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the Degree of Doctor of Philosophy in the Graduate
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By

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* * * * *

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1974

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INTRODUCTION

This paper is a classification of certain finite groups related to the Ree group $^{2}F_{4}(2)$. The general idea is to show that a basic assumption on the structure of a 2-group assumed to be a Sylow 2-subgroup of an unknown group $G$ forces the structure of centralizers of elements of order two, called involutions. Quoting known results allows us to conclude the identity of $G$. The basic assumption is in three parts. First, the maximal odd order normal subgroup of $G$, denoted $O(G)$, is trivial. Second the center of $G$ is trivial. Third $G$ does not possess subgroups of index two. Altogether this set of assumptions is called the fusion-simplicity hypothesis.

A few remarks on this hypothesis are in order. First, a result of Feit and Thompson [1] indicates that looking at 2-groups to understand non-abelian simple groups is worth while, for all such groups possess 2-groups. Secondly, these assumptions are reasonable, because there are theorems that describe what might be happening in the opposite case. Specifically, a result of Glauberman [2] states that if the center of a Sylow 2-subgroup is not in the center of our unknown group $G$, then all involutions have conjugates under $G$ in the Sylow 2-subgroup they are in. (Such conjugation is called fusion, and the
The fact that two elements $a$ and $b$ in $G$ are fused is denoted by $a \sim b$. We use Glauberman's notation. Thus $Z^*(G)$ is the inverse image of $Z(G/O(G))$. All other notation is standard and follows [4]. The fact there is no subgroup of index 2 allows us to use a result of Harada [7] that generalizes a result of Thompson, the so-called Thompson Transfer Lemma. In the form used the result says that given a maximal subgroup of a Sylow 2-subgroup and an involution outside it in the Sylow 2-subgroup, then the involution is fused into the maximal subgroup. (Hence the name fusion-simple.)

Other general concepts play an important role in this paper. The most basic idea is due to Brauer. To define it we assume $P$ is a Sylow $p$-subgroup of a finite group $G$ and $x \in P \setminus \{1\}$. If $y \in P$ and $x \sim y$, assume $|C_p(x)| \geq |C_p(y)|$. Then $x$ is called extremal in $P$ with respect to $G$. It is immediate from Sylow's theorems that if $x \sim y$ in $G$ and $x, y \in P$ with $x$ extremal, then there is some $g \in G$ such that $x = y^g$ and $C_p(y)^G \subseteq C_p(x)$.

The idea is used in essentially two ways. One case that arises concerns the possibility $C_p(y)^G = C_p(y)$. Various characteristic subgroups are then located, yielding good information about possible fusion patterns of involutions, since we work with $P$ a Sylow 2-subgroup of $G$. We remark that the establishment of fusion with elements in the center of $P$, hence extremal in $P$ with respect to $G$, is accomplished by the $Z^*$-theorem of Glauberman. The other case involves careful computation of various subgroups of $C_p(y)$.
that would be moved to the corresponding subgroup in $C_p(x)$; for example, members of the lower central series or even commutators themselves. When the sizes involved are fairly "equal" or grossly "unequal" ($C_p(y)$'s member is non-abelian and in $C_p(x)$ it is) or the fusion would violate known fusion patterns in $G$, useful results are again obtained.

Another important idea involves determination of fusion of various elementary 2-groups $E$. In all cases $m(E) = 5$, so detailed knowledge of the structure of $GL(5,2)$ is required [7].

In capsule form we note:

1. $|GL(5,2)| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$
2. A Sylow 31-normalizer is a Frobenius group of order $5 \cdot 31$
3. A Sylow 7-normalizer is the direct product of a non-abelian group of order 6 and a Frobenius group of order 21.
4. A Sylow 5-normalizer has order $3 \cdot 4 \cdot 5$ and a Sylow 5-centralizer is cyclic of order 15.
5. If $\tau$ has order 3, $C_G(\tau) \cong <\tau> \times A_5$ or $<\tau> \times L_3(2)$.

A Sylow 3-normalizer is a faithful split extension by $D_8$.

Also crucial to the determination of the centralizers of involutions will be the location of normal elementary 2-groups $E$ within the various centralizers. The importance of this is two-fold. First we have indicated the structure of $N(E)/C(E)$ is accessible. Second we note that for Chevalley groups of characteristic two fusion of involutions seems to be controlled by the structure of the centralizer of an involution in the center of a Sylow 2-subgroup and by the
normalizer of the second center of a Sylow 2-subgroup.

It turns out that the groups $E$ in question have the property of being strongly closed in a Sylow 2-subgroup of the centralizer of an involution or even in $G$ itself. (We recall if $E \leq F \leq G$ and $E$ is strongly closed in $F$ with respect to $G$, then for $e \in E$ and $e^g \in F$, we have $e^g \in E$.) This property has been investigated by Goldschmidt [3]. His main result states:

**Theorem.** If $E$ is a strongly closed abelian subgroup of a Sylow 2-subgroup $T$ of $G$, then $E^G_{/0(E^G)}$ is a central product of an abelian 2-group and quasi-simple groups with either abelian Sylow 2-subgroups or strongly-embedded subgroups. Further the image of $E$ in $E^G_{/0(E^G)}$ is equal to $O_2\left(E^G_{/0(E^G)}\right) \Omega_1\left(T \cdot E^G_{/0(E^G)}\right)$.

By using the structure of the Sylow 2-subgroup in question and known fusion, the normality of $E$ is established.

It is also appropriate to remark that use of Goldschmidt's result is not essential, but the work is greatly complicated.

The last general concept involves essentially showing that the centralizer of any involution possesses no non-trivial normal subgroup of odd order, the so-called core of the centralizer. The triviality of the core is by no means an easy task, since there exist groups with such substructures; for example, Chevalley groups of odd characteristic and alternating groups on $n$ letters, for $n$ odd and $n \geq 11$. All of the sporadic groups are known to have trivial cores, as are characteristic two Chevalley groups, with a few exceptions. A
full explanation of the ideas involved may be found in [5] and the references therein. The existence of the normal 2-groups in the centralizers modulo cores and the fact the 2-rank of a Sylow 2-subgroup is 5 in all cases allow use of various theorems in [5] that guarantee the core is trivial. Once this is known, the structure of the centralizer of an involution in the center of a Sylow 2-subgroup can be used to identify the group \( G \); [8], [9], [10].

Looking to the future various difficulties in use of the general approach become apparent. Recall the structure of centralizers of involutions was facilitated by the location of normal elementary 2-groups, because the structure of \( \text{GL}(n,2) \) was known for the relevant \( n = 5 \). For arbitrary \( n \) this problem is regrettably difficult. Also, location of normal 2-groups at the outset of the problem will be a source of difficulty. One need only look at examples like \( 2^{F_4}(q) \), \( q = 2^{2n+1} \) for \( n \geq 1 \), to see that fusion of all involutions in the second center of a Sylow 2-subgroup in \( 2^{F_4}(q) \) is not an easy matter to demonstrate. What seems to be called for are results that say when a 2-group is normal in a group \( G \) under conditions similar to the strong closure of abelian 2-groups. The advantages of classifying groups by properties of Sylow subgroups are too great to be lost now.

A flow chart of ideas for this paper looks like this:

1. Determine basic fusion in the second center of a Sylow 2-subgroup using extremality and the \( Z^* \)-theorem.
2. Use Thompson's Transfer Lemma to complete the basic analysis of fusion.

3. Determine what groups $G$ possess this pattern by determining the structure of all centralizers of involutions.

4. Hypothesize further fusion of involutions. Analyze the local structure of elementary groups. Identify $G$, as in Step 3.

The main results found here are that fusion-simple groups of the type found in $^{2}E_{6}(2)'$ and the recently discovered simple group of Rudvalis are up to isomorphism $^{2}F_{4}(2)'$, the Rudvalis group, or a 2-local subgroup found in the Rudvalis group. The method used, as indicated, is to determine the structure of the centralizers of all involutions. Then appropriate results of Parrott [8], [9], [10] are quoted. The structure for the 2-groups in question are taken from [8] and [9].
Throughout this section we denote by $T$ a 2-group isomorphic to a Sylow 2-subgroup found in $\text{Fr}_4(2)'$. More exactly we assume $T$ is generated by elements:

\begin{align*}
\alpha_1, \alpha_5, \alpha_3, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12} \text{ where } \\
(\alpha_1 \alpha_5)^2 = \alpha_2 \alpha_{12}; (\alpha_6 \alpha_7)^2 = \alpha_{10} \alpha_{11}; (\alpha_4 \alpha_5)^2 = \alpha_8 \alpha_9.
\end{align*}

These squares and remaining generators are involutions. We have also the following commutator relations:

\begin{align*}
\alpha_{12} &= [\alpha_7 \alpha_8] = [\alpha_6 \alpha_5 \alpha_9] = [\alpha_4 \alpha_5 \alpha_{10}] = [\alpha_3 \alpha_{11}] \\
\alpha_{11} &= [\alpha_7 \alpha_6 \alpha_5] = [\alpha_1 \alpha_5 \alpha_{10}] = [\alpha_1 \alpha_5 \alpha_7] = [\alpha_2 \alpha_{12} \alpha_9] \\
\alpha_{10} \alpha_{12} &= [\alpha_7 \alpha_4 \alpha_5] \\
\alpha_9 \alpha_{10} &= [\alpha_7 \alpha_3] = [\alpha_6 \alpha_5 \alpha_4 \alpha_5] \\
\alpha_9 \alpha_{12} &= [\alpha_6 \alpha_5 \alpha_3] \\
\alpha_8 &= [\alpha_4 \alpha_5 \alpha_3] \\
\alpha_7 \alpha_{10} &= [\alpha_1 \alpha_5 \alpha_6 \alpha_5] \\
\alpha_7 \alpha_{11} &= [\alpha_2 \alpha_{12} \alpha_4 \alpha_5] \\
\alpha_{10} \alpha_{11} \alpha_{12} &= [\alpha_2 \alpha_{12} \alpha_8] = [\alpha_1 \alpha_5 \alpha_9] \\
\alpha_9 \alpha_{11} \alpha_{12} &= [\alpha_1 \alpha_5 \alpha_8].
\end{align*}
\[ \alpha_6 \alpha_5 \alpha_1 \alpha \alpha_{12} = [\alpha_1 \alpha_5 \alpha_4 \alpha_3] \]
\[ \alpha_6 \alpha_5 \alpha_4 \alpha_9 \alpha \alpha_{12} = [\alpha_3 \alpha_2 \alpha_{12}] \]
\[ \alpha_4 \alpha_5 \alpha_9 \alpha_{10} \alpha \alpha_{12} = [\alpha_1 \alpha_5 \alpha_3] \]

all other commutators are trivial.

