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A COMPARATIVE STUDY OF THREE-WAVE, FOUR-WAVE
AND HIGHER-ORDER-WAVE PARAMETRIC PROCESSES

DISSERTATION

Presented in Partial Fulfillment of the Requirements
for the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

John Charles Corbin, Jr., B.E.E., M.S.

* * * * *

The Ohio State University

1974

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ACKNOWLEDGMENTS

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"The Influence of Magnetic Hysteresis on Skin Effect," ARL TR 65-167 (August 1965).


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Control Systems - Professor F. Carlin Weimer

Solid State Devices and Semiconductor Theory - Professor Marlin O. Thurston

Applied Mathematics - Professor Stefan Drobot

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CHAPTER I

INTRODUCTION

1.1 Review

In this thesis, a parametric process is defined as one in which some energy storage parameter of an oscillatory system is varied periodically so that different oscillation modes of the system are coupled together. This coupling allows energy to be exchanged among the different modes. The oscillatory system can have many physical realizations such as mechanical, acoustical, electromagnetic or combinations thereof.

In most cases of importance to date, the parametric process has been limited to three modes of the system. These modes are generally called the "pump," "signal," and "idler" modes and will be designated in this thesis by the subscripts p, s, and i, respectively. In the parametric oscillator, only the pump energy at an angular frequency \( \omega_p \) is supplied by an external source and both the signal and idler modes at angular frequencies \( \omega_s \) and \( \omega_i \) are generated internally by the system at the expense of the pump energy. In the parametric mixer, both the pump energy at \( \omega_p \) and the signal energy at \( \omega_s \) are supplied by an external source and the idler mode at \( \omega_i \) is generated internally. If both the signal and idler frequencies are the same,
the system is "degenerate." If both pump and signal frequencies are the same, we have second harmonic generation.

Recently, systems involving four or more coupled modes have become important. However, without introducing additional complexity, they can be treated using the same concepts as pump, signal, and idler modes to interpret the interactions.

Since 1961, with the advent of high intensity coherent radiation sources, namely the laser, major research efforts in parametric processes have been at optical frequencies. In 1961, Franken et al observed second harmonic generation in which the red light of a ruby laser was upconverted to the ultraviolet. Shortly after Franken's initial experiment, Maker et al and Giordmaine reported independently their observations on refractive index matching, a technique which greatly increased the conversion efficiencies of the optical nonlinear parametric process.

In 1962, it was first suggested by Kingston that electric susceptibilities that depend on the magnitude of the accompanying electric fields of the laser "pump" could lead to optical parametric oscillation. However, it was not until 1965 that the first experimental observation of parametric amplification was made by Wang and Racette in the material ADP (NH₄H₂PO₄). Since that time, many materials have been found to be suitable for three-wave parametric processes. Since 1970, many experiments have demonstrated tunable parametric oscillation as well as tunable parametric mixing.
1.2 Theory

In 1960, Jurkus and Robson treated the case of a traveling wave parametric amplifier in which an interaction between pump, signal, and idler voltages is achieved along a transmission line with distributed nonlinear capacitance where \( C = C_0 (1 \pm nv) \) per unit length and \( n \) is a constant of proportionality between capacitance change and applied voltage. Their solution showed that there was a periodic transfer of energy between the signal, idler, and pumping waves which could be expressed in terms of Jacobian elliptic functions.

In 1962, Armstrong et al. presented a comprehensive theory for interactions between forward traveling light waves in a nonlinear dielectric which took into account pump depletion, phase mismatch, and losses in the dielectric.

The case of backward traveling waves in a lossless dielectric was treated in a series of papers by Hsu and Yu and a comparison was made with three-wave forward traveling wave interactions. Their solutions were expressed in terms of Jacobian elliptic functions for the periodic transfer of energy between the waves.

1.3 Scope of Thesis

Although a comprehensive theory for traveling wave parametric processes exists, only a limited amount of work has been done in taking into account the effects of pump depletion on the interactions. Most analyses to date assume a constant pump excitation which is independent of the interaction length of the parametric process.
In this thesis, we examine the effects of second and higher order nonlinear polarizations on electromagnetic wave interactions. The pump wave will be allowed to freely interact with the other waves and will not be constrained to a fixed amplitude over the interaction length of the medium.

In Chapter II, the theory for both three-wave forward traveling wave (FTW) and backward traveling wave (BTW) interactions is reviewed. Three-wave interactions can occur as a result of the second order nonlinear polarization of the medium. We show that there are at most six possible systems which lead to independent solution equations in terms of the electric field amplitudes, power, and photon flux. A comprehensive comparison of energy conversion efficiencies for each system is graphically portrayed as a function of pump wave excitation for fixed ratios of input signal flux to input pump flux. The special case of second harmonic generation in which the frequency of the generated wave is twice the frequency of the pump wave is included for completeness.

In Chapter III, the effect of losses on three-wave FTW processes is analyzed. With loss in the system, the Manley-Rowe relations no longer hold and separate conversion efficiencies must be calculated for each of the three waves. The differential equations for three-wave BTW interactions are given but, with loss in the system, analytic solutions cannot be obtained.

Multiple three-wave processes are examined in Chapter IV. Simultaneous FTW down conversion and up conversion has been shown to be
feasible so this case as well as simultaneous BTW down conversion and up conversion are analyzed.

Four-wave processes that can occur as a result of the third order nonlinear polarization of the medium are treated in Chapter V. Because phase matching conditions are difficult to obtain with four separate waves, the analysis proceeds from the special case of third harmonic generation to the more difficult cases of second harmonically pumped FTW and BTW down conversion and up conversion, simultaneous second harmonically pumped processes, and the case with two separate pump waves at different frequencies.

In Chapter VI, higher-order-wave processes are discussed and system differential equations given. Only in a limited number of cases can solution equations be obtained.

The findings of the present work and their significance are discussed in Chapter VII.
2.1 Introduction

In this chapter, we review single three-wave processes that can occur as a result of the second order nonlinear polarization of the medium. For simplicity, we assume that we have three collinear plane traveling waves in a lossless quadratic medium. We further assume that interactions involving three traveling waves satisfy the phase matching conditions

\[ \omega_\ell = \omega_m + \omega_n \]  (2-1-1)

and

\[ k_\ell = k_m + k_n \]  (2-1-2)

where \( k \) is the wave vector (meters\(^{-1}\)).

For both forward and backward traveling waves, we find that we must solve three coupled differential equations which involve the wave amplitudes and the effective nonlinear coupling coefficients. The equations are of the form\(^{16}\)

\[ \frac{dA_\ell(z)}{dz} = \pm \alpha_\ell A_m(z)A_n(z) \]

\[ m = s, i, p \]

\[ n = i, p, s \]  (2-1-3)
where $A_p$, $A_s$, and $A_i$ are the scalar field amplitudes (volts/meter) of the pump, signal, and idler waves, respectively, at angular frequencies $\omega_p$, $\omega_s$, and $\omega_i$, $\alpha$ is the nonlinear coupling coefficient (volts$^{-1}$), and the sign of the right-hand expression is determined from physical considerations. As is shown in Appendix A, only the amplitude equations need be considered in the analysis because the phase relationship between the waves will remain invariant in space when one of the waves is taken as the idler.

Wave interactions which result in the generation of an idler wave at the difference frequency between the pump wave at $\omega_p$ and the signal wave at $\omega_s$ are called down conversion processes; interactions which result in the generation of a wave at the sum of the pump and signal wave frequencies are called up conversion processes. In down conversion, it is accepted practice to designate the input wave at the highest frequency (shortest wavelength) as the pump. Then the difference frequency is

\begin{equation}
(2-1-4) \quad \omega_i = \omega_p - \omega_s.
\end{equation}

Although $\omega_i$ may be higher in frequency than $\omega_s$, in most cases $\omega_i < \omega_s < \omega_p$. In up conversion, the sum frequency is

\begin{equation}
(2-1-5) \quad \omega_i = \omega_p + \omega_s.
\end{equation}

Although $\omega_s$ may be higher in frequency than $\omega_p$, in most cases $\omega_s < \omega_p < \omega_i$.

In down conversion, energy is initially transferred from the pump wave to both the signal and idler waves. If the interaction
length is sufficiently long, the process reverses after the pump is completely depleted. The process then becomes an up conversion since energy is transferred from the two waves at the lower frequencies to the wave at the highest frequency. In practice, the interaction length is usually sufficiently short that the reverse process does not take place. In up conversion, energy is initially transferred from both the pump and signal waves to the generated sum frequency wave. Again, the interaction length is usually sufficiently short in practice to preclude a process reversal.

In the analysis that follows, the interaction length (meters) is designated as \( z = L \). It is assumed that the initial amplitudes of both the pump and signal waves are known at the respective input boundaries of the medium and that the generated wave amplitude is initially zero at one of the boundaries. We choose its amplitude to be zero at \( z = 0 \) which permits a more unified analysis. Forward traveling pump and signal waves enter the medium at \( z = 0 \) whereas backward traveling pump and signal waves enter the medium at \( z = L \).

Solutions to the coupled amplitude equation (2-1-1) are given in terms of the Jacobian elliptic functions \( \text{sn}(u,k^2) \), \( \text{cn}(u,k^2) \), and \( \text{dn}(u,k^2) \) which are defined through the elliptic integral \( F(\phi,k^2) \) as follows:

\[
\begin{align*}
(2-1-6) \quad u &= F(\phi,k^2) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = \int_0^\phi \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} \\
(2-1-7) \quad K &= F(\pi/2,k^2) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}}
\end{align*}
\]
where

\[ \phi = \text{am}(u, k^2) \quad \text{(amplitude function)} \]

\[ \sin \phi = \text{sn}(u, k^2) \quad \text{(sine amplitude function)} \]

\[ \cos \phi = \text{cn}(u, k^2) \quad \text{(cosine amplitude function)} \]

\[ \sqrt{1 - k^2 \sin^2 \phi} = \text{dn}(u, k^2) \quad \text{(delta amplitude function)} \]

\[ k^2 + k'^2 = 1 \]

The parameter \( k^2 \) is called the modulus and \( k'^2 \) the complementary modulus. \( \phi \), the upper limit of the integral, is called the amplitude, \( u \) is called the argument, and its dependence on \( \phi \) is written \( u = \text{arg} \phi \).

The elliptic functions reduce to trigonometric functions when \( k^2 = 0 \) and to hyperbolic functions when \( k^2 = 1 \) as shown in Table 1 below. \( K \) is the real quarter period of the elliptic function.

<table>
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<th>( k^2 = 1 )</th>
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<td>( K )</td>
<td>( \pi/2 )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \text{sn} u )</td>
<td>( \sin u )</td>
<td>( \tanh u )</td>
</tr>
<tr>
<td>( \text{cn} u )</td>
<td>( \cos u )</td>
<td>( \text{sech} u )</td>
</tr>
<tr>
<td>( \text{dn} u )</td>
<td>( 1 )</td>
<td>( \text{sech} u )</td>
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Table 1. Elliptic Function Degeneracies
From the character of the elliptic functions, an interaction length of \( z = L = K/\Gamma \) (where \( \Gamma \) is a constant to be defined later) will result in maximum energy transfer from the pump wave to the generated wave. With the constraint that the amplitude of the generated wave must be zero at \( z = 0 \), only four possible independent conditions are allowed in terms of the elliptic functions. These conditions are shown in Fig. 1 for a representative value of the elliptic function parameter \( k^2 \). The pump, signal, and generated waves are labeled \( \omega_p \), \( \omega_s \), and \( \omega_i \), respectively. If we now relate these four conditions to possible FTW and BTW down and up conversion processes, we find that we can have six independent systems, one each for FTW down conversion and up conversion, and two each for BTW down conversion and up conversion.

In the sections that follow, we will treat each of the conversion processes in detail and relate each of the six systems to the appropriate condition depicted in Fig. 1. After we obtain solutions to the applicable set of coupled amplitude equations for each conversion process, we will compare the conversion efficiency for each system based on pump excitation required for fixed ratios of input signal to input pump energies.

2.2 FTW Down Conversion (System 1)

For FTW down conversion, Eq. (2-1-3) can be written as

\[
\begin{align*}
(2-2-1a) \quad \frac{dA_p(z)}{dz} &= -\alpha P S \frac{A_p(z)A_i(z)}{A_s(z)} \\
(2-2-1b) \quad \frac{dA_s(z)}{dz} &= \alpha P S \frac{A_p(z)A_i(z)}{A_s(z)}
\end{align*}
\]
Fig. 1. The Four Independent Allowed Conditions as Normalized Solutions to All Cases of Three-Wave FTW and BTW Interactions in Terms of the Jacobian Elliptic Functions.
\( \frac{dA_j(z)}{dz} = \alpha_j A_j(z) A_s(z) \)

since energy is transferred from the pump to both the signal and idler waves.

The solutions to Eq. (2-2-1) are:

\begin{align*}
\text{(2-2-2a)} & \quad A_p(z) = A_p(0) \text{sn}(u, k^2) \\
\text{(2-2-2b)} & \quad A_s(z) = \left[ A_s^2(0) + \frac{\alpha_s}{\alpha_p} A_p^2(0) \right]^{\frac{1}{2}} \text{dn}(u, k^2) \\
& \quad = k^{-1} \left( \frac{\alpha_s}{\alpha_p} \right)^{\frac{1}{2}} A_p(0) \text{dn}(u, k^2) \\
\text{(2-2-2c)} & \quad A_i(z) = (\alpha_i / \alpha_p)^{\frac{1}{2}} A_p(0) \text{cn}(u, k^2)
\end{align*}

where

\begin{align*}
\text{(2-2-3)} & \quad u = K - \Gamma z, \\
\text{(2-2-4)} & \quad k^2 = \frac{A_p^2(0)/\alpha_p}{A_s^2(0)/\alpha_s + A_p^2(0)/\alpha_p}, \quad 0 \leq k^2 \leq 1,
\end{align*}

and

\begin{align*}
\text{(2-2-5)} & \quad \Gamma = \left[ \alpha_i \alpha_p A_s^2(0) + \alpha_i \alpha_s A_p^2(0) \right]^{\frac{1}{2}} \\
& \quad = k^{-1} \left( \frac{\alpha_i}{\alpha_s} \right)^{\frac{1}{2}} A_p(0).
\end{align*}

From Eq. (2-2-2) above, we see that FTW down conversion (System 1) is depicted by Fig. 1a. All three waves travel from left to right.

Since \( \text{sn}(u, k^2) = 0 \) when \( u = 0 \), we see from Eq. (2-2-3) and from Fig. 1a that we can have complete pump depletion when \( \Gamma z = K \).
Thus, we must not exceed an interaction length \( L = \frac{K}{\Gamma} \) if we are to maintain down conversion.

In terms of power, Eq. (2-2-2) can be written as

\[
(2-2-6a) \quad P_p(z) = P_p(0) \, \text{sn}^2(K-Tz)
\]

\[
(2-2-6b) \quad P_s(z) = [P_s(0) + \frac{\omega_s}{\omega_p} P_p(0)] \, \text{dn}^2(K-Tz)
\]

\[
(2-2-6c) \quad P_i(z) = \frac{\omega_i}{\omega_p} P_p(0) \, \text{cn}^2(K-Tz)
\]

where

\[
(2-2-7) \quad k^2 = \frac{P_p(0)/\omega_p}{P_s(0)/\omega_s + P_p(0)/\omega_p}, \quad 0 \leq k^2 \leq 1,
\]

and

\[
(2-2-8) \quad \Gamma = \left\{ \frac{d^2}{2 \mu_0 \epsilon_0 n^2} \left[ \frac{\mu_0}{\epsilon_0} \right]^{\frac{3}{2}} \left[ \omega_i \omega_p P_s(0) + \omega_i \omega_s P_s(0) \right] \right\}^{\frac{1}{2}}.
\]

In Eq. (2-2-6), (2-2-7), and (2-2-8), we have used the relations

\[
(2-2-9) \quad \alpha = \frac{\omega d}{2 n} \left( \frac{\mu_0}{\epsilon_0} \right)^{\frac{1}{2}}
\]

and

\[
(2-2-10) \quad A^2(0) = \frac{2 S(0)}{n} \left( \frac{\mu_0}{\epsilon_0} \right)^{\frac{1}{2}}
\]

where \( d \) is the effective nonlinear coefficient (amp-sec/volts\(^2\)), \( n \) is the refractive index of the medium at a particular frequency, \( S(0) \) is the intensity of the input wave (watts/meter\(^2\)), and \( P(0) \) is the power of the input wave (watts).
If we use the relation

\[(2-2-11) \quad S(0) = \hbar \omega N(0)\]

where \(N(0)\) is the photon flux (no. of photons/meter\(^2\)/sec.) of the input wave, Eq. (2-2-6) becomes

\[(2-2-12a) \quad N_p(z) = N_p(0) \text{sn}^2(K-rz)\]

\[(2-2-12b) \quad N_s(z) = \left[N_s(0) + N_p(0)\right] \text{dn}^2(K-rz)\]

\[(2-2-12c) \quad N_\perp(z) = N_p(0) \text{cn}^2(K-rz)\]

where

\[(2-2-13) \quad k^2 = \frac{N_p(0)}{N_s(0) + N_p(0)}, \quad 0 \leq k^2 \leq 1,\]

and

\[(2-2-14) \quad \Gamma = \left\{\frac{\hbar \omega_p \omega_s}{2n_p n_s n_\perp} \left(\frac{u_0}{\epsilon_0}\right)^{3/2} \left[N_s(0) + N_p(0)\right] \right\}^{1/2}\]

If we differentiate Eq. (2-2-12) with respect to \(z\), we obtain the Manley-Rowe relations\(^\text{19}\) in terms of the rate of change of photon flux. Thus,

\[(2-2-15a) \quad \frac{dN_\perp(z)}{dz} = \frac{dN_s(z)}{dz} = -\frac{dN_p(z)}{dz}\]

\[(2-2-15b) \quad = 2\Gamma N_p(0) \text{sn}(K-rz) \text{cn}(K-rz) \text{dn}(K-rz)\]

The conversion efficiency for transferring energy from the forward traveling pump wave over the interaction length \(z = L\) can
be written as

\[
\begin{align*}
(2-2-16a) & \quad E = \frac{P_p(0) - P_p(L)}{P_p(0)} = \frac{P_p(L)/\omega_p}{P_p(0)/\omega_p} = \frac{[P_p(L) - P_p(0)]/\omega_p}{P_p(0)/\omega_p} \\
(2-2-16b) & \quad \frac{N_s(0) - N_s(L)}{N_s(0)} = \frac{N_s(L)}{N_s(0)} = \frac{N_s(L) - N_s(0)}{N_s(0)} \\
(2-2-16c) & \quad = c n^2 (K-\Gamma L), \quad \Gamma L \leq K.
\end{align*}
\]

Maximum conversion efficiency is achieved when the pump is completely depleted at an interaction length \( L = K/\Gamma \). From Eq. (2-2-12) we obtain

\[
(2-2-17) \quad \frac{N_s}{N_s(0)} = 1 + \frac{N_p(0)}{N_s(0)} = \frac{1}{k^{'2}}
\]

In Fig. 2, we have plotted conversion efficiency, \( E = c n^2 (K-\Gamma L) \), vs. pump excitation, \( k\Gamma L = \left[ \frac{\hbar \omega_p \omega_s \omega_i d^2}{2 n_s n_i n_s} \left( \frac{\hbar^2}{\epsilon_0} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \), at fixed ratios of input signal photon flux to input pump photon flux, \( N_s(0)/N_p(0) \). Since conversion efficiency is a direct measure of pump depletion, we see that we must have an input signal before we can transfer energy from the pump wave to either the signal or idler wave. For a given pump excitation, as we increase the input signal we increase the pump depletion and transfer additional energy to the \( \omega_s \) and \( \omega_i \) waves. If we increase \( N_s(0) \) to a sufficiently large value, we obtain total pump depletion or 100% conversion. The figure also shows that less pump excitation is required to achieve a given conversion efficiency as the input signal is increased. Finally, it should be noted that if total pump depletion is achieved for a given signal
input, increasing the signal input or increasing the pump excitation will change the down conversion process to that of up conversion. Energy will be transferred back to the pump thereby effectively reducing the conversion efficiency.

Since the pump excitation is proportional to the square root of the product $\omega_p \omega_s \omega_i$, it is instructive to see how $(\omega_p \omega_s \omega_i)^{1/2}$ varies when plotted against the ratio of $\omega_i$ to $\omega_p$. In Fig. 3, we have plotted $(\omega_p \omega_s \omega_i)^{1/2}$ (normalized to 1.0 for $\omega_i/\omega_p = 0.5$) vs. $\omega_i/\omega_p$. We see that the product is maximized when $\omega_i/\omega_p = 0.5$ or when the system is "degenerate."

As an example of down conversion, let us assume that we have a pump wave at a wavelength of $\lambda_p = 0.694 \mu m (\omega_p = 2.716 \times 10^{15} \text{ sec}^{-1})$ and a signal wave at $\lambda_s = 0.743 \mu m (\omega_s = 2.537 \times 10^{15} \text{ sec}^{-1})$ so that a wave at a difference frequency $\omega_i = 1.78 \times 10^{14} \text{ sec}^{-1}$ ($\lambda_i = 10.6\mu m$) is generated. If we assume that $d = 4.425 \times 10^{-22} \text{ amp-sec/volt}^2$ (which corresponds to the effective $d$ coefficient for HgS), $L = 0.01$ meter, $N_s(0) = 10^{29}$ (which corresponds to a signal intensity $S_s(0) = 2.675 \times 10^{10}$ watts/meter$^2$), $N_p(0) = 10^{30}$ (which corresponds to a pump intensity of $S_p(0) = 2.864 \times 10^{11}$ watts/meter$^2$), and $n_p = n_s = n_i = 2.5$, we find that we have a pump excitation $k \Gamma L = 2.082$. For the ratio of $N_s(0)/N_p(0) = 0.1$, we find from Fig. 2 that the conversion efficiency is 0.828.

As an example of process reversal from down conversion to up conversion, let us increase the input signal and pump photon fluxes. Let $N_s(0) = 2 \times 10^{29}$ and $N_p(0) = 2 \times 10^{30}$ which maintains the ratio...
Fig. 2. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for FTW Down Conversion (System 1).
Fig. 3. \((\omega_p \omega_s \omega_i)^{\frac{1}{3}}\) vs. \(\omega_i/\omega_p\) (with \(\omega_p \omega_s \omega_i\) normalized to 1.0 for \(\omega_i/\omega_p = 0.5\)) for \(\omega_i = \omega_p - \omega_s\).
\( \frac{N_s(0)}{N_p(0)} = 0.1 \). We calculate pump excitation \( kT_L = 2.94 \). We see from Fig. 2 that we now have a process reversal. Since \( T_L = 3.088 \) and \( K = 2.623 \) for \( k^2 = 0.909 \), we must use the equation

\[
E = cn^2(T_L - K, k^2)
\]

to calculate the conversion efficiency.\(^{20}\) We obtain \( E = 0.81 \). If we plot the mirror image of the curve for \( \frac{N_s(0)}{N_p(0)} = 0.1 \) around a vertical axis passing through the point where the \( \frac{N_s(0)}{N_p(0)} = 0.1 \) curve intersects the \( E = 1.0 \) line, we find that \( E = 0.81 \) at a pump excitation of 2.94.
2.3 FTW Up Conversion (System 2)

For FTW up conversion, Eq. (2-1-3) can be written as

\[
\begin{align*}
(2-3-1a) \quad \frac{dA_p(z)}{dz} &= -\alpha_p A_s(z)A_1(z) \\
(2-3-1b) \quad \frac{dA_s(z)}{dz} &= -\alpha_s A_p(z)A_1(z) \\
(2-3-1c) \quad \frac{dA_1(z)}{dz} &= \alpha_p A_s(z)A_s(z)
\end{align*}
\]

since energy is transferred from both the pump and signal waves to the sum frequency wave.

Two sets of solution equations are possible depending upon the relative magnitudes of the input signal and pump waves.

**Case 1.** \(A_s^2(0)/\alpha_s \leq A_p^2(0)/\alpha_p\).

The solutions to Eq. (2-3-1) are:

\[
\begin{align*}
(2-3-2a) \quad A_p(z) &= A_p(0) \text{dn}(u,k^2) \\
(2-3-2b) \quad A_s(z) &= A_s(0) \text{cn}(u,k^2) = k(\alpha_s/\alpha_p)^{1/2} A_p(0) \text{cn}(u,k^2) \\
(2-3-2c) \quad A_1(z) &= (\alpha_s/\alpha_p)^{1/2} A_s(0) \text{sn}(u,k^2) \\
&= k(\alpha_s/\alpha_p)^{1/2} A_p(0) \text{sn}(u,k^2)
\end{align*}
\]

where

\[
\begin{align*}
(2-3-3) \quad u &= \Gamma_1 z, \\
(2-3-4) \quad k^2 &= \frac{A_s^2(0)/\alpha_s}{A_p^2(0)/\alpha_p}, \quad 0 \leq k^2 \leq 1,
\end{align*}
\]
and

\[ 2-3-5 \]

\[ \Gamma_1 = (\alpha_1^2 \alpha_s^2) \frac{1}{2} A_p(0). \]

From this set of solutions, we see that FTW up conversion (System 2) for \( A_s^2(0) / \alpha_s \leq A_p^2(0) / \alpha_p \) is depicted by Fig. 1c. All three waves travel from left to right.

In terms of power, Eq. (2-3-2) can be written as

\[ 2-3-6a \]

\[ P_p(z) = P_p(0) \text{dn}^2(\Gamma_1 z) \]

\[ 2-3-6b \]

\[ P_s(z) = P_s(0) \text{cn}^2(\Gamma_1 z) \]

\[ 2-3-6c \]

\[ P_i(z) = \frac{\omega_i}{\omega_s} P_s(0) \text{sn}^2(\Gamma_1 z) \]

where

\[ 2-3-7 \]

\[ k^2 = \frac{P_s(0)/\omega_s}{P_p(0)/\omega_p}, \quad 0 \leq k^2 \leq 1, \]

and

\[ 2-3-8 \]

\[ \Gamma_1 = \left[ \frac{\omega_s \omega_i d^2}{2n_p n_s n_i} \left( \frac{\mu_0}{\varepsilon_0} \right)^{\frac{3}{2}} S_p(0) \right]^{\frac{1}{2}}. \]

In terms of photon flux, Eq. (2-3-6) becomes

\[ 2-3-9a \]

\[ N_p(z) = N_p(0) \text{dn}^2(\Gamma_1 z) \]

\[ 2-3-9b \]

\[ N_s(z) = N_s(0) \text{cn}^2(\Gamma_1 z) \]

\[ 2-3-9c \]

\[ N_i(z) = N_s(0) \text{sn}^2(\Gamma_1 z) \]

where
\[ \begin{align*}
(2-3-10) \quad k^2 &= \frac{N_s(0)}{N_p(0)}, \quad 0 \leq k^2 \leq 1, \\
(2-3-11) \quad \Gamma_1 &= \left[ \frac{\mu_0 \omega \omega_p d^2}{2 n p s_1} \left( \frac{\mu_0}{\varepsilon_0} \right)^{3/2} N_p(0) \right]^{1/2} \\
\end{align*} \]

If we differentiate Eq. (2-3-9) with respect to \( z \), we obtain the Manley-Rowe relations in terms of the rate of change of photon flux. Thus,

\[ (2-3-12a) \quad \frac{dN_s(z)}{dz} = -\frac{dN_p(z)}{dz} = -\frac{dN_s(z)}{dz} \]

\[ (2-3-12b) \quad = 2 \Gamma_1 N_s(0) \text{sn}(\Gamma_1 z) \text{cn}(\Gamma_1 z) \text{dn}(\Gamma_1 z). \]

The conversion efficiency for transferring energy from the forward traveling pump wave over the interaction length \( z = L \) can be written as

\[ (2-3-13a) \quad E = \frac{P_{p}(0)-P_{p}(L)}{P_{p}(0)} = \frac{P_{s}(L)/\omega_{p}}{P_{p}(0)/\omega_{p}} = \frac{[P_{s}(0)-P_{s}(L)]/\omega_{s}}{P_{p}(0)/\omega_{p}} \]

\[ (2-3-13b) \quad = \frac{N_p(0)-N_p(L)}{N_p(0)} = \frac{N_s(0)-N_s(L)}{N_p(0)} \]

\[ (2-3-13c) \quad = k^2 \text{sn}^2(\Gamma_1 L), \quad \Gamma_1 L \leq K. \]

Case 2. \( A_s^2(0)/\alpha_s > A_p^2(0)/\alpha_p \).

The solutions to Eq. (2-3-1) are:

\[ (2-3-14a) \quad A_p(z) = A_p(0) \text{cn}(u, k^{-2}) = k^{-1}(\alpha_p/\alpha_s)^{1/2} A_s(0) \text{cn}(u, k^{-2}) \]
(2-3-14b) \[ A_s(z) = A_s(0) \, \text{dn}(u, k^{-2}) \]

(2-3-14c) \[ A_{\perp}(z) = k^{-1} (\alpha_{\perp} / \alpha_s)^{1/2} A_s(0) \, \text{sn}(u, k^{-2}) \]

\[ = (\alpha_{\perp} / \alpha_p)^{1/2} A_p(0) \, \text{sn}(u, k^{-2}) \]

where

(2-3-15) \[ u = \Gamma_2 z, \]

(2-3-16) \[ k^{-2} = \frac{A_p^2(0) / \alpha_p}{A_s^2(0) / \alpha_s}, \quad 0 \leq k^{-2} \leq 1, \]

and

(2-3-17) \[ \Gamma_2 = k (\alpha_{\perp} \alpha_s)^{1/2} A_p(0) = (\alpha_{\perp} \alpha_p)^{1/2} A_s(0). \]

From this set of solutions, we see that FTW up conversion (System 2) for \( A_{s}^2(0) / \alpha_s > A_{p}^2(0) / \alpha_p \) is depicted by Fig. 1d. All three waves travel from left to right.

From Eq. (2-3-2) and (2-3-14) and from Fig. 1d, we see that we can have complete depletion only when \( A_{s}^2(0) / \alpha_s > A_{p}^2(0) / \alpha_p \).

Since \( \text{cn} u = 0 \) when \( u = K \), the interaction length for complete pump depletion is \( L = K / \Gamma_2 = K / (\alpha_{\perp} \alpha_p)^{1/2} A_s(0). \)

For \( p_s(0) / \omega_s > p_p(0) / \omega_p \), Eq. (2-3-6) becomes

(2-3-18a) \[ p_p(z) = p_p(0) \, \text{cn}^2(\Gamma_2 z, k^{-2}) \]

(2-3-18b) \[ p_s(z) = p_s(0) \, \text{dn}^2(\Gamma_2 z, k^{-2}) \]

(2-3-18c) \[ p_{\perp}(z) = \frac{\omega_{\perp}}{\omega_p} p_p(0) \, \text{sn}^2(\Gamma_2 z, k^{-2}) \]
where
\begin{equation}
(2-3-19) \quad k^{-2} = \frac{P_{p}(0)/\omega_{p}}{P_{s}(0)/\omega_{s}}, \quad 0 \leq k^{-2} \leq 1,
\end{equation}
and
\begin{equation}
(2-3-20) \quad \Gamma_{2} = \left[ \frac{\omega_{p}}{2n_{p}^{2}n_{s}^{2}n_{i}} \frac{d^{2}}{d\Omega_{0}} \left( \frac{\mu_{0}}{\varepsilon_{0}} \right) S_{s}(0) \right]^{1/2}.
\end{equation}

For \( N_{s}(0) > N_{p}(0) \), Eq. (2-3-9) becomes
\begin{align}
(2-3-21a) \quad N_{p}(z) &= N_{p}(0) \operatorname{cn}^{2}(\Gamma_{2}z, k^{-2}) \\
(2-3-21b) \quad N_{s}(z) &= N_{s}(0) \operatorname{dn}^{2}(\Gamma_{2}z, k^{-2}) \\
(2-3-21c) \quad N_{i}(z) &= N_{p}(0) \operatorname{sn}^{2}(\Gamma_{2}z, k^{-2})
\end{align}

where
\begin{equation}
(2-3-22) \quad k^{-2} = \frac{N_{p}(0)}{N_{s}(0)}, \quad 0 \leq k^{-2} \leq 1,
\end{equation}
and
\begin{equation}
(2-3-23) \quad \Gamma_{2} = \left[ \frac{\omega_{p}}{2n_{p}^{2}n_{s}^{2}n_{i}} \frac{d^{2}}{d\Omega_{0}} \left( \frac{\mu_{0}}{\varepsilon_{0}} \right) N_{s}(0) \right]^{1/2}.
\end{equation}

The Manley-Rowe relations become
\begin{align}
(2-3-24a) \quad dN_{i}(z)/dz &= -dN_{p}(z)/dz = -dN_{s}(z)/dz \\
(2-3-24b) \quad &= 2\Gamma_{2} N_{p}(0) \operatorname{sn}(\Gamma_{2}z, k^{-2}) \operatorname{cn}(\Gamma_{2}z, k^{-2}) \operatorname{dn}(\Gamma_{2}z, k^{-2})
\end{align}

and the conversion efficiency becomes
In Fig. 4, we have plotted conversion efficiency, \( E \), vs. pump excitation at fixed ratios of input signal photon flux to input pump photon flux, \( N_s(0)/N_p(0) \). For \( N_s(0) \leq N_p(0) \), \( E = k^2 \text{sn}^2(\Gamma_2 L, k^{-2}) \) and pump excitation is \( \Gamma_1 L = \left[ \frac{\hbar \omega_p \omega_s \omega_i d^2}{2 n_p n_s n_i} \left( \frac{\nu_0}{\epsilon_0} \right)^{3/2} N_p(0) \right]^{1/2} L \). For \( N_s(0) > N_p(0) \), \( E = \text{sn}^2(\Gamma_2 L, k^{-2}) \) and pump excitation is

\[
k^{-1} \Gamma_2 L = \left[ \frac{\hbar \omega_p \omega_s \omega_i d^2}{2 n_p n_s n_i} \left( \frac{\nu_0}{\epsilon_0} \right)^{3/2} N_p(0) \right]^{1/2} L.
\]

For \( N_s(0) \leq N_p(0) \), we see that we can never have total pump depletion. The dot-dash curve is a measure of the maximum pump excitation that can be applied before process reversal takes place. For \( N_s(0) > N_p(0) \), total pump depletion can be obtained. However, it should be noted that if total pump depletion is achieved for a given signal input, increasing either the signal or pump input will change the up conversion process to one of down conversion.

