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The Ohio State University, Ph.D., 1974
Mathematics

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GENERALIZED CLOSED SETS AND $T_{1/2}$-SPACES

DISSERATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

William Wade Dunham, B.S., M.S.

*****

The Ohio State University

1974

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ACKNOWLEDGMENTS

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INTRODUCTION

Generalized closed sets in a topological space were introduced by Levine in [12]. They comprise a family of sets which contains the closed sets, with the containment, in general, being proper. Yet many of the important and familiar theorems concerning closed sets remain true of generalized closed sets. Thus, the introduction into topology of generalized closed sets may be viewed as an attempt to extend the "nice" properties of closed sets to a broader class. It is the purpose of this dissertation to provide a comprehensive study of generalized closed sets.

In Chapter 1, the concept of a generalized closed set is defined and characterized, as is the complementary notion of a generalized open set.

Before developing the properties of such sets, we furnish, in Chapter 2, examples of generalized closed sets which occur "naturally" in topology. A number of common topological structures prove to be generalized closed sets - among them, the derived set of any subset of a topological space; complete subspaces of uniform or pseudometric spaces; and compact subsets or retracts of regular spaces.

In Chapter 3 we return to the topological properties of generalized closed and generalized open sets, examining their
behavior with respect to subspaces, products, and mappings. We also treat the problem of extending a continuous map from a generalized closed set to its closure.

Chapter 4 focuses on those "nice" properties of generalized closed sets mentioned above. For instance, it is shown that compactness, paracompactness, completeness, and normality are inherited by generalized closed subspaces. In Theorem 4.46 we prove a "generalized" Tietze Extension Theorem - namely, that a continuous, real-valued function on a generalized closed subset of a normal space can be extended continuously to the entire space. And, a number of corollaries are derived by combining the results of this chapter with the examples of Chapter 2.

The relationship between a dense, generalized closed set and the entire space (or, equivalently, between a generalized closed set and its closure) is explored in Chapter 5. We show that such properties as compactness, connectedness, and completeness are possessed by a dense, generalized closed set if and only if they are possessed by the space itself. At the end of the chapter, we prove that, under a weak regularity condition, a pseudometric, uniformity, or proximity may be extended from a dense, generalized closed set to the entire space.

Spaces in which every generalized closed set is, in fact, closed were called $T_{1/2}$-spaces by Levine in [12]. As the name suggests, the $T_{1/2}$ property falls strictly between the $T_0$ and
$T_1$ separation axioms. In Chapter 6, we study $T_{1/2}$-spaces, providing a point-wise characterization independent of generalized closed sets; examining subspaces, products, and images of such spaces; and investigating maximal and minimal $T_{1/2}$ topologies on a given set.

Finally, in Chapter 7 we define a "generalized" closure operator on an arbitrary topological space and consider the new topology it induces. This proves to be a $T_{1/2}$ topology, finer than the original, and having as a base the generalized open sets from the original space.

Throughout this dissertation, definitions and notation will be as in Kelley [10], except where certain ambiguities in the literature require qualifying remarks. Moreover, most of the results from Levine's original paper, [12], appear in the course of this work, particularly in Chapters 1, 3 and 4, and occasionally alternate proofs are provided.
CHAPTER 1
DEFINITIONS AND CHARACTERIZATIONS

Definition 1.1: A subset $A$ of a topological space is said to be \textit{generalized closed} (hereafter written as $g$-closed) iff $c(A) \subseteq 0$ whenever $A \subseteq 0$ and $0$ is open.

Remark 1.2: It is immediate that closed sets are necessarily $g$-closed. Examples of $g$-closed sets which are not closed will be provided by the following simple, yet useful, observation:

Theorem 1.3: In the topological space $(X, \mathcal{J})$, $A$ is $g$-closed if its only open superset is $X$ itself.

\textbf{Proof:} If $A \subseteq 0 \in \mathcal{J}$, then $0 = X$ and so $c(A) \subseteq 0$.

Example 1.4: Let $X = \{a, b\}$ and $\mathcal{J} = \{\emptyset, X\}$. Then $\{a\}$ is $g$-closed by Theorem 1.3, but $\{a\}$ is not closed.

We characterize $g$-closed sets in

Theorem 1.5: Let $(X, \mathcal{J})$ be a topological space and $A \subseteq X$. Then the following conditions are equivalent:

(a) $A$ is $g$-closed

(b) $A \subseteq 0 \in \mathcal{J}$ implies $A' \subseteq 0$ ($A'$ is the derived set)

(c) $c(A) \setminus A$ contains no non-empty closed set
(d) For every \( x \in c(A) \), \( c(x) \cap A \neq \emptyset \)

**Proof:** (a) implies (b): If \( A \subseteq \emptyset \in \mathcal{T} \), then \( A' \subseteq c(A) \subseteq \emptyset \).

(b) implies (c): Suppose \( F \subseteq c(A) \setminus A \) where \( F \) is closed. Then \( A \subseteq \mathcal{C} F \subseteq \mathcal{T} \) and so \( A' \subseteq \mathcal{C}F \) by hypothesis. Thus \( F \subseteq c(A) \setminus A \subseteq A' \subseteq \mathcal{C}F \) and so \( F = \emptyset \).

(c) implies (d): Let \( x \in c(A) \). If \( c(x) \cap A = \emptyset \), then \( \emptyset \neq c(x) \subseteq c(A) \setminus A \), contradicting (c).

(d) implies (a): Let \( A \subseteq \emptyset \in \mathcal{T} \) and let \( x \in c(A) \). By (d), \( a \in c(x) \cap A \) for some \( a \in A \). But then \( a \in \emptyset \), and \( a \in c(x) \) implies \( x \in \emptyset \). Thus, \( c(A) \subseteq \emptyset \) and \( A \) is \( g \)-closed.

**Corollary 1.6** [12; Theorem 5.3]: In a \( T_1 \)-space, \( g \)-closed sets are closed.

**Proof:** If \( A \) is \( g \)-closed and \( x \in c(A) \), then, since singletons are closed, \( \emptyset \neq c(x) \cap A = \{x\} \cap A \). It follows that \( x \in A \), and thus \( A \) is closed.

The next result concerns the structure of a \( g \)-closed set.

**Theorem 1.7:** \( A \) is \( g \)-closed iff \( A = F \setminus N \) where \( F \) is closed and \( N \) contains no non-empty closed sets.

**Proof:** Necessity: \( A = c(A) \setminus (c(A) \setminus A) \). Let \( F = c(A) \), \( N = c(A) \setminus A \), and refer to Theorem 1.5(c).

Sufficiency: Suppose \( A = F \setminus N \) where \( F \) is closed and \( N \) contains no non-empty closed set. Let \( A \subseteq \emptyset \) where \( \emptyset \) is open. Then \( F \cap \emptyset \) is closed and \( F \cap \emptyset \subseteq F \cap cA \subseteq N \).
Hence $F \cap CO = \emptyset$ and so $A = F \setminus N \subseteq F \subseteq 0$. It follows that $c(A) \subseteq F \subseteq 0$, and thus $A$ is $g$-closed.

**Example 1.8:** The decomposition of $g$-closed sets in the previous theorem is not unique. For, if $X = \{a,b,c\}$ and $\mathcal{F} = \{\emptyset, \{a\}, X\}$, then $\{c\}$ is $g$-closed by Theorem 1.3. But $\{c\} = X \setminus \{a,b\}$, where $X$ is closed and $\{a,b\}$ contains no non-empty closed set; and $\{c\} = \{b,c\} \setminus \{b\}$ where $\{b,c\}$ is closed and $\{b\}$ contains no non-empty closed set.

Complementing the concept of a $g$-closed set is the following:

**Definition 1.9:** A set $Q$ will be called **generalized open** (written $g$-open) iff $CQ$ is $g$-closed.

**Remark 1.10:** It is clear that every open set is $g$-open. That the converse is false follows from Example 1.4 by noting that $\{b\}$ is $g$-open but not open.

**Theorem 1.11:** Let $(X, \mathcal{F})$ be a topological space and $Q \subseteq X$. Then the following conditions are equivalent:

(a) $Q$ is $g$-open.

(b) $F \subseteq \text{int}(Q)$ whenever $F \subseteq Q$ and $F$ is closed.

(c) $0 = X$ whenever $0 \notin \mathcal{F}$ and $\text{int}(Q) \cup CQ \subseteq 0$.

(d) If $x \notin \text{int}(Q)$, then $c(x) \cap CQ \neq \emptyset$.

**Proof:** (a) implies (b): See Levine [12], Theorem 4.2.

(b) implies (c): Suppose $0 \in \mathcal{F}$ and $\text{int}(Q) \cup CQ \subseteq 0$. 

Since \( C_0 \subseteq Q \) and \( C_0 \) is closed, we have by (b) that
\[ C_0 \subseteq \text{int}(Q). \] Thus \( C_0 \subseteq \text{int}(Q) \cup CQ = O. \) It follows that
\[ C_0 = \emptyset \text{ and } O = X. \]

(c) implies (d): Suppose \( x \notin \text{int}(Q) \) but \( C(x) \cap CQ = \emptyset. \)
Since \( \text{int}(Q) \subseteq Cc(x) \) and \( CQ \subseteq Cc(x), \) we have
\[ \text{int}(Q) \cup CQ \subseteq Cc(x) \subseteq I. \] By the hypothesis of (c), \( Cc(x) = X \)
and consequently \( C(x) = \emptyset, \) a contradiction.

(d) implies (a): Let \( x \in C(CQ). \) Then \( x \notin Cc(CQ) = \text{int}(Q) \)
and so by (d), \( C(x) \cap CQ \neq \emptyset. \) Thus, by Theorem 1.5(d), \( CQ \)
is g-closed and so \( Q \) is g-open.

**Theorem 1.12:** \( Q \) is g-open iff \( Q = O \cup N \) where \( O \) is open
and \( N \) contains no non-empty closed set.

**Proof:** The proof follows immediately from Theorem 1.7 by complementation.

**Remark 1.13:** Letting \( Q = \{a,b\} \) is Example 1.8, we see that
the above structure theorem for g-open sets does not furnish a
unique decomposition.

**Corollary 1.14:** \( \{x\} \) is g-open iff either \( \{x\} \) is open or \( \{x\} \)
is not closed.

**Proof:** Necessity: If \( \{x\} \) is g-open, \( \{x\} = O \cup N \) where
\( O \) is open and \( N \) contains no non-empty closed set. If \( \{x\} = O, \)
\( \{x\} \) is open, while if \( \{x\} = N, \) \( \{x\} \) is not closed.

Sufficiency: If \( \{x\} \) is open, it is automatically g-open.
If \( \{x\} \) is not closed, let \( O = \emptyset \) and \( N = \{x\} \) in Theorem 1.12.

We conclude this chapter with two results linking \( g \)-closed and \( g \)-open sets.

**Theorem 1.15** [12; Theorem 4.9]: \( A \) is \( g \)-closed iff \( c(A) \setminus A \) is \( g \)-open.

**Proof:** Necessity: If \( A \) is \( g \)-closed, \( c(A) \setminus A \) contains no non-empty closed sets and thus is \( g \)-open by Theorem 1.12.

Sufficiency: If \( c(A) \setminus A \) is \( g \)-open, then \( c(A) \setminus A = 0 \cup N \), where \( O \) is open and \( N \) contains no non-empty closed sets.

But \( 0 \subseteq c(A) \setminus A \) implies \( 0 = \emptyset \), and thus \( A \) is \( g \)-closed by Theorem 1.5(c).

**Corollary 1.16:** \( Q \) is \( g \)-open iff \( \text{int}(Q) \cup CQ \) is \( g \)-closed.

**Proof:** \( Q \) is \( g \)-open iff \( CQ \) is \( g \)-closed iff \( c(CQ) \setminus CQ \) is \( g \)-open iff \( c(c(CQ) \setminus CQ) \) is \( g \)-closed iff \( \text{int}(Q) \cup CQ \) is \( g \)-closed.
CHAPTER 2

SOME IMPORTANT EXAMPLES OF G-CLOSED SETS

In Chapter 2 we shall provide a number of examples of topological structures which are g-closed (but not necessarily closed) and which occur "naturally" in topology. Such examples, it is hoped, will justify further study of generalized closed sets.

Derived Sets

Lemma 2.1: Suppose \((X, \mathcal{T})\) is a topological space and \(A \subseteq X\). Suppose also that for some \(0 \in \mathcal{J}, A' \subseteq 0\) where \(A'\) is the derived set of \(A\). Then \(A'' \subseteq 0\).

Proof: If we deny the conclusion, there is an \(x \in A''\) such that \(x \notin 0\). Hence \(x \notin A'\) and we can find a \(U \in \mathcal{J}\) such that \(x \in U\) and \(U \cap A \cap C(x) = \emptyset\). Consequently, \(U \cap A \subseteq \{x\}\). Now, \(x \in A''\) and \(x \in U\) implies \(y \in U \cap A' \cap C(x)\) for some \(y\). However, \(y \in A'\) and \(y \in U \cap A' \subseteq U \cap 0 \in \mathcal{J}\) implies \(\emptyset \neq U \cap 0 \cap A \cap C(y) \subseteq U \cap A \subseteq \{x\}\). Thus \(\{x\} = 0 \cap U \cap A \cap C(y)\) and so \(x \in 0\), a contradiction.

Theorem 2.2: If \((X, \mathcal{J})\) is a topological space and \(A \subseteq X\), then \(A'\) is g-closed.
Proof: Suppose $A' \subseteq 0 \in \mathcal{J}$. By Lemma 2.1, $(A')' = A'' \subseteq 0$ and so $A'$ is $g$-closed by Theorem 1.5(b).

Weakly Hausdorff and $R_0$-spaces

Before considering further examples of $g$-closed sets, we introduce the following two concepts which will be used frequently in this and succeeding chapters.

Definition 2.3: A space is said to be an $R_0$-space iff $c(x) \subseteq 0$ whenever $x \in 0$ and $0$ is open. (See Davis, [5].)

An $R_0$-space, then, is one in which singletons are $g$-closed.

Theorem 2.4: $(X, \mathcal{J})$ is an $R_0$-space iff for $x, y \in X$, $x \in c(y)$ implies $y \in c(x)$.

Proof: Necessity: Suppose $(X, \mathcal{J})$ is $R_0$ and $x \in c(y)$. Then, if $y \notin c(x)$, $y \in c(c(x)) \in \mathcal{J}$ and so $x \in c(y) \subseteq c(c(x))$, a contradiction.

Sufficiency: Let $x \in 0 \in \mathcal{J}$ and $y \in c(x)$. By hypothesis, $x \in c(y)$ and thus $y \in 0$. It follows that $c(x) \subseteq 0$.

Remark 2.5: In the literature, a space is said to be symmetric if and only if $x \in c(y)$ implies $y \in c(x)$. By virtue of the previous theorem, the $R_0$ property and symmetry are seen to be identical.

Definition 2.6: A space $(X, \mathcal{J})$ is weakly Hausdorff iff $c(x) = c(y)$ whenever there exists a net $S : D \to X$ for which
lim S = x and lim S = y. (See Levine, [14].)

The concepts of $R_0$ and weakly Hausdorff are related by:

**Theorem 2.7:** A weakly Hausdorff space is $R_0$.

**Proof:** Let $x \in O$ where $O$ is open, and let $y \in c(x)$. Then the net which is constantly equal to $x$ converges to both $x$ and $y$ and so $c(x) = c(y)$ by the weakly Hausdorff condition. Thus $x \in c(y) \cap O$ and consequently $y \in O$. It follows that $c(x) \subseteq O$ and the space is $R_0$.

**Example 2.8:** The implication of Theorem 2.7 is not reversible. For, letting $X = [0,1]$, the unit interval in the reals, with $\mathcal{J}$ the cofinite topology, then $(X, \mathcal{J})$ is $R_0$ since singletons are closed. However, $(X, \mathcal{J})$ is not weakly Hausdorff since the sequence $(\frac{1}{n})$ converges to both 0 and 1 (indeed, it converges to every point in the space), but $c(0) = \{0\} \neq \{1\} = c(1)$.

The utility of the study of weakly Hausdorff spaces arises from:

**Theorem 2.9:** If $(X, \mathcal{J})$ is regular or Hausdorff, it is weakly Hausdorff.

**Proof:** Suppose $(X, \mathcal{J})$ is regular and for some net $S: D \rightarrow X$, $\lim S = x$ and $\lim S = y$. We assert that $x \in c(y)$. For otherwise, $x \in c(c(y)) \in \mathcal{J}$ and by regularity there is an $O \in \mathcal{J}$ such that $x \in O \subseteq c(0) \subseteq c(c(y))$. Since $x \in O$ and $\lim S = x$, $S$ is eventually in $O$. Since $y \in c(c(0))$ and $\lim S = y$, $S$
is eventually in \( Cc(0) \). Thus, \( S \) is eventually in \( 0 \cap Cc(0) \), a contradiction. We conclude that \( x \in c(y) \), and a symmetric argument implies \( y \in c(x) \). Thus \( c(x) = c(y) \) and \((X, \mathcal{J})\) is weakly Hausdorff.

To see that a Hausdorff space is weakly Hausdorff, we need only recall that in a Hausdorff space net limits are unique.

Example 2.10: Again, neither implication from the previous theorem is reversible. For, a two-point space with the indiscrete topology is weakly Hausdorff but not \( T_2 \); and a \( T_2 \)-space that is not \( T_3 \) (see, for instance, Willard [19], p. 92) is weakly Hausdorff but not regular.

Compact subsets, Retracts, and Graphs

Theorem 2.11: A compact subset of a weakly Hausdorff space is \( g \)-closed.

Proof: Let \( A \) be a compact subset of the weakly Hausdorff space \((X, \mathcal{J})\). We shall use Theorem 1.5(d). If \( x \in c(A) \), there is a net \( S:D \to A \) such that \( \lim S = x \). By compactness, there is a subnet \( T:E \to A \) such that \( \lim T = a \) for some \( a \in A \). Since \( \lim T = x \) and \( \lim T = a \), we have \( c(x) = c(a) \). Thus \( a \in c(a) \cap A = c(x) \cap A \) and so \( A \) is \( g \)-closed.

Corollary 2.12: A compact subset of a \( T_2 \)-space is closed.

Proof: By Theorem 2.9, a \( T_2 \)-space is weakly Hausdorff and thus a compact subset is \( g \)-closed by the previous result. But by
Corollary 1.6, any $g$-closed set in a $T_2$-space is closed.

**Corollary 2.13** [12; Theorem 3.5]: A compact subset of a regular space is $g$-closed.

**Proof:** Apply Theorems 2.9 and 2.11.

**Theorem 2.14:** A retract of a weakly Hausdorff space is $g$-closed.

**Proof:** Let $(X, \tau)$ be weakly Hausdorff and let $A \subseteq X$ be a retract. That is, there is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Now if $x \in c(A)$, there is a net $S : D \to A$ such that $\lim S = x$. Thus, $\lim r \circ S = r(x)$ and $\lim r \circ S = \lim S = x$, which implies $c(x) = c(r(x))$. Hence $r(x) \in c(x) \cap A$, and so $A$ is $g$-closed by Theorem 1.5(d).

**Corollary 2.15:** A retract of a $T_2$-space is closed.

**Proof:** The result follows as in Corollary 2.12.

**Corollary 2.16:** A retract of a regular space is $g$-closed.

**Proof:** Combine Theorems 2.9 and 2.14.

**Theorem 2.17:** Let $f : (X, \tau) \to (Y, \mathcal{U})$ be continuous, where $(Y, \mathcal{U})$ is weakly Hausdorff. Let $G_f = \{(x, f(x)) : x \in X\}$ be the graph of $f$. Then $G_f$ is $g$-closed in $(X \times Y, \mathcal{J} \times \mathcal{U})$.

**Proof:** We shall use Theorem 1.5(d). Let $(x, y) \in c(G_f)$. Then there is a net $S : D \to G_f$, where $S(d) = (x_d, f(x_d))$, such that $\lim S = (x, y)$. Consequently, $\lim x_d = x$ and $\lim f(x_d) = y$. By the continuity of $f$, $\lim f(x_d) = f(x)$, and so the weak Hausdorff condition implies $c_y(f(x)) = c_y(y)$. But
$$(x,f(x)) \in c_X(x) \times c_Y(f(x)) = c_X(x) \times c_Y(y) = c_X \times c_Y((x,y))$$. Thus $c_X \times c_Y((x,y)) \cap G_f \neq \emptyset$.

**Corollary 2.18**: A continuous function whose range lies in a regular space has a $g$-closed graph.

**Proof**: Apply Theorems 2.9 and 2.17.

**Corollary 2.19**: The diagonal of a weakly Hausdorff space is $g$-closed.

**Proof**: The diagonal, $\{(x,x) : x \in X\}$, is the graph of the identity map.

**Complete Subspaces**

A well-known result of the theory of uniform spaces is that a complete subspace of a separated uniform space is closed. (See Kelley [10], p. 192.) In a more general situation, we have:

**Theorem 2.20**: Let $(X,\mathcal{U})$ be a uniform space and let $A \subset X$ such that $(A, A \times A \cap \mathcal{U})$ is a complete uniform space. Then $A$ is a $g$-closed subset of $(X, \mathcal{T}(\mathcal{U}))$.

**Proof**: If $x \in c(A)$, there is a net $S:D \rightarrow A$ such that $\lim S = x$ in $(X, \mathcal{T}(\mathcal{U}))$. But $S$ convergent in $\mathcal{T}(\mathcal{U})$ implies $S$ is $\mathcal{U}$-Cauchy, and, since $S(D) \subset A$, $S$ is also $A \times A \cap \mathcal{U}$-Cauchy. By completeness, there is an $a \in A$ such that $\lim S = a$, the convergence occurring in $(A, \mathcal{T}(A \times A \cap \mathcal{U}))$ and thus in $(X, \mathcal{T}(\mathcal{U}))$. Since $(X, \mathcal{T}(\mathcal{U}))$ is uniformizable,
it is completely regular, hence regular, and hence weakly Hausdorff. It follows that \( c(a) = c(x) \). Thus, \( a \in c(x) \cap A \), and \( A \) is \( g \)-closed by Theorem 1.5(d).

**Corollary 2.21:** Let \((X, d)\) be a pseudometric space and let \( A \subseteq X \) be a complete subspace. Then \( A \) is \( g \)-closed in \((X, \mathcal{F}(d))\).

**Proof:** Let \( \mathcal{U}(d) \) be the canonical uniformity on \( X \) generated by \( d \). Then \((X, \mathcal{F}(d)) = (X, \mathcal{F}(\mathcal{U}(d)))\), and it suffices to show \( A \) is \( g \)-closed in \((X, \mathcal{F}(\mathcal{U}(d)))\). By virtue of the previous theorem, we need only show that \((A, A \times A \cap \mathcal{U}(d))\) is a complete uniform space. By Kelley [10], Theorem 6.24, \((A, A \times A \cap \mathcal{U}(d))\) is complete if and only if every \( d \)-Cauchy sequence in \( A \) converges in \( A \). This latter condition holds since \( A \) is complete in the pseudometric sense.

The previous results show that complete subspaces of uniform or pseudometric spaces are \( g \)-closed. Immediately we can derive:

**Corollary 2.22:** A complete subspace of a separated uniform space is closed, and a complete subspace of a metric space is closed.

**Proof:** Refer to Corollary 1.6.

**Saturated Spaces**

\( G \)-closed sets appear in the following theorem of Levine:

**Theorem 2.23** [12; Theorem 2.10]: Let \((X, \mathcal{F})\) be a topological space with \( \mathcal{F} \) the family of closed sets. Then \( \mathcal{F} = \mathcal{J} \) iff
every subset of $X$ is $g$-closed.

**Definition 2.24:** A topological space is saturated iff arbitrary intersections of open sets are open. (See Lorrain [15].)

We now shall use Theorem 2.23 to prove the following result about saturated spaces, and thus about any space with a finite topology.

**Theorem 2.25:** Let $(X, \mathcal{J})$ be saturated. Then the following conditions are equivalent:

(a) $(X, \mathcal{J})$ is 0-dimensional
(b) $(X, \mathcal{J})$ is completely regular
(c) $(X, \mathcal{J})$ is regular
(d) $(X, \mathcal{J})$ is weakly Hausdorff
(e) $(X, \mathcal{J})$ is $R_0$
(f) Every subset of $X$ is $g$-closed
(g) $\mathcal{J} = \mathcal{J}$

**Proof:** That (a) implies (b) implies (c) implies (d) implies (e) follows from Theorem 2.9, Theorem 2.7, and standard results.

(e) implies (f): Suppose $(X, \mathcal{J})$ is $R_0$ and for an arbitrary subset $A$ of $X$, $A \subseteq 0 \in \mathcal{J}$. If $A = \emptyset$, $A$ is automatically $g$-closed. Otherwise, for all $a \in A$, $c(a) \subseteq 0$ by the $R_0$ property. Hence $A \subseteq \bigcup \{c(a): a \in A\} \subseteq 0$. But in a saturated space, arbitrary unions of closed sets are closed. Thus, $c(A) \subseteq \bigcup \{c(a): a \in A\} \subseteq 0$ and $A$ is $g$-closed.
(f) implies (g): See Theorem 2.23.

(g) implies (a): \( \mathcal{J} \) itself is an open-closed base for \( \mathcal{J} \).

Remark 2.26: The previous result shows that the different "regularity" conditions from \( R_o \) to 0-dimensionality coincide on saturated spaces, and thus on finite sets (in this regard, see Examples 2.8 and 2.10). Moreover, Theorem 2.25(g) provides a structure theorem for an \( R_o \) (or weakly Hausdorff, regular, completely regular, 0-dimensional) topology on a saturated space. Namely, for each open set \( 0 \), \( CO \) is also open. We draw a few more conclusions:

Corollary 2.27: A saturated, \( R_o \)-space whose topology is not indiscrete is disconnected.

Proof: There exists a proper, non-empty open set, and it and its complement provide a disconnection by Remark 2.26.

Corollary 2.28: If \( (X, \mathcal{J}) \) is \( R_o \) (or weakly Hausdorff, regular, completely regular, 0-dimensional) with \( |\mathcal{J}| < \infty \), then \( |\mathcal{J}| \) is even.

Proof: By Theorem 2.25, \( \mathcal{J} = \mathcal{J}' \). Thus, by pairing each \( 0 \in \mathcal{J} \) with \( CO \in \mathcal{J}' \), we exhaust the open sets. Hence \( |\mathcal{J}| \) is even.

