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Mathematics

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SIZE TWO GENERATORS AND GROUPS OF TRANSFORMATIONS

WITHOUT A FINITE INVARIANT MEASURE

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Amy Jean Kuntz, A.B., M.S.

****

The Ohio State University

1974

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ACKNOWLEDGEMENT

I would like to express my gratitude to my advisers, Professors Ulrich Krengel, Wolfgang Krieger and Louis Sucheston for their assistance.

Dr. Krieger and I have had many helpful discussions and his suggestions have been most valuable.

Dr. Krengel's direction and encouragement, and his critical reading of the results has been invaluable to me in producing this thesis.

Finally, I would like to thank Harry, Danny and Joshua Rosenberg for their patience and encouragement in this endeavor.
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146, February, 1974.

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Major Field: Mathematics

Studies in Ergodic Theory. Professors Ulrich Krengel, Wolfgang
Krieger and Louis Sucheston
Studies in Probability Theory. Professors Ulrich Krengel, Wolfgang Krieger and Louis Sucheston

Studies in Algebra. Professors Arnold Ross and Alan Woods

Studies in Analysis. Professor Bajanski
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INTRODUCTION

This paper contains results concerning generators and groups of transformations of a $\sigma$-finite measure space. This introduction contains basic definitions, to be found for example in Halmos [7], and some interpretations concerning generators which are found in Krengel [13] and a comparison of the results proved in this paper with those of Krengel which were known before.

A measurable space $(X, \mathcal{F})$ is a set $X$ and a class $\mathcal{F}$ of subsets of $X$, called a $\sigma$-algebra satisfying: i) $X \in \mathcal{F}$, ii) $F \in \mathcal{F}$ implies $F^c \in \mathcal{F}$, iii) $F_1 \in \mathcal{F}$ implies $\bigcup_{i=1}^{n} F_i \in \mathcal{F}$. If iii') $F_i \in \mathcal{F}$ implies $\bigcup_{i=1}^{n} F_i \in \mathcal{F}$ is replaced by iii''), $F_i \in \mathcal{F}$ implies $\bigcup_{i=1}^{n} F_i \in \mathcal{F}$ then $\mathcal{F}$ is an algebra. A measurable space $(X, \mathcal{F}, \mu)$ is a measurable space $(X, \mathcal{F})$ and a measure $\mu$, a nonnegative extended real valued function on $\mathcal{F}$ such that i) $\mu(\emptyset) = 0$, ii) $\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)$ where $E_i \in \mathcal{F}$ and $E_i$ are pairwise disjoint.

If iii') $\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)$, $E_i \in \mathcal{F}$, and $E_i$ disjoint is substituted for ii), then $\mu$ is called a finitely additive measure, or a charge. If $\mu$ and $\nu$ are two measures on a measurable space $(X, \mathcal{F})$ then $\mu$ is absolutely continuous with respect to $\nu$ (denoted $\mu \ll \nu$) if $F \in \mathcal{F}$ and $\nu(F) = 0$ implies $\mu(F) = 0$. The two measures $\mu$ and $\nu$ are equivalent if $\mu \ll \nu$ and $\nu \ll \mu$. All measure spaces will be assumed to be at most $\sigma$-finite; i.e., that there exists $F_i \in \mathcal{F}$ such that $\mu(F_i) < \infty$ and $\bigcup_{i=1}^{\infty} F_i = X$. 

