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The Ohio State University, Ph.D., 1974
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ON THE FIXED POINT PROPERTY FOR GRASSMANN MANIFOLDS

Dissertation

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Larkin S. O'Neill, B.S., M.S.

* * * * *

The Ohio State University
1974

Reading Committee:
J. Philip Huneke
Yung-Chen Lu
Henry H. Glover

Approved By
Adviser
Department of Mathematics
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UNIVERSITY MICROFILMS
VITA

September 2, 1938, Born - Columbus, Ohio

1956................. South High School
Columbus, Ohio

1962................. B.S. in Engineering Physics
The Ohio State University,
Columbus, Ohio

1962-1964........... Physicist, Battelle Memorial Institute
Columbus, Ohio

1964-1965......... Engineer, Eastman Kodak Co.
Rochester, New York

1965-1968........... Mathematics and Science Teacher
Dodson High School, Dodson, Montana

1968................. M.S. in Mathematics
The Ohio State University,
Columbus, Ohio

1968-1974........... Teaching Associate,
Department of Mathematics
The Ohio State University,
Columbus, Ohio

1974................. Visiting Instructor,
Department of Mathematics
The Ohio State University,
Mansfield, Ohio

FIELDS OF STUDY

Algebra: Professor Harold Brown

Analysis: Professors Bogdan Baishanski and Ranko Bojanic

Topology: Professor Norman Levine
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INTRODUCTION

A topological space, $X$, is said to have the fixed point property if every self-map of $X$ has a fixed point. That is, for every map $f: X \to X$ there exists $x \in X$ such that $f(x) = x$. Numerous examples of topological spaces with the fixed point property have been given (see, e.g., Bing[1], Fadell[4]), but manifolds with the fixed point property are relatively scarce. Indeed the following is an exhaustive list of those manifolds known to possess the fixed point property:

(a) the closed disks, $D^n$, $n \geq 0$,

(b) the even-dimensional real and complex projective spaces, $\mathbb{R}P^{2n}$, $\mathbb{C}P^{2n}$, $n \geq 0$,

(c) all the quaternionic projective spaces, $\mathbb{H}P^n$, except $\mathbb{H}P^1 \cong S^4$,

(d) the Cayley plane, which is the homogeneous space $F_4/\text{Spin}(9)$, where $F_4$ is the Lie group which is the quotient group $\text{SO}(8)/\text{U}(4)$. The Cayley plane may almost be defined as the projective plane of the Cayley numbers (see Steenrod[8], page 108), but the non-associativity of the Cayley numbers prevents this,

(e) compact smooth manifolds with the same cohomology
algebras as those of the manifolds in (a) through (d) above. Such manifolds may be obtained, for instance, by mixing homotopy types (see [6]).

That the spaces in (a) have the fixed point property is the classic Brouwer fixed point theorem. The other spaces are proved to have the fixed point property by means of the Lefschetz fixed point theorem. It is relatively easy to apply the Lefschetz theorem there because these spaces have such simple cohomology algebras.

In this paper we give a new class of smooth manifolds, M, with the fixed point property (written M ∈FPP) which are a natural generalization of the projective spaces above. Let $F_{G_{2,q}}$ denote the Grassmann manifold of 2-dimensional subspaces in $F^{q+2}$, $F = R, C, or H$. We prove the following results. (Note that we use numbers from sections in the text.)

**Theorem 4.1.** $F_{G_{2,q}} ∈FPP$, for all $q > 2$ and $F = C$ or $H$.

**Theorem 5.1.** $R_{G_{2,q}} ∈FPP$, for all $q = 4k$, or $q = 4k+1$, $k = 1, 2, 3, ...$.

Notice that Theorems 4.1 and 5.1 and examples (b) and (c) are special cases of the following conjecture:

**Conjecture 0.1.** $F_{G_{p,q}} ∈FPP$, if and only if $q ∉p$ and $p·q$ is even, $F = R$ or $C$. $H_{G_{p,q}} ∈FPP$ if and only if $q ∉p$. 
Recall that
\[ H^*(FG, R; Z) = R \langle \overline{c_1}, \ldots, \overline{c_p}, \overline{\tilde{c}_1}, \ldots, \overline{\tilde{c}_q} \rangle / (\overline{c\tilde{c}} = 1), \]
where 
\[ c = 1 + c_1 + \cdots + c_p, \quad \overline{\tilde{c}} = 1 + \overline{\tilde{c}_1} + \cdots + \overline{\tilde{c}_q} \]
and \( \overline{c_k}, \overline{\tilde{c}_k} \) are of degree \( d_F k \), and \( R \) is an arbitrary commutative ring with unit for \( F = C \) or \( H \) and \( R = Z_2 \), for \( F = R \). Our theorems then follow easily from the Lefschetz theorem as consequences of corollaries to the following theorems concerning the structures of the endomorphism rings of the cohomology algebras of the spaces in question:

**Theorem 4.2.** For all \( q > 2 \) and \( F = C \) or \( H \), every endomorphism \( \varphi \) of \( H^*(FG, q; Z) \) is an Adams mapping. That is, if \( \varphi(c_1) = \alpha c_1 \) then \( \varphi(c_2) = \alpha^2 c_2 \).

**Corollary 4.3.** For every endomorphism \( \varphi \) of \( H^*(FG, q; Z) \), \( L(\varphi; Z) = \sum_{k=1}^{2q} \dim H^k(FG, q; Z) \alpha^k \), where \( L(\varphi; Z) \) is the Lefschetz number of \( \varphi \).

**Theorem 5.2.** Every endomorphism of \( H^*(RG, q; Z_2) \) is an Adams mapping if \( q = 4k \) or \( q = 4k+1 \), \( k = 1, 2, 3, \ldots \).

**Corollary 5.3.** For every endomorphism \( \varphi \) of \( H^*(RG, q; Z_2) \), \( L(\varphi; Z_2) \equiv 0 \pmod{2} \).

Conjecture 0.1 would follow from the corresponding conjecture about endomorphisms:

**Conjecture 0.2.** Every endomorphism \( \varphi \) of \( H^*(FG, q; Z) \) is an Adams mapping, for all \( p \neq q \) and \( F = C \) or \( H \). That is, if \( \varphi(c_1) = \alpha c_1 \) then
\[ \phi(c_k) = \alpha^k c_k \text{ for all } k = 1, \ldots, p. \] Every endomorphism of \( H^*(RG_{p, q}; Z_2) \) is an Adams mapping, for all \( p \neq q \).

The plan of the paper follows. In section 1 we give the definition and well-known properties of the Grassmann manifolds. In section 2 we give examples of fixed point free maps that prove half of conjecture 0.1. In section 3 we prove the Lefschetz theorem for manifolds. In sections 4 and 5 we prove theorems 4.1 and 5.1 respectively.
1. THE GRASSMANN MANIFOLDS

In this section we define the Grassmann manifolds \( \mathcal{G}_{p,q} (\mathbb{F} = \mathbb{R}, \mathbb{C}, \text{or } \mathbb{H}) \), show that \( \mathcal{G}_{p,q} \) is a smooth manifold of \( \mathbb{F} \)-dimension \( pq \), give a canonical cell decomposition for \( \mathcal{G}_{p,q} \), and recall the cohomology ring \( H^*(\mathcal{G}_{p,q}, R) \) where \( R \) is an arbitrary commutative ring with unit if \( \mathbb{F} = \mathbb{C} \) or \( \mathbb{H} \) and \( R = \mathbb{Z}_2 \) if \( \mathbb{F} = \mathbb{R} \).

The Grassmann manifolds are generalizations of projective spaces. We recall that \( \mathbb{F}P^n \) can be defined as the orbit space of the action of the units of \( \mathbb{F} \) on the \((dn+d-1)\)-sphere \((d = \dim_{\mathbb{R}} \mathbb{F})\). It is clear that any orbit under this action determines a unique \( \mathbb{F} \)-line in \( \mathbb{F}^{n+1} \), so \( \mathbb{F}P^n \) can be described as the space of \( \mathbb{F} \)-lines in \( \mathbb{F}^{n+1} \). Since \( \mathbb{F} \)-lines are \( \mathbb{F} \)-1-dimensional subspaces of \( \mathbb{F}^{n+1} \), to generalize projective spaces we may consider the space of \( \mathbb{F} \)-p-dimensional subspaces of \( \mathbb{F}^{p+q} \), for some \( p, q \geq 0 \). We will denote by \( \mathcal{G}_{p,q} \) the set of \( \mathbb{F} \)-p-planes in \( \mathbb{F}^{p+q} \). Observe that we then have \( \mathbb{F}P^n = \mathcal{G}_{1,n} \).

Recall that the projective spaces are given the quotient topology. We proceed in a similar manner to define the topology on \( \mathcal{G}_{p,q} \). For this, recall the Stiefel manifold of orthonormal p-frames in \( \mathbb{F}^{p+q} \), which we will
denote by \( FV_{p,q} \). More precisely, we have

\[
FV_{p,q} = \{ v \in (S^d(p+q)-1)p \mid (v_i | v_j) = \delta_{i,j} \}
\]

where \((v_i | v_j)\) denotes the usual inner product of \(v_i\) and \(v_j\) in \(F^{p+q}\) and \(\delta_{i,j}\) is the Kronecker delta function. Note that \(FV_{p,q}\) is closed in \((S^d(p+q)-1)p\), hence is compact.

Also, there is a canonical projection \(\pi_{d}^{p,q} : FV_{p,q} \to FG_{p,q}\) given by \(v \mapsto \langle v_1, \ldots, v_p \rangle\), where \(\langle v_1, \ldots, v_p \rangle\) denotes the subspace of \(F^{p+q}\) generated by \(\{v_1, \ldots, v_p\}\). From the definition of \(FV_{p,q}\) it is clear that \(\{v_1, \ldots, v_p\}\) is linearly independent, so \(\langle v_1, \ldots, v_p \rangle\) is a \(p\)-plane in \(F^{p+q}\) and \(\pi_{d}^{p,q}\) is well-defined as a set morphism.

Thus, let \(FG_{p,q}\) have the topology induced by \(\pi_{d}^{p,q}\) (hence \(\pi_{d}^{p,q}\) becomes an identification mapping). For any \(p\)-plane \(x \in FG_{p,q}\), \((\pi_{d}^{p,q})^{-1}(x)\) is the set of all orthonormal \(p\)-frames in the \(p\)-plane \(x\) and

\[
(\pi_{d}^{p,q})^{-1}(x) = FV_{p,0}, \quad \text{whence} \quad (FV_{p,q}, \pi_{d}^{p,q}, FG_{p,q}) \text{ is a fiber bundle with fiber } FV_{p,0} \quad \text{(of course } FV_{p,0} \text{ is the } p\text{-orthogonal, } p\text{-unitary or } p\text{-symplectic group respectively for } F = \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H} \text{ respectively).}
\]

We remark that the compactness of \(FV_{p,q}\) implies that of \(FG_{p,q}\).

