}$ (see section 8). $D_{11}$ irreducible (Case N14) $\Rightarrow$ $D_8$ irreducible. Now applying Lemma 7.7, we have only to check $\{(1,4), (1,x), (4,x)\}$, but by $S$

$$(1,4) \sim (3,6)$$

$$(1,x) \sim (3,x)$$

$$(4,x) \sim (6,x)$$

So all edges are irreducible.

Case N34

$C_8$:

![Diagram](image_url)

Figure 7.33

$S$: $[(1,2), (x,y)]$

Note that $S_{x}(p,1) (C_8) = (p,1) = D_{17}$ is irreducible (Case N26). So apply Lemma 7.7, we need only check the edges

$\{(x,p), (x,1), (p,1)\}$. Now by $S$

$$(x,p) \sim (y,p)$$

$$(x,1) \sim (y,2)$$

$$(p,1) \sim (p,2)$$

So all are irreducible.
Case N35

\( B_{12} : \)

![Figure 7.34](image)

![Figure 7.35](image)

\( S_1 : [(4,5), (2,3)] \)

\( S_2 : [(5,6), (1,2)] \)

Note that \( S_{5;(4,5)(B_{12}) - (4,5) = C_9 \) is irreducible (Case N33). So by Lemma 7.7, we only have to check three edges \([(4,5), (6,5), (6,4)]\). Now \( S_1, S_2 \Rightarrow (4,5) \sim (4,6) \sim (5,6) \), but Figure 7.35 shows \((4,5)\) is irreducible.

Case N36

\( B_{11} : \)

![Figure 7.36](image)

\( S : [(C,E), (A',y), (A,z), (B,D), (t,x)] \)

Note that \( S_{A;(C,D)(B_{11}) - \{(C,D), (x,y)\} = D_{10} \). \( D_{10} \) is irreducible (Case N29). Now by Lemma 7.8 we only have to check
six edges \( \delta = \{(A,C), (A,D), (D,D), (D,x), (D,y), (x,y)\} \) but by \( S \)

\[
\begin{align*}
(A,C) & \sim (E,z) \\
(A,D) & \sim (B,z) \\
(C,D) & \sim (B,E) \\
(D,x) & \sim (B,t) \\
(D,y) & \sim (A',B) \\
(x,y) & \sim (A',t)
\end{align*}
\]

So all edges are irreducible.

**Case N37**

\( B_9: \)

\( S_1: \ [(p,A), (D,C), (A',E)] \)

\( S_2: \ [(x,y)] \)

Note that \( S_{A'(x,E)}(B_9) = \{(B,x), (E,y)\} \) = \( B_9 \). \( D_9 \) is irreducible (by Case N30). So by Lemma 7.8 we need only check

six edges \( \delta = \{(A',x), (A',B), (B,x), (B,E), (B,y), (y,E)\} \). But

\( (A',x) = (A',y) \) is irreducible by \( S_2 \), \( (y,E) \sim (x,E) \) is irreducible. \( (A',B), (B,y) \) are irreducible by placing \( B \) in \( \mathcal{R} \)}
on Figure 7.38 and \((B,E) \sim (A',B)\), \((B,x) \sim (B,y)\) by \(S_1\), so are irreducible.

**Case N\,38**

\(A_4:\)

\[
\begin{matrix}
4 & 1 & 5 \\
2 & x & 6 \\
3 & & 5
\end{matrix}
\]

**Figure 7.39**

\(S_1: [(1,6)]\)

\(S_2: [(2,5)]\)

Note that \(S_1\cdot (2,3)(A_4) \sim (2,3) = B_{12}\) which is irreducible by Case N\,35. So applying Lemma 7.7, we check only three edges \([(1,2), (1,3), (2,3)]\), but by \(S_1, S_2\)

\[
(1,2) \sim (2,6) \\
(1,3) \sim (3,6) \\
(2,3) \sim (3,5)
\]