1
A measurable transformation $s$ is a mapping of a measurable space $(X, \mathcal{F})$ into a measurable space $(Y, \mathcal{G})$ such that if $G \in \mathcal{F}$ then $s^{-1} G \in \mathcal{F}$. A transformation $s$ is nonsingular if $s$ is a measurable transformation of $(X, \mathcal{F}, m)$ into $(Y, \mathcal{G}, n)$ and if $n(G) = 0$ then $m(s^{-1}G) = 0$. If $s$ is a one-to-one, onto transformation of $X$ to $Y$ for which $s$ and $s^{-1}$ are both measurable then $s$ is invertible. If, in addition to being invertible, $s$ and $s^{-1}$ are nonsingular then $s$ is nonsingular invertible. A measurable transformation $s$ of $X$ into $Y$ is measure preserving if $m(s^{-1}G) = n(G)$ for all $G \in \mathcal{F}$. If $s$ is measure preserving then clearly it is nonsingular, and if $s$ is also invertible then $F \in \mathcal{F}$ implies that $m(F) = m(s^{-1}sF) = n(sF)$; i.e., $s^{-1}$ is also measure preserving.

A measurable function $f$ on $(X, \mathcal{F}, m)$ is a mapping of $X$ into the real numbers such that $f^{-1}(B) \in \mathcal{F}$ for every Borel set $B$. $L_p(X)$ is the set of equivalence classes of real valued measurable functions such that $\left(\int_X |f|^p \, dm\right)^{1/p} = \|f\|_p < \infty$. $L_p(X)$ is a Banach space with this norm and $L_2(X)$ is a Hilbert space with inner product of $f$ and $g$ equal to $\int_X fg \, dm$.

The notation $E^c$ stands for the complement of $E$ and the notation $E\setminus F$ stands for $E \cap F^c$. Denote $(E \cap F^c) \cup (E^c \cap F)$, the symmetric difference of $E$ and $F$, by $E \Delta F$. $E = F(m)$ if $m(E \Delta F) = 0$.

A measure algebra $(\mathcal{F}^*, m^*)$ of a measure space $(X, \mathcal{F}, m)$ is a set $\mathcal{F}^*$ of equivalence classes of sets in $\mathcal{F}$ and a nonnegative extended real valued function $m^*$ defined on $\mathcal{F}^*$. The elements $E$ and $F$ of $\mathcal{F}$ are in the same equivalence class if and only if $E = F(m)$, and
$*$ is defined on $E^* \in \mathcal{E}^*$ by $m^*(E^*) = m(E)$ for $E \in E^*$. Note that $m^*$ is well defined since for any $E_1 \in E^*$, $m(E \triangle E_1) = 0$ and hence $m(E_1) = m^*(E^*) = m(E)$. Define $\bigcup_{i=1}^\infty E_i^* = \left( \bigcup_{i=1}^\infty E_i \right)^*$ where $E_i \in E_i^*$, and define $(E^*)^C = (E^C)^*$ where $E \in E^*$. A homomorphism $T^*$ of a measure algebra $(\mathcal{E}^*, m^*)$ into a measure algebra $(\mathcal{E}^*, m^*)$ is a mapping of $\mathcal{E}^*$ into $\mathcal{E}^*$ such that $T^*(\bigcup_{i=1}^\infty E_i^*) = \bigcup_{i=1}^\infty T^*(E_i^*)^C$, $T^*(H^*)^C = (T^*H^*)^C$ and $m^*(T^*H^*) = m^*(H^*)$. If $T^*$ is also one-to-one and onto then $T^*$ is an isomorphism. Two measure spaces are isomorphic if there is an isomorphism of their associated measure algebras.