One of the most significant aspects of Grassmann manifolds is that they can be viewed as homogeneous spaces. Let the lie group \(G\) act transitively on the smooth manifold \(M\) and for some \(x \in M\), let \(K\) be the isotropy
subgroup of $G$ at $x$. We recall the basic theorem concerning homogeneous spaces (see, e.g., Wolf[9], page 12):

**Proposition 1.1.** $K$ is a closed subgroup of $G$ and the left coset space $G/K$ carries a natural differentiable manifold structure such that (i) $G$ acts differentiably and transitively on $G/K$ by left translations, (ii) the projection $G \rightarrow G/K$ is a differentiable map of maximal rank, (iii) $\dim G/K = \dim G - \dim K$, and (iv) $G/K$ is diffeomorphic to $M$.

Note that the diffeomorphism in (iv) is obtained by means of the map $G \rightarrow M$ given by $g \mapsto g(x)$. This map induces a one-to-one differentiable map of maximal rank from $G/K$ onto $M$ which is a diffeomorphism by the Inverse Function Theorem. Proposition 1.1 combined with the following proposition will allow us to view $FG_{p,q}$ as a manifold.

**Proposition 1.2.** $FG_{p,q} \cong GL(p+q,F) / \begin{pmatrix} GL(p,F) & \ast \\ 0 & GL(q,F) \end{pmatrix}$ (a homeomorphism).

**Proof:** Let $G = GL(p+q,F)$ and let

$H = \left\{ \begin{pmatrix} R & S \\ 0 & T \end{pmatrix} \in G : R \in GL(p,F), S \in M(p \times q,F), \text{ and } T \in GL(q,F) \right\}$.

We will obtain a homeomorphism $f:FG_{p,q} \rightarrow G/H$ with $f$ defined as follows: for any $p$-plane $x \in FG_{p,q}$ let $\{x_1, \ldots, x_p\}$ be a basis for $x$ which is extended to a basis for $F^{p+q}$ by $\{x_{p+1}, \ldots, x_{p+q}\}$, then take $f(x) = xH$ where $x$ is the matrix whose column vectors are the $x_i$'s.
Suppose \( \{x_i^\prime, \ldots, x_p^\prime\} \) is another basis and extension to \( F^{p+q} \) for \( x \), then we must show \( \tilde{x}H = \tilde{x'}H \) in order that \( f \) be well-defined. Let \( a \) be the matrix of the transformation from the basis of \( x^i \)'s to that of \( x_i^\prime \)'s, i.e. \( x_i^\prime = \sum_{j=1}^{p+q} x_j a_{ji} \). Now when \( 1 \leq i \leq p \), \( x_i \) is a linear combination of \( x_1^\prime, \ldots, x_p^\prime \), so for any such \( i \) we must have \( a_{ji} = 0 \) when \( j > p \). But then it is seen that \( a \in H \) and \( \tilde{x'} = \tilde{x}a \) implies \( \tilde{x'}H = \tilde{x}H \), so \( f \) is well-defined. Since the first \( p \) columns in an arbitrary matrix of \( G \) determine a \( p \)-plane in \( F^{p+q} \) it is clear that \( f \) is onto. Suppose \( f(x) = f(y) \). Then \( \tilde{x}H = \tilde{y}H \), or \( \tilde{x} = \tilde{y}h \) for some \( h \in H \). But since the lower left block in \( h \) is zero, the first \( p \) columns in the product \( \tilde{y}h \) are linear combinations of the first \( p \) columns of \( \tilde{y} \). As \( \tilde{y}h = \tilde{x} \), it follows that \( y = x \) and \( f \) is one-to-one.

Next let us show that \( f \) is continuous. If \( W \) is open in \( G/H \), then the inverse image \( U \) of \( W \) under the canonical projection of \( G \) onto \( H \) is open in \( G \). As \( G \) is open in \( F^{(p+q)^2} \), if we say in the usual way that

\[
F_{p,q} \subseteq (S^d(p+q)-1)^p \subseteq F^{(p+q)p} \subseteq F^{(p+q)^2},
\]

then \( U' = U \cap F_{p,q} \) is open in \( F_{p,q} \). But

\[
(f^{-1}(W')) = U', \text{ so } f^{-1}(W) \text{ is open in } FG_{p,q}
\]

since \( \pi_{d_{p,q}} \) is an identification map, hence \( f \) is continuous.

Since \( FG_{p,q} \) is compact and \( G/H \) is Hausdorff, the
bijectivity of \( f \) makes it a homeomorphism, q. e. d.

If we identify \( \text{FG}_{p,q} \) with \( G/H \) by means of the homeomorphism given above, then Proposition 1.1 immediately gives us:

**Corollary 1.3.** \( \text{FG}_{p,q} \) is a smooth manifold of \( F \)-dimension \( pq \).

Note also that if \( K \) is the subgroup of \( G \) defined in the same way as \( H \) but with \( p \) and \( q \) interchanged, it is clear that \( G/H \cong G/K \). Hence, \( \text{FG}_{p,q} \cong \text{FG}_{q,p} \). This corresponds to the intuitively obvious fact that there is a homeomorphism defined between the space of \( p \)-planes in \( F^{p+q} \) and the space of \( q \)-planes in \( F^{p+q} \) by mapping a \( p \)-plane to its orthogonal complement.

To discuss the homology and cohomology of the Grassmann manifolds let us give a cellular decomposition of \( \text{FG}_{p,q} \). To do this we note that we can represent each \( p \)-plane in \( F^{p+q} \) by a unique \( px(p+q) \) matrix in a canonical row echelon form where each row of the matrix corresponds to a basis vector of the given plane. The \( px(p+q) \) matrices have the following representative forms:
where the parenthesized matrices have arbitrary entries from \( F \) and the indicated sizes. By induction down we may choose constants to replace the zeroes under each of the indicated 1's so as to make the matrix have orthogonal rows. If we then normalize each row it is clear that the resulting matrix corresponds naturally to a unique element of \( \mathbb{F} \mathcal{V}_{p,q} \). We may classify each such matrix by a strictly increasing \( p \)-tuple of positive integers, 
\[
\mathbf{r} = (r_1, \ldots, r_p),
\]
where \( r_i \) gives the column of the \( i \)-th indicated 1. Letting \( L = \sum_{i=1}^{p} \lambda_i (p-i+1) \) and denoting the set of all matrices of type \( \mathbf{r} \) by \( \sigma_{\mathbf{r}} \), it is clear that \( \sigma_{\mathbf{r}} \) is an open \( L \)-cell in \( \mathbb{F} \mathcal{V}_{p,q} \). Milnor and Stasheff([7], page 76) show that its closure in \( \mathbb{F} \mathcal{V}_{p,q} \), \( \bar{\sigma}_{\mathbf{r}} \), is a closed \( L \)-cell and that the restriction of \( \pi_{p,q}^d \) to \( \bar{\sigma}_{\mathbf{r}} \) serves as a cellular attaching map. Note that there are as many cells in this decomposition as there are strictly increasing \( p \)-tuples of positive integers whose last entry is at most \( p+q \). Thus \( \mathbb{F} \mathcal{G}_{p,q} \)

\[
\begin{pmatrix}
\lambda_1 x^p & \lambda_2 x(p-1) & \cdots & \lambda_p x(p-1) & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]
has \( \binom{p+q}{p} \) cells in this decomposition. There is exactly one zero-cell for any \( \mathbb{F}G_{p,q} : \sigma \) \( = \{ [I_p \ 0] \} \), where
\( I_p \) denotes the \( p \times p \) identity matrix; exactly one \( d \)-cell:
\[
\sigma(1, \ldots, p-1, p+1) = \left\{ \begin{bmatrix} I_{p-1} & 0_{(p-1) \times (q+1)} \\ 0_{1 \times (p-1)} & v \end{bmatrix} : v \in \mathbb{F} \right\};
\]
and exactly one \( dpq \)-cell:
\[
\sigma(q+1, \ldots, q+p) = \left\{ \begin{bmatrix} A_{pxq} & I_p \end{bmatrix} : A \in M(pxq, \mathbb{F}) \right\}.
\]

In sum:

**Proposition 1.4.** \( \mathbb{F}G_{p,q} \) is a CW-complex with as many cells in dimension \( dk \) as there are \( p \)-tuples \( \lambda \) of non-negative integers of weight \( k \) (i.e., \( \sum_{i=1}^{p} i \cdot \lambda_i = k \)).

Applying a fixed permutation to the columns of each matrix in \( \sigma(q+1, \ldots, q+p) \) we would obtain another open \( dpq \)-cell in \( \mathbb{F}G_{p,q} \) and by taking the cells resulting from all possible permutations we would obtain a system of coordinate neighborhoods for \( \mathbb{F}G_{p,q} \). When \( F = \mathbb{C} \) or \( \mathbb{H} \) we are attaching only even real dimensional cells. Thus it follows that \( \mathbb{F}G_{p,q} \) is simply connected, hence orientable for any \( (p,q) \) when \( F = \mathbb{C} \) or \( \mathbb{H} \). In this case the lack of odd real dimensional cells also gives the homology completely and we have \( H_k(\mathbb{F}G_{p,q}; \mathbb{R}) = \#k\text{-cells} \mathbb{R} \), for \( F = \mathbb{C} \) or \( \mathbb{H} \).

Next, let us consider a generalization of the canonical line bundle over a projective space. Thinking of \( q \)-dimensional \( F \)-projective space as lines in \( \mathbb{F}^{q+1} \), the
canonical line bundle consists of pairs of lines and vectors in those lines. Similarly, there is a canonical p-plane bundle over $\mathbb{F}_G^p,q$ consisting of pairs of planes and vectors in those planes. Denoting this canonical p-plane bundle by $\mathbb{F}_\theta^p,q$, we have $\mathbb{F}_\theta^p,q = \{ (\gamma, v') : \gamma \in \mathbb{F}_G^p,q \text{ and } v' \in \gamma \subseteq \mathbb{F}^{p+q} \}$. There is also the bundle determined by the orthogonal complement, $\mathbb{F}_\theta^\perp_p,q = \{ (\gamma, v'') : \gamma \in \mathbb{F}_G^p,q \text{ and } v'' \in \gamma \perp \subseteq \mathbb{F}^{p+q} \}$, where $\gamma \perp$ denotes the orthogonal complement of $\gamma$ in $\mathbb{F}^{p+q}$. Given a linear subspace $\gamma$ of $\mathbb{F}^{p+q}$, any vector $v$ in $\mathbb{F}^{p+q}$ may be represented uniquely as the sum of a vector $v'$ in $\gamma$ and a vector $v-v' = v''$ which is orthogonal to $\gamma$. Thus, $\mathbb{F}_\theta^p,q \oplus \mathbb{F}_\theta^\perp_p,q = \{ (\gamma, v'), (\gamma, v'') : \gamma \in \mathbb{F}_G^p,q \text{, } v' \in \gamma \text{ and } v'' \in \gamma \perp \}$. the Whitney sum of $\mathbb{F}_\theta^p,q$ and $\mathbb{F}_\theta^\perp_p,q$, is easily seen to be trivial.

In order to discuss the cohomology of the Grassmann manifolds we introduce the following notation. Let $c_i$ denote the $i$-th Stiefel-Whitney class of $\mathbb{F}_\theta^p,q$ when $F = \mathbb{R}$, let it denote the $i$-th Chern class when $F = \mathbb{C}$, and let it denote the $(2i)$-th Chern class when $F = \mathbb{H}$ (and we consider $\mathbb{H}^p,q$ to be a complex vector bundle in the natural way). Let $\tilde{c}_i$ denote the corresponding classes of the bundles $\mathbb{F}_\theta^\perp_p,q$. Let $c = \sum_{i=0}^{q} c_i$ and $\tilde{c} = \sum_{i=0}^{q} \tilde{c}_i$ denote the corresponding total classes. As $\mathbb{F}_\theta^p,q \oplus \mathbb{F}_\theta^\perp_p,q$ is trivial, $\tilde{c} = 1$. Borel ([2], sections
21 and 22) has shown that the $c_i$'s generate $H^*(\mathcal{F}_p, q; R)$ and that they are subject only to the relation $cc = 1$.