It is easy to see that \( J = \langle \alpha_3, \alpha_4 \alpha_5, \alpha_6 \alpha_5, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12} \rangle \) is normal in \( T \). Further we see

\( J' = \mathfrak{N} = < \alpha_{12}, \alpha_{11}, \alpha_{10}, \alpha_9, \alpha_8 > \) is elementary; and finally

\( Z(J) = L_3 J = Z(T) = < \alpha_{12} > \). Other subgroups of \( T \) that are useful will be listed in the appendix. Here we simply note the following:

F1. There are 3 maximal non-isomorphic subgroups of \( T \) and only the one isomorphic to \( C_T(\alpha_{11}) \) has a center not of order 2.

F2. \( T \) possesses no subgroup of order \( 2^9 \) with a center of order 8.

F3. If a subgroup of order \( 2^9 \) has a center of order 4, then it is isomorphic to \( C_T(\alpha_{10}) \) or a subgroup of \( C_T(\alpha_{11}) \).

F4. There is a unique third maximal subgroup of \( T \) with center of order 8, namely \( \mathfrak{S} = C_T(\alpha_{11}) \) with center \( < \alpha_{12}, \alpha_{11}, \alpha_{10} > \).

F5. In \( J/J' \), exactly 5 cosets contain involutions, namely

\( \alpha_7 J', \alpha_3 J', \alpha_3 \alpha_6 \alpha_5 J', \alpha_7 \alpha_6 \alpha_5 \alpha_4 \alpha_3 J', \) and \( \alpha_3 \alpha_4 \alpha_5 J' \).

F6. The sectional 2-rank of \( T \) is exactly 5.
F7. Useful Information on the Centralizers of Involutions in $T$.

1. $C_T(a_{11}) = \langle J', a_1a_3, a_4a_5, a_6a_5, a_7 \rangle$

   $C_T(a_{11})' = \langle a_{12}, a_{11}, a_{10}, a_9, a_7, a_6a_5 \rangle$

   $L_3(C_T(a_{11})) = \langle a_{12}, a_{11}, a_{10}, a_9, a_7 \rangle$

   $Z(C_T(a_{11})) = \langle a_{12}, a_{11}, a_{10} \rangle$

   $Z(C_T(a_{11})) = \langle a_{12}, a_{11} \rangle$

2. $C_T(a_{10}) = \langle J', a_6a_5, a_3, a_2 \rangle$

   $C_T(a_{10})' = \langle a_{12}, a_{11}, a_{10}, a_9, a_8 \rangle$

   $C_T(a_{10})'' = \langle a_{12} \rangle$

   $L_3(C_T(a_{10})) = \langle a_{12}, a_{11}, a_{10} \rangle$

   $Z(C_T(a_{10})) = \langle a_{12}, a_{10} \rangle$

3. $C_T(a_9) = \langle J', a_7, a_4a_5, a_3 \rangle$

   $C_T(a_9)' = \langle a_{12}, a_{10}a_8, a_9 \rangle$

   $Z(C_T(a_9)) = \langle a_{12}, a_{10}, a_9, a_8 \rangle$

   $L_3(C_T(a_9)) = \langle a_{12} \rangle$

   $Z(C_T(a_9)) = \langle a_{12}, a_9 \rangle$

4. $C_T(a_8) = \langle J', a_6a_5, a_4a_5, a_3 \rangle$

   $C_T(a_8)' = \langle a_{12}, a_{10}, a_9, a_8 \rangle$
\[ \xi(\alpha_8^2) = J \]

\[ L_3(c_T(\alpha_8)) = \langle \alpha_{12} \rangle \]

\[ Z(c_T(\alpha_8)) = \langle \alpha_{12}, \alpha_8 \rangle \]

5. \[ c_T(\alpha_8 \alpha_{11}) = \langle J, \alpha_4 \alpha_5, \alpha_6 \alpha_5, \alpha_3 \alpha_7 \rangle \]

\[ c_T(\alpha_8 \alpha_{11})' = \langle \alpha_{12}, \alpha_{11} \alpha_9, \alpha_{10} \alpha_8 \rangle \]

\[ L_3(c_T(\alpha_8 \alpha_{11})) = \langle \alpha_{12} \rangle \]

\[ Z(c_T(\alpha_8 \alpha_{11})) = \langle \alpha_{12}, \alpha_8 \alpha_{11} \rangle \]

6. \[ c_T(\alpha_7) = \langle \alpha_{12}, \alpha_{11}, \alpha_{10}, \alpha_9, \alpha_7, \alpha_4 \alpha_5 \alpha_6 \alpha_5, \alpha_2 \rangle \]

\[ c_T(\alpha_7)' = \langle \alpha_{10}, \alpha_{11} \rangle \]

\[ L_3(c_T(\alpha_7)) = \langle \alpha_{11} \rangle \]

\[ Z(c_T(\alpha_7)) = \langle \alpha_7, \alpha_{11}, \alpha_{12} \rangle \]

7. \[ c_T(\alpha_2) = \langle \alpha_{12}, \alpha_{11}, \alpha_{10}, \alpha_7, \alpha_6 \alpha_5, \alpha_1 \alpha_5 \rangle \]

\[ c_T(\alpha_2)' = \langle \alpha_{11}, \alpha_{10} \alpha_7 \rangle \]

\[ L_3(c_T(\alpha_2)) = \langle \alpha_{11} \rangle \]

\[ Z(c_T(\alpha_2)) = \langle \alpha_{12}, \alpha_{11}, \alpha_2 \rangle \]

8. \[ c_T(\alpha_2 \alpha_{12}) = c_T(\alpha_2) \]

9. \[ c_T(\alpha_3) = \langle \alpha_{12}, \alpha_{10}, \alpha_9, \alpha_8, \alpha_3 \rangle \]

10. Within T all conjugate classes of involutions are represented by
We now investigate the fusion pattern of involutions under the assumption any group $G$ that contains a Sylow 2-subgroup isomorphic to $T$ is fusion-simple.

**Lemma 1.** Within $G$ $\alpha_{12} \sim \alpha_{11}$ by an element of $g$ of order 3.

**Proof.** We proceed by contradiction, showing in the contrary case $\alpha_{12} \notin Z^*(G)$. To this end assume $\alpha_{12} \notin \alpha_{11}$ in $G$.

Suppose $\alpha_{12} \sim \alpha_{10}$ in $G$. Then there is an element $x \in G$ such that $\alpha_{12}^x = \alpha_{12}$ and $C_T(\alpha_{10})^x \subseteq T$. Since $C_T(\alpha_{10})^x = \langle \alpha_{12} \rangle$, we see $x$ cannot normalize $C_T(\alpha_{10})$. By statement F3 we see $\langle \alpha_{10}, \alpha_{12} \rangle^x = \langle \alpha_{11}, \alpha_{12} \rangle$, forcing $\alpha_{10} \sim \alpha_{12}$ and $\alpha_{12} \sim \alpha_{11}\alpha_{11}\alpha_{12}$, a contradiction.

Suppose $\alpha_9 \sim \alpha_{12}$ in $G$. Since $|C_T(\alpha_9)| = 2^8$ we may find a 2-group $X \supseteq C_T(\alpha_9)$ such that $\alpha_9 \in Z(X)$ and $[X : C_T(\alpha_9)] = 2$. Since $L_3C_T(\alpha_9) = \langle \alpha_{12} \rangle$ we see $Z(X) = \langle \alpha_9, \alpha_{12} \rangle$. Using F3 again $\langle \alpha_9, \alpha_{12} \rangle$ is fused to either $\langle \alpha_{10}, \alpha_{12} \rangle$ or $\langle \alpha_{11}, \alpha_{12} \rangle$, again impossible.

Next note $|C_T(\alpha_9)| = |C_T(\alpha_8)| = |C_T(\alpha_8\alpha_{11})|$ and $L_3C_T(\alpha_9) = L_3C_T(\alpha_8\alpha_{11})$. Hence we may use the previous argument to show $\alpha_8 \not\sim \alpha_{12}$ and $\alpha_8\alpha_{11} \not\sim \alpha_{12}$ in $G$, if $\alpha_{12} \not\sim \alpha_{11}$ in $G$.

Now assume $\alpha_7 \sim \alpha_{12}$ in $G$. Note $|C_T(\alpha_7)| = 2^7$ and $ZC_T(\alpha_7) = \langle \alpha_{12}, \alpha_{11}, \alpha_7 \rangle$. Choose a 2-group $X_1 \supseteq C_T(\alpha_7)$ s.t. $[X_1 : C_T(\alpha_7)] = 2$, with $\alpha_7 \in Z(X_1)$. 

\[ \alpha_{12}, \alpha_{11}, \alpha_{10}, \alpha_9, \alpha_8, \alpha_8\alpha_{11}, \alpha_7, \alpha_3, \alpha_2, \text{ or } \alpha_2\alpha_{12}. \]
Since \( L_3 C_T(\alpha_7) = \langle \alpha_{11} \rangle \) we see \( Z(X_1) \) is \( \langle \alpha_7, \alpha_{11} \rangle \) or \( \langle \alpha_7, \alpha_{11}, \alpha_{12} \rangle \). In the latter case statement F4 implies
\( \langle \alpha_{10}, \alpha_{11}, \alpha_{12} \rangle \sim \langle \alpha_7, \alpha_{11}, \alpha_{12} \rangle \) in \( G \). Counting conjugates of \( \alpha_{12} \) yields a contradiction. Hence \( Z(X_1) = \langle \alpha_7, \alpha_{11} \rangle \). Now choose a 2-group \( X_2 \supseteq X_1 \) such that \([X_2 : X_1] = 2\). By F2 we see
\( Z(X_2) = \langle \alpha_7 \rangle \) or \( \langle \alpha_7, \alpha_{11} \rangle \). If \( Z(X_2) = \langle \alpha_7 \rangle \), we conclude
\( \alpha_{11} \sim \alpha_7 \) in \( G \), a contradiction. Now use F3 to see \( Z(X_2) \) is fused to \( \langle \alpha_{11}, \alpha_{12} \rangle \) or \( \langle \alpha_{10}, \alpha_{12} \rangle \), again impossible.

Since \( |C_T(\alpha_2)| = |C_T(\alpha_2 \alpha_{12})| = |C_T(\alpha_7)| \) and \( L_3 C_T(\alpha_2) = L_3 C_T(\alpha_2 \alpha_{12}) = L_3 C_T(\alpha_7) \), while all three have centers of order 8, the above argument eliminates the cases involving \( \alpha_2 \) and \( \alpha_2 \alpha_{12} \).