If we change the abscissa of Fig. 3 from \( \omega_i/\omega_p \) to \( \omega_i/\omega_i \), the product \( \omega_i \omega_s \omega_p \) is maximized when \( \omega_i = 2\omega_p \) (second harmonic generation).

As an example of up conversion, if we assume that we have a pump wave at \( \lambda_p = 0.743 \mu m(\omega_p = 2.537 \times 10^{15} \text{ sec}^{-1}) \) and a signal wave at \( \lambda_s = 10.6 \mu m(\omega_s = 1.78 \times 10^{14} \text{ sec}^{-1}) \), we can generate a sum frequency wave at \( \lambda_4 = 0.694 \mu m(\omega_4 = 2.716 \times 10^{15} \text{ sec}^{-1}) \). If we assume the same values for \( d, n, \) and \( L \) as in the FTW down conversion
example but let $N_s(0) = 2 \times 10^{29}$ and $N_p(0) = 10^{29}$, we find that
$S_s(0) = 3.75 \times 10^9$ watts/meter$^2$ and $S_p(0) = 2.675 \times 10^{10}$ watts/meter$^2$.
Since $N_s(0)/N_p(0) = 2$, we determine that the pump excitation is
$k^{-1}T_2L = 0.658$. From Fig. 4, we find that the conversion efficiency
is 0.588.

As an example of process reversal from up conversion to down
conversion, let us increase the input signal and pump photon fluxes.
Let $N_s(0) = 2 \times 10^{30}$ and $N_p(0) = 10^{30}$ which maintains the ratio
$N_s(0)/N_p(0) = 2$. We calculate pump excitation $k^{-1}T_2L = 2.08$. We see
from Fig. 4 that we now have a process reversal. Since $\Gamma_2L = 2.942$
and $K = 1.854$ for $k^{-2} = 0.5$, we must use the equation
$E = s_n^2(2K-\Gamma_2L,k^{-2})$ to calculate the conversion efficiency.$^{20}$ We
obtain $E = 0.448$. If we plot the mirror image of the curve for
$N_s(0)/N_p(0) = 2$ around a vertical axis passing through the point
where the $N_s(0)/N_p(0) = 2$ curve intersects the $E = 1.0$ line, we find
that $E = 0.448$ at a pump excitation of 2.08.
Fig. 4. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for FTW Up Conversion (System 2).
2.4 Second Harmonic Generation

In the preceding section, we treated FTW up conversion in which all three waves were at different frequencies and satisfied the phase matching conditions given by Eq. (2-1-1) and (2-1-2). However, if both the pump and signal frequencies are the same, we have second harmonic generation at frequency $\omega_i$, i.e.

\begin{equation}
\omega_i = \omega_p + \omega_s = 2\omega_p.
\end{equation}

For this special case of up conversion, it is possible to satisfy the phase matching conditions with but a single wave input—that of the pump. The amplitude equations (2-3-1) reduce to

\begin{align}
(2-4-2a) \quad & \frac{dA_p(z)}{dz} = -\alpha_p A_p(z)A_i(z) \\
(2-4-2b) \quad & \frac{dA_i(z)}{dz} = \alpha_i A_p^2(z)
\end{align}

since $A_p(z) = A_s(z)$.

The solutions to Eq. (2-4-2) are:

\begin{align}
(2-4-3a) \quad & A_p(z) = A_p(0) \text{ sech}(\Gamma z) \\
(2-4-3b) \quad & A_i(z) = (\alpha_i/\alpha_p)^{1/2} A_p(0) \text{ tanh}(\Gamma z)
\end{align}

where

\begin{equation}
\Gamma = (\alpha_i/\alpha_p)^{1/2} A_p(0).
\end{equation}

In terms of power, Eq. (2-4-3) becomes
(2-4-5a) \[ P_p(z) = P_p(0) \operatorname{sech}^2(\Gamma z) \]

(2-4-5b) \[ P_1(z) = \frac{\omega_1}{\omega_p} P_p(0) \tanh^2(\Gamma z) \]

where

(2-4-6) \[ \Gamma = \left[ \frac{\hbar \omega_p^2 \omega_1}{2 n_p^2 n_1} \left( \frac{\nu_0}{\epsilon_0} \right)^{3/2} \right]^{1/2}. \]

In terms of the photon flux, Eq. (2-4-5) can be written as

(2-4-7a) \[ N_p(z) = N_p(0) \operatorname{sech}^2(\Gamma z) \]

(2-4-7b) \[ N_1(z) = N_p(0) \tanh^2(\Gamma z) \]

where

(2-4-8) \[ \Gamma = \left[ \frac{\hbar \omega_p^2 \omega_1}{2 n_p^2 n_1} \left( \frac{\nu_0}{\epsilon_0} \right)^{3/2} N_p(0) \right]^{1/2}. \]

If we differentiate Eq. (2-4-7) with respect to \( z \), we obtain the Manley-Rowe relations in terms of the rate of change of photon flux. Thus,

(2-4-9a) \[ \frac{dN_1(z)}{dz} = - \frac{dN_p(z)}{dz} \]

(2-4-9b) \[ = 2 \Gamma N_p(0) \operatorname{sech}^2(\Gamma z) \tanh(\Gamma z). \]

The conversion efficiency over the interaction length \( z = L \) is

(2-4-10a) \[ E = \frac{P_p(0) - P_p(L)}{P_p(0)} = \frac{P_1(L)/\omega_1}{P_p(0)/\omega_p}. \]
From Table 1, we see that \( \tanh^2(\Gamma L) \) is a degenerate case of \( k^2 \text{sn}^2(\Gamma L, k^2) \) where \( k^2 = 1.0 \). Since \( k^2 = N_s(0)/N_p(0) \), we see from Fig. 4 that conversion efficiency vs. pump excitation for second harmonic generation is represented by the \( N_s(0)/N_p(0) = 1.0 \) curve. Since the curve is asymptotic to \( E = 1.0 \), total energy transfer from the pump to the generated second harmonic wave cannot be obtained.
2.5 BTW Down Conversion

Unlike FTW interactions where all wave inputs are at \( z = 0 \) and the outputs at \( z = L \), BTW interactions involve at least one backward traveling wave with its input at \( z = L \) and its output at \( z = 0 \). To be consistent with our previous analysis of FTW interactions, we arbitrarily choose the internally-generated idler wave to be a FTW with zero amplitude at \( z = 0 \).

There are two possible systems for obtaining BTW down conversion (and hence signal amplification). We can have either (1) both pump and signal as backward traveling waves with inputs at \( z = L \), or (2) we can have a FTW pump with input at \( z = 0 \) and a BTW signal with input at \( z = L \). Consistent with the accepted practice that \( \omega_p > \omega_s \), this precludes a third possibility— that of a FTW signal and a BTW pump.

System 3. BTW at both \( \omega_p \) and \( \omega_s \).

Eq. (2-1-3) can be written as

\[
\begin{align*}
(2-5-1a) & \quad \frac{dA_p(z)}{dz} = \alpha_p A_s(z)A_i(z) \\
(2-5-1b) & \quad \frac{dA_s(z)}{dz} = -\alpha_s A_p(z)A_i(z) \\
(2-5-1c) & \quad \frac{dA_i(z)}{dz} = \alpha_i A_p(z)A_s(z)
\end{align*}
\]

since energy is transferred from the BTW pump to both the BTW signal and the FTW idler.

The solutions to Eq. (2-5-1) are:
(2-5-2a) \[ A_p(z) = \left( \frac{\alpha_p}{\alpha_s} \right) A_s^2(L) + A_p^2(L) \] \[ \frac{1}{\sqrt{1 - k^2}} \] \[ \text{dn}(u, k^2) \]

(2-5-2b) \[ A_s(z) = A_s(0) \text{sn}(u, k^2) \]

(2-5-2c) \[ A_i(z) = \left( \frac{\alpha_i}{\alpha_s} \right)^{\frac{1}{2}} A_s(0) \text{cn}(u, k^2) \]

where

(2-5-3) \[ u = K - Tz, \]

(2-5-4) \[ k^2 = \frac{A_s^2(0)/\alpha_s}{A_s^2(L)/\alpha_s + A_p^2(L)/\alpha_p}, \quad 0 < k^2 \leq 1, \]

and

(2-5-5) \[ \Gamma = \left[ \alpha_i \alpha_p A_s^2(L) + \alpha_i \alpha_s A_p^2(L) \right]^{\frac{1}{2}}. \]

In this system, \( k^2 \) cannot be greater than unity since the denominator, \( A_s^2(L)/\alpha_s + A_p^2(L)/\alpha_p \), represents the sum of both the signal and pump input energies.

From the above solutions, we see that BTW down conversion (System 3) is depicted by Fig. 1b. The signal and pump waves travel from right to left and the generated idler wave travels from left to right.

If we examine Eq. (2-5-2), we see that we do not know \( A_s(0) \), the amplitude of the output signal. Similarly, we cannot immediately determine the modulus, \( k^2 \), in Eq. (2-5-4) since we do not know \( A_s(0) \).

Thus, we must use an iterative trial and error method to find the correct value of \( A_s(0) \) that will give us our known BTW input \( A_s(L) \) at \( z = L \).
We also note that, since the solution for $A_p(z)$ is in terms of $dn(K-rz)$, we cannot obtain complete pump depletion since $dn$, the delta amplitude function, is always positive.

In terms of power, Eq. (2-5-2) can be written as

\begin{align}
(2-5-6a) \quad P_p(z) & = \left[ \frac{\omega_p}{\omega_s} P_s(L) + P_p(L) \right] dn^2(K-rz) \\
(2-5-6b) \quad P_s(z) & = P_s(0) \, sn^2(K-rz) \\
(2-5-6c) \quad P_i(z) & = \frac{i \omega_i}{\omega_s} P_s(0) \, cn^2(K-rz)
\end{align}

where

\begin{align}
(2-5-7) \quad k^2 & = \frac{P_s(0)/\omega_s}{P_s(L)/\omega_s + P_p(L)/\omega_p} \quad 0 \leq k^2 \leq 1,
\end{align}

and

\begin{align}
(2-5-8) \quad \Gamma & = \left\{ \frac{d^2}{2n_p n_s n_i (\frac{\mu_0}{\varepsilon_0})^{3/2}} \left[ \omega_i \omega_p S_s(L) + \omega_i \omega_s S_p(L) \right] \right\}^{1/2}
\end{align}

In terms of photon flux, Eq. (2-5-6) becomes

\begin{align}
(2-5-9a) \quad N_p(z) & = [N_s(L) + N_p(L)] \, dn^2(K-rz) \\
(2-5-9b) \quad N_s(z) & = N_s(0) \, sn^2(K-rz) \\
(2-5-9c) \quad N_i(z) & = N_s(0) \, cn^2(K-rz)
\end{align}

where

\begin{align}
(2-5-10) \quad k^2 & = \frac{N_s(0)}{N_s(L) + N_p(L)}, \quad 0 \leq k^2 \leq 1,
\end{align}
The Manley-Rowe relations in terms of the rate of change of photon flux are:

\[(2-5-12a) \frac{dN_s(z)}{dz} = \frac{dN_p(z)}{dz} = -\frac{dN_s(z)}{dz} = 2\Gamma N_s(0) \text{sn}(K-Iz) \text{cn}(K-Iz) \text{dn}(K-Iz).\]

The conversion efficiency for transferring energy from the backward traveling pump wave over the interaction length \(z = L\) can be written as

\[(2-5-13a) E = \frac{P_p(L) - P_p(0)}{P_p(L)} = \frac{N_p(L) - N_p(0)}{N_p(L)} = \frac{N_i(L)}{N_p(L)} = \frac{N_s(0) - N_s(L)}{N_p(L)} = k^2 \text{sn}^2(\Gamma L, k^2).\]

In Fig. 5, we have plotted conversion efficiency, \(E = k^2 \text{sn}^2(\Gamma L),\) vs. pump excitation, \(\Gamma L \left(\frac{N_p(L)}{N_s(L) + N_p(L)}\right)^{1/2}\)

\[= \left[\frac{\hbar \omega_p \omega_s}{2 n_p n_s n_i} \left(\frac{\mu_0}{\varepsilon_0}\right)^{3/2} N_p(L)\right]^{1/2} L,\]

at fixed ratios of input signal photon flux to input pump photon flux, \(N_s(L)/N_p(L).\) This is a particularly interesting system insofar as it is impossible to achieve complete pump depletion as all curves are asymptotic to \(E = 1.0\) for all ratios of \(N_s(L)/N_p(L).\) However, for small input signals, the system is much
more efficient than FTW down conversion in transferring energy from the pump wave to the other waves for a given pump excitation. The dot-dash curve represents the condition for system oscillation with no signal wave present. For \( N_s(L)/N_p(L) = 0 \), the threshold for oscillation is at a pump excitation of \( \pi/2 \).

As an example of BTW down conversion, let us choose the same values for \( \omega_p, \omega_s, \omega_i, d, n, L \), input signal photon flux, and input pump photon flux as we assumed for FTW down conversion. It follows then that we have a pump excitation of 2.082. For the ratio of \( N_s(L)/N_p(L) = 0.1 \), we find from Fig. 5 that the conversion efficiency for this system is 0.879 compared to 0.828 for FTW down conversion.

System 4. BTW at \( \omega_s \); FTW at \( \omega_p \).

Eq. (2-1-3) can be written as

\[
\begin{align*}
(2-5-14a) \quad \frac{dA_p(z)}{dz} &= -\alpha A_s(z)A_i(z) \\
(2-5-14b) \quad \frac{dA_s(z)}{dz} &= -\alpha A_p(z)A_i(z) \\
(2-5-14c) \quad \frac{dA_i(z)}{dz} &= \alpha A_p(z)A_s(z)
\end{align*}
\]

since the energy is transferred from the FTW pump to both the BTW signal and the FTW idler waves.

We see that Eq. (2-5-14) is identical to Eq. (2-3-1) for FTW up conversion. Therefore, the same solution equations (2-3-2 through 2-3-25) which apply for FTW up conversion also apply for this system. However, it must be remembered that \( A_s(0) \) now refers to the output
Fig. 5. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for BTW Down Conversion (System 3) with BTW at Both $\omega_p$ and $\omega_s$. 

\[ \frac{N_s(L)}{N_p(L)} \] 

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for the case of a BTW signal wave in these equations. Since we do not
know $A_s(0)$, the amplitude of the output signal, an iterative trial and
error method must be used to find the proper value of $A_s(0)$ that will
satisfy the known input condition $A_s(L)$ at $z = L$.

If we examine the solution equations, we see that BTW down
conversion (System 4) for $\frac{A_s^2(0)}{\alpha_s} \leq \frac{A_p^2(0)}{\alpha_p}$ is depicted by Fig. 1c;
for $\frac{A_s^2(0)}{\alpha_s} > \frac{A_p^2(0)}{\alpha_p}$, the system is depicted by Fig. 1d. In both
cases, the signal wave propagates from right to left and the pump and
idler waves propagate from left to right.

In Fig. 6, we have plotted conversion efficiency, $E$, vs. pump
excitation for fixed ratios of input signal photon flux to input pump
photon flux, $N_s(L)/N_p(0)$. Between the two dot-dash curves,
$N_s(0) \leq N_p(0)$ so that $E = k^2sn^2(T_1L,k^2)$ and the pump excitation is

$$T_1L = \frac{n_p \omega_p \omega_s d^2}{2 n_p n_s n_1} \left( \frac{\mu_0}{\epsilon_0} \right)^{3/2} N_p(0)^{1/2}.$$

Above the left dot-dash curve,

$N_s(0) > N_p(0)$ so that $E = sn^2(T_2L,k^{-2})$ and the pump excitation is

$$k^{-1}T_2L = \frac{n_p \omega_p \omega_s d^2}{2 n_p n_s n_1} \left( \frac{\mu_0}{\epsilon_0} \right)^{3/2} N_p(0)^{1/2}.$$

For very small signals such

that $N_s(0) \leq N_p(0)$, we can never have total pump depletion. However,
for $N_s(0) > N_p(0)$, total pump depletion is obtainable. If total
pump depletion is achieved for a given signal input, increasing either
the signal or pump input will reverse the conversion process. The
dot-dash curve on the right represents the condition for system
oscillation with no signal wave present. For $N_s(L)/N_p(0) = 0$, the
threshold for oscillation is a pump excitation of $\pi/2$. 
Fig. 6. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for BTW Down Conversion (System 4) with a BTW at $\omega_s$ and a FTW at $\omega_p$. 
If we again choose as our example the same values for $\omega_p$, $\omega_s$, $\omega_i$, $d$, $n$, $L$, input signal photon flux and input pump photon flux as we previously assumed for FTW down conversion, we obtain a pump excitation of 2.082. For a ratio of $N_s(L)/N_p(0) = 0.1$, we find from Fig. 6 that the conversion efficiency is 0.977 for this system compared to 0.879 for the other BTW down conversion system.

If we compare this BTW down conversion system (System 4) to FTW down conversion (System 1) and BTW down conversion (System 3) for a particular ratio of input signal to input pump photon flux and a particular pump excitation, we find that this system (System 4) provides the highest conversion efficiency in every instance.
2.6 BTW Up Conversion

There are two possible systems for obtaining BTW up conversion. We can have either (1) a FTW pump with input at \( z = 0 \) and a BTW signal with input at \( z = L \) or (2) we can have a FTW signal with input at \( z = 0 \) and a BTW pump with input at \( z = L \). In both systems we have a sum frequency FTW generated with zero amplitude at \( z = 0 \).

System 5. BTW at \( \omega_s \); FTW at \( \omega_p \).

Eq. (2-1-3) can be written as

\[
\begin{align*}
(2-6-la) \quad \frac{dA_p(z)}{dz} & = - \alpha_p A_s(z)A_i(z) \\
(2-6-lb) \quad \frac{dA_s(z)}{dz} & = \alpha_s A_p(z)A_i(z) \\
(2-6-1c) \quad \frac{dA_i(z)}{dz} & = \alpha_i A_p(z)A_s(z)
\end{align*}
\]

since energy is transferred from both the FTW pump and BTW signal to the sun frequency FTW.

We see that Eq. (2-6-1) is identical to Eq. (2-2-1) for FTW down conversion. Therefore, the same solution equations (2-2-2 through 2-2-16) which apply for FTW down conversion also apply for this system. However, it must be remembered that \( A_s(0) \) now refers to the output for the case of a BTW signal wave in these equations. Since we do not know \( A_s(0) \), the output signal amplitude, an iterative trial and error method must be used to find the correct value of \( A_s(0) \) that will satisfy the known input condition \( A_s(L) \) at \( z = L \).
Eq. (2-2-17) does not apply to this system since the signal wave is depleted in the BTW up conversion process.

From the solution equations, we see that BTW up conversion (System 5) is depicted by Fig. 1a. In this system, the signal wave travels from right to left and the pump and sum frequency waves travel from left to right.

In Fig. 7, we have plotted conversion efficiency, \( E = cn^2(K\Gamma L) \), vs. pump excitation, \( k\Gamma L = \left[ \frac{n_s \omega \omega_1 d^2}{2 n_p n_s n_i} \left( \frac{\nu_0}{\epsilon_0} \right) N_s(0) \right] L \), at fixed ratios of input signal photon flux to input pump photon flux, \( N_s(L)/N_p(0) \). We see from the figure that for \( 0 < N_s(L)/N_p(0) \leq 1 \), we cannot have complete pump depletion. In fact, the curves are asymptotic to the value of conversion efficiency that equals the ratio of \( N_s(L)/N_p(0) \). For \( N_s(L) > N_p(0) \), complete pump depletion is possible. However, if total pump depletion is achieved for a particular signal input, increasing either the signal input or pump input will reverse the conversion process.

If we compare this BTW up conversion system to that of FTW up conversion for a given ratio of input signal to input pump photon flux and a given pump excitation, we find that this system is less efficient in transferring energy from the pump wave to the up converted wave in every instance.

System 6. FTW at \( \omega_s \); BTW at \( \omega_p \).

Eq. (2-1-3) can be written as
Fig. 7. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for BTW Up Conversion (System 5) with a BTW at \( \omega_s \) and a FTW at \( \omega_p \).
\begin{align}
(2-6-2a) \quad & \frac{dA_p(z)}{dz} = \alpha_p A_s(z) A_i(z) \\
(2-6-2b) \quad & \frac{dA_s(z)}{dz} = -\alpha_s A_p(z) A_i(z) \\
(2-6-2c) \quad & \frac{dA_i(z)}{dz} = \alpha_i A_p(z) A_s(z)
\end{align}

since energy is transferred from both the BTW pump and FTW signal to the sum frequency FTW.

Since Eq. \((2-6-2)\) is identical to Eq. \((2-5-1)\) for BTW down conversion (System 3), the same solution equations \((2-5-2\) through \(2-5-13\)) also apply for this system. However, if we examine the equations, we see that we do not know \(A_s(L)\) and similarly \(P_s(L)\) and \(N_s(L)\). Thus, we must use an iterative trial and error method to find the correct value of \(A_s(L)\), the amplitude of the output signal, that will give us our known FTW input \(A_s(0)\) at \(z = 0\) for this system of up conversion.

It should be noted that
\begin{align}
(2-6-3a) \quad & \frac{A_p^2(L)}{\alpha_p} + \frac{A_s^2(L)}{\alpha_s} = \frac{A_p^2(0)}{\alpha_p} + \frac{A_s^2(0)}{\alpha_s} \\
(2-6-3b) \quad & P_p(L)/\omega_p + P_s(L)/\omega_s = P_p(0)/\omega_p + P_s(0)/\omega_s \\
(2-6-3c) \quad & N_p(L) + N_s(L) = N_p(0) + N_s(0)
\end{align}

so that solution equations \((2-5-2\) through \(2-5-13\)) can be modified accordingly.

From the solution equations, we see that this system is depicted by Fig. 1b. The pump wave propagates from right to left and
In Fig. 8, we have plotted conversion efficiency, \( E = k^2 \text{sn}^2 (\Gamma L) \), vs. pump excitation, \( \Gamma L \left( \frac{N_p(L)}{N_s(L) + N_p(L)} \right)^{1/2} = \left[ \frac{\hbar \omega_p \omega_s d^2}{2 \hbar n_p n_s} \left( \frac{\epsilon_0}{\mu_0} \right)^{3/2} \frac{N_p(L)}{N_p(L)} \right]^{1/2} L \), at fixed ratios of input signal photon flux to input pump photon flux, \( N_s(0)/N_p(L) \), for this system. For \( 0 < N_s(0)/N_p(L) \leq 1 \), the curves are asymptotic to the value of the conversion efficiency that equals the ratio of \( N_s(0)/N_p(L) \). For \( N_s(0)/N_p(L) > 1 \), the curves are asymptotic to \( E = 1.0 \). Thus, total pump depletion cannot be obtained using this system. The dot-dash curve represents the maximum pump excitation that can be applied before up conversion reverts to down conversion.

If we compare the two BTW up conversion systems to the FTW up conversion system, we find that the FTW up conversion system provides the highest conversion efficiency for a given ratio of input signal to input pump photon flux and a given pump excitation in every instance.
Fig. 8. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for BTW Up Conversion (System 6) with a FTW at $\omega_s$ and a BTW at $\omega_p$. 
3.1 Introduction

In this chapter, we will include the effect of energy losses due to absorption, scattering, and other mechanisms in our amplitude equations for both FTW and BTW processes and determine solutions to these equations.

The amplitude equation with the loss term now included can be written as

\[
\frac{dA_\ell(z)}{dz} \pm \delta_\ell A_\ell(z) = \pm \alpha_\ell A_m(z)A_n(z)
\]

where \( \delta_\ell \) (meters\(^{-1}\)) is the loss coefficient. For purposes of analysis, the assumption will be made that \( \delta_\ell = \delta_m = \delta_n = \delta \) over the frequency (wavelength) range of interest. Since

\[
\delta = \frac{\sigma}{2n} \left( \frac{\mu_0}{\epsilon_0} \right)^{\frac{1}{2}}
\]

where \( \sigma \) is the conductivity of the medium (amps/volt/meter) and \( n \) is the refractive index, the assumption is fairly good since \( n \) is usually only weakly dependent on frequency over the transmission range.
3.2 FTW Down Conversion (System 1)

Eq. (3-1-1) for FTW down conversion can be written as

\begin{align*}
(3-2-1a) \quad \frac{dA_p(z)}{dz} + \delta A_p(z) &= - \alpha A_s(z)A_l(z) \\
(3-2-1b) \quad \frac{dA_s(z)}{dz} + \delta A_s(z) &= \alpha A_p(z)A_l(z) \\
(3-2-1c) \quad \frac{dA_l(z)}{dz} + \delta A_l(z) &= \alpha A_p(z)A_s(z).
\end{align*}

In order to solve for \( A_p(z) \), \( A_s(z) \), and \( A_l(z) \), we must transform Eq. (3-2-1) into an integrable form. Let

\begin{align*}
(3-2-2a) \quad B_p(z) &= e^{\delta z} A_p(z) \\
(3-2-2b) \quad B_s(z) &= e^{\delta z} A_s(z) \\
(3-2-2c) \quad B_l(z) &= e^{\delta z} A_l(z).
\end{align*}

If we differentiate Eq. (3-2-2) with respect to \( z \), we obtain

\begin{align*}
(3-2-3a) \quad \frac{dB_p(z)}{dz} &= e^{\delta z} \left[ \frac{dA_p(z)}{dz} + \delta A_p(z) \right] \\
(3-2-3b) \quad \frac{dB_s(z)}{dz} &= e^{\delta z} \left[ \frac{dA_s(z)}{dz} + \delta A_s(z) \right] \\
(3-2-3c) \quad \frac{dB_l(z)}{dz} &= e^{\delta z} \left[ \frac{dA_l(z)}{dz} + \delta A_l(z) \right].
\end{align*}

Now let

\begin{align*}
(3-2-4) \quad y &= \frac{1 - e^{-\delta z}}{\delta} \quad \text{so that} \quad \frac{dy}{dz} = e^{-\delta z}.
\end{align*}

Then
(3-2-5a) \[ \frac{dB_p(z)}{dy} = (\frac{dB_p}{dz})(\frac{dz}{dy}) = e^{2\delta z}[dA_p(z)/dz + \delta A_p(z)] \]

(3-2-5b) \[ \frac{dB_s(z)}{dy} = (\frac{dB_s}{dz})(\frac{dz}{dy}) = e^{2\delta z}[dA_s(z)/dz + \delta A_s(z)] \]

(3-2-5c) \[ \frac{dB_i(z)}{dy} = (\frac{dB_i}{dz})(\frac{dz}{dy}) = e^{2\delta z}[dA_i(z)/dz + \delta A_i(z)]. \]

If we substitute Eq. (3-2-2) and (3-2-5) into Eq. (3-2-1), we obtain

(3-2-6a) \[ \frac{dB_p(z)}{dy} = -\alpha B^p(z)B_i(z) \]

(3-2-6b) \[ \frac{dB_s(z)}{dy} = \alpha B_p(z)B_i(z) \]

(3-2-6c) \[ \frac{dB_i(z)}{dy} = \alpha B_p(z)B_s(z) \]

The solutions to Eq. (3-2-6) with the boundary condition that \( B_i(z) = 0 \) at \( y = 0 \) are:

(3-2-7a) \[ B_p(z) = B_p(0) \text{sn}(K-\Gamma y, k^2) \]

(3-2-7b) \[ B_s(z) = \left[B_s^2(0) + \frac{\alpha_s}{\alpha_p} B_p^2(0)\right]^{1/2} \text{dn}(K-\Gamma y, k^2) \]

(3-2-7c) \[ B_i(z) = (\alpha_i/\alpha_p)^{1/2} B_p(0) \text{cn}(K-\Gamma y, k^2) \]

where

(3-2-8) \[ k^2 = \frac{B_p^2(0)/\alpha_p}{B_s^2(0)/\alpha_s + B_p^2(0)/\alpha_p}, \quad 0 \leq k^2 \leq 1, \]

and

(3-2-9) \[ \Gamma = [\alpha_i \alpha_p B_s^2(0) + \alpha_i B_s B_p^2(0)]^{1/2}. \]
If we now substitute Eq. (3-2-2) and (3-2-4) into Eq. (3-2-7), we obtain the solutions for \( A_p(z) \), \( A_s(z) \), and \( A_{\perp}(z) \) which reflect the effect of losses in the system. Thus,

\begin{align*}
(3-2-10a) \quad A_p(z) &= e^{-\frac{\delta z}{\delta}} A_p(0) \text{ sn}[K - \Gamma \left(\frac{1-e^{-\delta z}}{\delta}\right), k^2] \\
(3-2-10b) \quad A_s(z) &= e^{-\frac{\delta z}{\delta}} [A_s^2(0) + \frac{\alpha_s}{\alpha_p} A_p^2(0)]^{\frac{1}{2}} \text{ dn}[K - \Gamma \left(\frac{1-e^{-\delta z}}{\delta}\right), k^2] \\
(3-2-10c) \quad A_{\perp}(z) &= e^{-\frac{\delta z}{\delta}} (\alpha_p/\alpha_p) A_p(0) \text{ cn}[K - \Gamma \left(\frac{1-e^{-\delta z}}{\delta}\right), k^2]
\end{align*}

where \( k^2 \) is given by Eq. (2-2-4) and \( \Gamma \) by Eq. (2-2-5).

The effect of loss in the system is reflected in the solution equation (3-2-10) by the addition of a damping term, \( e^{-\frac{\delta z}{\delta}} \), and a modification of the argument of the elliptic function from \( K - \Gamma z \) for the lossless case to \( K - \Gamma \left(\frac{1-e^{-\delta z}}{\delta}\right) \) for this case.

With loss in the system, we can no longer express the conversion efficiency as in Eq. (2-2-16) for the lossless system. We find that we must write three separate equations to express the various conversion efficiencies. Thus,

\begin{align*}
(3-2-11a) \quad E_p &= \frac{N_p(0) - N_p(L)}{N_p(0)} = 1 - e^{-2\delta L} \text{ sn}^2[K - \Gamma \left(\frac{1-e^{-\delta L}}{\delta}\right), k^2] \\
(3-2-11b) \quad E_s &= \frac{N_s(L) - N_s(0)}{N_p(0)} = e^{-2\delta L} \text{ cn}^2[K - \Gamma \left(\frac{1-e^{-\delta L}}{\delta}\right), k^2] \\
&\quad - \frac{N_s(0)}{N_p(0)} (1-e^{-2\delta L}) \\
(3-2-11c) \quad E_{\perp} &= \frac{N_{\perp}(L)}{N_p(0)} = e^{-2\delta L} \text{ cn}^2[K - \Gamma \left(\frac{1-e^{-\delta L}}{\delta}\right), k^2]
\end{align*}
where \( k^2 \) is given by Eq. (2-2-13) and \( \Gamma \) by Eq. (2-2-14).