To be precise:

**Proposition 1.5.** $H^*(\mathcal{F}_p, q; R) = R[c_1, \ldots, c_p]/(cc = 1)$

and $H^{dk}(\mathcal{F}_p, q; R)$ has

$$\left\{ \pi^p c_i \left| \sum_{i=1} r_i \geq k \text{ and } \sum_{i=1}^p r_i \leq q, r \in \mathbb{N} \right. \right\}$$

as free $R$-basis where $R$ can be taken to be an arbitrary commutative ring with unit when $F = \mathbb{C}$ or $H$, and $R = \mathbb{Z}_2$ when $F = \mathbb{R}$.

Up to dimension $dq$, the relation $cc = 1$ simply serves to express the $\tilde{c}_i$'s in terms of the $c_i$'s; above dimension $dq$ it provides certain relations among the $c_i$'s.
2. FIXED POINT FREE MAPS OF GRASSMANN MANIFOLDS

In this section we give some examples of fixed point free maps of Grassmann manifolds. Our examples are generalizations of known examples for projective spaces, which are in turn generalizations of the fixed point free antipodal maps of $S^1 \cong RG_{1,1}$ and $S^2 \cong CG_{1,1}$. It will be seen that we can produce these examples only in the real and complex cases. We will point out where the process fails in the quaternionic case.

We begin with a generalization of the antipodal map $a: F^{p+q} \to F^{p+q}$ given by $a(u_1, \ldots, u_{p+q}) = (-\tilde{u}_2, \tilde{u}_1, \ldots, -\tilde{u}_{2l}, \tilde{u}_{2l-1}, \ldots, -\tilde{u}_{p+q}, \tilde{u}_{p+q-1})$ where $p+q$ is even and $F = R$, $C$, or $H$. If we respectively think of $R^{p+q}$ as $C^{(p+q)/2}$ and $C^{p+q}$ as $H^{(p+q)/2}$ (with $(z_1, z_2)$ corresponding to $z_1 + z_2j$), then we observe that $a$ is respectively the linear transformation induced through multiplication by the imaginary number $i$ when $F = R$ and the conjugate-linear transformation induced by left-multiplication by the quaternion $j$ when $F = C$. We use $a$ to obtain a map of the Stiefel manifolds $a_V: FV_{p,q} \to FV_{p,q}$ with $a_V(v_1, \ldots, v_p) = (av_1, \ldots, av_p)$ when $F = R$ or $C$. We must check that $a_V$ is well-defined,
i.e. that \((av_1, \ldots, av_p) \in FV_{p,q}\). But \((av_1 | av_j)\)
\[\sum_{k=1}^{p+q} (av_i)_k (av_j)_k = \sum_{k=1}^{p+q} (\tilde{v_i})_k (\tilde{v_j})_k = (\tilde{v_i} | \tilde{v_j}) = \delta_{i,j},\]
so \(a_v\) is well-defined. (Note that for quaternions
\((v_i | v_j) = \delta_{i,j} \iff (\tilde{v}_i | \tilde{v}_j) = \delta_{i,j}\). Thus \(a_H: H^{p+q} \to H^{p+q}\)
does not induce a map \(a_V:HV_{p,q} \to HV_{p,q}\).) Clearly, \(a\) is
a homeomorphism, and \(FV_{p,q} \subseteq (F^{p+q})^p\), so \(a_v\) is a homeo-
morphism also. Next we use \(a\) to induce a map
\[A^d_{p,q}: FG_{p,q} \to FG_{p,q}\]
by \(A^d_{p,q}(\langle v_1, \ldots, v_p \rangle) = \langle av_1, \ldots, av_p \rangle\).
Again, we must check that \(A^d_{p,q}\) is well-defined, i.e.
\[\langle v_1, \ldots, v_p \rangle = \langle w_1, \ldots, w_p \rangle \implies \langle av_1, \ldots, av_p \rangle = \langle aw_1, \ldots, aw_p \rangle; \text{ we have } x = \sum_{i=1}^{p} a_i (av_i) \iff ax \]
\[= \sum_{i=1}^{p} \tilde{a_i} v_i = -\sum_{i=1}^{p} \tilde{b_i} w_i \iff x = \sum_{i=1}^{p} b_i (aw_i). \] By definition
of \(A^d_{p,q}\), it is clear that the following diagram of set
maps is commutative:

\[
\begin{array}{cccc}
FV_{p,q} & \xrightarrow{a_V} & FV_{p,q} \\
\pi^d_{p,q} \downarrow & & \downarrow \pi^d_{p,q} \\
FG_{p,q} & \xrightarrow{A^d_{p,q}} & FG_{p,q}
\end{array}
\]

If \(U\) is open in \(FG_{p,q}\), then \(a_V^{-1}(\pi^{-1}(U))\) is open, but
\(a_V^{-1}((\pi^d_{p,q})^{-1}(U)) = (\pi^d_{p,q})^{-1}(A^d_{p,q})^{-1}(U))\), so since \(FG_{p,q}\)
has topology induced by \(\pi^d_{p,q}\), \((A^d_{p,q})^{-1}(U)\) is open and
\(A^d_{p,q}\) is continuous. Further, \(a^2 = -v\) implies \((A^d_{p,q})^2\)
is the identity on $FG_{p,q}$, so $A^d_{p,q}$ is a homeomorphism.

Finally, we must show that $A^d_{p,q}$ is fixed point free for $p$ and $q$ odd.

Proposition 2.1. $A^d_{p,q}:FG_{p,q} \to FG_{p,q}$ is defined for $F = \mathbb{R}$ or $\mathbb{C}$ and $p+q$ even and is fixed point free when $p$ and $q$ are both odd.

Proof: Only the fixed point free part remains to be shown. When $F = \mathbb{R}$ the fact that $\mathbb{R}$ is not algebraically closed allows us to give a simple proof by using techniques of linear algebra. In this case, the matrix of the linear transformation $a$ is the direct sum of matrices of the form $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From this it follows that the characteristic equation of $a$ is $(\lambda^2+1)(p+q)/2 = 0$, which is seen to have no real roots. Thus $a$ has no eigenvectors. Now suppose $W$ is an odd-dimensional linear subspace of $\mathbb{R}^{p+q}$ such that $a(W) = W$. Then we have a linear map $a_{W}: W \to W$ defined by restricting both the domain and range of $a$. Now since $W$ is odd-dimensional, $a_{W}$ must have at least one eigenvalue $\lambda \neq 0$ with corresponding eigenvector $x$. But this means $a_{W}(x) = \lambda x = a(x)$, so $x$ would also be an eigenvector of $a$, a contradiction. Thus if $W$ is an odd-dimensional linear subspace of $\mathbb{R}^{p+q}$ we must have $a(W) \neq W$. Hence, when $p$ and $q$ are odd, we have $A^1_{p,q}(W) = a(W) \neq W$, for any $W \in R_{FG}$, and $A^1_{p,q}$ is fixed point free, q. e. d.
(for $F = R$).

When $F = \mathbb{C}$ we need a different proof. We remark that the following argument applies equally well when $F = R$. We will use the representation of $FG_{p,q}$ as a set of matrices which was given in Proposition 1.4. If we refer to the matrix for an element $\langle v_1, \ldots, v_p \rangle \in FG_{p,q}$, it was seen that for any such element we can define unambiguously a set of lengths $(r_1, \ldots, r_p)$ where $r_i$ is the length of the $i$-th row vector in the sense that $r_i$ is the number of the last coordinate whose value is non-zero. Note that $r_1 < r_2 < \cdots < r_p$. We will say that two consecutive lengths $r_i$ and $r_{i+1}$ are adjacent if $r_{i+1} = r_i + 1$ and $r_i$ is odd. Let $(\tilde{r}_1, \ldots, \tilde{r}_p)$ be the set of lengths for $A_{p,q}^d(\langle v_1, \ldots, v_p \rangle)$ where we will assume that $v_1, \ldots, v_p$ are the row vectors of the canonical matrix for $\langle v_1, \ldots, v_p \rangle$ and $\tilde{v}_1, \ldots, \tilde{v}_p$ those for $A_{p,q}^d(\langle v_1, \ldots, v_p \rangle)$. From the definition of $A_{p,q}^d$ it is seen that the canonical matrix of $A_{p,q}^d(\langle v_1, \ldots, v_p \rangle)$ is obtained from that of $\langle v_1, \ldots, v_p \rangle$ by first conjugating, next negating each even column, then interchanging each even column with the preceding column, and lastly reducing this pre-reduced matrix to canonical echelon form. Looking at the process of reducing the pre-reduced matrix we will see that $\tilde{r}_i$ differs from $r_i$ by at most 1. Starting with the bottom
row, if \( r_p \) is non-adjacent to \( r_{p-1} \), then \( \tilde{r}_p = r_{p+1} \) and the \( p \)-th row is unchanged when \( r_p \) is odd, while when \( r_p \) is even the new \( p \)-th row is negated if originally the component preceding the \( r_p \)-component was 0 (so \( \tilde{r}_p = r_{p-1} \)) and is divided by this preceding component to make it equal to 1 if it was not zero (so \( \tilde{r}_p = r_p \)). If \( r_p \) is adjacent to \( r_{p-1} \) we simply interchange the \( p \)-th and \((p-1)\)-th rows, negating the new \((p-1)\)-th row. We continue upward in this manner until we finish the first row. To put this matrix in reduced echelon form we need only to do row operations to have zeroes under the last non-zero component of each row. Since \( p \) is odd, there must be at least one \( r_i \) which is non-adjacent to \( r_{i+1} \). For such an \( r_i \) which is odd the above process produces \( \tilde{r}_i = r_{i+1} \neq r_i \). But since the reduced echelon matrix corresponds uniquely to \( \langle v_1, \ldots, v_p \rangle \) it follows from \( \tilde{r}_i \neq r_i \) that 

\[
A_{p,q}^d(\langle v_1, \ldots, v_p \rangle) \neq \langle v_1, \ldots, v_p \rangle.
\]

If such an \( r_1 \) is even, the \( i \)-th row of the matrix we started with looks like \((*, \ldots, *, z, 1, 0, \ldots, 0)\), where 1 is the \( i \)-th component. The \( i \)-th row of the image matrix looks like \((*, \ldots, *, -1, z, 0, \ldots, 0)\), so if \( z = 0 \) we have \( \tilde{r}_i = r_{i-1} \) and 

\[
A_{p,q}^d(\langle v_1, \ldots, v_p \rangle) \neq \langle v_1, \ldots, v_p \rangle.
\]

But if \( z \neq 0 \) we must multiply this row by \( 1/z \) to put the matrix in echelon form, so the \( i \)-th row of the reduced matrix will
be \((*,\ldots,*,1/\tilde{z},1,0,\ldots,0)\) which is seen to differ from \((*,\ldots,*,z,1,0,\ldots,0)\). For if they were equal we would have \(z = -1/\tilde{z}\), contradicting \(z\tilde{z} > 0\), for any \(0 \neq z \in \mathbb{F}\). Hence the \(i\)-th rows of the matrix being different gives \(A_{p,q}^d(\langle v_1,\ldots,v_p \rangle) \neq \langle v_1,\ldots,v_p \rangle\). Thus for any \(\langle v_1,\ldots,v_p \rangle \in \mathbb{F}_{p,q}^G\) we have shown \(A_{p,q}^d(\langle v_1,\ldots,v_p \rangle) \neq \langle v_1,\ldots,v_p \rangle\), so \(A\) is fixed point free, q.e.d.