Remark 2.29: The contrapositive of Corollary 2.28 states that a space with an odd number of open sets can not be \( R_o \) (nor weakly Hausdorff, regular, completely regular, or 0-dimensional).
This yields a quick method (i.e., counting the open sets) of showing that a finite topology is not regular.

**Embeddings**

Our final three examples of $g$-closed sets will be somewhat more specialized.

**Remark 2.30**: If $X$ is a non-void set and $(Y,d)$ is a pseudometric space, we can define  

$$B(X,Y) = \{ f: X \to Y : f[X] \text{ is bounded} \}.$$  

Then, defining  

$$\sigma: B(X,Y) \times B(X,Y) \to [0,\infty) \text{ by } \sigma(f,g) = \sup \{ d(f(x),g(x)) : x \in X \},$$

we can prove that $\sigma$ is a pseudometric on $B(X,Y)$. It is known that, if $X$ is a topological space, the family of all continuous, bounded functions is a closed subset of $(B(X,Y), \sigma)$ (see Cullen [4], Theorem 21.20).

On the other hand, the family of all constant functions need not be closed in $B(X,Y)$. For example, let $X = Y = \{a,b\}$ and $d(a,a) = d(a,b) = d(b,a) = d(b,b) = 0$. Then the two constant functions form a non-empty, proper subset of $B(X,Y)$ which cannot be closed since $B(X,Y)$ is indiscrete.

We treat the constant functions a bit more formally by defining  

$$\tilde{\phi}: Y \to B(X,Y) \text{ by } \tilde{\phi}(y)(x) = y \text{ for all } x \in X.$$  

can be easily shown that $\tilde{\phi}$ is an isometry from $(Y,d)$ to $(B(X,Y), \sigma)$, and thus the constant functions provide an isometric copy of $Y$, namely $\tilde{\phi}[Y]$, in $B(X,Y)$. 
Theorem 2.31: Using the terminology of the previous remark, \( \bar{\mathcal{Y}}[Y] \) is a \( g \)-closed subset of \( (B(X,Y), \mathcal{T}(\sigma)) \).

Proof: We shall use Theorem 1.5(d). Let \( f \in c(\bar{\mathcal{Y}}[Y]) \), the closure being taken in \( (B(X,Y), \mathcal{T}(\sigma)) \). Then for each natural number \( n \), there is a \( y_n \in Y \) such that
\[
\sigma(f, \delta(y_n)) < \frac{1}{n}.
\]
Fix \( x_0 \in X \). Then \( f(x_0) \in Y \) and we assert \( \sigma(f, \delta(f(x_0))) = 0 \). It suffices to prove that
\[
d(f(x), f(x_0)) = 0 \quad \text{for all} \quad x \in X.
\]
So, letting \( x \in X \) and \( \varepsilon > 0 \), we choose a natural number \( N \) such that \( \frac{1}{N} < \frac{1}{2} \varepsilon \). Then
\[
d(f(x), f(x_0)) \leq d(f(x), y_N) + d(y_N, f(x_0))
\]
\[
\leq \sigma(f, \delta(y_N)) + \sigma(\delta(y_N), f)
\]
\[
< \frac{1}{N} + \frac{1}{N} < \varepsilon.
\]
Hence the assertion is proved, and we conclude that \( \delta(f(x_0)) \in c(f) \cap \bar{\mathcal{Y}}[Y] \). By Theorem 1.5(d), \( \bar{\mathcal{Y}}[Y] \) is \( g \)-closed in \( B(X,Y) \).

A similar result concerning embeddings will be found when we shift our attention to uniform spaces.

Remark 2.32: Given a uniform space \((Y, \mathcal{U})\), a non-empty set \( X \), and \( \mathcal{F} \), the family of all functions from \( X \) to \( Y \), Kelley [10] defines the uniformity of uniform convergence \( \mathcal{U} \) on \( \mathcal{F} \) as follows:

For each \( V \in \mathcal{U} \), define \( W(V) \subset \mathcal{F} \times \mathcal{F} \) by
\[
W(V) = \{(f,g): f,g \in \mathcal{F} \text{ and } (f(x), g(x)) \in V \text{ for all } x \in X\}.
\]
Then \( \mathcal{U} = \{ U : \mathcal{F} \times \mathcal{F} \supseteq U \supseteq W(V) \text{ for some } V \in \mathcal{V} \} \). Kelley [10] proves that \((\mathcal{F}, \mathcal{U})\) is a uniform space. If we define \( \hat{\phi} : Y \to \mathcal{F} \) by \( \hat{\phi}(y)(x) = y \) for all \( x \in X \), then it can be shown that \( \hat{\phi} : (Y, \mathcal{V}) \to (\mathcal{F}, \mathcal{U}) \) is a unimorphism into. (That is, \( \hat{\phi} \) is a uniformly continuous bijection such that \( \hat{\phi}^{-1} : (\hat{\phi}(Y), \hat{\phi}(Y) \times \hat{\phi}(Y) \cap \mathcal{U}) \to (Y, \mathcal{V}) \) is also uniformly continuous.)

**Theorem 2.33:** Using the terminology and hypotheses of the previous remark, \( \hat{\phi}[Y] \) is a \( g \)-closed subset of \((\mathcal{F}, \mathcal{J}(\mathcal{U}))\).

**Proof:** Let \( f \in c(\hat{\phi}[Y]) \), the closure being taken in \((\mathcal{F}, \mathcal{J}(\mathcal{U}))\). Then there is a net \( S : D \to \hat{\phi}[Y] \), denoted \( S(d) = \hat{\phi}(y_d) \), such that \( \lim S = f \). Fix \( x_0 \in X \). Then \( f(x_0) \in Y \), and we assert that \( \hat{\phi}(f(x_0)) \in c(f) \). Since \( c(f) = \cap[W(V)[f] : V \in \mathcal{V}] \) (see Kelley [10], p. 179), it suffices to show that \((f, \hat{\phi}(f(x_0))) \in W(V)\) for each \( V \in \mathcal{V} \). So, we let \( V \in \mathcal{V} \) and find \( V^* \in \mathcal{V} \) such that \( V^* \circ V^* \subseteq V \) and \( V^* \) is symmetric. Now, \( \lim \hat{\phi}(y_d) = f \) and \( W(V^*)[f] \) is a neighborhood of \( f \), and so there is a \( d^* \in D \) such that \( (f, \hat{\phi}(y_d)) \in W(V^*) \) for all \( d \geq d^* \). Hence, for any \( x \in X \),

\[
(f(x), y_*) = (f(x), \hat{\phi}(y_*) (x)) \in V^*, \quad \text{and, by symmetry,}
\]

\[
(d_*, f(x_0)) \in V^*. \quad \text{It follows that}
\]

\[
(f(x), \hat{\phi}(f(x_0))(x)) = (f(x), f(x_0)) \in V^* \circ V^* \subseteq V, \quad \text{and thus}
\]

\[
(f, \hat{\phi}(f(x_0))) \in W(V) \quad \text{as asserted. We have shown, then, that}
\]
$(f(x_0)) \in c(f) \cap \hat{\phi}[Y]$ and we conclude that $\hat{\phi}[Y]$ is 
g-closed in $(\hat{\mathfrak{F}}, \mathfrak{F}(\hat{\mathcal{U}}))$ by Theorem 1.5(d).

The last example of a g-closed set arising "naturally"
will concern the embedding of a uniform space into its associated
hyperuniform space.

Remark 2.34: Many authors define hyperuniformities (or hyper-
topologies) on the family of all non-empty, closed subsets of
$X$ because of the nice separation properties which arise in
the hyperspace. However, one can define a hyperuniformity on
all of $\hat{X}$, the power set of $X$, as follows:

If $(X, \mathcal{U})$ is a uniform space, let $\hat{\mathcal{U}}$ be the uniformity on
$\hat{X}$ with base $\{H(U): U \in \mathcal{U}\}$ where
$H(U) = \{(A,B): A \subseteq U[B] \text{ and } B \subseteq U[A]\}$ (see Bourbaki [1],
p. 206). The natural embedding $i: X \rightarrow \hat{X}$ is defined by
$i(x) = \{x\}$ and is seen to be a unimorphism into. If we let $\hat{c}$
denote the closure operator in $(\hat{X}, \mathcal{F}(\hat{\mathcal{U}}))$ and let $A \subseteq X$,
then Caulfield [3], p. 25 shows that $\hat{c}(i[A]) = \{B: \emptyset \neq B \subseteq c(x) \text{ for }
\text{some } x \in c(A)\}$. Using the above notation and results, we prove:

Theorem 2.35: $i[X]$ is a g-closed subset of $(\hat{X}, \mathcal{F}(\hat{\mathcal{U}}))$.

Proof: Let $i[X] \subseteq \hat{\mathcal{U}} \subseteq \mathcal{F}(\hat{\mathcal{U}})$ and assert $\hat{c}(i[X]) \subseteq \hat{\mathcal{U}}$.

Now, if $B \in \hat{c}(i[X])$, then $\emptyset \neq B \subseteq c(x)$ for some $x \in c(X) = X$
by Caulfield's result. Then $i(x) = \{x\} \in \hat{\mathcal{U}} \subseteq \mathcal{F}(\hat{\mathcal{U}})$ and so we
can find $U^* \in \mathcal{U}$ such that $U^*$ is symmetric and
\[ H(U^*)[[x]] \subseteq \hat{\emptyset}. \] Since, in the uniform space \((X, \mathcal{U})\),
\[ c(x) = \bigcap\{U[x]: U \in \mathcal{U}\}, \] we have \( B \subseteq c(x) \subseteq U^*[x] \). Moreover,
\[ x \in U^*[B] \text{ since } B \neq \emptyset \text{ and } U^* \text{ is symmetric}. \] Thus,
\[ ([x], B) \in H(U^*) \text{ and so } B \in H(U^*)[[x]] \subseteq \hat{\emptyset}. \] It follows that
\[ c(i[X]) \subseteq \hat{\emptyset} \text{ and so } i[X] \text{ is } g\text{-closed}. \]

Remark 2.36: The preceding three examples, in which the images of certain natural embeddings are shown to be \( g \)-closed, will be considered again in Chapter 4. (See Corollaries 4.11, 4.25, 4.28, 4.29, 4.32, and 4.41.)
CHAPTER 3

BASIC PROPERTIES

In this chapter we shall develop a number of important properties of $g$-closed (and $g$-open) sets in an arbitrary topological space. We shall consider their behavior under subspaces and products and compare them with their closed (and open) counterparts. We shall investigate images and inverse images of $g$-closed and $g$-open sets under various types of transformations. Finally, it will be proved that any continuous map from a $g$-closed set to a locally compact, Hausdorff space has a continuous extension to its closure.

Unions and Intersections

**Theorem 3.1** [12; Theorem 2.4]: If $A$ and $B$ are $g$-closed sets, then $A \cup B$ is $g$-closed.

**Corollary 3.2**: If $Q$ and $R$ are $g$-open, then $Q \cap R$ is $g$-open.

**Proof**: Use Theorem 3.1 and complementation.

**Remark 3.3**: The previous corollary shows that the family of $g$-open sets in a topological space serves as a base for a new topology on the space. The topology generated by the $g$-open sets will occupy our attention in Chapter 7.
Intersections of $g$-closed sets (and unions of $g$-open sets) need not be $g$-closed ($g$-open) as seen in the following example:

Example 3.4 [12; Example 2.5]: Let $X = \{a,b,c\}$ and $J = \{\emptyset, \{a\}, X\}$. If $A = \{a,b\}$ and $B = \{a,c\}$, then $A$ and $B$ are $g$-closed by Theorem 1.3, but $A \cap B = \{a\}$ which is not $g$-closed since $\{a\} \subseteq \{a\} \in J$ but $c(\{a\}) \notin \{a\}$. Moreover, $CA$ and $CB$ are two $g$-open sets whose union is not $g$-open.

Although arbitrary intersections of $g$-closed sets may fail to be $g$-closed, the next two theorems provide conditions under which the intersections will be $g$-closed.

Theorem 3.5 [12; Corollary 4.4]: Suppose $A$ and $B$ are $g$-closed with $CA$ and $CB$ separated. Then $A \cap B$ is $g$-closed.

Proof: Since $CA$ and $CB$ are separated, we have $CA \cap c(CB) = \emptyset$ and $CB \cap c(CA) = \emptyset$, or, equivalently, $CA \subseteq \text{int}(B)$ and $CB \subseteq \text{int}(A)$. Now let $A \cap B \subseteq O$ where $O$ is open. We assert $A \subseteq O \cup \text{int}(A)$. For, given $x \in A$, if $x \in B$, then $x \in A \cap B \subseteq O$; while if $x \notin B$, then $x \in CB \subseteq \text{int}(A)$. Thus, $A \subseteq O \cup \text{int}(A)$, an open set, and so $c(A) \subseteq O \cup \text{int}(A)$ since $A$ is $g$-closed. Similarly, we can show that $c(B) \subseteq O \cup \text{int}(B)$. Hence

$$c(A \cap B) \subseteq c(A) \cap c(B) \subseteq (O \cup \text{int}(A)) \cap (O \cup \text{int}(B)) \subseteq O \cup (A \cap B) \subseteq O.$$ 

It follows that $A \cap B$ is $g$-closed.
Theorem 5.6 [12; Corollary 2.7]: Suppose $A$ is $g$-closed and $F$ is closed. Then $A \cap F$ is $g$-closed.

Proof: By the necessity of Theorem 1.7, $A = G \setminus N$ where $G$ is closed and $N$ contains no non-empty closed sets. Thus $A \cap F = (F \cap G) \setminus N$, and, since $F \cap G$ is closed, the result follows from the sufficiency of Theorem 1.7.

The two previous theorems yield complementary results for $g$-open sets which we state without proof.

Corollary 5.7: If $Q$ and $R$ are separated, $g$-open sets, then $Q \cup R$ is $g$-open.

Corollary 5.8: If $Q$ is $g$-open and $O$ is open, then $Q \cup O$ is $g$-open.

Closures and Interiors

Theorem 5.9 [12; Theorem 2.8]: If $A$ is $g$-closed and $A \subseteq B \subseteq c(A)$, then $B$ is $g$-closed.

Corollary 5.10: If $Q$ is $g$-open and $\text{int}(Q) \subseteq R \subseteq Q$, then $R$ is $g$-open.

Proof: The result follows by complementation.

Subspaces

By definition of the subspace topology, we know that if $Z \subseteq Y \subseteq X$, then $Z$ is $Y$-open ($Y$-closed) if and only if
Z = Y ∩ S where S is X-open (X-closed). The analogous statement for g-open (g-closed) sets will hold in only one direction.

**Theorem 3.11:** Suppose Q ⊆ Y ⊆ X where Q is g-open in Y. Then \( Q = Y ∩ Q^* \) for some \( Q^* \) g-open in X.

**Proof:** By Theorem 1.12, \( Q = O ∪ N \) where \( O \) is Y-open and N contains no non-empty Y-closed sets, and thus no non-empty X-closed sets. Now \( O = Y ∩ O^* \) for some \( O^* \) open in X. If we define \( Q^* = O^* ∪ N \), \( Q^* \) is g-open in X by Theorem 1.12 and \( Q = O ∪ N = (Y ∩ O^*) ∪ N = (Y ∪ N) ∩ (O^* ∪ N) = Y ∩ Q^* \).

**Corollary 3.12:** Suppose \( A ⊆ Y ⊆ X \) where A is g-closed in Y. Then \( A = Y ∩ A^* \) for some \( A^* \) g-closed in X.

**Proof:** \( C_Y A \) is g-open in Y, and so by the previous theorem \( C_Y A = Y ∩ Q^* \) where \( Q^* \) is g-open in X. Defining \( A^* = C_X Q^* \), we have \( A^* \) is g-closed in X and \( A = C_Y (C_Y A) = C_Y (Y ∩ Q^*) = Y ∩ C_X Q^* = Y ∩ A^* \).

The converses of the two previous results are false as seen in:

**Example 3.13:** Let \( X = \{a, b, c\} \), \( J = \{\emptyset, \{a\}, X\} \) and \( Y = \{a, b\} \).

We observe that the relative topology on Y is \( Y ∩ J = \{\emptyset, \{a\}, Y\} \). Now if \( Q^* = \{b\} \), \( Q^* \) is g-open in X since \( \{b\} \) is not X-closed (see Corollary 1.14). And, if \( A^* = \{a, c\} \), then \( A^* \) is g-closed in X by Theorem 1.3. But \( Y ∩ Q^* = \{b\} \), which is not g-open in Y by Corollary 1.14; and,
\[ Y \cap A^* = \{a\}, \text{ which is not } g\text{-closed in } Y \text{ since } \{a\} \subset \{a\} \in Y \cap \mathcal{F}, \]

but \( c_Y\{a\} = Y \).

Although Example 3.13 shows that \( g\)-closed sets do not always behave like closed sets with respect to subspaces, the following two \( g\)-closed analogues of theorems about closed sets hold true.

**Theorem 3.14** [12; Theorem 2.6]: If \( A \subseteq B \subseteq X \) where \( A \) is \( g\)-closed with respect to \( B \) and \( B \) is \( g\)-closed with respect to \( X \), then \( A \) is \( g\)-closed with respect to \( X \).

**Theorem 3.15** [12; Theorem 2.9]: If \( A \subseteq B \subseteq X \) where \( A \) is \( g\)-closed with respect to \( X \), then \( A \) is \( g\)-closed with respect to \( B \).

If we reformulate the two previous results in terms of \( g\)-open sets, the one corresponding to Theorem 3.14 will hold while the one corresponding to Theorem 3.15 will not:

**Theorem 3.16** [12; Theorem 4.6]: If \( Q \subseteq R \subseteq X \) where \( Q \) is \( g\)-open in \( R \) and \( R \) is \( g\)-open in \( X \), then \( Q \) is \( g\)-open in \( X \).

**Proof:** Since \( Q \) is \( g\)-open in \( R \), \( Q = R \cap Q^* \) for some \( Q^* \) \( g\)-open in \( X \) by Theorem 3.11. But then \( R \) and \( Q^* \) are both \( g\)-open in \( X \) and so \( Q \) is \( g\)-open in \( X \) by Corollary 3.2.

**Example 3.17** [12; Example 4.7]: Let \( X = \{a,b,c\} \) and \( \mathcal{F} = \{\emptyset, \{a\}, X\} \). Let \( Q = \{b\} \) and \( R = \{a,b\} \). Then \( Q \subseteq R \subseteq X \) with \( Q \) \( g\)-open in \( X \) by Corollary 1.14. But \( Q \) is
not \( g \)-open in \((R, R \cap \mathcal{J})\) since \{b\} is \( R \)-closed but not \( R \)-open.

**Products**

\( G \)-closed sets behave well with respect to products as seen in:

**Theorem 3.18:** Let \((X, \mathcal{J}) = \chi((X_\alpha, \mathcal{J}_\alpha): \alpha \in \Delta)\) and let \( \emptyset \neq A_\alpha \subset X_\alpha \) for each \( \alpha \in \Delta \). Then \( \chi(A_\alpha: \alpha \in \Delta) \) is \( g \)-closed in \( X \) iff \( A_\alpha \) is \( g \)-closed in \( X_\alpha \) for each \( \alpha \in \Delta \).

**Proof:** Necessity: Suppose \( \chi(A_\alpha: \alpha \in \Delta) \) is \( g \)-closed in \((X, \mathcal{J})\). Let \( \beta \in \Delta \) be fixed and suppose \( A_\beta \subset O_\beta \in \mathcal{I}_\beta \).

Now, \( \chi(A_\alpha: \alpha \in \Delta) \subset P_\beta^{-1}[A_\beta] \subset P_\beta^{-1}[O_\beta] \in \mathcal{J} \) and so \( \chi(c_\alpha(A_\alpha): \alpha \in \Delta) = \chi(c_\alpha(A_\alpha): \alpha \in \Delta) \subset P_\beta^{-1}[O_\beta] \). Since \( A_\alpha \neq \emptyset \) for all \( \alpha \), we have \( c_\beta(A_\beta) \subset P_\beta^{-1}[\chi(c_\alpha(A_\alpha): \alpha \in \Delta)] \subset O_\beta \) and thus \( A_\beta \) is \( g \)-closed in \((X_\beta, \mathcal{J}_\beta)\).

Sufficiency [12; Theorem 7.1]: Suppose \( A_\alpha \) is \( g \)-closed in \((X_\alpha, \mathcal{J}_\alpha)\) for each \( \alpha \in \Delta \). To show \( \chi(A_\alpha: \alpha \in \Delta) \) is \( g \)-closed in \((X, \mathcal{J})\), we shall use Theorem 1.5(d). Let \( x \in c(\chi(A_\alpha: \alpha \in \Delta)) = \chi(c_\alpha(A_\alpha): \alpha \in \Delta) \). Then for each \( \alpha \in \Delta \), \( P_\alpha(x) \subset c_\alpha(A_\alpha) \), and thus by Theorem 1.5(d) there is an \( a_\alpha \in c(P_\alpha(x)) \cap A_\alpha \). Consider \( a \in X \) defined by \( a(\alpha) = a_\alpha \) for each \( \alpha \in \Delta \). Then \( a \in \chi(A_\alpha: \alpha \in \Delta) \). Moreover, \( a \in \chi(c_\alpha(P_\alpha(x)): \alpha \in \Delta) = c(x) \). It follows that \( c(x) \subset c(\chi(A_\alpha: \alpha \in \Delta) \neq \emptyset \) and so \( \chi(A_\alpha: \alpha \in \Delta) \) is \( g \)-closed.
Remark 3.19: Arbitrary products of g-open sets need not be g-open. See Levine [12], Example 7.2.

However, finite products of g-open sets are g-open:

Theorem 3.20 [12, Theorem 7.3]: Suppose $Q$ is g-open in $(X, \mathcal{J})$ and $R$ is g-open in $(Y, \mathcal{U})$. Then $Q \times R$ is g-open in $(X \times Y, \mathcal{J} \times \mathcal{U})$.

Proof: Since $Q$ is g-open in $X$, $\mathcal{C}_X Q$ is g-closed in $X$, and thus $\mathcal{C}_X Q \times Y$ is g-closed in $X \times Y$ by Theorem 3.18. Similarly, $X \times \mathcal{C}_Y R$ is g-closed in $X \times Y$ and so, by Theorem 3.1, $\mathcal{C}_X \times_Y (Q \times R) = (\mathcal{C}_X Q \times Y) \cup (X \times \mathcal{C}_Y R)$ is g-closed in $X \times Y$. Consequently, $Q \times R$ is g-open.

A simple inductive argument establishes:

Corollary 3.21: If $Q_i$ is g-open in $(X_i, \mathcal{J}_i)$ for $1 \leq i \leq n$, then $\chi(Q_i, 1 \leq i \leq n)$ is g-open in $\chi((X_i, \mathcal{J}_i), 1 \leq i \leq n)$.

Unlike the situation for open sets, where we have that for $0 \neq \emptyset$ and $U \neq \emptyset$, $0 \times U$ is open in the product space if and only if $0$ and $U$ are both open in the factor spaces, we have no converse for Theorem 3.20, as the following example shows:

Example 3.22: Let $X = \{a, b\}$, $\mathcal{J} = \{\emptyset, X\}$, and $\mathcal{U} = \{\emptyset, \{a\}, X\}$. Let $Q = \{a\} \subset X$ and $R = \{b\} \subset X$. Then $Q \times R = \{(a, b)\}$ is g-open in $(X \times X, \mathcal{J} \times \mathcal{U})$ by Corollary 1.14. But $R$ is not g-open in $(X, \mathcal{U})$, again by Corollary 1.14.
G-closed Sets and the Lattice of Topologies

Example 3.23: Let \( X = \{a, b, c\} \), \( \mathcal{T} = \{\emptyset, \{a\}, X\} \) and 
\[ \mathcal{U} = \{\emptyset, \{b\}, X\}. \]
Then \( A = \{a, b\} \) is g-closed in \((X, \mathcal{T})\) and
in \((X, \mathcal{U})\) by Theorem 1.3. But \( \mathcal{T} \lor \mathcal{U} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\} \)
and \( A \) is not g-closed in \((X, \mathcal{T} \lor \mathcal{U})\) since \( \{a,b\} \subseteq \{a,b\} \in \mathcal{T} \lor \mathcal{U} \),
but \( c(\{a,b\}) \not\subseteq \{a,b\} \).

Example 3.24: Let \( X = \{a, b, c\} \), \( \mathcal{T} = \{\emptyset, \{a, b\}, \{c\}, X\} \), and 
\[ \mathcal{U} = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, X\}. \]
Then \( B = \{b\} \) is g-closed in 
\((X, \mathcal{T})\) and in \((X, \mathcal{U})\). But \( \mathcal{T} \land \mathcal{U} = \{\emptyset, \{a,b\}, X\} \), and \( \{b\} \)
is not g-closed in \((X, \mathcal{T} \land \mathcal{U})\) since \( \{b\} \subseteq \{a,b\} \in \mathcal{T} \land \mathcal{U} \) but 
\( c \in c(\{b\}) \).

Remark 3.25: The previous examples illustrate that a set may
be g-closed with respect to a family of topologies yet not be
g-closed with respect to the supremum or infimum of the family.
Thus, of course, a set g-closed with respect to one topology need
not be g-closed with respect to a finer or a coarser topology.

Taking complements in the above examples yields similar negative
results for g-open sets.

Transformations

Theorem 3.26 [12; Theorems 6.1 and 6.3]: Suppose \( f: (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}) \)
is continuous and closed. Then (a) If \( A \subseteq X \) is g-closed,
\( f[A] \subseteq Y \) is g-closed; and (b) If \( B \subseteq Y \) is g-closed,
\( f^{-1}[B] \subseteq X \) is g-closed.
Corollary 3.27 [12; Theorem 6.3]: If \( f: (X, \tau) \to (Y, \nu) \) is continuous and \( Q \subseteq Y \) is g-open, then \( f^{-1}[Q] \) is g-open in \( X \).

Remark 3.28: Contrasting to Theorem 3.26(a), the image of a g-open set under a continuous, closed mapping need not be g-open. See Levine [12], Example 6.2.