As an example of the effect of loss in FTW down conversion, let us assume that we have a pump wave at \( \lambda_p = 0.694\mu\text{m} \), a signal wave at \( \lambda_s = 0.743\mu\text{m} \), and an idler wave at \( \lambda_i = 10.6\mu\text{m} \). If we assume that \( d = 4.425 \times 10^{-22} \text{amp-sec/volt}^2 \), \( N_s(0) = N_p(0) = 10^{29} \), and \( n_p = n_s = n_i = 2.5 \), then our initial wave amplitudes at \( z = 0 \) are \( A_p(0) = 2.939 \times 10^6 \text{volts/meter} \), \( A_s(0) = 2.84 \times 10^6 \text{volts/meter} \), and \( A_i(0) = 0 \). In Fig. 9, we have plotted the solutions to Eq. (3-4-1) using the above initial conditions for \( \delta = 0 \) (solid curves for the lossless case) and for \( \delta = 5 \text{ meters}^{-1} \) (dashed curves for the lossy system). The curves extend over several wave periods so that the effects of the loss coefficient \( \delta \) can more readily be seen.
Fig. 9. Pump, Signal, and Idler Wave Amplitudes for FTW Down Conversion with $\delta = 0$ (solid curves) and $\delta = 5 \text{ meters}^{-1}$ (dashed curves).
3.3 FTW Up Conversion (System 2)

An analysis similar to the one made for FTW down conversion can be made for FTW up conversion. Eq. (3-1-1) can be written as

\(3-3-1a\) \[ \frac{dA_p(z)}{dz} + \delta A_p(z) = -\alpha_s A_s(z) A_i(z) \]  
\(3-3-1b\) \[ \frac{dA_s(z)}{dz} + \delta A_s(z) = -\alpha_s A_p(z) A_i(z) \]  
\(3-3-1c\) \[ \frac{dA_i(z)}{dz} + \delta A_i(z) = \alpha_s A_p(z) A_s(z) \]

Two sets of solution equations are possible depending upon the relative magnitudes of the input signal and pump waves.

Case 1. \( A_s^2(0)/\alpha_s < A_p^2(0)/\alpha_p \).

The solutions to Eq. (3-3-1) are:

\(3-3-2a\) \[ A_p(z) = e^{-\delta z} A_p(0) \text{dn}[\Gamma_1 \left( \frac{1-e^{-\delta z}}{\delta} \right), k^2] \]

\(3-3-2b\) \[ A_s(z) = e^{-\delta z} A_s(0) \text{cn}[\Gamma_1 \left( \frac{1-e^{-\delta z}}{\delta} \right), k^2] \]

\[ = e^{-\delta z} k(\alpha_s/\alpha_p)^{1/2} A_p(0) \text{cn}[\Gamma_1 \left( \frac{1-e^{-\delta z}}{\delta} \right), k^2] \]

\(3-3-2c\) \[ A_i(z) = e^{-\delta z} (\alpha_i/\alpha_s)^{1/2} A_s(0) \text{sn}[\Gamma_1 \left( \frac{1-e^{-\delta z}}{\delta} \right), k^2] \]

\[ = e^{-\delta z} (\alpha_i/\alpha_p)^{1/2} k A_p(0) \text{sn}[\Gamma_1 \left( \frac{1-e^{-\delta z}}{\delta} \right), k^2] \]

where \( k^2 \) is given by Eq. (2-3-4) and \( \Gamma_1 \) by Eq. (2-3-5).

We find that we must write three separate equations to express conversion efficiencies with loss in the system. Thus,
where is given by Eq. (2-3-10) and \( \Gamma_i \) by Eq. (2-3-11).

Second harmonic generation can be treated as a special example of Case 1 where \( A_s^2(0)/\alpha_s = A_p^2(0)/\alpha_p \) and \( \omega_i = 2\omega_p \). For \( \omega_i = 2\omega_p \), Eq. (3-1-1) can be written as

\[
(3-3-4a) \quad \frac{dA_p(z)}{dz} + \delta A_p(z) = -\alpha_p A_p(z) A_\perp(z)
\]

\[
(3-3-4b) \quad \frac{dA_\perp(z)}{dz} + \delta A_\perp(z) = \alpha_p A_p^2(z)
\]

The solutions to Eq. (3-3-4) are:

\[
(3-3-5a) \quad A_p(z) = e^{-\delta z} A_p(0) \text{sech}(\Gamma \frac{1-e^{-\delta z}}{\delta})
\]

\[
(3-3-5b) \quad A_\perp(z) = e^{-\delta z} \left( \frac{\alpha_\perp}{\alpha_p} \right)^{1/2} A_p(0) \text{tanh}(\Gamma \frac{1-e^{-\delta z}}{\delta})
\]

where \( \Gamma \) is given by Eq. (2-4-4).

Two separate equations must be written to express conversion efficiencies with loss in the system. We have

\[
(3-3-6a) \quad E_p = \frac{N_p(0) - N_p(L)}{N_p(0)} = 1 - e^{-2\delta L} \text{sech}^2 \left( \Gamma \frac{1-e^{-\delta L}}{\delta} \right)
\]

\[
(3-3-6b) \quad E_\perp = \frac{N_\perp(L)}{N_p(0)} = e^{-2\delta L} \text{tanh}^2 \left( \Gamma \frac{1-e^{-\delta L}}{\delta} \right)
\]
where \( \Gamma \) is given by Eq. (2-4-8).

As an example of the effect of energy loss in a system on converting input pump energy to second harmonic output energy, in Fig. 10 we have plotted conversion efficiency, \( E_1 = e^{-2\delta L} \frac{\tanh^2[\Gamma(1-e^{-\delta L})]}{\tanh \left[ \frac{\Delta \omega}{\delta} \right]} \), vs. pump excitation, \( \Gamma L = \left[ \frac{\hbar \omega_p^2 \omega_i^2 d^2 \mu_0}{2 n_p^2 n_i^2 \varepsilon_0} \right] N_p(0) \frac{1}{L} \)

(for \( L = 0.01 \) meter), at fixed values of the loss coefficient, \( \delta \). The curves are asymptotic to a value of \( E_1 = e^{-2\delta L} \). The quantity, \( \frac{1-e^{-\delta L}}{\delta} \), in the argument of the tanh function effectively acts as a reduced length over which energy transfer can take place.

Case 2. \( \frac{\alpha_2(0)}{\alpha_s} > \frac{\alpha_2(0)}{\alpha_p} \).

The solutions to Eq. (3-3-1) are:

\[
(3-3-7a) \quad A_p(z) = e^{-\delta z} A_p(0) \cn [\Gamma_2 \left( \frac{1-e^{-\delta z}}{\delta} \right), k^{-2}]
\]

\[
(3-3-7b) \quad A_s(z) = e^{-\delta z} A_s(0) \dn [\Gamma_2 \left( \frac{1-e^{-\delta z}}{\delta} \right), k^{-2}]
\]

\[
(3-3-7c) \quad A_1(z) = e^{-\delta z} \left( \frac{\alpha_1}{\alpha_p} \right)^{\frac{1}{2}} A_p(0) \sn [\Gamma_2 \left( \frac{1-e^{-\delta z}}{\delta} \right), k^{-2}]
\]

where \( k^{-2} \) is given by Eq. (2-3-16) and \( \Gamma_2 \) by Eq. (2-3-17).

Three separate equations must be written to express conversion efficiencies with loss in the system. The equations are:

\[
(3-3-8a) \quad E_p = \frac{N_p(0)-N_p(L)}{N_p(0)} = 1-e^{-2\delta L} \frac{1-e^{-\delta L}}{\delta} \cn^2 [\Gamma_2 \left( \frac{1-e^{-\delta L}}{\delta} \right), k^{-2}]
\]
Fig. 10. Photon Conversion Efficiency vs. Pump Excitation at Fixed Values of the Loss Coefficient, $\delta$, for Second Harmonic Generation, $\omega_1 = 2\omega_p$, with $L = 0.01$ meter.
where \( k^2 \) is given by Eq. (2-3-22) and \( \Gamma_2 \) by Eq. (2-3-23).

The effect of system loss for FTW up conversion is much the same as for FTW down conversion. A damping term, \( e^{-\delta z} \), is added to each amplitude solution equation and the argument of each of the elliptic functions is changed from \( \Gamma z \) for the lossless case to \( \Gamma \left( \frac{1-e^{-\delta z}}{\delta} \right) \) for the lossy case.
3.4 BTW Down Conversion

System 3. BTW at both $\omega_p$ and $\omega_s$.

Eq. (3-1-1) can be written as

$$\frac{dA_p(z)}{dz} - \delta A_p(z) = \alpha_A p(z)A_1(z)$$ (3-4-la)

$$\frac{dA_s(z)}{dz} - \delta A_s(z) = -\alpha_A p(z)A_1(z)$$ (3-4-lb)

$$\frac{dA_1(z)}{dz} + \delta A_1(z) = \alpha_A p(z)A_s(z)$$ (3-4-1c)

Let

$$B_p(z) = e^{-\delta z}A_p(z)$$ (3-4-2a)

$$B_s(z) = e^{-\delta z}A_s(z)$$ (3-4-2b)

$$B_1(z) = e^{\delta z}A_1(z)$$ (3-4-2c)

If we differentiate Eq. (3-4-2) with respect to $z$, we obtain

$$\frac{dB_p(z)}{dz} = e^{-\delta z}[dA_p(z)/dz - \delta A_p(z)]$$ (3-4-3a)

$$\frac{dB_s(z)}{dz} = e^{-\delta z}[dA_s(z)/dz - \delta A_s(z)]$$ (3-4-3b)

$$\frac{dB_1(z)}{dz} = e^{\delta z}[dA_1(z)/dz + \delta A_1(z)]$$ (3-4-3c)

Now let

$$y_1 = \frac{1-e^{-\delta z}}{\delta} \text{ so that } dy_1/dz = e^{-\delta z}$$ (3-4-4a)
\[(3-4-4b) \quad y_2 = \frac{e^{3\delta z} - 1}{3\delta} \quad \text{so that} \quad dy_2/dz = e^{3\delta z}\]

Then

\[(3-4-5a) \quad dB_p(z)/dy_1 = (dB_p/dz)(dz/dy_1) = dA_p(z)/dz - \delta A_p(z)\]

\[(3-4-5b) \quad dB_s(z)/dy_1 = (dB_s/dz)(dz/dy_1) = dA_s(z)/dz - \delta A_s(z)\]

\[(3-4-5c) \quad dB_i(z)/dy_2 = (dB_i/dz)(dz/dy_2) = e^{-2\delta z}[dA_i(z)/dz + \delta A_i(z)].\]

If we substitute Eq. (3-4-2) and (3-4-5) into Eq. (3-4-1), we obtain

\[(3-4-6a) \quad dB_p(z)/dy_1 = \alpha_p B_s(z)B_i(z)\]

\[(3-4-6b) \quad dB_s(z)/dy_1 = -\alpha_s B_p(z)B_i(z)\]

\[(3-4-6c) \quad dB_i(z)/dy_2 = \alpha_i B_p(z)B_s(z)\]

Unfortunately, analytic solutions to Eq. (3-4-6) cannot be obtained. Therefore, we are unable to obtain solutions for \(A_p(z)\), \(A_s(z)\), and \(A_i(z)\) which reflect the effect of losses in the system and to express the conversion efficiencies \(E_p\), \(E_s\), and \(E_i\) as was done for the FTW systems.

**System 4. BTW at \(\omega_s\); FTW at \(\omega_p\)**

Eq. (3-1-1) can be written as
Analytic solutions to Eq. (3-4-7) cannot be obtained.

As an example of the effect of loss in a BTW system, in Fig. 11 we have plotted the solutions to Eq. (3-4-7) for BTW down conversion (System 4) using the same parameter values chosen to plot Fig. 9 except for the values of \( N_s(0) \) and hence \( A_s(0) \). For this example, we let \( N_s(0) = 2 \times 10^{29} \) and \( A_s(0) = 4.016 \times 10^6 \) volts/meter. The solid curves are for \( \delta = 0 \) (lossless case) and the dashed curves are for \( \delta = 5 \text{ meters}^{-1} \) (lossy case). The curves extend over several wave periods so that the effects of the loss coefficient \( \delta \) can more readily be seen.
Fig. 11. Pump, Signal, and Idler Wave Amplitudes for BTW Down Conversion (System 4) with $\delta = 0$ (solid curves) and $\delta = 5$ meters$^{-1}$ (dashed curves).
3.5 BTW Up Conversion

System 5. BTW at $\omega_s$; FTW at $\omega_p$.

Eq. (3-1-1) can be written as

\[
(3-5-1a) \quad \frac{dA_p(z)}{dz} + \delta A_p(z) = -\alpha s' A_p(z)A_i(z)
\]

\[
(3-5-1b) \quad \frac{dA_s(z)}{dz} - \delta A_s(z) = \alpha s' A_p(z)A_i(z)
\]

\[
(3-5-1c) \quad \frac{dA_i(z)}{dz} + \delta A_i(z) = \alpha s' A_p(z)A_s(z)
\]

Analytic solutions to Eq. (3-5-1) cannot be obtained.

System 6. FTW at $\omega_s$; BTW at $\omega_p$.

Eq. (3-1-1) can be written as

\[
(3-5-2a) \quad \frac{dA_p(z)}{dz} - \delta A_p(z) = \alpha s' A_p(z)A_i(z)
\]

\[
(3-5-2b) \quad \frac{dA_s(z)}{dz} + \delta A_s(z) = -\alpha s' A_p(z)A_i(z)
\]

\[
(3-5-2c) \quad \frac{dA_i(z)}{dz} + \delta A_i(z) = \alpha i' A_p(z)A_s(z)
\]

Analytic solutions to Eq. (3-5-2) cannot be obtained.
4.1 Introduction

In Chapter II, we determined the effects of pump depletion in single FTW and BTW down conversion and up conversion three-wave processes that can occur as a result of the second order nonlinear polarization of the medium. In this chapter, we will examine the effects of pump depletion in several cases where we have two three-wave processes occurring simultaneously. This happening was first observed by Andrews, Rabin, and Tang\textsuperscript{21} in 1970 when they coupled parametric down conversion and up conversion with simultaneous phase matching (collinear as well as noncollinear) in ammonium dihydrogen phosphate (ADP).

As was the case for single three-wave parametric processes in lossless media, we assume that the phase matching conditions are satisfied. The phase matching conditions for simultaneous down conversion and up conversion are:

\begin{align}
\omega_{il} &= \omega_p - \omega_s \\
\hat{k}_{il} &= \hat{k}_p - \hat{k}_s
\end{align}

and
This coupling of two simultaneous three-wave processes can best be visualized by means of an energy level diagram as shown in Fig. 12.

(4-1-2a) \[ \omega_{12} = \omega_p + \omega_s \]

(4-1-2b) \[ \vec{k}_{12} = \vec{k}_p + \vec{k}_s. \]

Because simultaneous down conversion and up conversion involves four waves, we now have four coupled equations to solve. Furthermore, since in general simultaneous three-wave processes are not collinearly phase matched, the nonlinear coefficients will include cosine terms relating the \( \vec{k} \) vectors with the z-axis. For simplicity, we will assume collinear phase matching in our analysis.
4.2 Simultaneous FTW Down Conversion and Up Conversion

The four coupled amplitude equations which must be solved are:

\[(4-2-1a) \quad \frac{dA_p(z)}{dz} = -\alpha_p A_p(z) [A_{11}(z) + A_{12}(z)]\]

\[(4-2-1b) \quad \frac{dA_s(z)}{dz} = \alpha_s A_p(z) [A_{11}(z) - A_{12}(z)]\]

\[(4-2-1c) \quad \frac{dA_{11}(z)}{dz} = \alpha_{11} A_p(z) A_s(z)\]

\[(4-2-1d) \quad \frac{dA_{12}(z)}{dz} = \alpha_{12} A_p(z) A_s(z)\]

Both the pump and signal waves are supplied by external sources, and the two idler waves, initially zero at \(z = 0\), are generated internally.

If we solve for \(A_p(z)\), \(A_s(z)\), and \(A_{12}(z)\) in terms of \(A_{11}(z)\), we obtain

\[(4-2-2a) \quad A_p(z) = \left[ A_p^2(0) - \frac{\alpha_p}{\alpha_{11}} \left( \frac{\alpha_{12}}{\alpha_{11}} + 1 \right) A_{11}^2(z) \right]^{1/2}\]

\[(4-2-2b) \quad A_s(z) = \left[ A_s^2(0) - \frac{\alpha_s}{\alpha_{11}} \left( \frac{\alpha_{12}}{\alpha_{11}} - 1 \right) A_{11}^2(z) \right]^{1/2}\]

\[(4-2-2c) \quad A_{12}(z) = \left( \frac{\alpha_{12}}{\alpha_{11}} \right) A_{11}(z)\]

where we have assumed that \(\alpha_{12} > \alpha_{11}\) since \(\omega_{12} > \omega_{11}\) as seen in Fig. 12. The relationship between \(\alpha\) and \(\omega\) is given by Eq. (2-2-9).

If we substitute Eq. (4-2-2a) and (4-2-2b) into Eq. (4-2-1c), we obtain
\[ (4-2-3) \quad \frac{dA_{11}(z)}{dz} = \alpha_{11} \left\{ \left[ A_p^2(0) - \frac{\alpha_p}{\alpha_{11}} \left( \frac{\alpha_{12}}{\alpha_{11}} + 1 \right) A_{11}^2(z) \right] \right\}^{1/2} \]

It follows that

\[ (4-2-4) \quad A_{11}(z) \]

\[ \int_0^{A_{11}(z)} \frac{dA_{11}(z)}{dz} \left\{ \left[ A_p^2(0) - \frac{\alpha_p}{\alpha_{11}} \left( \frac{\alpha_{12}}{\alpha_{11}} + 1 \right) A_{11}^2(z) \right] \right\}^{1/2} = \alpha_{11} \int_0^z dz \]

Eq. (4-2-4) can be transformed into the standard Legendre form of the elliptic integral and integrated to obtain an analytic solution for \( A_{11}(z) \). The solution for \( A_{11}(z) \) can then be substituted into Eq. (4-2-2) to obtain \( A_p(z), A_s(z), \) and \( A_{12}(z) \).

Two sets of solution equations are possible depending upon the relative magnitudes of the input signal and pump waves.

Case 1. \( A_s^2(0)/\alpha_s \leq \frac{\alpha_{12} - \alpha_{11}}{\alpha_{12} + \alpha_{11}} \frac{A_p^2(0)}{\alpha_p} \).

The solution equations to Eq. (4-2-1) are:

\[ (4-2-5a) \quad A_p(z) = A_p(0) \, \text{dn}(\Gamma_1 z, k^2) \]

\[ (4-2-5b) \quad A_s(z) = A_s(0) \, \text{cn}(\Gamma_1 z, k^2) \]

\[ (4-2-5c) \quad A_{11}(z) = \left( \frac{\alpha_{11}}{\alpha_s} \right)^{1/2} \left( \frac{\alpha_{11}}{\alpha_{12} - \alpha_{11}} \right)^{1/2} A_s(0) \, \text{sn}(\Gamma_1 z, k^2) \]

\[ (4-2-5d) \quad A_{12}(z) = \left( \frac{\alpha_{12}}{\alpha_s} \right)^{1/2} \left( \frac{\alpha_{12}}{\alpha_{12} - \alpha_{11}} \right)^{1/2} A_s(0) \, \text{sn}(\Gamma_1 z, k^2) \]
where

\[ k^2 = \left( \frac{\alpha_{12} + \alpha_{11}}{\alpha_{12} - \alpha_{11}} \right) \frac{A_s^2(0)/\alpha_s}{A_p^2(0)/\alpha_p}, \quad 0 \leq k^2 \leq 1, \]

and

\[ \Gamma_1 = \left[ \frac{\alpha_s}{\alpha_{12} - \alpha_{11}} \right]^{1/2} A_p(0). \]

In terms of power, Eq. (4-2-5) can be written as

\[ P_p(z) = P_p(0) \, dn^2(\Gamma_1 z) \]  
\[ P_s(z) = P_s(0) \, cn^2(\Gamma_1 z) \]  
\[ P_{i1}(z) = \frac{\omega_{11}}{\omega_s} \left( \frac{n_{11} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) P_s(0) \, sn^2(\Gamma_1 z) \]  
\[ P_{i2}(z) = \frac{\omega_{12}}{\omega_s} \left( \frac{n_{11} \omega_{12}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) P_s(0) \, sn^2(\Gamma_1 z) \]

where

\[ k^2 = \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) \frac{P_s(0)/\omega_s}{P_p(0)/\omega_p}, \quad 0 \leq k^2 \leq 1, \]

and

\[ \Gamma_1 = \left[ \frac{\omega_s(n_{11} \omega_{12} - n_{12} \omega_{11})d^2}{2n_{11}n_{12}} \left( \frac{\mu_0}{\epsilon_0} \right)^{3/2} S_p(0) \right]^{1/2}. \]

In terms of photon flux, Eq. (4-2-8) becomes

\[ N_p(z) = N_p(0) \, dn^2(\Gamma_1 z) \]  
\[ N_s(z) = N_s(0) \, cn^2(\Gamma_1 z) \]
(4-2-11c) \[ N_{11}(z) = \left( \frac{n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) N_s(0) \text{ sn}^2(\Gamma_1 z) \]

(4-2-11d) \[ N_{12}(z) = \left( \frac{n_{11} \omega_{12}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) N_s(0) \text{ sn}^2(\Gamma_1 z) \]

where

(4-2-12) \[ k^2 = \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) \frac{N_s(0)}{N_p(0)}, \quad 0 \leq k^2 \leq 1, \]

and

(4-2-13) \[ \Gamma_1 = \left[ \frac{\pi \omega_p (n_{11} \omega_{12} - n_{12} \omega_{11})}{2 \pi n_p n_s n_{11} n_{12}} \left( \frac{\mu_0}{\varepsilon_0} \right) \frac{N_s(0)}{N_p(0)} \right]^{1/2} \]

If we differentiate Eq. (4-2-11) with respect to z, we obtain the Manley-Rowe relations in terms of the rate of change of photon flux. Thus,

(4-2-14a) \[ \frac{d}{dz} [N_{11}(z) + N_{12}(z)] = - \frac{dN_p(z)}{dz} \]

\[ = - \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) \frac{dN_s(z)}{dz} \]

(4-2-14b) \[ = 2\Gamma_1 N_s(0) \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) \text{ sn}(\Gamma_1 z) \text{ cn}(\Gamma_1 z) \text{ dn}(\Gamma_1 z). \]

The conversion efficiency is

(4-2-15a) \[ E = \frac{P_p(0) - P_p(L)}{P_p(0)} = \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) \frac{[P_s(0) - P_s(L)]/s}{P_p(0)/w_p} \]

\[ = \frac{P_{11}(L)/\omega_{11} + P_{12}(L)/\omega_{12}}{P_p(0)/w_p}. \]
\[
(4-2-15b) \quad \frac{N_p(0) - N_p(L)}{N_p(0)} = \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) \left( \frac{N_s(0) - N_s(L)}{N_p(0)} \right)
\]

\[
= \frac{N_{11}(L) + N_{12}(L)}{N_p(0)}
\]

\[
(4-2-15c) \quad = k^2 \text{ sn}^2(\Gamma_1 L), \quad \Gamma_1 L \leq k.
\]

Case 2. \( A_s^2(0)/\alpha_s > \left( \frac{\alpha_{12} - \alpha_{11}}{\alpha_{12} + \alpha_{11}} \right) \frac{A_p^2(0)/\alpha_p}{\alpha_s}. \)

The solutions to Eq. (4-2-1) are:

\[
(4-2-16a) \quad A_p(z) = A_p(0) \text{ cn}(\Gamma_2 z, k^{-2})
\]

\[
(4-2-16b) \quad A_s(z) = A_s(0) \text{ dn}(\Gamma_2 z, k^{-2})
\]

\[
(4-2-16c) \quad A_{11}(z) = \left( \frac{\alpha_{11}}{\alpha_p} \right)^{1/2} \left( \frac{\alpha_{11}}{\alpha_{12} + \alpha_{11}} \right)^{1/2} A_p(0) \text{ sn}(\Gamma_2 z, k^{-2})
\]

\[
(4-2-16d) \quad A_{12}(z) = \left( \frac{\alpha_{12}}{\alpha_p} \right)^{1/2} \left( \frac{\alpha_{12}}{\alpha_{12} + \alpha_{11}} \right)^{1/2} A_p(0) \text{ sn}(\Gamma_2 z, k^{-2})
\]

where

\[
(4-2-17) \quad k^{-2} = \left( \frac{\alpha_{12} - \alpha_{11}}{\alpha_{12} + \alpha_{11}} \right) \frac{A_p^2(0)/\alpha_p}{A_s^2(0)/\alpha_s}, \quad 0 \leq k^2 \leq 1
\]

and

\[
(4-2-18) \quad \Gamma_2 = \left[ \alpha_p \left( \frac{\alpha_{12} + \alpha_{11}}{\alpha_{12} + \alpha_{11}} \right) \right]^{1/2} A_s(0).
\]

From Eq. (4-2-5) and (4-2-16), we see that we can have complete pump depletion only when

\[
\frac{A_s^2(0)}{\alpha_s} > \left( \frac{\alpha_{12} - \alpha_{11}}{\alpha_{12} + \alpha_{11}} \right) \frac{A_p^2(0)}{\alpha_p}.
\]
Since \( \text{cn}(\Gamma z) = 0 \) when \( \Gamma z = K \), the interaction length for complete pump depletion is \( L = K / \Gamma_z = K / [\alpha_p (\alpha_{12} + \alpha_{11})]^{1/2} A_s(0) \).

In terms of power, Eq. (4-2-16) can be written as

\[
\begin{align*}
(4-2-19a) \quad P_p(z) &= P_p(0) \text{cn}^2(\Gamma_z, k^{-2}) \\
(4-2-19b) \quad P_s(z) &= P_s(0) \text{dn}^2(\Gamma_z, k^{-2}) \\
(4-2-19c) \quad P_{11}(z) &= \left(\frac{\omega_{11}}{\omega_p}\right) \left(\frac{n_{12} \omega_{11}}{n_{11} \omega_{12} + n_{12} \omega_{11}}\right) P_p(0) \text{ sn}^2(\Gamma_z, k^{-2}) \\
(4-2-19d) \quad P_{12}(z) &= \left(\frac{\omega_{12}}{\omega_p}\right) \left(\frac{n_{11} \omega_{12}}{n_{11} \omega_{12} + n_{12} \omega_{11}}\right) P_p(0) \text{ sn}^2(\Gamma_z, k^{-2})
\end{align*}
\]

where

\[
(4-2-20) \quad k^{-2} = \left(\frac{n_{12} \omega_{12} - n_{12} \omega_{11}}{n_{11} \omega_{12} + n_{12} \omega_{11}}\right) \frac{P_p(0)/\omega_p}{P_s(0)/\omega_s}, \quad 0 \leq k^{-2} \leq 1,
\]

and

\[
(4-2-21) \quad \Gamma_z = \left[\frac{2 \omega_p (n_{11} \omega_{12} + n_{12} \omega_{11})}{n_{11} \omega_{12} + n_{12} \omega_{11}} \frac{d^2}{n_{11} \omega_{12} + n_{12} \omega_{11}} \left(\frac{\mu_0}{\varepsilon_0}\right) \text{ sn}^2(\Gamma_z, k^{-2}) \left[\frac{s_0(0)}{s_0}\right]^{3/2}\right]^{1/2}
\]

In terms of photon flux, Eq. (4-2-19) can be written as

\[
\begin{align*}
(4-2-22a) \quad N_p(z) &= N_p(0) \text{cn}^2(\Gamma_z, k^{-2}) \\
(4-2-22b) \quad N_s(z) &= N_s(0) \text{dn}^2(\Gamma_z, k^{-2}) \\
(4-2-22c) \quad N_{11}(z) &= \left(\frac{n_{12} \omega_{11}}{n_{11} \omega_{12} + n_{12} \omega_{11}}\right) N_p(0) \text{ sn}^2(\Gamma_z, k^{-2}) \\
(4-2-22d) \quad N_{12}(z) &= \left(\frac{n_{11} \omega_{12}}{n_{11} \omega_{12} + n_{12} \omega_{11}}\right) N_p(0) \text{ sn}^2(\Gamma_z, k^{-2})
\end{align*}
\]
where

\[
\begin{align*}
(4-2-23) & \quad k^{-2} = \left( \frac{n_{i1} \omega_{12} - n_{i2} \omega_{11}}{n_{i1} \omega_{12} + n_{i2} \omega_{11}} \right) \left( \frac{N_p(0)}{N_s(0)} \right), \quad 0 \leq k^{-2} \leq 1, \\
(4-2-24) & \quad \Gamma_2 = \left( \frac{\hbar \omega_p (n_{i1} \omega_{12} + n_{i2} \omega_{11})}{2} \right)^{\frac{3}{2}} \left( \frac{\mu_0}{\varepsilon_0} \right) \left( \frac{N_p(0)}{N_s(0)} \right) ^{\frac{1}{2}}.
\end{align*}
\]

The Manley-Rowe Relations become

\[
(4-2-25a) \quad \frac{d}{dz} [N_{11}(z) + N_{12}(z)] = - \left( \frac{\mu_0}{\varepsilon_0} \right) \left( \frac{N_p(0)}{N_s(0)} \right) \frac{dN_s(z)}{dz}
\]

\[
(4-2-25b) \quad = 2\Gamma_2 N_p(0) \, \text{sn}(\Gamma_2 z, k^{-2}) \, \text{cn}(\Gamma_2 z, k^{-2}) \, \text{dn}(\Gamma_2 z, k^{-2})
\]

and the conversion efficiency is

\[
(4-2-26) \quad E = \text{sn}^2(\Gamma_2 L, k^{-2}), \quad \Gamma_2 L \leq K.
\]

Let us now compare the simultaneous FTW down conversion and up conversion process with single three-wave interactions. We see that the solution equations for this multiple three-wave process are similar to those for FTW up conversion but with noticeable differences. Whereas \( k^2 = \frac{N_s(0)}{N_p(0)} \) for FTW up conversion, the elliptic function modulus is now modified by a constant which depends on the frequencies of the two generated waves and their refractive indices, i.e.,

\[
k^2 = \left( \frac{n_{i1} \omega_{12} + n_{i2} \omega_{11}}{n_{i1} \omega_{12} - n_{i2} \omega_{11}} \right) \left( \frac{N_s(0)}{N_p(0)} \right)
\]
Also, the factor $\Gamma$ in the pump excitation is modified to reflect the interaction of the competing processes.

Whereas total pump depletion is possible when $N_s(0) > N_p(0)$ for FTW up conversion, the condition now becomes

$$N_s(0) > \left( \frac{n_{12}(\omega) - n_{12}(\omega_1)}{n_{11}(\omega) - n_{12}(\omega_1)} \right) N_p(0)$$

for this multiple process. We see that as $\omega_{12} \rightarrow \omega_{11}$, we have

$$\lim_{\omega_{12} \rightarrow \omega_{11}} \frac{N_s(0)}{N_p(0)} = 0.$$  

On the other hand, for $\omega_{12} \gg \omega_{11}$, we have

$$\lim_{\omega_{12} \gg \omega_{11}} \frac{N_s(0)}{N_p(0)} = 1.$$  

Thus, the relative frequencies of the two generated waves can drastically change the condition for possible total pump depletion.

We will provide two examples which demonstrate the above conditions: (1) $\omega_{12} \rightarrow \omega_{11}$ and (2) $\omega_{12} \gg \omega_{11}$.

Example 1. $\omega_{12} \rightarrow \omega_{11}$.