Another proof of Proposition 2.1 using the properties of symmetric spaces is given by Wolf([9], page 304) for the case \(\mathbb{F} = \mathbb{C}\).

When \(p = q\) define the orthogonal complementation map \(\perp : \mathbb{F}_{p,p} \rightarrow \mathbb{F}_{p,p}\) by \(\perp(\langle v_1,\ldots,v_p \rangle) = \langle v_1,\ldots,v_p \rangle^\perp\)

where \(\langle v_1,\ldots,v_p \rangle^\perp\) is the orthogonal complement in \(\mathbb{F}^{2p}\) of \(\langle v_1,\ldots,v_p \rangle\), \(\mathbb{F} = \mathbb{R}, \mathbb{C}\) or \(\mathbb{H}\). In this case, \(\langle v_1,\ldots,v_p \rangle^\perp\) is also a \(p\)-plane which is uniquely determined by \(\langle v_1,\ldots,v_p \rangle\), so \(\perp\) is well-defined and continuous. It is then immediate that:

**Proposition 2.2.** The map \(\perp : \mathbb{F}_{p,p} \rightarrow \mathbb{F}_{p,p}\) is fixed point free for all \(p \geq 1\) and \(\mathbb{F} = \mathbb{R}, \mathbb{C}\) or \(\mathbb{H}\).
3. THE LEFSCHETZ FIXED POINT THEOREM FOR MANIFOLDS

In order to show that a space has the fixed point property we must show that every self-map of this space has a fixed point. The most usual way to do this is to apply the Lefschetz fixed point theorem. In its most general form, as given by Brown([3], page 42), the Lefschetz theorem applies to compact absolute neighborhood retracts. In this form the theorem is proved first for simplicial complexes and then for compact ANR's since they can be approximated by simplicial complexes. But we are interested only in manifolds. In this case, following Greenberg([5], section 30), we will prove the Lefschetz theorem as a consequence of the properties of the fundamental class and the Thom class of a manifold. Throughout this section our (co-)homology will have coefficients in a principal ideal domain, \( R \).

The slant product will play a major role in our discussion so we begin with a review of the properties of the slant product. Recall that the slant product is the bilinear pairing of a cohomology class \( \gamma \in H^{p+q}(X \times Y) \) with a homology class \( \sigma \in H_p(X) \) to the cohomology class denoted \( \gamma/\sigma \in H^q(Y) \) such that \( [\theta, \gamma/\sigma] = [\sigma x \theta, \gamma] \), for any \( \theta \in H_q(Y) \) (\( [\ , \ ] \) denotes the Kronecker product).
Note that for $\gamma = 1_{XXY} \in H^0(XXY)$ and any generator $\sigma_i \in H_0(X)$, for any generator $\theta_j$ of $H_0(Y)$ we have $[\theta_j, 1/\sigma_i] = [\sigma_i \times \theta_j, 1] = 1 \in R$. From the bilinearity of the slant product we can thus conclude:

Lemma 3.1. For any $\sigma \in H_0(X), 1/\sigma = [\sigma, 1] \gamma \in H^0(Y)$.

We also note that the slant product is related to the various other products as follows:

Lemma 3.2. If $\xi \in H^p(X), \eta \in H^q(Y), \gamma \in H^F(XXY)$, and $\sigma \in H_s(X)$, then $((\xi \times \eta) \cup \gamma)/\sigma = (-1)^p(p+q+r+s)\eta \cup (\gamma/\sigma \cap \xi)$. To illustrate this consider $\theta \in H_{p+q+r-s}(Y)$, then $[\theta, \eta \cup (\gamma/\sigma \cap \xi)] = [\theta \cap \eta, \gamma/\sigma \cap \xi]$, since cap is adjoint to cup,

$= [(\theta \cap \eta) \times (\sigma \cap \xi), \gamma], \text{ by definition of the slant product,}$

$= (-1)^p(p+q+r-s)[(\sigma \times \theta) \cap (\xi \times \eta), \gamma], \text{ by the relation of cap product to cross product,}$

$= (-1)^p(p+q+r+s)[\sigma \times \theta, (\xi \times \eta) \cup \gamma], \text{ by cap adjoint to cup,}$

$= (-1)^p(p+q+r+s)[\theta, ((\xi \times \eta) \cup \gamma)/\sigma], \text{ by definition of slant.}$

Lastly, we note that slant is functorial, that is:

Lemma 3.3. For maps $f: X \to X', g: Y \to Y'$, and for $\gamma' \in H^{p+q}(XX'), \theta \in H_p(Y)$, we have $H^{p+q}(fx \cap g)(\gamma')/\theta = H^q(f)(\gamma'/H_p(g)(\theta))$. 

Let $M$ be an $n$-dimensional compact connected $\mathbb{R}$-orientable manifold. We recall that $H_n(M) \cong \mathbb{R}$ and that the fundamental class of $M$ is a generator $\zeta_M$ of $H_n(M)$, so we have $H_n(M) = \mathbb{R} \zeta_M$. Letting $j_x : (M, \phi) \hookrightarrow (M, M-x)$ be the inclusion, where $x \in M$, we also know that $H_n(j_x) (\zeta_M) = \zeta_x$ is a generator of $H_n(M, M-x) = \mathbb{R}$. Hence $\zeta_x = \zeta_M \mod(M-x)$ provides a local orientation of $M$ at $x$, while $\zeta_M$ thus gives a global orientation of $M$. In addition we know that $\zeta_M$ furnishes the Poincaré duality isomorphism between the homology and cohomology of $M$ by means of the cap product, i.e. $H^q(M) \cong H_{n-q}(M)$ under $z \mapsto \zeta_M \cap z$.

Let $\theta_M$ denote the Thom class of the manifold $M$. Letting $i_x : (M, M-x) \hookrightarrow (M \times M, M \times M - \Delta)$ be given by $y \mapsto (y, x)$, we recall that $\theta_M$ is the unique element of $H^n(M \times M, M \times M - \Delta)$ possessing the following property: for any $x \in M$, $H^n(i_x)(\theta_M) = \theta_x$ is that generator of $H^n(M, M-x) = \mathbb{R}$ such that $[\zeta_x, \theta_x] = 1$. To be more precise we should say that $\theta_M$ is the Thom class of the $\mathbb{R}$-orientation $\zeta_M$ which was originally chosen. It will be appreciated as we proceed that the Thom class is related to the product $M \times M$ because it is involved with various (co-)homological products. For instance, letting $j_M : (M \times M, \phi) \hookrightarrow (M \times M, M \times M - \Delta)$ be the inclusion and $H^n(j_M)(\theta_M) = \theta_M^\ast$, it is known that $\theta_M^\ast / \zeta_M = 1$. Thus $\theta_M^\ast$
provides the inverse of the Poincaré duality isomorphism $H^qM \cong H_{n-q}M$ (given above by means of the slant product) under $\sigma \mapsto (-1)^q \theta^*_M / \sigma$ where $\sigma \in H_{n-q}M$.

Next, let us relate our knowledge of the fundamental and Thom classes to mappings of manifolds. Let $M_1$ and $M_2$ be $n$-dimensional compact connected R-orientable manifolds. For any map $f: M_1 \to M_2$, by means of the map $f \times \text{id}: M_1 \times M_2 \to M_2 \times M_2$ we define the cohomology class of the graph of $f$ to be $\theta_f = H^n(f \times \text{id})(\theta^*_M)$ in $H^n(M_1 \times M_2)$. The following proposition tells us that $\theta_f$ determines $H^k(f)$:

**Lemma 3.4.** $\forall z \in H^qM_2, H^q(f)(z) = (-1)^q \theta_f / \mathcal{Z}_{M_2} \cap z$

**Proof:** $\theta_f / \mathcal{Z}_{M_2} \cap z = h^n(f \times \text{id})(\theta^*_M) / \mathcal{Z}_{M_2} \cap z$, by definition of $\theta_f$

$= h^n(f)(\theta^*_M) / h^n(\text{id})(\mathcal{Z}_{M_2} \cap z)$, by the functoriality of the slant product

$= h^n(f) / \mathcal{Z}_{M_2} \cap z$,

$= h^n(f)(-1)^q z$, since slant product by $\theta^*_M$ is the inverse of the Poincaré duality isomorphism.

Letting $M = M_1 = M_2$, this allows us to obtain a condition for $f$ to have a fixed point:
Proposition 3.5. If $\theta_f \neq 0$, then $f$ has a fixed point.

Proof: This proposition can be considered to be a form of the Lefschetz theorem. Thus, as is usually done in proving the Lefschetz theorem, we suppose that $f$ has no fixed point; so we must show that $\theta_f = 0$. Since $f$ has no fixed point, $f \times \text{id}$ may be factored through the deleted product by $\phi$:

![Diagram](MxM - \Delta \xrightarrow{\phi} MxM, where $i_M$ is the inclusion.]

Considering $MxM - \Delta \xrightarrow{i_M} MxM \xrightarrow{j_M} (MxM, MxM - \Delta)$ we have $H^n(i_M)H^n(j_M) = 0$ from the exact cohomology sequence of $(MxM, MxM - \Delta)$ at $H^n(MxM)$. But $\theta_M^* = H^n(j_M)(\theta_M)$, so $H^n(i_M)(\theta_M^*) = H^n(i_M)H^n(j_M)(\theta_M) = 0$. Thus as $\theta_f = H^n(f \times \text{id})(\theta_M^*)$ by definition, it follows that $\theta_f = H^n(i_M)(\theta_M)(\theta_M^*) = H^n(\phi)H^n(i_M)(\theta_M^*) = H(\phi)(0) = 0$, q.e.d.

From this we wish to obtain a fixed point condition more directly related to the cohomology of $M$ itself. Thus we use the diagonal map $d: M \rightarrow MxM$ given by $x \mapsto (x,x)$, to define the Lefschetz class of $f$, denoted $\lambda_f$, to be $\lambda_f = H^n(d)(\theta_f)$ in $H^n(M)$. Note that if the Kronecker product of the fundamental class of $M$ with the Lefschetz class of $f$ is non-zero, i.e. if
0 \neq [\zeta_M^*,\lambda_f] = [\zeta_M^*,\theta^*(d)(\theta_f)], then we must have \theta_f \neq 0, whence \( f \) has a fixed point by Proposition 3.5. When we express this fact by means of the cohomology of \( M \) we will have the Lefschetz theorem.