It remains to treat the case \( \alpha_3 \sim \alpha_{12} \) in \( G \). Note that the set
\( J' \cap C_T(\alpha_3) = \langle \alpha_{12} \rangle \) is normal in \( N_G(C_T(\alpha_3)) \), so \( \langle \alpha_{12} \rangle < N_G(C_T(\alpha_3)) \). Checking the classes of elementary 2-groups of order 2^5 in \( T \) we see \( C_T(\alpha_3) \) is fused to \( J' \) or \( L_3 C_T(\alpha_{11}) \) or \( \langle \alpha_{12}, \alpha_{11}, \alpha_{10}, \alpha_7, \alpha_2 \rangle \). Counting conjugates yields a contradiction.

We apply the \( Z^* \)-theorem of Glauberman [2] to conclude \( \alpha_{12} \sim \alpha_{11} \) in \( G \). Using extremality and \( C_T(\alpha_{11}) \char T \), we have an element \( g \in G \) such that \( C_T(\alpha_{11})^g = C_T(\alpha_{11}) \) and \( \alpha_{11}^g = \alpha_{12} \). Since
\( Z(C_T(\alpha_{11})) = \langle \alpha_{11}, \alpha_{12} \rangle \), we may assume \( g \) has order 3.

QED.
The existence of such an element $g$ helps considerably.

**Lemma 2.** In $G$ we have $a_9 \sim a_7$, the set \{a_8, a_8a_{11}\} \sim \{a_2, a_2a_{12}\}$. Also $J' \sim <a_{12}, a_{11}, a_{10}, a_7, a_2>$.

**Proof.** We use the element $g$ found in Lemma 1. Since $Z \in C_T(a_{11}) = <a_{12}, a_{11}, a_{10}>$, $g$ must fix some conjugate of $a_{10}$.

Since $L_3C_T(a_{11})$ char $C_T(a_{11})$, $<g>$ normalizes $L_3C_T(a_{11})$, so $<g>$ acts on $L_3C_T(a_{11})/Z \notin C_T(a_{11})$. Assume $<g>$ acts trivially on this quotient. Then $<g>$ acts on $K = <a_{12}, a_{11}, a_{10}, a_9>$.

So $<g>$ acts on $C_T(K) \cap C_T(a_{11}) = <J', a_7>$. But $<J', a_7> = <a_{12}>$, so $a_{12}^g = a_{12}$, a contradiction. So $a_9 \sim a_7$ within $C_T(a_{11}) \cdot <g>$.

Now consider $<g>$ acting on $\phi C_T(a_{11})/L_3C_T(a_{11})$.

Note that this quotient equals $<a_8, a_{12}, a_8a_{12}>$ where $L = L_3C_T(a_{11})$. If $<g>$ acts on $<a_{12}, L>$ or $<a_8, L>$, then $<g>$ acts on the respective commutator subgroups, $<a_{11}>$ and $<a_{12}>$, which is not so. We see as sets $\{a_2, a_2a_{12}\}$ is fused to $\{a_8, a_8a_{11}\}$.

That $J' \sim <a_{12}, a_{11}, a_{10}, a_7, a_2>$ is now clear.

QED.
We need the following trivial result.

**Lemma 3.** Within $G$ $\alpha_{12} \not\sim \alpha_{10}$.

**Proof.** Suppose not. Say $\tilde{g} \in G$ fuses $\alpha_{10}$ to $\alpha_{12}$ and $C_T(\alpha_{10})^{\tilde{g}} \subseteq T$. (Such an element exists by extremality of $\alpha_{12}$.) Since $C_T(\alpha_{10})'' = \langle \alpha_{12} \rangle$ cannot normalize $C_T(\alpha_{10})$. Using statement $F_3$ we conclude $C_T(\alpha_{10})^{\tilde{g}}$ is a maximal subgroup of $C_T(\alpha_{11})$. But the maximal subgroups of $C_T(\alpha_{11})$ have abelian commutator subgroups equal to $L_3 C_T(\alpha_{11})$. Thus $\alpha_{12} \not\sim \alpha_{10}$ in $G$.

We investigate the fusion of $\alpha_8$.

**Lemma 4.** If $\alpha_8$ is not extremal in $T$, then $\alpha_8 \sim \alpha_{12}$ in $G$.

**Proof.** Suppose $\alpha_{10} \sim \alpha_8$ in $G$. Choose $g \in G$ so that $\alpha_8^g = \alpha_{10}$ and $C_T(\alpha_8)^g \subseteq C_T(\alpha_{10})$. Since $|C_T(\alpha_8)| = |C_T(\alpha_{10})|$, they are isomorphic. But $C_T(\alpha_8) = J^*$ is elementary, while $C_T(\alpha_{10})$ is not. We conclude $\alpha_{10} \not\sim \alpha_8$ in $G$, forcing $\alpha_8 \sim \alpha_{12}$ in $G$.

We complete the fusion analysis, using $G = 0^2(G)$ and Thompson's Transfer Lemma. We consider the fusion of elementary groups of order $2^5$.

**Lemma 5.** In $G$ $C_T(\alpha_3)$ is a normal subgroup of some Sylow 2-subgroup.

**Proof.** By Lemma 2 $J^* \sim \langle \alpha_{12}, \alpha_{11}, \alpha_{10}, \alpha_7, \alpha_2 \rangle$. Since
$C_T(\alpha_3)$, $J'$, $L_3C_T(\alpha_{11})$, and $<\alpha_{12}, \alpha_{11}, \alpha_{10}, \alpha_7, \alpha_2>$ are representatives of all classes of elementary groups of order $2^5$ in $T$, we conclude Lemma 5 is correct or $J \in \text{Syl}_2 \left( N_G(C_T(\alpha_3)) \right)$.

Note $N_G(C_T(\alpha_3))/C_T(\alpha_3) \subseteq \text{GL}(5,2)$. We shall use the bar convention on this quotient, which we denote by $\bar{Y}$. We see $J \cong Z_2 \times D_8$, while $J' = \bar{J} = <\bar{\alpha}_{11}>$ and $Z(J) = <\bar{\alpha}_{11}, \bar{\alpha}_4\bar{\alpha}_5>$. Note that $\alpha_{11}C_{T}(\alpha_3)$ is the only coset in $J$ with exactly 16 involutions. By the $Z^*$-theorem of Glauberman [2], we conclude $\bar{Y} = 0(\bar{Y})C_{T}(\bar{\alpha}_{11})$. By the $p$-local structure of $\text{GL}(5,2)$ we conclude $0(\bar{Y})$ has order at most 3. Suppose $0(\bar{Y})$ has order exactly 3. Let $\bar{N} = 0(\bar{Y})$. Now either $<\bar{\alpha}_7, \bar{\alpha}_6\bar{\alpha}_5> \subseteq C_{\bar{T}}(\bar{N})$ or $<\bar{\alpha}_7\bar{\alpha}_{11}, \bar{\alpha}_6\bar{\alpha}_5\bar{\alpha}_{11}> \subseteq C_{\bar{T}}(\bar{N})$. In either case the four group in question centralizes in $C_T(\alpha_3)$ exactly $<\alpha_{12}, \alpha_{10}>$. Since $\alpha_{10} \neq \alpha_{12}$ in $G$, we conclude $\bar{N}$ acts trivially on $<\alpha_{12}, \alpha_{10}>$, a contradiction to the $A \times B$-lemma. We conclude $0(\bar{Y}) = 1$. Note $C_T(\alpha_3) \cap C_T(\alpha_{11}) = <\alpha_{12}, \alpha_{10}, \alpha_9, \alpha_8>$. By Maschke's Theorem $\alpha_3$ is not fused to any involution in $<\alpha_{12}, \alpha_{10}, \alpha_9, \alpha_8>$ within $\bar{Y}$. By Thompson's Transfer Lemma $G$ has a subgroup of index 2, a contradiction.

QED.

Lemma 6. $\alpha_{12} \sim \alpha_{11} \sim \alpha_8$ and $\alpha_{10} \sim \alpha_9 \sim \alpha_8\alpha_{11} \sim \alpha_7 \sim \alpha_3$ in $G$.

Proof. Note $C_T(\alpha_3)$ has at least 16 conjugates of $\alpha_3$ since $J \subseteq N_G(C_T(\alpha_3))$. If $x \in J'$, then $[T : C_T(x)] \leq 8$. We conclude $C_T(\alpha_3) \not\subseteq J'$ in $G$, and $C_T(\alpha_3) \sim L_3C_T(\alpha_{11})$. In $L_3C_T(\alpha_{11})$ we have
3 classes of involutions with representatives \( \alpha_{12}, \alpha_{10}, \) and \( \alpha_9 \). Also, in \( L_3C_T(\alpha_{11}) \) we have 24 conjugates of \( \alpha_9 \). Hence \( \alpha_9 \sim \alpha_3 \) and \( \alpha_9 \sim \alpha_8 \alpha_{11} \sim \alpha_8 \alpha_{11} \) in \( G \). If \( \alpha_9 \) were extremal in \( T \), then there exists an element \( g \in G \) such that \( C_T(\alpha_8 \alpha_{11}) \subseteq C_T(\alpha_9) \). But \( \alpha_8 \) could not be fused to \( \alpha_3 \), hence \( \alpha_9 \) (looking at the respective commutator subgroups). Now look at the respective Frattini subgroups to conclude \( \alpha_9 \) is not extremal in \( T \). The proof of Lemma 4 shows \( \alpha_8 \not\sim \alpha_{10} \) in \( G \). We conclude \( \alpha_9 \sim \alpha_{10} \) in \( G \) counting conjugates of \( \alpha_{10} \) in \( L_3C_T(\alpha_{11}) \). In \( L_3C_T(\alpha_{11}) \) there are now 2 classes of involutions. We conclude \( \alpha_9 \sim \alpha_{12} \) in \( G \), using the proof of Lemma 4.

QED.

Since the \( T \)-classes of involutions have representatives \( \alpha_{12}, \alpha_{11}, \alpha_{10}, \alpha_9, \alpha_8, \alpha_8 \alpha_{11}, \alpha_7, \alpha_3, \alpha_2, \) and \( \alpha_2 \alpha_{12} \), we have determined all necessary fusion within \( J \). Denote by \( H \) the subgroup \( C_G(\alpha_{12}) \), and fix this notation for the rest of the section. We investigate the fusion of involutions in \( H \).

Lemma 7. Within \( H \) \( \alpha_2 \) has no conjugates in \( J \). Thus 
\[ [H : \theta^2(H)] \geq 4. \]

Proof. Let \( \theta \) be an involution in \( J \) such that \( \alpha_2 \sim \theta \) in \( H \). So there is some \( g \in H \) such that \( (\alpha_2 \alpha_{12})^g = \theta \alpha_{12} \). Since \( \theta \sim \theta \alpha_{12} \) in \( T \), we see \( \alpha_2 \) and \( \alpha_2 \alpha_{12} \) are fused in \( G \). But this contradicts Lemma 2 and Lemma 5. Now use Thompson's Transfer Lemma.