Assume $\omega_p = 2.716 \times 10^{15}$ sec.$^{-1}$ ($\lambda_p = 0.694\mu m$) and $\omega_s = 1.778 \times 10^{14}$ sec.$^{-1}$ ($\lambda_s = 10.6\mu m$). Then, since $\omega_{12} = \omega_p + \omega_s$ and $\omega_{11} = \omega_p - \omega_s$, we have $\omega_{12} = 2.894 \times 10^{15}$ sec.$^{-1}$ ($\lambda_{12} = 0.651\mu m$), and $\omega_{11} = 2.538 \times 10^{15}$ sec.$^{-1}$ ($\lambda_{11} = 0.743\mu m$). If we use the refractive indices for proustite ($Ag_3AsS_3$) which transmits over these wavelengths, we have $n_{12} \approx 2.997$ and $n_{11} \approx 2.915$. Thus, we find that

$$k^2 = 19.3 \frac{N_s(0)}{N_p(0)}$$

and the condition for total pump depletion becomes

$$N_s(0) > 0.0517 N_p(0).$$
In Fig. 13, we have plotted conversion efficiency, $E$, vs. pump excitation at fixed ratios of input signal photon flux to input pump photon flux, $N_s(0)/N_p(0)$. For $N_s(0) \leq 0.0517 N_p(0)$,

$$E = k^2 s n^2 (\Gamma_1, k^2)$$ and pump excitation is

$$\Gamma_1 = \left[ \frac{\hbar \omega_n (n_{11} \omega_{12} - n_{12} \omega_{11})}{2 n_p n_s n_{11} n_{12}} \right]^{3/2} \frac{\mu_0}{\epsilon_0} N_p(0)^{1/2} L.$$ For $N_s(0) > 0.0517 N_p(0)$,

$$E = s n^2 (\Gamma_2, k^2)$$ and pump excitation is

$$k^{-1} \Gamma_2 = \left[ \frac{\hbar \omega_n (n_{11} \omega_{12} - n_{12} \omega_{11})}{2 n_p n_s n_{11} n_{12}} \right]^{3/2} \frac{\mu_0}{\epsilon_0} N_p(0)^{1/2} L.$$ We see that Fig. 13 is similar to Fig. 4 for FTW up conversion. The principal difference in the two figures is the ratio of $N_s(0)/N_p(0)$ required before complete pump depletion is possible and the definition of pump depletion. The simultaneous FTW down conversion and up conversion interaction process permits a lower ratio of $N_s(0)/N_p(0)$ for total pump depletion.

**Example 2.** $\omega_{12} \gg \omega_{11}$.

Assume $\omega_{12} = 2.716 \times 10^{15}$ sec.$^{-1}$ ($\lambda_{12} = 0.694 \mu m$) and $\omega_{11} = 1.778 \times 10^{14}$ sec.$^{-1}$ ($\lambda_{11} = 10.6 \mu m$). Since $\omega_{12} = \omega_p + \omega_s$ and $\omega_{11} = \omega_p - \omega_s$, we obtain $\omega_p = 1.447 \times 10^{15}$ sec.$^{-1}$ ($\lambda_p = 1.303 \mu m$) and $\omega_s = 1.269 \times 10^{15}$ sec.$^{-1}$ ($\lambda_s = 1.485 \mu m$). The refractive indices for proustite at these frequencies are $n_{12} = 2.955$ and $n_{11} = 2.703$. Thus, we find that $k^2 = 1.15 \frac{N_s(0)}{N_p(0)}$ and the condition for total pump depletion becomes $N_s(0) > 0.866 N_p(0)$. For a given ratio of
Fig. 13. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for Simultaneous FTW Down Conversion and Up Conversion where $\omega_{12} \rightarrow \omega_{11}$ (Example 1).
\( \frac{N_s(0)}{N_p(0)} \), conversion efficiencies for this example would be less than for Example 1 for the same pump excitation.
4.3 Simultaneous BTW Down Conversion and Up Conversion

In this case, we choose to have a BTW signal wave at \( \omega_s \) with input at \( z = L \), a FTW pump at \( \omega_p \) with input at \( z = 0 \), and two forward traveling idler waves, initially zero at \( z = 0 \), generated internally.

The four coupled amplitude equations which must be solved are:

\[
\begin{align*}
(4-3-1a) & \quad \frac{dA_p(z)}{dz} = -\alpha_p A_s(z) [A_{11}(z) + A_{12}(z)] \\
(4-3-1b) & \quad \frac{dA_s(z)}{dz} = -\alpha_s A_p(z) [A_{11}(z) - A_{12}(z)] \\
(4-3-1c) & \quad \frac{dA_{11}(z)}{dz} = \alpha_{11} A_p(z) A_s(z) \\
(4-3-1d) & \quad \frac{dA_{12}(z)}{dz} = \alpha_{12} A_p(z) A_s(z)
\end{align*}
\]

If we solve for \( A_p(z), A_s(z), \) and \( A_{12}(z) \) in terms of \( A_{11}(z) \), we obtain

\[
\begin{align*}
(4-3-2a) & \quad A_p(z) = [A_p^2(0) - \frac{\alpha_p}{\alpha_{11}} \left( \frac{\alpha_{12}}{\alpha_{11}} + 1 \right) A_{11}^2(z)]^{1/2} \\
(4-3-2b) & \quad A_s(z) = [A_s^2(0) + \frac{\alpha_s}{\alpha_{11}} \left( \frac{\alpha_{12}}{\alpha_{11}} - 1 \right) A_{11}^2(z)]^{1/2} \\
(4-3-2c) & \quad A_{12}(z) = \left( \frac{\alpha_{12}}{\alpha_{11}} \right) A_{11}(z)
\end{align*}
\]

If we substitute Eq. (4-3-2a) and (4-3-2b) into Eq. (4-3-1c), we obtain

\[
\begin{align*}
(4-3-3) & \quad \frac{dA_{11}(z)}{dz} = \alpha_{11} \left\{ \left[ A_p^2(0) - \frac{\alpha_p}{\alpha_{11}} \left( \frac{\alpha_{12}}{\alpha_{11}} + 1 \right) A_{11}^2(z) \right] + \frac{\alpha_s}{\alpha_{11}} \left( \frac{\alpha_{12}}{\alpha_{11}} - 1 \right) A_{11}^2(z) \right\}^{1/2}
\end{align*}
\]
Eq. (4-3-4) can be transformed into the standard Legendre form of the elliptic integral and integrated to obtain an analytic solution for $A_{11}(z)$. The solution for $A_{11}(z)$ can then be substituted into Eq. (4-3-2) to obtain $A_p(z)$, $A_s(z)$, and $A_{12}(z)$.

The solutions to Eq. (4-3-1) are:

$$A_p(z) = \frac{\alpha_p}{\alpha_{11}} \left( \frac{\alpha_{12}}{\alpha_{12} + \alpha_{A_1}} \right) A_{11}(z)$$

$$A_s(z) = \left[ A_s^2(0) + \left( \frac{\alpha_s}{\alpha_p} \left( \frac{\alpha_{12} - \alpha_{11}}{\alpha_{12} + \alpha_{11}} \right) A_p^2(0) \right)^{1/2} \right] dn(K-z, k^2)$$

$$A_{11}(z) = \frac{\alpha_{11}}{\alpha_p} \left( \frac{\alpha_{11}}{\alpha_{12} + \alpha_{11}} \right)^{1/2} A_p(0) \csc(K-z, k^2)$$

$$A_{12}(z) = \left( \frac{\alpha_{12}}{\alpha_p} \right)^{1/2} \left( \frac{\alpha_{12}}{\alpha_{12} + \alpha_{11}} \right)^{1/2} A_p(0) \csc(K-z, k^2)$$

where

$$k^2 = \frac{(\alpha_{12} - \alpha_{11})A_p^2(0)/\alpha_p}{(\alpha_{12} + \alpha_{11})A_s^2(0)/\alpha_s + (\alpha_{12} - \alpha_{11})A_p^2(0)/\alpha_p}, \quad 0 \leq k^2 \leq 1,$$

and

$$\Gamma = [\alpha_p (\alpha_{12} + \alpha_{11}) A_s^2(0) + \alpha_s (\alpha_{12} - \alpha_{11}) A_p^2(0)]^{1/2}.$$

If we examine Eq. (4-3-5), (4-3-6), and (4-3-7), we note that we do not know $A_s(0)$, the boundary condition at $z = 0$. Hence, we
must use an iterative trial and error method to find the correct value of $A_s(0)$ that will give us our known boundary condition $A_s(L)$ at $z = L$.

Since $sn(K - \Gamma z) = 0$ when $K - \Gamma z = 0$, we see from Eq. (4-3-5) that we can have complete pump depletion when $\Gamma z = K$.

In terms of power, Eq. (4-3-5) can be written as

\[ (4-3-8a) \quad P_p(z) = P_p(0) \cdot sn^2(K - \Gamma z) \]

\[ (4-3-8b) \quad P_s(z) = \left[ P_s(0) + \frac{\omega_s}{\omega_p} \left( \frac{n_{12}^2 - n_{12}^2 w_{11}}{n_{11}^2 + n_{12}^2 w_{11}} \right) P_p(0) \right] \cdot cn^2(K - \Gamma z) \]

\[ (4-3-8c) \quad P_{11}(z) = \frac{\omega_{11}}{\omega_p} \left( \frac{n_{12}^2 - n_{12}^2 w_{11}}{n_{11}^2 + n_{12}^2 w_{11}} \right) P_p(0) \cdot cn^2(K - \Gamma z) \]

\[ (4-3-8d) \quad P_{12}(z) = \frac{\omega_{12}}{\omega_p} \left( \frac{n_{11}^2 - n_{12}^2 w_{11}}{n_{11}^2 + n_{12}^2 w_{11}} \right) P_p(0) \cdot cn^2(K - \Gamma z) \]

where

\[ (4-3-9) \]

\[ k^2 = \frac{(n_{12}^2 - n_{12}^2 w_{11}) P_p(0)/\omega_p}{(n_{11}^2 + n_{12}^2 w_{11}) P_s(0)/\omega_s + (n_{12}^2 - n_{12}^2 w_{11}) P_p(0)/\omega_p}, \quad 0 \leq k^2 \leq 1, \]

and

\[ (4-3-10) \]

\[ \Gamma = \left( \frac{d^2 \left( \frac{\mu_0}{\varepsilon_0} \right)^{3/2}}{2 n_p n_s n_{11} n_{12}} \right) \left[ (n_{12}^2 + n_{12}^2 w_{11}) P_s(0) + (n_{11}^2 + n_{12}^2 w_{11}) P_p(0) \right] \]

In terms of photon flux, Eq. (4-3-8) can be written as

\[ (4-3-11a) \quad N_p(z) = N_p(0) \cdot sn^2(K - \Gamma z) \]
\[ N_s(z) = \left[ N_s(0) + \left( \frac{n_{11}^{\omega_1} \omega_{12} - n_{12}^{\omega_1} \omega_{11}}{n_{11}^{\omega_1} \omega_{12} + n_{12}^{\omega_1} \omega_{11}} \right) N_p(0) \right] \text{dn}^2(K-Gz) \]

\[ N_{11}(z) = \left( \frac{n_{12}^{\omega_1}}{n_{11}^{\omega_1} \omega_{12} + n_{12}^{\omega_1} \omega_{11}} \right) N_p(0) \text{cn}^2(K-Gz) \]

\[ N_{12}(z) = \left( \frac{n_{11}^{\omega_1} \omega_{12}}{n_{11}^{\omega_1} \omega_{12} + n_{12}^{\omega_1} \omega_{11}} \right) N_p(0) \text{cn}^2(K-Gz) \]

where

\[ k^2 = \frac{(n_{11}^{\omega_1} \omega_{12} - n_{12}^{\omega_1} \omega_{11}) N_p(0)}{(n_{11}^{\omega_1} \omega_{12} + n_{12}^{\omega_1} \omega_{11}) N_s(0) + (n_{11}^{\omega_1} \omega_{12} - n_{12}^{\omega_1} \omega_{11}) N_p(0)}, \quad 0 \leq k^2 \leq 1, \]

and

\[ \Gamma = \left\{ \begin{array}{l} \frac{\kappa \omega_p \omega_s}{2n_p n_s n_{11} n_{12}} \left( \frac{\mu_0}{\varepsilon_0} \right) \left[ (n_{11}^{\omega_1} \omega_{12} + n_{12}^{\omega_1} \omega_{11}) N_s(0) + (n_{11}^{\omega_1} \omega_{12} - n_{12}^{\omega_1} \omega_{11}) N_p(0) \right] \end{array} \right\}^{1/2} \]

If we differentiate Eq. (4-3-11) with respect to \( z \), we obtain the Manley-Rowe relations in terms of the rate of change of photon flux. Thus,

\[ \frac{d}{dz} (N_{11}(z) + N_{12}(z)) = -\frac{dN_s(z)}{dz} \]

\[ = \left( \frac{n_{11}^{\omega_1} \omega_{12} + n_{12}^{\omega_1} \omega_{11}}{n_{11}^{\omega_1} \omega_{12} - n_{12}^{\omega_1} \omega_{11}} \right) \frac{dN_s(z)}{dz} \]

\[ = 2\Gamma N_p(0) \text{sn}(K-Gz) \text{cn}(K-Gz) \text{dn}(K-Gz) \]

The conversion efficiency is

\[ E = \frac{P_p(0) - P_p(L)}{P_p(0)} = \left( \frac{n_{11}^{\omega_1} \omega_{12} + n_{12}^{\omega_1} \omega_{11}}{n_{11}^{\omega_1} \omega_{12} - n_{12}^{\omega_1} \omega_{11}} \right) \frac{[P_s(L) - P_s(0)]/\omega_s}{P_p(0)/\omega_p} \]

\[ = \frac{P_{11}(L)/\omega_{11} + P_{12}(L)/\omega_{12}}{P_p(0)/\omega_p} \]
If we compare the process of simultaneous BTW down conversion and up conversion with single three-wave interactions, we see that the solution equations for this multiple three-wave process are similar to those for FTW down conversion but with noticeable differences.

Whereas \( k^2 = \frac{N_p(0)}{N_s(0) + N_p(0)} \) for FTW down conversion, the modulus becomes

\[
k^2 = \frac{(n_{11} \omega_2 - n_{12} \omega_{11})N_p(0)}{(n_{11} \omega_2 + n_{12} \omega_{11})N_s(0) + (n_{11} \omega_2 - n_{12} \omega_{11})N_p(0)}
\]

for simultaneous BTW down and up conversion. The factor \( \Gamma \) in the pump excitation is now

\[
\Gamma = \left\{ \frac{\hbar \omega_b d^2}{2n_s n_{11} n_{12}} \left( \frac{\mu_0}{\epsilon_0} \right)^{3/2} \left[ (n_{11} \omega_2 + n_{12} \omega_{11})N_s(0) + (n_{11} \omega_2 - n_{12} \omega_{11})N_p(0) \right] \right\}^{1/2}
\]

Since \( k^2 \) now depends on the frequencies of the two generated waves and their refractive indices, we see that as \( \omega_2 \rightarrow \omega_{11} \), we have

\[
\lim_{\omega_2 \rightarrow \omega_{11}} k^2 = 0.
\]

For \( \omega_2 \gg \omega_{11} \), we have

\[
\lim_{\omega_2 \gg \omega_{11}} k^2 = \frac{N_p(0)}{N_s(0) + N_p(0)}.
\]

Thus, the relative frequencies of the two generated waves can drastically affect the value of \( k^2 \) and the condition for total pump depletion.
As in the case of simultaneous FTW down conversion and up conversion, we will provide two examples which demonstrate the above conditions; namely, (1) $\omega_{i2} \rightarrow \omega_{i1}$ and (2) $\omega_{i2} \gg \omega_{i1}$. We will use the same assumed values for $\omega_p$, $\omega_s$, $\omega_{i1}$, $\omega_{i2}$, $n_{i1}$, and $n_{i2}$ as we used in the previous FTW examples.

**Example 1.** $\omega_{i2} \rightarrow \omega_{i1}$.

We find that $k^2 = \frac{0.0517 \, N_p(0)}{N_s(0) + 0.0517 \, N_p(0)}$. In Fig. 14, we have plotted conversion efficiency, $E = cn^2 \left( k - \Gamma_L, k^2 \right)$, vs. pump excitation,

$$k \Gamma L = \left[ \frac{n_p \omega_s \left( n_i \omega_{i1} - n_{i2} \omega_{i1} \right)}{2 \, n_p \, n_s \, n_{i1} \, n_{i2}} \right] \left( \frac{\mu_0}{\epsilon_0} \right)^{\frac{1}{2}} \frac{1}{N_p(0)} \frac{1}{L},$$

at fixed ratios of input signal photon flux to input pump photon flux, $N_s(L)/N_p(0)$, for this value of $k^2$. We see that Fig. 14 is similar to Fig. 7 for BTW up conversion (System 5). The principal difference in the two figures is the ratio of $N_s(L)/N_p(0)$ required before total pump depletion is possible and the definition of pump excitation. The simultaneous BTW down and up conversion interaction process allows a lower ratio of $N_s(L)/N_p(0)$ for total pump depletion. For this example, $N_s(L) > 0.0517 \, N_p(0)$ is the condition for complete energy transfer.

**Example 2.** $\omega_{i2} \gg \omega_{i1}$.

In this example, we find that $k^2 = \frac{0.866 \, N_p(0)}{N_s(0) + 0.866 \, N_p(0)}$.

A plot of $E$ vs. pump excitation at fixed ratios of $N_s(L)/N_p(0)$ for this example would show that the condition for total pump depletion is
\[ N_s(L) > 0.866 N_p(0). \] For a given ratio of \( N_s(L)/N_p(0) \), conversion efficiencies for this example would be less than for Example 1 for the same pump excitation.
Fig. 14. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for Simultaneous BTW Down Conversion and Up Conversion where $\omega_{12} \rightarrow \omega_{11}$ (Example 1).
5.1 Introduction

In previous chapters, we examined in detail the effects of pump depletion in three-wave processes that can occur as a result of the second order nonlinear polarization of the medium. In this chapter, we will examine the effects of pump depletion in four-wave processes that can occur due to the effects of a third order nonlinear polarization of the medium.

We assume that interactions involving four traveling waves satisfy one or the other of two sets of phase matching conditions:

\[ \omega_1 = \omega_2 + \omega_3 + \omega_4 \]  
(5-1-1a)

\[ \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 \]  
(5-1-1b)

or

\[ \omega_1 + \omega_2 = \omega_3 + \omega_4 \]  
(5-1-2a)

\[ \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 \]  
(5-1-2b)

Since it is very difficult to satisfy the above conditions with four separate waves, we will limit our discussion and analysis to a number of special cases. In Section 5.2, we will examine third
harmonic generation which is a process most easily achieved in practice. In Sections 5.3 through 5.6 we will analyze second harmonically pumped FTW and BTW down conversion and up conversion processes since they are analogous to three-wave processes. In Sections 5.7 and 5.8, simultaneous second harmonically pumped FTW and BTW down and up conversion processes will be examined. In Section 5.9, we will treat FTW up conversion involving four separate waves.

In our analysis, we will again assume that we have plane collinear traveling waves in a lossless medium. In the general case involving four separate waves, four coupled equations of the form

\[
\frac{dA_k(z)}{dz} = \pm \beta_{k\ell\mu\nu} A_\ell(z)A_\mu(z)A_\nu(z) \quad k = 1, 2, 3, 4
\]

\[
\ell = 2, 3, 4, 1
\]

\[
\mu = 3, 4, 1, 2
\]

\[
\nu = 4, 1, 2, 3
\]

must be solved. We show in Appendix B that only the amplitude equations are required in the analysis. In the above equation, \( \beta \) is the nonlinear coupling coefficient (volts\(^{-2}\) meters).
5.2 Third Harmonic Generation

Third harmonic generation is a special case of FTW up conversion in which Eq. (5-1-1) is satisfied with but a single wave input—that of the pump. To meet this condition, \( \omega_2, \omega_3, \) and \( \omega_4 \) must equal the pump frequency. Then \( \omega_1 \), the generated sum frequency, is

\[
\omega_1 = \omega_1 = 3\omega_p.
\]

With \( \omega_2 = \omega_3 = \omega_4 = \omega_p \), the amplitude equations (5-1-3) reduce to

\[
\begin{align*}
\frac{dA_p(z)}{dz} &= -\beta_p A_p^2(z)A_1(z) \\
\frac{dA_1(z)}{dz} &= \beta_p A_p^3(z).
\end{align*}
\]

If we assume that \( A_1(z) \) is zero at \( z = 0 \), the solutions to Eq. (5-2-2) are:

\[
\begin{align*}
A_p(z) &= \frac{A_p(0)}{(1 + \beta_p A_p^4(0)z^2)^{\frac{1}{2}}} \\
A_1(z) &= \frac{\beta_p A_p^3(0)z}{(1 + \beta_p A_p^4(0)z^2)^{\frac{1}{2}}}.
\end{align*}
\]

In terms of power, Eq. (5-2-3) can be written as

\[
\begin{align*}
P_p(z) &= P_p(0) \left( 1 - \frac{\Gamma^2 z^2}{1 + \Gamma^2 z^2} \right) \\
P_1(z) &= \frac{\omega_1}{\omega_p} P_p(0) \left( \frac{\Gamma^2 z^2}{1 + \Gamma^2 z^2} \right)
\end{align*}
\]
where

\( \Gamma^2 = \frac{\omega_P \omega_1 c^2}{n_p^3 n_1} \left( \frac{\mu_0}{\varepsilon_0} \right)^2 \mathcal{S}_p (0) \). (5-2-5)

In Eq. (5-2-4) and (5-2-5), we have used the relation

\( \beta = \frac{\omega c}{2n} \left( \frac{\mu_0}{\varepsilon_0} \right)^{1/2} \) (5-2-6)

where \( c \) is the effective third order nonlinear coefficient (amp–sec–meters/volts\(^3\)) described in Appendix B.

In terms of photon flux, Eq. (5-2-4) becomes

\[ N_p(z) = N_p(0) \left( 1 - \frac{\Gamma^2 z^2}{1 + \Gamma^2 z^2} \right) \] (5-2-7a)

\[ N_1(z) = N_p(0) \left( \frac{\Gamma^2 z^2}{1 + \Gamma^2 z^2} \right) \] (5-2-7b)

where

\[ \Gamma^2 = \frac{\hbar^2 \omega_1^3 c^2}{n_p^3 n_1} \left( \frac{\mu_0}{\varepsilon_0} \right)^2 N_p^2 (0). \] (5-2-8)

If we differentiate Eq. (5-2-7) with respect to \( z \), we obtain the Manley-Rowe relations in terms of the rate of change of photon flux. We have

\[ \frac{dN_1(z)}{dz} = - \frac{dN_p(z)}{dz} \] (5-2-9a)

\[ = \frac{2\Gamma^2 N_p(0) z}{(1 + \Gamma^2 z^2)^2} \] (5-2-9b)

The conversion efficiency over the interaction length \( z = L \) is:
In Fig. 15, we have plotted conversion efficiency, $E = \frac{\Gamma^2 L^2}{1 + \Gamma^2 L^2}$, vs. pump excitation for third harmonic generation where we have defined pump excitation as the product $\Gamma L = \frac{\hbar^2}{n_p} \frac{\omega_0}{c} \frac{\omega_i}{\varepsilon_0} \left[ \frac{\mu_0}{\varepsilon_0} \right]^2 N_p^2(0) \text{L}$. We see that the curve is asymptotic to $E = 1.0$ so that the pump can never be totally depleted.

As an example of third harmonic generation, if we assume that we have a pump wave at a wavelength of $10.6\,\mu\text{m} (\omega_p = 1.78 \times 10^{14}\,\text{sec}^{-1})$, we will generate a sum frequency wave at $3.53\,\mu\text{m} (\omega_i = 5.34 \times 10^{14}\,\text{sec}^{-1})$. If we assume that $c$ is $10^{-5}$ as large as the value of $d = 4.425 \times 10^{-22}$, $L = 0.01\,\text{meter}$, $n_i = n_p = 2.5$, and $N_p(0) = 10^{30}$ (which corresponds to a pump intensity of $1.88 \times 10^{10}\,\text{watts/meter}^2$), we find that we have a pump excitation of $\Gamma L = 5.82$. From Fig. 15, we see that the conversion efficiency is 0.971.
Fig. 15. Photon Conversion Efficiency vs. Pump Excitation for Third Harmonic Generation, $\omega_i = 3\omega_p$. 
5.3 Second Harmonically Pumped FTW Down Conversion (System 1)

The second harmonically pumped down conversion process is a special case of a four-wave interaction which satisfies Eq. (5-1-2). To meet this condition, \( \omega_1 \) and \( \omega_2 \) are set equal to the pump frequency, \( \omega_3 \) equal to the signal frequency, and \( \omega_4 \) equal to the generated wave frequency. We then have

\[
2 \omega_p = \omega_s + \omega_i
\]

With \( \omega_1 = \omega_2 = \omega_p \), the amplitude equations (5-1-3) reduce to

\[
\frac{dA_p(z)}{dz} = -\beta_p A_p(z)A_s(z)A_i(z)
\]

\[
\frac{dA_s(z)}{dz} = \beta_s A_p^2(z)A_i(z)
\]

\[
\frac{dA_i(z)}{dz} = \beta_i A_p^2(z)A_s(z)
\]

Both the pump and signal waves are supplied by external sources and the idler wave, initially zero at \( z = 0 \), is generated internally.

If we solve for \( A_p(z) \) and \( A_s(z) \) in terms of \( A_i(z) \), we obtain

\[
A_p(z) = \left[ A_p^2(0) - \frac{\beta_p}{\beta_i} A_i^2(z) \right]^{1/2}
\]

\[
A_s(z) = \left[ A_s^2(0) + \frac{\beta_s}{\beta_i} A_i^2(z) \right]^{1/2}
\]

If we substitute Eq. (5-3-3a) and (5-3-3b) into Eq. (5-3-2c), we obtain

\[
\frac{dA_i(z)}{dz} = \beta_i \left[ A_p^2(0) - \frac{\beta_p}{\beta_i} A_i^2(z) \right] \left[ A_s^2(0) + \frac{\beta_s}{\beta_i} A_i^2(z) \right]^{1/2}
\]
It follows that

\[ \frac{A_1(z)}{\frac{A_p^2(0)}{\beta_p} - \frac{A_s^2(0)}{\beta_s} + \frac{B_s A_s^2(z)}{\beta_s}} \int_0^z dz = \frac{A_1(z)}{A_s(0)} \]

If we let \( y = (\beta_s / \beta_i)^{1/2} \frac{A_1(z)}{A_s(0)} \), we obtain

\[ \frac{1}{\left(\beta_i \beta_s \right)^{1/2} A_p^2(0)} \int_0^y \frac{dy}{\left(1 - \frac{A_p^2(z)}{\beta_s A_p^2(0)} \right)^{1/2}} = z \]

If we now let \( \sinh \theta = y \), we obtain

\[ \frac{1}{\left(\beta_i \beta_s \right)^{1/2} A_p^2(0)} \int_0^{\sinh^{-1} y} \frac{d\theta}{\left(1 - \frac{A_p^2(0)}{\sinh^2 \theta} \right)} = z \]

Eq. (5-3-7) can be integrated. If \( A_1^2(z)/\beta_i < A_p^2(0)/\beta_p \), we obtain

\[ \Gamma^{-1} \tanh^{-1} \left\{ k^{-1} \tanh \left[ \sinh^{-1} \left( \frac{\beta_i}{\beta_s} \right)^{1/2} \frac{A_1(z)}{A_s(0)} \right] \right\} = z \]

where

\[ k^2 = \frac{A_p^2(0)/\beta_p}{A_s^2(0)/\beta_s + A_p^2(0)/\beta_p} \]

and

\[ \Gamma = \left[ \beta_i \beta_p A_s^2(0) + \beta_i \beta_s A_p^2(0) \right]^{1/2} A_s(0). \]
We can solve Eq. (5-3-8) for $A^z_i$. The solution for $A^z_i$ is then substituted into Eq. (5-3-3a) and (5-3-3b) to obtain solutions for $A^z_p(z)$ and $A^z_s(z)$. The three solutions are:

(5-3-11a) \[ A^z_p(z) = A^z_p(0) \left\{ 1 - \frac{\beta^2 A^2_s(0)}{\beta^2 A^2_p(0)} \sinh^2 \left[ \tanh^{-1} \left( k \tanh (\Gamma z) \right) \right] \right\}^{\frac{1}{2}} \]

(5-3-11b) \[ A^z_s(z) = A^z_s(0) \cosh \left[ \tanh^{-1} \left( k \tanh (\Gamma z) \right) \right] \]

(5-3-11c) \[ A^z_i(z) = \left( \frac{\beta^2_i}{\beta^2_s} \right)^{\frac{1}{2}} A^z_s(0) \sinh \left[ \tanh^{-1} \left( k \tanh (\Gamma z) \right) \right] \]

In terms of power, Eq. (5-3-11) becomes

(5-3-12a) \[ P^z_p(z) = P^z_p(0) \left\{ 1 - \frac{\omega_i P^2_s(0)}{\omega_s P^2_p(0)} \sinh^2 \left[ \tanh^{-1} \left( k \tanh (\Gamma z) \right) \right] \right\} \]

(5-3-12b) \[ P^z_s(z) = P^z_s(0) \cosh^2 \left[ \tanh^{-1} \left( k \tanh (\Gamma z) \right) \right] \]

(5-3-12c) \[ P^z_i(z) = \left( \frac{\omega_i}{\omega_s} \right) P^z_s(0) \sinh^2 \left[ \tanh^{-1} \left( k \tanh (\Gamma z) \right) \right] \]

where

(5-3-13) \[ k^2 = \frac{P^z_p(0)/\omega_p}{P^z_s(0)/\omega_s + P^z_p(0)/\omega_p} \]

and

(5-3-14) \[ \Gamma = \left\{ \frac{c^2}{n^2p_n n^2_s n^2_i} \left( \frac{\mu_0}{\epsilon_0} \right)^2 \left[ \omega_i \omega_p S^z_p(0) + \omega_i \omega_s S^z_p(0) \right] S^z_p(0) \right\}^{\frac{1}{2}}. \]

In terms of photon flux, Eq. (5-3-12) becomes

(5-3-15a) \[ N^z_p(z) = N^z_p(0) \left\{ 1 - \frac{N^z_s(0)}{N^z_p(0)} \sinh^2 \left[ \tanh^{-1} \left( k \tanh (\Gamma z) \right) \right] \right\} \]
(5-3-15b) \[ N_s(z) = N_s(0) \cosh^2 \{ \tanh^{-1} [k \tanh(\Gamma z)] \} \]

(5-3-15c) \[ N_1(z) = N_s(0) \sinh^2 \{ \tanh^{-1} [k \tanh(\Gamma z)] \} \]

where

(5-3-16) \[ k^2 = \frac{N_p(0)}{N_s(0) + N_p(0)} \]

and

(5-3-17) \[ \Gamma = \left\{ \frac{\hbar^2}{2} \frac{\omega_p^2}{\omega_s^2} \frac{c^2}{n_p n_s n_s} \left( \frac{\mu_0}{\epsilon_0} \right)^2 \frac{N_s(0) + N_p(0)}{N_p(0)} \right\}^{\frac{1}{2}} \]

If we differentiate Eq. (5-3-15) with respect to \( z \), we obtain the Manley-Rowe relations in terms of the rate of change of photon flux. Thus,

(5-3-18a) \[ \frac{dN_s(z)}{dz} = \frac{dN_1(z)}{dz} = - \frac{dN_p(z)}{dz} \]

(5-3-18b) \[ = 2k \Gamma N_s(0) \sinh \{ \tanh^{-1} [k \tanh(\Gamma z)] \} \cosh \{ \tanh^{-1} [k \tanh(\Gamma z)] \} \frac{\text{sech}^2(\Gamma z)}{1 - k^2 \tanh^2(\Gamma z)} \]

The conversion efficiency over the interaction length \( z = L \) is

(5-3-19a) \[ E = \frac{P_p(0) - P_p(L)}{P_p(0)} = \frac{P_1(L) - \omega_i}{\omega_p} = \frac{[P_s(L) - P_s(0)]/\omega}{P_p(0)/\omega_p} \]

(5-3-19b) \[ = \frac{N_p(0) - N_p(L)}{N_p(0)} = \frac{N_1(L) - N_1(0)}{N_p(0)} = \frac{N_s(L) - N_s(0)}{N_p(0)} \]
\[ (5-3-19c) \quad \frac{N_s(0)}{N_p(0)} \sinh^2 \{ \tanh^{-1} [k \tanh(\Gamma L)] \}. \]

In Fig. 16, we have plotted conversion efficiency, \( E = \frac{N_s(0)}{N_p(0)} \), \( \sinh^2 \{ \tanh^{-1} [k \tanh(\Gamma L)] \} \), vs. pump excitation, \( k \Gamma L = \frac{n_p^2 \omega^2 \omega_s c^2}{n_s n_i n_s} \left( \frac{\mu_0}{\varepsilon_0} \right)^2 N_p^2(0) \) \( \frac{1}{2} \) \( L \), for fixed ratios of input signal photon flux to input pump photon flux, \( \frac{N_s(0)}{N_p(0)} \). We see that the curves in Fig. 16 are similar to those in Fig. 2 for three-wave FTW down conversion. However, for all ratios of \( \frac{N_s(0)}{N_p(0)} \) in second harmonically pumped FTW down conversion, the curves are asymptotic to \( E = 1.0 \) so that total pump depletion is not possible.