Under mappings which are only continuous, or even continuous and open, images and inverse images of g-closed (g-open) sets need not be g-closed (g-open) as the following examples show:

Example 3.29: Let \( X = \{a, b\} \) with topology \( \mathcal{T} = \{\emptyset, \{a\}, X\} \).
Let \( Y = \{x, y, z\} \) with topology \( \mathcal{U} = \{\emptyset, \{x\}, \{x, y\}, Y\} \). Define \( f: X \to Y \) by \( f(a) = x \) and \( f(b) = y \). Then \( f \) is continuous and open, but

(a): \( \{b\} \) is g-closed in \( X \) (in fact, \( \{b\} \) is closed), but \( \{f(b)\} = \{y\} \) is not g-closed in \( Y \).

(b): \( \{x, z\} \) is g-closed in \( Y \) by Theorem 1.3, but
\( f^{-1}[\{x, z\}] = \{a\} \) is not g-closed in \( X \).

(c): \( \{y\} \) is g-open in \( Y \) by Corollary 1.14, but
\( f^{-1}[\{y\}] = \{b\} \) is not g-open in \( X \).

Example 3.30: Let \( X = \{a, b, c\} \) with topology \( \mathcal{T} = \{\emptyset, \{a\}, X\} \).
Let \( Y = \{x, y\} \) with topology \( \mathcal{U} = \{\emptyset, \{x\}, Y\} \). Define \( f: X \to Y \) by \( f(a) = x \) and \( f(b) = f(c) = y \). Then \( f \) is continuous and open, but \( \{b\} \) is g-open in \( X \) by Corollary 1.14, while \( \{f(b)\} = \{y\} \) is not g-open in \( Y \).
We next address ourselves to the problem of extending a continuous function on a $g$-closed set to the closure of the set. Our fundamental tool will be the following theorem, whose elegant proof can be found in Engelking [6], p. 110.

**Theorem 3.31:** Suppose $A$ is a dense subset of the topological space $(X, \mathcal{T})$ and $f: (A, A \cap \mathcal{T}) \rightarrow (Y, \mathcal{U})$ is continuous for $(Y, \mathcal{U})$ a compact Hausdorff space. Then $f$ has a continuous extension to all of $X$ iff for every $G_1, G_2 \subseteq Y$ where $G_1$ and $G_2$ are closed and disjoint, $c_X(f^{-1}[G_1]) \cap c_X(f^{-1}[G_2]) = \emptyset$.

**Theorem 3.32:** Suppose $A$ is a $g$-closed subset of $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ is a compact Hausdorff space. Suppose further that $f: (A, A \cap \mathcal{T}) \rightarrow (Y, \mathcal{U})$ is continuous. Then there is a continuous mapping $F: (c(A), c(A) \cap \mathcal{T}) \rightarrow (Y, \mathcal{U})$ such that $F|_A = f$.

**Proof:** Let $G_1$ and $G_2$ be disjoint, closed subsets of $Y$ and consider $c(A) \cap c(f^{-1}[G_1]) \cap c(f^{-1}[G_2])$, a subset of $X$:

(a): $c(A) \cap c(f^{-1}[G_1]) \cap c(f^{-1}[G_2])$ is clearly $X$-closed.

(b): $c(A) \cap c(f^{-1}[G_1]) \cap c(f^{-1}[G_2]) \subseteq c(A)$, trivially.

(c): $c(A) \cap c(f^{-1}[G_1]) \cap c(f^{-1}[G_2]) \subseteq CA$. For, otherwise, there is an $a \in A$ such that $a \in c(A) \cap c(f^{-1}[G_1]) \cap c(f^{-1}[G_2])$.

But then $a \in A \cap c(f^{-1}[G_1]) = c_A(f^{-1}[G_1]) = f^{-1}[G_1]$ since $f$ is continuous. Similarly, $a \in f^{-1}[G_2]$. Thus, $a \in f^{-1}[G_1] \cap f^{-1}[G_2]$, contradicting the disjointness of $G_1$ and $G_2$. 
So, by (a) - (c) above, \( c(A) \cap c(f^{-1}[G_1]) \cap c(f^{-1}[G_2]) \) is an \( X \)-closed subset of \( c(A) \setminus A \). By Theorem 1.5(c),
\[
\emptyset = c(A) \cap c(f^{-1}[G_1]) \cap c(f^{-1}[G_2]) = c_c(A)(f^{-1}[G_1]) \cap c_c(A)(f^{-1}[G_2]).
\]
Hence, applying Theorem 3.31 to the space \((c(A), c(A) \cap \mathcal{J})\), we conclude that there is a mapping \( F: c(A) \to Y \) such that \( F \) is continuous and \( F|_A = f \).

The next result shows that Theorem 3.32 remains true if the compactness assumption on \((Y, \mathcal{U})\) is replaced by the weaker assumption of local compactness.

**Theorem 3.33:** Suppose \( A \) is a \( g \)-closed subset of \((X, \mathcal{J})\) and that \((Y, \mathcal{U})\) is a locally compact, Hausdorff space. Suppose further that \( f: (A, A \cap \mathcal{J}) \to (Y, \mathcal{U}) \) is continuous. Then there is a continuous map \( F: (c(A), c(A) \cap \mathcal{J}) \to (Y, \mathcal{U}) \) such that \( F|_A = f \).

**Proof:** If \((Y, \mathcal{U})\) is compact, we apply Theorem 3.32 directly. Otherwise, let \((Y^*, \mathcal{U}^*)\) be the one-point compactification of \((Y, \mathcal{U})\). That is, let \( Y^* = Y \cup \{\alpha\} \) where \( \alpha \notin Y \) and let \( \mathcal{U}^* = \{U^*: U^* \in \mathcal{U}, \text{ or } U^* = \{\alpha\} \cup U \} \) where \( U \in \mathcal{U} \) and \( C_{YU} \) is compact. Since \((Y, \mathcal{U})\) is locally compact and Hausdorff, \((Y^*, \mathcal{U}^*)\) is compact and Hausdorff. Considering \( f: (A, A \cap \mathcal{J}) \to (Y^*, \mathcal{U}^*) \) as a continuous mapping from \( A \) to the space \((Y^*, \mathcal{U}^*)\), we apply Theorem 3.32 to get a continuous map \( F: (c(A), c(A) \cap \mathcal{J}) \to (Y^*, \mathcal{U}^*) \) such that \( F|_A = f \). We reason that \( F^{-1}[\{\alpha\}] = \emptyset \) as follows:
(a): \( \{ \infty \} \) is closed in \((Y^*, \mathcal{U}^*)\) and so, by continuity, \( F^{-1}[\{ \infty \}] \) is closed in \( c(A) \) and thus closed in \( X \).

(b): \( F^{-1}[\{ \infty \}] \subseteq c(A) \) trivially.

(c): \( F^{-1}[\{ \infty \}] \subseteq CA \), for if \( a \in A \), then \( F(a) = F|_A(a) = f(a) \in Y \).

So, by (a) - (c), \( F^{-1}[\{ \infty \}] \) is an \( X \)-closed subset of \( c(A) \setminus A \), and thus, by Theorem 1.5(c), \( F^{-1}[\{ \infty \}] = \emptyset \). It follows that \( F[c(A)] \subseteq Y \). Hence we have \( F: (c(A), c(A) \cap \mathcal{U}) \to (Y, \mathcal{U}) \) is continuous with \( F|_A = f \).

This proves the result.

**Corollary 3.34:** Any continuous, real-valued function defined on a \( g \)-closed set \( A \) has a continuous extension to \( c(A) \).

**Proof:** The real line with the usual topology is a locally compact, Hausdorff space. Use Theorem 3.33.
CHAPTER 4

G-CLOSED SETS, INHERITANCE, AND AN EXTENSION THEOREM

In this chapter we consider the question of when we can replace "closed" by "g-closed" in a number of well-known theorems. Primarily, we shall examine the inheritance of topological and uniform properties by g-closed subspaces. It will be shown that many "compactness-type" properties (e.g., compactness, sequential compactness, paracompactness); local compactness; completeness in uniform, pseudometric, and proximity spaces; and normality will be inherited by g-closed subspaces. That these properties are inherited by closed subspaces then becomes an immediate corollary. We shall also look at a "generalized" version of Urysohn's Lemma and the Tietze Extension Theorem.

It should be noted that the general results of this chapter may be applied to the specific examples of g-closed sets in Chapter 2 to yield a number of additional corollaries. We shall include some, but not all of them in this chapter.

Compactness and Related Properties

Theorem 4.1 [12; Theorem 3.1]: If A is a g-closed subset of the compact space (X, J), then A is compact.

Corollary 4.2: If X is compact and A ⊆ X, then A' is compact.
Proof: Combine Theorems 2.2 and 4.1.

The next two results are proved similarly to Theorem 4.1, and we omit the proofs.

Theorem 4.3: If $A$ is a $g$-closed subset of the countably compact space $(X, J)$, then $A$ is countably compact.

Theorem 4.4: If $A$ is a $g$-closed subset of the Lindelof space $(X, J)$, then $A$ is Lindelof.

Theorem 4.5: If $A$ is a $g$-closed subset of the sequentially compact space $(X, J)$, then $A$ is sequentially compact.

Proof: Let $(a_n)$ be a sequence in $A$. By the sequential compactness of $X$, there is an $x \in X$ and a subsequence $(a_{n_k})$ such that $\lim a_{n_k} = x$. Thus $x \in c(A)$, and by Theorem 1.5(d) there is an $a \in c(x) \cap A$. Clearly $\lim a_{n_k} = a$, and so $A$ is sequentially compact.

Our next theorem will treat inheritance of paracompactness, metacompactness, hypocompactness, and full normality by $g$-closed subspaces, and we review these definitions here:

Definition 4.6: A space is paracompact iff every open cover has an open, locally finite refinement which is also a cover. (This is the definition found in Gaal [8]. Kelley [10] also requires that the space be regular, while Willard [19] requires that the
space be Hausdorff.)

**Definition 4.7:** A space is **metacompact** iff every open cover has an open, point finite refinement which is also a cover. (See Kelley [10].)

**Definition 4.8:** A space is **hypocompact** iff every open cover has an open, star finite refinement which is also a cover, where a star finite family is one in which each member of the family intersects at most finitely many of the other members. (See Gaal [8].)

**Definition 4.9:** A refinement \( \{V_\beta : \beta \in \Gamma \} \) of \( \{0_\alpha : \alpha \in \Delta \} \) is a **star refinement** iff for each point \( x \) there is an \( \alpha^* \in \Delta \) such that \( \bigcup \{V_\beta : x \in V_\beta \} \subset 0_{\alpha^*} \). A space is **fully normal** iff every open cover has an open star refinement which is also a cover. (See Kelley [10].)

Each of these properties is inherited by closed subspaces. We prove a stronger result in:

**Theorem 4.10:** If \( A \) is a g-closed subset of the paracompact (respectively, metacompact, hypocompact, fully normal) space \((X, \mathcal{J})\), then \((A, A \cap \mathcal{J})\) is paracompact (respectively, metacompact, hypocompact, fully normal).

*Proof:* Let \( A = \bigcup \{A \cap 0_\alpha : \alpha \in \Delta \} \) where \( 0_\alpha \in \mathcal{J} \) for all \( \alpha \in \Delta \). Then \( A \subset \bigcup \{0_\alpha : \alpha \in \Delta \} \) and so \( c(A) \subset \bigcup \{0_\alpha : \alpha \in \Delta \} \).
Thus $X = \mathcal{C}(A) \cup \bigcup \{O_\alpha : \alpha \in \Delta \}$ and so there is a family 
\( \{V_\beta : \beta \in \Gamma \} \) of $X$-open sets which covers $X$ and which is a 
locally finite (respectively, point finite, star finite, star) 
refinement of \( \{\mathcal{C}(A)\} \cup \{O_\alpha : \alpha \in \Delta \} \). Then \( \{A \cap V_\beta : \beta \in \Gamma \} \)
is an $A$-open cover of $A$, and we assert \( \{A \cap V_\beta : \beta \in \Gamma \} \)
refines \( \{A \cap O_\alpha : \alpha \in \Delta \} \). For, if $\beta \in \Gamma$, then either $V_\beta \subseteq O_{\alpha^\#}$
for some $\alpha^\# \in \Delta$, or $V_\beta \subseteq \mathcal{C}(A)$. In the first case,
$A \cap V_\beta \subseteq A \cap O_{\alpha^\#}$, and in the second case $A \cap V_\beta = \emptyset$. Finally,
we note that \( \{A \cap V_\beta : \beta \in \Gamma \} \) is a locally finite (respectively, 
point finite, star finite, star) family since \( \{V_\beta : \beta \in \Gamma \} \) is,
and thus $(A, A \cap \mathcal{Q})$ is paracompact (respectively, metacompact, 
hypocompact, fully normal).

**Corollary 4.11:** If $(X, \mathcal{U})$ is a uniform space and $(\hat{X}, \mathcal{U})$ is 
the associated hyperspace as defined in Remark 2.34, and if 
$(\hat{X}, \mathcal{U})$ is compact (countably compact, Lindelof, sequentially 
compact, paracompact, metacompact, hypocompact, fully normal),
then so is $(X, \mathcal{U})$.

**Proof:** By Theorem 2.35, $i[X]$ is $g$-closed in $(\hat{X}, \mathcal{P}(\mathcal{U}))$
and thus $i[X]$ will inherit each of these compactness properties 
by the previous five theorems. The result follows since $i$ is a
unimorphism.

**Remark 4.12:** The preceding results show that many variants of 
compactness are inherited by $g$-closed subspaces. We shall now
mention two compactness-type properties which will not be inherited
by g-closed sets. As in Thron [18], we define a space to be
B-W compact (for Bolzano-Weierstrass) if and only if every
infinite subset has at least one limit point. This very weak form
of compactness is easily seen to be inherited by closed subspaces,
but it will not be inherited by g-closed subspaces. (Let
\((X, \mathcal{J})\) be infinite and discrete in Theorem 4.13.) The second
property, pseudocompactness, appears in Willard [19]. A space
is pseudocompact if and only if every continuous, real-valued
mapping on the space is bounded. Example 4.14 treats pseudo-
compactness. These two properties will be encountered again in
Chapter 5.

**Theorem 4.13**: Any topological space is homeomorphic to a dense,
g-closed subspace of a B-W compact space.

**Proof**: Let \((X, \mathcal{J})\) be a space; let \(Y = \{a, b\}\) with \(\mathcal{U}
the indiscrete topology; and let \((Z, \mathcal{K}) = (X \times Y, \mathcal{J} \times \mathcal{U})\).
We assert:

(a): \((Z, \mathcal{K})\) is B-W compact. For, suppose \(E\) is an
infinite subset of \(Z\) and \((x, a) \in E\) for some \(x \in X\). Then
\((x, b) \in O \times U \in \mathcal{J} \times \mathcal{U}\) implies \(U = Y\), and so
\((x, a) \in E \cap O \times U \cap \mathcal{C}_{Z}((x, b))\). Consequently, \((x, b) \in E'\). If,
for no \(x \in X\) is \((x, a) \in E\), then there is an \(x^* \in X\) such
that \((x^*, b) \in E\) and a symmetric argument shows \((x^*, a) \in E'\).

(b): \((X, \mathcal{J})\) is homeomorphic to \(X \times \{a\}\) with the relative
topology; \(X \times \{a\}\) is g-closed in \(Z\) by Theorem 1.3; and
$X \times \{a\}$ is clearly dense.

**Example 4.14:** A $g$-closed subspace of a pseudocompact space need not be pseudocompact. For if we let $X$ be the set of all natural numbers and $\mathcal{J} = \{O: O = \emptyset \text{ or } 1 \in O\}$, then $(X, \mathcal{J})$ is pseudocompact since any continuous, real-valued map on $X$ is constant. However, if we let $A = X \setminus \{1\}$, then $A$ is $g$-closed - in fact, closed - in $X$; but, since $(A, A \cap \mathcal{J})$ is discrete, the unbounded mapping $f: A \to \mathbb{R}$ given by $f(n) = n$ is continuous.

**Remark 4.15:** In [14], Levine defines a C-space to be one in which closures of compact sets are compact, and he shows that closed subspaces of C-spaces are C-spaces. We generalize that result in the following:

**Theorem 4.16:** If $A$ is a $g$-closed subset of the C-space $(X, \mathcal{J})$, then $(A, A \cap \mathcal{J})$ is a C-space.

**Proof:** Let $K \subseteq A$ be $A$-compact. Then $K$ is $X$-compact and thus $c(K)$ is $X$-compact as well. But $A \cap c(K)$ is $g$-closed in $X$ by Theorem 3.6, and so $A \cap c(K)$ is $g$-closed in $c(K)$ by Theorem 3.15. Consequently, by Theorem 4.1, $c_A(K) = A \cap c(K)$ is compact, and it follows that $(A, A \cap \mathcal{J})$ is a C-space.

Although the intersection of a closed set and a compact set is compact, the analogous statement for $g$-closed sets is false, as the following example shows:
Example 4.17: Let \( X \) be the real numbers and \( \mathcal{J} = \{ \emptyset \} \cup \{(a, \infty): a \in X\} \). Let \( K = [0,1] \) and \( A = (-\infty,0) \cup (0,1] \). Then \( K \) is clearly compact, and \( A \) is \( g \)-closed by Theorem 1.3; but \( A \cap K = (0,1] \) which is not compact since \( \{ \frac{1}{n}, \infty): n \geq 1 \} \) is an open cover of \( (0,1] \) with no finite subcover.

**Local Compactness**

Remark 4.18: Using Kelley's definition of local compactness (i.e., each point has a compact neighborhood), Levine proved in [12] that a \( g \)-closed subspace of a locally compact, regular space is locally compact. In fact, the theorem will remain true if "regular" is replaced by "R_0" (see Definition 2.3). But first we need a lemma:

**Lemma 4.19:** Suppose \((X, \mathcal{J})\) is an \( R_0 \)-space with \( A \subseteq X \) \( g \)-closed and \( K \subseteq X \) compact. Then if we let \( K^* = A \cap \bigcup \{c(x): x \in K\} \), \( K^* \) is also compact.

**Proof:** Let \( \{0_\alpha: \alpha \in \Delta\} \) be an open cover of \( K^* \). We assert that \( c(A) \cap K \subseteq \bigcup 0_\alpha: \alpha \in \Delta\). For, if \( x \in c(A) \cap K \), then, by Theorem 1.5(d), there is an \( a \in A \) such that \( a \in c(x) \). Since \( a \in A \cap c(x) \subseteq K^* \subseteq \bigcup 0_\alpha: \alpha \in \Delta\), \( a \in 0_{\alpha^*} \) for some \( \alpha^* \in \Delta \) and thus \( x \in 0_{\alpha^*} \), establishing the assertion. But, \( c(A) \cap K \) is compact, and so \( c(A) \cap K \subseteq 0_{\alpha_1} \cup \cdots \cup 0_{\alpha_n} \). We claim that \( K^* \subseteq 0_{\alpha_1} \cup \cdots \cup 0_{\alpha_n} \). For, if \( a \in K^* \), then \( a \in A \cap c(x) \) for some \( x \in K \). By Theorem 2.4, \( a \in c(x) \) implies
Thus, \( a \in 0_{\alpha_1} \cup \ldots \cup 0_{\alpha_n} \). We conclude that \( K^* \leq 0_{\alpha_1} \cup \ldots \cup 0_{\alpha_n} \) and thus \( K^* \) is compact.

**Theorem 4.20:** If \( A \) is a g-closed subset of the locally compact, \( R_0 \)-space \((X, \mathcal{J})\), then \((A, A \cap \mathcal{J})\) is locally compact.

**Proof:** Let \( a \in A \). Then \( a \in 0 \leq K \) for some \( 0 \in \mathcal{J} \) and some \( K \) compact in \( X \). Using the notation of Lemma 4.19,

\[
a \in A \cap 0 \subseteq A \cap K \subseteq A \cap \bigcup \{c(x) : x \in K\} = K^* \subseteq A
\]

where \( K^* \) is compact in \( X \) and thus in \( A \). Hence \((A, A \cap \mathcal{J})\) is locally compact.

**Remark 4.21:** Another local compactness property requires that the topology have a base consisting of compact neighborhoods. In fact, Willard [19] actually defines local compactness in this manner.

**Theorem 4.22:** Suppose \((X, \mathcal{J})\) is \( R_0 \) and \( \mathcal{J} \) has a base consisting of compact neighborhoods. Then, if \( A \) is a g-closed subset of \( X \), \( A \cap \mathcal{J} \) has a base of compact neighborhoods.

**Proof:** Let \( a \in A \cap 0 \) where \( 0 \in \mathcal{J} \). Then for some \( U \in \mathcal{J} \) and some \( K \) compact in \( X \), \( a \in U \subseteq K \subseteq 0 \). Hence \( a \in A \cap U \subseteq A \cap K \subseteq A \cap \bigcup \{c(x) : x \in K\} = K^* \subseteq A \), where \( K^* \) is compact by Lemma 4.19. \( K^* \) is thus an \( A \)-compact neighborhood of \( a \), and it suffices to show that \( K^* \subseteq A \cap 0 \). So, for \( b \in K^* \), \( b \in c(x) \) for some \( x \in K \). By the \( R_0 \) property,
Remark 4.23: Yet a third variant of local compactness is called strong local compactness by Cullen [4]. A space is strongly locally compact if and only if each point has an open neighborhood whose closure is compact. Here again, the literature is ambiguous, for Royden [17] calls this property local compactness. A truly confusing situation is averted since these three different concepts of local compactness agree on Hausdorff or regular spaces.

Theorem 4.24: If \( A \) is a \( g \)-closed subset of a strongly locally compact space \((X, \mathcal{J})\), then \((A, A \cap \mathcal{J})\) is strongly locally compact.

Proof: Let \( a \in A \). Then for some \( 0 \in \mathcal{J}, a \in 0 \) and \( c(0) \) is \( X \)-compact. Since \( A \) is \( g \)-closed in \( X \), \( A \cap c(0) \) is \( g \)-closed in \( X \) by Theorem 3.6, and so \( A \cap c(0) \) is \( g \)-closed in \( c(0) \) by Theorem 3.15. But \( c(0) \) is compact, and so \( A \cap c(0) \) is compact in \( c(0) \), and thus in \((A, A \cap \mathcal{J})\), by Theorem 4.1.

Hence \( a \in A \cap 0 \subseteq A \cap c(0) \) where \( A \cap c(0) \) is a closed, compact subset of \((A, A \cap \mathcal{J})\). It follows that \( a \in A \cap 0 \subseteq c_A(A \cap 0) \subseteq A \cap c(0) \) and that \( c_A(A \cap 0) \) is \( A \)-compact. Thus, \( A \cap 0 \) is an \( A \)-neighborhood of \( a \) whose closure in \( A \) is compact.

Corollary 4.25: If \((X, \mathcal{U})\) is a uniform space and \((\hat{X}, \hat{\mathcal{U}})\) is the
associated hyperspace as in Remark 2.34, then \((X, \mathcal{U})\) is locally compact, strongly locally compact, or has a compact neighborhood base if \((\hat{X}, \hat{\mathcal{U}})\) possesses the corresponding property.

**Proof:** Combine the three previous theorems and Theorem 2.35, while noting that a uniform space, being completely regular, must be \(R_0\) as well.

**Completeness**

**Theorem 4.26** [12; Theorem 3.4]: If \(A\) is a \(g\)-closed subset of the complete uniform space \((X, \mathcal{U})\), then \((A, A \times A \cap \mathcal{U})\) is complete.

**Proof:** Let \(S: D \to A\) be an \(A \times A \cap \mathcal{U}\)-Cauchy net. Then \(S\) is \(\mathcal{U}\)-Cauchy as well, and so there is an \(x \in X\) such that \(\lim S = x\) in \(\mathcal{J}(\mathcal{U})\). Consequently, \(x \in c(A)\) and there is an \(a \in c(x) \cap A\) by Theorem 1.5(d). Clearly \(\lim S = a\) in \((A, A \cap \mathcal{J})\), and thus \((A, A \times A \cap \mathcal{U})\) is complete.

**Remark 4.27:** We summarize the previous theorem and Theorem 2.20 by stating that a complete subspace of a uniform space is \(g\)-closed and a \(g\)-closed subspace of a complete uniform space is complete.

**Corollary 4.28:** If \((X, \mathcal{U})\) is a uniform space and its hyperspace \((\hat{X}, \hat{\mathcal{U}})\) is complete, then \((X, \mathcal{U})\) is complete.

**Proof:** Apply Theorems 4.26 and 2.35.
Corollary 4.29: Under the definitions and notation of Remark 2.32, 
\((Y, \mathcal{V})\) is complete iff \((\mathcal{J}, \mathcal{U})\) is complete.

Proof: Necessity: This is proved in Kelley [10], p. 227.

Sufficiency: By Theorem 2.33, \(\phi[Y]\) is g-closed in
\((\mathcal{J}, \mathcal{J}(\mathcal{U}))\) and so \(\phi[Y], \phi[Y] \times \phi[Y] \cap \mathcal{U}\) is complete by
Theorem 4.26. Since \(\phi\) is a unimorphism, \((Y, \mathcal{V})\) is complete
as well.

Theorem 4.30: If \(A\) is a g-closed subset of the complete
pseudometric space \((X, d)\), then \((A, d\big|_A \times A)\) is complete.

Proof: If \(\mathcal{U}(d)\) is the uniformity generated by \(d\), then
\((X, \mathcal{U}(d))\) is a complete uniform space by Kelley [10], Theorem
6.24. Moreover, since \(\mathcal{J}(\mathcal{U}(d)) = \mathcal{J}(d), A\) is g-closed in
\((X, \mathcal{J}(\mathcal{U}(d)))\) and thus \((A, A \times A \cap \mathcal{U}(d))\) is a complete uniform
space by Theorem 4.26. Again applying the result of Kelley,
we conclude that \((A, d\big|_A \times A)\) is a complete pseudometric space.

Remark 4.31: Summarizing Corollary 2.21 and Theorem 4.30, we
have that a complete subspace of a pseudometric space is g-closed
and a g-closed subspace of a complete pseudometric space is
complete.

Corollary 4.32: Under the definitions and notation of Remark
2.30 and Theorem 2.31, \((Y, d)\) is complete iff \((B(X,Y), \sigma)\) is
complete.

Proof: Necessity: This is a standard result. See Cullen [4],
Sufficiency: By Theorem 2.31, \( \bar{\phi}(Y) \) is a \( g \)-closed subspace of \((B(X,Y), \sigma)\) and thus is complete by Theorem 4.30. Since \( \bar{\phi} \) is an isometry, \((Y, d)\) is also complete.

Remark 4.33: Proximity spaces are discussed in Willard [19] and Engelking [6]. The following definition of completeness in proximity spaces is found in Nachman [16].