QED.
Lemma 8. Within $H$ we have $\alpha_{11} \sim \alpha_8$; $\alpha_7 \sim \alpha_3$; $\alpha_{10} \sim \alpha_9 \sim \alpha_8 \alpha_{11}$.

Proof. By Lemma 6 $\alpha_{10} \sim \alpha_9$ in $G$. Since $\alpha_{10}$ is extremal in $T$ there is some $g \in G$ such that $\alpha_9^g = \alpha_{10}$ and $C_T(\alpha_9)^g \subseteq C_T(\alpha_{10})$. Comparing orders we see $C_T(\alpha_9)^g$ is a maximal subgroup of $C_T(\alpha_{10})$.

Note $Z(C_T(\alpha_{10})) = \langle \alpha_{12}, \alpha_{10} \rangle$, while $\Lambda_3 C_T(\alpha_9) = \langle \alpha_{12} \rangle$. Since $\alpha_{12} \not\sim \alpha_{10}$ in $G$, we see $g \in H$. Since $(\Omega_1 \not\subseteq C_T(\alpha_{10})) = \langle \alpha_{12}, \alpha_{11}, \alpha_{10}, \alpha_9 \rangle$ and $C_T(\alpha_9) = \langle \alpha_{12}, \alpha_{10}, \alpha_9, \alpha_8 \rangle$, we may use the known fusion in $G$ involving $\alpha_8$ to conclude $\alpha_8 \sim \alpha_{11}$ or $\alpha_8 \sim \alpha_{11} \alpha_{12}$ via $g$. So $(\alpha_8^g)^g = \alpha_{10} \alpha_{11}$ or $\alpha_{10}^g \alpha_8 \alpha_{11} \alpha_{12}$. Since $\alpha_8^g \alpha_9 = \alpha_8 \alpha_{11}$ in $T$, we see $\alpha_{10} \sim \alpha_8 \alpha_{11}$ in $H$. From the fusion in $G$ we see that all conjugates of $\alpha_{12}$ that lie in $J$ actually fall in $J'$. In particular $\alpha_{11}$ must be mapped by $g$ into $J'$. Hence $\langle g \rangle$ must normalize $J'$.

Now consider the map induced on the quotients $C_T(\alpha_9)/J' \rightarrow C_T(\alpha_{10})/J'$. Using Lemma 7 we see the action of $\langle g \rangle$ on these quotients is to map $C_T(\alpha_9)/J'$ into $J \cap C_T(\alpha_{10})/J'$.

Suppose $\alpha_7 \not\sim \alpha_3$ in $H$. If $\langle \alpha_3, J' \rangle$ is normalized by $\langle g \rangle$, we see $\alpha_3^g = \alpha_3 x$ for some $x \in J'$. Then $[\alpha_3, \alpha_8]^g = 1 = [\alpha_3 x, \alpha_{11}]$ or $[\alpha_3 x, \alpha_{11} \alpha_{12}]$, both of which equal $\alpha_{12}$. If $\langle \alpha_7, J' \rangle^g = \langle \alpha_7, J' \rangle^g$, then $\alpha_7^g = \alpha_7 \beta$ for some $\beta \in J'$. Then $[\alpha_7, \alpha_8]^g = \alpha_{12} = [\alpha_7 \beta, \alpha_{11}]$ or $[\alpha_7 \beta, \alpha_{11} \alpha_{12}]$, which happen to be trivial. So $\langle g \rangle$ cannot normalize $\langle \alpha_7, J' \rangle$ or $\langle \alpha_3, J' \rangle$. Now use statement F5 to conclude $\alpha_7 \sim \alpha_3$ in $H$.

QED.
Before considering the structure of $H/\mathcal{O}(H)$, we need a trivial result.

**Lemma 9.** In $H$, $\alpha_{10} \not\equiv \alpha_7$.

**Proof.** Suppose not. Choose $g \in H$ such that $\alpha_7^g = \alpha_{10}$ and $C_T(\alpha_7)^g \subseteq C_T(\alpha_{10})$. Note $\hat{C}_T(\alpha_7) = \langle \alpha_{11}, \alpha_{10}, \alpha_2 \alpha_7 \alpha_{12} \rangle$, while $\hat{C}_T(\alpha_{10}) = \langle \alpha_{12}, \alpha_{11}, \alpha_{10}, \alpha_9 \rangle$. Thus $g$ sends a conjugate of $\alpha_2$ or $\alpha_2\alpha_{12}$ into $J$, a contradiction to Lemma 7.

QED.

We let $\overline{H} = H/\mathcal{O}(H)$ and use the bar convention throughout the rest of this section.

**Lemma 10.** $O_2(\overline{H}) = \overline{J}$ and $\overline{H}/\overline{J} \cong F_{20}$. If $\overline{F} \in Syl_5(\overline{H})$ then $C_{\overline{J}}(\overline{F}) = \langle \overline{\alpha_{12}} \rangle$.

**Proof.** Note that the element $g$ used in Lemma 8 normalized $J'$, so $g \not\in \mathcal{O}(H)$. Also, Lemmas 7 and 9 imply $\overline{J}/\overline{J'}$ is strongly closed in $\overline{T}/\overline{J'}$. Since we know $\alpha_3 \overline{J'} \sim \alpha_7 \overline{J'}$ in $\overline{H}$, we see $\alpha_3 \overline{J'} \sim \alpha_7 \overline{J'}$ in $N(\overline{J}/\overline{J'})$.

This implies $N_{\overline{H}}(\overline{J}/\overline{J'}) \cong F_{20} \cdot E_{16}$. Let $\overline{F} \in Syl_5(N_{\overline{H}}(\overline{J}/\overline{J'}))$. If $\overline{F}$ acts trivially on $\overline{J'}$ then $N_{\overline{H}}(\overline{J}/\overline{J'})$ stabilizes $\overline{J'} = \langle 1 \rangle$, implying this group is abelian. Hence $\overline{F}$ acts non-trivially, so $\overline{J}' = \langle \overline{\alpha_{12}} \rangle$. This shows $C_{\overline{J}}(\overline{F}) = \langle \overline{\alpha_{12}} \rangle$. We must finally show $\overline{J} < \overline{H}$. 

First note $\tilde{J}' = \tilde{E}$ is strongly closed in $\tilde{T}$ with regard to $\tilde{H}$. By a result of Goldschmidt [3] we see $\tilde{E}^H = \tilde{E}$, using the structure of $\tilde{J}$. Since $C_T(E) = E$ we see $C_H(E) = \tilde{E}$, showing $\tilde{H}/\tilde{E}$ is a subgroup of $\text{GL}(5,2)$. By the structure of $\text{GL}(5,2)$ we may conclude $\tilde{H}/\tilde{E} \cong F_{20} \cdot E_{16}$ or $<\alpha_2\alpha_{12}>$ can centralize an element $\tilde{x}$ of order 3. Note $<\alpha_2\alpha_{12}>$ centralizes in $\tilde{E}$ exactly $<\alpha_{12}, \alpha_{11}, \alpha_{10}>$. Since $\alpha_{10} \neq \alpha_{11}$ in $G$ we conclude $\tilde{x}$ must act trivially on this elementary group, so by the $A \times B$-lemma $\tilde{x}$ acts trivially on $\tilde{E}$. Hence $\tilde{J} \triangleleft \tilde{H}$.

QED.

Lemma 11. $C_G(\alpha_{10})$ is solvable.

Proof. By the action of $\tilde{H}$ in Lemma 10 we conclude $\tilde{H}$ fuses $L_3C_T(\alpha_{11})$ to $C_T(\alpha_3)$. Referring to the proof on p. 23 of [ ] we conclude $\alpha_3$ is not fused into $<J', \alpha_7, \alpha_6\alpha_5, \alpha_2>$ under the action of $C_G(\alpha_{10})$. Now suppose $\alpha_{12} \sim \alpha_8$ in $C_G(\alpha_{10})$. Let $B = C_T(\alpha_{10}) \cap C_T(\alpha_8)$. Then there is some $g \in C_G(\alpha_{10})$ such that $B^g \subseteq C_T(\alpha_{10})$ while $\alpha_8^g = \alpha_{12}$ (Recall $\alpha_{10} \neq \alpha_{12}$ in $G$ by Lemma 3.) Note $L_3(B) = <\alpha_{12}>$, so $g \notin N_G(B)$. Since $L_3C_T(\alpha_{10}) = <\alpha_{12}, \alpha_{11}, \alpha_{10}>$ we conclude $\alpha_{12}^g = \alpha_{11}$ or $\alpha_{11}^g = \alpha_{12}$. Also, there is some $x \in <J', \alpha_7, \alpha_6\alpha_5, \alpha_2>$ such that $\alpha_3^g = \alpha_3x$ by the above. We have then $[\alpha_{12}, \alpha_3]^g = [\alpha_{11}, \alpha_3]^g = \alpha_{12} = 1$, which is absurd. Using Lemma 2 we see $\alpha_2$ or $\alpha_2\alpha_{12}$ is not fused to $\alpha_{12}$ in $C_G(\alpha_{10})$. We conclude $<\alpha_{12}, \alpha_{11}>$ is strongly closed in $C_T(\alpha_{10})$ with respect
to $C_G(\alpha_{10})$. Using Goldschmidt's result [3], we see there is a normal subgroup $N$ of $C_G(\alpha_{10})$ such that $\langle \alpha_{12}, \alpha_{11} \rangle$ is a Sylow 2-subgroup of $N$. By a Frattini argument $C_G(\alpha_{10}) = [N_G(\langle \alpha_{12}, \alpha_{11} \rangle) \cap C_G(\alpha_{10})] \cdot O(N)$. In particular $C_G(\alpha_{10})$ is solvable.

QED.

We are now done.

**Lemma 12.** $G \cong ^2F_4(2)'$.

**Proof.** We see from the proof of Lemma 10 and Lemma 11 that all centralizers of involutions are solvable. By Theorem B of [5] the cores are all trivial. Now apply the main result of [10].

QED.
The recently discovered simple group of Rudvalis is known to be a rank three extension of $^{2}F_{4}(2)$. With regard to a Sylow 2-subgroup $S$ of Rudvalis type, it is known that $S$ has order $2^{14}$ and a 2-group of type $^{2}F_{4}(2)$ is a normal subgroup of index 4 in $S$. Hence the fusion pattern of involutions in $S$ strongly reflects that in a 2-group of type $^{2}F_{4}(2)$. Also given in this section is an explicit embedding of the 2-group of type $^{2}F_{4}(2)$ within $S$. 
The Generators and Relations of a 2-group of Rudvalis-type

1. Let \( z, t, v, w, w_1, a, b, c, d, y, y_1, \) and \( u \) be involutions.
   Let \( x, x_1 \) be of order 4 such that \( x^2 = x_1^2 = z \).