If we compare Fig. 16 with Fig. 2, we see that we need greater pump excitation to achieve the same conversion efficiency for the same ratio of \( \frac{N_s(0)}{N_p(0)} \) in second harmonically pumped FTW down conversion. Thus, the three-wave process is more efficient in transferring energy from the pump wave to the signal and idler waves than this four-wave process.
Fig. 16. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for Second Harmonically Pumped FTW Down Conversion, $2\omega_p = \omega_s + \omega_1$ (System 1).
5.4 Second Harmonically Pumped FTW Up Conversion (System 2)

To have second harmonically pumped up conversion, we must satisfy the equation

\[ 2\omega_p + \omega_s = \omega_1. \]

The amplitude equations (5-1-3) reduce to

\[ \frac{dA_p(z)}{dz} = -\beta_p A_p(z) A_s(z) A_i(z) \]  \hspace{1cm} (5-4-2a)

\[ \frac{dA_s(z)}{dz} = -\beta_s A_p^2(z) A_i(z) \]  \hspace{1cm} (5-4-2b)

\[ \frac{dA_i(z)}{dz} = \beta_i A_p^2(z) A_s(z) \]  \hspace{1cm} (5-4-2c)

Both the pump and signal waves are supplied by external sources and the sum frequency wave, initially zero at \( z = 0 \), is generated internally.

If we solve for \( A_p(z) \) and \( A_s(z) \) in terms of \( A_i(z) \), we obtain

\[ A_p(z) = [A_p^2(0) - \frac{\beta_p}{\beta_i} A_i^2(z)]^{1/2} \]  \hspace{1cm} (5-4-3a)

\[ A_s(z) = [A_s^2(0) - \frac{\beta_s}{\beta_i} A_i^2(z)]^{1/2}. \]  \hspace{1cm} (5-4-3b)

If we substitute Eq. (5-4-3a) and (5-4-3b) into Eq. (5-4-2c), we obtain

\[ \frac{dA_i(z)}{dz} = \beta_i [A_p^2(0) - \frac{\beta_p}{\beta_i} A_i^2(z)] [A_s^2(0) - \frac{\beta_s}{\beta_i} A_i^2(z)]^{1/2}. \]  \hspace{1cm} (5-4-4)

It follows that
If we let $y = \left(\frac{\beta_s}{\beta_1}\right)^{\frac{1}{2}} \frac{A_1(0)}{A_s(0)}$, we obtain

$$
(5-4-6) \quad \left[\left(\frac{\beta_s}{\beta_1}\right)^{\frac{1}{2}} A_p^2(0)\right]^{-1} \int_0^y \frac{dy}{(1 - \frac{\beta_p}{\beta_s} \frac{A_s^2(0)}{A_p^2(0)} y^2)(1-y^2)^{\frac{1}{2}}} = z
$$

If we now let $\sin \theta = y$, we obtain

$$
(5-4-7) \quad \left[\left(\frac{\beta_s}{\beta_1}\right)^{\frac{1}{2}} A_p^2(0)\right]^{-1} \int_0^{\sin^{-1}y} \frac{d\theta}{1 - \frac{\beta_p}{\beta_s} \frac{A_s^2(0)}{A_p^2(0)} \sin^2 \theta} = z
$$

Eq. (5-4-7) can be integrated. Two solution sets are possible depending upon the relative magnitudes of the input signal and pump waves.

Case 1. $A_s^2(0)/\beta_s < A_p^2(0)/\beta_p$.

For this case, we obtain

$$
(5-4-8) \quad T_1^{-1} \tan^{-1} \{k_1^{-1} \tan \left[\sin^{-1} \left(\frac{\beta_s}{\beta_1}\right)^{\frac{1}{2}} \frac{A_1(z)}{A_s(0)}\right]\} = z
$$

where

$$
(5-4-9) \quad k_1^2 = \frac{A_p^2(0)/\beta_p}{A_p^2(0)/\beta_p - A_s^2(0)/\beta_s}
$$
and

\[ (5-4-10) \quad \Gamma_1 = \left[ \beta_1 \beta_s A_p^2(0) - \beta_1 \beta_s A_s^2(0) \right]^{1/2} A_p(0). \]

We can solve Eq. (5-4-8) for \( A_1(z) \). The solution for \( A_1(z) \) is then substituted into Eq. (5-4-3a) and (5-4-3b) to obtain solutions for \( A_p(z) \) and \( A_s(z) \). The three solution equations are:

\[ (5-4-11a) \quad A_p(z) = A_p(0) \left\{ 1 - \frac{\beta_p A_s^2(0)}{\beta_s A_p^2(0)} \sin^2 \left[ \tan^{-1} \{ k_1 \tan(\Gamma_1 z) \} \right] \right\}^{1/2} \]

\[ (5-4-11b) \quad A_s(z) = A_s(0) \cos \left\{ \tan^{-1} \left[ k_1 \tan(\Gamma_1 z) \right] \right\} \]

\[ (5-4-11c) \quad A_1(z) = (\beta_1/\beta_s)^{1/2} A_s(0) \sin \left\{ \tan^{-1} \left[ k_1 \tan(\Gamma_1 z) \right] \right\} \]

In terms of power, Eq. (5-4-11) can be written as

\[ (5-4-12a) \quad P_p(z) = P_p(0) \left\{ 1 - \frac{\omega_p P_s(0)}{\omega_s P_p(0)} \sin^2 \left[ \tan^{-1} \{ k_1 \tan(\Gamma_1 z) \} \right] \right\} \]

\[ (5-4-12b) \quad P_s(z) = P_s(0) \cos^2 \left\{ \tan^{-1} \left[ k_1 \tan(\Gamma_1 z) \right] \right\} \]

\[ (5-4-12c) \quad P_1(z) = (\omega_1/\omega_s) P_s(0) \sin^2 \left\{ \tan^{-1} \left[ k_1 \tan(\Gamma_1 z) \right] \right\} \]

where

\[ (5-4-13) \quad k_1^2 = \frac{P_p(0)/\omega_p}{P_p(0)/\omega_p - P_s(0)/\omega_s} \]

and

\[ (5-4-14) \quad \Gamma_1 = \left\{ \frac{c_n^2}{n_p n_s n_i} \frac{\omega_p^0}{\varepsilon_0} \left[ \omega_1 \omega_s P_p(0) - \omega_1 \omega_p P_s(0) \right] S_p(0) \right\}^{1/2}. \]
In terms of photon flux, Eq. (5-4-12) can be written as

\begin{equation}
N_p(z) = N_p(0) \left\{ 1 - \frac{N_s(0)}{N_p(0)} \sin^2 \left[ \tan^{-1} \left\{ k_1 \tan(\Gamma_1 z) \right\} \right] \right\}
\end{equation}

(5-4-15a)

\begin{equation}
N_s(z) = N_s(0) \cos^2 \left[ \tan^{-1} \left\{ k_1 \tan(\Gamma_1 z) \right\} \right]
\end{equation}

(5-4-15b)

\begin{equation}
N_1(z) = N_s(0) \sin^2 \left[ \tan^{-1} \left\{ k_1 \tan(\Gamma_1 z) \right\} \right]
\end{equation}

(5-4-15c)

where

\begin{equation}
k_1^2 = \frac{N_p(0)}{N_p(0) - N_s(0)}
\end{equation}

(5-4-16)

and

\begin{equation}
\Gamma_1 = \left\{ \frac{\hbar^2 \omega^2 \omega_1 c^2}{N_p N_s} \left( \frac{\epsilon_0}{\epsilon_p} \right)^2 \left[ N_p(0) - N_s(0) \right] N_p(0) \right\}^{1/2}
\end{equation}

(5-4-17)

If we differentiate Eq. (5-4-15) with respect to z, we obtain the Manley-Rowe relations in terms of the rate of change of photon flux. Thus,

\begin{equation}
dN_1(z)/dz = -dN_p(z)/dz = -dN_s(z)/dz
\end{equation}

(5-4-18a)

\begin{equation}
= 2k_1 \Gamma_1 N_s(0) \sin \left\{ \tan^{-1} \left\{ k_1 \tan(\Gamma_1 z) \right\} \right\} \sin \left\{ \tan^{-1} \left\{ k_1 \tan(\Gamma_1 z) \right\} \right\}
\end{equation}

(5-4-18b)

\begin{equation}
\cos \left\{ \tan^{-1} \left\{ k_1 \tan(\Gamma_1 z) \right\} \right\} \left[ \frac{\sec^2(\Gamma_1 z)}{1 + k_1^2 \tan^2(\Gamma_1 z)} \right]
\end{equation}

The conversion efficiency over the interaction length z = L is:

\begin{equation}
E = \frac{P_p(0) - P_p(L)}{P_p(0)} = \frac{P_1(L)/\omega_1}{P_p(0)/\omega_p} = \frac{[P_s(0) - P_s(L)]/\omega_s}{P_p(0)/\omega_p}
\end{equation}

(5-4-19a)
Case 2. $A_s^2(0)/\beta_s > A_p^2(0)/\beta_p^{25}$

For this case, we obtain as a solution to Eq. (5-4-7) the equation

\[
(5-4-20) \quad \Gamma_2^{-1} \tanh^{-1} \{k_2^{-1}\tan \left[ \sin^{-1} \left( \frac{\beta_s}{\beta_p} \right) \frac{1}{2} A_1(z) \right] \} = z
\]

where

\[
(5-4-21) \quad k_2^2 = \frac{A_p^2(0)/\beta_p}{A_s^2(0)/\beta_s - A_p^2(0)/\beta_p}
\]

and

\[
(5-4-22) \quad \Gamma_2 = \left[ \beta_p A_s^2(0) - \beta_s A_p^2(0) \right]^{1/2} A_p(0).
\]

We can solve Eq. (5-4-20) for $A_1(z)$ and then substitute the solution into Eq. (5-4-3a) and (5-4-3b) to obtain solutions for $A_p(z)$ and $A_s(z)$. The three solution equations are:

\[
(5-4-23a) \quad A_p(z) = A_p(0) \left[ 1 - \frac{\beta_p A_s^2(0)}{\beta_s A_p^2(0)} \sin^2 \left[ \tan^{-1} \left( k_2 \tanh(\Gamma_2 z) \right) \right] \right]^{1/2}
\]

\[
(5-4-23b) \quad A_s(z) = A_s(0) \cos \left[ \tan^{-1} \left( k_2 \tanh(\Gamma_2 z) \right) \right]
\]

\[
(5-4-23c) \quad A_1(z) = \left( \frac{\beta_p}{\beta_s} \right)^{1/2} A_s(0) \sin \left[ \tan^{-1} \left( k_2 \tanh(\Gamma_2 z) \right) \right]
\]
In terms of power, Eq. (5-4-23) becomes

\[ P_p(z) = P_p(0) \left\{ 1 - \frac{\omega_s P_s(0)}{\omega_p P_p(0)} \sin^2 \left[ \tan^{-1} \left( k_2 \tanh(\Gamma_2 z) \right) \right] \right\} \]

\[ P_s(z) = P_s(0) \cos^2 \left\{ \tan^{-1} \left( k_2 \tanh(\Gamma_2 z) \right) \right\} \]

\[ P_\perp(z) = P_\perp(0) \sin^2 \left\{ \tan^{-1} \left( k_2 \tanh(\Gamma_2 z) \right) \right\} \]

where

\[ k_2^2 = \frac{P_p(0)/\omega_p}{P_s(0)/\omega_s - P_p(0)/\omega_p} \]

and

\[ \Gamma_2 = \left\{ \frac{c^2}{n_p^2 n_s n_\perp} \left[ \omega_i \omega_p S_s(0) - \omega_i \omega_s S_p(0) \right] S_p(0) \right\}^2 \]

In terms of photon flux, Eq. (5-4-24) becomes

\[ N_p(z) = N_p(0) \left\{ 1 - \frac{N_s(0)}{N_p(0)} \sin^2 \left[ \tan^{-1} \left( k_2 \tanh(\Gamma_2 z) \right) \right] \right\} \]

\[ N_s(z) = N_s(0) \cos^2 \left\{ \tan^{-1} \left( k_2 \tanh(\Gamma_2 z) \right) \right\} \]

\[ N_\perp(z) = N_\perp(0) \sin^2 \left\{ \tan^{-1} \left( k_2 \tanh(\Gamma_2 z) \right) \right\} \]

where

\[ k_2^2 = \frac{N_p(0)}{N_s(0) - N_p(0)} \]

and
The Manley–Rowe relations become

(5-4-30a) \( \frac{dN_i}{dz} = -\frac{dN_p}{dz} = -\frac{dN_s}{dz} \)

(5-4-30b) \( = 2k_2 \Gamma_2 N_s(0) \sin \left\{ \tan^{-1} \left[ k_2 \tanh(\Gamma_2 z) \right] \right\} \cos \left\{ \tan^{-1} \left[ k_2 \tanh(\Gamma_2 z) \right] \right\} \left( \frac{\text{sech}^2(\Gamma_2 z)}{1 + k_2^2 \tan^2(\Gamma_2 z)} \right) \)

The conversion efficiency becomes

(5-4-31) \( E = \frac{N_s(0)}{N_p(0)} \sin^2 \left\{ \tan^{-1} \left[ k_2 \tanh(\Gamma_2 L) \right] \right\} \)

In Fig. 17, we have plotted conversion efficiency, \( E \), vs. pump excitation at fixed ratios of input signal photon flux to input pump photon flux, \( N_s(0)/N_p(0) \). For \( N_s(0) \leq N_p(0) \),

\( E = \frac{N_s(0)}{N_p(0)} \sin^2 \left\{ \tan^{-1} \left[ k_1 \tan(\Gamma_1 L) \right] \right\} \) and pump excitation is

\[ k_1 \Gamma_1 L = \left( \frac{\hbar^2 \omega_p \omega_s c^2}{n_p n_s n_i} \right) \left( \frac{\mu_0}{\varepsilon_0} \right)^2 N_p^2(0) \]

For \( N_s(0) > N_p(0) \),

\( E = \frac{N_s(0)}{N_p(0)} \sin^2 \left\{ \tan^{-1} \left[ k_2 \tanh(\Gamma_2 L) \right] \right\} \) and pump excitation is

\[ k_2 \Gamma_2 L = \left( \frac{\hbar^2 \omega_p \omega_s c^2}{n_p n_s n_i} \right) \left( \frac{\mu_0}{\varepsilon_0} \right)^2 N_p^2(0) \]

We see that the curves in
Fig. 17 are similar to those in Fig. 3 for three-wave FTW up conversion. However, total pump depletion cannot be achieved for any ratio of $N_s(0)/N_p(0)$ in second harmonically pumped FTW up conversion. For $N_s(0) \geq N_p(0)$, the curves are asymptotic to $E = 1.0$. The dot-dash curve is a measure of the maximum pump excitation that can be applied before process reversal occurs. Process reversal can only take place for $N_s(0) < N_p(0)$. The curve for third harmonic generation shown in Fig. 15 appears as the $N_s(0)/N_p(0) = 1$ curve in Fig. 17.

If we compare Fig. 17 with Fig. 3, we see that we need greater pump excitation to achieve the same conversion efficiency for the same ratio of $N_s(0)/N_p(0)$ in second harmonically pumped FTW up conversion. Thus, the three-wave process is more efficient in transferring energy from the pump wave to the generated sum frequency wave than this four-wave process.
Fig. 17. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for Second Harmonically Pumped FTW Up Conversion, $2\omega_p + \omega_s = \omega_i$ (System 2).
5.5 Second Harmonically Pumped BTW Down Conversion

There are two possible systems for obtaining second harmonically pumped BTW down conversion if we arbitrarily choose the internally-generated idler wave to be a FTW with zero amplitude at \( z = 0 \) and assume that \( 2\omega_p > \omega_s > \omega_i \) so that energy is transferred from the pump wave to the other two waves. We can have (1) both pump and signal as backward traveling waves with inputs at \( z = L \), or we can have (2) a FTW pump with input at \( z = 0 \) and a BTW signal with input at \( z = L \). In both systems, we must satisfy the condition that

\[
2\omega_p = \omega_s + \omega_i. \tag{5-5-1}
\]

System 3. BTW at both \( 2\omega_p \) and \( \omega_s \).

The amplitude equations (5-1-3) can be written as

\[
(5-5-2a) \quad \frac{dA_p(z)}{dz} = \beta_p A_p(z)A_s(z)A_i(z)
\]

\[
(5-5-2b) \quad \frac{dA_s(z)}{dz} = -\beta_s A_p^2(z)A_i(z)
\]

\[
(5-5-2c) \quad \frac{dA_i(z)}{dz} = \beta_i A_p^2(z)A_s(z)
\]

Both the signal and pump waves are supplied by an external source at \( z = L \) and the idler wave, initially zero at \( z = 0 \), is generated internally.

If we solve for \( A_p(z) \) and \( A_s(z) \) in terms of \( A_i(z) \), we obtain

\[
A_p(z) = [A_p^2(0) + \frac{\beta_p}{\beta_i} A_i^2(z)]^{1/2} \tag{5-5-3a}
\]
(5-5-3b) \[ A_s(z) = [A_s^2(0) - \frac{\beta_s}{\beta_1} A_1^2(z)]^{1/2} \]

If we substitute Eq. (5-5-3a) and (5-5-3b) into Eq. (5-5-2c), we obtain

(5-5-4) \[ \frac{dA_1(z)}{dz} = \beta_1 [A_p^2(0) + \frac{\beta_p}{\beta_1} A_1^2(z)] [A_s^2(0) - \frac{\beta_s}{\beta_1} A_1^2(z)]^{1/2} \]

It follows that

(5-5-5) \[ \int_0^z \frac{dA_1(z)}{\beta_1 [A_p^2(0) + \frac{\beta_p}{\beta_1} A_1^2(z)] [A_s^2(0) - \frac{\beta_s}{\beta_1} A_1^2(z)]^{1/2}} = \int_0^1 \frac{dA_1(z)}{\beta_1 [A_p^2(0) + \frac{\beta_p}{\beta_1} A_1^2(z)] [A_s^2(0) - \frac{\beta_s}{\beta_1} A_1^2(z)]^{1/2}} \]

If we let \( y = (\beta_s/\beta_1)^{1/2} \frac{A_1(z)}{A_s(0)} \), we obtain

(5-5-6) \[ [(\beta_1\beta_s)^{1/2} A_p^2(0)]^{-1} \int_0^y \frac{dy}{(1 + \frac{\beta_p}{\beta_s} \frac{A_s^2(0)}{A_p^2(0)} y^2)(1 - y^2)} = \int_0^1 \frac{d\theta}{1 + \frac{\beta_p}{\beta_s} \frac{A_s^2(0)}{A_p^2(0)} \sin^2 \theta} \]

If we now let \( \sin \theta = y \), we obtain

(5-5-7) \[ [(\beta_1\beta_s)^{1/2} A_p^2(0)]^{-1} \int_0^{\sin^{-1} y} \frac{d\theta}{1 + \frac{\beta_p}{\beta_s} \frac{A_s^2(0)}{A_p^2(0)} \sin^2 \theta} = \int_0^1 \frac{d\theta}{1 + \frac{\beta_p}{\beta_s} \frac{A_s^2(0)}{A_p^2(0)}} \]

Eq. (5-5-7) can be integrated. If \( A_1^2(z)/\beta_1 < A_p^2(L)/\beta_p \), we obtain
(5-5-8) \[ \Gamma^{-1} \tan^{-1}\{k^{-1}\tan[\sin^{-1}(\beta_s/\beta_p)^{1/2} A_s(z)/A_s(0)]\} = z \]

where

(5-5-9) \[ k^2 = \frac{A_p^2(0)/\beta_p}{A_s^2(L)/\beta_s + A_p^2(L)/\beta_p} \]

and

(5-5-10) \[ \Gamma = [\beta_p A_p^2(L) + \beta_s A_p A_s(L)]^{1/2} A_p(0) \]

We can solve Eq. (5-5-8) for \( A_s(z) \). We then substitute the solution for \( A_s(z) \) into Eq. (5-5-3a) and (5-5-3b) to obtain solutions for \( A_p(z) \) and \( A_s(z) \). The three solution equations are:

(5-5-11a) \[ A_p(z) = A_p(0)\left\{1 + \frac{A_p^2(0)}{A_s^2(0)} \sin^2\{\tan^{-1}[k \tan(\Gamma z)]\}\right\}^{1/2} \]

(5-5-11b) \[ A_s(z) = A_s(0) \cos\{\tan^{-1}[k \tan(\Gamma z)]\} \]

(5-5-11c) \[ A_s(z) = (\beta_1/\beta_s)^{1/2} A_s(0) \sin\{\tan^{-1}[k \tan(\Gamma z)]\} \]

In the above equations, it must be remembered that \( A_p(0) \) and \( A_s(0) \) are unknown boundary conditions.

In terms of power, Eq. (5-5-11) becomes

(5-5-12a) \[ P_p(z) = P_p(0)\left\{1 + \frac{\omega_p}{\omega_s} \frac{P_p(0)}{P_s(0)} \sin^2\{\tan^{-1}[k \tan(\Gamma z)]\}\right\} \]

(5-5-12b) \[ P_s(z) = P_s(0) \cos^2\{\tan^{-1}[k \tan(\Gamma z)]\} \]
(5-5-12c) \[ P_i(z) = \left( \frac{\omega_s}{\omega_s} \right) P_s(0) \sin^2 \left\{ \tan^{-1} [k \tan (\Gamma z)] \right\} \]

where

(5-5-13) \[ k^2 = \frac{P_p(0)/\omega_p}{P_s(L)/\omega_s + P_p(L)/\omega_p} \]

and

(5-5-14) \[ \Gamma = \left\{ \frac{\sqrt{c^2}}{n_p n_s n_i} \left( \frac{\mu_0}{\epsilon_0} \right)^2 \left[ \omega_s \omega_s \omega_s S_s(L) + \omega_s \omega_s \omega_s S_p(L) \right] N_p(0) \right\}^{1/2} \]

In terms of photon flux, Eq. (5-5-12) becomes

(5-5-15a) \[ N_p(z) = N_p(0) \left\{ 1 + \frac{N_s(0)}{N_p(0)} \sin^2 \left\{ \tan^{-1} [k \tan (\Gamma z)] \right\} \right\} \]
(5-5-15b) \[ N_s(z) = N_s(0) \cos^2 \left\{ \tan^{-1} [k \tan (\Gamma z)] \right\} \]
(5-5-15c) \[ N_i(z) = N_s(0) \sin^2 \left\{ \tan^{-1} [k \tan (\Gamma z)] \right\} \]

where

(5-5-16) \[ k^2 = \frac{N_p(0)}{N_s(L) + N_p(L)} \]

and

(5-5-17) \[ \Gamma = \left\{ \frac{\sqrt{\omega_p \omega_s \omega_i c^2}}{n_p n_s n_i} \left( \frac{\mu_0}{\epsilon_0} \right)^2 \left[ N_s(L) + N_p(L) \right] N_p(0) \right\}^{1/2} \]

The Manley-Rowe relations are:

(5-5-18a) \[ \frac{dN_i(z)}{dz} = \frac{dN_p(z)}{dz} = - \frac{dN_s(z)}{dz} \]
(5-5-18b) \[ = 2kTN_s(0) \sin \{\tan^{-1} [k \tan(\Gamma z)]\}. \]

\[ \cos \{\tan^{-1} [k \tan(\Gamma z)]\} \left(\frac{\sec^2(\Gamma z)}{1 + k^2 \tan^2(\Gamma z)}\right). \]

The conversion efficiency is

\[ P_p(L) - P_p(0) = \frac{N_p(0) - N_p(L)}{N_p(L)} \]

\[ P_p(L) = \frac{N_s(0) - N_s(L)}{N_s(L)} \]

\[ \frac{N_s(0) \sin^2 \{\tan^{-1}[k \tan(\Gamma L)]\}}{N_p(0) + N_s(0) \sin^2 \{\tan^{-1}[k \tan(\Gamma L)]\}}. \]

In Fig. 18, we have plotted conversion efficiency,

\[ E = \frac{N_s(0) \sin^2 \{\tan^{-1}[k \tan(\Gamma L)]\}}{N_p(0) + N_s(0) \sin^2 \{\tan^{-1}[k \tan(\Gamma L)]\}}, \text{ vs. pump excitation,} \]

\[ k^{-1} \Gamma L \left[ \frac{N_p(L)}{N_s(L) + N_p(L)} \right] = \frac{\left(\frac{\mu_0}{\varepsilon_0}\right)^2 N_p^2(L)}{n_p^2 n_s n_i} \left(\frac{\omega_p}{\omega_s}\right)^2, \text{ for fixed ratios of input signal photon flux to input pump photon flux,} \]

\[ N_s(L)/N_p(L). \] We see that the curves in Fig. 18 are similar to those in Fig. 5 for three-wave BTW down conversion where both pump and signal are backward traveling waves. All curves in Fig. 18 are asymptotic to \( E = 1.0 \) for all ratios of \( N_s(L)/N_p(L) \). The dot-dash curve represents the condition for system oscillation with no signal wave present. For \( N_s(L)/N_p(L) = 0 \), the threshold for oscillation is a pump excitation of \( \pi/2 \).
If we compare Fig. 18 with Fig. 5, we see that we need a larger pump excitation to achieve the same conversion efficiency for the same ratio of \( N_s(L)/N_p(L) \) in second harmonically pumped BTW down conversion. Thus, the three-wave process is more efficient in transferring energy from the pump wave to the signal and idler waves than this four-wave process.

System 4. BTW at \( \omega_s \); FTW at \( 2\omega_p \).

The amplitude equations (5-1-3) can be written as

\[
\begin{align*}
(5-5-20a) \quad & \frac{dA_p(z)}{dz} = -\beta A_p(z)A_s(z)A_i(z) \\
(5-5-20b) \quad & \frac{dA_s(z)}{dz} = -\beta A_p^2(z)A_i(z) \\
(5-5-20c) \quad & \frac{dA_i(z)}{dz} = \beta A_p^2(z)A_s(z)
\end{align*}
\]

We see that Eq. (5-5-20) is identical to Eq. (5-4-2) for second harmonically pumped FTW up conversion. Therefore, the same solution equations (5-4-3 through 5-4-31) which apply for second harmonically pumped FTW up conversion also apply for this system. It must be remembered that \( A_s(0) \) now refers to the amplitude of the signal wave at the output boundary in the case of a BTW signal in these equations.

In Fig. 19, we have plotted conversion efficiency, \( E \), vs. pump excitation for fixed ratios of input signal photon flux to input pump photon flux, \( N_s(L)/N_p(0) \) for this system. Between the two dot-dash curves, \( N_s(0) \leq N_p(0) \), so that
Fig. 18. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for Second Harmonically Pumped BTW Down Conversion, $2\omega_p = \omega_s + \omega_s$, with BTW at both $\omega_p$ and $\omega_s$ (System 3).
\[ E = \frac{N_s(0)}{N_p(0)} \sin^2 \{\tan^{-1} [k_1 \tan(\Gamma_1 L)]\} \] and pump excitation is

\[ k_1 \Gamma_1 L = \left[ \frac{\hbar^2 \omega_p^2 \omega_s \omega_i c^2}{n_p^2 n_s n_i} \left( \frac{\nu_0}{\epsilon_0} \right)^2 N_p^2(0) \right] \frac{1}{L}. \]

Above the left dot-dash curve, \( N_s(0) > N_p(0) \), so that \( E = \frac{N_s(0)}{N_p(0)} \sin^2 \{\tan^{-1} [k_2 \tanh(\Gamma_2 L)]\} \)

and pump excitation is

\[ k_2 \Gamma_2 L = \left[ \frac{\hbar^2 \omega_p^2 \omega_s \omega_i c^2}{n_p^2 n_s n_i} \left( \frac{\nu_0}{\epsilon_0} \right)^2 N_p^2(0) \right] \frac{1}{L}. \]

We see that the curves in Fig. 19 are similar to those in Fig. 6 for three-wave BTW down conversion where the signal is a backward traveling wave and the pump a forward traveling wave. All the curves in Fig. 19 are asymptotic to \( E = 1.0 \) for all ratios of \( N_s(L)/N_p(0) \) so that total pump depletion is not possible. The dot-dash curve on the right represents the condition for system oscillation with no signal wave present. For \( N_s(L)/N_p(0) = 0 \), the threshold for oscillation is a pump excitation of \( \pi/2 \).

If we compare Fig. 19 with Fig. 6, we find that a larger pump excitation is required to obtain the same conversion efficiency for the same ratio of \( N_s(L)/N_p(0) \) in second harmonically pumped BTW down conversion. Thus, we see that this four-wave process is less efficient than the three-wave process in transferring energy from the pump wave to the signal and idler waves.

If we compare the efficiencies of the three four-wave down conversion systems as depicted in Fig. 16, 18, and 19, we see that this second harmonically pumped BTW down conversion system (System 4)
provides the highest conversion efficiency up to approximately 90 percent for a specified ratio of input signal/pump photon flux and a specified pump excitation. Above 90 percent, the second harmonically pumped FTW down conversion system (System 1) appears to be slightly more efficient even though total pump depletion cannot be realized in any of the three systems.
Fig. 19. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for Second Harmonically Pumped BTW Down Conversion, $2\omega_p = \omega_s + \omega_1$, with a BTW at $\omega_s$ and a FTW at $\omega_p$ (System 4).
5.6 Second Harmonically Pumped BTW Up Conversion

There are two possible systems for obtaining second harmonically pumped BTW up conversion. We can have (1) a FTW pump with input at \( z = 0 \) and a BTW signal with input at \( z = L \), or we can have (2) a FTW signal with input at \( z = 0 \) and a BTW pump with input at \( z = L \). In both systems, we have a sum frequency FTW generated with zero amplitude at \( z = 0 \).

For second harmonically pumped BTW up conversion, we must satisfy the condition that

\[
2\omega_p + \omega_s = \omega_i.
\]  

System 5. BTW at \( \omega_i \); FTW at \( 2\omega_p \).

The amplitude equations (5-1-3) can be written as

\[
(5-6-2a) \quad \frac{dA_p(z)}{dz} = -\beta_p A_p(z)A_s(z)A_i(z)
\]

\[
(5-6-2b) \quad \frac{dA_s(z)}{dz} = \beta_s A_p^2(z)A_i(z)
\]

\[
(5-6-2c) \quad \frac{dA_i(z)}{dz} = \beta_i A_p^2(z)A_s(z)
\]

We see that Eq. (5-6-2) is identical to Eq. (5-3-2) for second harmonically pumped FTW down conversion. Therefore, the same solution equations (5-3-3 through 5-3-19) which apply for second harmonically pumped FTW down conversion also apply for this system. It must be remembered that \( A_s(0) \) now refers to the output amplitude for the case of a BTW signal in these equations.
Fig. 20 shows conversion efficiency,

\[ E = \frac{N_s(0)}{N_p(0)} \sinh^2 \left\{ \tan^{-1} \left[ k \tanh(tL) \right] \right\}, \text{ vs. pump excitation}, \]

\[ kTL = \left[ \frac{\hbar^2 \omega_p^2 \omega_s \omega_i c^2}{n_p^2 n_s n_i} \left( \frac{\mu_0^2}{\varepsilon_0} \right)^2 N_p^2(0) \right]^\frac{1}{2}, \]

for fixed ratios of input signal/pump photon flux, \( N_s(L)/N_p(0) \), for this system. We see that the curves in Fig. 20 are similar to those in Fig. 7 for three-wave BTW up conversion where the signal is a backward traveling wave and the pump a forward traveling wave. However, all the curves in Fig. 20 for \( N_s(L)/N_p(0) \geq 1 \) are asymptotic to \( E = 1.0 \) at a finite value of pump excitation. For ratios of \( N_s(L)/N_p(0) < 1 \), the curves are asymptotic to the value of the conversion efficiency that equals the ratio of \( N_s(L)/N_p(0) \) at infinite pump excitation.

If we compare Fig. 20 with Fig. 7, we see that greater pump excitation is required to achieve a particular conversion efficiency at a given ratio of \( N_s(L)/N_p(0) \) for the second harmonically pumped BTW up conversion system. We must conclude that the three-wave process is the more efficient in transferring energy from the pump wave to the generated sum frequency wave than this four-wave process.
Fig. 20. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for Second Harmonically Pumped BTW Up Conversion, $2\omega_p + \omega_s = \omega_i$, with a BTW at $\omega_s$ and a FTW at $\omega_p$ (System 5).
System 6. FTW at $\omega_s$; BTW at $2\omega_p$.

The amplitude equations (5-1-3) can be written as

\begin{align*}
(5-6-3a) \quad dA_p(z)/dz &= \beta_p A_p(z)A_s(z)A_i(z) \\
(5-6-3b) \quad dA_s(z)/dz &= -\beta_s A_p^2(z)A_i(z) \\
(5-6-3c) \quad dA_i(z)/dz &= \beta_i A_p^2(z)A_s(z)
\end{align*}

Since Eq. (5-6-3) is identical to Eq. (5-5-2) for second harmonically pumped BTW down conversion (System 3), the same solution equations (5-5-3 through 5-5-19) also apply to this system. However, $A_p(0)$ now refers to the amplitude of the BTW pump at the output boundary in these equations.