Definition 4.34: If \( \delta \) is a proximity on \( X \), let \( P(\delta) \) be the union of all uniformities on \( X \) which induce \( \delta \). (See Willard [19], section 40.)

Definition 4.35: Let \((X, \delta)\) be a proximity space and \( S: D \to X \) be a net. Then we say \( S \) is \( \delta \)-Cauchy iff \( S \) is \( \U \)-Cauchy for all \( \U \in P(\delta) \).

Definition 4.36: The proximity space \((X, \delta)\) is complete iff every \( \delta \)-Cauchy net converges in \((X, \tau(\delta))\).

We shall need the following routine lemma:

Lemma 4.37: Let \((X, \delta)\) be a proximity space, and let \( A \subseteq X \) with \( \delta_A \) the relative proximity on \( A \) (see Willard [19], p. 272). Then if a net \( S: D \to A \) is \( \delta_A \)-Cauchy, it is \( \delta \)-Cauchy.

Proof: Let \( U \in \U \in P(\delta) \). Then \( A \times A \cap U \subseteq A \times A \cap \U \), and \( A \times A \cap \U \) is a uniformity on \( A \) which induces \( \delta_A \). Thus,
by Cauchyness, there is a $d^* \in D$ such that $d, d' \geq d^*$ implies $(S(d), S(d')) \in A \times A \cap U \subseteq U$. Thus $S$ is $U$-Cauchy and so, by Definition 4.35, $S$ is $\delta$-Cauchy.

Theorem 4.38: If $(X, \delta)$ is a complete proximity space and $A$ is a $g$-closed subset of $(X, \tau(\delta))$, then $(A, \delta_A)$ is complete, where $\delta_A$ is the relative proximity on $A$.

Proof: Let $S: D \to A$ be $\delta^-A$-Cauchy. By Lemma 4.37, $S$ is $\delta$-Cauchy and thus $\lim S = x$ for some $x \in X$. By Theorem 1.5(d), there is an $a \in A \cap c(x)$, and it is clear that $\lim S = a$ in $(A, \tau(\delta_A))$. Hence $(A, \delta_A)$ is complete.

Corollary 4.39: A retract of a complete uniform (pseudometric, proximity) space is complete.

Proof: A uniform, pseudometric, or proximity space is regular, and thus a retract is $g$-closed by Corollary 2.16. The result follows from Theorems 4.26, 4.30, and 4.38.

Normality

Theorem 4.40 [12; Theorem 3.3]: If $A$ is a $g$-closed subset of the normal space $(X, \tau)$, then $(A, A \cap \tau)$ is normal.

Corollary 4.41: If $(X, \mathcal{U})$ is a uniform space and its hyperspace $(\hat{X}, \mathcal{U})$ is normal, then $(X, \mathcal{U})$ is normal.

Proof: Apply Theorems 2.35 and 4.40.

Corollary 4.42: If $A$ is a $g$-closed subset of the perfectly
normal space \((X, \mathcal{T})\), then \((A, A \cap \mathcal{T})\) is perfectly normal.

**Proof:** Since \((X, \mathcal{T})\) is perfectly normal, it is normal, and thus \((A, A \cap \mathcal{T})\) is normal by Theorem 4.40. Moreover, if \(G\) is an \(A\)-closed set, \(G = A \cap F\) where \(F\) is \(X\)-closed.

Consequently, \(F = \bigcap\{G_i : i \geq 1\}\) where \(G_i \in \mathcal{T}\), and so \(G = \bigcap\{A \cap G_i : i \geq 1\}\) is a \(G_\delta\) set in \(A\).

**Theorem 4.43** [12; Theorem 3.7]: Suppose \((X, \mathcal{T})\) is normal with \(A \cap F = \emptyset\), where \(A\) is \(g\)-closed and \(F\) is closed. Then there are disjoint, open sets \(O\) and \(U\) such that \(A \subseteq O\) and \(F \subseteq U\).

The following example shows that disjoint, \(g\)-closed sets in a normal space can not generally be separated by disjoint open sets.

**Example 4.44:** Let \(X = \{a, b\}\) and \(\mathcal{T}\) be indiscrete. Then \(\{a\}\) and \(\{b\}\) are disjoint and are \(g\)-closed by Theorem 1.3, but they can not be separated by disjoint, open sets. Moreover, \(\{a\}\) and \(\{b\}\) can not be functionally separated, since the only continuous mappings on \(X\) are constant.

In light of the previous example, the best we can do in attempting to generalize Urysohn's Lemma is:

**Theorem 4.45:** Let \((X, \mathcal{T})\) be normal and \(A \cap F = \emptyset\), where \(A\) is \(g\)-closed and \(F\) is closed. Then there is a continuous mapping
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\( f : (X, \mathcal{J}) \rightarrow [0,1] \) such that \( f[A] \subseteq \{0\} \) and \( f[F] \subseteq \{1\} \).

**Proof:** If \( A \subseteq CF \in \mathcal{J} \) implies \( c(A) \subseteq CF \). Thus \( c(A) \) and \( F \) are disjoint, closed sets, and Urysohn's Lemma provides a separating function for \( c(A) \) and \( F \), hence for \( A \) and \( F \).

We have more success with the Tietze Extension theorem, getting a complete generalization in:

**Theorem 4.46:** Let \( A \) be a \( g \)-closed subset of the normal space \((X, \mathcal{J})\). Then if \( f : (A, A \cap \mathcal{J}) \rightarrow \mathbb{R} \) is a continuous, real-valued mapping, there is a continuous mapping \( F^* : (X, \mathcal{J}) \rightarrow \mathbb{R} \) such that \( F^*|_A = f \).

**Proof:** By Corollary 3.34 there is an \( F : (c(A), c(A) \cap \mathcal{J}) \rightarrow \mathbb{R} \) such that \( F \) is continuous and \( F|_A = f \). By the Tietze Extension theorem, there is a continuous function \( F^* : (X, \mathcal{J}) \rightarrow \mathbb{R} \) such that \( F^*|_{c(A)} = F \). Thus \( F^*|_A = F|_A = f \).

**Corollary 4.47:** Any continuous, real-valued function defined on a complete subspace of a pseudometric space has a continuous extension to the whole space.

**Proof:** A pseudometric space is normal, and a complete subspace is \( g \)-closed by Corollary 2.21. The result follows from Theorem 4.46.

**Corollary 4.48:** If \((X, \mathcal{U})\) is a uniform space such that \((X, \mathcal{F}(\mathcal{U}))\) is normal, then any continuous, real-valued function
defined on a complete subspace has a continuous extension to the whole space.

**Proof:** Apply Theorems 2.20 and 4.46.

**Example 4.49:** Generally in a normal space, a continuous, real-valued function defined on a compact subset can not be extended continuously to the entire space. For example, let \( X = \{a,b,c,d\} \) and \( \mathcal{J} = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, X\} \). Then \((X, \mathcal{J})\) is normal since there are no non-empty, disjoint closed sets.

Let \( K = \{b,c\} \), a compact set, whose relative topology is discrete. Then \( f: K \to \mathbb{R} \) by \( f(b) = 0 \) and \( f(c) = 1 \) is continuous. But \( f \) can not be extended continuously to all of \( X \). For, if \( F: (X, \mathcal{J}) \to \mathbb{R} \) is continuous with \( F|_K = f \), then \( b \in F^{-1}[(-\frac{1}{2}, \frac{1}{2})] \in \mathcal{J} \), and so \( a \in F^{-1}[(-1, \frac{1}{2})] \). Similarly, \( c \in F^{-1}[(\frac{1}{2}, 2)] \in \mathcal{J} \), and so \( a \in F^{-1}[(\frac{1}{2}, 2)] \) as well, a contradiction.

However, we can prove:

**Theorem 4.50:** If \((X, \mathcal{J})\) is normal and weakly Hausdorff, a continuous, real-valued map on a compact subset has a continuous extension to the entire space.

**Proof:** By Theorem 2.11, a compact subset of a weakly Hausdorff space is \(g\)-closed. Apply Theorem 4.46.

**Corollary 4.51:** A continuous, real-valued map on a compact
subset of a normal, Hausdorff space has a continuous extension to the entire space.

Proof: Combine Theorems 2.9 and 4.50.

Corollary 4.52: A continuous, real-valued mapping on a compact subset of a normal, regular space has a continuous extension to the entire space.

Proof: Combine Theorems 2.9 and 4.50.
The following question arises naturally in topology:
Suppose $A$ is a dense subset of $(X, \mathcal{J})$ and $(A, A \cap \mathcal{J})$ has a certain topological property. Then does $(X, \mathcal{J})$ have the property as well?

Generally, little can be said about the transfer of properties from a dense subspace to the entire space. For instance, if $X$ is an uncountable set with $x^* \in X$ fixed, we can define a topology $\mathcal{J} = \{O: O = \emptyset, \text{ or } x^* \in O\}$. Then $\{x^*\}$ is dense in $(X, \mathcal{J})$ and $\{x^*\}$ is $T_2$, regular, normal, compact, and second axiom, but $X$ shares none of these properties.

Conversely, knowing that a space has a certain property, one can not generally conclude that a dense subspace has the same property. For instance, the closed unit interval in the reals is compact and connected, but the set of rationals in the unit interval is not.

However, if we require that the dense subspace also be $g$-closed, many topological and uniform properties will be shared by the dense set and the entire space. This chapter will show that, for the most part, a dense, $g$-closed set differs very little topologically from the whole space.
Some Preliminaries

Remark 5.1: We observe that if, for some property $P$, we have shown that a dense, $g$-closed subspace possesses property $P$ if and only if the entire space possesses this property, then we can immediately conclude that, for any $g$-closed set $A$ in $(X, J)$ and for any set $B$ such that $A \subseteq B \subseteq c(A)$, $(A, A \cap J)$ has property $P$ if and only if $(B, B \cap J)$ has property $P$. For, $A$ is dense in $(B, B \cap J)$, and, since $A$ is $g$-closed in $X$, $A$ will be $g$-closed in $B$ by Theorem 3.15. We note, then, that the results of this chapter could just as well have been formulated in terms of $g$-closed sets and their closures as in terms of dense, $g$-closed sets and the entire space.

Lemma 5.2: If $A$ is $g$-closed and dense in $X$, then for any open set $0$, $A \subseteq 0$ iff $X \subseteq 0$.

Proof: If $A \subseteq 0$, then $X = c(A) \subseteq 0$ since $A$ is $g$-closed and dense. If $X \subseteq 0$, $A \subseteq 0$ trivially.

Remark 5.3: In [11], Levine defines an equivalence relation on subsets of a topological space as follows: $A \equiv B$ if and only if, for each open set $0$, $A \subseteq 0$ iff $B \subseteq 0$. He then proves that a number of topological and uniform properties are shared by all members of an equivalence class. This becomes pertinent to what follows by virtue of Lemma 5.2, which states that $A \equiv X$ if $A$ is dense and $g$-closed. Thus, many of the theorems of this
chapter will be corollaries of the results of [11] and will be noted as such, even if we also supply a proof independent of this equivalence relation.

Lemma 5.4: Let $A$ be a dense subset of $(X, \mathcal{J})$, and let $0$ be open and $F$ be closed in $X$. Then if $A \cap 0 \subseteq A \cap F$, $0 \subseteq F$.

Proof: Let $x \in 0$. If $x \notin F$, then $x \in 0 \cap CF \in \mathcal{J}$ and so $0 \cap CF \cap A \neq \emptyset$, contradicting $A \cap 0 \subseteq A \cap F$.

Lemma 5.5: Let $A$ be a $g$-closed, dense subset of $(X, \mathcal{J})$, and let $0$ be open and $F$ be closed in $X$. Then if $A \cap F \subseteq A \cap 0$, $F \subseteq 0$.

Proof: Let $x \in F$ and suppose $x \notin 0$. Then $x \in F \cap CO \subseteq CA = X \setminus A = c(A) \setminus A$, where $F \cap CO$ is closed, contradicting Theorem 1.5(c).

Lemma 5.6: If $A$ is $g$-closed and dense in $(X, \mathcal{J})$, and if $A \cap 0 = A \cap F$ where $0$ is open and $F$ is closed in $X$, then $0 = F$.

Proof: Combine the results of the two previous lemmas.

Lemma 5.7: Suppose $A$ is a dense, $g$-closed subset of the $R_0$-space $(X, \mathcal{J})$. Suppose $A \cap 0 \subseteq A \cap U$ where $0$ and $U$ are in $\mathcal{J}$. Then $0 \subseteq U$.

Proof: Let $x \in 0$ and suppose $x \notin U$. Then $x \in CU$,
a closed set, and so \( c(x) \subseteq C U \). Moreover, \( c(x) \subseteq 0 \) by the \( R_0 \) property. Since \( A \) is dense, \( x \in c(A) \), and so, by Theorem 1.5(d), \( \emptyset \neq c(x) \cap A \subseteq C U \cap 0 \cap A = \emptyset \), a contradiction.

**Compactness and Related Properties**

**Theorem 5.8**: Let \( A \) be dense and \( g \)-closed in \((X, \mathcal{J})\). Then \( A \) is compact iff \( X \) is compact.

**Proof**: Necessity: Suppose \( A \) is compact and let \( X = \bigcup \{ O_\alpha : \alpha \in \Delta \} \). Then \( A \subseteq \bigcup \{ O_\alpha : \alpha \in \Delta \} \) and by compactness \( A \subseteq \bigcup \bigcup \bigcup O_\alpha \). Hence \( X = c(A) \subseteq \bigcup \bigcup \bigcup O_\alpha \), and \( X \) is compact.

Sufficiency: If \( X \) is compact, \( A \) is compact by Theorem 4.1.

(Alternately, use Remark 5.3 and Levine [11], Theorem 4.1.)

**Corollary 5.9**: If \( A \) is \( g \)-closed and \( A \subseteq B \subseteq c(A) \), then \( A \) is compact iff \( B \) is compact.

**Proof**: Use Remark 5.1 and Theorem 5.8.

**Corollary 5.10**: In a weakly Hausdorff space, closures of compact sets are compact.

**Proof**: If \( K \) is compact in a weakly Hausdorff space, then \( K \) is \( g \)-closed by Theorem 2.11, and thus \( c(K) \) is compact by Corollary 5.9.

**Remark 5.11**: The previous result says that a weakly Hausdorff space is a \( C \)-space (see Remark 4.15). As an immediate consequence
of Theorem 2.9 we have:

**Corollary 5.12:** A regular or a Hausdorff space is a C-space.

**Theorem 5.13:** Let \( A \) be dense and g-closed in \((X, \mathcal{J})\). Then \( A \) is countably compact iff \( X \) is countably compact.

**Proof:** Apply Remark 5.3 and Levine [11], Theorem 4.1.

**Theorem 5.14:** Let \( A \) be dense and g-closed in \((X, \mathcal{J})\). Then \( A \) is Lindelof iff \( X \) is Lindelof.

**Theorem 5.15:** Let \( A \) be dense and g-closed in \((X, \mathcal{J})\). Then \( A \) is sequentially compact iff \( X \) is sequentially compact.

**Proof:** Apply Remark 5.3 and Levine [11], Theorem 4.2.

The following lemma will be useful in our discussion of paracompactness and hypocompactness:

**Lemma 5.16:** Suppose \( A \) is a dense, g-closed subset of \((X, \mathcal{J})\) such that, for every \( A \)-open cover of \( A \), there is a refinement of \( A \)-open sets which covers \( A \). Then, for every open cover of \( X \), there is a refinement of \( X \)-open sets which covers \( X \).

**Proof:** Let \( X = \bigcup \{ O_\alpha : \alpha \in \Delta \} \) where \( O_\alpha \in \mathcal{J} \) for all \( \alpha \in \Delta \). Then \( A = \bigcup \{ A \cap O_\alpha : \alpha \in \Delta \} \) is an \( A \)-open cover, and by hypothesis there is a family \( \{ V_\beta : \beta \in \Gamma \} \) such that

(a): \( V_\beta \in \mathcal{J} \) and thus \( A \cap V_\beta \in A \cap \mathcal{J} \).

(b): \( A = \bigcup \{ A \cap V_\beta : \beta \in \Gamma \} \).

and (c): For each \( \beta \in \Gamma \), there is an \( \alpha(\beta) \in \Delta \) such that
A \cap V_\beta \subseteq A \cap 0_\alpha(\beta) \\

We consider the family \( \{ V_\beta \cap 0_\alpha(\beta) : \beta \in \Gamma \} \). Clearly this is a family of \( X \)-open sets which refines \( \{ 0_\alpha : \alpha \in \Delta \} \), and it remains only to show it is a cover of \( X \). But, for \( x \in X = c(A) \), there is an \( a \in c(x) \cap A \) by Theorem 1.5(d), and thus \( a \in A \cap V_\beta \) for some \( \beta \in \Gamma \). Hence \( a \in V_\beta \cap 0_\alpha(\beta) \) by (c) above and \( a \in c(x) \) implies \( x \in V_\beta \cap 0_\alpha(\beta) \). It follows that \( X = \bigcup \{ V_\beta \cap 0_\alpha(\beta) : \beta \in \Gamma \} \), and the result is proved.

Theorem 5.17: Let \( A \) be a dense, \( g \)-closed subset of \( (X, \mathcal{J}) \). Then \( (A, A \cap \mathcal{J}) \) is paracompact iff \( (X, \mathcal{J}) \) is paracompact.

Proof: Necessity: Suppose \( (A, A \cap \mathcal{J}) \) is paracompact and \( X = \bigcup \{ 0_\alpha : \alpha \in \Delta \} \) where \( 0_\alpha \in \mathcal{J} \) for all \( \alpha \in \Delta \). Then \( \{ A \cap 0_\alpha : \alpha \in \Delta \} \) is an \( A \)-open cover of \( A \), and so there is an \( A \)-open, locally finite refinement \( \{ A \cap V_\beta : \beta \in \Gamma \} \) which covers \( A \). The family \( \{ V_\beta \cap 0_\alpha(\beta) : \beta \in \Gamma \} \) as constructed in Lemma 5.16 is an open cover of \( X \) which refines \( \{ 0_\alpha : \alpha \in \Delta \} \), and we need only show this family is locally finite. But for \( x \in X \), there is an \( a \in c(x) \cap A \) by Theorem 1.5(d), and thus for some \( U \in \mathcal{J} \), \( a \in A \cap U \) and \( (A \cap U) \cap (A \cap V_\beta) = \emptyset \) for \( \beta \neq \beta_1, \ldots, \beta_n \). Since \( a \in c(x) \cap U \), \( x \in U \). Moreover, for \( \beta \neq \beta_1, \ldots, \beta_n \), \( A \cap U \subseteq A \cap CV_\beta \) and so \( U \subseteq CV_\beta \) by Lemma 5.4. That is, \( U \) is an \( X \)-open neighborhood of \( x \) such that, for \( \beta \neq \beta_1, \ldots, \beta_n \), \( U \cap (V_\beta \cap 0_\alpha(\beta)) \subseteq U \cap V_\beta = \emptyset \). Thus \( \{ V_\beta \cap 0_\alpha(\beta) : \beta \in \Gamma \} \) is the required locally finite
refinement, and \((X, \mathcal{J})\) is paracompact.

Sufficiency: If \((X, \mathcal{J})\) is paracompact and \(A\) is \(g\)-closed, then \((A, A \cap \mathcal{J})\) is paracompact by Theorem 4.10.

**Theorem 5.18:** Let \(A\) be a dense, \(g\)-closed subset of \((X, \mathcal{J})\). Then \((A, A \cap \mathcal{J})\) is hypocompact iff \((X, \mathcal{J})\) is hypocompact.

**Proof:** Necessity: Suppose \((A, A \cap \mathcal{J})\) is hypocompact and let \(X = \bigcup \{ O_\alpha : \alpha \in \Delta \}\) be an open cover. Then \(\{ A \cap O_\alpha : \alpha \in \Delta \}\) is an \(A\)-open cover of \(A\), and thus there is an \(A\)-open, star finite refinement \(\{ A \cap V_\beta : \beta \in \Gamma \}\) which covers \(A\). We need only show that the family \(\{ V_\beta \cap O_\alpha(\beta) : \beta \in \Gamma \}\) as constructed in Lemma 5.16 is star finite. But, if \(\beta^* \in \Gamma\), there exist \(\beta_1, \ldots, \beta_n \in \Gamma\) such that \((A \cap V_{\beta^*}) \cap (A \cap V_\beta) = \emptyset\) for \(\beta \neq \beta_1\). Thus for \(\beta \neq \beta_1\), \(A \cap V_{\beta^*} \subseteq A \cap CV_\beta\), and it follows from Lemma 5.4 that \(V_{\beta^*} \subseteq CV_\beta\). Hence, for \(\beta \neq \beta_1, \ldots, \beta_n\), \((V_{\beta^*} \cap O_\alpha(\beta^*)) \cap (V_\beta \cap O_\alpha(\beta)) \subseteq V_{\beta^*} \cap V_\beta = \emptyset\) and so \(\{ V_\beta \cap O_\alpha(\beta) : \beta \in \Gamma \}\) is star finite. Consequently, \((X, \mathcal{J})\) is hypocompact.

Sufficiency: If \((X, \mathcal{J})\) is hypocompact and \(A\) is \(g\)-closed, then \((A, A \cap \mathcal{J})\) is hypocompact by Theorem 4.10.

We next treat the two types of compactness which were mentioned in Remark 4.12.

**Theorem 5.19:** Let \(A\) be a dense, \(g\)-closed subset of \((X, \mathcal{J})\).

Then if \((A, A \cap \mathcal{J})\) is \(B\)-\(W\) compact, \((X, \mathcal{J})\) is \(B\)-\(W\) compact.
Proof: Suppose \( B \subseteq X \) is an infinite subset. We assert \( B' \neq \emptyset \).

Case (i): Suppose \( A \cap B \) is infinite. Then by B-W compactness, there is an \( a \in A \) such that \( a \in (A \cap B)' \), the derived set being taken in the space \( (A, A \cap J) \). But then \( a \in (A \cap B)' \subseteq B' \), where the derived set is taken in \( (X, J) \).

Case (ii): Suppose \( A \cap B \) is a finite set. Then, if we define \( B^* = B \cap \text{c}A \), \( B^* \) will be an infinite subset of \( X \setminus A = \text{c}(A) \setminus A \). We claim that in \( (X, J) \), \( (B^*)' \neq \emptyset \). For otherwise, \( \text{c}(B^*) = B^* \cup (B^*)' = B^* \), and thus \( B^* \) is a non-empty, closed subset of \( \text{c}(A) \setminus A \), contradicting Theorem 1.5(c). Hence \( B' = (B^*)' \neq \emptyset \).

In either case, \( B' \neq \emptyset \), and so \( (X, J) \) is B-W compact.

Remark 5.20: The converse of Theorem 5.19 is false. See Theorem 4.13.

Theorem 5.21: Let \( A \) be a dense, g-closed subset of \( (X, J) \). Then \( (A, A \cap J) \) is pseudocompact iff \( (X, J) \) is pseudocompact.

Proof: Necessity: This implication will hold for any dense set \( A \), whether or not it is g-closed. For, if \( f: (X, J) \rightarrow \mathbb{R} \) is continuous, then \( f|_A: (A, A \cap J) \rightarrow \mathbb{R} \) is also continuous and thus bounded. But \( f[X] = f[c(A)] \subseteq c(f[A]) \), which is also bounded in \( \mathbb{R} \).

Sufficiency: Suppose \( (X, J) \) is pseudocompact and \( f: (A, A \cap J) \rightarrow \mathbb{R} \) is continuous. By Corollary 3.34, there
is a continuous mapping \( F : (X, \mathfrak{I}) \to \mathbb{R} \) such that \( F|_A = f \).

But then \( f[A] = F|_A[A] = F[A] \subseteq F[X] \), a bounded subset of the reals. Thus \( f \) is bounded and \((A, A \cap \mathfrak{I})\) is pseudocompact.

**Remark 5.22:** We note that, in light of Example 4.14, the sufficiency condition in the previous theorem could not have been proved using inheritance of pseudocompactness by \( g \)-closed sets.

**Theorem 5.23:** Let \( A \) be a dense, \( g \)-closed subset of \((X, \mathfrak{I})\).

Then \((A, A \cap \mathfrak{I})\) is strongly locally compact iff \((X, \mathfrak{I})\) is strongly locally compact (see Remark 4.23).

**Proof:** Necessity: Suppose \((A, A \cap \mathfrak{I})\) is strongly locally compact and \( x \in X \).

By Theorem 1.5(d) there is an \( a \in c(x) \cap A \), and so, for some \( O \in \mathfrak{I} \), \( a \in A \cap O \in A \cap \mathfrak{I} \) with \( c_A(A \cap O) \) compact. Now, \( a \in O \cap c(x) \) implies \( x \in O \). Moreover, \( c_A(A \cap O) = A \cap c(A \cap O) = A \cap c(0) \) since \( A \) is dense, and thus \( A \cap c(0) \) is compact in \( X \). But \( A \cap c(0) \) is also \( g \)-closed in \( X \) by Theorem 3.6, and so \( c(A \cap c(0)) \) is compact in \( X \) by Corollary 5.9. Finally, since \( A \) is dense, \( 0 \subseteq c(0) = c(A \cap O) \subseteq c(A \cap c(0)) \), and so \( c(0) \) is compact as well. Consequently, \( x \in O \subseteq c(0) \), a compact set, implies \((X, \mathfrak{I})\) is strongly locally compact.

Sufficiency: This follows immediately from Theorem 4.24.
Completeness

Theorem 5.24: Let \((X, \mathcal{U})\) be a uniform space and let \(A \subset X\) be dense and \(g\)-closed in \((X, \mathcal{J}(\mathcal{U}))\). Then \((A, A \times A \cap \mathcal{U})\) is complete iff \((X, \mathcal{U})\) is complete.

Proof: Necessity: Suppose \((A, A \times A \cap \mathcal{U})\) is complete and let \(S: D \rightarrow X\) be a \(\mathcal{U}\)-Cauchy net. Then for each \(d \in D\), \(S(d) \in X = c(A)\), and so by Theorem 1.5(d) there is an \(a(d) \in c(S(d)) \cap A\). We define a net \(T: D \rightarrow A\) by \(T(d) = a(d)\) and assert \(T\) is \(A \times A \cap \mathcal{U}\)-Cauchy. For, if \(U \in \mathcal{U}\), we select \(V \in \mathcal{U}\) such that \(V\) is symmetric and \(V \circ V \circ V \subseteq U\).