2. Suppose the following commutator values hold:

\[
\begin{align*}
[w, y] &= t & [w, y_1] &= vt & [w, b] &= z \\
[w_1, y] &= vt & [w_1, y_1] &= vz & [w_1, a] &= z \\
[u, w_1] &= w & [c, v] &= z & [d, t] &= z & [u, v] &= t \\
[x, x_1] &= z & [x, y_1] &= tz & [x, a] &= t & [x, b] &= v \\
[x, u] &= z & [x, c] &= w & [x, d] &= w_1 \\
[x_1, y] &= z & [x_1, y_1] &= vtz & [x_1, a] &= vz & [x_1, b] &= vt \\
[x_1, c] &= ztw_1 & [x_1, u] &= xz & [x_1, d] &= vw \\
[a, b] &= vt & [a, y_1] &= t & [a, d] &= w \\
[b, y] &= vz & [b, c] &= w_1 & [b, u] &= axwv \\
[c, y] &= a & [c, y_1] &= w_1x_1xba \\
[y, d] &= abx_1vt & [y_1, d] &= bv & [u, d] &= c \\
[u, y_1] &= y
\end{align*}
\]

Assume all unstated commutators are trivial. Then

\( S = \langle z, t, v, w, w_1, x, x_1, a, b, c, d, u, y_1, y \rangle \) is a 2-group of Rudvalis type.
Summary of Useful Facts Concerning the 2-group $S$

Throughout this section let $E = <z, t, v, w, w_1>$

1. $C_S(t) = <E, x, x_1, a, b, c, y_1, y, u>$

   $C_S(t)' = <E, x, a, bx_1, y>$

   $\Omega_1(C_S(t)') = <E, a, bx_1, y>$

   $L_3(C_S(t)) = <a, y, v, t, z>$

   $C_S(t)'' = <z, t, v>$

   $Z(C_S(t)) = <z, t>$

2. $C_S(v) = <E, x, x_1, a, b, d, y, y_1>$

   $C_S(v)' = <E, b, ax_1>$

   $\Omega_1(C_S(v)') = <E, b>$

   $\Omega_1(C_S(v)') = <z>$

   $L_3(C_S(v)) = <z, t, v>$

   $Z(C_S(v)) = <z, v>$

3. $C_S(w) = <E, x, x_1, a, c, d, u>$

   $C_S(w)' = <E, x, c>$

   $L_3(C_S(w)) = <z, t, w, w_1>$

   $\Omega_1(C_S(w)') = <E, c>$
\[ \Omega_1(c_g(w')') = <z> \]
\[ z(c_g(w)) = <z, w> \]

4. \[ c_s(w_1) = <E, x, x_1, b, c, d> \]
   \[ c_s(w_1)' = E \]
   \[ L_3(c_s(w_1)) = <z> \]
   \[ z(c_s(w_1)) = <z, w_1> \]

5. \[ c_s(a) = <z, t, v, w, a, c, y_1x, y, bx_1w_1y_1, u> \]
   \[ c_s(a)' = <z, t, v, a, y, wbxx_1w_1> \]
   \[ u'(c_s(a)') = <t> \]
   \[ z(c_s(a)) = <z, t, a> \]

6. \[ c_s(b) = <z, t, v, w_1, ax_1, b, yxw, y_1, d> \]
   \[ c_s(b)' = <z, t, v, ax_1w_1, b> \]
   \[ u'(c_s(b)') = <vz> \]
   \[ z(c_s(b)) = <z, v, b> \]

7. \[ c_s(y) = <z, t, v, a, x, w_1wx_1b, y, y_1, u> \]
   \[ c_s(y)' = <a, y, v, t, z> \]
   \[ L_3 c_s(y) = <t> \]
   \[ z(c_s(y)) = <z, t, y> \]
8. \( C_S(y_1) = \langle z, t, v, w_1x_1, w_1x, b, y_1, y \rangle \)
   \( C_S(y_1)' = \langle z, t, v \rangle \)
   \( \Omega_1 C_S(y_1) = \langle z, t, v, b, y_1, y \rangle \)
   \( (\Omega_1 C_S(y_1))' = \langle vz \rangle \)
   \( Z C_S(y_1) = \langle z, t, v, y_1 \rangle \)

9. \( C_S(c) = \langle z, t, w, w_1, a, c, d, u \rangle \)
   \( C_S(c)' = \langle z, w, c \rangle \)
   \( Z C_S(c) = \langle z, w, c \rangle \)

10. \( C_S(d) = \langle z, v, w, w_1, b, d, c \rangle \)
    \( C_S(d)' = \langle z, w_1 \rangle \)
    \( Z C_S(d) = \langle z, w_1, d \rangle \)

11. \( C_S(u) = \langle z, t, w, a, c, y, u \rangle \)
    \( C_S(u)' = \langle a, t \rangle \)
    \( Z C_S(u) = \langle z, t, a, u \rangle \)

12. \( C_S(yx_1) = C_S(y_1abx) = C_S(yx_1y_1abx) = \)
    \( \langle yx_1, y_1abx \rangle \times \langle ywy_1, bw_1y_1, abw_1 \rangle \)
    \( Z C_S(yx_1) = \langle yx_1, y_1abx, z, t, v \rangle \)
    \( C_S(yx_1)' = \langle z, t, v \rangle \)
13. If $T$ is a 2-group contained in $S$ and $|T| = 2^{11}$ and $|Z(T)| = 2^3$, then $T \simeq C_S(v,t)$ and $Z(T) = (v, t, z)$.

14. There are no 2-groups $T \subseteq S$ with $|T| = 2^9$ and $|Z(T)| \geq 2^4$.

15. Within $S$ all conjugate classes of involutions are represented by $z, t, v, w, w_1, a, b, c, d, u, y, y_1, y_1 abx, yx_1$, or $yx_1 y_1 abx$. 
The maximal subgroups of a 2-group $S$

Rudvalis type and of $C_S(t)$

1. $\delta S = \langle E, a, b, c, x, x_1, y \rangle$ and $S/\delta S = \langle d_S, u_S, y_1 \delta S \rangle$

We get 7 maximal subgroups of $S$ and check only $\langle u, y_1, \delta S \rangle = C_S(t)$ has a non-cyclic center. In this case $Z C_S(t) = \langle z, t \rangle$.

2. $\delta C_S(t) = \langle E, x, a, bx_1, y \rangle$ and $C_S(t)/\delta C_S(t) = \langle \overline{x_1}, \overline{y_1}, \overline{c}, \overline{u} \rangle$, using the bar convention. Checking those maximal subgroups of $C_S(t)$ that could be isomorphic to $C_S(v)$, we see there are exactly two possibilities.

$M_1 = \langle x_1, y_1, c, \delta C_S(t) \rangle$ and $M_2 = \langle x_1, y_1, u, \delta C_S(t) \rangle$

$M_1' = \langle E, a, bx_1 \rangle$ and $M_2' = \langle z, t, v, w, a, x, y \rangle$

$\Omega_1(M_1') = \langle E, a \rangle$ and $\Omega_1(M_2') = \langle z, t, v, w, a, y \rangle$

$[\Omega_1(M_1')]' = \langle z \rangle$ and $[\Omega_1(M_2')]' = \langle t \rangle$

$L_3 M_1 = \langle z, t, v \rangle$ and $L_3 M_2 = \langle z, t, v \rangle$
The Embedding of a 2-group of Type $^{2}F_{4}\langle 2 \rangle$

in a 2-group of Type Rudvalis

We use the generators and relations given for each section. The correspondence is as follows:

\[
\begin{align*}
\alpha_{12} & \quad z & \quad \alpha_{6} & \quad bxx_{1}w_{1} \\
\alpha_{11} & \quad t & \quad \alpha_{5} & \quad x \\
\alpha_{10}\alpha_{12} & \quad v & \quad \alpha_{4} & \quad bcw_{1} \\
\alpha_{9}\alpha_{10} & \quad w & \quad \alpha_{3} & \quad d \\
\alpha_{8} & \quad w_{1} & \quad \alpha_{1} & \quad uy_{1}yax \\
\alpha_{7} & \quad a
\end{align*}
\]

We remark this 2-group is a normal subgroup of $S$. 
Fusion-Simple Groups of Rudvalis Type

We begin with a determination of some basic fusion of involutions.

Lemma 1. In $G$ $z \sim t$ by an element of order 3.

Proof. Suppose $z \not\sim t$ in $G$. We shall show $z \in Z^*(G)$.

1. Suppose there is some $g \in G$ such that $\nu^g = z$ and $C_S(v)^G \subseteq S$. Since $C_S(v)^G = <z>$, we see $g \notin N_G(C_S(v))$. So $C_S(v)^G$ is a maximal subgroup of $C_S(t)$. Comparing centers and counting $z$-conjugates, we would force $t \sim z$. So in $G$ $v \not\sim z$.

2. Suppose $w \sim z$ in $G$. Choose a 2-group $T \supseteq C_S(w)$ such that $[T : C_S(w)] = 2$ and $Z(T) = <w, z>$. Since $|T| = 2^{12}$, we conclude $Z(T)$ is fused to $<z, t>$ or $<z, v>$, which yields a contradiction in either case.

3. Now assume $w_1 \sim z$ in $G$. Choose a 2-group $T$ of order $2^n$ such that $T \supseteq C_S(w_1)$ and $Z(T) \supsetneq <z, w_1>$. Now, if $Z(T) \simeq Z_2 \times Z_2 \times Z_2$, then $Z(T) \sim <z, t, v>$ in $G$. Counting conjugates yields a contradiction. Otherwise $T$ is fused to $C_S(w)$ or a subgroup of $C_S(t)$ or a subgroup of $C_S(v)$. Counting conjugates of $z$ in the centers yields a contradiction.

4. Note that $|C_S(a)| = 2^{10}$, $Z(C_S(a)) = <z, t, a>$ and $U'(C_S(a))' = <t>$. Suppose that $a \sim z$ in $G$. Choose a 2-group $T \supseteq C_S(a)$ such that $[T : C_S(a)] = 2$ and $Z(T) \supsetneq <a, t>$. As in Step 3 we rule out $Z(T) \simeq Z_2 \times Z_2 \times Z_2$. As in Step 2 $|Z(T)| = 4$ is impossible.

5. Suppose $y \sim z$ in $G$. Choose the 2-group $T$ in the usual way. Note $Z(T) \subseteq <z, t, y>$ since in $S$ there are no 2-groups of
order $2^9$ with centers of order $2^4$ or more. Using Steps 1-3 we rule out $<z, t, y>$ being fused to any other elementary group in $E$, so $Z(T) \neq <z, t, y>$. Since $L_2C_S(y) = <t>$, we see $Z(T) = <t, y>$. Using Steps 1-4, this is clearly impossible.