In order to compare this system with other systems, in Fig. 19 we have plotted conversion efficiency,

$$E = \frac{N_s(0) \sin^2 \{\tan^{-1} [k \tan(\Gamma L)]\}}{N_p(0) + N_s(0) \sin^2 \{\tan^{-1} [k \tan(\Gamma L)]\}}, \text{ vs. pump excitation,}$$

$$k^{-1} \Gamma \left[ \frac{N_p(L)}{N_s(L) - N_p(L)} \right] L = \frac{h^2 \omega_p^2 \omega_i c^2 \left( \frac{\mu_0}{\epsilon_0} \right)^2 N_p^2(L)}{n_p^2 n_s^2 n_i} L, \text{ for fixed}$$

ratios of input signal/pump photon flux, $N_s(0)/N_p(L)$. We see that the curves in Fig. 21 are similar to those in Fig. 8 for three-wave BTW up conversion where the signal is a forward traveling wave and the pump a backward traveling wave. For $N_s(0)/N_p(L) \geq 1$, the curves in Fig. 21 are asymptotic to $E = 1.0$ at an infinite pump excitation. For
\[ \frac{N_s(0)}{N_p(L)} < 1, \] the curves are asymptotic to the value of the conversion efficiency that equals the ratio of \[ \frac{N_s(0)}{N_p(L)}. \] The dot-dash curve represents the maximum pump excitation that can be applied before up conversion reverts to down conversion.

If we compare Fig. 21 with Fig. 8, it is apparent that greater pump excitation is needed to achieve a particular conversion efficiency for a given ratio of \[ \frac{N_s(0)}{N_p(L)} \] for the second harmonically pumped BTW up conversion system. Thus, the three-wave process is more efficient than this four-wave process in transferring energy from the pump wave to the generated sum frequency wave.

If we now compare the efficiencies of the three four-wave up conversion systems as shown in Fig. 17, 20, and 21, we see that the second harmonically pumped FTW up conversion system (System 2) provides the highest conversion efficiency for a given pump excitation and ratio of input signal/pump photon flux in all instances.
Fig. 21. Photon Conversion Efficiency vs. Pump Excitation at Fixed Ratios of Signal/Pump Input Photon Flux for Second Harmonically Pumped BTW Up Conversion, $2\omega_p + \omega_s = \omega_i$, with a FTW at $\omega_s$ and a BTW at $\omega_p$ (System 6).
5.7 Simultaneous Second Harmonically Pumped FTW Down and Up Conversion

In Chapter IV, we examined simultaneous down and up conversion three-wave interactions. In this section and the next, we will follow a similar procedure in treating simultaneous second harmonically pumped down and up conversion four-wave interactions. In this section, we will examine the FTW system.

We assume that the phase matching conditions are satisfied. The phase matching conditions for simultaneous second harmonically pumped down conversion and up conversion are:

\[(5-7-1a) \quad \omega_{i1} = 2\omega_p - \omega_s\]

\[(5-7-1b) \quad k_{i1} = 2k_p - k_s\]

and

\[(5-7-2a) \quad \omega_{i2} = 2\omega_p + \omega_s\]

\[(5-7-2b) \quad k_{i2} = 2k_p + k_s.\]

The energy level diagram for the process is shown in Fig. 22.
In the analysis that follows, we will assume collinear phase matching.

The four coupled amplitude equations which must be solved are:

\[(5-7-3a) \quad \frac{dA_p(z)}{dz} = - \beta_p A_p(z) A_s(z) [A_{11}(z) + A_{12}(z)]\]

\[(5-7-3b) \quad \frac{dA_s(z)}{dz} = \beta_s A_p^2(z) [A_{11}(z) - A_{12}(z)]\]

\[(5-7-3c) \quad \frac{dA_{11}(z)}{dz} = \beta_{11} A_p(z) A_s(z)\]

\[(5-7-3d) \quad \frac{dA_{12}(z)}{dz} = \beta_{12} A_p(z) A_s(z)\]

Both the pump and signal waves are supplied externally and the two idler waves, initially zero at \(z = 0\), are generated internally.

If we solve for \(A_p(z)\), \(A_s(z)\), and \(A_{12}(z)\) in terms of \(A_{11}(z)\), we obtain

\[(5-7-4a) \quad A_p(z) = \left[ A_p^2(0) - \frac{\beta_p}{\beta_{11}} \left( \frac{\beta_{12}}{\beta_{11}} + 1 \right) A_{11}^2(z) \right]^{1/2}\]
(5-7-4b) \[ A_8(z) = \left[ A_s^2(0) - \frac{\beta_s}{\beta_{11}} \left( \frac{\beta_{12}}{\beta_{11}} - 1 \right) A_{11}^2(z) \right]^{1/2} \]

(5-7-4c) \[ A_{12}(z) = (\beta_{12}/\beta_{11}) A_{11}(z) \]

where we have assumed that \( \beta_{12} > \beta_{11} \) since \( \omega_{12} > \omega_{11} \). The relationship between \( \beta \) and \( \omega \) is given by Eq. (5-2-6).

If we substitute Eq. (5-7-4a) and (5-7-4b) into Eq. (5-7-3c), we obtain

(5-7-5) \[ \frac{dA_{11}(z)}{dz} = \beta_{11} \left[ A_p^2(0) - \frac{\beta_p}{\beta_{11}} \left( \frac{\beta_{12}}{\beta_{11}} + 1 \right) A_{11}^2(z) \right]^{1/2} \]

(5-7-6) \[ A_{11}(z) = \int_0^z \frac{dA_{11}(z)}{dz} \frac{dz}{\beta_{11} \left[ A_p^2(0) - \frac{\beta_p}{\beta_{11}} \left( \frac{\beta_{12}}{\beta_{11}} + 1 \right) A_{11}^2(z) \right]^{1/2}} \]

If we let \( y = \left[ \frac{\beta_s}{\beta_{11}} \left( \frac{\beta_{12}}{\beta_{11}} - 1 \right) \right]^{1/2} \frac{A_1(z)}{A_s(0)} \), we obtain

(5-7-7) \[ \left\{ \left( \frac{\beta_{12} - \beta_{11}}{\beta_s} \right) A_p^2(0) \right\}^{-1} \int_0^y \frac{dy}{1 - \frac{\beta_p}{\beta_s} \left( \frac{\beta_{12} + \beta_{11}}{\beta_{12} - \beta_{11}} \right) A_p^2(0)} \left[ 1 - y^2 \right]^{1/2} = z \]
If we now let $\sin \theta = y$, we obtain

$$(5-7-8)$$

$$\left\{ \left( \frac{\beta_{12} - \beta_{11}}{\beta_{12} + \beta_{11}} \right)^{\frac{1}{2}} A_p^2(0) \right\}^{-1} \int_{0}^{\sin^{-1} y} \frac{d\theta}{1 - \frac{\beta_p}{\beta_s} \left( \frac{\beta_{12} + \beta_{11}}{\beta_{12} - \beta_{11}} \right) A_s^2(0) \sin^2 \theta} = z$$

Eq. (5-7-8) can be integrated. Two solution sets are possible depending upon the relative magnitudes of the input signal and pump waves.

Case 1. $A_s^2(0)/\beta_s \leq \left( \frac{\beta_{12} - \beta_{11}}{\beta_{12} + \beta_{11}} \right) A_p^2(0)/\beta_p$.

For this case, we obtain

$$(5-7-9) \quad \Gamma_1^{-1} \tan^{-1} \left[ k_1^{-1} \tan \left( \sin^{-1} \left( \frac{\beta_s}{\beta_{11}} \left( \frac{\beta_{12} - \beta_{11}}{\beta_{12} + \beta_{11}} \right) \right) \right) \right] = z$$

where

$$(5-7-10) \quad k_1^2 = \frac{A_p^2(0)/\beta_p}{A_p^2(0)/\beta_p - \left( \frac{\beta_{12} + \beta_{11}}{\beta_{12} - \beta_{11}} A_s^2(0)/\beta_s \right)}$$

and

$$(5-7-11) \quad \Gamma_1 = [\beta_s (\beta_{12} - \beta_{11}) A_p^2(0) - \beta_p (\beta_{12} + \beta_{11}) A_s^2(0)]^{\frac{1}{2}} A_p(0).$$

We can solve Eq. (5-7-9) for $A_{11}(z)$. We can then substitute the solution for $A_{11}(z)$ into Eq. (5-7-4a), (5-7-4b), and (5-7-4c) to obtain
solutions for \( A_p(z) \), \( A_s(z) \), and \( A_{12}(z) \). The four solution equations are:

\[
(5-7-12a) \quad A_p(z) = A_p(0) \left\{ 1 - \frac{\beta_p}{\beta_s} \left( \frac{\beta_{12} + \beta_{11}}{\beta_{12} - \beta_{11}} \right) \frac{A_s^2(0)}{A_p^2(0)} \sin^2 \left[ \tan^{-1}[k_1 \tan(\Gamma_1 z)] \right] \right\}^{1/2}
\]

\[
(5-7-12b) \quad A_s(z) = A_s(0) \cos \left[ \tan^{-1}[k_1 \tan(\Gamma_1 z)] \right]
\]

\[
(5-7-12c) \quad A_{11}(z) = \left( \frac{\beta_{11}}{\beta_s} \right)^{1/2} \left( \frac{\beta_{11}}{\beta_{12} - \beta_{11}} \right)^{1/2} A_s(0) \sin \left[ \tan^{-1}[k_1 \tan(\Gamma_1 z)] \right]
\]

\[
(5-7-12d) \quad A_{12}(z) = \left( \frac{\beta_{12}}{\beta_s} \right)^{1/2} \left( \frac{\beta_{12}}{\beta_{12} - \beta_{11}} \right)^{1/2} A_s(0) \sin \left[ \tan^{-1}[k_1 \tan(\Gamma_1 z)] \right]
\]

In terms of power, Eq. (5-7-12) can be written as

\[
(5-7-13a) \quad P_p(z) = P_p(0) \left\{ 1 - \frac{\omega_p}{\omega_s} \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) \frac{P_s(0)}{P_p(0)} \sin^2 \left[ \tan^{-1}[k_1 \tan(\Gamma_1 z)] \right] \right\}
\]

\[
(5-7-13b) \quad P_s(z) = P_s(0) \cos^2 \left[ \tan^{-1}[k_1 \tan(\Gamma_1 z)] \right]
\]

\[
(5-7-13c) \quad P_{11}(z) = \frac{\omega_{11}}{\omega_s} \left( \frac{n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) P_s(0) \sin^2 \left[ \tan^{-1}[k_1 \tan(\Gamma_1 z)] \right]
\]
(5-7-13d)
\[ P_{12}(z) = \frac{\omega_{12}}{w_s} \left( \frac{n_{11} \omega_{12} - n_{12} \omega_{11}}{n_{11} \omega_{12} + n_{12} \omega_{11}} \right) P_s(0) \sin^2 \{ \tan^{-1} \{ k_1 \tan(\Gamma_1 z) \} \} \]

where

(5-7-14)
\[ k_1^2 = \frac{P_p(0)/\omega_p}{P_p(0)/\omega_p - \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) P_s(0)/\omega_s} \]

and

(5-7-15)
\[ \Gamma_1 = \left\{ \frac{c^2}{n_p^2 n_s^2} \left( \varepsilon_0^2 \right)^2 \left[ \left( \frac{n_{11} \omega_{12} - n_{12} \omega_{11}}{n_{11}^2 n_{12}} \right) \omega_p S_p(0) - \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11}^2 n_{12}} \right) \omega_s S_s(0) \right]^2 \right\}^{\frac{1}{2}} \]

In terms of photon flux, Eq. (5-7-13) becomes

(5-7-16a)
\[ N_p(z) = N_p(0) \left\{ 1 - \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) \frac{N_s(0)}{N_p(0)} \sin^2 \{ \tan^{-1} \{ k_1 \tan(\Gamma_1 z) \} \} \right\} \]

(5-7-16b)
\[ N_s(z) = N_s(0) \cos^2 \{ \tan^{-1} \{ k_1 \tan(\Gamma_1 z) \} \} \]

(5-7-16c)
\[ N_{11}(z) = \left( \frac{n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) N_s(0) \sin^2 \{ \tan^{-1} \{ k_1 \tan(\Gamma_1 z) \} \} \]

(5-7-16d)
\[ N_{12}(z) = \left( \frac{n_{11} \omega_{12}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) N_s(0) \sin^2 \{ \tan^{-1} \{ k_1 \tan(\Gamma_1 z) \} \} \]
where

\[ (5-7-17) \quad k_1^2 = \frac{N_p(0)}{N_p(0) - \left(\frac{n_{11} - n_{12}}{n_{11} + n_{12}}\right) N_s(0)} \]

and

\[ (5-7-18) \quad \Gamma_1 = \left\{ \frac{\hbar^2 \omega_p^2 \omega_s c^2}{n_p^2 n_s} \left(\frac{\nu_0}{\varepsilon_0}\right)^2 \left[ \left(\frac{n_{11} + n_{12}}{n_{11} - n_{12}}\right) N_p(0) \right. \right. \\
\left. \left. - \left(\frac{n_{11} + n_{12}}{n_{11} - n_{12}}\right) N_s(0) \right] N_p(0) \right\}^{1/2}. \]

The Manley-Rowe relations are:

\[ (5-7-19a) \quad \frac{d}{dz} [N_{11}(z) + N_{12}(z)] = - \frac{d}{dz} N_p(z) = - \left(\frac{n_{11} + n_{12}}{n_{11} - n_{12}}\right) \frac{d}{dz} N_s(z) \]

\[ (5-7-19b) \quad = 2k_1 \Gamma_1 N_s(0) \left(\frac{n_{11} + n_{12}}{n_{11} - n_{12}}\right) \sin^{-1}[k_1 \tan(\Gamma_1 z)]. \]

\[ \cos^{-1} \left[ k_1 \tan(\Gamma_1 z) \right] \left[ \sec^2(\Gamma_1 z) \right] \left[ \frac{1}{1 + k_1^2 \tan^2(\Gamma_1 z)} \right]. \]

The conversion efficiency is

\[ (5-7-20a) \quad E = \frac{P_p(0) - P_p(L)}{P_p(0)} = \left(\frac{n_{11} + n_{12}}{n_{11} - n_{12}}\right) \frac{P_s(0) - P_s(L)}{P_p(0) \omega_s} \]

\[ = \frac{P_{11}(L)}{\omega_{11}} + \frac{P_{12}(L)}{\omega_{12}} \]

\[ \frac{P_p(0) / \omega_p}{P_p(0) / \omega_s}. \]
Case 2. As 

\[ (5-7-20b) \]

\[
\frac{N_p(0)-N_p(L)}{N_p(0)} = \left(\frac{n_{11} n_{12}^2 + n_{12} n_{11}^2}{n_{11} n_{12} + n_{12} n_{11}}\right) \left(\frac{N_s(0)-N_s(L)}{N_s(0)}\right)
\]

\[
= \frac{N_{11}(L) + N_{12}(L)}{N_p(0)}
\]

\[ (5-7-20c) \]

\[
\frac{N_s(0)}{N_p(0)} \sin^2\{\tan^{-1}[k_1 \tan(\Gamma_1 L)]\}.
\]

Case 2. \( A_s^2(0)/\beta_s > \left(\frac{\beta_{12} - \beta_{11}}{\beta_{12} + \beta_{11}}\right) A_p^2(0)/\beta_p \)

For this case, we obtain as a solution to Eq. (5-7-8) the equation

\[ (5-7-21) \]

\[
\Gamma_2^{-1} \tan^{-1}\left[k_2^{-1} \tan\left[\sin^{-1}\left\{\frac{\beta_{12} - \beta_{11}}{\beta_{11}} \left(\frac{A_{11}(z)}{A_s(0)}\right)^{1/2}\right\}\right]\right] = z
\]

where

\[ (5-7-22) \]

\[
k_2^2 = \frac{A_p^2(0)/\beta_p}{\left(\frac{\beta_{12} + \beta_{11}}{\beta_{12} - \beta_{11}}\right) A_s^2(0)/\beta_s - A_p^2(0)/\beta_p}
\]

and

\[ (5-7-23) \]

\[
\Gamma_2 = \left[\beta_p (\beta_{12} + \beta_{11}) A_s^2(0) - \beta_s (\beta_{12} - \beta_{11}) A_p^2(0)\right]^{1/2} A_p(0).
\]

We can solve Eq. (5-7-21) for \( A_{11}(z) \). We can then obtain the solutions for \( A_p(z), A_s(z), \) and \( A_{12}(z) \) by substituting the solution for \( A_{11}(z) \) into Eq. (5-7-4a), (5-7-4b), and (5-7-4c). The four solution equations are:
\begin{align}
(5-7-24a) \quad A_p(z) &= A_p(0) \left\{ 1 - \frac{\beta}{\beta_p} \left( \frac{\beta_{12}^2 + \beta_{11}^2}{\beta_{12}^2 - \beta_{11}^2} \right) \frac{A_s^2(0)}{A_p^2(0)} \sin^2 \left[ \tan^{-1} \{ k_2 \tanh(\Gamma_2 z) \} \right] \right\}^{\frac{1}{2}} \\
(5-7-24b) \quad A_s(z) &= A_s(0) \cos \left[ \tan^{-1} \{ k_2 \tanh(\Gamma_2 z) \} \right] \\
(5-7-24c) \quad A_{11}(z) &= \left( \frac{\beta_{11}}{\beta} \right)^{\frac{1}{2}} \left( \frac{\beta_{11}}{\beta_{12} - \beta_{11}} \right) A_s(0) \sin \left[ \tan^{-1} \{ k_2 \tanh(\Gamma_2 z) \} \right] \\
(5-7-24d) \quad A_{12}(z) &= \left( \frac{\beta_{12}}{\beta} \right)^{\frac{1}{2}} \left( \frac{\beta_{12}}{\beta_{12} - \beta_{11}} \right) A_s(0) \sin \left[ \tan^{-1} \{ k_2 \tanh(\Gamma_2 z) \} \right] \\

\text{In terms of power, Eq. (5-7-24) can be written as}

(5-7-25a) \quad P_p(z) &= P_p(0) \left\{ 1 - \frac{\omega}{\omega_s} \left( \frac{n_{11} \omega_{11} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) \frac{P_s(0)}{P_p(0)} \sin^2 \left[ \tan^{-1} \{ k_2 \tanh(\Gamma_2 z) \} \right] \right\}^{\frac{1}{2}} \\
(5-7-25b) \quad P_s(z) &= P_s(0) \cos^2 \left[ \tan^{-1} \{ k_2 \tanh(\Gamma_2 z) \} \right] \\
(5-7-25c) \quad P_{11}(z) &= \frac{\omega_{11}}{\omega} \left( \frac{n_{11} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) P_s(0) \sin^2 \left[ \tan^{-1} \{ k_2 \tanh(\Gamma_2 z) \} \right] \\
(5-7-25d) \quad P_{12}(z) &= \frac{\omega_{12}}{\omega} \left( \frac{n_{11} \omega_{12}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) P_s(0) \sin^2 \left[ \tan^{-1} \{ k_2 \tanh(\Gamma_2 z) \} \right]

\text{where}

(5-7-26) \quad k_2^2 &= \frac{P_p(0)/\omega}{\left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) P_s(0)/\omega_s - P_p(0)/\omega_p}
and

\[
(5-7-27) \quad \Gamma_2 = \left\{ \frac{c^2}{n_p n_s} \left( \frac{\mu_0}{\varepsilon_0} \right)^2 \left[ \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} n_{12}} \right) \omega_p s_s(0) \right. \right. \\
- \left. \left. \left( \frac{n_{11} \omega_{12} - n_{12} \omega_{11}}{n_{11} n_{12}} \right) \omega_s s_p(0) \right] s_p(0) \right\}^{\frac{1}{2}}.
\]

In terms of photon flux, Eq. (5-7-25) becomes

\[
(5-7-28a) \quad N_p(z) = N_p(0) \left\{ 1 - \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) \frac{N_s(0)}{N_p(0)} \sin^2 \left\{ \tan^{-1} \left[ k_2 \tanh(T_2 z) \right] \right\} \right\}
\]

\[
(5-7-28b) \quad N_s(z) = N_s(0) \cos^2 \left\{ \tan^{-1} \left[ k_2 \tanh(T_2 z) \right] \right\}
\]

\[
(5-7-28c) \quad N_{11}(z) = \left( \frac{n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) N_s(0) \sin^2 \left\{ \tan^{-1} \left[ k_2 \tanh(T_2 z) \right] \right\}
\]

\[
(5-7-28d) \quad N_{12}(z) = \left( \frac{n_{11} \omega_{12}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) N_s(0) \sin^2 \left\{ \tan^{-1} \left[ k_2 \tanh(T_2 z) \right] \right\}
\]

where

\[
(5-7-29) \quad k_2^2 = \frac{N_p(0)}{\left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) N_s(0) - N_p(0)}
\]

\[
(5-7-30) \quad \Gamma_2 = \left\{ \frac{\hbar^2 \omega_p^2 \omega_s c^2}{n_p^2 n_s} \left( \frac{\mu_0}{\varepsilon_0} \right)^2 \left[ \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} n_{12}} \right) N_s(0) \right. \right. \\
- \left. \left. \left( \frac{n_{11} \omega_{12} - n_{12} \omega_{11}}{n_{11} n_{12}} \right) N_p(0) \right] N_p(0) \right\}^{\frac{1}{2}}.
\]
The Manley-Rowe relations become

\begin{align}
\frac{d}{dz} [N_{11}(z) + N_{12}(z)] &= - \frac{d}{dz} N_{p}(z) = - \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) \frac{d}{dz} N_{s}(z) \\
\end{align}

(5-7-31b)

\begin{align}
= 2 k_{2} \Gamma_{2} N_{s}(0) \left( \frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}} \right) \sin\{\tan^{-1}[k_{2} \tanh(\Gamma_{2}z)]\} \times \\
\cos\{\tan^{-1}[k_{2} \tanh(\Gamma_{2}z)]\} \left[ \frac{\text{sech}^2(\Gamma_{2}z)}{1 + k_{2}^2 \tan^2(\Gamma_{2}z)} \right].
\end{align}

The conversion efficiency becomes

\begin{align}
(5-7-32) \quad E &= \frac{N_{s}(0)}{N_{p}(0)} \sin^2\{\tan^{-1}[k_{2} \tanh(\Gamma_{2}L)]\}.
\end{align}
5.8 Simultaneous Second Harmonically Pumped BTW Down and Up Conversion

In this section, we examine the BTW system. We choose to have a BTW signal at \( \omega_s \) with input at \( z = L \), a FTW pump at \( \omega_p \) with input at \( z = 0 \), and two FTW idlers at \( \omega_{11} \) and \( \omega_{12} \) which are generated internally with \( A_{11} = A_{12} = 0 \) at \( z = 0 \).

The four coupled amplitude equations which must be solved are:

\[
\begin{align*}
(5-8-1a) & \quad \frac{dA_p(z)}{dz} = -\beta_p A_p(z)A_s(z)[A_{s1}(z) + A_{s2}(z)] \\
(5-8-1b) & \quad \frac{dA_s(z)}{dz} = -\beta_s A_p^2(z)[A_{s1}(z) - A_{s2}(z)] \\
(5-8-1c) & \quad \frac{dA_{s1}(z)}{dz} = \beta_{11} A_p^2(z)A_s(z) \\
(5-8-1d) & \quad \frac{dA_{s2}(z)}{dz} = \beta_{12} A_p^2(z)A_s(z)
\end{align*}
\]

If we solve for \( A_p(z) \), \( A_s(z) \), and \( A_{s2}(z) \) in terms of \( A_{s1}(z) \), we obtain

\[
\begin{align*}
(5-8-2a) & \quad A_p(z) = \left[A_p^2(0) - \frac{\beta_p}{\beta_{11}} \left( \frac{\beta_{12}}{\beta_{11}} + 1 \right) A_{s1}^2(z) \right]^{1/2} \\
(5-8-2b) & \quad A_s(z) = \left[A_s^2(0) + \frac{\beta_s}{\beta_{11}} \left( \frac{\beta_{12}}{\beta_{11}} - 1 \right) A_{s1}^2(z) \right]^{1/2} \\
(5-8-2c) & \quad A_{s2}(z) = \left( \frac{\beta_{12}}{\beta_{11}} \right) A_{s1}(z)
\end{align*}
\]

If we substitute Eq. (5-8-2a) and (5-8-2b) into Eq. (5-8-1c), we obtain
\[
\frac{d}{dz} A_{11}(z) = \beta_{11} \left[ A_p^2(0) - \frac{\beta_p}{\beta_{11}} \left( \frac{\beta_{12}}{\beta_{11}} + 1 \right) A_{11}^2(z) \right] \\
\left[ A_s^2(0) + \frac{\beta_s}{\beta_{11}} \left( \frac{\beta_{12}}{\beta_{11}} - 1 \right) A_{11}^2(z) \right]^{1/2}
\]

It follows that

\[
A_{11}(z) = \int_0^z \frac{dA_{11}(z)}{\beta_{11} \left[ A_p^2(0) - \frac{\beta_p}{\beta_{11}} \left( \frac{\beta_{12}}{\beta_{11}} + 1 \right) A_{11}^2(z) \right] \left[ A_s^2(0) + \frac{\beta_s}{\beta_{11}} \left( \frac{\beta_{12}}{\beta_{11}} - 1 \right) A_{11}^2(z) \right]^{1/2}} = \int_0^z \frac{dy}{y} \left[ 1 - \frac{\beta_p (\beta_{12} + \beta_{11}) A_s^2(0)}{\beta_s (\beta_{12} - \beta_{11}) A_p^2(0)} y^2 \right] [1 + y^2]
\]

If we let \( y = \left[ \frac{\beta_s}{\beta_{11}} \left( \frac{\beta_{12}}{\beta_{11}} - 1 \right) \right]^{1/2} A_{11}(z) \), we obtain

\[
\left[ (\beta_{12} - \beta_{11}) \beta_s \right] A_p^2(0) \left[ 1 - \frac{\beta_p (\beta_{12} + \beta_{11}) A_s^2(0)}{\beta_s (\beta_{12} - \beta_{11}) A_p^2(0)} y^2 \right] [1 + y^2]
\]

If we now let \( \sinh \theta = y \), we obtain

\[
\sinh^{-1} y \left[ (\beta_{12} - \beta_{11}) \beta_s \right] A_p^2(0) \left[ 1 - \frac{\beta_p (\beta_{12} + \beta_{11}) A_s^2(0)}{\beta_s (\beta_{12} - \beta_{11}) A_p^2(0)} \sinh^2 \theta \right] = z
\]

Eq. (5-8-6) can be integrated. If \( \left( \frac{\beta_{12}}{\beta_{11}} + 1 \right) \frac{A_{11}^2(z)}{\beta_{11}} < \frac{A_p^2(0)}{\beta_p} \), we obtain
where

\[(5-8-8) \quad k^2 = \frac{A_p^2(0)/\beta_p}{\left(\frac{\beta_{12} + \beta_{11}}{\beta_{12} - \beta_{11}}\right) \frac{A_s^2(0)}{\beta_s} + \frac{A_p^2(0)}{\beta_p}}\]

and

\[(5-8-9) \quad \Gamma = \left[\left(\frac{\beta_{12} + \beta_{11}}{\beta_{12} - \beta_{11}}\right) \frac{A_p^2(0)}{\beta_p} + \left(\frac{\beta_{12} - \beta_{11}}{\beta_{12} + \beta_{11}}\right) \frac{A_s^2(0)}{\beta_s}\right]^{1/2} A_p(0).\]

We can solve Eq. (5-8-7) for \(A_{11}(z)\). If we then substitute the solution for \(A_{11}(z)\) in Eq. (5-8-2a), (5-8-2b), and (5-8-2c), we obtain the solutions for \(A_p(z), A_s(z),\) and \(A_{12}(z)\). The four solution equations are:

\[(5-8-10a) \quad A_p(z) = A_p(0) \left[1 - \frac{\beta_p}{\beta_s} \frac{A_{12}(0)}{A_{11}(0)} \frac{A_{11}(0)}{A_{12}(0)} \right]^{1/2} \sinh^2\left(\tanh^{-1}[k \tanh(\Gamma z)]\right)\]

\[(5-8-10b) \quad A_s(z) = A_s(0) \cosh \left(\tanh^{-1}[k \tanh(\Gamma z)]\right)\]

\[(5-8-10c) \quad A_{11}(z) = \left(\frac{\beta_{11}}{\beta_s}\right)^{1/2} \left(\frac{\beta_{11}}{\beta_{12} - \beta_{11}}\right)^{1/2} A_s(0) \sinh\left(\tanh^{-1}[k \tanh(\Gamma z)]\right)\]

\[(5-8-10d) \quad A_{12}(z) = \left(\frac{\beta_{12}}{\beta_s}\right)^{1/2} \left(\frac{\beta_{12}}{\beta_{12} - \beta_{11}}\right)^{1/2} A_s(0) \sinh\left(\tanh^{-1}[k \tanh(\Gamma z)]\right)\]
In terms of power, Eq. (5-8-10) can be written as

\[(5-8-11a)\]
\[P_p(z) = P_p(0)
\left(1 - \frac{\omega}{\omega_p}\frac{n_{11\omega_{12}^{n_{12\omega_{11}}}}}{n_{11\omega_{12}^{n_{12\omega_{11}}}}}
\frac{P_s(0)}{P_p(0)} \sinh^2\left[k \tanh^{-1}\left(k \tanh(\Gamma z)\right)\right]\right)\]

\[(5-8-11b)\]
\[P_s(z) = P_s(0) \cosh^2\left[k \tanh^{-1}\left(k \tanh(\Gamma z)\right)\right]\]

\[(5-8-11c)\]
\[P_{11}(z) = \frac{\omega_{11}}{\omega_s}\frac{n_{11\omega_{11}}}{n_{11\omega_{12}^{n_{12\omega_{11}}}}}
\frac{P_s(0)}{P_p(0)} \sinh^2\left[k \tanh^{-1}\left(k \tanh(\Gamma z)\right)\right]\]

\[(5-8-11d)\]
\[P_{12}(z) = \frac{\omega_{12}}{\omega_s}\frac{n_{11\omega_{12}}}{n_{11\omega_{12}^{n_{12\omega_{11}}}}}
\frac{P_s(0)}{P_p(0)} \sinh^2\left[k \tanh^{-1}\left(k \tanh(\Gamma z)\right)\right]\]

where

\[(5-8-12)\]
\[k^2 = \frac{P_p(0)/\omega_p}{\frac{n_{11\omega_{12}^{n_{12\omega_{11}}}}}{n_{11\omega_{12}^{n_{12\omega_{11}}}}}}
\frac{P_s(0)}{\omega_s} + \frac{P_s(0)}{\omega_p}\]

and

\[(5-8-13)\]
\[\Gamma = \left(\frac{c^2}{\omega_2}\right)^2\left(\frac{n_{11\omega_{12}^{n_{12\omega_{11}}}}}{n_{11\omega_{12}^{n_{12\omega_{11}}}}}ight)\omega_p S(0) + \left(\frac{n_{11\omega_{12}^{n_{12\omega_{11}}}}}{n_{11\omega_{12}^{n_{12\omega_{11}}}}}ight)\omega_p S_p(0)\]

In terms of photon flux, Eq. (5-8-11) becomes

\[(5-8-14a)\]
\[N_p(z) = N_p(0)
\left(1 - \frac{n_{11\omega_{12}^{n_{12\omega_{11}}}}}{n_{11\omega_{12}^{n_{12\omega_{11}}}}}
\frac{N_s(0)}{N_p(0)} \sinh^2\left[k \tanh^{-1}\left(k \tanh(\Gamma z)\right)\right]\right)\]
\begin{align}
\text{(5-8-14b)} \\
N_s(z) &= N_s(0) \cosh^2\{\tanh^{-1}[k \tanh(\Gamma z)]\} \\
\text{(5-8-14c)} \\
N_{11}(z) &= \left(\frac{n_{12}^{+} \omega_{l1}}{n_{11}^{+} \omega_{l1} - n_{12}^{+} \omega_{l1}}\right) N_s(0) \sinh^2\{\tanh^{-1}[k \tanh(\Gamma z)]\} \\
\text{(5-8-14d)} \\
N_{12}(z) &= \left(\frac{n_{11}^{+} \omega_{l1}}{n_{11}^{+} \omega_{l1} - n_{12}^{+} \omega_{l1}}\right) N_s(0) \sinh^2\{\tanh^{-1}[k \tanh(\Gamma z)]\}
\end{align}

where

\begin{align}
\text{(5-8-15)} \\
k^2 &= \frac{N_p(0)}{\left(\frac{n_{11}^{+} \omega_{l1} + n_{12}^{+} \omega_{l1}}{n_{11}^{+} \omega_{l1} - n_{12}^{+} \omega_{l1}}\right) N_s(0) + N_p(0)}
\end{align}

and

\begin{align}
\text{(5-8-16)} \\
\Gamma &= \left\{\frac{\pi^2 c^2 \mu_0^2}{n_p^2 n_s^2 n_{11}^{+} n_{12}^{+}}\right\} \left[ (n_{11}^{+} \omega_{l1} + n_{12}^{+} \omega_{l1}) N_s(0) + (n_{11}^{+} \omega_{l1} - n_{12}^{+} \omega_{l1}) N_p(0) \right] \frac{k^2}{N_s(0) + N_p(0)}
\end{align}