There is a \(d^{*} \in D\) such that \(d, d' \geq d^{*}\) implies \((S(d), S(d')) \in V\).

Now, for any \(d \geq d^{*}\), \(V[T(d)]\) is a \(\mathcal{J}(\mathcal{U})\)-neighborhood of \(T(d)\), and, since \(T(d) = a(d) \in c(S(d))\), we have \(S(d) \in V[T(d)]\).

That is, \((T(d), S(d)) \in V\) for \(d \geq d^{*}\). Similarly, using the symmetry of \(V\), we have \((S(d'), T(d')) \in V\) for \(d' \geq d^{*}\). Hence, for \(d, d' \geq d^{*}\), \((T(d), T(d')) \in V \circ V \circ V \subseteq U\) and thus \((T(d), T(d')) \in A \times A \cap U\). So, \(T\) is \(A \times A \cap \mathcal{U}\)-Cauchy as claimed, and by completeness, \(\lim T = a\) for some \(a \in A\). We need only show \(\lim S = a\) in \((X, \mathcal{J}(\mathcal{U}))\). So, let \(a \in \mathcal{U} \in \mathcal{J}(\mathcal{U})\). Then \(a \in A \cap \mathcal{U} = A \cap \mathcal{J}(\mathcal{U})\) and thus for some \(d^{\#} \in D\), \(T(d) \in A \cap \mathcal{U}\) for all \(d \geq d^{\#}\). But \(T(d) = a(d) \in c(S(d))\), and so \(S(d) \in \mathcal{U}\) for all \(d \geq d^{\#}\).

Thus \(\lim S = a\) and \((X, \mathcal{U})\) is complete.

Sufficiency: This follows directly from Theorem 4.26.
(Alternately, use Remark 5.3 and Levine [11], Theorem 5.1.)

**Corollary 5.25:** The closure of a complete subspace of a uniform space is complete.

**Proof:** If \((A, A \times A \cap \mathcal{U})\) is a complete subspace of \((X, \mathcal{U})\), then \(A\) is g-closed in \((X, \pi(\mathcal{U}))\) by Theorem 2.20. Thus \(A\) is g-closed and dense in \(c(A)\) by Theorem 3.15, and so \(c(A)\) is complete by Theorem 5.24.

**Theorem 5.26:** Let \((X, d)\) be a pseudometric space and let \(A \subseteq X\) be dense and g-closed in \((X, \mathcal{U}(d))\). Then \((A, d|_{A \times A})\) is complete iff \((X, d)\) is complete.

**Proof:** Necessity: Suppose \((A, d|_{A \times A})\) is complete and consider \(\mathcal{U}(d)\), the uniformity on \(X\) generated by \(d\). We know that \(A \times A \cap \mathcal{U}(d) = \mathcal{U}(d|_{A \times A})\) and Kelley [10], Theorem 6.24 implies \((A, \mathcal{U}(d|_{A \times A}))\) is a complete uniform space. Thus, since \(\mathcal{U}(\mathcal{U}(d)) = \mathcal{I}(d)\), we apply Theorem 5.24 and conclude \((X, \mathcal{U}(d))\) is complete. By another application of Kelley's result, it follows that \((X, d)\) is a complete pseudometric space.

Sufficiency: This follows immediately from Theorem 4.30.

**Corollary 5.27:** The closure of a complete subspace of a pseudometric space is complete.

**Proof:** If \((A, d|_{A \times A})\) is a complete subspace of \((X, d)\), then \(A\) is g-closed in \((X, \mathcal{I}(d))\) by Corollary 2.21. Thus \(A\) is g-closed and dense in \(c(A)\) by Theorem 3.15, and so
c(A) is complete by Theorem 5.26.

Connectedness and Normality

Theorem 5.28: Let A be a dense, g-closed subset of (X, J). Then A is connected iff X is connected.

Proof: Necessity: The closure of any connected set is connected.

Sufficiency: Suppose X is connected but A is not. Then A = (A ∩ 0) U (A ∩ U) where 0, U ∈ J, A ∩ 0 ≠ 0 ≠ A ∩ U, and A ∩ 0 ∩ U = 0. Then 0 and U are non-empty, and, since A ∩ 0 ⊂ A ∩ U, it follows from Lemma 5.4 that 0 ⊂ CU. Moreover A ⊂ 0 U U ∈ J implies X = c(A) ⊂ 0 U U. Thus 0 and U provide a disconnection for X, a contradiction.

Corollary 5.29: If A is g-closed and A ⊂ B ⊂ c(A), then A is connected iff B is connected.

Proof: Combine Remark 5.1 and Theorem 5.28.

Remark 5.30: Willard [19] defines a space to be extremally disconnected if and only if the closure of every open set is open. With this in mind, we prove:

Theorem 5.31: Let A be a dense, g-closed subset of (X, J). Then (A, A ∩ J) is extremally disconnected iff (X, J) is extremally disconnected.

Proof: Necessity: Suppose (A, A ∩ J) is extremally
disconnected and let \( 0 \in \mathcal{J} \). Then \( A \cap 0 \in A \cap \mathcal{J} \) and so 
\[
c_\mathcal{A}(A \cap 0) = A \cap U \text{ for some } U \in \mathcal{J}. \quad \text{But}
\]
\[
c_\mathcal{A}(A \cap 0) = A \cap c(A \cap 0) = A \cap c(0) \quad \text{since } A \text{ is dense. Thus}
\]
we have \( A \cap c(0) = A \cap U \), and it follows from Lemma 5.6 that 
\[
c(0) = U \in \mathcal{J}. \quad \text{Thus } (X, \mathcal{J}) \text{ is extremally disconnected.}
\]

Sufficiency: Any dense subspace of an extremally disconnected space is extremally disconnected. (See Willard [19], p. 107.)

**Theorem 5.32:** Let \( A \) be a dense, \( g \)-closed subset of \((X, \mathcal{J})\). Then \((A, A \cap \mathcal{J})\) is normal iff \((X, \mathcal{J})\) is normal.

**Proof:** Necessity: Suppose \((A, A \cap \mathcal{J})\) is normal and \( F \) and \( G \) are disjoint, \( X \)-closed sets. Then \( A \cap F \) and \( A \cap G \) are disjoint, \( A \)-closed sets, and so, for some \( 0, U \in \mathcal{J} \), 
\[
A \cap F \subseteq A \cap 0, \quad A \cap G \subseteq A \cap U, \quad \text{and } A \cap 0 \cap U = \emptyset. \quad \text{By Lemma 5.5, } F \subseteq 0 \text{ and } G \subseteq U. \quad \text{Moreover, } 0 \cap U = \emptyset, \text{ for if there is an } x \in 0 \cap U \in \mathcal{J}, \text{ then } A \cap 0 \cap U \neq \emptyset \text{ since } A \text{ is dense. This contradiction implies } (X, \mathcal{J}) \text{ is normal.}
\]

Sufficiency: This is an immediate consequence of Theorem 4.40.

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**Dense, \( G \)-closed Sets in \( R_0 \)-spaces**

We shall now consider some topological properties which, in order to be transferred from dense, \( g \)-closed subspaces to the entire space, require the additional assumption that the space be \( R_0 \). The first of these properties is regularity, and we provide the following example to show the \( R_0 \) assumption is needed:
Example 5.33: Let \((\mathbb{R}, \mathcal{U})\) be the reals with the usual topology, and let \(p \notin \mathbb{R}\). Define \(X = \mathbb{R} \cup \{p\}\) and \(\mathcal{J} = \{U^*: U^* \in \mathcal{U} \text{ or } U^* = \{p\} \cup U \text{ where } 0 \in U \in \mathcal{U}\}\). One can verify that \((X, \mathcal{J})\) is a topological space. We prove the following results about \((X, \mathcal{J})\):

(a) Let \(x \in \mathbb{R}\) with \(x > 0\). Then if \(x \in U^* \in \mathcal{J}\), there exist \(0 < a < x < b\) such that \(x \in (a, b) \subseteq U^*\).

Proof: If \(U^* \in \mathcal{U}\), the result is immediate. Otherwise, \(U^* = \{p\} \cup U\) for some \(0 \in U \in \mathcal{U}\). Then \(x \in U\) and the result follows.

(b) Let \(x \in \mathbb{R}\) and \(x < 0\). Then if \(x \in U^* \in \mathcal{J}\), there exist \(a < x < b < 0\) such that \(x \in (a, b) \subseteq U^*\).

Proof: The proof is similar to (a).

(c): If \(p \in U^* \in \mathcal{J}\), there exist \(a < 0 < b\) such that \(p \in \{p\} \cup (a, b) \subseteq U^*\).

Proof: If \(p \in U^* \in \mathcal{J}\) implies \(U^* = \{p\} \cup U\) where \(0 \in U \in \mathcal{U}\).

(d): Suppose \(0 < c < d\) in \(\mathbb{R}\). Then \(c_X((c, d)) \subseteq [c, d]\).

Proof: Let \(x \in X \setminus [c, d]\). If \(x > d\), \(x \in (d, \infty) \in \mathcal{J}\) with \((d, \infty) \cap (c, d) = \emptyset\). If \(x < c\), \(x \in (-\infty, c) \in \mathcal{J}\) with \((-\infty, c) \cap (c, d) = \emptyset\). If \(x = p\), then \(x \in \{p\} \cup (-\frac{1}{2}c, \frac{1}{2}c) \in \mathcal{J}\) with \((\{p\} \cup (-\frac{1}{2}c, \frac{1}{2}c)) \cap (c, d) = \emptyset\). So, if \(x \in X \setminus [c, d]\), then \(x \notin c_X((c, d))\).

(e): Suppose \(c < d < 0\) in \(\mathbb{R}\). Then \(c_X((c, d)) \subseteq [c, d]\).
Proof: The proof is similar to (d).

(f): Suppose \( c < 0 < d \) in \( \mathbb{R} \). Then
\[
c_X(\{p\} \cup (c,d)) \subset \{p\} \cup [c,d].
\]
Proof: The proof is similar to (d).

We now define \( A = X \setminus \{0\} \).

(g): \( A \) is dense in \( (X, \mathcal{J}) \), since if \( 0 \in U^* \in \mathcal{J} \), then \( 0 \in U^* \cap \mathbb{R} \in \mathcal{U} \) and so \( \emptyset \neq U^* \cap (\mathbb{R} \setminus \{0\}) \subset U^* \cap A \).

(h): \( A \) is \( g \)-closed in \( (X, \mathcal{J}) \), since \( A \subset U^* \in \mathcal{J} \) implies \( p \in U^* \) and thus \( 0 \notin U^* \). Hence \( U^* = X \) and \( A \) is \( g \)-closed by Theorem 1.3.

(i): \( (A, A \cap \mathcal{J}) \) is regular.

Proof: Let \( x \in A \cap U^* \) where \( U^* \in \mathcal{J} \). We shall find \( 0^* \in \mathcal{J} \) such that \( x \in A \cap 0^* \subset c_A(A \cap 0^*) \subset A \cap U^* \). But, since \( A \) is dense by (g), \( c_A(A \cap 0^*) = A \cap c_X(A \cap 0^*) = A \cap c_X(0^*) \) and so it suffices to show \( x \in A \cap 0^* \subset A \cap c_X(0^*) \subset A \cap U^* \).

Case (i): If \( x \in \mathbb{R} \) and \( x > 0 \), then \( x \in (a,b) \subset U^* \) for some \( 0 < a < x < b \) by (a). Choose \( c \) and \( d \) such that \( a < c < x < d < b \) and let \( 0^* = (c,d) \in \mathcal{J} \). Then, using (d),
\[
x \in A \cap 0^* \subset A \cap c_X(0^*) = A \cap c_X((c,d)) \subset A \cap [c,d] \subset A \cap (a,b) \subset A \cap U^*.
\]

Case (ii): If \( x \in \mathbb{R} \) and \( x < 0 \), modify the proof of case (i), using (b) and (e) instead of (a) and (d).

Case (iii): If \( x = p \), then \( p \in \{p\} \cup (a,b) \subset U^* \) for some \( a < 0 < b \) by (c) above. Choose \( c \) and \( d \) such that
\[ a < c < 0 < d < b \] and let \( O^* = \{p\} \cup (c,d) \in \mathcal{J} \). Then, using part (f) above, \( p \in A \cap O^* \subseteq A \cap c_x(O^*) = A \cap c_x(\{p\} \cup (c,d)) \subseteq A \cap (\{p\} \cup [c,d]) = \{p\} \cup [c,0) \cup (0,d] \subseteq \{p\} \cup (a,0) \cup (0,b) = A \cap ([p] \cup (a,b)) \subseteq A \cap U^* . \]

So, by cases (i)-(iii), \((A, A \cap \mathcal{J})\) is regular.

(j): \((X, \mathcal{J})\) is not regular. In fact, \((X, \mathcal{J})\) is not even \(R_0\). For \( \emptyset \in (-1,1) \in \mathcal{J} \), but \( p \in c_x(0) \) and \( p \notin (-1,1) \).

The previous example shows that a space may fail to be regular even though it possesses a dense, \( g \)-closed subspace which is regular. We shall need the \( R_0 \) assumption on \((X, \mathcal{J})\) in order to continue:

**Theorem 5.34**: Let \((X, \mathcal{J})\) be \( R_0 \) and let \( A \) be a dense, \( g \)-closed subset of \( X \). Then \((A, A \cap \mathcal{J})\) is regular iff \((X, \mathcal{J})\) is regular.

**Proof**: Necessity: Suppose \((A, A \cap \mathcal{J})\) is regular and let \( x \notin G \) where \( x \in X \) and \( G \) is \( X \)-closed. By the \( R_0 \) property, \( c(x) \subseteq CG \). Moreover, by Theorem 1.5(d), there is an \( a \in c(x) \cap A \subseteq CG \cap A \). Thus \( a \notin A \cap G \), an \( A \)-closed set, and so there are \( O \) and \( U \) in \( \mathcal{J} \) such that \( a \in A \cap O \), \( A \cap G \subseteq A \cap U \), and \( A \cap O \cap U = \emptyset \). Now, \( x \in O \) since \( a \in c(x) \cap O \), and \( G \subseteq U \) by Lemma 5.5. Finally, \( O \cap U = \emptyset \), for \( y \in O \cap U \in \mathcal{J} \) and \( y \in X = c(A) \) implies \( A \cap O \cap U \neq \emptyset \),
Thus \((X, \tau)\) is regular.

**Sufficiency:** Subspaces of regular spaces are regular.

(Alternately, use Remark 5.3 and Levine [11], Theorem 8.1.)

**Theorem 5.3.5:** Let \((X, \mathcal{J})\) be \(R_0\) and let \(A\) be dense and \(g\)-closed in \(X\). Then \((A, A \cap \mathcal{J})\) is completely regular iff \((X, \mathcal{J})\) is completely regular.

**Proof:** Necessity: Suppose \((A, A \cap \mathcal{J})\) is completely regular and \(x \notin G\) where \(x \in X\) and \(G\) is \(X\)-closed. Then \(a \in c(x)\) for some \(a \in A\) by Theorem 1.5(d), and \(c(x) \subseteq CG\) by the \(R_0\) property. Thus \(a \notin A \cap G\), an \(A\)-closed set, and so there is a continuous map \(f: (A, A \cap \mathcal{J}) \to [0,1]\) such that \(f(a) = 0\) and \(f[A \cap G] \subseteq \{1\}\). By Corollary 3.34 there is an \(F: (X, \mathcal{J}) \to [0,1]\) which is continuous and such that \(F|_A = f\). We claim that \(F(x) = 0\). For otherwise \(F(x) = \xi > 0\) and, since \(F|_A = f\), \(a \in F^{-1}([0,\xi]) \in \mathcal{J}\). But since \(a \in c(x)\), \(x \in F^{-1}([0,\xi])\), a contradiction. Moreover, we assert that \(F[G] \subseteq \{1\}\). For otherwise \(F(y) = 1 - \xi < 1\) for some \(y \in G\) and some \(\xi > 0\). But, because \(F[A \cap G] = f[A \cap G] \subseteq \{1\}\), we have \(A \cap G \subseteq A \cap F^{-1}([1 - \xi,1])\) and so \(G \subseteq F^{-1}([1 - \xi,1])\) by Lemma 5.5. Consequently, \(F(y) > 1 - \xi\), a contradiction. So, \(F(x) = 0\) and \(F[G] \subseteq \{1\}\), and thus \((X, \mathcal{J})\) is completely regular.

**Sufficiency:** Subspaces of completely regular spaces are completely regular.

(Alternately, use Remark 5.3 and Levine [11], Theorem 8.3.)
Corollary 5.36: Let \((X, \mathcal{J})\) be \(R_q\) and let \(A\) be dense and \(g\)-closed in \(X\). Then \((A, A \cap \mathcal{J})\) is uniformizable (proximizable) iff \((X, \mathcal{J})\) is uniformizable (proximizable).

Proof: A space is uniformizable or proximizable if and only if it is completely regular. Use Theorem 5.35.

Remark 5.37: At the end of this chapter we shall show not only that \((X, \mathcal{J})\) is uniformizable (proximizable) if \((A, A \cap \mathcal{J})\) is, but also that the required uniformity (proximity) on \(X\) can be an extension of the uniformity (proximity) on \(A\).

Theorem 5.38: Let \((X, \mathcal{J})\) be \(R_q\) and let \(A\) be a dense, \(g\)-closed subset of \(X\). Then \((A, A \cap \mathcal{J})\) is completely normal iff \((X, \mathcal{J})\) is completely normal.

Proof: Use Remark 5.3 and Levine [11], Corollary 8.6.

Theorem 5.39: Let \((X, \mathcal{J})\) be \(R_q\) and let \(A\) be a dense, \(g\)-closed subset of \(X\). Then \((A, A \cap \mathcal{J})\) is perfectly normal iff \((X, \mathcal{J})\) is perfectly normal.

Proof: Necessity: Suppose \((A, A \cap \mathcal{J})\) is perfectly normal. Then, in particular, \((A, A \cap \mathcal{J})\) is normal, and so \((X, \mathcal{J})\) is normal by Theorem 5.32 (where \(R_q\) was not assumed). Moreover, if \(F\) is an \(X\)-closed set, then \(A \cap F\) is \(A\)-closed and so

\[ A \cap F = \bigcap\{A \cap O_n : n \geq 1\} \text{ where } O_n \in \mathcal{J} \text{ for all } n. \]

Thus

\[ A \cap F \subseteq A \cap O_n \text{ for all } n, \text{ and so } F \subseteq O_n \text{ by Lemma 5.5. Hence } F \subseteq \bigcap\{O_n : n \geq 1\}. \]

We assert \(\bigcap\{O_n : n \geq 1\} \subseteq F\). For, if there
were an \( x \in \bigcap \{ O_n : n \geq 1 \} \) such that \( x \not\in F \), then by the \( R_0 \)
property, \( c(x) \subseteq O_n \) for each \( n \) and \( c(x) \subseteq CF \). It would follow
that \( \emptyset \neq c(x) \subseteq \bigcap \{ O_n : n \geq 1 \} \cap CF \subseteq X \setminus A = c(A) \setminus A \), contradicting
Theorem 1.5(c). Thus, the assertion holds and so \( F = \bigcap \{ O_n : n \geq 1 \} \).

Sufficiency: This follows immediately from Corollary 4.42.

**Theorem 5.40:** Let \((X, \mathcal{J})\) be \( R_0 \) and let \( A \) be a dense,
g-closed subset of \( X \). Then \((A, A \cap \mathcal{J})\) is separable iff
\((X, \mathcal{J})\) is separable.

**Proof:** Combine Remark 5.3 and Levine [11], Theorem 9.1.

**Theorem 5.41:** Let \((X, \mathcal{J})\) be \( R_0 \) and let \( A \) be a dense,
g-closed subset of \( X \). Then \((A, A \cap \mathcal{J})\) is first axiom iff
\((X, \mathcal{J})\) is first axiom.

**Proof:** Necessity: Suppose \((A, A \cap \mathcal{J})\) is first axiom
and \( x \in X \). By Theorem 1.5(d), there is an \( a \in A \cap c(x) \).
By hypothesis, there is a countable family \( \{ O_n : n \geq 1 \} \subseteq \mathcal{J} \)
such that \( \{ A \cap O_n : n \geq 1 \} \) is a base at \( a \) for \( A \cap \mathcal{J} \). We
assert that \( \{ O_n : n \geq 1 \} \) is a base at \( x \) for \( \mathcal{J} \). Firstly,
\( a \in A \cap O_n \) for each \( n \) and \( a \in c(x) \) implies \( x \in O_n \) for each
\( n \). Moreover, if \( x \in O \in \mathcal{J} \), then, by \( R_0 \),
\( a \in A \cap c(x) \subseteq A \cap O \subseteq A \cap \mathcal{J} \), and thus for some \( m \),
\( a \in A \cap O_m \subseteq A \cap O \). Hence by Lemma 5.7, \( x \in O_m \subseteq O \), and so
\((X, \mathcal{J})\) is first axiom.

Sufficiency: Subspaces of first axiom spaces are first axiom.
Theorem 5.42: Let $(X, \mathcal{J})$ be $R_0$ and let $A$ be dense and $g$-closed in $X$. Then $(A, A \cap \mathcal{J})$ is second axiom iff $(X, \mathcal{J})$ is second axiom.

Proof: Combine Remark 5.3 and Levine [11], Theorem 9.3.

Theorem 5.43: Let $(X, \mathcal{J})$ be $R_0$ and let $A$ be dense and $g$-closed in $X$. Then $(A, A \cap \mathcal{J})$ is locally connected iff $(X, \mathcal{J})$ is locally connected.

Proof: Use Remark 5.3 and Levine [11], Theorem 7.6.

Theorem 5.44: Let $(X, \mathcal{J})$ be $R_0$ and let $A$ be dense and $g$-closed in $X$. Then $(A, A \cap \mathcal{J})$ is pathwise connected iff $(X, \mathcal{J})$ is pathwise connected.

Proof: Use Remark 5.3 and Levine [11], Theorem 7.5.

Theorem 5.45: Let $(X, \mathcal{J})$ be $R_0$ and let $A$ be dense and $g$-closed in $X$. Then $(A, A \cap \mathcal{J})$ is metacompact iff $(X, \mathcal{J})$ is metacompact.

Proof: Necessity: Suppose $(A, A \cap \mathcal{J})$ is metacompact and let $X = \bigcup \{0_\alpha: \alpha \in \Delta\}$ where $0_\alpha \in \mathcal{J}$ for all $\alpha \in \Delta$. Then $(A \cap 0_\alpha: \alpha \in \Delta)$ is an $A$-open cover of $A$ and so there exists an $A$-open, point finite refinement $(A \cap V_\beta: \beta \in \Gamma)$ which covers $A$. Let $(V_\beta \cap 0_{\alpha(\beta)}: \beta \in \Gamma)$ be as in Lemma 5.16. Then this family is an open cover of $X$ which refines $(0_\alpha: \alpha \in \Delta)$ by Lemma 5.16, and it remains only to show it is point finite. So, let $x \in X$. By Theorem 1.5(d) there is an $a \in c(x) \cap A$ and thus there is a
finite set \( \{ \beta_1, \ldots, \beta_n \} \) \( \subseteq \Gamma \) such that \( a \not\in A \cap V_\beta \) for \( \beta \neq \beta_1, \ldots, \beta_n \). We note that \( x \not\in V_\beta \cap O_{\alpha(\beta)} \) for \( \beta \neq \beta_1, \ldots, \beta_n \). For, if \( x \in V_\beta \cap O_{\alpha(\beta)} \) for some \( \beta \neq \beta_1, \ldots, \beta_n \), then, by \( R_o \), \( a \in A \cap c(x) \subseteq A \cap V_\beta \cap O_{\alpha(\beta)} \subseteq A \cap V_\beta \), a contradiction. Hence \( \{ V_\beta \cap O_{\alpha(\beta)} : \beta \in \Gamma \} \) is the required point finite refinement and \( (X, \mathcal{J}) \) is metacompact.

Sufficiency: This follows immediately from Theorem 4.10.

Theorem 5.46: Let \( (X, \mathcal{J}) \) be \( R_o \) and let \( A \) be dense and \( g \)-closed in \( X \). Then \( (A, A \cap \mathcal{J}) \) is fully normal iff \( (X, \mathcal{J}) \) is fully normal.

Proof: Necessity: Suppose \( (A, A \cap \mathcal{J}) \) is fully normal and let \( X = \bigcup \{ O_\alpha : \alpha \in \Delta \} \) where \( O_\alpha \in \mathcal{J} \) for all \( \alpha \in \Delta \). Then \( \{ A \cap O_\alpha : \alpha \in \Delta \} \) is an \( A \)-open cover of \( A \), and so there is an \( A \)-open, star refinement \( \{ A \cap V_\beta : \beta \in \Gamma \} \) which also covers \( A \).

Letting \( \{ V_\beta \cap O_{\alpha(\beta)} : \beta \in \Gamma \} \) be as in Lemma 5.16, we know that this family is an open cover of \( X \) which refines \( \{ O_\alpha : \alpha \in \Delta \} \), and we need only show \( \{ V_\beta \cap O_{\alpha(\beta)} : \beta \in \Gamma \} \) is a star refinement.

So, let \( x \in X \). By Theorem 1.5(d) there is an \( a \in c(x) \cap A \) and by hypotheses there is an \( \alpha^* \in \Delta \) such that \( \bigcup \{ A \cap V_\beta : \beta \in \Gamma \} \cap \alpha^* \). Thus \( A \cap \bigcup \{ V_\beta : \beta \in \Gamma \} \cap A \cap \alpha^* \), and it follows from Lemma 5.7 that \( \bigcup \{ V_\beta : \beta \in \Gamma \} \cap A \cap V_\beta \cap \alpha^* \). Now, we assert that 

\( a \in A \cap V_\beta \) iff \( x \in V_\beta \). For, if \( a \in A \cap V_\beta \), \( x \in V_\beta \) since \( a \in c(x) \). Conversely, if \( x \in V_\beta \), then \( c(x) \subseteq V_\beta \) by \( R_o \).
and so $a \in A \cap c(x) \subseteq A \cap V_\beta$. Consequently, $\cup\{V_\beta \cap O_\alpha(\beta) : \beta \in \Gamma\}$ and $x \in V_\beta \cap O_\alpha(\beta)$, $x \in V_\beta \subseteq \cup\{V_\beta : \beta \in \Gamma \}$ and $a \in A \cap V_\beta \subseteq O_\alpha^*$.