Since \( M \) is a compact manifold, the cohomology of \( M \) is finitely generated and for the remainder of this section we will also assume that it has free homology over \( R \). Thus, as \( R \) is a PID, the universal coefficient theorem tells us that the cohomology of \( M \) is a free finitely generated, graded \( R \)-module. Hence we have a basis \( \{\alpha_i : i \in I\} \) for \( H^*M \) such that \( I \) is finite and \( \alpha_i \in H^qM \) for \( 0 \leq q \leq n \). Also, since \( H^*(f) \) is a graded map of free \( R \)-modules, we can speak of the trace of \( H^*(f) \) in each degree. We use this to define the Lefschetz number of \( f, L(f) \), to be the alternating sum of the traces:

\[ L(f) = \sum_{q=0}^{n} (-1)^q \text{Trace} H^q(f). \]

We are now ready for the main result of this section:

**Corollary 3.6 (The Lefschetz fixed point Theorem for manifolds).** If \( L(f) \neq 0 \), then \( f \) has a fixed point.

**Proof:** By the remarks of the preceding paragraph, it is sufficient to show that \( L(f) = (-1)^n [\zeta_M^*,\lambda_f] \). It will be seen that this follows because \( \lambda_f \) is defined by means of \( \theta_M^* \), and \( \theta_M^* \) determines the inverse to the Poincaré duality isomorphism, which is in turn induced by \( \zeta_M^* \). To proceed we need to express \( \theta_M^* \) in terms of a basis for
$H^*(M \times M)$, but since $R$ is PID and we have free cohomology the Künneth formula tells us that $\{\alpha_i x \alpha_j : (i,j) \in I^2\}$ is such a basis. We write $\theta^*_M = \sum_{i,j} \alpha_i x \alpha_j \in H^n(M \times M)$, where $c_{ij} \in R$ and as $\deg(\alpha_i x \alpha_j) = \deg \alpha_i + \deg \alpha_j$, $c_{ij} = 0$ if $\deg \alpha_i + \deg \alpha_j \neq n$. We also need to express the Kronecker product in terms of elements of $H^*_M \times H^*_M$.

Noting that Poincaré duality tells us that $\zeta_M \land \alpha_i, i \in I$ is a basis for $H^*_M$, we have $[\zeta_M \land \alpha_i, \alpha_j] = y_{ij}$, where $y_{ij} = 0$ if $\deg \alpha_i + \deg \alpha_j \neq n$ as the Kronecker product is zero for elements of different degree. As $[\zeta_M \land \alpha_i, \alpha_j] = [\zeta_M, \alpha_i \cup \alpha_j]$ by definition of the cap product, the anticommutativity of the cup product gives us $y_{ji} = (-1)^{\deg \alpha_i(n-\deg \alpha_j)} y_{ij}$. Using this in conjunction with $\theta^*_M / \zeta_M \land$ we obtain:

$$(-1)^{\deg \alpha_k} \alpha_k = \theta^*_M / \zeta_M \land \alpha_k$$

$$= (\sum c_{ij} \alpha_i x \alpha_j) / \zeta_M \land \alpha_k,$$

expressing $\theta^*_M$ in terms of basis elements,

$$= (\sum c_{ij} \alpha_i x \alpha_j) \lambda_M \lambda M / \zeta_M \land \alpha_k,$$

$$= \sum c_{ij} \alpha_j \cup (1/(\zeta_M \land \alpha_k) \land \alpha_i)(-1)^{\deg \alpha_i \deg \alpha_k},$$

by 3.2,

$$= \sum c_{ij} \alpha_j \cup (1/\zeta_M \land (\alpha_k \cup \alpha_i))(1)^{\deg \alpha_i \deg \alpha_k},$$

since the cap product is adjoint to the cup product.
by 3.1,

\[ \Sigma c_{ij} a_j \] \[ C_M \cap (x_k U a_i), 1_M) l_M(-1) \]

\[ \deg a_i \deg a_k, \]

by definition of the cap product,

\[ (n-\deg a_k) \deg a_k. \]

But since the \( a_i \)'s are a basis we may conclude that

\[ \Sigma y_{ki} c_{ij} = (-1)^{n \deg a_k} \delta_{jk}. \]

Let \( s_{ij} = (-1)^{n \deg a_i} \delta_{ij} \), then we have \( N \times N \) matrices

\[ Y = (y_{ij}), C = (c_{ij}), S = (s_{ij}), \] where \( N = \dim_R H^\ast M \). The latter equation may then be written \( YC = S \). If \( n \) is even, \( S \) is the identity matrix, so by linear algebra \( CY = S \). If \( n \) is odd, then \( Y \) is easily seen to be

symmetric, so on multiplying we find \( YS = -SY \). From this, \( YCY = SY = -YS \) gives \( CY = -S \). In sum, \( CY = (-1)^n S \).

Finally, to compute \( [C_M \lambda_f] \), we let

\[ H^\ast(f)(x_i) = \Sigma_{k \in I} a_k x_i, \] where \( a_{ki} = 0 \) if \( \deg x_i \neq \deg x_k \) since \( H^\ast(f) \) is a graded map. Thus,

\[ [C_M \lambda_f] = [C_M, H^n(d)(\theta_f)] = [C_M, H^n(d)H^n(f \times \text{id})(\theta_M^n)] \]

\[ = [C_M, H^n(d)H^n(f \times \text{id})(\Sigma c_{ij} a_i \times a_j)] \]

\[ = [C_M, H^n(d)(\Sigma c_{ij} H^{\deg a_i}(f)(a_i) \times a_j)], \] by 3.3,

\[ = [C_M, \Sigma c_{ij} H^{\deg x_i(f)(a_i) \cup a_j}], \] since the diagonal carries cross product to cup product,
\[ \begin{align*}
&= [\zeta_1^i, \zeta_2^j a_k a_l \cup a_j] \\
&= \Sigma(c_{ij} a_k \Sigma_{M} a_k \cup a_j) \\
&= \Sigma c_{ij} a_k y_{kj} \\
&= \Sigma a_k (\Sigma c_{ij} y_{kj}), \text{ summing over } j \text{ first,} \\
&\quad \deg a_k (n-\deg a_k) \\
&= \Sigma a_k (-1)^n (\Sigma c_{ij} y_{jk}), \text{ by the} \\
&\quad \text{anti-symmetry of the } y_{ij}'s, \\
&\quad \deg a_k (n-\deg a_k) \\
&= \Sigma a_k (-1)^n s_{ik}, \text{ as} \\
&\quad C Y = (-1)^n S, \\
&= (-1)^n \Sigma a_k \delta_{ik} (-1) \\
&\quad \deg a_k (n-\deg a_k) \\
&= (-1)^n \Sigma (-1)^n a_k \delta_{ik} \\
&\quad \deg a_k a_k \\
&= (-1)^n \Sigma (-1)^n a_{kk} \\
&\quad \Sigma_{q=0}^n (-1)^q \deg a_k = q a_{kk} \\
&= (-1)^n \Sigma_{q=0}^n (-1)^q \text{Trace } H^q(f) = (-1)^n L(f), \text{ q.e.d.}
\end{align*} \]

In applications of the Lefschetz theorem to the fixed point property, one needs some sort of universal restrictions on the form of \( H^k(f) \). It is then shown that \( H^k(f) \) having the given form is sufficient to imply \( L(f) \neq 0 \). This being true for any map \( f \), the fixed point property follows.
4. THE FIXED POINT PROPERTY FOR $\mathbb{C}G_2,q$ AND $\mathbb{H}G_2,q$

In this section we prove:

**Theorem 4.1.** $\mathbb{F}G_2,q \in FPP$, for all $q > 2$ and $F = \mathbb{C}$ or $\mathbb{H}$.

This is done by showing that the relations among the generators in the cohomology algebra of $\mathbb{F}G_2,q$ sharply restrict the ring of endomorphisms of this algebra, whence also the Lefschetz numbers of these endomorphisms. To wit, we first prove Theorem 4.2 which concerns endomorphisms of $H^*\mathbb{F}G_2,q$ (we will use integral (co-)homology throughout this section):

**Theorem 4.2.** For any endomorphism $f$ of $H^*\mathbb{F}G_2,q$, if $f(c_1) = \alpha c_1$, $\alpha \in \mathbb{Z}$, then $f(c_2) = \alpha^2 c_2$ ($q > 2$).

Granting Theorem 4.2, we can quickly show:

**Corollary 4.3.** $L(f) = \sum_{k=0}^{2q}(\dim H^k\mathbb{F}G_2,q)\alpha^k$. ($d = 2$ respectively 4, as $F = \mathbb{C}$ respectively $\mathbb{H}$.)

**Proof:** $H^d\mathbb{F}G_2,q$ has the basis $\left\{c_1^{r_1}c_2^{r_2} \mid r_1 + 2r_2 = k \text{ and } r_1 + r_2 \leq q \right\}$. Thus, for any basis element $c_1^{r_1}c_2^{r_2}$ of $H^d\mathbb{F}G_2,q$, we have $f(c_1^{r_1}c_2^{r_2}) = (\alpha c_1^{r_1})(\alpha^2 c_2^{r_2}) = \alpha^k c_1^{r_1}c_2^{r_2}$, by Theorem 4.2. As $H^p\mathbb{F}G_2,q = 0$ if $p \notin dk$ or $p > 2dq$, the corollary follows by definition of $L(f)$, q.e.d.
Theorem 4.1 now follows easily from Corollary 4.3:

**Proof of Theorem 4.1.** From Corollary 4.3 it is clear that \( L(f) \equiv 1 \pmod{\alpha} \). Thus \( L(f) \neq 0 \), unless possibly in the case \( \alpha = \pm 1 \). When \( \alpha = 1 \), \( L(f) > 0 \), so the only case left is \( \alpha = -1 \). But in this case,  
\[
L(f) = \sum_{k=0}^{2q} (-1)^k \dim H^k_{FG_2,q} = \sum_{k=0}^{2q} (-1)^k \beta_k(RG_2,q)
\]
well-known that \( \gamma(RG_2,q) \), since \( \beta_k(RG_2,q) = \beta_{dk}(FG_2,q) \). Now it is well-known that \( \gamma(RG_2,q) > 0 \), for \( q \neq 2 \) (see Wolf[9], page 303) so \( L(f) \neq 0 \), for any endomorphism \( f \) of \( H^*_{FG_2,q} \), whence by the Lefschetz theorem we are done, \( \text{q.e.d.} \)

We turn now to the proof of Theorem 4.2 and list some facts and notations which will be used in the course of the proof.

(a) \( \mathbb{N} = \{0,1,2,\ldots\} \).

(b) \( \mathbb{N}_q^p = \{r \in \mathbb{N}^p \mid \sum_{i=1}^p r_i = q \} \) = "\( p \)-tuples of weight \( q \),

(c) \( |r| = \sum_{i=1}^p r_i, \ \forall r \in \mathbb{Z}_p. \)

(d) \( M(r) = \begin{cases} 
|r|! \div \prod_{i=1}^p r_i! & \text{if } r \in \mathbb{N}^p, \\
0 & \text{otherwise}, 
\end{cases} \)

= "the multinomial coefficient", \( \forall r \in \mathbb{Z}_p^p \)

(e) \( e_i \in \mathbb{N}^p \) is such that \( (e_i)_k = \delta_{i,k} \)

(f) In the usual way, we say \( \mathbb{N}^0 = \{(0)\} \leq \mathbb{N}^1 \leq \mathbb{N}^2 \leq \mathbb{N}^3 \leq \ldots \), so \( e_2 \in \mathbb{N}^i \), for instance.