The Manley-Rowe relations are:

\begin{align}
\text{(5-8-17a)} \\
\frac{d}{dz} [N_{11}(z) + N_{12}(z)] &= -\frac{d}{dz} N_p(z) = \left(\frac{n_{11}^{+} \omega_{l1} + n_{12}^{+} \omega_{l1}}{n_{11}^{+} \omega_{l1} - n_{12}^{+} \omega_{l1}}\right) \frac{d}{dz} N_s(z) \\
\text{(5-8-17b)} \\
&= 2k \Gamma N_s(0) \left(\frac{n_{11}^{+} \omega_{l1} + n_{12}^{+} \omega_{l1}}{n_{11}^{+} \omega_{l1} - n_{12}^{+} \omega_{l1}}\right) \sinh\{\tanh^{-1}[k \tanh(\Gamma z)]\}
\end{align}

cosh\{\tanh^{-1}[k \tanh(\Gamma z)]\} \left[\frac{\text{sech}^2(\Gamma z)}{1 - k^2 \tanh^2(\Gamma z)}\right]
The conversion efficiency is

\[(5-8-18a) \quad E = \frac{P_p(0) - P_p(L)}{P_p(0)} = \left(\frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}}\right) \frac{[P_s(L) - P_s(0)]/\omega_s}{P_p(0)/\omega_p}\]

\[= \frac{P_{11}(L)/\omega_{11} + P_{12}(L)/\omega_{12}}{P_p(0)/\omega_p}\]

\[(5-8-18b) \quad \frac{N_p(0) - N_p(L)}{N_p(0)} = \left(\frac{n_{11} \omega_{12} + n_{12} \omega_{11}}{n_{11} \omega_{12} - n_{12} \omega_{11}}\right) \left(\frac{N_s(L) - N_s(0)}{N_p(0)}\right)\]

\[= \frac{N_{11}(L) + N_{12}(L)}{N_p(0)}\]

\[(5-8-18c) \quad = \frac{N_s(0)}{N_p(0)} \sinh^2\{\tanh^{-1}[k \tanh(\Gamma z)]\}\]
5.9 FTW Up Conversion with $\omega_{p1} \neq \omega_{p2}$

In this section, we consider FTW up conversion in which we have two separate pump sources at different frequencies, a signal source, and a sum frequency wave generated internally with zero amplitude at $z = 0$. We assume that the following phase matching conditions are satisfied:

\[(5-9-la) \quad \omega_{p1} + \omega_{p2} + \omega_s = \omega_i.\]

\[(5-9-1b) \quad k_{p1} + k_{p2} + k_s = k_i.\]

The coupled amplitude equations (5-1-3) are written as

\[(5-9-2a) \quad \frac{dA_{p1}(z)}{dz} = - \beta_{p1} A_{p2}(z) A_s(z) A_i(z)\]

\[(5-9-2b) \quad \frac{dA_{p2}(z)}{dz} = - \beta_{p2} A_{p1}(z) A_s(z) A_i(z)\]

\[(5-9-2c) \quad \frac{dA_s(z)}{dz} = - \beta_s A_{p1}(z) A_{p2}(z) A_i(z)\]

\[(5-9-2d) \quad \frac{dA_i(z)}{dz} = \beta_i A_{p1}(z) A_{p2}(z) A_s(z)\]

If we solve for $A_{p1}(z)$, $A_{p2}(z)$, and $A_s(z)$ in terms of $A_i(z)$, we obtain

\[(5-9-3a) \quad A_{p1}(z) = [A_{p1}^2(0) - \frac{\beta_{p1}}{\beta_{i}} A_i^2(z)]^{\frac{1}{2}}\]

\[(5-9-3a) \quad A_{p2}(z) = [A_{p2}^2(0) - \frac{\beta_{p2}}{\beta_{i}} A_i^2(z)]^{\frac{1}{2}}\]

\[(5-9-3c) \quad A_s(z) = [A_s^2(0) - \frac{\beta_s}{\beta_{i}} A_i^2(z)]^{\frac{1}{2}}\]
If we substitute Eq. (5-9-3a), (5-9-3b), and (5-9-3c) into Eq. (5-9-2d), we obtain

\[
\frac{d}{dz} A_1(z) = \beta_1 \left[ A_{p1}^2(0) - \frac{\beta_{p1}}{\beta_1} A_1^2(z) \right] \left[ A_{p2}^2(0) - \frac{\beta_{p2}}{\beta_1} A_1^2(z) \right] [A_s^2(0) - \frac{\beta_s}{\beta_1} A_1^2(z)]^{1/2}
\]

It follows that

\[
A_1(z) = \int_0^z \frac{dA_1(z)}{\beta_1 \left[ A_{p1}^2(0) - \frac{\beta_{p1}}{\beta_1} A_1^2(z) \right] \left[ A_{p2}^2(0) - \frac{\beta_{p2}}{\beta_1} A_1^2(z) \right] [A_s^2(0) - \frac{\beta_s}{\beta_1} A_1^2(z)]^{1/2}}
\]

If we let \( y = \left( \frac{\beta_s}{\beta_1} \right)^{1/2} \frac{A_1(z)}{A_s(0)} \), we obtain

\[
\left( \frac{\beta_s}{\beta_1} \right)^{1/2} A_{p1}(0) A_{p2}(0) \int_0^y \frac{dy}{\left[ [1 - \frac{\beta_{p1}}{\beta_s} A_s^2(0) y^2] [1 - \frac{\beta_{p2}}{\beta_s} A_s^2(0) y^2] [1 - y^2] \right]^{1/2}} = z
\]

If we now let \( \sin \theta = y \), we obtain

\[
\sin^{-1} y = \left( \frac{\beta_s}{\beta_1} \right)^{1/2} A_{p1}(0) A_{p2}(0) \int_0^{\sin^{-1} y} \frac{d\theta}{\left[ [1 - \frac{\beta_{p1}}{\beta_s} A_s^2(0) \sin^2 \theta] [1 - \frac{\beta_{p2}}{\beta_s} A_s^2(0) \sin^2 \theta] \right]^{1/2}} = z
\]
For $0 < \frac{\beta_{p1} A_s^2(0)}{\beta_s A_s^2(0)} < \frac{\beta_{p2} A_s^2(0)}{\beta_s A_s^2(0)} < 1$ and 

$$0 < \sin^{-1}\left(\frac{\beta_s}{\beta_1} A_1(z) \frac{1}{A_s(0)}\right) \leq \pi/2,$$

we obtain

$$F(\phi, k^2) = \Gamma z$$

where

$$F(\phi, k^2) = \text{elliptic integral of the first kind},$$

$$\phi = \sin^{-1}\left[\frac{A_1(z)}{A_s(0)} \left(\frac{\beta A_{p1}^2(0) - \beta_{p1} A_s^2(0)}{\beta_{p1} A_1^2(z)}\right)^\frac{1}{2}\right]$$

and

$$k^2 = \frac{A_s^2(0)}{A_{p2}^2(0)} \left(\frac{\beta_{p2} A_{p1}^2(0) - \beta_{p1} A_{p2}^2(0)}{\beta_{p1} A_{p1}^2(0) - \beta_{p1} A_{s}^2(0)}\right)^\frac{1}{2}$$

$$\Gamma = \left[\beta_{p1} A_{p1}^2(0) - \beta_{p1} A_{s}^2(0)\right]^\frac{1}{2} A_{p2}(0).$$
6.1 Introduction

In Chapter V, we examined the effects of pump depletion in four-wave processes. We examined in detail third harmonic generation and second harmonically pumped FTW and BTW down conversion and up conversion processes. In this chapter, we will examine the effects of pump depletion in higher-order-wave processes where the nonlinear coefficient in the amplitude equations is determined by a fourth order or higher nonlinear polarization of the medium.

The general phase matching conditions which we assume are satisfied can be written as

\[ \omega_1 = \sum_{j}^{n} \omega_j \]

\[ \omega_1 + \omega_2 = \sum_{j}^{n} \omega_j \]

(6-1-1)

\[ \omega_1 + \omega_2 + \omega_3 = \sum_{j}^{n} \omega_j \]

\[ \omega_1 + \omega_2 + \ldots + \omega_k = \sum_{j}^{n} \omega_j \]

\[ \omega_1 + \omega_2 + \ldots + \omega_k = \sum_{j}^{n} \omega_j \]
where \( n \) is the order of the wave process and the number of equations required is the integer value of \( n/2 \). For example, only two equations are required for a five-wave process since \( \omega_1 + \omega_2 = \omega_3 + \omega_4 + \omega_5 \) can be equally well expressed as \( \omega_1 + \omega_2 + \omega_3 = \omega_4 + \omega_5 \).

For phase-matched \( n \)-wave parametric interactions of both forward and backward plane collinear traveling waves in a lossless medium, \( n \) coupled amplitude equations of the form

\[
(6-1-3) \quad \frac{dA_{j_1}(z)}{dz} = \pm \zeta_{j_1} A_{j_2}(z) A_{j_3}(z) \ldots A_{j_n}(z) \quad j_1 = 1, 2, \ldots, n \\
\quad \quad \quad j_2 = 2, 3, \ldots, n, 1 \\
\quad \quad \quad j_3 = 3, 4, \ldots, n, 1, 2 \\
\quad \quad \quad \ldots \ldots \ldots \\
\quad \quad \quad j_n = n, 1, 2, \ldots, (n-1)
\]

must be solved. \( \zeta \) is the nonlinear coupling coefficient.
We will limit our analysis to special cases of harmonic generation since in practice Eq. (6-1-1) and (6-1-2) cannot be satisfied for the general cases of higher order wave processes. In Section 6.2, we will examine higher order harmonic generation. In Sections 6.3 through 6.6, we will examine third and higher order harmonically pumped FTW and BTW down conversion and up conversion processes.
6.2 Higher Order Harmonic Generation

In Section 5.2, we found that third harmonic generation was a special case of up conversion in which the phase matching conditions were satisfied with but a single wave input—that of the pump—such that \( \omega_i = 3\omega_p \). In the general case, we have

\[ \omega_i = (n-1)\omega_p \tag{6-2-1} \]

where \( n \) is the order of the wave process.

The general form of the amplitude equations (6-1-3) reduce to

\[ \frac{dA_p(z)}{dz} = -\zeta_p A_p^{n-2}(z)A_i(z) \tag{6-2-2a} \]

\[ \frac{dA_i(z)}{dz} = \zeta_i A_i^{n-1}(z) \tag{6-2-2b} \]

If we solve for \( A_p(z) \) in terms of \( A_i(z) \), we obtain

\[ A_p(z) = [A_p^2(0) - \frac{\zeta_p}{\zeta_i} A_i^2(z)]^{1/2} \tag{6-2-3} \]

where \( A_i(0) = 0 \) at \( z = 0 \).

If we substitute Eq. (6-2-3) into Eq. (6-2-2b), we obtain

\[ \frac{dA_i(z)}{dz} = \zeta_i [A_p^2(0) - \frac{\zeta_p}{\zeta_i} A_i^2(z)]^{\frac{n-1}{2}} \]

\[ = \frac{\zeta_p^2}{\zeta_i^{n-3}} [\frac{\zeta_i}{\zeta_p} A_p^2(0) - A_i^2(z)]^{\frac{n-1}{2}} \]

It follows that
Eq. (6-2-5) can be integrated. For \( n = 5 \) (fourth harmonic generation), we obtain

\[
\frac{A_1(z)}{2\frac{\zeta_1}{\zeta_p}A_p^2(0)\left[\frac{\zeta_1}{\zeta_p}A_p^2(0) - A_1^2(z)\right]} + \frac{1}{4\frac{\zeta_1}{\zeta_p}A_p^2(0)} = \frac{\zeta_p}{\zeta_1} \frac{2}{z}.
\]

For \( n = 6 \) (fifth harmonic generation), we obtain

\[
\frac{\zeta_1}{\zeta_p}A_p^2(0)\left[\frac{\zeta_1}{\zeta_p}A_p^2(0) - A_1^2(z)\right]^{-\frac{1}{2}} \left\{ \frac{A_1(z)}{3\frac{\zeta_1}{\zeta_p}A_p^2(0) - A_1^2(z)} \right\}^{\frac{3}{2}} = \frac{\zeta_p}{\zeta_1} \frac{5/2}{z}.
\]

For \( n = 7 \) and higher, the solutions to the integral equation (6-2-5) become increasingly complex and will not be given.
6.3 Third and Higher Order Harmonically Pumped FTW Down Conversion

In Section 5.3, we determined that second harmonically pumped
FTW down conversion required that we satisfy the relation $2\omega_p = \omega_s + \omega_1$.

In the general case, we have

$$(n-2)\omega_p = \omega_s + \omega_1$$

where $n$ is the order of the wave process.

The general form of the amplitude equations (6-1-3) reduce to

$$dA_p(z)/dz = -\zeta_p A_p^{n-3}(z)A_s(z)A_1(z)$$

$$dA_s(z)/dz = \zeta_s A_s^{n-2}(z)A_1(z)$$

$$dA_1(z)/dz = \zeta_1 A_1^{n-2}(z)A_s(z)$$

If we solve for $A_p(z)$ and $A_s(z)$ in terms of $A_1(z)$, we obtain

$$A_p(z) = [A_p^2(0) - \frac{\zeta_p}{\zeta_1} A_1^2(z)]^{1/2}$$

$$A_s(z) = [A_s^2(0) + \frac{\zeta_s}{\zeta_1} A_1^2(z)]^{1/2}$$

where $A_1(0) = 0$ at $z = 0$.

If we substitute Eq. (6-3-3a) and (6-3-3b) into Eq. (6-3-2c),
we obtain

$$dA_1(z)/dz = \zeta_1 [A_p^2(0) - \frac{\zeta_p}{\zeta_1} A_1^2(z)] \frac{n-2}{2} [A_s^2(0) + \frac{\zeta_s}{\zeta_1} A_1^2(z)]^{1/2}$$
It follows that

\[ A_1(z) = \int_0^z \frac{dA_1(z)}{\frac{\zeta_1}{\zeta_s} \frac{A_p^2(0)}{A_1^2(z)} - \frac{\zeta_p}{\zeta_1} \frac{A_s^2(0)}{A_1^2(z)} + \frac{\zeta_s}{\zeta_1} \frac{A_1^2(z)}{A_s^2(0)}} = \int_0^z dz \]

(6-3-5)

If we let \( y = \left( \frac{\zeta_s}{\zeta_1} \right)^{1/2} \frac{A_1(z)}{A_s(0)} \), we obtain

\[
\left( \frac{\zeta_1}{\zeta_s} \right)^{1/2} \frac{A_p^2(0)}{2} \int_0^y \frac{dy}{1 - \frac{\zeta_p}{\zeta_s} \frac{A_s^2(0)}{A_p^2(0)} y^2} = z
\]

(6-3-6)

If we now let \( \sinh \theta = y \), we obtain

\[
\left( \frac{\zeta_1}{\zeta_s} \right)^{1/2} \frac{A_p^2(0)}{2} \int_0^{\sinh^{-1} y} \frac{d\theta}{1 - \frac{\zeta_p}{\zeta_s} \frac{A_s^2(0)}{\sinh^2 \theta} \frac{A_1^2(z)}{A_p^2(0)}} = z
\]

(6-3-7)

Eq. (6-3-7) can be integrated for a limited number of cases.

For \( n = 5 \) (third harmonically pumped FTW down conversion), no solution is given. For \( n = 6 \) (fourth harmonically pumped FTW down conversion), we obtain

\[
\frac{\zeta_p A_p(0) A_1(z) \cosh \left[ \sinh^{-1} \left( \frac{\zeta_s}{\zeta_1} \frac{A_1(z)}{A_s(0)} \right) \right]}{2 A_p^2(0) [\zeta_p A_p^2(0) + \zeta_s A_p^2(0)] [\zeta_1 A_1^2(0) - \zeta_p A_1^2(z)]} + \\
\frac{\zeta_p A_s^2(0) + 2 \zeta_s A_p^2(0)}{2 A_p^2(0) [\zeta_p A_s^2(0) + \zeta_s A_p^2(0)]} \left[ k^{-1} \tanh^{-1} \left[ k^{-1} \tanh \left( \sinh^{-1} \left( \frac{\zeta_s}{\zeta_1} \frac{A_1(z)}{A_s(0)} \right) \right) \right] \right] = z
\]

(6-3-8)
where $k^2$ is given by Eq. (5-3-9) and $\Gamma$ by Eq. (5-3-10).

For $n = 7$, no solution is given. For $n = 8$ (sixth harmonically pumped FTW down conversion), the solution to the integral equation (6-3-7) is considerably more complex than Eq. (6-3-8) and will not be given.
6.4 Third and Higher Order Harmonically Pumped FTW Up Conversion

In Section 5.4, we determined that second harmonically pumped FTW up conversion required that we satisfy the relation $2\omega_p + \omega_s = \omega_1$.

In the general case, we have

$$(n-2)\omega_p + \omega_s = \omega_1.$$  

where $n$ is the order of the wave process.

The general form of the amplitude equations (6-1-3) reduce to

$$(6-4-2a) \quad \frac{dA_p(z)}{dz} = -\frac{\zeta_p}{\zeta_1} A_p^{n-3}(z) A_s(z) A_1(z)$$

$$(6-4-2b) \quad \frac{dA_s(z)}{dz} = -\frac{\zeta_s}{\zeta_1} A_p^{n-2}(z) A_1(z)$$

$$(6-4-2c) \quad \frac{dA_1(z)}{dz} = \frac{\zeta_1}{\zeta_p} A_s^{n-2}(z) A_s(z)$$

If we solve for $A_p(z)$ and $A_s(z)$ in terms of $A_1(z)$, we obtain

$$(6-4-3a) \quad A_p(z) = [A_p^2(0) - \frac{\zeta_p}{\zeta_1} A_1^2(z)]^{1/2}$$

$$(6-4-3b) \quad A_s(z) = [A_s^2(0) - \frac{\zeta_s}{\zeta_1} A_1^2(z)]^{1/2}$$

where $A_1(0) = 0$ at $z = 0$.

If we substitute Eq. (6-4-3a) and (6-4-3b) into Eq. (6-4-2c), we obtain

$$(6-4-4) \quad \frac{dA_1(z)}{dz} = \frac{\zeta_1}{\zeta_p} [A_p^2(0) - \frac{\zeta_p}{\zeta_1} A_1^2(z)]^{n-2/2} [A_s^2(0) - \frac{\zeta_s}{\zeta_1} A_1^2(z)]^{1/2}$$
It follows that

\[
A_1(z) \frac{dA_1(z)}{dz} = \int_0^z \frac{dz}{[A_p^2(0)\frac{\zeta_p}{\zeta_1} - \frac{\zeta_p}{\zeta_1}A_1(z)]^2 \frac{2}{\zeta_1^2 A_1^2(z)} [A_p^2(0) - \frac{\zeta_p}{\zeta_1}A_1^2(z)]^2}.
\]

If we let \( y = \left(\frac{\zeta_s}{\zeta_1}\right)^{\frac{1}{2}} \frac{A_1(z)}{A_s(0)} \), we obtain

\[
\left(\zeta_1 \zeta_s \right)^{\frac{1}{2}} [A_p^2(0)]^{\frac{n-2}{2}} \int_0^y \frac{dy}{\left[1 - \frac{\zeta_p}{\zeta_s}A_s^2(0) - y^2\right]^2 [1-y^2]^{\frac{1}{2}}} = z
\]

If we now let \( \sin \theta = y \), we obtain

\[
\left(\zeta_1 \zeta_s \right)^{\frac{1}{2}} [A_p^2(0)]^{\frac{n-2}{2}} \int_0^{\sin^{-1} y} \frac{d\theta}{\left[1 - \frac{\zeta_p}{\zeta_s}A_s^2(0) \sin^2 \theta\right]^2 [1-\sin^2 \theta]^{\frac{1}{2}}} = z
\]

Eq. (6-4-7) can be integrated for a limited number of cases.

For \( n = 5 \) (third harmonically pumped FTW up conversion), we obtain

\[
\left(\zeta_1 \zeta_s \right)^{\frac{1}{2}} \left[A_p^2(0)^{\frac{3}{2}} \right] \left[\frac{\zeta_s A_p^2(0)}{\zeta_s A_s^2(0) - \zeta_p A_p^2(0)}\right] E(\phi, k^2) - \left[\frac{\zeta_s A_p^2(0)}{\zeta_s A_s^2(0) - \zeta_p A_p^2(0)}\right] \left[\frac{\zeta_1 A_s^2(0) - \zeta_p A_p^2(0)}{\zeta_1 A_s^2(0) - \zeta_1 A_p^2(0)}\right] = z.
\]
where

\[(6-4-9)\quad E(\phi, k^2) = \text{elliptic integral of the second kind},\]

\[(6-4-10)\quad \phi = \sin^{-1}\left[\frac{\zeta_s}{\zeta_i} \frac{A_i(z)}{A_s(0)}\right]\]

and

\[(6-4-11)\quad k^2 = \frac{\zeta_p A_s^2(0)}{\zeta_s A_p^2(0)}.\]

For \(n = 6\) (fourth harmonically pumped FTW up conversion), we obtain two solutions depending upon the relative magnitudes of the input signal and pump waves.\(^{31}\)

**Case 1.** \(A_s^2(0)/\zeta_s < A_p^2(0)/\zeta_p.\(^{24}\)

For this case, the solution is

\[(6-4-12)\]

\[
\frac{2\zeta_s A_p^2(0) - \zeta_p A_s^2(0)}{2A_p^2(0)\left[\zeta_s A_p^2(0) - \zeta_p A_s^2(0)\right]} - \Gamma_1^{-1}\tan^{-1}\left[\frac{1}{\tan^{-1}\left[\frac{\zeta_s}{\zeta_i} \frac{A_i(z)}{A_s(0)}\right]}\right] - \frac{\zeta_p A_s(0) A_i(z) \cos\left[\sin^{-1}\left[\frac{\zeta_i}{\zeta_i} \frac{A_i(z)}{A_s(0)}\right]\right]}{2A_p^2(0)\left[\zeta_s A_p^2(0) - \zeta_p A_s^2(0)\right]\left[\zeta_i A_p^2(0) - \zeta_p A_i^2(z)\right]} = z
\]

where \(k_1^2\) is given by Eq. (5-4-9) and \(\Gamma_1\) by Eq. (5-4-10).

**Case 2.** \(A_s^2(0)/\zeta_s > A_p^2(0)/\zeta_p.\(^{25}\)
In this case, the solution is

\[(6-4-13)\]

\[
\frac{2\zeta_s A_p^2(0) - \zeta_p A_p^2(0)}{2A_p^2(0) [\zeta_s A_p^2(0) - \zeta_p A_p^2(0)]} \Gamma_2^{-1} \tanh^{-1} [k_2^{-1} \tan \{\sin^{-1} \left(\frac{\zeta_s A_p(z)}{A_s(0)}\right)\}] = z
\]

\[
-\frac{\zeta_p A_s(0)A_1(z) \cos \{\sin^{-1} \left(\frac{\zeta_s A_p(z)}{A_s(0)}\right)\}}{2A_p^2(0) (\zeta_s A_p^2(0) - \zeta_p A_p^2(0)) (\zeta_1 A_p^2(0) - \zeta_p A_p^2(0))} = z
\]

where \(k_2^2\) is given by Eq. (5-4-21) and \(\Gamma_2\) by Eq. (5-4-22).

For \(n = 7\) and higher, the solutions to the integral equation \((6-4-7)\) are considerably more complex than Eq. (6-4-12) and (6-4-13) and will not be given.
6.5 Third and Higher Order Harmonically Pumped BTW Down Conversion

In Section 5.5, we determined that there are two possible systems for obtaining second harmonically pumped BTW down conversion if the condition $2\omega_p = \omega_s + \omega_1$ is satisfied. In the general case, we must satisfy

\[(6-5-1)\quad (n-2)\omega_p = \omega_s + \omega_1\]

where $n$ is the order of the wave process.

System 3. BTW at both $(n-2)\omega_p$ and $\omega_s$.

The general form of the amplitude equations (6-1-3) reduce to

\[(6-5-2a)\quad \frac{dA_p(z)}{dz} = \zeta_p A_p^{n-3}(z)A_s(z)A_1(z)\]

\[(6-5-2b)\quad \frac{dA_s(z)}{dz} = -\zeta_s A_p^{n-2}(z)A_1(z)\]

\[(6-5-2c)\quad \frac{dA_1(z)}{dz} = \zeta_1 A_p^{n-2}(z)A_s(z)\]

If we solve for $A_p(z)$ and $A_s(z)$ in terms of $A_1(z)$, we obtain

\[(6-5-3a)\quad A_p(z) = \left[A_p^2(0) + \frac{\zeta_p}{\zeta_1} A_1^2(z)\right]^{1/2}\]

\[(6-5-3b)\quad A_s(z) = \left[A_s^2(0) - \frac{\zeta_s}{\zeta_1} A_1^2(z)\right]^{1/2}\]

where $A_1(0) = 0$ at $z = 0$. 
If we substitute Eq. (6-5-3a) and (6-5-3b) into Eq. (6-5-2c), we obtain

\begin{equation}
(6-5-4) \quad \frac{dA_1(z)}{dz} = \zeta_1 [A_p^2(0) + \frac{\zeta_p}{\zeta_1} A_1^2(z)] \frac{n-2}{2} \left[ A_s^2(0) - \frac{\zeta_s}{\zeta_1} A_1^2(z) \right]^{\frac{1}{2}}.
\end{equation}

It follows that

\begin{equation}
(6-5-5) \quad \frac{dA_1(z)}{dz} \bigg|_{0}^{z} = \int_{0}^{z} dz = \int_{0}^{z} \left( \frac{A_1(z)}{\zeta_1 [A_p^2(0) + \frac{\zeta_p}{\zeta_1} A_1^2(z)] \frac{n-2}{2} \left[ A_s^2(0) - \frac{\zeta_s}{\zeta_1} A_1^2(z) \right]^{\frac{1}{2}}} \right) dy
\end{equation}

If we let \( y = (\zeta_s/\zeta_1)^{\frac{1}{2}} \frac{A_1(z)}{A_s(0)} \), we obtain

\begin{equation}
(6-5-6) \quad \left\{ (\zeta_1 \zeta_s)^{\frac{1}{2}} [A_p^2(0)]^\frac{n-2}{2} \right\} \int_{0}^{z} dy = \int_{0}^{z} \left[ 1 + \frac{\zeta_p}{\zeta_s} \frac{A_p^2(0)}{A_p^2(0)} - y^2 \right]^{\frac{1}{2}} \left[ 1 - y^2 \right]^{\frac{1}{2}}
\end{equation}

If we now let \( \sin \theta = y \), we obtain

\begin{equation}
(6-5-7) \quad \left\{ (\zeta_1 \zeta_s)^{\frac{1}{2}} [A_p^2(0)]^\frac{n-2}{2} \right\} \int_{0}^{z} d\theta = \int_{0}^{z} \left[ 1 + \frac{\zeta_p}{\zeta_s} \frac{A_p^2(0)}{A_p^2(0)} \sin^2 \theta \right]^{\frac{1}{2}} \left[ 1 - \sin^2 \theta \right]^{\frac{1}{2}}
\end{equation}

Eq. (6-5-7) can be integrated for a limited number of cases. For \( n = 5 \) (third harmonically pumped BTW down conversion), we obtain

\begin{equation}
(6-5-8) \quad \left\{ (\zeta_1 \zeta_s A_p^2(0) + \zeta_1 \zeta_p A_s^2(0))^{\frac{1}{2}} A_p^2(0) \right\} \int_{0}^{z} d\phi = \int_{0}^{z} \left[ 1 + \frac{\zeta_p}{\zeta_s} \frac{A_p^2(0)}{A_p^2(0)} \sin^2 \theta \right]^{\frac{1}{2}} \left[ 1 - \sin^2 \theta \right]^{\frac{1}{2}}
\end{equation}

where
\[
(6-5-9) \quad \phi = \sin^{-1} \left[ \frac{A_1(z)}{A_s(0)} \left( \frac{\zeta_s A_p^2(0) + \zeta_p A_s^2(0)}{\zeta_p A_s^2(0) + \zeta_p A_s^2(z)} \right)^{1/2} \right]
\]

and

\[
(6-5-10) \quad k^2 = \frac{\zeta_p A_s^2(0)}{\zeta_s A_p^2(0) + \zeta_p A_s^2(0)}.
\]

For \( n = 6 \) (fourth harmonically pumped BTW down conversion), we obtain \(^{31,24}\)

\[
(6-5-11)
\]

\[
\frac{2\zeta_s A_p^2(0) + \zeta_p A_s^2(0)}{2A_p^2(0)(\zeta_s A_p^2(0) + \zeta_p A_s^2(0))} \left\{ \Gamma^{-1} \tan^{-1} \left[ k^{-1} \tan^{-1} \left( \frac{\zeta_s^2}{A_s(0)} \right) \right] \right\} \\
+ \frac{\zeta_p A_s^2(0)A_1(z) \cos \left[ \sin^{-1} \left( \frac{\zeta_s}{\zeta_p} \right) \right]}{2A_p^2(0)[\zeta_s A_p^2(0) + \zeta_p A_s^2(0)][\zeta_p A_p^2(0) + \zeta_p A_1^2(z)]} = z
\]

where \( k \) is given by Eq. (5-5-9) and \( \Gamma \) by Eq. (5-5-10).

For \( n = 7 \) and higher, the solutions to the integral equation (6-5-7) are considerably more complex than Eq. (6-5-8) and (6-5-11) and will not be given. In the above equations, it must be remembered that \( A_s(0) \) and \( A_p(0) \) refer to the wave amplitudes at the output boundary.

System 4. BTW at \( \omega_s \); FTW at \( (n-2)\omega_p \).
The general form of the amplitude equations (6-1-3) reduces to

\[ (6-5-12a) \quad \frac{dA_p(z)}{dz} = - \zeta_p A_p^{n-3}(z)A_s(z)A_1(z) \]

\[ (6-5-12b) \quad \frac{dA_s(z)}{dz} = - \zeta_s A_p^{n-2}(z)A_1(z) \]

\[ (6-5-12c) \quad \frac{dA_1(z)}{dz} = \zeta_1 A_p^{n-2}(z)A_s(z) \]

Since Eq. (6-5-12) is identical to Eq. (6-4-2) for third and higher order harmonically pumped FTW up conversion, the same solution equations (6-4-3 through 6-4-13) apply. It must be remembered that \( A_s(0) \) is now the amplitude of the signal wave at the output boundary.
6.6 Third and Higher Order Harmonically Pumped BTW Up Conversion

In Section 5.6, we determined that there are two possible systems for obtaining second harmonically pumped BTW up conversion and that the condition $2\omega_p + \omega_s = \omega_i$ must be satisfied. In the general case, we must satisfy the condition

$$(n-2)\omega_p + \omega_s = \omega_i$$

where $n$ is the order of the wave process.

System 5. BTW at $\omega_s$; FTW at $(n-2)\omega_p$.

The general form of the amplitude equations can be written as

(6-6-2a) \[ \frac{dA_p(z)}{dz} = -\zeta_p A_p^{n-3}(z)A_s(z)A_i(z) \]

(6-6-2b) \[ \frac{dA_s(z)}{dz} = \zeta_s A_p^{n-2}(z)A_i(z) \]

(6-6-2c) \[ \frac{dA_i(z)}{dz} = \zeta_i A_p^{n-2}(z)A_s(z) \]

Since Eq. (6-6-2) is identical to Eq. (6-3-2) for third and higher order harmonically pumped FTW down conversion, the same solution equations (6-3-3 through 6-3-8) apply. However, $A_s(0)$ now refers to the amplitude of the signal wave at the output boundary.