Thus $\{V_\beta \cap O_\alpha(\beta) : \beta \in \Gamma\}$ is the required star refinement and $(X, \mathcal{J})$ is fully normal.

Sufficiency: This follows directly from Theorem 4.10.

**Theorem 5.47**: Let $(X, \mathcal{J})$ be $R_0$ and let $A$ be dense and $g$-closed in $X$. Then $(A, A \cap \mathcal{J})$ is locally compact iff $(X, \mathcal{J})$ is locally compact.

**Proof**: Apply Remark 5.3 and Levine [11], Theorem 10.3.

**Theorem 5.48**: Let $(X, \mathcal{J})$ be $R_0$ and let $A$ be dense and $g$-closed in $X$. Then $A \cap \mathcal{J}$ has a base consisting of compact neighborhoods iff $\mathcal{J}$ has a base consisting of compact neighborhoods.

**Proof**: Necessity: Let $x \in O \in \mathcal{J}$. By Theorem 1.5(d) there is an $a \in c(x) \cap A$ and by the $R_0$ property, $a \in A \cap O$. Hence there is a $U \in \mathcal{J}$ and a $K \subseteq A$ compact such that $a \in A \cap U \subseteq K \subseteq A \cap O$. We define $K^* = \cup\{c(b) : b \in K\}$. Then we assert:

(a): $K^*$ is compact.

**Proof**: If $K^* \subseteq \cup\{O_\alpha : \alpha \in \Delta\}$ where $O_\alpha \in \mathcal{J}$, then $K \subseteq \cup\{O_\alpha : \alpha \in \Delta\}$ and so $K \subseteq O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$. But for $b \in K$, $c(b) \subseteq O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$ by the $R_0$ property, and so $K^* = \cup\{c(b) : b \in K\} \subseteq O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$.

(b): $x \in U \subseteq K^* \subseteq O$. 


Proof: Firstly, \( x \in U \) since \( a \in c(x) \cap U \). Secondly, if \( y \in U \), then for some \( a^* \in A \), \( a^* \in c(y) \) by Theorem 1.5(d). Hence \( a^* \in A \cap c(y) \subseteq A \cap U \subseteq K \) by \( R_0 \), and thus, by Theorem 2.4, \( y \in c(a^*) \subseteq \cup\{c(b) : b \in K\} = K^* \). It follows that \( U \subseteq K^* \). Finally, if \( z \in K^* \), then \( z \in c(b) \) for some \( b \in K \subseteq A \cap O \). Hence \( z \in c(b) \subseteq 0 \). This proves \( K^* \subseteq O \).

By (a) and (b) above, \( \mathfrak{J} \) has a base of compact neighborhoods.

Sufficiency: This follows directly from Theorem 4.22.

Extending Pseudometrics, Uniformities, and Proximities

In this final section of Chapter 5, we shall treat the problem of "extending" a pseudometric, uniformity, or proximity from a dense, \( g \)-closed subspace of an \( R_0 \)-space to the entire space. First, we shall need a remark to clarify the notation and terminology, and two preparatory lemmas.

Remark 5.49: If \( X \) is any non-empty set, we write:

\[
P(X) = \{d : d \text{ is a pseudometric on } X\}
\]

\[
U(X) = \{\mathcal{U} : \mathcal{U} \text{ is a uniformity on } X\}
\]

\[
\pi(X) = \{\delta : \delta \text{ is a proximity on } X\}
\]

Moreover, subspaces will be denoted as follows:

If \( d \in P(Y) \) and \( d^* \in P(X) \), then \( (Y, d) \subseteq (X, d^*) \) iff \( Y \subseteq X \) and \( d = d^* |_{Y \times Y} \).

If \( \mathcal{U} \in U(Y) \) and \( \mathcal{U}^* \in U(X) \), then \( (Y, \mathcal{U}) \subseteq (X, \mathcal{U}^*) \) iff \( Y \subseteq X \) and \( \mathcal{U} = \{Y \times Y \cap U^* : U^* \in \mathcal{U}^*\} \).
If $\delta \in \pi(Y)$ and $\delta^* \in \pi(X)$, then $(Y, \delta) \subseteq (X, \delta^*)$ iff $Y \subseteq X$ and for all $A, B \subseteq Y$, $A \delta B$ iff $A \delta^* B$.

**Lemma 5.50**: Let $(X, \mathcal{J})$ be a topological space with $A \subseteq X$.

Suppose there is a $d \in P(A)$ such that $\mathcal{J}(d) \subseteq A \cap \mathcal{J}$. Further, let $x \in c(A)$ and let $a_1 \in c(x) \cap A$ and $a_2 \in c(x) \cap A$.

Then $d(a_1, a_2) = 0$.

**Proof**: Let $\varepsilon > 0$. Then $a_1 \in \{b \in A: d(a_1, b) < \frac{1}{2} \varepsilon\} \in \mathcal{J}(d) \subseteq A \cap \mathcal{J}$. Thus $\{b \in A: d(a_1, b) < \frac{1}{2} \varepsilon\} = A \cap O_1$ for some $O_1 \in \mathcal{J}$. Similarly, $\{b \in A: d(a_2, b) < \frac{1}{2} \varepsilon\} = A \cap O_2$ for some $O_2 \in \mathcal{J}$. Now, $a_1 \in O_1 \cap c(x)$ implies $x \in O_1$. Likewise, $a_2 \in O_2$ implies $x \in O_2$, and so $x \in O_1 \cap O_2 \in \mathcal{J}$. But $x \in c(A)$ implies there is an $a \in A \cap O_1 \cap O_2$. Since $a \in A \cap O_1$, $d(a_1, a) < \frac{1}{2} \varepsilon$ and since $a \in A \cap O_2$, $d(a_2, a) < \frac{1}{2} \varepsilon$. Hence $d(a_1, a_2) \leq d(a_1, a) + d(a, a_2) < \varepsilon$, and, since $\varepsilon$ was arbitrary, it follows that $d(a_1, a_2) = 0$.

**Lemma 5.51**: Let $(X, \mathcal{J})$ be $R_0$ and let $A$ be a dense, $g$-closed subset of $X$. Suppose there is a $d \in P(A)$ such that $\mathcal{J}(d) \subseteq A \cap \mathcal{J}$. Define $d^*: X \times X \to [0, \infty)$ by $d^*(x, y) = d(a, b)$ where $a \in c(x) \cap A$ and $b \in c(y) \cap A$. Then

(a): $d^* \in P(X)$.

(b): $d^* |_{A \times A} = d$ and

(c): $\mathcal{J}(d^*) \subseteq \mathcal{J}$.

**Proof**: (a): We first must show $d^*$ is well-defined. So,
let $x, y \in X$. By Theorem 1.5(d), $c(x) \cap A \neq \emptyset \neq c(y) \cap A$.

Suppose $a_1, a_2 \in c(x) \cap A$ and $b_1, b_2 \in c(y) \cap A$. Then by Lemma 5.50, $d(a_1, b_1) \leq d(a_1, a_2) + d(a_2, b_2) + d(b_2, b_1) = 0 + d(a_2, b_2) + 0 = d(a_2, b_2)$. Symmetrically, $d(a_2, b_2) \leq d(a_1, b_1)$ and thus $d(a_1, b_1) = d(a_2, b_2)$. Consequently, $d^*$ is well-defined. Now, $d^*: X \times X \to [0, \infty)$ is clear. Moreover, if $x \in X$,

$b(a, a) = 0$ where $a \in c(x) \cap A$. Symmetry and the triangle inequality for $d^*$ follow immediately from the corresponding properties of $d$.

(b): If $a, b \in A$, then $a \in c(a) \cap A$ and $b \in c(b) \cap A$ and so $d^*(a, b) = d(a, b)$.

(c): Let $x \in \{y \in X: d^*(x, y) < \varepsilon\} \in \mathcal{F}(d^*)$, and let $a \in c(x) \cap A$. Then by hypothesis,

$a \in \{b \in A: d(a, b) < \varepsilon\} \subset \mathcal{F}(d) \subset A \cap \mathcal{F}$ and so

$\{b \in A: d(a, b) < \varepsilon\} = A \cap 0$ for some $0 \in \mathcal{F}$. Now, $a \in 0 \cap c(x)$ implies $x \in 0$. Further, we assert

$0 \subset \{y \in X: d^*(x, y) < \varepsilon\}$. For, if $y \in 0$, then $c(y) \subset 0$ by the $R_0$ property and thus there is a $b^* \in c(y) \cap A \subset A \cap 0$. Hence $d^*(x, y) = d(a, b^*) < \varepsilon$. It follows that $x \in 0 \subset \{y \in X: d^*(x, y) < \varepsilon\}$ and so $\mathcal{F}(d^*) \subset \mathcal{F}$.

Theorem 5.52: Let $(X, \mathcal{F})$ be $R_0$ and let $A$ be a dense, $\sigma$-closed subset of $X$. Suppose further that there is a $d \in P(A)$ such that $\mathcal{F}(d) = A \cap \mathcal{F}$. Then there is a $d^* \in P(X)$ such that
(a): \( \mathcal{J} = \mathcal{J}(d^*) \) and
(b): \((A, d) \subseteq (X, d^*)\).

**Proof:** Since \( \mathcal{J}(d) \subseteq A \cap \mathcal{J} \), we can let \( d^* \) be as defined in Lemma 5.51. Using the results of that lemma and Remark 5.49, we need only show \( \mathcal{J} \subseteq \mathcal{J}(d^*) \) in order to complete the theorem.

So, let \( x \in \Omega \in \mathcal{J} \). Then, by \( R_0 \), \( c(x) \subseteq 0 \), and so there is an \( a \in c(x) \cap A \subseteq A \cap 0 \subseteq A \cap \mathcal{J} = \mathcal{J}(d) \). Hence,

\[
a \in \{ b \in A : d(a, b) < \varepsilon \} \subseteq A \cap 0 \quad \text{for some } \varepsilon > 0.
\]

We assert that \( x \in \{ y \in X : d^*(x, y) < \varepsilon \} \subseteq 0 \). For, if \( y \in X \) with \( d^*(x, y) < \varepsilon \), then by Lemma 5.51, \( d(a, b') < \varepsilon \) for some \( b' \in c(y) \cap A \). Thus \( b' \in \{ b \in A : d(a, b) < \varepsilon \} \subseteq A \cap 0 \) and, since \( b' \in c(y) \) as well, it follows that \( y \in 0 \), verifying the assertion. Consequently, \( \mathcal{J} \subseteq \mathcal{J}(d^*) \) and the theorem is proved.

**Corollary 5.53:** Let \((X, \mathcal{U})\) be \( R_0 \) and let \( A \) be a dense, \( g \)-closed subset of \( X \). Then \((A, A \cap \mathcal{J})\) is pseudometrizable iff \((X, \mathcal{J})\) is pseudometrizable.

**Proof:** The necessity follows from Theorem 5.52, while any subspace of a pseudometrizable space is pseudometrizable.

We shall now prove the analogue of Theorem 5.52 for uniformities. Here we let \( \mathcal{U}_A(\mathcal{U}) = \{ d \in P(A) : 2d(d) \subseteq \mathcal{U} \} \) denote the gage of a uniformity \( \mathcal{U} \in U(A) \) (see Kelley [10], p. 188).

**Theorem 5.54:** Let \((X, \mathcal{J})\) be \( R_0 \) and let \( A \) be a dense,
g-closed subset of $X$. Suppose further that there is a
\[ \mathcal{U} \in \mathcal{U}(A) \] such that \( \mathcal{J}(\mathcal{U}) = A \cap \mathcal{I} \). Then there is a
\( \mathcal{U}^* \in \mathcal{U}(X) \) such that
\[
(a): \quad \mathcal{J} = \mathcal{J}(\mathcal{U}^*) \quad \text{and} \quad \\
(b): \quad (A, \mathcal{U}) \subseteq (X, \mathcal{U}^*).
\]

**Proof:** Let \( \delta_A(\mathcal{U}) = \{ \delta: \alpha \in \Delta \} \) be the gage of \( \mathcal{U} \).

Then \( \mathcal{U} = \sup_A \{ \mathcal{U}(\delta): \alpha \in \Delta \} \), the supremum in \( \mathcal{U}(A) \) of the
uniformities generated by the \( \delta \). So, for each \( \alpha \in \Delta \),
\[
\mathcal{J}(\delta) = \mathcal{J}(\mathcal{U}(\delta)) \subseteq \mathcal{J}(\mathcal{U}) = A \cap \mathcal{I}.
\]
Thus, if we define
\[
d^*_{\alpha}: X \times X \to [0, \infty) \quad \text{by} \quad d^*_{\alpha}(x, y) = (\alpha, a, b) \quad \text{where}
\]
a \( \in \mathcal{C}(x) \cap A \) and \( b \in \mathcal{C}(y) \cap A \), then we have by Lemma 5.51
that \( d^*_{\alpha} \in \mathcal{P}(X) \), \( (A, d_{\alpha}) \subseteq (X, d^*_{\alpha}) \), and \( \mathcal{J}(d^*_{\alpha}) \subseteq \mathcal{I} \).
We assert that \( \mathcal{J} = \sup_X \{ \mathcal{J}(d^*_{\alpha}): \alpha \in \Delta \} \). On the one hand,
\[
\sup_X \{ \mathcal{J}(d^*_{\alpha}): \alpha \in \Delta \} \subseteq \mathcal{I} \quad \text{is immediate. Conversely, let}
\]
x \( \in 0 \in \mathcal{I} \). By the \( R_o \) property, \( \mathcal{C}(x) \subseteq 0 \) and so there exists
an \( a' \in \mathcal{C}(x) \cap A \subseteq A = A \cap 0 \in A \cap \mathcal{I} = \mathcal{J}(\mathcal{U}) \). But
\[
\mathcal{J}(\mathcal{U}) = \mathcal{J}(\sup_A \{ \mathcal{U}(d^*_{\alpha}): \alpha \in \Delta \}) = \sup_A \{ \mathcal{J}(\mathcal{U}(d^*_{\alpha})): \alpha \in \Delta \} = \\
\sup_A \{ \mathcal{J}(d^*_{\alpha}): \alpha \in \Delta \}.
\]
Thus \( a' \in A \cap 0 \in \sup_A \{ \mathcal{J}(d^*_{\alpha}): \alpha \in \Delta \} \)
and so there are \( \alpha_1, \ldots, \alpha_n \) in \( \Delta \) and \( \varepsilon_1, \ldots, \varepsilon_n \) positive
such that \( a' \in S_{\alpha_1} \cap \cdots \cap S_{\alpha_n} \subseteq A \cap 0 \) where
\[
S_{\alpha} = \{ b \in A: d_{\alpha}(a', b) < \varepsilon_1 \}.
\]
We claim that
\[
x \in S^*_1 \cap \cdots \cap S^*_n \subseteq 0 \quad \text{where} \quad S^*_i = \{ y \in X: d^*_{\alpha}(x, y) < \varepsilon_1 \}.
\]
For, let \( y \in S^*_1 \cap \cdots \cap S^*_n \). By Theorem 1.5(d) and Lemma 5.51,
there is a \( b' \in \mathcal{C}(y) \cap A \) such that \( d_{\alpha}(a', b') = d^*_{\alpha}(x, y) < \varepsilon_1 \).
for \( i = 1, \ldots, n \). Thus \( b'_i \in S_1 \cap \cdots \cap S_n \subseteq A \cap 0 \). Since \( b'_i \in \mathcal{C}(y) \cap 0 \), \( y \in 0 \), and we have thereby proved that \( x \in S_1^* \cap \cdots \cap S_n^* \subseteq 0 \). But this implies that \( 0 \in \sup_{X} \{ \mathcal{I}(d^*): \alpha \in \Delta \} \), and our initial assertion is proved - i.e., \( \mathcal{I} = \sup_{X} \{ \mathcal{I}(d^*_\alpha): \alpha \in \Delta \} \).

Now we let \( \mathcal{U}^* = \sup_{X} \{ \mathcal{U}(d^*_\alpha): \alpha \in \Delta \} \in \mathcal{U}(X) \). Then

(a): \( \mathcal{I} = \sup_{X} \{ \mathcal{I}(d^*_\alpha): \alpha \in \Delta \} = \sup_{X} \{ \mathcal{I}(d^*_\alpha): \alpha \in \Delta \} = \mathcal{I}(\sup_{X} \{ \mathcal{U}(d^*_\alpha): \alpha \in \Delta \}) = \mathcal{I}(\mathcal{U}^*) \).

and

(b): \( A \times A \cap \mathcal{U}^* = A \times A \cap \sup_{X} \{ \mathcal{U}(d^*_\alpha): \alpha \in \Delta \} = A \times A \cap \sup_{X} \{ \mathcal{U}(d^*_\alpha): \alpha \in \Delta \} = \mathcal{U} \).

Thus, \( (A, \mathcal{U}) \subseteq (X, \mathcal{U}^*) \).

We conclude this chapter by proving the analogous result for proximities:

**Theorem 5.55:** Let \((X, \mathcal{I})\) be \( R_0 \) and let \( A \) be a dense, \( g \)-closed subset of \( X \). Suppose further that there is a \( \delta \in \pi(A) \) such that \( \mathcal{I}(\delta) = A \cap \mathcal{I} \). Then there is a \( \delta^* \in \pi(X) \) such that

(a): \( \mathcal{I} = \mathcal{I}(\delta^*) \) and

(b): \( (A, \delta) \subseteq (X, \delta^*) \).

**Proof:** By Willard [19], Theorem 40.15, there is a totally bounded uniformity \( \mathcal{U} \in \mathcal{U}(A) \) which induces \( \delta \). Thus
\( \mathcal{I}(\mathcal{U}) = \mathcal{I}(\delta) = A \cap \mathcal{I} \), and so by the previous theorem, there
is a \( \mathcal{U}^* \in \mathcal{U}(X) \) such that \( \mathcal{J} = \mathcal{J}(\mathcal{U}^*) \) and \( (A, \mathcal{U}) \subseteq (X, \mathcal{U}^*) \).

Let \( \delta^* \in \pi(X) \) be the proximity induced by \( \mathcal{U}^* \). Then

(a): \( \mathcal{J} = \mathcal{J}(\mathcal{U}^*) = \mathcal{J}(\delta^*) \) and

(b): \( (A, \delta) \subseteq (X, \delta^*) \)

since \( (A, \mathcal{U}) \subseteq (X, \mathcal{U}^*) \) (see Willard [19], problem 40 E3).
CHAPTER 6

$T_{1/2}$-SPACES

In [12], Levine defined a $T_{1/2}$-space to be one in which every $g$-closed set is closed. We have already noted in Corollary 1.6 that any $T_1$-space has this property. The remainder of this paper will be devoted to investigating $T_{1/2}$-spaces, beginning with a formal definition and the theorem which will be the cornerstone of the discussion in Chapter 6.

**Definition 6.1:** A space is $T_{1/2}$ iff every $g$-closed set is closed.

**Remark 6.2:** There is an ambiguity in the literature surrounding the term "$T_{1/2}$-space." In contrast to the definition of Levine given above, Bruns [2] defines $(X, T)$ to be a $T_{1/2}$-space if and only if, for each $x \in X$, $\{x\} = O \cap F$ for some $O$ open and some $F$ closed. We shall show that these two definitions of $T_{1/2}$-spaces are not equivalent. On the other hand, Thron [18] defines a space to be a $T_D$-space if and only if the derived set of each singleton is closed. Bruns' $T_{1/2}$ condition is equivalent to Thron's $T_D$ condition as proved in Thron [18], Theorem 14.3. Thus, the terminology "$T_{1/2}$-space" will be used
in the sense of Definition 6.1, while Bruns' terminology will be replaced by Thron's equivalent "$T_D$-space."

**Theorem 6.3:** A space $(X, \mathcal{J})$ is $T_{1/2}$ iff for every $x \in X$, either $\{x\}$ is open or $\{x\}$ is closed.

**Proof:** Necessity: Suppose $(X, \mathcal{J})$ is a $T_{1/2}$-space, and let $x \in X$ where $\{x\}$ is not closed. Then $\mathcal{C}(x)$ is not open, and so the only open superset of $\mathcal{C}(x)$ is $X$ itself. By Theorem 1.3, $\mathcal{C}(x)$ is $g$-closed and thus closed, since $(X, \mathcal{J})$ is $T_{1/2}$. Hence $\{x\}$ is open.

Sufficiency: Suppose every singleton is either open or closed and $A \subseteq X$ is $g$-closed. We assert that $A$ is closed. For, if $x \in \mathcal{C}(A)$ and $\{x\}$ is open, then $\{x\} \cap A \neq \emptyset$. If $\{x\}$ is not open, then $\{x\}$ is closed by hypothesis, and so, using Theorem 1.5(d), we have $\emptyset \neq \mathcal{C}(x) \cap A = \{x\} \cap A$. In either case, $x \in A$ and so $A$ is closed.

**Corollary 6.4:** $(X, \mathcal{J})$ is $T_{1/2}$ iff $\{x\}$ is open whenever $x \in X$ and $\{x\}$ is $g$-open.

**Proof:** Necessity: If $(X, \mathcal{J})$ is $T_{1/2}$ and $\{x\}$ is $g$-open, then $\mathcal{C}(x)$ is $g$-closed and thus closed. It follows that $\{x\}$ is open.

Sufficiency: Let $x \in X$ and suppose $\{x\}$ is not closed. Then, by Corollary 1.14, $\{x\}$ is $g$-open and thus open by hypothesis. By Theorem 6.3, $(X, \mathcal{J})$ is $T_{1/2}$. 
Corollary 6.5: \((X, \mathcal{J})\) is \(T_{1/2}\) iff every subset of \(X\) is the intersection of all open sets and all closed sets containing it.

Proof: Necessity: Let \((X, \mathcal{J})\) be \(T_{1/2}\) and \(B \subset X\) be arbitrary. Then \(B = \cap \{C(x) : x \not\in B\}\), an intersection of open sets and closed sets by Theorem 6.3. The result follows.

Sufficiency: For each \(x \in X\), \(C(x)\) is the intersection of all open sets and all closed sets containing it. Thus \(C(x)\) itself is either open or closed, and so \((X, \mathcal{J})\) is \(T_{1/2}\).

Remark 6.6: To place \(T_{1/2}\) in the sequence of separation axioms, we recall that a space is a \(T_D\)-space if and only if \(\{x\}'\) is closed for each \(x\) in the space. We also note that Kelley [10] defines a door space as one in which every subset is either open or closed.

Theorem 6.7: If \((X, \mathcal{J})\) is a door space, then \((X, \mathcal{J})\) is \(T_{1/2}\).

Proof: In a door space, singletons are either open or closed. Apply Theorem 6.3.

Theorem 6.8: If \((X, \mathcal{J})\) is \(T_{1/2}\), then \((X, \mathcal{J})\) is \(T_D\).

Proof: Let \((X, \mathcal{J})\) be \(T_{1/2}\) and let \(x \in X\). By Theorem 2.2, \(\{x\}'\) is \(g\)-closed, and thus closed. Hence \((X, \mathcal{J})\) is \(T_D\).

The following diagram summarizes the two previous theorems, Corollary 1.6, and Theorem 14.4 of Thron [18]:

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Diagram 6-1. $T_{1/2}$ and Separation Properties.

None of the above implications is reversible. Consider Thron [18], Example 14.1 as well as the following two examples:

**Example 6.9:** Let $X = \{a, b, c\}$ and $\mathcal{J} = \{\emptyset, \{a\}, \{a,b\}, X\}$. Then $(X, \mathcal{J})$ is T$_D$ since $\{a\}' = \{b, c\}$, $\{b\}' = \{c\}$, and $\{c\}' = \emptyset$, all of which are closed. But $(X, \mathcal{J})$ is not $T_{1/2}$ since $\{b\}'$ is neither open nor closed.

**Example 6.10:** Let $X = \{a, b, c, d\}$ and
$\mathcal{J} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$. Then $(X, \mathcal{J})$ is $T_{1/2}$ by Theorem 6.3, but $(X, \mathcal{J})$ is not T$_1$ since $\{a\}$ is not closed, nor is $(X, \mathcal{J})$ a door space since $\{a, c\}$ is neither open nor closed.

The following result shows that distinctions vanish upon adding the R$_o$ property.

**Theorem 6.11** [12; Theorem 8.6]: If $(X, \mathcal{J})$ is R$_o$, then $(X, \mathcal{J})$ is T$_o$ iff $(X, \mathcal{J})$ is T$_D$ iff $(X, \mathcal{J})$ is $T_{1/2}$ iff $(X, \mathcal{J})$ is T$_1$.

**Proof:** In light of Diagram 1, it suffices to show that R$_o$
and $T_0$ together imply $T_1$. So, let $(X, T)$ be $R_0$ and $T_0$, and let $x \in X$. If $\{x\}$ is not closed, then for some $y \neq x$, $y \in c(x)$. By the $T_0$ property, $c(x) \neq c(y)$, yet $x \in c(y)$ by Theorem 2.4, a contradiction. Hence $\{x\}$ is closed for all $x \in X$.

**Subspaces, Transformations, and Identifications**

**Theorem 6.12:** If $(X, T)$ is $T_{1/2}$ and $Y \subseteq X$, then $(Y, Y \cap T)$ is $T_{1/2}$.

**Proof:** Let $A \subseteq Y$ be $g$-closed in $Y$. By Corollary 3.12

$$A = Y \cap A^*$$

where $A^*$ is $g$-closed in $X$. But then $A^*$ is closed in $X$, and so $A$ is closed in $Y$, proving the theorem.

Before providing two conditions under which the image of a $T_{1/2}$-space is $T_{1/2}$, we consider the following motivating example:

**Example 6.13:** Let $X = \{1, 2, 3, 4, \ldots\}$ be the natural numbers with topology $\mathcal{T} = \{\emptyset, \{1\}\} \cup \{0 : 1 \in 0 \text{ and } 0 \text{ is finite}\}$. Then $(X, \mathcal{T})$ is a $T_{1/2}$-space by Theorem 6.3 since $\{1\} \in \mathcal{T}$ and, if $n \neq 1$, $\{n\}$ is closed. Also, let $Y = \{a, b, c\}$ with topology $\mathcal{U} = \{\emptyset, \{a\}, Y\}$. Then $(Y, \mathcal{U})$ is not $T_{1/2}$ since $\{b\}$ is neither $Y$-open nor $Y$-closed. Now define $f : X \to Y$ by

$$f(1) = a,$$

$$f(2n) = b \quad \text{for } n = 1, 2, \ldots$$

$$f(2n + 1) = c \quad \text{for } n = 1, 2, \ldots$$

Then $f$ is continuous, onto, and open, but the image under $f$
of the $T_{1/2}$-space $(X, \mathcal{T})$ is not $T_{1/2}$.