Hence every edge is irreducible.
8. SOURCES AND SINKS OF A GRAPH

By the definitions of a source and a sink, any graph $K$ in the directed set $I(P)$ is either a source or follows a source with the same Betti number as $K$, and also is either a sink or is followed by a sink with the same Betti number as $K$. Sources and sinks related to $K$ in this fashion are the sources and sinks of the graph $K$. This section includes some branches of $I(P)$ demonstrating some sources and sinks for each graph in $I(P)$ which is not both a source and a sink. More specifically we will use the notation

$$K^1 \xrightarrow{v_0:(v_1,\ldots,v_k)} K^2 \longrightarrow \ldots \longrightarrow K^n$$

to indicate the following information:

(1) $K^1$ is a source of $K^i$;

(2) $K^n$ is a sink of $K^i$;

(3) $K^2 = \mathcal{S}_{v_0:(v_1,\ldots,v_k)}K^1$ and similarly for $K^i, K^{i-1}$;

(4) The notation for vertices $v_0, v_1, \ldots, v_k$ agree with labelling of $K^1$ as appears in the case discussing $K^1$ in section 6; and

(5) If $v_0:(v_1,\ldots,v_k)$ is replaced by $P$ then $P$ is assumed to be the cutpoint of $K^1$ and the splitting

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is left to the reader to compare.

Figure 8.1
Figure 8.2
Figure 8.3
Figure 8.4
Figure 8.5
9. CONJECTURES AND RELATED QUESTIONS

We have shown that \( I(P) \) contains 103 graphs. However, we conjecture that \( I(P) \) is the set consisting of exactly the 103 graphs of the Appendix.

The proof of this conjecture should be made in the following way. In [3] Huneke and Glover will show that \( A_1, A_4, B_2, B_3, D_20 \) are the maximal elements in \( I(P) \). In [11] the writer will show that the 103 graphs above are all graphs below these five.

A first question that can be asked about the partially ordered set \( I(P) \) is whether it is essentially a lattice. That is, after introducing a maximal and minimal element in \( I(P) \) do we have a lattice. The answer to this question is no. The following partially order subset of \( I(P) \) shows this.

\[ \text{Figure 9.1} \]

More particularly \( E_{15} \) and \( E_{41} \) do not have a supremum in \( I(P) \).

Another natural question one may ask about \( I(M) \) is whether for \( K \in I(M) \) there exists an immersion (a local embedding) \( f: K \rightarrow M \) with a single double point interior to two edges. This
appears to be correct for \( M = P \) and \( K \) any one of the 103 graphs of our list. There is no proof of this observation about \( I(P) \), or general \( I(M) \).

It is conceivable that \( |I(M)| \) increases rapidly as a function of the genus of \( M \). In particular, one may ask: Is \( I(M) \) finite for all \( M \)? The question is not yet answered.

In view of the last paragraph, a more practical way of attacking the problem of characterizing \( I(M) \) would be to look for an effective algorithm for generating graphs in \( I(M) \). There has been very little success with the exception of \( I(\mathbb{R}^2) \).

Rather than enumerating all elements in \( I(M) \) it seems to be more practical to look for all maximal elements in \( I(M) \) or to derive a characterization for all maximal elements in \( I(M) \).
APPENDIX: THE 103 GRAPHS IN \( I(P) \)

The following 103 graphs are given by the order \( A_i \), \( B_i \), \( C_i \), \( D_i \), \( E_i \), \( F_i \), and \( G \).

\( A_i \)'s are graphs with Betti number \( \beta = 12 \). Similarly, \( B_i \)'s, \( C_i \)'s, \( D_i \)'s, \( E_i \)'s, \( F_i \)'s, \( G \) are graphs with \( \beta = 11, 10, 9, 8, 7, 6 \) respectively.

The valency sequence of each graph is given directly below the picture. The general form of the valency sequence is

\[ s_0(s_1, s_2, \ldots, s_n) \]

where \( s_0 = n \) is the number of vertices of the graph and \( s_i \)'s are valency of vertices arranged in non-increasing order.
$A_1:$

$9(8, 4, 4, 4, 4, 4, 4, 4)$

$A_2:$

$10(6, 4, 4, 4, 4, 4, 4, 4)$

$A_3:$

$11(4, 4, 4, 4, 4, 4, 4, 4)$
$A_4 : \quad 7(6, 5, 5, 5, 5, 5, 5)$

$B_1 : \quad 10(4, 4, 4, 4, 4, 4, 4, 4)$

$B_2 : \quad 8(6, 6, 4, 4, 4, 4, 4, 4)$
$B_3 : \quad 9(6, 4, 4, 4, 4, 4, 4, 4, 4)$