Recall the multinomial identity (Pascal's triangle for \( p = 2 \)).
Proposition 4.4. \( \forall r \in \mathbb{N}_p, M(r) = \sum_{i=1}^{p} M(r-e_i) \), where \( r-e_i \) indicates the usual vector addition.

Also recall the structure of the integral cohomology of the Grassmann manifolds.

Proposition 1.5. \( H^*_{FG_p,q} = \mathbb{Z}[c_1, \ldots, c_p, \tilde{c}_1, \ldots, \tilde{c}_q] / (cc = 1) \), where \( c = \sum_{j=0}^{\infty} c_j \), \( \tilde{c} = \sum_{j=0}^{\infty} \tilde{c}_j \) and \( c_0 = \tilde{c}_0 = 1, c_i = \tilde{c}_j = 0 \)

if \( i > p, j > q, i < 0 \) or \( j < 0 \). (\( F = \mathbb{C} \) or \( H, q > p. \))

We note that since the cohomology algebra is graded by weight, for each \( k > 0, cc = 1 \) implies that in degree \( k \) we have \( \sum_{i=0}^{k} c_i \tilde{c}_{k-i} = 0 \). We can use these relations to eliminate the \( \tilde{c}_j \)'s in \( H^*_{FG_p,q} \) as follows:

Corollary 4.5. \( \tilde{c}_k = \sum_{r \in \mathbb{N}_p} (-1)^{|r|} M(r) c^r, \ k > 0, \)

where \( c^r = \prod_{i=1}^{p} c_i^{r_i} \).

Proof: By induction on \( k \).

\( k = 1 \): In degree 1 of \( cc = 1, c_1 + \tilde{c}_1 = 0 \), or as \( \mathbb{N}_1 = \{e_1\}, \tilde{c}_1 = -c_1 = \sum_{r \in \mathbb{N}_1} (-1)^{|r|} M(r) c^r \).

\( k > 1 \): \( \tilde{c}_k = -\sum_{i=1}^{p} c_i \tilde{c}_{k-i} \), in degree \( k \) of \( cc = 1 (\tilde{c}_{-i} = 0) \),

\( = -\sum_{i=1}^{p} c_i (\sum_{r \in \mathbb{N}_{k-i}} (-1)^{|r|} M(r) c^r), \) by the induction hypothesis (Note
\[ N_0 = \{(0)\}, \Sigma \phi = 0 \).
\[ \sum_{i=1}^{p} \sum_{r \in \mathbb{N}_{k-1}} (-1)^{1+|r|} M(r) c^{r+e_i}, \text{ by multiplication,} \]
\[ = \sum_{i=1}^{p} \sum_{r-e_i \in \mathbb{N}_{k-1}} (-1)^{|r|} M(r-e_i) c^r, \text{ by change of index } r, \text{ and so now } r \in \mathbb{N}_k, \]
\[ = \sum_{r \in \mathbb{N}_k} (-1)^{|r|} \left[ \sum_{i=1}^{p} M(r-e_i) c^r \right], \text{ summing over } i \text{ first}, \]
\[ = \sum_{r \in \mathbb{N}_k} (-1)^{|r|} M(r) c^r, \text{ by the multinomial identity, q.e.d.} \]

When \( 1 \leq k \leq q \), the corollary simply tells us how to express \( \tilde{c}_k \) in terms of the \( c_i \)'s. When \( q+1 \leq k \leq p+q \), we get a set of relations which determine the cohomology algebra, i.e.,
\[ H^*_{FG_{p,q}} = \mathbb{Z}[c_1, \ldots, c_p]/(\sum_{r \in \mathbb{N}_{q+k}} (-1)^{|r|} M(r) c^r = 0)_{k=1}^p. \]

In \( H^*_{FG_{2,q}} \), we can use this to express any monomial in terms of additive basis elements as follows:

**Lemma 4.6.** In \( H^{d(q+k)}_{FG_{2,q}} \),
\[ \sum_{r \in \mathbb{N}_{q+k}} (-1)^{|r|} M(r-v) c^r = 0, \]
\[ \forall v \in \mathbb{N}^2 \] such that \( |v| = k-1, (1 \leq k \leq q) \).

**Proof:** By induction on \( k \).

For \( k = 1 \) as \( |v| = k-1 = 0 \), in this case we must have \( v = (0,0) \) and need only to show
\[ \sum_{r \in N_{q+1}} (-1)^{|r|} M(r)c^r = 0. \] But this is precisely the same as \( \tilde{c}_{q+1} = 0 \) which was shown in the preceding corollary.

**k = 2** Here \(|v| = 1\), so \( v = e_1 \) or \( v = e_2 \).

From the cases \( \tilde{c}_{q+1} = \tilde{c}_{q+2} = 0 \) of the preceding corollary we have

\[
c_1 \tilde{c}_{q+1} = c_1 \sum_{r \in N_{q+1}} (-1)^{|r|} M(r)c^r
= \sum_{r \in N_{q+1}} (-1)^{|r|+|e_1|} M(r)e_1c^r
= \sum_{r \in N_{q+2}} (-1)^{|r|} M(r-e_1)c^r = 0
\]

(Note that if \((0, \frac{q+2}{2}) = r_0 \in N_{q+2}\), then \(M(r_0-e_1) = 0\), so we are allowed to change from \(N_{q+1}\) to \(N_{q+2}\) as done.) And

\[
\tilde{c}_{q+2} - c_1 \tilde{c}_{q+1}
= \sum_{r \in N_{q+2}} (-1)^{|r|} M(r)c^r - \sum_{r \in N_{q+2}} (-1)^{|r|} M(r-e_1)c^r
= \sum_{r \in N_{q+2}} (-1)^{|r|} [M(r) - M(r-e_1)]c^r
= \sum_{r \in N_{q+2}} (-1)^{|r|} M(r-e_2)c^r = 0, \text{ by the multinomial identity, and we are done when } k = 2.
\]

**k > 2** True for \( k-1 \) by induction, so \( \forall w \in N^2 \) such
that \( w = k-2 \) we have
\[
\sum_{r \in \mathbb{N}_{q+k-1}} c_1 \left\langle -1 \right\rangle^r M(r-w)c^r = 0
\]
which covers all \( v \) such that \(|v| = k-1 = |w+e_1|\) when \( v = w+e_1 \) and to finish in this case we need only to prove for \( v = (k-1)e_2 \). For this we use true for \( k-2 \) by induction, so
\[
\sum_{r \in \mathbb{N}_{q+k-2}} c_2 (-1)^r M(r-(k-3)e_2)c^r = 0,
\]
combining this with \( v = e_1+(k-2)e_2 \) which is already done in this case, we have:
\[
\sum_{r \in \mathbb{N}_{q+k}} (-1)^r \left[ M(r-(k-2)e_2)-M(r-e_1-(k-2)e_2) \right]c^r = 0, \text{ q.e.d.}
\]

Recall once more that \( H^*FG_{2,q} \) has an additive basis of all elements \( c^r \) such that \(|r| \leq q\). So for any \( r \in \mathbb{N}_{q+k} \) such that \(|r| > q\), we must have \( r_2 \leq k-1 \). For if \( r_2 > k \), we would have
\[
q+k = r_1+2r_2 \geq r_1+r_2+k = |r|+k > q+k, \text{ a contradiction.}
\]
Thus if we let \( v = (k-1-r_2)e_1+r_2e_2 \), it is seen that
\( v \in \mathbb{N}^2 \) and \( |v| = k-1 \). So by the lemma,
\[
\Sigma_{r \in \mathbb{N}^2_{q+k}} (-1)^{|r|} M(r-v) c^r = 0,
\]
and we note that in this sum the coefficient of \( c^r \) is
\[
(-1)^{|r|} M(r-v) = (-1)^{|r|} M(n,0) = (-1)^{|r|},
\]
while any monomial \( c^r \) with \( r_2 < r_2 \) has coefficient \( M(r-v) = 0 \). Hence in \( H^d(q+k)_{FG_2,q} \) we have \( k \) equations such that the terms \( c_1^{q+k}, c_1^{q+k-2}c_2, \ldots, c_1^{q-k+2}c_2^{k-1} \) occur with coefficients \( \pm 1 \), while in the corresponding equation no term occurs with a higher power of \( c_1 \) than the indicated term. In other words, we have a set of equations in echelon form which are sufficient to solve for any element in terms of basis elements. The solution to these equations in the highest dimension of \( H^d_{FG_2,q} \) as given in the following lemma is the key to the proof of Theorem 4.2:

**Lemma 4.7.** In \( H^d_{FG_2,q} \),
\[
c_1^{2n} c_2^{q-n} = \frac{1}{n+1} M(n,n)c_2^q,
\]
\( 1 \leq n \leq q \).

**Proof:** By induction.

\( n = 1 \) From the previous lemma,
\[
\Sigma_{r \in \mathbb{N}^2_{2q}} (-1)^{|r|} M(r-(q-1)e_2) c^r = 0.
\]
Now since \( r \in \mathbb{N}^2_{2q} \Rightarrow r_2 \leq q \) and
\( r_2 < q-1 \Rightarrow M(r-(q-1)e_2) = 0 \) we must have \( q-1 \leq r_2 \leq q \) in the above sum, or
\[
(-1)^{2+q-1} M((2,q-1)-(0,q-1))c_1^2 c_2^{q-1} + (-1)^q M((0,q) - (0,q-1))c_2^q = 0 \text{ or } c_1^2 c_2^{q-1} = c_2^q \text{ and this} \]
case is done.

\( n > 1 \) Again from the previous lemma,
\[
\sum_{r \in \mathbb{N}_2^q} (-1)^r M(r-(n-1)e_1-(q-n)e_2)c^r = 0.
\]
As before \( q-n \leq r \) and \( n-1 \leq r \) and since
\[2r = 2q-r_1 \leq 2q-n+1 \] we have
\[q-n \leq r \leq q-\frac{n-1}{2}, \]
so we may rewrite the above sum in the form
\[
\sum_{q-n \leq r \leq q-\frac{n-1}{2}} (-1)^r M(2q-2r-n+1,r-q+n)c_1^2(q-r_2)r_2 = 0, \] or
\[
\sum_{0 \leq s \leq \frac{n+1}{2}} (-1)^s M(n-2s+1,s)c_1^{2(n-s)}c_2^{q-n+s} = 0,
\]
by index change, or \( c_1^{2n}c_2^{q-n} \)
\[= \sum_{1 \leq s \leq \frac{n+1}{2}} (-1)^s M(n-2s+1,s) \cdot \frac{M(n-s,n-s)}{n-s+1}c_2^{q-n+s} \]
by induction so we must show
\[\frac{1}{n+1} M(n,n) = \sum_{1 \leq s \leq \frac{n+1}{2}} (-1)^s M(n-2s+1,s) \cdot \frac{M(n-s,n-s)}{n-s+1}
\]
which is the content of the following technical lemma.

**Lemma 4.8.** For \( n \geq 1 \), \( D_n = \frac{M(n,n)}{n+1} \), where

\[D_n = \sum_{1 \leq s \leq \frac{n+1}{2}} (-1)^{s+1} M(n-2s+1,s) \cdot \frac{M(n-s,n-s)}{n-s+1}.
\]

**Proof:** By induction on \( n \).

\( n = 1 \text{ or } 2 \). By inspection.