**Theorem 6.14**: If $(X, \mathcal{T})$ is a $T_{1/2}$-space and $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is continuous, closed, and onto, then $(Y, \mathcal{U})$ is $T_{1/2}$.

**Proof**: Let $B \subseteq Y$ be $g$-closed. Then $f^{-1}[B]$ is $g$-closed in $X$ by Theorem 3.26(b) and thus closed in $X$ since $(X, \mathcal{T})$ is $T_{1/2}$. But $f$ is closed and onto, and so $B = f[f^{-1}[B]]$ is $Y$-closed. Thus $(Y, \mathcal{U})$ is $T_{1/2}$. (Note that the mapping in Example 6.13 is not closed.)

**Theorem 6.15**: Let $(X, \mathcal{T})$ be a $T_{1/2}$-space, and let $f : X \to Y$ be an open, onto map (not necessarily continuous) such that for each $y \in Y$, $f^{-1}([y])$ is a finite set. Then $(Y, \mathcal{U})$ is $T_{1/2}$.

**Proof**: We shall use Theorem 6.3. Let $y \in Y$. By hypothesis, $f^{-1}([y]) = \{x_1, \ldots, x_n\}$. If, for some $i$, $\{x_i\} \in \mathcal{T}$, then $\{y\} = \{f(x_i)\} \in \mathcal{U}$ since $f$ is open. Otherwise, by the $T_{1/2}$ property, $\mathcal{C}(x_i) \in \mathcal{T}$ for all $i = 1, 2, \ldots, n$ and thus

$\mathcal{C}(x_1) \cap \mathcal{C}(x_2) \cap \ldots \cap \mathcal{C}(x_n) \in \mathcal{T}$. Hence

$\mathcal{C}(y) = f[\mathcal{C}(x_1) \cap \ldots \cap \mathcal{C}(x_n)] \in \mathcal{U}$ and so $(Y, \mathcal{U})$ is $T_{1/2}$.

(Compare this theorem to Example 6.13 where both $f^{-1}([b])$ and $f^{-1}([c])$ are infinite sets.)

**Corollary 6.16**: The homeomorphic image of a $T_{1/2}$-space is $T_{1/2}$.

**Proof**: This follows immediately from either Theorem 6.14 or 6.15.
Remark 6.17: Example 6.13 also shows that identifications of $T_{1/2}$-spaces need not be $T_{1/2}$.

Products

Theorem 6.18: Let $(X, \mathcal{S}) = \chi\{(X_\alpha, \mathcal{S}_\alpha): \alpha \in \Delta\}$. Then if $(X, \mathcal{S})$ is $T_{1/2}$, $(X_\alpha, \mathcal{S}_\alpha)$ is $T_{1/2}$ for all $\alpha \in \Delta$.

Proof: For each $\alpha \in \Delta$, $(X, \mathcal{S})$ contains a subspace homeomorphic to $(X_\alpha, \mathcal{S}_\alpha)$. Apply Theorem 6.12 and Corollary 6.16.

In contrast to the $T_0$, $T_1$, and $T_2$ separation axioms, the converse of Theorem 6.18 is false as the following example shows:

Example 6.19 [12; Example 7.4]: Let $X = \{a, b\}$ and $\mathcal{S} = \{\emptyset, \{a\}, X\}$. Then $(X, \mathcal{S})$ is $T_{1/2}$ by Theorem 6.3, but $(X \times X, \mathcal{S} \times \mathcal{S})$ is not $T_{1/2}$ since $\{(a, b)\}$ is neither open nor closed in $\mathcal{S} \times \mathcal{S}$.

We shall now derive necessary and sufficient conditions that a product space be $T_{1/2}$. There will be two cases, depending on whether the product has an infinite or a finite number of factors. In what follows, we shall insist that by an infinite number of factors we mean an infinite number of non-singleton factors. We first look at the infinite cases by proving two simple lemmas:

Lemma 6.20: Let $(X, \mathcal{S}) = \chi\{(X_\alpha, \mathcal{S}_\alpha): \alpha \in \Delta\}$ where $\Delta$ is
infinite. Then \((X, \mathcal{G})\) contains no isolated points.

**Proof:** Suppose \(\{x\} \in \mathcal{G}\) for some \(x \in X\). Then \(x \in P^{-1}[0_{\alpha_1}] \cap \ldots \cap P^{-1}[0_{\alpha_n}] \subseteq \{x\}\), and we can choose \(\beta \in \Delta \setminus \{\alpha_1, \ldots, \alpha_n\}\) such that \(X_\beta\) is not a singleton. Let \(y_\beta \in X_\beta\) such that \(y_\beta \neq P_\beta(x)\) and define \(y: \Delta \to \bigcup_{\alpha \in \Delta} [x_\alpha: \alpha \in \Delta]\) by

\[
y(\alpha_i) = P_{\alpha_i}(x) \quad \text{for } i = 1, 2, \ldots, n
\]

\[
y(\beta) = y_\beta
\]

\[
y(\gamma) \in X_\gamma \quad \text{for } \gamma \in \Delta \setminus \{\alpha_1, \ldots, \alpha_n, \beta\}
\]

Then \(y \in X\) and \(y \in P^{-1}[0_{\alpha_1}] \cap \ldots \cap P^{-1}[0_{\alpha_n}]\), but \(y \neq x\) since \(P_\beta(y) \neq P_\beta(x)\). This contradiction proves the lemma.

**Lemma 6.21:** Let \((X, \mathcal{G}) = \chi((x_\alpha, \mathcal{G}_\alpha): \alpha \in \Delta)\) where \(\Delta\) is infinite. Then \((X, \mathcal{G})\) is \(T_{1/2}\) iff \((X, \mathcal{G}_1)\) is \(T_1\).

**Proof:** Necessity: If \((X, \mathcal{G})\) is \(T_{1/2}\) and \(x \in X\), then either \(\{x\}\) is open or \(\{x\}\) is closed by Theorem 6.3. However, the first possibility is eliminated by Lemma 6.20, and thus each singleton is closed in \((X, \mathcal{G})\). Hence \((X, \mathcal{G})\) is \(T_1\).

Sufficiency: This is Corollary 1.6.

**Theorem 6.22:** Let \((X, \mathcal{G}) = \chi((x_\alpha, \mathcal{G}_\alpha): \alpha \in \Delta)\) where \(\Delta\) is infinite. Then \((X, \mathcal{G})\) is \(T_{1/2}\) iff \((x_\alpha, \mathcal{G}_\alpha)\) is \(T_1\) for all \(\alpha \in \Delta\).

**Proof:** Use Lemma 6.21 and the fact that a product is \(T_1\) if and only if all of the factors are \(T_1\).
Remark 6.23: The previous results show that the distinction between $T_1$ and $T_{1/2}$ vanishes in infinite product spaces. A different situation exists in the case of finite products, where we can relax the $T_1$ condition on one of the factors if we put severe restrictions upon the others:

**Theorem 6.24:** Let $(X, \mathcal{J}) = \chi((X_i, \mathcal{J}_i): i = 1, \ldots, n)$. Then $(X, \mathcal{J})$ is $T_{1/2}$ iff one of the following conditions hold:

(a): $(X_i, \mathcal{J}_i)$ is $T_1$ for all $i$.

or

(b): For some $k$, $(X_k, \mathcal{J}_k)$ is $T_{1/2}$ but not $T_1$, while $(X_i, \mathcal{J}_i)$ is discrete for all $i \neq k$.

**Proof:** Necessity: Suppose $(X, \mathcal{J})$ is $T_{1/2}$ and (a) does not hold. Then for some $k \in \{1, 2, \ldots, n\}$, $(X_k, \mathcal{J}_k)$ is not $T_1$, but $(X_k, \mathcal{J}_k)$ is $T_{1/2}$ by Theorem 6.18. Fix $i \neq k$.

We assert that $(X_i, \mathcal{J}_i)$ is discrete. For otherwise, $\{x_i\} \notin \mathcal{J}_i$ for some $x_i \in X_i$. Moreover, for some $x_k \in X_k$, $\{x_k\}$ is not closed in $(X_k, \mathcal{J}_k)$ since this space is not $T_1$. Finally, for each $j \neq i, k$, choose $x_j \in X_j$ arbitrary. Then define $x \in X$ by

\[
x(1) = x_i \\
x(k) = x_k \\
x(j) = x_j \quad \text{for } j \neq i, k
\]

Now, if $\{x\} \in \mathcal{J}$, then $\{x_i\} = \pi_i(x) \in \mathcal{J}_i$, a contradiction. On the other hand, if $\{x\}$ is $\mathcal{J}$-closed, then $\{x_k\}$ is closed in $(X_k, \mathcal{J}_k)$, again a contradiction. We have thus contradicted
Theorem 6.3 and conclude that for all \( i \neq k \), \((X_k', J_1')\) is discrete. This establishes (b).

Sufficiency: Suppose (a) holds. Then \((X, J)\) will be \(T_1\) and thus \(T_{1/2}\). On the other hand, if (b) holds, then for some \( k \), \((X_k', J_k')\) is \(T_{1/2}\) but not \(T_1\), while \((X_1', J_1')\) is discrete for all \( i \neq k \). Let \( x \in X \).

Case (i): If \( \{x(k)\} \notin J_k' \), then \( \{x(k)\} \) is \(J_k'\)-closed by Theorem 6.3, and, since \( \{x(i)\} \) is \(J_1'\)-closed for all \( i \neq k \), we have \( \{x\} = \chi(\{x(j)\}: 1 \leq j \leq n) \) is \( J \)-closed.

Case (ii): If \( \{x(k)\} \in J_k' \), then, by the discreteness of \((X_1', J_1')\), \( \{x\} = \chi(\{x(j)\}: 1 \leq j \leq n) \in J \).

So, by cases (i) and (ii) and Theorem 6.3, \((X, J)\) is \(T_{1/2}\).

The One-Point Compactification

Theorem 6.25: Let \((X^*, J^*)\) be the one-point compactification of \((X, J)\). (See Theorem 3.33.) Then \((X, J)\) is \(T_{1/2}\) iff \((X^*, J^*)\) is \(T_{1/2}\).

Proof: Necessity: Suppose \((X, J)\) is \(T_{1/2}\) and let \( x \in X \).

If \( \{x\} \in J \), then \( \{x\} \in J^* \). Otherwise, \( C_X(\{x\}) \in J^* \) and so \( C_{X^*}(\{x\}) = \{\infty\} \cup C_X(\{x\}) \in J^* \). Moreover, \( \{\infty\} \) is \(J^*\)-closed since \( C_{X^*}(\{\infty\}) = X \in J^* \). So, \((X^*, J^*)\) is \(T_{1/2}\) by Theorem 6.3.

Sufficiency: Note that \((X, J)\) is a subspace of \((X^*, J^*)\) and apply Theorem 6.12.
The $T_{1/2}$ Property and the Lattice of Topologies

**Theorem 6.26:** If $(X, \mathcal{J})$ is $T_{1/2}$ and $\mathcal{J} \subseteq \mathcal{U}$, then $(X, \mathcal{U})$ is $T_{1/2}$.

**Proof:** Let $x \in X$. If $\{x\} \in \mathcal{J}$, then $\{x\} \in \mathcal{U}$.
Otherwise, $C(x) \in \mathcal{J}$ and so $C(x) \in \mathcal{U}$. It follows that $(X, \mathcal{U})$ is $T_{1/2}$.

**Corollary 6.27:** Let $\{ \mathcal{J}_\alpha : \alpha \in \Delta \}$ be a family of $T_{1/2}$ topologies on $X$. Then $(X, \sup \{ \mathcal{J}_\alpha : \alpha \in \Delta \})$ is $T_{1/2}$.

**Proof:** The result follows immediately from Theorem 6.26.

**Example 6.28:** The previous results do not hold for coarser topologies or infima. For, if $X = \{a, b\}$, $\mathcal{J} = \{\emptyset, \{a\}, X\}$, and $\mathcal{U} = \{\emptyset, \{b\}, X\}$, then $(X, \mathcal{J})$ and $(X, \mathcal{U})$ are $T_{1/2}$ but $(X, \mathcal{J} \cap \mathcal{U})$ is not.

However, we can prove:

**Theorem 6.29:** If $(X_\alpha, \mathcal{J}_\alpha)$ is $T_{1/2}$ for all $\alpha \in \Delta$, and if $\{ \mathcal{J}_\alpha : \alpha \in \Delta \}$ is a totally ordered family with respect to inclusion, then $(X, \inf \{ \mathcal{J}_\alpha : \alpha \in \Delta \})$ is $T_{1/2}$.

**Proof:** Let $x \in X$ and suppose $\{x\} \notin \inf \{ \mathcal{J}_\alpha : \alpha \in \Delta \}$. Then $\{x\} \notin \mathcal{J}_\beta$ for some $\beta \in \Delta$ and so $C(x) \in \mathcal{J}_\beta$ by Theorem 6.3. We assert that $C(x) \in \mathcal{J}_\alpha$ for all $\alpha \in \Delta$.

For, if $\alpha \in \Delta$ and $\mathcal{J}_\beta \subseteq \mathcal{J}_\alpha$, then $C(x) \in \mathcal{J}_\beta$ implies $C(x) \in \mathcal{J}_\alpha$. Otherwise, by total ordering, $\mathcal{J}_\alpha \subseteq \mathcal{J}_\beta$. Now,
if $C(x) \notin \mathcal{S}_\alpha$, then $(x) \in \mathcal{S}_\alpha$ and so $(x) \in \mathcal{S}_\beta$, a contradiction. We have thus proved that $C(x) \in \mathcal{S}_\alpha$ for all $\alpha \in \Delta$, and thus $C(x) \in \inf\{\mathcal{S}_\alpha: \alpha \in \Delta\}$. It follows from Theorem 6.3 that $(X, \inf\{\mathcal{S}_\alpha: \alpha \in \Delta\})$ is $T_{1/2}$.

**Corollary 6.30:** If $\pi$ is any topology on $X$, then there is a topology $\mathcal{U}$ on $X$ such that:

(a): $\mathcal{S} \subseteq \mathcal{U}$

(b): $(X, \mathcal{U})$ is $T_{1/2}$ and

(c): If $\mathcal{S} \subseteq \mathcal{V} \subseteq \mathcal{U}$ and $(X, \mathcal{V})$ is $T_{1/2}$, then $\mathcal{U} = \mathcal{V}$.

**Proof:** Let $\mathcal{A} = \{\mathcal{S}_\alpha: \alpha \in \Delta\}$ be the indexed family of $T_{1/2}$ topologies on $X$ that are finer than $\mathcal{S}$. Then $\mathcal{A} \neq \emptyset$ since the discrete topology is $T_{1/2}$. Further, if $\{\mathcal{S}_\alpha: \alpha \in \Delta^*\}$ is a subset of $\mathcal{A}$ totally ordered with respect to inclusion and $\mathcal{S}^* = \inf\{\mathcal{S}_\alpha: \alpha \in \Delta^*\}$, then $\mathcal{S} \subseteq \mathcal{S}^*$ clearly and $(X, \mathcal{S}^*)$ is $T_{1/2}$ by Theorem 6.29. So, $\mathcal{S}^* \in \mathcal{A}$ and by Zorn's Lemma, $\mathcal{A}$ contains a minimal element $\mathcal{U}$ which satisfies properties (a) - (c) above.

**Minimal $T_{1/2}$ Topologies**

**Remark 6.31:** If we let $\mathcal{S} = \{\emptyset, X\}$ in Corollary 6.30, we see that on any set $X$ there is at least one topology minimal with respect to the property of being $T_{1/2}$. In the spirit of Girhinny [9], we shall completely determine the structure of a minimal $T_{1/2}$ topology. In so doing, the cases where $X$ is
infinite and $X$ is finite must be considered separately. But first we need four lemmas:

**Lemma 6.32:** Suppose $X$ contains more than one point and $\tau$ is the discrete topology on $X$. Then $\tau$ is not a minimal $T_{1/2}$ topology.

**Proof:** Since $X$ is not a singleton, there exist points $x, y \in X$ such that $x \neq y$. Let $\mathcal{U} = \{U: U = \emptyset$ or $x \in U\}$. Then $(X, \mathcal{U})$ is a $T_{1/2}$-space since $\{x\}$ is open and any other singleton is closed. But $\mathcal{U} \not= \tau$ since $\{y\} \not\in \mathcal{U}$.

**Lemma 6.33:** Let $X$ be a finite set and suppose $(X, \mathcal{U})$ is $T_{1/2}$. Moreover, suppose that for some $c \in X$, $\{c\}$ is $\mathcal{U}$-closed and $\{x: \{x\} \in \mathcal{U}\} \subseteq \{c\}$. Then $(X, \mathcal{U})$ is discrete.

**Proof:** If $x \neq c$, $\{x\} \not\in \mathcal{U}$ and so $\{x\}$ is closed by Theorem 6.3, while $\{c\}$ is closed by hypothesis. It follows that $(X, \mathcal{U})$ is a $T_1$-space and thus is discrete since $X$ is finite.

**Lemma 6.34:** Let $X \neq \emptyset$ with $A \subseteq X$ and define $\mathcal{U} = \{U: U \subseteq A$, or $A \subseteq U$ and $CU$ is finite$\}$. Then $\mathcal{U}$ is a topology on $X$ and $(X, \mathcal{U})$ is $T_{1/2}$.

**Proof:** It is routine to show that $\mathcal{U}$ is a topology. To show $(X, \mathcal{U})$ is $T_{1/2}$, let $x \in X$. If $x \in A$, $\{x\} \in \mathcal{U}$. If $x \not\in A$, $A \subseteq C\{x\}$ and $C(C\{x\})$ is finite, implying that $\{x\}$ is closed. The result follows from Theorem 6.3.
Lemma 6.35: Let \((X, \mathcal{J})\) be a minimal \(T_{1/2}\) space where \(X\) contains more than one point. Define

\[
A = \{x: \{x\} \in \mathcal{J} \text{ and } C(x) \notin \mathcal{J}\}
\]

\[
B = \{x: \{x\} \notin \mathcal{J} \text{ and } C(x) \in \mathcal{J}\}
\]

\[
C = \{x: \{x\} \in \mathcal{J} \text{ and } C(x) \notin \mathcal{J}\}
\]

Then we conclude:

(a): \(X = A \cup B \cup C\)

(b): \(B \neq \emptyset\)

(c): \(C = \emptyset\)

Proof: (a): This is just a restatement of Theorem 6.3.

(b): If \(B = \emptyset\), then \(X = A \cup C\) by part (a), and thus \((X, \mathcal{J})\) is discrete, contradicting Lemma 6.32.

(c): Suppose \(C \neq \emptyset\). Then for some \(c \in X\), \(\{c\}\) is both open and closed. Let us define

\[A^* = \{x: \{x\} \in \mathcal{J} \text{ and } x \neq c\} = (A \cup C) \setminus \{c\}.\]

Now, if we let \(\mathcal{U} = \{U: U \subseteq A^*, \text{ or } A^* \subseteq U \text{ and } \mathcal{C}U \text{ is finite}\}\), we have by Lemma 6.34 that \((X, \mathcal{U})\) is a \(T_{1/2}\)-space. We assert that \(\mathcal{U} \subseteq \mathcal{J}\).

For, letting \(U \in \mathcal{U}\), if \(U \subseteq A^*\), then \(U = \bigcup\{\{x\}: x \in A^* \cap U\} \in \mathcal{J}\).

On the other hand, if \(U \supseteq A^*\), then \(A^* \subseteq U\) with \(\mathcal{C}U = \{x_1, \ldots, x_n\}\). Now, for each \(i = 1, 2, \ldots, n\), \(x_i \notin U\) and so \(x_i \notin A^*\). Hence either \(x_i = c\) or \(\{x_i\} \notin \mathcal{J}\), and in either case, \(\{x_i\}\) is \(\mathcal{J}\)-closed by Theorem 6.3. Thus

\[U = C(x_1) \cap \ldots \cap C(x_n) \in \mathcal{J}\]

and the assertion that \(\mathcal{U} \subseteq \mathcal{J}\) is proved. But since \(\mathcal{J}\) is a minimal \(T_{1/2}\) topology, we conclude \(\mathcal{U} = \mathcal{J}\). By hypothesis, \(\{c\} \in \mathcal{J}\), and thus \(\{c\} \in \mathcal{U}\). Hence, by
definition of \( \mathcal{U} \), either \( \{c\} \subseteq A^* \) or \( A^* \subseteq \{c\} \) and \( C(c) \) is finite. The first possibility is dismissed by the definition of \( A^* \), and so the second possibility must hold. But then
\[
X = \{c\} \cup C(c) \text{ implies } X \text{ is a finite set. Moreover, } A^* \subseteq \{c\} \text{ implies } A^* = \emptyset \text{ since } c \not\in A^* \text{, and thus } (A \cup C) \setminus \{c\} = \emptyset.
\]
Consequently, \([x : \{x\} \in \mathcal{U}] = [x : \{x\} \in \mathcal{J}] = A \cup C \subseteq \{c\}\), and we conclude \((X, \mathcal{U})\) is discrete by Lemma 6.32. But then \((X, \mathcal{J})\) is also discrete, contradicting Lemma 6.32. We thus abandon the original hypothesis and conclude \( C = \emptyset \).

We now prove the structure theorem for a minimal \( T_{1/2} \) topology on an infinite set \( X \):

**Theorem 6.36:** Suppose \( X \) is an infinite set. Then \( \mathcal{J} \) is a minimal \( T_{1/2} \) topology on \( X \) iff there is an \( A \subseteq X \) such that
\[
\mathcal{J} = \{0 : 0 \subseteq A, \text{ or } A \subseteq \emptyset \text{ and } \emptyset \text{ is finite}\}.
\]

**Proof:** Necessity: Let \((X, \mathcal{J})\) be a minimal \( T_{1/2} \)-space and, as in the previous lemma, define
\[
A = \{x : \{x\} \in \mathcal{J} \text{ and } C(x) \notin \mathcal{J}\}
\]
\[
B = \{x : \{x\} \notin \mathcal{J} \text{ and } C(x) \in \mathcal{J}\}
\]
\[
C = \{x : \{x\} \in \mathcal{J} \text{ and } C(x) \in \mathcal{J}\}
\]
Then by Lemma 6.35, \( C = \emptyset \), \( B \neq \emptyset \), and \( X = A \cup B \). Hence \( A \nsubseteq X \). If we define \( \mathcal{U} = \{U : U \subseteq A, \text{ or } A \subseteq U \text{ and } CU \text{ is finite}\} \), then \((X, \mathcal{U})\) is \( T_{1/2} \) by Lemma 6.34, and we have proved the result if we show \( \mathcal{J} = \mathcal{U} \). But by minimality of \( \mathcal{J} \), it suffices to show \( \mathcal{U} \subseteq \mathcal{J} \). So, let \( U \in \mathcal{U} \). If \( U \subseteq A \), then
U = \bigcup\{\{x\}: x \in U\} \in \mathcal{J}$. Otherwise, \( A \subseteq U \) and 
\[ \mathcal{C}U = \{x^1, \ldots, x^n\}. \] Then for each \( i \), \( x_i \notin A \) and so \( x_i \notin B \). Thus \( U = \mathcal{C}\{x^1\} \cap \cdots \cap \mathcal{C}\{x^n\} \in \mathcal{J} \), showing \( \mathcal{U} \subseteq \mathcal{J} \) and completing the proof.

Sufficiency: Suppose for some \( A \subsetneq X \), 
\[ \mathcal{J} = \{0: 0 \subsetneq A, \text{ or } A \subsetneq 0 \text{ and } C0 \text{ is finite}\}. \] By Lemma 6.34, \((X, \mathcal{J})\) is \( T_{1/2} \), and it remains only to show minimality. So, let \((X, \mathcal{U})\) be \( T_{1/2} \) with \( \mathcal{U} \subseteq \mathcal{J} \). We assert \( \mathcal{U} \subseteq \mathcal{J} \). To prove this, we first define \( A^* \subseteq X \) by \( A^* = \{x: \{x\} \in \mathcal{U}\} \).

We claim that \( A^* \subseteq A \). For if \( x \in A^* \), then \( \{x\} \in \mathcal{U} \) and thus \( \{x\} \in \mathcal{J} \). Hence either \( \{x\} \subseteq A \), or \( A \subseteq \{x\} \) and \( C\{x\} \) is finite. The latter possibility must be dismissed since \( \{x\} \cup C\{x\} = X \), an infinite set. Thus, \( x \in A \) and so \( A^* \subseteq A \) is proved. Next we claim that \( A \subseteq A^* \). For otherwise there is an \( x \in A \) such that \( x \notin A^* \). Thus \( \{x\} \notin \mathcal{U} \) and so \( C\{x\} \in \mathcal{U} \) by Theorem 6.3. Consequently, \( C\{x\} \in \mathcal{J} \) and so either \( C\{x\} \subseteq A \), or \( A \subseteq C\{x\} \). But if \( C\{x\} \subseteq A \), then, since \( x \in A \), we have \( A = X \), contradicting \( A \not\subseteq X \). And, if \( A \subseteq C\{x\} \), then 
\[ x \in A \subseteq C\{x\}, \] also a contradiction. So, by contradiction we have shown \( A \subseteq A^* \), and, combining inequalities, we have \( A = A^* \).

We now return to the matter of showing \( \mathcal{J} \subseteq \mathcal{U} \). Let \( 0 \in \mathcal{J} \). If \( 0 \subseteq A \), then \( 0 \subseteq A^* \), and thus \( 0 \in \mathcal{U} \). Otherwise, \( A \subseteq 0 \) and \( C0 = \{x_1, \ldots, x_n\} \). Then for each \( i \), \( x_i \notin 0 \) and so \( x_i \notin A = A^* \). Consequently, \( \{x_i\} \notin \mathcal{U} \) and so \( C\{x_i\} \in \mathcal{U} \).
by Theorem 6.3. It follows that \( 0 = c \{ x_1 \} \cap \ldots \cap c \{ x_n \} \in \mathcal{U} \), proving \( \tau \subseteq \mathcal{U} \). \((X, \mathcal{J})\) is thus a minimal \( T_{1/2} \)-space.