$B_4 : \quad 10(4, 4, 4, 4, 4, 4, 4, 4, 4)$

$B_5 : \quad 7(6, 6, 6, 4, 4, 4, 4)$
\begin{align*}
\mathcal{B}_9 & : \\
& 9(6, 6, 4, 4, 4, 4, 3, 3) \\
\mathcal{B}_{10} & : \\
& 9(6, 4, 4, 4, 4, 4, 4, 4) \\
\mathcal{B}_{11} & : \\
& 10(4, 4, 4, 4, 4, 4, 4, 4) 
\end{align*}
$B_{12}$:

$8(6, 5, 5, 5, 4, 4, 4, 3)$

$c_1$:

$10(7, 4, 4, 4, 4, 3, 3, 3, 3, 3)$

$c_2$:

$11(5, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3)$
$c_3 :$ 

$11(6, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3)$

$\quad$

$\quad$

$c_4 :$ 

$12(4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3)$

$c_5 :$ 

$9(6, 6, 4, 4, 4, 3, 3, 3, 3)$
$c_6 : \quad 10(6, 4, 4, 4, 4, 3, 3, 3, 3)$

$\therefore c_7 : \quad 11(4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3)$

$\therefore c_8 : \quad 9(6, 5, 4, 4, 4, 4, 3, 3, 3)$
$c_9:$

$9(6, 4, 4, 4, 4, 4, 4, 3, 3)$

$c_{10}:$

$8(5, 5, 4, 4, 4, 4, 4, 4)$

$d_1:$

$8(5, 5, 5, 4, 4, 3, 3, 3)$
\[ D_2 : \]
\[ (5, 5, 4, 4, 3, 3, 3, 3) \]

\[ D_3 : \]
\[ (5, 5, 4, 4, 3, 3, 3, 3) \]

\[ D_4 : \]
\[ (5, 5, 4, 4, 3, 3, 3, 3) \]
$D_5:$

$9(5, 5, 4, 4, 4, 3, 3, 3, 3)$

$D_6:$

$10(5, 4, 4, 4, 4, 3, 3, 3, 3, 3)$

$D_7:$

$10(5, 5, 4, 4, 3, 3, 3, 3, 3, 3)$
$D_8$:
$10(5, 4, 4, 4, 3, 3, 3, 3, 3)$

$D_9$:
$10(5, 5, 4, 4, 3, 3, 3, 3, 3)$

$D_{10}$:
$11(4, 4, 4, 4, 4, 3, 3, 3, 3)$
$D_{11}: 11(5, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3)$

$D_{12}: 9(5, 5, 4, 4, 4, 3, 3, 3, 3, 3)$

$D_{13}: 10(5, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3)$
$D_{14}: 11(4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3)$

$D_{15}: 10(6, 6, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$

$D_{16}: 9(5, 4, 4, 4, 4, 3, 3, 3, 3)$
$D_{17}: 10(6, 4, 4, 4, 3, 3, 3, 3, 3, 3)$

$D_{18}: 11(4, 4, 4, 4, 4, 3, 3, 3, 3, 3)$

$D_{19}: 8(4, 4, 4, 4, 4, 4, 4)$
$D_{20} :$ 

10(5, 5, 4, 4, 3, 3, 3, 3, 3, 3)
$E_1$:

$11(6,3,3,3,3,3,3,3,3,3,3,3)$

$E_2$:

$12(5,3,3,3,3,3,3,3,3,3,3,3)$

$E_3$:

$13(4,3,3,3,3,3,3,3,3,3,3,3)$
\( E_4 : \)

\( 10(5, 5, 3, 3, 3, 3, 3, 3, 3, 3, 3) \)

\( E_5 : \)

\( 11(5, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3) \)

\( E_6 : \)

\( 12(4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3) \)
$E_7$:

$9(4, 4, 4, 4, 4, 3, 3, 3, 3)$

$E_8$:

$9(4, 4, 4, 4, 4, 3, 3, 3, 3)$

$E_9$:

$10(4, 4, 4, 4, 4, 3, 3, 3, 3, 3)$
$E_{10}$:

$8(5, 5, 5, 3, 3, 3, 3, 3)$

$E_{11}$:

$9(5, 5, 4, 3, 3, 3, 3, 3, 3)$

$E_{12}$:

$10(5, 5, 3, 3, 3, 3, 3, 3, 3)$
$E_{13}$:

10(5, 4, 4, 3, 3, 3, 3, 3, 3, 3)

$E_{14}$:

10(5, 4, 4, 3, 3, 3, 3, 3, 3, 3)

$E_{15}$:

11(4, 4, 4, 3, 3, 3, 3, 3, 3, 3)
$E_{16}$:

11(4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3)

$E_{17}$:

11(5, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)

$E_{18}$:

12(5, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)
$E_{19} :$

$12(4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3)$

$E_{20} :$

$8(4, 4, 4, 4, 4, 4, 3, 3)$

$E_{21} :$

$9(4, 4, 4, 4, 4, 3, 3, 3, 3)$
$E_{22}$:

$10(4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3)$

$E_{23}$:

$10(4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3)$

$E_{24}$:

$11(4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3)$
$E_{25}$:

$12(4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$

$E_{26}$:

$11(6, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$

$E_{27}$:

$9(5, 5, 4, 3, 3, 3, 3, 3, 3)$
$E_{28}: 10(5, 5, 3, 3, 3, 3, 3, 3, 3, 3)$

$E_{29}: 10(5, 4, 4, 3, 3, 3, 3, 3, 3, 3)$

$E_{30}: 11(4, 4, 4, 3, 3, 3, 3, 3, 3, 3)$
$E_{31} : \quad 11(4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3)$

$E_{32} : \quad 11(5, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$

$E_{33} : \quad 12(4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$
$E_{34}$:

$13(4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$

$E_{35}$:

$9(4, 4, 4, 4, 3, 3, 3, 3)$

$E_{36}$:

$10(4, 4, 4, 4, 3, 3, 3, 3, 3, 3)$
$E_{37}$:

11(4, 4, 4, 3, 3, 3, 3, 3, 3, 3)

$E_{38}$:

10(4, 4, 4, 4, 3, 3, 3, 3, 3, 3)

$E_{39}$:

10(5, 4, 4, 3, 3, 3, 3, 3, 3, 3)
$E_{40}: 10(4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3)$

$E_{41}: 11(4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3)$

$F_{1}: 9(4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3)$
$F_2:$

$10(4, 4, 3, 3, 3, 3, 3, 3, 3, 3)$

$F_3:$

$10(4, 4, 3, 3, 3, 3, 3, 3, 3, 3)$

$F_4:$

$10(4, 4, 3, 3, 3, 3, 3, 3, 3, 3)$
$F_5 :$

$10(4, 3, 3, 3, 3, 3, 3, 3, 3)$

$F_6 :$

$10(4, 4, 3, 3, 3, 3, 3, 3, 3)$

$F_7 :$

$11(4, 3, 3, 3, 3, 3, 3, 3, 3)$
$F_8$: $11(4, 3, 3, 3, 3, 3, 3, 3, 3, 3)$

$F_9$: $11(4, 3, 3, 3, 3, 3, 3, 3, 3, 3)$

$F_{10}$: $11(4, 3, 3, 3, 3, 3, 3, 3, 3, 3)$
$F_{11}$:

$12(3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$

$F_{12}$:

$12(3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$

$F_{13}$:

$12(3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$
\( F_{14} : \)

\[ 12(3, 3, 3, 3, 3, 3, 3, 3, 3, 3) \]

\( F_{15} : \)

\[ 12(3, 3, 3, 3, 3, 3, 3, 3, 3, 3) \]

\( G : \)

\[ 10(3, 3, 3, 3, 3, 3, 3, 3, 3) \]
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E_21, 22, 70, 98
E_22, 23, 70, 99
E_23, 23, 70, 99
E_24, 24, 70, 99
E_25, 24, 50, 70, 100
E_26, 21, 35, 58, 70, 100
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