System 6. FTW at $\omega_s$; BTW at $(n-2)\omega_p$.

The general form of the amplitude equations can be written as
(6-6-3a) \[ \frac{dA_p(z)}{dz} = \zeta_p A_p^{n-3}(z)A_s(z)A_i(z) \]

(6-6-3b) \[ \frac{dA_s(z)}{dz} = -\zeta_s A_p^{n-2}(z)A_i(z) \]

(6-6-3c) \[ \frac{dA_i(z)}{dz} = \zeta_i A_p^{n-2}(z)A_s(z) \]

Since Eq. (6-6-3) is identical to Eq. (6-5-2) for third and higher order harmonically pumped BTW down conversion (System 3), the same solution equations (6-5-3 through 6-5-11) apply. In these equations, \( A_p(0) \) now refers to the amplitude of the pump wave at the output boundary.
A comparative study of three-wave, four-wave, and higher order-wave processes has been made taking into account pump depletion. During the course of the work, it was determined that for single three-wave interactions there are six possible systems that permit wave mixing due to the second order nonlinear polarizability of the medium. It was found possible to make a direct comparison of the energy conversion efficiencies of these six systems based on a set of parameters that was common to each system. Thus, for a common set of parameters consisting of the input pump excitation and the ratio of input signal to pump photon flux, it was possible to assess the characteristics of each of the systems. These characteristics were graphically displayed in Fig. 2, 4, 5, 6, 7, and 8. The figures showed that it was possible to achieve total pump depletion in some systems and not in others. The figures showed at what level of pump excitation or ratio of signal/pump flux that process reversal could occur in some systems. Fig. 5 and 6 showed that BTW down conversion systems 3 and 4 could operate as oscillators with a threshold pump excitation of $\pi/2$ and that energy transfer to the generated wave could take place without a signal wave input if pump excitation were
increased above π/2. In terms of conversion efficiency, the figures showed that of the three down conversion systems, the system consisting of a BTW at ω_s and a FTW at ω_p would transfer energy from the pump wave most efficiently for any given level of pump excitation or ratio of signal/pump photon flux. On the other hand, the figures showed that of the three up conversion processes, the system consisting of two forward traveling waves at ω_p and ω_s would be most effective in transferring pump energy to the sum frequency generated wave.

The effect of loss in the system was then examined. It was found that FTW system operation could be assessed in terms of the same set of common parameters that was used for the lossless system. However, the BTW systems were not amenable to analysis so comparisons could not be made. For FTW systems, loss introduced a damping factor in the solution equations and effectively decreased the interaction length over which energy transfer can take place.

The effects of pump depletion on multiple FTW and BTW three-wave processes were assessed in terms of a similar set of common parameters. Although pump excitation was redefined to take into account competing down conversion and up conversion processes, the resulting figures revealed many similarities in system characteristics with single three-wave processes. However, it was found that ratios of signal/pump input flux could be considerably reduced for total energy transfer and that this reduction depended on the relative frequencies of the two internally-generated waves and the refractive indices of the medium at these frequencies.
Second harmonically pumped four-wave processes that can occur as a result of the third order nonlinear polarizability of the medium were found to have system characteristics similar to those for three-wave processes. It was possible to develop a common set of system parameters so that a comparison of system performance could be made among the four-wave processes as well as with the various three-wave interactions. System characteristics were shown in Fig. 16, 17, 18, 19, 20, and 21. The figures revealed that total pump depletion could not be achieved in most cases and that the conversion efficiencies of the six second harmonically pumped four-wave systems were always less than their three-wave system counterparts. Fig. 18 and 19 revealed that second harmonically pumped BTW down conversion systems could operate as oscillators with an initial pump excitation threshold of $\pi/2$, but that increased pump excitation would be required to obtain the same conversion efficiency as their three-wave BTW down conversion counterparts above threshold.

Conversion efficiency equations were developed for simultaneous second harmonically pumped FTW down conversion and up conversion as well as BTW down conversion and up conversion. Although system characteristics were not plotted, the equations indicate system performance somewhat similar to that revealed by Fig. 13 and 14 for particular examples of simultaneous down conversion and up conversion three-wave processes.

The interaction equations for higher-order-wave processes that could result from higher order nonlinear polarizabilities were
examined, but solution equations could not be obtained that would enable one to make system performance comparisons.

Although this work has dealt primarily with idealized systems requiring only steady-state, plane wave analysis, a priority of system performance has been revealed for the first time that should serve as a guide for the designer and builder of practical systems. The study has shown that pump depletion should be taken into account for pump and signal intensities that can be achieved with state-of-the-art energy sources.
APPENDIX A

DERIVATION OF THE COUPLED AMPLITUDE EQUATIONS
FOR THREE-WAVE NONLINEAR INTERACTIONS

Electromagnetic waves propagating in a lossy, non-magnetic medium are governed by Maxwell's equations (in MKS units);

\begin{align*}
\n\n(A-1) & \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial}{\partial t} \mathbf{P}_{NL} \\
(A-2) & \quad \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}
\end{align*}

where the conductivity is \( \sigma = \sigma(\omega) \), \( \mu = \mu_0 \), \( \mathbf{D} = \varepsilon \mathbf{E} + \mathbf{P}_{NL} \), the dielectric constant is \( \varepsilon = \varepsilon_0 (1 + \chi_L) \), \( \chi_L \) is the linear polarizability, and the components of \( \mathbf{P}_{NL} \) are given by

\begin{align*}
(A-3) & \quad \mathbf{P}_{NL} = d^{2} e^{2} + c d^{3} e^{3} + \ldots \ldots + \text{higher order nonlinearities.}
\end{align*}

The equation

\begin{align*}
(A-4) & \quad P_{21} = d_{ijk} E_j E_k,
\end{align*}

which assumes a summation over repeated indices, relates the three components of the second order nonlinear polarization \( d^{2} e^{2} \) to the nine possible combinations of the applied field \( E_j E_k \). The nonlinear tensor \( d_{ijk} \) is composed of 27 components known as the nonlinear coefficients. Due to symmetry considerations, all components corresponding to any
rearrangement of the indices are equal which limits the number of independent nonlinear coefficients to 10 at most. Since several of these 10 coefficients are zero for most crystals and by a proper choice of electric field polarizations, at most one or two coefficients contribute to second order polarization. Since $d_{ijk}$ is a polar tensor of odd rank, its components are identically zero for any medium possessing inversion symmetry.

If we take the curl of Eq. (A-2), differentiate Eq. (A-1) with respect to time, and combine the resulting equations, we obtain

$$\nabla^2 \vec{E} = \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} + \mu \frac{\partial^2 \vec{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} + P_{NL}$$

We now make the following assumptions: (1) harmonic time variation; (2) only three frequencies exist, with $\omega_3 = \omega_1 + \omega_2$; and (3) plane wave propagation in the $z$-direction (thus $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$).

This allows us to write the instantaneous field as

$$E(z,t) = E_1(z,t) + E_2(z,t) + E_3(z,t)$$

where

$$E_j(z,t) = \frac{1}{2}[A_j(z)e^{i(\omega_j t - k_j z + \phi_j(z))} + \text{c.c.}] \quad j = 1,2,3$$

If we further assume that the field amplitudes are slowly varying functions of $z$; i.e., $d^2A(z)/dz^2 < k dA(z)/dz$, we may neglect all second order derivatives resulting from the substitution of Eq. (A-6) and (A-7) into Eq. (A-5). We then obtain
(A-8a) \[ \frac{dA_1}{dz} + i \frac{d\phi_1}{dz} A_1 = - \frac{\sigma_1}{2n_1} \sqrt{\frac{\mu_0}{\epsilon_0}} A_1 - \frac{i \omega_1 d}{2n_1} \sqrt{\frac{\mu_0}{\epsilon_0}} A_2 A_3 e^{-i[\Delta k z - \phi(z)]} \]

(A-8b) \[ \frac{dA_2}{dz} + i \frac{d\phi_2}{dz} A_2 = - \frac{\sigma_2}{2n_2} \sqrt{\frac{\mu_0}{\epsilon_0}} A_2 - \frac{i \omega_2 d}{2n_2} \sqrt{\frac{\mu_0}{\epsilon_0}} A_1 A_3 e^{-i[\Delta k z - \phi(z)]} \]

(A-8c) \[ \frac{dA_3}{dz} + i \frac{d\phi_3}{dz} A_3 = - \frac{\sigma_3}{2n_3} \sqrt{\frac{\mu_0}{\epsilon_0}} A_3 - \frac{i \omega_3 d}{2n_3} \sqrt{\frac{\mu_0}{\epsilon_0}} A_1 A_2 e^{-i[\Delta k z - \phi(z)]} \]

where the substitutions \( \Delta k = k_3 - k_1 - k_2 \), \( \phi(z) = \phi_3 - \phi_1 - \phi_2 \),

\( k^2 = \omega^2 \mu_0 \epsilon \), and \( n = \sqrt{\frac{\epsilon}{\epsilon_0}} \) have been made and \( d \) is the effective nonlinear coefficient. We can write Eq. (A-8) as

(A-9a) \[ \frac{dA_1}{dz} + i \frac{d\phi_1}{dz} A_1 = - \delta_1 A_1 - i \alpha_1 A_2 A_3 e^{-i[\Delta k z - \phi(z)]} \]

(A-9b) \[ \frac{dA_2}{dz} + i \frac{d\phi_2}{dz} A_2 = - \delta_2 A_2 - i \alpha_2 A_1 A_3 e^{-i[\Delta k z - \phi(z)]} \]

(A-9c) \[ \frac{dA_3}{dz} + i \frac{d\phi_3}{dz} A_3 = - \delta_3 A_3 - i \alpha_3 A_1 A_2 e^{-i[\Delta k z - \phi(z)]} \]

where \( \delta = \frac{\sigma}{2n} \sqrt{\frac{\mu_0}{\epsilon_0}} \) and \( \alpha = \frac{\omega d}{2n} \sqrt{\frac{\mu_0}{\epsilon_0}} \).

Since \( e^{\pm i[\Delta k z - \phi(z)]} = \cos[\Delta k z - \phi(z)] \pm i \sin[\Delta k z - \phi(z)] \),

we can rewrite Eq. (A-9) in terms of its real and imaginary components. Thus,

(A-10a) \[ \frac{dA_1}{dz} + \delta_1 A_1 + \alpha_1 A_2 A_3 \sin \phi(z) = 0 \]

(A-10b) \[ \frac{dA_2}{dz} + \delta_2 A_2 + \alpha_2 A_1 A_3 \sin \phi(z) = 0 \]
where $\Phi(z) = Akz - \phi(z)$. Since $d\phi/dz = Ak - d\phi_1/dz$ and $\phi = \phi_3 - \phi_1 - \phi_2$, Eq. (A-11) can be written as

\[(A-12) \quad \Delta k + \frac{\alpha_2 A_3}{A_1} \cos \Phi(z) = 0\]

Let us assume that $\Delta k = 0$ and that $\delta_1 = \delta_2 = \delta_3 = 0$. If we now divide Eq. (A-12) by Eq. (A-10), we obtain

\[(A-13) \quad \tan \Phi d\Phi = dA_1/A_1 + dA_2/A_2 + dA_3/A_3\]

If we integrate Eq. (A-13), we obtain

\[(A-14) \quad A_1(z) A_2(z) A_3(z) \cos \Phi(z) = \text{constant}\]

If we assume that $A_1(z)$ is the amplitude of the idler wave and that $A_1(0) = 0$ at $z = 0$, then for any $z \neq 0$, $\cos \Phi = 0$. If $\cos \Phi = 0$, then $\Phi = \pm \pi/2$. If we designate $A_3(z)$ to be the pump amplitude and $A_2(z)$ the signal amplitude, we choose $\sin \Phi = -1$ since the signal and idler waves are amplified at the expense of the pump.
Although we will not go through the analysis, Eq. (A-14) is obtained not only for FTW interactions but also for BTW interactions. Hence, our use of only the amplitude equations is justified.
APPENDIX B

DERIVATION OF THE COUPLED AMPLITUDE EQUATIONS

FOR FOUR-WAVE NONLINEAR INTERACTIONS

The nonlinear polarization of a medium can be expressed by the equation

\[ P_{nl} = dE^2 + cE^3 + bE^4 + \ldots + \text{higher order nonlinearities}. \]

The equation

\[ P_{31} = c_{ijkl} E_j E_k E_l \]

which assumes a summation over repeated indices, relates the three components of the third order nonlinear polarization \( cE^3 \) to the 27 possible combinations of the applied field \( E_j E_k E_l \). The nonlinear tensor \( c_{ijkl} \) is composed of 81 elements or coefficients. The permutation symmetry between the last three indices reduces the number of independent nonlinear coefficients to 30 at most. For most crystals, the number of coefficients which effectively contribute to third order polarization is further reduced to at most three or four coefficients.

Our analysis is similar to that employed in Appendix A. We start with Eq. (A-5):

\[ \nabla^2 E = \mu_0 \frac{\partial E}{\partial t} \mu_0 \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial^2 E}{\partial t^2} P_{nl} \]
We make the following assumptions: (1) harmonic time variation; (2) only four frequencies exist, with \( \omega_4 = \omega_1 + \omega_2 + \omega_3 \); and (3) plane wave propagation in the z-direction (thus \( \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 \)).

The instantaneous field is then

\[
E(z,t) = E_1(z,t) + E_2(z,t) + E_3(z,t) + E_4(z,t)
\]

where

\[
E_j(z,t) = \frac{1}{2}[A_j(z)e^{i[\omega_j t - k_j z + \phi_j(z)]} + \text{c.c.}] \quad j = 1,2,3,4
\]

Again, we make the assumption that the field amplitudes are slowly varying functions of z; i.e., \( d^2A(z)/dz^2 \ll k \, dA(z)/dz \). This permits us to neglect all second order derivatives resulting from the substitution of Eq. (B-4) and (B-5) into Eq. (B-3). We obtain

\[
\begin{align*}
d\frac{A_1}{dz} + i\frac{d\phi_1}{dz}A_1 &= -\delta_1 A_1 - i\beta_1 A_2 A_3 A_4 e^{-i[\Delta k z - \phi(z)]} \\
d\frac{A_2}{dz} + i\frac{d\phi_2}{dz}A_2 &= -\delta_2 A_2 - i\beta_2 A_1 A_3 A_4 e^{-i[\Delta k z - \phi(z)]} \\
d\frac{A_3}{dz} + i\frac{d\phi_3}{dz}A_3 &= -\delta_3 A_3 - i\beta_3 A_1 A_2 A_4 e^{-i[\Delta k z - \phi(z)]} \\
d\frac{A_4}{dz} + i\frac{d\phi_4}{dz}A_4 &= -\delta_4 A_4 - i\beta_4 A_1 A_2 A_3 e^{-i[\Delta k z - \phi(z)]}
\end{align*}
\]

where the substitutions \( \Delta k = k_4 - k_1 - k_2 - k_3 \), \( \phi(z) = \phi_4 - \phi_1 - \phi_2 - \phi_3 \),

\[
k^2 = \omega_0^2 \varepsilon_0, \quad n = \sqrt{\frac{\varepsilon}{\varepsilon_0}}, \quad \delta = \sigma \frac{\mu_0}{2n \varepsilon_0}, \quad \text{and} \quad \beta = \frac{\omega c}{2n} \sqrt{\frac{\mu_0}{\varepsilon_0}}
\]

have been made and \( c \) is the effective third order nonlinear coefficient.
If we write Eq. (B-6) in terms of its real and imaginary components, we obtain

\[
\begin{align*}
\text{(B-7a)} \quad \frac{dA_1}{dz} + \delta_1 A_1 + \beta_1 A_2 A_3 A_4 \sin \phi(z) &= 0 \\
\text{(B-7b)} \quad \frac{dA_2}{dz} + \delta_2 A_2 + \beta_2 A_1 A_3 A_4 \sin \phi(z) &= 0 \\
\text{(B-7c)} \quad \frac{dA_3}{dz} + \delta_3 A_3 + \beta_3 A_1 A_2 A_4 \sin \phi(z) &= 0 \\
\text{(B-7d)} \quad \frac{dA_4}{dz} + \delta_4 A_4 - \beta_4 A_1 A_2 A_3 \sin \phi(z) &= 0
\end{align*}
\]

and

\[
\begin{align*}
\text{(B-8a)} \quad \frac{d\phi_1}{dz} A_1 + \beta_1 A_2 A_3 A_4 \cos \phi(z) &= 0 \\
\text{(B-8b)} \quad \frac{d\phi_2}{dz} A_2 + \beta_2 A_1 A_3 A_4 \cos \phi(z) &= 0 \\
\text{(B-8c)} \quad \frac{d\phi_3}{dz} A_3 + \beta_3 A_1 A_2 A_4 \cos \phi(z) &= 0 \\
\text{(B-8d)} \quad \frac{d\phi_4}{dz} A_4 + \beta_4 A_1 A_2 A_3 \cos \phi(z) &= 0
\end{align*}
\]

where \( \phi(z) = \Delta k z - \phi(z) \). Since \( d\phi/dz = \Delta k - d\phi/dz \) and

\[
\phi = \phi_4 - \phi_1 - \phi_2 - \phi_3,
\]

Eq. (B-8) can be written as

\[
\text{(B-9)} \quad \frac{d\phi}{dz} - \Delta k + \left[ \beta_1 \frac{A_2 A_3 A_4}{A_1} + \beta_2 \frac{A_1 A_3 A_4}{A_2} + \beta_3 \frac{A_1 A_2 A_4}{A_3} - \beta_4 \frac{A_1 A_2 A_3}{A_4} \right] \cos \phi = 0.
\]

Let us assume that \( \Delta k = 0 \) and that \( \delta_1 = \delta_2 = \delta_3 = \delta_4 = 0 \). If we divide Eq. (B-9) by Eq. (B-7), we obtain
If we integrate Eq. (B-10), we obtain

\begin{equation}
\tan \phi d\phi = \frac{dA_1}{A_1} + \frac{dA_2}{A_2} + \frac{dA_3}{A_3} + \frac{dA_4}{A_4}
\end{equation}

If we integrate Eq. (B-10), we obtain

\begin{equation}
A_1(z) A_2(z) A_3(z) A_4(z) \cos \phi(z) = \text{constant}
\end{equation}

If we assume that $A_1(z)$ is the amplitude of the idler wave and that $A_1(0) = 0$ at $z = 0$, then for any $z \neq 0$, $\cos \phi = 0$. If $\cos \phi = 0$, then $\phi = \pm \pi/2$. If we designate $A_4(z)$ to be the pump amplitude and $A_2(z)$ and $A_3(z)$ to be the two signal amplitudes, we choose $\sin \phi = -1$ since the signal and idler waves are amplified at the expense of the pump.

It should be noted that Eq. (B-11) is obtained not only for FTW interactions but also for BTW interactions.
APPENDIX C

A REVIEW OF PARAMETRIC OSCILLATORS AND MIXERS AND AN EVALUATION OF MATERIALS FOR 2 - 6 \( \mu \text{m} \) APPLICATIONS

by

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(Abstract)

A tunable source of narrow line radiation in the 2-6\( \mu \text{m} \) band and, to a lesser extent, the 8-12\( \mu \text{m} \) band would be useful in several military applications. Only recently have nonlinear optical materials become available to permit construction of optical parametric oscillators (OPO's) and mixers operating at these wavelengths. In this report, a review of the progress in wavelength coverage and output of OPO's and mixers is given, the theoretical performance of eleven nonlinear materials is discussed in detail, and, as an example, the materials are compared for use in operating a singly resonant OPO in the 3-6\( \mu \text{m} \) band. For this application, the outstanding materials are HgS pumped at 1.06\( \mu \text{m} \) and ZnGeP\(_2\) pumped at 2.1\( \mu \text{m} \). Refractive index data on the nonlinear materials and details of the FORTRAN computer program used in the calculations are included in the Appendices.
APPENDIX D

PROGRAMMED CALCULATIONS

Program sequences used in obtaining numerical data for plotting the figures which relate conversion efficiency to pump excitation and to ratios of input signal/pump photon flux for the various three-wave and four-wave interactions are listed. A Hewlett Packard Model 9820A calculator, a Texas Instruments Model SR-50 calculator, and tables of Jacobian elliptic functions\textsuperscript{20} were used in making the calculations.
FTW Down Conversion (System 1) - Figure 2

\[ N_s(0) \]

\[ N_p(0) \]

CALC \[ \frac{N_p(0)}{N_s(0) + N_p(0)} = k^2 \]

\[ N_s(L) \]

CALC \[ \frac{(N_s(L) - N_s(0))}{N_p(0)} = \frac{E}{k^2} \]

CALC \[ \text{cn}(K-\Gamma L, k^2) = \frac{E}{k^2} \]

\[ K, U \text{ (from Table) } \]

CALC \[ \Gamma L = K - U \]

CALC \[ k \Gamma L = \text{Pump Exc.} \]

PRT \[ E, \text{Pump Exc., } \frac{N_s(0)}{N_p(0)} \]

FTW Up Conversion (System 2) - Figure 4

Case 1. \( N_s(0) \leq N_p(0) \)

\[ N_s(0) \]

\[ N_p(0) \]

CALC \[ \frac{N_s(0)}{N_p(0)} = k^2 \]

\[ N_s(L) \]

CALC \[ \frac{(N_s(0) - N_s(L))}{N_p(0)} = \frac{E}{k^2} \]

CALC \[ \text{sn}(\Gamma L, k^2) = \left(\frac{E}{k^2}\right)^{\frac{1}{2}} \]

\[ \Gamma L = U \text{ (from Table) } = \text{Pump Exc.} \]

PRT \[ E, \text{Pump Exc., } \frac{N_s(0)}{N_p(0)} \]
Case 2. \( N_s(0) > N_p(0) \)

\[
\begin{align*}
\text{ENT} & \quad N_s(0) \\
\text{ENT} & \quad N_p(0) \\
\text{CALC} & \quad N_s(0)/N_p(0) = k^2 \\
\text{CALC} & \quad k^{-2} = 1/k^2 \\
\text{ENT} & \quad N_s(L) \\
\text{CALC} & \quad (N_s(0) - N_s(L))/N_p(0) = E \\
\text{CALC} & \quad sn(\Gamma_2L, k^{-2}) = E^{\frac{1}{2}} \\
\text{ENT} & \quad \Gamma_2L = U \quad (\text{from Table}^{20}) \\
\text{CALC} & \quad k^{-1}\Gamma_2L = \text{Pump Exc.} \\
\text{PRT} & \quad E, \text{ Pump Exc.}, N_s(0)/N_p(0)
\end{align*}
\]

BTW Down Conversion (System 3) - Figure 5

\[
\begin{align*}
\text{ENT} & \quad N_s(L) \\
\text{ENT} & \quad N_p(L) \\
\text{CALC} & \quad N_s(L)/N_p(L) \\
\text{ENT} & \quad k^2 = N_s(0)/(N_s(L) + N_p(L)) \\
\text{CALC} & \quad N_s(0) = k^2*(N_s(L) + N_p(L)) \\
\text{CALC} & \quad (N_s(0) - N_s(L))/N_p(L) = E \\
\text{CALC} & \quad sn(\Gamma L, k^2) = (E/k^2)^{\frac{1}{2}} \\
\text{ENT} & \quad \Gamma L = U \quad (\text{from Table}^{20}) \\
\text{CALC} & \quad \Gamma L*(N_p(L)/(N_s(L) + N_p(L)))^{\frac{1}{2}} = \text{Pump Exc.} \\
\text{PRT} & \quad E, \text{ Pump Exc.}, N_s(L)/N_p(L)
\end{align*}
\]
Case 1. \( N_s(0) \leq N_p(0) \)

ENT \( N_s(L) \)
ENT \( N_p(0) \)
CALC \( N_s(L)/N_p(0) \)
ENT \( k^2 = N_s(0)/N_p(0) \)
CALC \( k^2 - (N_s(L)/N_p(0)) = E \)
CALC \( \sin(\Gamma_1L, k^2) = (E/k^2)^{1/2} \)
ENT \( \Gamma_1L = U \) (from Table 20) = Pump Exc.
PRT \( E, \) Pump Exc., \( N_s(L)/N_p(0) \)

Case 2. \( N_s(0) > N_p(0) \)

ENT \( N_s(L) \)
ENT \( N_p(0) \)
CALC \( N_s(L)/N_p(0) \)
ENT \( k^2 = N_s(0)/N_p(0) \)
CALC \( k^{-2} = 1/k^2 \)
CALC \( k^2 - (N_s(L)/N_p(0)) = E \)
CALC \( \sin(\Gamma_2L, k^{-2}) = E^{1/2} \)
ENT \( \Gamma_2L = U \) (from Table 20)
CALC \( k^{-1} \Gamma_2L = \) Pump Exc.
PRT \( E, \) Pump Exc., \( N_s(L)/N_p(0) \)
BTW Up Conversion (System 5) - Figure 7

ENT $N_s(L)$

ENT $N_p(0)$

CALC $N_s(L)/N_p(0)$

ENT $k^2 = N_p(0)/(N_s(0) + N_p(0))$

CALC $(N_s(L)/N_p(0)) - ((1 - k^2)/k^2) = E$

CALC $cn(K - \Gamma L, k^2) = E^{1/2}$

ENT $K, U$ (from Table 20)

CALC $\Gamma L = K - U$

CALC $k*\Gamma L = $ Pump Exc.

PRT $E, Pump Exc., N_s(L)/N_p(0)$

BTW Up Conversion (System 6) - Figure 8

ENT $N_s(0)$

ENT $N_p(L)$

CALC $N_s(0)/N_p(L)$

ENT $k^2 = N_s(0)/(N_s(L) + N_p(L))$

CALC $N_s(L) + N_p(L) = N_s(0)/k^2$

CALC $(N_s(0)/N_p(L))*(1-(1/k^2)) + 1 = E$

CALC $sn(\Gamma L, k^2) = (E/k^2)^{1/2}$

ENT $\Gamma L = U$ (from Table 20)

CALC $\Gamma L*(N_p(L)/(N_s(L) - N_p(L)))^{1/2} = $ Pump Exc.

PRT $E, Pump Exc., N_s(0)/N_p(L)$
Second Harmonically Pumped FTW Down Conversion (System 1) - Figure 16

\[ \begin{align*}
\text{ENT} & \quad N_s(0) \\
\text{ENT} & \quad N_p(0) \\
\text{CALC} & \quad \frac{N_s(0)}{N_p(0)} \\
\text{CALC} & \quad k = \left(\frac{N_p(0)}{N_s(0)} + N_p(0)\right)^{\frac{1}{2}} \\
\text{ENT} & \quad \Gamma L \\
\text{CALC} & \quad \left(\frac{N_s(0)}{N_p(0)}\right) \sinh^2\left(\tanh^{-1}(k \tanh(\Gamma L))\right) = E \\
\text{CALC} & \quad k \Gamma L = \text{Pump Exc.} \\
\text{PRT} & \quad E, \text{ Pump Exc.}, \frac{N_s(0)}{N_p(0)}
\end{align*} \]

Second Harmonically Pumped FTW Up Conversion (System 2) - Figure 17

Case 1. \( N_s(0) < N_p(0) \)

\[ \begin{align*}
\text{ENT} & \quad N_s(0) \\
\text{ENT} & \quad N_p(0) \\
\text{CALC} & \quad \frac{N_s(0)}{N_p(0)} \\
\text{CALC} & \quad k_1 = \frac{N_p(0)}{N_s(0) - N_p(0)} \\
\text{ENT} & \quad \Gamma_1 L \\
\text{CALC} & \quad \left(\frac{N_s(0)}{N_p(0)}\right) \sin^2\left(\tan^{-1}(k_1 \tan(\Gamma_1 L))\right) = E \\
\text{CALC} & \quad k_1 \Gamma_1 L = \text{Pump Exc.} \\
\text{PRT} & \quad E, \text{ Pump Exc.}, \frac{N_s(0)}{N_p(0)}
\end{align*} \]
Case 2. \( N_s(0) > N_p(0) \)

ENT \( N_s(0) \)
ENT \( N_p(0) \)
CALC \( N_s(0)/N_p(0) \)
CALC \( k_2 = N_p(0)/(N_s(0) - N_p(0)) \)
ENT \( T_2L \)
CALC \( (N_s(0)/N_p(0))^2 \sin^2(\tan^{-1}(k_2 \tanh(T_2L))) = E \)
CALC \( k_2 \Gamma_2L = \text{Pump Exc.} \)

PRT \( E, \text{Pump Exc.}, N_s(0)/N_p(0) \)

---

Second Harmonically Pumped BTW Down Conversion (System 3) – Figure 18

ENT \( N_s(L) \)
ENT \( N_p(L) \)
CALC \( N_s(L)/N_p(L) \)
ENT \( N_p(0) \)
CALC \( N_s(0) = N_s(L) + N_p(L) - N_p(0) \)
CALC \( k^{-1} = ((N_s(L) + N_p(L))/N_p(0))^{1/2} \)
CALC \( \Gamma L = \tan^{-1}(k^{-1} \tan \cos^{-1}(N_s(L)/N_s(0))^{1/2}) \)
CALC \( (N_p(L)/(N_s(L) + N_p(L)))^{1/2} k^{-1} \Gamma L = \text{Pump Exc.} \)
CALC \( (N_p(L) - N_p(0))/N_p(L) = E \)
PRT \( E, \text{Pump Exc.}, N_s(L)/N_p(L) \)
Case 1. $N_s(0) < N_p(0)$

ENT $N_s(L)$

ENT $N_p(0)$

CALC $N_s(L)/N_p(0)$

ENT $N_p(L)$

CALC $N_s(0) = N_s(L) + N_p(0) - N_p(L)$

CALC $k_1 = (N_p(0)/(N_p(0) - N_s(0)))^{1/2}$

CALC $k_1^{-1} = 1/k_1$

CALC $\Gamma_1L = \tan^{-1}(k_1^{-1}\tan^{-1}(N_s(L)/N_s(0))^{1/2})$

CALC $k_1\Gamma_1L = \text{Pump Exc.}$

CALC $(N_p(0) - N_p(L))/N_p(0) = E$

PRT $E$, Pump Exc., $N_s(L)/N_p(0)$

Case 2. $N_s(0) > N_p(0)$

ENT $N_s(L)$

ENT $N_p(0)$

CALC $N_s(L)/N_p(0)$

ENT $N_p(L)$

CALC $N_s(0) = N_s(L) + N_p(0) - N_p(L)$

CALC $k_2 = (N_p(0)/(N_s(0) - N_p(0)))^{1/2}$

CALC $k_2^{-1} = 1/k_2$

CALC $\Gamma_2L = \tanh^{-1}(k_2^{-1}\tan^{-1}(N_s(L)/N_s(0))^{1/2})$

CALC $k_2\Gamma_2L = \text{Pump Exc.}$

CALC $(N_p(0) - N_p(L))/N_p(0) = E$

PRT $E$, Pump Exc., $N_s(L)/N_p(0)$
Second Harmonically Pumped BTW Up Conversion (System 5) - Figure 20

ENT \( N_s(L) \)

ENT \( N_p(0) \)

CALC \( N_s(L)/N_p(0) \)

ENT \( N_p(L) \)

CALC \( N_s(0) = N_s(L) + N_p(L) - N_p(0) \)

CALC \( k = (N_p(0)/(N_s(0) + N_p(0)))^{1/2} \)

CALC \( k^{-1} = 1/k \)

CALC \( \Gamma_L = \tanh^{-1}(k^{-1} \cdot \tanh \cos^{-1}(N_s(L)/N_s(0))^{1/2}) \)

CALC \( k \cdot \Gamma_L = \text{Pump Exc.} \)

CALC \( (N_p(0) - N_p(L))/N_p(0) = E \)

PRT \( E, \text{Pump Exc.}, N_s(L)/N_p(0) \)

Second Harmonically Pumped BTW Up Conversion (System 6) - Figure 21

ENT \( N_s(0) \)

ENT \( N_p(L) \)

CALC \( N_s(0)/N_p(L) \)

ENT \( N_p(0) \)

CALC \( N_s(L) = N_s(0) + N_p(0) - N_s(L) \)

CALC \( k^{-1} = ((N_s(L) + N_p(L))/N_p(0))^{1/2} \)

CALC \( \Gamma_L = \tan^{-1}(k^{-1} \cdot \tan \cos^{-1}(N_s(L)/N_s(0))^{1/2}) \)

CALC \( (N_p(L)/(N_s(L) + N_p(L))) \cdot k^{-1} \cdot \Gamma_L = \text{Pump Exc.} \)

CALC \( (N_p(L) - N_p(0))/N_p(L) = E \)

PRT \( E, \text{Pump Exc.}, N_s(0)/N_p(L) \)
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