Remark 6.37: The previous result shows that the minimal \( T_{1/2} \) topologies are composed of some "very small" open sets (i.e., subsets of \( A \)) and some "very large" ones (i.e., supersets of \( A \) with finite complements). We also see that, if \( X \) is infinite, there is a one-to-one correspondence between the minimal \( T_{1/2} \) topologies on \( X \) and the proper subsets \( A \) of \( X \). We have not ruled out the case where \( A = \emptyset \), a case which yields the cofinite topology as a minimal \( T_{1/2} \) topology. However, when we shift our attention to finite sets, we must eliminate the cofinite, or equivalently, discrete, topology by virtue of Lemma 6.32.

Fortunately, this is the only modification necessary:

Theorem 6.38: Suppose \( X \) is a finite set containing more than one point. Then \( \mathcal{J} \) is a minimal \( T_{1/2} \) topology on \( X \) iff there is an \( \emptyset \neq A \subseteq X \) such that

\[ \mathcal{J} = \{0: 0 \subseteq A, \text{ or } A \not\subseteq \emptyset \text{ and } \complement_0 \text{ is finite}\}. \]

Proof: Necessity: Defining \( A, B, \) and \( C \) as in Theorem 6.36, we again conclude from Lemma 6.35 that \( X = A \cup B \), where \( B \neq \emptyset \). Thus, \( A \not\subseteq X \). Moreover, \( A \neq \emptyset \), for otherwise \( \mathcal{J} \) is discrete, contradicting Lemma 6.32. The remainder of the necessary condition follows exactly as in Theorem 6.36.

Sufficiency: Suppose there is an \( \emptyset \neq A \not\subseteq X \) such that \( \mathcal{J} = \{0: 0 \subseteq A, \text{ or } A \subseteq \emptyset \text{ and } \complement_0 \text{ is finite}\} \). Then by
Lemma 6.34, \((X, \mathcal{F})\) is \(T_{1/2}\). As in the previous theorem, let 
\((X, \mathcal{U})\) be \(T_{1/2}\) with \(\mathcal{U} \subseteq \mathcal{F}\), and define \(A^* = \{x: \{x\} \in \mathcal{U}\}\).
We assert that \(A^* \subseteq A\). For, if \(x \in A^*\), then \(\{x\} \in \mathcal{U} \subseteq \mathcal{F}\) and so either \(\{x\} \subseteq A\), or \(A \subseteq \{x\}\). In the first case, \(x \in A\), and in the second case \(\emptyset \neq A \subseteq \{x\}\) and so again \(x \in A\). This proves the assertion that \(A^* \subseteq A\). We now show \(A \subseteq A^*\) and the
minimality of \(\mathcal{F}\) exactly as in Theorem 6.36.

Remark 6.39: The previous theorem shows that if \(X\) is finite, there is a one-to-one correspondence between the minimal \(T_{1/2}\) topologies on \(X\) and the proper, non-empty subsets of \(X\).

Corollary 6.40: Let \(X\) contain more than one point and let \(\mathcal{F}\) be a minimal \(T_{1/2}\) topology on \(X\). Then

(a): If \(X\) is finite, \((X, \mathcal{F})\) is not \(T_1\).

(b): If \(X\) is infinite, \((X, \mathcal{F})\) is \(T_1\) iff \(\mathcal{F}\) is the
cofinite topology.

Proof: Part (a) follows from Lemma 6.32. For part (b) suppose \(X\) is infinite and \((X, \mathcal{F})\) is \(T_1\). Then for some \(A \subseteq X, \mathcal{F} = \{0: 0 \subseteq A, \text{ or } A \subseteq 0\text{ and } C_0\text{ is finite}\}\). We assert \(A = \emptyset\). For otherwise, choose \(a \in A\) and \(x \in X \setminus A\). Then \(x \in 0 \in \mathcal{F}\) implies \(a \in A \subseteq 0\), contradicting the \(T_1\) property. We conclude that \(A = \emptyset\) and \(\mathcal{F}\) is cofinite. The converse is immediate.

Remark 6.41: For the following corollary, we adopt the definition
of Cullen [4] and call a space a continuum if it is compact and connected. (Willard [19], for instance, also requires that the space be Hausdorff.)

**Corollary 6.42:** If $\mathcal{J}$ is a minimal $T_{1/2}$ topology on $X$, then $(X, \mathcal{J})$ is a continuum.

**Proof:** (a): Compactness: If $X$ is finite, $(X, \mathcal{J})$ is automatically compact. Otherwise, by Theorem 6.36,

$\mathcal{J} = \{0: 0 \subseteq A, \text{ or } A \subseteq 0 \text{ and } CO \text{ is finite} \} \text{ for some } A \not\subseteq X$. Now, if $X = \bigcup \{0_\alpha: \alpha \in \Delta \}$ is an open cover, then, choosing $x \in X \setminus A$, $x \in 0_\beta$ for some $\beta \in \Delta$. Since $x \notin A$, $0_\beta \notin A$, and so $A \subseteq 0_\beta$ with $CO_\beta = \{x_1, \ldots, x_n\}$. For each $i = 1, 2, \ldots, n$, choose $\alpha_i \in \Delta$ such that $x_i \in 0_{\alpha_i}$. Then

$X = 0_\beta \cup 0_{\alpha_1} \cup \cdots \cup 0_{\alpha_n}$ is a finite subcover.

(b): Connectedness: Suppose $X = 0_1 \cup 0_2$ where $0_1$ and $0_2$ are non-empty, disjoint, and open. Based on the structure of $\mathcal{J}$ in Theorems 6.36 and 6.38, one of four cases can occur:

Case (i): If $0_1 \subseteq A$ and $0_2 \subseteq A$, then $X = 0_1 \cup 0_2 \subseteq A \neq X$, a contradiction.

Case (ii): If $0_1 \subseteq A$ and $A \subseteq 0_2$ with $CO_2$ finite, then $\emptyset \neq 0_1 = 0_1 \cap 0_2$, a contradiction.

Case (iii): If $0_2 \subseteq A$ and $A \subseteq 0_1$ with $CO_1$ finite, then $\emptyset \neq 0_2 = 0_1 \cap 0_2$, a contradiction.

Case (iv): If $A \subseteq 0_1$ with $CO_1$ finite, and $A \subseteq 0_2$...
with \( C_0 \) finite, then \( X = C\emptyset = C(0_1 \cap 0_2) = C0_1 \cup C0_2 \) implies \( X \) is finite, and thus we deduce from Theorem 6.38 that \( A \neq \emptyset \). But then \( \emptyset \neq A \subseteq 0_1 \cap 0_2 \), again a contradiction.

From cases (i) - (iv), we conclude that \((X, \mathcal{J})\) is connected.

### Maximal \( T_{1/2} \) Topologies

**Remark 6.43:** By a maximal \( T_{1/2} \) topology we shall mean a non-discrete, \( T_{1/2} \) topology such that any finer \( T_{1/2} \) topology is discrete. We shall prove that the maximal \( T_{1/2} \) topologies are simply the ultratopologies - i.e., maximal, non-discrete topologies - as defined in Frohlich [7]. There Frohlich gives a structure theorem for ultratopologies which is used by Girhinny in [9] to show that any ultratopology is a door space. (See Remark 6.6.) Since this is the only property of ultratopologies concerning us here, we offer the following alternate, and much simpler, proof:

**Theorem 6.44:** If \( \mathcal{J} \) is an ultratopology, \((X, \mathcal{J})\) is a door space.

**Proof:** Proving the contrapositive, we suppose \((X, \mathcal{J})\) is not a door space. Then there is an \( A \subseteq X \) such that \( A \notin \mathcal{J} \) and \( CA \notin \mathcal{J} \). Consider the simple extension of topology \( \mathcal{J} \) over \( A \) as defined in Levine [13]. That is, let \( \mathcal{J}(A) = \{0 \cup (0^* \cap A): 0^* \in \mathcal{J}\} \). Then \( \mathcal{J} \subsetneq \mathcal{J}(A) \) since \( A \notin \mathcal{J} \) but \( A \in \mathcal{J}(A) \). Moreover, \( CA \notin \mathcal{J}(A) \) because \( CA = 0 \cup (0^* \cap A) \) implies \( CA = 0 \in \mathcal{J} \), a contradiction. Since \( \mathcal{J} \subsetneq \mathcal{J}(A) \subsetneq \mathcal{P}(X) \),
J is not an ultratopology, and the theorem is proved.

Corollary 6.45: J is a maximal $T_{1/2}$ topology on X iff J is an ultratopology.

Proof: Necessity: Suppose J is a maximal $T_{1/2}$ topology on X and $J \neq \mathcal{U}$. Then $(X, \mathcal{U})$ is $T_{1/2}$ by Theorem 6.26 and thus $\mathcal{U}$ is discrete. It follows that J is an ultratopology.

Sufficiency: If J is an ultratopology, $(X, J)$ is a door space by Theorem 6.44 and thus $(X, J)$ is $T_{1/2}$ by Theorem 6.7. Thus J is a maximal $T_{1/2}$ topology on X.
CHAPTER 7

THE $T_{1/2}$ EXTENSION OF A TOPOLOGY

In this chapter we shall define a closure operator on an
arbitrary topological space by means of the g-closed sets and
consider the topology it generates. This topology will be the
same as that whose base consists of the g-open sets from the
original space and will provide an extension of the original
topology to a $T_{1/2}$ topology. Finally, some properties of
this $T_{1/2}$ extension will be examined.

The $T_{1/2}$ Extension

Definition 7.1: Let $(X, \mathcal{J})$ be a topological space and
define for each subset $E$ of $X$,
\[ c^*(E) = \bigcap \{ A: E \subseteq A \text{ and } A \text{ is g-closed} \} . \]
We call $c^*$ the
generalized closure operator induced by $\mathcal{J}$.

Theorem 7.2: For all $E \subseteq X$, $E \subseteq c^*(E) \subseteq c(E)$.

Proof: Clearly $E \subseteq c^*(E)$. Moreover, since any closed
set is g-closed, $c^*(E) = \bigcap \{ A: E \subseteq A \text{ and } A \text{ is g-closed} \} \subseteq
\bigcap \{ F: E \subseteq F \text{ and } F \text{ is closed} \} = c(E)$.

Example 7.3: Both containment relations in the previous theorem
may be proper. For, if $X = \{a,b,c\}$ with topology
\[ J = (\emptyset, (a), (a,b), X), \text{ then } c^*(a) = \{a,c\} \text{ since the only } \]
g-closed supersets of \( \{a\} \) are \( \{a,c\} \) and \( X \) itself. But \( c(a) = X \). So \( \emptyset \subseteq c^*(a) \subseteq c(a) \).

**Theorem 7.4:** If \( A \) is g-closed, \( c^*(A) = A \).

**Proof:** The result follows immediately from Definition 7.1.

**Example 7.5:** The converse of Theorem 7.4 is false. For, if \( X = \{a,b,c\} \) and \( J = (\emptyset, (a), X) \), then \( \{a,b\} \) and \( \{a,c\} \) are g-closed supersets of \( \{a\} \) by Theorem 1.3 and thus \( c^*([a]) = \{a\} \). But \( \{a\} \) is not g-closed in \((X, J)\). This example also shows that the generalized closure of a set need not be g-closed.

We now prove the fundamental result of this chapter:

**Theorem 7.6:** \( c^* \) as defined above is a Kuratowski closure operator.

**Proof:** (a): Since \( \emptyset \) is g-closed, \( c^*(\emptyset) = \emptyset \) by Theorem 7.4.

(b): If \( E \subseteq X \), then \( E \subseteq c^*(E) \) by Theorem 7.2.

(c): Let \( E \) and \( F \) be subsets of \( X \). We assert first that \( c^*(E) \cup c^*(F) \subseteq c^*(E \cup F) \). For, if \( A \) is g-closed and \( E \cup F \subseteq A \), then \( E \subseteq A \) and \( F \subseteq A \), implying \( c^*(E) \subseteq A \) and \( c^*(F) \subseteq A \). Hence, \( c^*(E) \cup c^*(F) \subseteq A \) and it follows that \( c^*(E) \cup c^*(F) \subseteq \cap \{A: E \cup F \subseteq A \text{ and } A \text{ is g-closed}\} = c^*(E \cup F) \).

Secondly, we claim that \( c^*(E \cup F) \subseteq c^*(E) \cup c^*(F) \). Otherwise, for some \( x \in c^*(E \cup F) \), \( x \notin c^*(E) \cup c^*(F) \). Since \( x \notin c^*(E) \),
x ∉ A for some A ⊆ E with A g-closed. Since x ∉ c*(F),
x ∉ B for some B ⊇ F with B g-closed. Thus x ∉ A ∪ B
where A ∪ B ⊆ E ∪ F, and A ∪ B is g-closed by Theorem 3.1.
This contradicts x ∈ c*(E ∪ F), and we conclude that
c*(E ∪ F) ⊆ c*(E) ∪ c*(F). Since both containments hold, we
have c*(E ∪ F) = c*(E) ∪ c*(F).

(d): Let E ⊆ X. We assert c*(E) = c*(c*(E)). By
part (a) above, it suffices to show c*(c*(E)) ⊆ c*(E). So,
let E ⊆ B where B is g-closed in X. Then by definition,
c*(E) ⊆ B and thus
c*(c*(E)) = ∩{A: c*(E) ⊆ A and A is g-closed} ⊆ B. Hence,
c*(c*(E)) ⊆ ∩{B: E ⊆ B and B is g-closed} = c*(B) and the
result is proved.

By (a) - (d) above, c* is a closure operator.

Definition 7.7: Let τ* be the topology on X generated by
c* in the usual manner. That is, τ* = {O: cO = c*(cO)}.

Theorem 7.8: τ ⊆ τ*

Proof: If F is closed, and hence g-closed in τ, then
c*(F) = F by Theorem 7.4, and thus F is closed in τ*.

We next obtain a necessary and sufficient condition that
equality hold in Theorem 7.8:

Theorem 7.9: τ = τ* iff (X, τ) is T1/2.
Proof: Necessity: Let $\mathcal{J} = \mathcal{J}^*$ and suppose $A \subseteq X$ is $g$-closed in $(X, \mathcal{J})$. Then $c^*(A) = A$ by Theorem 7.4 and so $A$ is closed in $(X, \mathcal{J}^*)$ and hence in $(X, \mathcal{J})$. Thus $(X, \mathcal{J})$ is $T_{\frac{1}{2}}$.

Sufficiency: If $(X, \mathcal{J})$ is $T_{\frac{1}{2}}$, the family of closed sets and the family of $g$-closed sets coincide. Thus the generalized closure operator agrees with the usual closure operator, and so $\mathcal{J} = \mathcal{J}^*$.

Theorem 7.10: If $x \neq y$, then $c^*(x) \neq c^*(y)$.

Proof: If both $\{x\}$ and $\{y\}$ are $g$-closed, then $c^*(x) = \{x\}$ and $c^*(y) = \{y\}$ by Theorem 7.4. Otherwise, we may assume without loss of generality that $\{x\}$ is not $g$-closed. Hence, $\{x\}$ is not closed and it follows from Corollary 1.14 that $\{x\}$ is $g$-open. Then $\cdot \in \mathcal{C}\{x\}$, a $g$-closed set, and so $c^*(y) \subseteq \mathcal{C}\{x\}$, implying $c^*(y) \neq c^*(x)$.

Remark 7.11: The previous theorem shows that $(X, \mathcal{J}^*)$ is a $T_o$-space. We shall prove a stronger result after first giving an alternate characterization of $\mathcal{J}^*$.

Theorem 7.12: $\mathcal{J}^*$ is the topology whose base is the family of $g$-open sets in $(X, \mathcal{J})$.

Proof: By Corollary 3.2 and Remark 3.3, we know that the family of $g$-open sets in $(X, \mathcal{J})$ does provide a base for a topology on $X$. Let us call this topology $\mathcal{U}$ and show $\mathcal{J}^* = \mathcal{U}$.
Now, if \( Q \) is \( g \)-open in \((X, \mathcal{J})\), then \( CQ \) is \( g \)-closed and thus \( c^*(CQ) = CQ \) by Theorem 7.14. Hence \( Q \in \mathcal{J}^* \), and so \( \mathcal{U} \subseteq \mathcal{J}^* \).

Conversely, let \( x \in 0 \subseteq \mathcal{J}^* \). Then
\[
0 = c^*(0) = \cap \{ A : C0 \subseteq A \text{ and } A \text{ is } g \text{-closed in } (X, \mathcal{J}) \}.
\]
So, \( x \in 0 = \cup \{ CA : CA \subseteq 0 \text{ and } CA \text{ is } g \text{-open in } (X, \mathcal{J}) \} \).
Thus \( x \in CA^* \subseteq 0 \) for some \( CA^* \) \( g \)-open in \((X, \mathcal{J})\), and consequently, \( 0 \in \mathcal{U} \). If follows that \( \mathcal{J}^* \subseteq \mathcal{U} \), and the theorem is proved.

**Theorem 7.13:** If \((X, \mathcal{J})\) is a topological space and \( \mathcal{J}^* \) is as in Definition 7.7, then \((X, \mathcal{J}^*)\) is \( T_{1/2} \).

**Proof:** Let \( x \in X \) with \( \{x\} \) \( g \)-open in \((X, \mathcal{J}^*)\). We assert \( \{x\} \in \mathcal{J}^* \). For otherwise, \( \{x\} \) is not \( \mathcal{J}^* \)-closed by Corollary 1.14, and thus \( \{x\} \) is not \( \mathcal{J} \)-closed by Theorem 7.8. Again by Corollary 1.14 we conclude that \( \{x\} \) is \( g \)-open in \((X, \mathcal{J})\), and thus, being a basic \( \mathcal{J}^* \)-open set by Theorem 7.12, \( \{x\} \in \mathcal{J}^* \), a contradiction. Hence, we have shown that \( \{x\} \) \( g \)-open in \((X, \mathcal{J}^*)\) implies \( \{x\} \in \mathcal{J}^* \), and it follows from Corollary 6.4 that \((X, \mathcal{J}^*)\) is \( T_{1/2} \).

**Remark 7.14:** The results of Theorems 7.8 and 7.13 justify the designation of \( \mathcal{J}^* \) as the \( T_{1/2} \) extension of \( \mathcal{J} \).

**Corollary 7.15:** If \((X, \mathcal{J})\) is a topological space, then \((\mathcal{J}^*)^* = \mathcal{J}^* \).
Proof: $(X, \mathcal{J}^*)$ is $T_{1/2}$ by Theorem 7.13, and so $(\mathcal{J}^*)^* = \mathcal{J}^*$ by Theorem 7.9.

Some Properties of $(X, \mathcal{J}^*)$

Example 7.16: The $T_{1/2}$ extension neither preserves nor reverses inclusion of topologies. For instance, let $X = \{a, b\}$, $\mathcal{T} = (\emptyset, X)$, $\mathcal{U} = (\emptyset, \{a\}, X)$, and $\mathcal{V} = (\emptyset, \{a\}, \{b\}, X)$. Then $\mathcal{U}^* = \mathcal{U}$ and $\mathcal{V}^* = \mathcal{V}$ by virtue of Theorem 7.9 since $(X, \mathcal{U})$ and $(X, \mathcal{V})$ are both $T_{1/2}$. Also, $\mathcal{J}^* = \mathcal{V}$ since singletons in an indiscrete space are $g$-open. Thus, $\mathcal{J} \subseteq \mathcal{U}$ but $\mathcal{U}^* \subseteq \mathcal{J}^*$, while $\mathcal{U} \subseteq \mathcal{V}$ and $\mathcal{U}^* \subseteq \mathcal{V}^*$.

The previous example shows that the $T_{1/2}$ extension does not behave well with respect to finer and coarser topologies. One problem is that the $T_{1/2}$ extension of both the discrete and the indiscrete topologies is discrete. We investigate the discreteness of $\mathcal{J}^*$ in the next two results:

Theorem 7.17: Let $(X, \mathcal{J})$ be a topological space with $(X, \mathcal{J}^*)$ its $T_{1/2}$ extension. Then $(X, \mathcal{J}^*)$ is not discrete iff there is an $x \in X$ such that $\{x\}$ is $\mathcal{J}$-closed but not $\mathcal{J}$-open.

Proof: Using Theorem 7.12 and Corollary 1.14, we note that $(X, \mathcal{J}^*)$ is not discrete iff $\{x\} \notin \mathcal{J}^*$ for some $x \in X$ iff $\{x\}$ is not $g$-open in $(X, \mathcal{J})$ for some $x \in X$ iff $\{x\}$ is $\mathcal{J}$-closed but not $\mathcal{J}$-open for some $x \in X$. 


Corollary 7.18: The following conditions are equivalent:

(a): \((X, \mathcal{J}^*)\) is discrete.

(b): If \(\{x\}\) is \(\mathcal{J}\)‐closed, \(\{x\}\) is \(\mathcal{J}\)‐open.

(c): For all \(x \in X\), \(\{x\}\) is \(g\)‐open in \((X, \mathcal{J})\).

Proof: (a) implies (b): Apply Theorem 7.17

(b) implies (c): Apply Corollary 1.14

(c) implies (d): Apply Theorem 7.12

Remark 7.19: We conclude this chapter by noting that as we put increasingly stronger "regularity" conditions on \((X, \mathcal{J})\), we induce increasingly stronger separation properties on \((X, \mathcal{J}^*)\).

Theorem 7.20: If \((X, \mathcal{J})\) is \(R_0\), then \((X, \mathcal{J}^*)\) is \(T_1\).

Proof: If \((X, \mathcal{J})\) is \(R_0\), \(\{x\}\) is \(g\)‐closed for all \(x \in X\), and thus \(c^\star(\{x\}) = \{x\}\) for all \(x \in X\).

Theorem 7.21: If \((X, \mathcal{J})\) is weakly Hausdorff, \((X, \mathcal{J}^*)\) is \(T_2\).

Proof: Let \(S: D \rightarrow X\) be a net such that \(\lim S = x\) and \(\lim S = y\) in \((X, \mathcal{J}^*)\). We assert \(x = y\). For, by Theorem 7.8, \(\lim S = x\) and \(\lim S = y\) in \((X, \mathcal{J})\), a weakly Hausdorff space, and so \(c(x) = c(y)\), the closure being taken in \((X, \mathcal{J})\). If \(\{x\}\) is \(\mathcal{J}\)‐closed, then \(y \in c(y) = c(x) = \{x\}\), and thus \(y = x\). Similarly, if \(\{y\}\) if \(\mathcal{J}\)‐closed, then \(x = y\). If neither \(\{x\}\) nor \(\{y\}\) is \(\mathcal{J}\)‐closed, then, by Corollary 1.14, \(\{x\}\) and \(\{y\}\) are \(g\)‐open in \(\mathcal{J}\), and thus \(\{x\} \in \mathcal{J}^*\) and \(\{y\} \in \mathcal{J}^*\) by Theorem 7.12. Hence \(S\) is eventually in both \(\{x\}\) and \(\{y\}\),
and so again \( x = y \). It follows that \((X, T^*)\) is Hausdorff.

**Theorem 7.22:** If \((X, T)\) is regular, then \((X, T^*)\) is \(T_1\).

**Proof:** Since \((X, T^*)\) is \(T_{1/2}\), it suffices to show that \((X, T^*)\) is regular. So, let \( x \notin F^* \), where \( F^* \) is \(T^*\)-closed.

Case (i): Suppose \( \{x\} \) is \(T\)-closed. Since \( F^* = c^*(F^*) = \bigcap \{ A : F^* \subseteq A \text{ and } A \text{ is } g\text{-closed in } (X, T) \} \), we have \( x \notin A^* \) for some \( A^* \supseteq F^* \) with \( A^* \) \(g\)-closed in \((X, T)\). By Theorem 1.7, \( A^* = F \setminus N \) where \( F \) is \(T\)-closed and \( N \) contains no non-empty, \(T\)-closed sets. Thus \( x \in C(A^*) = C F \cup N \), and it follows that \( x \in C F \). Since \((X, T)\) is regular, there are disjoint, \(T\)-open sets \( O \) and \( U \) such that \( x \in O \) and \( F \subseteq U \). By Theorem 7.8, \( O \) and \( U \) are \(T^*\)-open with \( x \in O \) and \( F^* \subseteq A^* = F \setminus N \subseteq F \subseteq U \).

Case (ii): Suppose \( \{x\} \) is not \(T\)-closed. Then by Corollary 1.14, \( \{x\} \) is \(g\)-open in \( T \), and so \( x \in \{x\} \in T^* \) by Theorem 7.12. We assert that \( F^* \subseteq Cc^*(x) \). For, given \( y \in F^* \), if \( \{y\} \) is \(g\)-open in \( T \), then \( y \in \{y\} \in T^* \) with \( x \notin \{y\} \), and thus \( y \in Cc^*(x) \). On the other hand, if \( \{y\} \) is not \(g\)-open, then \( \{y\} \) is \(T\)-closed, again by Corollary 1.14.

But \( x \notin \{y\} \), and by the regularity of \((X, T)\), there are disjoint, \(T\)-open sets \( O \) and \( U \) such that \( x \in O \) and \( \{y\} \subseteq U \). Thus \( y \in U \in T^* \) with \( x \notin U \), and so again \( y \in Cc^*(x) \). Hence, \( x \in \{x\} \in T^* \) and \( F^* \subseteq Cc^*(x) \subseteq T^* \) with \( \{x\} \cap Cc^*(x) = \emptyset \).
By cases (i) and (ii), \((X, \mathcal{T}^*)\) is regular and the theorem is proved.

We can summarize the three previous theorems, as well as Theorems 2.7 and 2.9 in the following diagram:

\[
\begin{array}{ccc}
(X, \mathcal{J}): & \text{regular} & \text{weakly Hausdorff} \quad \rightarrow \quad \mathcal{R}_0 \\
\downarrow & & \downarrow \\
(X, \mathcal{J}^*): & T_3 & T_2 \quad \rightarrow \quad T_1
\end{array}
\]

Diagram 7-1. \((X, \mathcal{J})\) and \((X, \mathcal{J}^*)\).

**Example 7.23:** Converges of the three previous results are false. For, if \(X = \{a, b, c\}\) with \(\mathcal{J} = \{\emptyset, \{a\}, X\}\), then \((X, \mathcal{J}^*)\) is discrete, and thus \(T_3\) by Corollary 7.18, while \((X, \mathcal{J})\) is not even \(\mathcal{R}_0\) by Remark 2.29.
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