\( n > 2 \). Note that by the induction hypothesis we may
write \( D_n = \sum_{1 \leq s \leq \frac{n-2h+1}{2}} (-1)^{s+1} M(n-2s+1,s) D_{n-s} \);
also, note the fact that for \( t < n \), \( D_t = \frac{4t-2}{t+1} D_{t-1} \). This case will follow by proving that
\[
\sum_{k+1 \leq s \leq h} (-1)^{s+1} M(n-2s+1,s) D_{n-s} = (-1)^{h+1} M(n-2h+1,h) D_{n-h},
\]
where \( h = \left\lfloor \frac{n+1}{2} \right\rfloor \). This will be done by a downwards induction on \( k \).

\( k = h-1 \). So the sum is over \( s = h \) and we need:

\[
(-1)^{h+1} M(n-2h+1,h) D_{n-h} = (-1)^{h-1} M(n-2h+3,h-1) D_{n-h+1} \frac{(n-2h+3)(n-2h+2)}{n(n+1)},
\]
or on using the relation between \( D_t \) and \( D_{t-1} \) noted above and cancelling \( D_{n-h} \),
we need:

\[
M(n-2h+3,h-1) 2(2n-2h+1)(n-2h+3)(n-2h+2) \frac{(n-h+2)}{n(n+1)}
\]

When \( n \) is odd this becomes:

\[
M(0,h) = M(2,h-1) 2(2n-2h+1)(n-2h+3)(n-2h+2) \frac{(n-h+2)}{n(n+1)(n-h+2)}
\]

\[
= \frac{(h+1)h}{2} \cdot \frac{2n \cdot 2 \cdot 1}{n(n+1)(h+1)} = 1, \text{ which checks, as } h = \frac{n+1}{2}.
\]

When \( n \) is even this becomes:

\[
M(1,h) = M(3,h-1) \cdot \frac{2(n+1) \cdot 3 \cdot 2}{(h+2)n(n+1)}
\]

\[
= \frac{(h+2)(h+1)h}{3 \cdot 2} \cdot \frac{2 \cdot 3 \cdot 2}{(h+2)n} = h+1, \text{ which checks.} \]
$1 \leq k < h-1$. \[ \sum_{k+1 \leq s \leq h} (-1)^{s+1} M(n-2s+1,s) D_{n-s} \]

\[= \sum_{k+2 \leq s \leq h} (-1)^{s+1} M(n-2s+1,s) D_{n-s} \]

\[+ (-1)^k M(n-2k-1,k+1) D_{n-k-1} \]

\[= (-1)^{k+1} M(n-2k-1,k+1) D_{n-k-1} \frac{(n-2k-1)(n-2k-2)}{n(n+1)} \]

\[+ (-1)^k M(n-2k-1,k+1) D_{n-k-1} \]

by downward induction at $k+1$,

\[= (-1)^k M(n-2k-1,k+1) D_{n-k} \frac{n-k+1}{4n-4k+2} \times \]

\[\times \left[ 1 - \frac{(n-2k-1)(n-2k-2)}{n(n+1)} \right], \text{ by combining} \]

and the relation of $D_{t-1}$ to $D_t$,

\[= (-1)^k M(n-2k+1,k) D_{n-k} \frac{(n-2k+1)(n-2k)}{2(k+1)(2n-2k-1)} \times \]

\[\times \left[ \frac{n(n+1)-(n-2k-1)(n-2k-2)}{n(n+1)} \right] \]

by algebraic rearrangement, so the induction on $k$ is done.

Now \[D_n = \sum_{1 \leq s \leq h} (-1)^{s+1} M(n-2s+1,s) D_{n-s} \]

\[= M(n-1,1) D_{n-1} + \sum_{2 \leq s \leq h} (-1)^{s+1} M(n-2s+1,s) D_{n-s} \]

\[= M(n-1,1) D_{n-1} - M(n-1,1) D_{n-1} \frac{(n-1)(n-2)}{n(n+1)} \]

by the case $k = 1$ above,

\[= \frac{nD_{n-1}}{n(n+1)} \left[ n(n+1) - (n-1)(n-2) \right] \]

\[= \frac{(2n-2)! \frac{4n-2}{(n+1)(n-1)!} (n-1)! n}{2n} \cdot 2n \]

\[= \frac{(2n)!}{(n+1)(n+1)!} = \frac{1}{n+1} M(n,n), \text{ q.e.d.} \]
We are now ready to prove Theorem 4.2.

**Proof of Theorem 4.2.** Let $f$ be an endomorphism of $H^*_{\mathbb{Z}}G_2,q$. Since $f$ is an algebra homomorphism and $H^*_{\mathbb{Z}}G_2,q$ has $c_1$ and $c_2$ as multiplicative generators, $f$ is characterized by how it maps $c_1$ and $c_2$. Further, as $f$ is a morphism of graded algebras, we must have $f(c_1) = \alpha c_1$ and $f(c_2) = \beta_1 c_1^2 + \beta_2 c_2$. We also have $\alpha, \beta_1 \in \mathbb{Z}$ because $H^*_{\mathbb{Z}}G_2,q$ is a free graded $\mathbb{Z}$-module.

The proof is divided into four cases.

1° $\beta_1 = 0$. Applying $f$ to the relation $c_1^{2q} = \frac{M(q,q)}{q+1} c_2$ obtained in the Lemma 4.7, $\alpha_1^{2q} c_1^{2q} = \alpha_1^{2q} \frac{M(q,q)}{q+1} c_2 = \frac{M(q,q)}{q+1} \beta_2 c_2$, so $\beta_2 = \alpha^{2q}$ as $c_2^{q}$ is an additive basis element.

Similarly, the relation $c_1^2 c_2^{q-1} = c_2^{q}$ gives $\alpha_2^{2q-1} = \beta_2^{q+1}$ which combines with $\beta_2^{q} = \alpha^{2q}$ to give $\beta_2 = \alpha^2$ and we are done in this case.

2° $\beta_2 = 0$. Proceeding as in the first case, $c_1^{2q-1}$ gives $\alpha_1^{2q-1} = \beta_1^{q+1}$, while $c_1^{2q} = \frac{M(q,q)}{q+1} c_2$ gives $\alpha_1^{2q} = \frac{M(q,q)}{q+1} \beta_1^{q+1}$. From these equations we deduce that when $\alpha \beta_1 \neq 0$ we must have $\beta_1 = \alpha^2$ whence $\frac{M(q,q)}{q+1} = 1$, a contradiction. Thus $\alpha \beta_1 = 0$, but the above equations imply $\alpha = 0 \iff \beta_1 = 0$, so $\alpha = \beta_1 = 0$ and we are done in this case.
\[ 3^\circ \quad \alpha \beta_1 \neq 0. \] From Lemma 4.7, we use the relations
\[ c_2^q = \frac{q+1}{M(q,q)} c_1^2, \quad c_2^{q-2} = \frac{q}{M(q-1,q-1)} c_1^{2q-2}, \quad c_2^{q-4} = \frac{q-1}{M(q-2,q-2)} c_1^{2q-4}, \quad c_2^2 = \frac{q-2}{M(q-3,q-3)} c_1^{2q-6}. \]

Applying \( f \) to these relations, we have:
\[ f(c_2^q) = \frac{q+1}{M(q,q)} \alpha^{2q} M(q,q) c_2^q \]
\[ = \frac{q}{M(q-1,q-1)} \alpha^{2q-4} \left[ M(q,q) \beta_1 + \frac{M(q-1,q-1)}{q} \beta_2 \right] c_2^q \]
\[ = \frac{q-1}{M(q-2,q-2)} \alpha^{2q-4} \left[ M(q,q) \beta_1^2 + \frac{M(q-1,q-1)}{q} \beta_2 \right] c_2^q \]
\[ = \frac{q-2}{M(q-3,q-3)} \alpha^{2q-6} \left[ M(q,q) \beta_1^3 + \frac{M(q-1,q-1)}{q} \beta_2 \right] \]
\[ + \frac{M(q-2,q-2)}{q-2} \beta_1 \beta_2 \beta_2. \]

Thus we have:
\[ \alpha^{2q} \left[ \frac{qM(q,q)}{(q+1)M(q-1,q-1)} \beta_1 + \beta_2 \right] = \alpha^{2q-4} \]
\[ = \frac{(q-1)M(q,q)}{(q+1)M(q-2,q-2)} \beta_1^2 + \frac{(q-1)M(q-1,q-1)}{qM(q-2,q-2)} \beta_1 \beta_2 + \beta_2^2 \]
which gives \( \alpha^2 = \frac{4q-2}{q+1} \beta_1 + \beta_2 \) from (4.9), and from
\[ \alpha^4 - \beta_2^2 = \frac{4(q-1)(2q-3)}{q(q+1)} \beta_1^2 + \frac{4(2q-3)}{q} \beta_1 \beta_2 \]
\[ = \frac{4q-2}{q+1} \beta_1 \left( \frac{4q-2}{q+1} \beta_1 + 2 \beta_2 \right), \text{ using (4.9).} \]

From the last two equations, it follows that
\[ \beta_1 = \frac{q+1}{2q-1} \beta_2, \quad \text{so} \quad \beta_2 = -\alpha^2. \quad \text{and} \quad \beta_1 = \frac{q+1}{2q-1} \alpha^2. \]
Substituting in (4.11) and cancelling, we obtain:
\[
1 = \frac{(q-2)M(q,q)}{(q+1)M(q-3,q-3)} \left( \frac{q+1}{2q-1} \right)^3 - \frac{3(q-2)M(q-1,q-1)}{qM(q-3,q-3)} \left( \frac{q+1}{2q-1} \right)^2 + \frac{3(q-2)M(q-2,q-2)(q+1)}{(q-1)M(q-3,q-3)(2q-1)} - 1;
\]

but this equation reduces to 12q + 30 = 0, contradicting \( q \) a positive integer, so we cannot have \( \alpha \beta_1 \neq 0 \) and we are done in this case. This case also shows that the only remaining case is \( \alpha = 0 \).

4° \( \alpha = 0 \). Here we have two subcases.