6. We see $b \not\sim z$ in $G$ is forced using Step 5's argument along with the fact $\nu'(C_S(b))' = <vz>$.

7. Suppose $y_1 \sim z$ in $G$. Choose $T$ in the usual manner. Note $[\Omega_1(C_S(y_1))]' = <vz>$, so $vz \in Z(T)$. Hence $Z(T)$ contains at least 2 conjugates of $z$, application of the standard fusion considerations rule out the fusion $y_1 \sim z$ in $G$.

8. Suppose $c \sim z$ in $G$. Choose $T$ as usual. Note $Z(T) \subseteq <z, w, c>$ with $c \in Z(T)$, since any 2-group in $S$ of order at least $2^9$ has a center of order at most $2^3$. Counting conjugates yields an introduction. A similar argument will prevent $z$ from fusing to $yx_1, y_1abx$, or $yx_1y_1abx$.

9. If $d$ or $u$ is fused to $z$, then there exists a 2-group $T$ with at least 2 conjugates of $z$ and $|T| = 2^3$. Checking fusion now shown impossible forces a contradiction.

Hence $z \in Z(G)$ if $z \not\sim t$ in $G$. So $z \sim t$ in $G$. Since $z$ is extremal there is some $g \in G$ such that $z = t^g$ and $C_S(t)^g \subseteq S$. Since $C_S(t)$ is the only maximal subgroup of $S$ with non-cyclic center, we see $g \in N_G(C_S(t))$, giving $g \in N_G(Z(C_S(t))) = N_G(z, t)$. So $g$ can be taken to have order 3 in $G$.

QED.
We examine the $N_G(<z, t>)$ next.

**Lemma 2.** In $N_G(<z, t>)$ we have $w \sim a$ and $w_1 \sim y$. Further $E \sim <a, y, v, t, z>$ in $N_G(<z, t>)$.

**Proof.** We denote by $g$ the element of order 3 found in Lemma 1. Since $C_S(t)'' = <z, t, v>$, $g$ fixes $v$. Consider now the action of $<g>$ on $L_3C_S(t)/C_S(t)$ where $C = C_S(t)''$. If $<g>$ acts trivially on this quotient, then $<g>$ must centralize some conjugate of $w$. So $<g>$ must act on $A = C_g(w) \cap C_S(t) = <E, x, x_1, a, c, u>$. Note that $\Omega_1A = <E, a, c, u>$, and $(\Omega_1A)' = <z, w>$, so that $<g>$ acts on $<z, w> \cap C = <z>$, which is not so. So $w \sim a$ in $N(<t, z>)$.

Finally note $\Omega_1(C_S(t)')/L_3C_S(t)$ is elementary of order 8. In it are exactly 3 cosets that contain involutions: $yL_3C_S(t)$, $wL_3C_S(t)$, and $ybxL_3C_S(t)$. If $<g>$ acted on $<w_1, L_3C_S(t)> = A$, then $<g>$ acts on $A' = <z>$, which is not so. Hence $y \sim w_1$ in $N_G(<t, z>)$. It is now clear $E \sim <a, y, v, t, z>$ in $N_G(<t, z>)$.

**QED.**

Before showing $z \sim t \sim v$ we need some preliminary results.

**Lemma 3.** Assume $z \sim t \sim v$ in $G$. Then $\frac{N_G(<z, t, v>)}{C_G(<z, t, v>)} \cong L_3(2)$.

**Proof.** By Sylow and Burnside fusion within $<z, t, v>$ must occur in $N_G(<z, t, v>)$, since $<z, t, v>$ is the third center in $S$. The proof of Lemma 2 shows that the element $g$ of order 3 used
there lies in \( N_G(<z, t, v>) \). Hence \( N_G(<z, t, v>) / G( <z, t, v>) \)
contains a group isomorphic to \( S_4 \).

Since \( v \sim z \) there is a 2-group \( T \) such that \( v \in Z(T) \),
\( T \cong C_S(v) \) and \([T : C_S(v)] = 2\). \( L_3 C_S(v) = <z, t, v> \) forces
\( T \subset N_G(<z, t, v>) \). No 2-group of order \( 2^{13} \) in \( S \) has a center of
order eight, so \( T \not\subset G( <z, t, v>) \). Hence our result follows.

QED.

Lemma 4. Assume \( z \sim t \sim v \) in \( G \). Then in \( G \) we have the
following fusion of involutions:

\[
\begin{align*}
z &\sim t \sim v \\
a &\sim b \sim y \sim y_1 \sim w \sim w_1 \\
c &\sim d \sim u \\
yx_1 &\sim y_1 abx \sim yx_1 y_1 abx
\end{align*}
\]

Further all this fusion occurs in an extension of \( C_S(t, v) \) by
\( L_3(2) \).

Proof. Denote the above extension by \( K \). By Lemma 3
\( N_G(<z, t, v>) / G( <z, t, v>) \cong L_3(2) \). Note a result of Burnside and
of Sylow show \( z \sim t \sim v \) in \( K \), since \( <z, t, v> \) is the third center of \( S \) and
is abelian. Since \( z \) is extremal in \( S \) there is some \( g \in G \) such that
\( z = v^g \) and \( C_S(v)^g \subseteq S \). Since \( [\Omega_1(C_S(v))'] = <z> \), \( g \notin N_G(C_S(v)) \).
Hence \( C_S(v)^g \) is a maximal subgroup of \( C_S(t) \). A check of the list of
maximal subgroups of \( C_S(t) \) shows that \( C_S(v)^g = <x_1, y_1, c, C_S(t)'> = M \).
Note \( L_3 M = <z, t, v> = L_3 C_S(v) \), so \( g \in K \).

Next \( \Omega_1(C_S(v))' = <E, b> \) must fuse to \( <a, y, w, v, t, z> \). Since
\( <a, w, v, t, z> \) char \( S \), a result of Burnside shows \( E \not\subset <a, w, v, t, z> \).
in $G$. Since $<a, y, v, t, z>$ is the only other elementary group of order $2^5$ in $<a, y, w, v, t, z>$, we conclude $E < a, y, v, t, z>$ in $K$. Comparing quotients we see $b \sim y$ in $K$. Next $w_1$ is fused to $a$ or $y$. If $w_1 \sim y$, then $w_1 \sim b$ in $K$. Since $C_S(w_1)' = E$ and $y'(C_S(b)') = <vz>$, we conclude $w_1$ is not extremal in $K$; hence $w \sim w_1 \sim y \sim b$ and $w \sim a$ by the fusion of the groups $E \sim <a, y, v, t, z>$ in $K$. In the other case $y \sim w$ and $w_1 \sim a \sim b$ is forced. In either case $w \sim w_1 \sim a \sim b \sim y$ in $K$ is obtained.

Let $R \in Syl_7(K)$. Suppose $[y_1, R] \neq 1$. Then $R$ acts on $[o_1(C_S(y_1))]' = <vz>$, which it does not. (Note, of course, $C_S(y_1) \subseteq C_S(t, v)$). Since $[C_S(t, v): C_S(y_1)] = 2^3$, some conjugate of $y_1$ must be fused to a non $S$-conjugate. Since $|Z(C_S(y_1))| = |Z(C_S(y_1 abx))| = |Z(C_S(y_1 y_1 abx))| = 32 > |Z(C_S(y_1))| = 16$, while the $S$-centralizers of $y_1, y_1 abx, y_1 x_1,$ and $yx_1 y_1 abx$ have the same order, we see $y_1$ cannot be extremal in $S$ in $K$. Hence $w \sim w_1 \sim a \sim b \sim y \sim y_1$ is obtained in $K$.

Note $S$ fuses all conjugates of $u$ in $<u, C_S(t, v)>$. Since $L^2_3(2)$ has one class of involutions, we conclude $c \sim d \sim u$ in $K$.

We consider fusion involving $y_1$. Note $C_S(y_1) \subseteq C_S(t, v)$, so there are 8 $S$-conjugates of $y_1$ in $C_S(t, v)$; hence $2^4$ $S$-conjugates of involutions of elements in $<y_1, y_1 abx>$. Since $R$ fixes no conjugate of $y_1$, we conclude $R$ fixes some conjugate of $y_1$.

(Note $C_S(y_1) = C_S(y_1 abx) = C_S(y_1 y_1 abx)$). Now choose $Q \in Syl_3(K)$ such that $Q$ normalizes $R$. Without loss $Q$ may be assumed to act
on \( \langle c, u, c_S(t, v) \rangle = A \). Suppose \( Q \) centralizes \( yx_1 \). Then \( Q \) acts on \( \langle A, yx \rangle = \langle t \rangle \). But \( Q \) acts on \( Z(A) = \langle z, t \rangle \), so \( Q \) centralizes \( \langle z, t \rangle \). Hence \( \langle <z, t, v>, Q \rangle = 1 \), a contradiction. We conclude \( yx_1 \sim y_1abx \sim y_1bx \) \( yx_1 \) in \( K \).

\[ \text{QED.} \]

**Lemma 5.** \( C_S(w_1) \) is not a Sylow 2-subgroup of \( C_G(w_1) \cap C_G(z) \).

**Proof.** Suppose the lemma is false. By Thompson's Transfer Lemma \( C_S(d) \) is not a Sylow 2-subgroup of \( C_G(d) \). Hence there is some \( \lambda \in G \) such that \( [\lambda, d] = 1 \) and \( \lambda^2 \in C_S(d) \), \( \lambda \notin C_S(d) \). Since \( C_S(d)' = \langle w_1, z \rangle \), our assumption forces \( [\lambda, z] = 1 \).

**Case 1.** Suppose \( w_1^\lambda = w_1z \). Consider \( \langle t, \lambda, C_S(d) \rangle \). Note that \( N(<w_1, d, z>)/C(<w_1, z, d>) \) is divisible by 8, since \( x, \lambda \)
and \( t \) lie in \( N(<w_1, d, z>) \). Note that the matrix representation of \( \lambda \) and \( t \) on \( <w_1, z, d> \) show they commute mod \( C_G(<w_1, d, z>) \).

Now consider \( A = \langle t, C_S(d) \rangle \). \( Z_5(A) = E \) so \( \lambda \) acts on \( E \). Since \( [\lambda, w] \neq 1 \), \( [E, \lambda] \neq 1 \). We know \( a \) acts on \( A \) and so \( \lambda a \) acts on \( A \). Since \( w_1 \) is extremal \( \lambda a \) has odd order. But \( \lambda a \) must act on the iterated centers of \( A \), and so \( \lambda a \) acts trivially on the chain \( A \supseteq E_7(A) \supseteq E_6(A) \supseteq E \supseteq Z_4(A) \supseteq Z_3(A) \supseteq Z_2(A) \supseteq <z> \supseteq 1 \).