4°(i) \( q \) odd. In \( H^d(q+1)_{FG_2,q} \) when we apply \( f \) to

\[
\sum_{r \in \mathbb{N}_{q+1}} (-1)^{|r|} M(r) c^r = 0 \]

we obtain

\[
f(c_2^2) = 0 = (\beta_1 c_1^2 + \beta_2 c_2) \frac{q+1}{2}.\]

Multiplying and adding,

\[
\left( \beta_1 c_1^2 + \beta_2 c_2 \right)^2 - \beta_1 \frac{q+1}{2} \sum_{r \in \mathbb{N}_{q+1}} (-1)^{|r|} M(r) c^r = 0.\]

Since \( c_1^{q+1} \) does not appear in this equation all coefficients must be zero by linear independence. In particular, the coefficients of \( c_2^{q+1} \) and \( c_1^2 c_2^2 \), so we have:

\[
\beta_2^2 - (-1)^{\frac{q-1}{2}} \beta_1^2 = 0 \quad \text{(whence } |\beta_2| = |\beta_1|\text{) and}
\]

\[
\frac{q+1}{2} \beta_1 \beta_2^2 - (-1)^{\frac{q-1}{2}} \frac{q+1}{2} M(2, \frac{q-1}{2}) = 0, \quad \text{whence as } |\beta_2| = |\beta_1| \text{ we must have } M(2, \frac{q-1}{2}) = \frac{q+1}{2} \text{ which implies } q = 1, \text{ if } \beta_1 \neq 0 \text{ contrary to } q \geq 3, \text{ so this subcase is done.}
4. Let $q$ be even. In $H^d(q+2)_{FG_2,q}$ we apply $f$ to

$$
\sum_{r \in N_{q+2}^2} (-1)^r M(r)c^r = 0
$$

to obtain

$$(\beta_1 c_1^2 + \beta_2 c_2)^{\frac{q}{2}+1} = 0,$$

and we also combine with

$$
\sum_{r \in N_{q+1}^2} (-1)^r M(r)c_1c^r = 0
$$

to obtain:

$$(\beta_1 c_1^2 + \beta_2 c_2)^{\frac{q}{2}+1} + \left(\frac{q+2}{2} \beta_1 \beta_2 + q \beta_1^2\right) \sum_{r \in N_{q+2}^2} (-1)^r M(r)c^r
$$

$$
+ \left(\frac{q+2}{2} \beta_1^{q/2} \beta_2 + (q+1) \beta_1^2\right) \sum_{r \in N_{q+1}^2} (-1)^r c_1^{r+2} = 0.
$$

As $c_1^{q+2}$ and $c_1^q c_2$ do not appear in this equation, the coefficients of $c_2^{q/2+1}$ and $c_1^2 c_2^{q/2}$ are zero and we have:

$$
\frac{q}{2} + 1 + \left(\frac{q+2}{2} \beta_1^{q/2} \beta_2 + q \beta_1^2\right)(-1)^{\frac{q}{2}} = 0
$$

and

$$
\frac{q+2}{2} \beta_1^{q/2} + \left(\frac{q+2}{2} \beta_1^{q/2} \beta_2 + q \beta_1^2\right)(-1)^{q/2}M(2,q/2)
$$

$$
+ \left(\frac{q+2}{2} \beta_1^{q/2} \beta_2 + (q+1) \beta_1^2\right)(-1)^{q/2+1} \cdot \frac{q}{2} = 0
$$

Let $b_1 = \beta_1 / (\beta_1, \beta_2)$, so $(b_1, b_2) = 1$ (when $\beta_1 \beta_2 \neq 0$) and dividing the first equation by $(\beta_1, \beta_2)^{q/2+1}$ we have

$$
\frac{q+2}{2} b_1^{q/2} b_2 + q b_1^2 = (-1)^{q/2} b_2^{q/2+1}
$$

which implies $|b_1| = 1$. Substituting this in the second equation gives
\[ \frac{a+2}{2} b_1 b_2^{q/2+1} + M(\gamma, q/2) b_2^{q/2+1} - \frac{a+2}{2} \left( b_2^{q/2+1} - (-b_1)^{q/2+1} \right) = 0 \]

or

\[ b_1 b_2^{q/2+1} + \frac{a}{4} b_2^{q/2+1} + (-b_1) b_2^{q/2+1} = 0 \]

which implies

\[ |b_2| = 1. \]

But then by integrality and \( b_1 b_2 \neq 0 \) we must have \( q = 8 \), which is also impossible as the last equation then becomes \( 2b_2 = 0 \). Thus we must have \( \beta_1 = 0 = \beta_2 \) and we are done, Q.E.D.
5. THE FIXED POINT PROPERTY FOR $\mathbb{R}G_2, q$

In this section we prove:

Theorem 5.1. $\mathbb{R}G_2, q \in \text{FPP}$, for all $q = 4k$ or $q = 4k+1$, $k = 1, 2, 3, \ldots$.

Because of Poincaré duality the middle dimension of a manifold has especial significance, and when the middle dimension has odd-dimensional cohomology it turns out that we can use $\mathbb{Z}_2$ coefficients. Recall that, except for a variation in grading corresponding to $d = \dim_k F$, $H^*(\mathbb{R}G_2, q; \mathbb{Z}_2)$ is the same, regardless of our choice of $F$. In particular, $H^*(\mathbb{R}G_2, q; \mathbb{Z}_2) = \mathbb{Z}_2[c_1, c_2, c_1, \ldots, c_q]/(c_1 = 1)$. We also note that the properties of the relations which were proved in section 4 still hold when we take coefficients modulo 2. In fact, our $\mathbb{Z}_2$-proof of the fixed point property is similar in many ways to our previous $\mathbb{Z}$-proof. Thus, let us grant:

Theorem 5.2. For any endomorphism $f$ of $H^*(\mathbb{R}G_2, q; \mathbb{Z}_2)$ if $f(c_1) = \alpha c_1$, $f(c_2) = \alpha c_2$, $q = 4k$ or $q = 4k+1$, $k = 1, 2, 3, \ldots$.

We can then easily demonstrate Theorem 5.1.

Proof of Theorem 5.1. When $\alpha = 0$. Theorem 5.2 gives $L(f; \mathbb{Z}_2) = 1$. When $\alpha = 1$, it gives $L(f; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^q(\mathbb{R}G_2, q; \mathbb{Z}_2)(\text{mod } 2)$ by the duality of the
dimensions of the cohomology groups and the definition of $L(f; Z_2)$. But $\dim_{Z_2} H^q(RG_2, q; Z_2) = \lfloor q/2 \rfloor + 1$ since $H^q(RG_2, q; Z_2)$ has basis $\left\{ c_1 r_1^1 c_2 r_2^2 : r_1 + 2r_2 = q, r_1 \in \mathbb{N} \right\}$, so $L(f; Z_2) = 1$ in this case when $q = 4k$ or $q = 4k+1$. Hence by the Lefschetz theorem, $RG_2, q$ has the fixed point property for the indicated values of $q$, q.e.d.

We now go to Theorem 5.2:

Proof of Theorem 5.2: As before, we start by letting $f^*(c_1) = \alpha c_1$ and $f^*(c_2) = \beta_1 c_1^2 + \beta_2 c_2$ with $\alpha, \beta_1 \in Z_2$. We will first use Steenrod squares to obtain some restrictions on the values of the coefficients. For this we need to determine $Sq^1 c_2$, which is done by use of the splitting principle and properties of the Steenrod squares.

Letting $E$ be the canonical 2-plane bundle over $RG_2, q$, we recall that the splitting principle tells us that there is a map $g: B \longrightarrow RG_2, q$ such that $g^* \xi$ is a sum of 2 line bundles, $\gamma_1 = \gamma_1 \oplus \gamma_2$, and $g^*: H^*(RG_2, q; Z_2) \longrightarrow H^*(B, Z_2)$ is a monomorphism. Now the Stiefel-Whitney class of $\xi$ is $w(\xi) = c$ and letting those of $\gamma_1$, $\gamma_2$ be $w(\gamma_1) = 1 + x_1$, we have $w(\gamma) = w(\gamma_1)w(\gamma_2) = 1 + x_1 + x_2 + x_1x_2$. As $w(g^* \xi) = g^*(w(\xi))$, we have $g^*(x_1 + x_2) = c_1$ and $g^*(x_1x_2) = c_2$. Now $Sq^1(x_1x_2) = Sq^0(x_2) + Sq^0(x_1)Sq^1(x_2) = x_1x_2 + x_1x_2^2 = x_1x_2(x_1+x_2)$, so $g^*Sq^1 c_2 = Sq^1 g^*(c_2) = Sq^1(x_1x_2) = (x_1x_2(x_1+x_2))$
\[ r^* (c_1 c_2). \] \[ \therefore \text{Sq}^1 c_2 = c_1 c_2. \] Now naturality of the Steenrod squares gives the following commutative diagram:

\[
\begin{array}{ccc}
H^2(RG_2, q; \mathbb{Z}_2) & \xrightarrow{f^*} & H^2(RG_2, q; \mathbb{Z}_2) \\
\text{Sq}^1 \downarrow & & \downarrow \text{Sq}^1 \\
H^3(RG_2, q; \mathbb{Z}_2) & \xrightarrow{f^*} & H^3(RG_2, q; \mathbb{Z}_2)
\end{array}
\]

Looking at what happens to \( c_2 \) in this square, we see that

\[ \text{Sq}^1 f^* c_2 = \text{Sq}^1 (\beta_1 c_1^2 + \beta_2 c_2) = \beta_1 \text{Sq}^1 c_1^2 + \beta_2 \text{Sq}^1 c_2 = \beta_2 c_1 c_2, \]
while \( f^* \text{Sq}^1 c_2 = f^*(c_1 c_2) = \alpha \beta_1 c_1^3 + \alpha \beta_2 c_1 c_2. \) Thus

\[ \alpha \beta_1 c_1^3 + \alpha \beta_2 c_1 c_2 = \beta_2 c_1 c_2 \]
and as \( q \geq 3 \) implies \( c_1^3 \) and \( c_1 c_2 \) are a basis for \( H^3(RG_2, q; \mathbb{Z}_2) \), \( \alpha \beta_1 = 0 \) and \( \alpha \beta_2 = \beta_2. \)

Now we are ready to take cases on the value of \( \alpha \):

**Case 1.** \( \alpha = 1. \) So \( \alpha \beta_1 = 0 \) gives \( \beta_1 = 0 \) immediately.

Applying \( f^* \) to \( \sum_{r} M(r) c^r = 0 \) gives

\[ c_1^{q+1} + \beta_2 \sum_{r \in \mathbb{N}_{q+1}} M(r) c^r = 0 \text{ or } (q+1,0) \]

\[ (1 + \beta_2) c_1^{q+1} + \beta_2 \sum_{r \in \mathbb{N}_{q+1}} M(r) c^r = (1+\beta_2)c_1^{q+1} = 0 \text{ and as } \]

\( c_1^{q+1} \neq 0 \) if \( q \neq 4k+2 \) it follows that \( \beta_2 = 1 \) and we are done in this case.

**Case 2.** \( \alpha = 0. \) Here, \( \alpha \beta_2 = \beta_2 \) gives \( \beta_2 = 0 \) immediately. When \( q \) is odd, applying \( f^* \) to \( \sum_{r} M(r) c^r = 0 \)

\[ r \in \mathbb{N}_{q+1} \]

gives \( \beta_1 c_1^{q+1} = 0 \) which implies \( \beta_1 = 0 \) as \( c_1^{q+1} \neq 0 \).
when $q$ is odd. When $q$ is even, applying $f^*$ to  
\[ \sum_{r \in N^2_{q+2}} M(r)c^r = 0 \]  
gives $\beta_1 c_1^{q+2} = 0$ and as $c_1^{q+2} \neq 0$ 
(consider $\sum_{r \in N^2_{q+2}} M(r)c^r = 0 = \sum_{r \in N^2_{q+2}} M(r-e_1)c^r$) we have $\beta_1 = 0$, q.e.d.

Note that the above proof also works for $CG_2,q$ and $HG_2,q$ because of the isomorphism in cohomology up to grading. The only difference comes in computing the value of $Sq^d c_2$. Chern classes must be used instead of Stiefel-Whitney classes and the canonical 2-dimensional bundle over $HG_2,q$ must be considered as a 4-dimensional complex bundle, but we still obtain $Sq^d c_2 = c_1 c_2$. Of course we have already proved more than this in Theorem 4.2.
BIBLIOGRAPHY