Hence \( [\lambda a, A] = 1 \). In particular \( 1 = [\lambda a, d] = [a, d] = w \), a contradiction.
Case 2. Assume $w_1^\lambda = w_1$. Hence $\lambda \in C(w_1, z, d)$. We see $2^{10}$ divides the order of $A = N(<w_1, z, d>) \cap C(w_1) \cap C(z)$. (Note that now $x$, $t$, and $\lambda$ lie in $A$.) So $A$ is in fact a 2-group, hence is a Sylow 2-subgroup of $C_G(w_1, z)$. But in $A$ $d$ has exactly 4 conjugates while in $C_S(w_1) \setminus <w_1, z>$ an involution has 2 or 8 conjugates. Hence $N(<w_1, z, d>) \not\subset C_S(w_1)$, a contradiction.

The Lemma is proved.

QED.

Lemma 6. In $G$ $z \sim t \sim v$.

Proof. We know $z \sim t$ in $G$. It suffices to show $t \sim v$ in $G$. By Lemma 5 $w_1$ is not extremal. Let $y$ be a 2-element such that $y \not\in C_S(w_1)$ and $y^2 \in C_S(w_1)$; $[y, w_1] = 1$. Since $E = C_S(w_1)$, $y$ acts on $E$. By the structure of $S$, we see $[E, y] \neq 1$. Since $E < S$, we conclude $N(E)/N(E) \cap C(w_1) \cap C(z)$ is not a 2-group. Let $B = N(E) \cap C(w_1) \cap C(z)$. Suppose $t$ has exactly 2 conjugates within $B$. Then all elements of odd order in $B$ must centralize $t$. Suppose $\sigma$ is an element of odd order in $B$. Then $\sigma$ must centralize $<z, t, w_1>$ and hence must act non-trivially on the quotient group $E/<z, t, w_1>$. We conclude the coset $v<z, t, w_1>$ is fused to $w<z, t, w_1>$. A simple check of possible fusion between the two cosets establishes $v$ is fused to $w$. Assume $v$ is extremal in $S$ with respect to $G$. Then $L_3 C_S(w) \cong Z_2 \times Z_2 \times Z_2$ is mapped into $L_3 C_S(v) \cong Z_2 \times Z_2 \times Z_2$, which is impossible. We
conclude \( v \sim t \sim z \) in \( G \). We assume now \( t \) has more than two conjugates in \( B \).

Consider now the group \( N(E) \cap C(z) = N \). Note \( S \subseteq N \). We assume \( v \not/ t \) in \( G \). If \( t \sim w \sim w_1 \) in \( N \), then \( t \) has a total of 26 conjugates. But \( 26 \nmid |L_3(2)| \). Suppose \( t \sim w \) and \( w \sim w_1 \). Then \( t \) has exactly 10 conjugates in \( N \), showing an element of order 5 acts on \( \langle t^N \rangle = \langle z, t, v, w \rangle \), which is not so. Again we may assume \( w \not/ v \) in \( G \). Finally, if \( t \sim w_1 \) only in \( N \), we see there are 18 conjugates of \( t \) in \( N \). But the set of all \( S \)-conjugates of \( v \) is now normalized, a contradiction.

QED.

Lemma 7. Within \( G \) \( yx_1 \) is not fused to \( z \) or \( w \) or \( c \).

Proof. Suppose there is a 2-group \( T \supseteq C_S(yx_1) \) such that \( [T : C_S(yx_1)] = 2 \) and \( yx_1 \in Z(T) \). Since \( C_S(yx_1)' = \langle z, t, v \rangle \), we see \( T \subseteq N_G(<v, t, z>) \). If \( T \subseteq C_G(v, t, z) \) then in \( S \) is a 2-group of order \( 2^9 \) with center of order at least \( 2^{14} \), which is not so. We conclude \( T \not\subseteq C_G(v, t, z) \). If \( N = N_G(v, t, z) \) and \( C = C_G(<v, t, z>) \), we know by the proof of Lemma 4 that

\[
\left| \frac{C_N(yx_1)}{C_C(yx_1)} \right| = 7.
\]

Since we have shown 2 also divides the order of this quotient, the structure of \( L_3(2) \) forces \( L_3(2) \cong \frac{C_N(yx_1)}{C_C(yx_1)} \). This contradicts the fact \( 3 \mid \left| \frac{C_N(yx_1)}{C_C(yx_1)} \right| \), as shown in the proof of Lemma 4.

QED.
Collecting the important fusion results obtained so far, we have the following in \( G \):

\[
\begin{align*}
z \sim t \sim v & \quad c \sim d \sim u \\
y_{x_1} \sim y_1abx & \sim y_{x_1}y_{1abx} \\
a \sim b \sim y \sim y_1 & \sim w \sim w_1 \\
y_{x_1} \not\sim z & \quad y_{x_1} \not\sim w & \quad y_{x_1} \not\sim c
\end{align*}
\]

**Lemma 8.** \( C_G(<v, t, z>) \) has a normal 2-complement.

**Proof.** Let \( V = <v, t, z> \) and \( C = C_S(<v, t>) \). Note \( \Omega_1(C) = C \) and \( C / _V \) is elementary of order \( 2^3 \). It suffices to show \( C / _V \) has a normal 2-complement in \( C_G(v) / _V \). If the lemma were false, then by Burnside there is a subgroup \( K / _V \) that normalizes but does not centralize \( C / _V \); and \( K / _V \) has odd order.

Recall from Lemma 4 \( w \) has 168 conjugates in \( C_S(v) \). Hence \( w \) has 21 conjugates in \( N(V) / _V \) and all these conjugates occur under the action of \( N(V) / _C(V) \). Similarly \( yx_1V \) can have only 24 conjugates and all these occur under the action of \( N(V) / _C(V) \). Since \( y_{x_1} \not\sim w \) in \( G \) by Lemma 7, we conclude \( K / _V \) fixes all cosets with involutions. Since \( \Omega_1(C) = C \), we conclude \([K / _V , C / _V] = 1\). Hence \( C_G(<v, t, z>) \) has a normal 2-complement.

QED.
Lemma 9. $C_G(<t, z>)$ has a normal 2-complement.

Proof. We show $C_G(<t, z>)/<t, z>$ has a normal 2-complement. We claim $v <t, z>$ is isolated in $C_S(t)/<t, z>$. A quick computation yields $C_S(b,t)' = C_S(y_1,t)' = <z, t, v>$. Also $C_S(t,v)' = <v, t, z>$. Note that for $\theta$ equal to any of $w$, $w_1$, $a$, or $y$ we have $C_S(\theta,t)'$ has order at least 16. Since $C_S(c,t)' = <z, w>$ and $C_S(u,t)' = <a, t>$, we conclude by the extremality of $v$ in $C_G(<t, z>)$ that $v <t, z>$ is isolated in $C_G(<t, z>)/<t, z>$.

By Glauberman's Z*-theorem and Lemma 8, we conclude $C_G(<t, z>)$ has a normal 2-complement. QED.

Lemma 10. Assume we have the fusion pattern of involutions found so far. Then $G$ is an extension of $C_S(t,v)$ by $L_3(2)$.

Proof. By assumption $<z, t, v>$ is strongly closed in $S$ with respect to $G$. By Goldschmidt's theorem [3] and the structure of $S$ we conclude $G$ has a normal subgroup $N$ with a Sylow 2-subgroup $<z, t, v>$ and a normal 2-complement. Since $O(G) = 1$, $N = <v, t, z>$. By Lemma 8 the result follows. QED.

We remark such a group occurs in Rudvalis' group.

Lemma 11. If any further fusion of involutions occurs, then there are exactly two classes with representatives $z$ and $yx_1$. 
Proof. By Lemma 7 only fusion between $z$, $w$, and $c$ can now occur. Suppose $w \sim z$ in $G$. Since $z$ is extremal in $S$ there is some $g \in G$ such that $w^g = z$ and $C_S(w)^g \subseteq S$. Note $[\Omega_1(C_S(w)')]' = <z>$, so $g \notin N_G(C_S(w))$. Since $L_3(C_S(v)) = <v,t,z>$ and $L_3(C_S(w)) = <z,v,w,w_1>$, we conclude $C_S(w)^g \subseteq C_S(t)$. Since $c \in C_S(w)'$ and $\Omega_1(C_S(t)') = <E,a,y,bx_1>$, we conclude $z \sim w \sim c$ in $G$.

Suppose $w \sim c$ in $G$. Assume $w$ is extremal in $S$. Then there is some $g \in G$ such that $c^g = w$ and $C_S(c)^g \subseteq C_S(w)$. Note $C_S(c)$ is special with $C_S(c)' = <z,w,c>$. Also note $\Omega_1(C_S(w)') = <E,c>$. Since $w$ is extremal in $S$, $z^g \in <v,t,z>$. Since $\Omega_1(C_S(c)) = C_S(c)$, we are forced to conclude that for some $e \in E$, $w^g = c \cdot e$.

But $1 = [z,w] = [z,w]^g = [z^g,c]$, showing $z^g \in <z,t>$. A similar argument with $t$ shows $g \in N(<z,t>)$. By Lemma 9, $g \notin C(<t,z>)$.

By the proof of Lemma 2 we see $g \in N_G(<v,t,z>)$. But then $z = [c,v]^g = [w,\theta] = 1$ for some $\theta \in <v,t,z>$, a contradiction. Thus $w \sim c \sim z$ in $G$.

Finally suppose $c \sim z$ in $G$. Choose $g \in G$ so that $c^g = z$ and $C_S(c)^g \subseteq S$. Assume $w$ is extremal in $S$. Then $<w,z> \not< N_G(c,w,z)$ and $<z>$ char $<w,z>$ by extremality. Hence $g \notin N_G(<z,w,c>)$. Observe $C_S(c)$ is a special 2-group of order $2^g$ and $\Omega_1(C_S(c)) = C_S(c)$. By the structure of $S$, $C_S(c)^g$ must be a subgroup of $C_S(t)$, $C_S(v)$, $C_S(w)$, $C_S(a)$, or $C_S(w_1)$.
Note \([\Omega_1(C_G(w_1))]' = <z, w>\), so \(C_G(c)^G \not\subseteq C_G(w_1)\). Since \(\Omega_1(C_S(a)') = <z, t, a>\), fusion of \(C_S(c)\) into \(C_S(a)\) contradicts extremality of \(w\). If \(C_S(c)^G \subseteq \Omega_1(C_S(t, w))\), then \(<z, t, w> \sim <z, w, c>\). Counting \(z\)-conjugates contradicts extremality of \(w\). If \(<z, t, v> \sim <z, w, c>\), again counting yields a contradiction. Since one of the above must occur the result follows.

QED.

For the remainder of the paper let \(H = C_G(z)\). We investigate fusion of involutions in \(H\).

**Lemma 12.** Within \(H\) we have \(v \sim t \sim w \sim w_1 \mid a \sim b\) \(c \sim d \mid y_1 \sim y\).

**Proof.** Recall that \(v \sim t\) within \(N_G(<v, t, z>) \cap H\) by an element of order 3. In \(H\) \(t\) is extremal in \(S\), so there is some \(g \in H\) such that \(v^g = t\) and \(C_S(v)^G \subseteq C_S(t)\). Since \(Z(C_S(v)) = <v, z>\), we see \(g \notin N_G(C_S(v))\). Inspecting the lattice of subgroups of \(C_S(t)\), we find \(C_S(v)^G = <c, x, y_1, C_S(t)'> = M\). Note \(\Omega_1(M') = <E, a>\) and \(\Omega_1(C_S(v)') = <E, b>\). Note that the fifth center of \(C_S(v)\) and \(M\) is \(E\), so \(E^G = E\). Checking centers of \(<E, a>\) and \(<E, b>\) we see \(w \sim w_1\) in \(H\) and \(a \sim b\) in \(H\).

Since \(C_S(v)^G / C_S(t, v) = M / C_S(t, v)\) and since all involutions in the respective quotients are fused under \(S\), we conclude \(c \sim d\) in \(H\).

We claim \(t \sim w\) in \(H\). Indeed \(w \sim z\) in \(G\). Choose \(g \in G\)
so that \( w^g = z \) and \( C_S(w)^g \subseteq S \). Since \([\Omega_1(C_S(w))]' = <z>\), we conclude \( C_S(w)^g \subseteq C_S(t) \). The lattice of subgroups forces \( Z(C_S(w)) = <w, z> \) to be fused to \( <z, t> \). Now either \( z^g = t \) or \( z^g = t z \).

By Lemma 2 there is an element \( g_1 \) such that \( t^g = z \) and \( z^{-1} = t z \).

We conclude \( w^{-1} = z^{-1} = t z \) and \( z = z^{-1} = t \). In the latter case use the element \( g d g_1 \) instead of \( g g_1 \). We conclude \( t \sim w \) in \( H \).

We show finally \( y_1 \sim y \) in \( H \). Suppose \( <g> \) acts on \( <E, x, x_1, a, b, y_1> \). Counting conjugates we see some conjugate of \( y_1 \) is fixed by \( <g> \). Note \( C_S(y_1) \subseteq C_S(t, v) \) and \([\Omega_1(C_S(y_1))]' = <v z>\). Hence \( <g> \) centralizes \( <v, z> \), so \( g \in C_G(v, t, z) \), a contradiction. Using Lemma 7 we conclude \( y \sim y_1 \) in \( H \).

**QED.**

**Lemma 13.** Within \( H \) we have \( a \not\sim t \) and \( a \not\sim y \).

**Proof.** Suppose \( a \sim t \) in \( H \). Choose a 2-group \( T \supseteq C_S(a) \) such that \([T : C_S(a)] = 2 \) and \( Z(T) \supseteq <a, z> \). Since \( [\Omega'(C_S(a))]' = <t> \), we conclude \( Z(T) = <a, t, z> \). By the structure of \( S \)

\( <a, t, z> \sim <z, t, v> \). But \( C_S(t, v) \) is a special 2-group with \( C_S(t, v)' = <z, t, v> \), while \( |C_S(a)'| = 2^6 \). We conclude \( a \not\sim t \) in \( H \).

Now suppose \( y \sim t \) in \( H \). Recall \( L_3(C_S(y)) = <t> \). We may find a 2-group \( T \supseteq C_S(y) \) such that \( Z(T) = <z, t, v> \) and \([T : C_S(y)] = 2 \). By the above \( <z, t, y> \not\sim <z, t, a> \) in \( H \). If
<z, t, v> ~ <z, t, y>, comparison of commutator subgroup orders yields a contradiction. A similar comparison shows \(<z, t, y> \sim <z, t, w>\) in \(H\), since \(a \not\sim t\) in \(H\) by the above. Thus \(y \not\sim t\) in \(H\).

Assume \(y \sim a\) in \(H\). By the above \(a\) is extremal in \(S\) in \(H\). Choose \(g \in H\) so that \(y^g = a\) and \(C_S(y)^g \subseteq C_S(a)\). Since \(L_3(C_S(y)) = <t>\) and \(C_S(y)^g\) is a maximal subgroup of \(C_S(a)\), we conclude \(t^g \in Z(C_S(a)) = <a, t, z>\). Since \(a \not\sim t\) in \(H\), we conclude \(g \in N_H(<t, z>)\). By Lemma 9 \(C_G(<t, z>)\) has a normal 2-complement. If \(t^g = tz\), then \(g = \zeta d\) for \(\zeta \in C_G(<t, z>)\).

Using \(gd = \zeta\), if necessary, we may assume \(g \in C_G(<t, z>)\). Note that comparison of commutator subgroups shows \(A = <a, y, v, t, z>\) is normalized by \(<g>\). But by a result of Frobenius we have \(N_C(A) / C_G(A)\) is a 2-group; here \(C = C_G(<t, z>)\). We conclude \(a \not\sim y\) in \(H\).

Assume \(u \sim y\) in \(H\). By the above \(y\) is extremal in \(S\) with respect to \(H\). Choose \(g \in H\) so that \(u^g = y\) and \(C_S(u)^g \subseteq C_S(y)\). Note \(C_S(u)' = <a, t>\) and \([\Omega_1(C_S(y))]' = <t, y>\). We conclude \(a \sim t\) or \(a \sim y\) in \(H\), both shown impossible. Hence \(u \not\sim y\) in \(H\).

Suppose \(u \sim a\) in \(H\). Choose in \(H\) a 2-group \(T \supseteq C_S(u)\) such that \([T : C_S(u)] = 2\) and \(u \in Z(T)\). Since \(a \not\sim t\) in \(H\) and \(C_S(u)' = <a, t>\), we conclude \(<z, t, u \subseteq Z(T)\), so \(T \subseteq C_G(t, z)\). By Lemma 9 \(C_G(t, z)\) has a normal 2-complement, so fusion involving \(u\) must already occur in \(C_S(t)\). We conclude \(u \not\sim a\) in \(H\). The same
argument also shows \( u \not\sim t \) in \( H \).

**Lemma 14.** In \( H \) \( a \sim b \sim c \sim d \).

**Proof.** We have \( w \sim t \) in \( H \) by Lemma 12. By the extremality of \( t \) we conclude \( C_S(w)' \) is fused into \( C_S(t)' \). By Lemma 13 this fusion normalizes \( E \). Suppose that this fusion sends \( c \) to \( y \). By Lemma 13 \( y \) is extremal, so \( C_S(c) = \Omega_1(C_S(c)) \) is sent into a group of order \( 2^7 \), which is impossible (\( |C_S(c)| = 2^8 \)). Since \( c \) must fuse to either \( y \) or something conjugate to \( a \) we conclude \( a \sim b \sim c \sim d \) in \( H \).

**QED.**

**Lemma 15.** \( E \cdot O(H) \triangleleft H \).

**Proof.** By Lemma 13 \( E \) is strongly closed in \( S \) with respect to \( H \). By the structure of \( S \) and Goldschmidt's theorem [3] we conclude \( E \cdot O(H) \triangleleft H \).

**QED.**

Let \( \overline{H} = H / O(H) \) and apply the bar convention.

**Lemma 16.** \( \langle \overline{x}, \overline{x}_1, \overline{E} \rangle \triangleleft \overline{H} \).

**Proof.** By Lemma 15 \( E \cdot O(H) \triangleleft H \). By a Frattini argument \( H = N_H(E) \cdot O(H) \supset C_H(E) \cdot O(H) \). By Lemma 9 \( C_H(E) \) has a normal 2-complement, so \( C_H(E) \cdot O(H) = \langle x, x_1, E \rangle \cdot O(H) \). The result follows.

**QED.**
Lemma 17. \( \overline{J} \triangleleft \overline{H} \).

Proof. \( \overline{H} / C_{\overline{H}}(\overline{E}) \) is a subgroup of \( GL(5,2) \). By the structure of \( S / C_{\overline{H}}(\overline{E}) \) we conclude \( \overline{J} \triangleleft \overline{H} \).

QED.

Lemma 18. \( O(H) = O(C_G(yx_1)) = 1 \).

Proof. Since the 2-rank of \( G \geq 5 \), \( G \) is connected. By Lemmas 16 and 17 we conclude \( \overline{H} \) is 2-constrained and 2-generated. (See [5]).

If \( C_G(yx_1) \) is solvable, application of Theorem A of [5] yields the desired conclusion. Assume \( C_G(yx_1) = K \) is non-solvable. By Lemma 7 we have \( z \notin yx_1 \) in \( G \). By this known fusion pattern of involutions we see also \( C_S(yx_1) \) is a Sylow 2-subgroup of \( C_G(yx_1) \). By Burnside's theorem we conclude \( <yx_1, y_1abx> \subseteq Z^*(K) \). Note \( C_S(yx_1)/<yx_1, y_1abx> \) is a Suzuki 2-group of order \( 64 \).

Since \( K \) is assumed non-solvable, we conclude by a result of Collins [0] \( K/<yx_1, y_1abx, O(K)> \cong S_2(8) \). Note that this quotient has a trivial core, since this core would centralize \( \overline{yx_1}, \overline{y_1abx} \) where the bars refer to \( \overline{K} = K/O(K) \). Hence \( \overline{K} \) is 2-generated. By Theorem C of [5] our result follows.

QED.

Lemma 19. \( J = O_2(H) \) and \( H/J \cong S_5 \).

Proof. By Lemma 18 \( O(H) = 1 \), so \( J \subseteq O_2(H) \). By Lemma 14
a ~ b ~ c ~ d in H, so $H/J$ properly contains a group isomorphic to $S_4$. $H/J$ must contain a subgroup of index 2 since $u$ is not fused to any element in $<E, x, x_1, y, y_1, a, b, c, d>$. By the structure of $GL(5,2)$ we conclude $H/J \cong S_5$. Our result follows.

QED.

**Lemma 20.** If $P \in Syl_3(H)$ then $C_J(P)$ is a quaternion group of order 8.

**Proof.** Say $<p> = P$. Without loss $P$ maybe chosen so that $y^P = y$. Note $C_{<E, x, x_1>} (y) = <z, t, v, x>$. Also $C_{<E, x, x_1>} (y_1) = <z, t, v, w_1 x>$. We conclude $P$ acts on $<E, x>$. Hence $P$ centralizes $<E, x, x_1>_{E}$; thus $P$ must centralize a quaternion group of order 8. Since $a ~ b ~ c ~ d$ in H by Lemma 11, Burnside's theorem and the structure of $H/J$ show $P$ fixes no element conjugate to $a$ in H. Similarly $t ~ v ~ w ~ w_1$ in H shows $P$ fixes no conjugate of $t$ in H. Our result follows.

QED.

**Lemma 21.** $G$ is isomorphic to Rudvalis' simple group.

**Proof.** Apply a result of Parrott [9].

QED.
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