Every measure preserving transformation $t$ of $(X, \mathcal{J}, m)$ induces a homomorphism $T^*$ of $(\mathcal{E}^*, m^*)$ into $(\mathcal{E}^*, m^*)$ in the following way: $T^*F^* = (t^{-1}F)^*$ for $F \in F^* \in \mathcal{E}^*$. $T^*$ is well defined and one-to-one since $F_1 = F_2(m)$ is equivalent to $t^{-1}F_1 = t^{-1}F_2(n)$ for all $F_1, F_2 \in \mathcal{J}$. Also $m^*(T^*F^*) = m(t^{-1}F) = m(F) = m^*(F^*)$, and $T^*(\bigcup_{i=1}^\infty F_i^*) = T^*(\bigcup_{i=1}^\infty t^{-1}F_i) = (\bigcup_{i=1}^\infty t^{-1}F_i)^* = (\bigcup_{i=1}^\infty T^*(F_i^*)^C)$, and $T^*(F^*)^C = (t^{-1}F^*)^C = (t^{-1}F)^* = (T^*(F^*)^C)$. So the homomorphism $T^*$ induced by a measure preserving transformation $t$ is an isomorphism if and only if $T^*$ is onto; i.e., if for all $G \in \mathcal{J}$ there exists an $F \in \mathcal{J}$ such that $G = t^{-1}F(m)$. Two $\sigma$-algebras $\mathcal{J}$ and $\mathcal{J}$ on $X$ are said to be equal mod $m$, denoted $\mathcal{J} = \mathcal{J}(m)$, if $(X, \mathcal{J}, m)$ and $(X, \mathcal{J}, m)$ are isomorphic under the homomorphism induced by the identity transformation on $X$; that is, if $\mathcal{J} = \mathcal{J}$. The metric space associated with $(X, \mathcal{J}, m)$ is the set of elements in the measure algebra $(\mathcal{E}^*, m^*)$ such that $m^*(E^*)$ is finite, and a distance function $d(E^*, F^*) = m^*(E \Delta F^*)$ for $E \in E^*$ and...
F ∈ F*. A metric space is separable if and only if it has a countable dense subset, and a measure space is separable if its associated metric space is separable. Note that if $\mathcal{J}$ is a sub-$\sigma$-algebra of $\mathcal{J}$ and if $\mathcal{J}$ is separable then so is $\mathcal{J}$. The metric space associated with a $\sigma$-algebra is complete. Call a set of elements $\{F_a : F_a \in \mathcal{J}, a \in A\}$ a generating set for a $\sigma$-algebra $\mathcal{J}$ if the smallest $\sigma$-algebra which contains $\{F_a : a \in A\}$, $\sigma\{F_a : a \in A\}$, is equal to $\mathcal{J}(m)$. 

$\{F_a : a \in A\}$ is a $S$-generating set if $\{s^{-1}F_a : s \in S, a \in A\}$ is a generating set. A size-$c(I)$-$S$-generator for $(X, \mathcal{J}, m)$ is a partition of $X$, $\{A_i : i \in I\}$, which is an $S$-generating set for $\mathcal{J}$ where $c(I)$ is the cardinality of $I$. A sub-$\sigma$-algebra $\mathcal{J}$ of $\mathcal{J}$ will be called an $S$-exhaustive sub-$\sigma$-algebra of $\mathcal{J}$ if $s^{-1} \in S^{-1}$ is $\mathcal{J}$-measurable and if $\sigma\{s^{-1}F : s \in S, F \in \mathcal{J}\} = \mathcal{J}(m)$. In this terminology then a strong generator, (as used by Krengel in [11], [12] and [13]), for a nonsingular invertible transformation $T$ becomes an $S$-generator with $S = \{T^{-1} : i \geq 0\}$, and an increasing exhaustive sub-$\sigma$-algebra of $\mathcal{J}$ becomes an $S^{-1}$ exhaustive sub-$\sigma$-algebra for the above $S$.

Let $\{A_i : i \in I\}$ be a $\mathcal{J}$-measurable partition of $X$ where $I$ is at most countable. Let $S$ be a semigroup of measurable nonsingular transformations on $(X, \mathcal{J})$. Let $I^S = \{f : f \text{ is a function from } S \text{ to } I\}$. Define a shift function $T_s$ of $I^S$ into itself by $T_s f(t) = f(ts)$ where $s, t \in S$. Notice that $T_{rs} f(t) = f(trs) = (T_s f)(tr) = T_T f(t)$.

Define $q$ from $X$ into $I^S$ by $q(x) = f_x \in I^S$ where $f_x(s) = 1$ if and only if $sx \in A_i$. $T_s q(x)(t) = T_s f_x(t) = f_x(ts) = q(sx)(t)$. So
the following diagram commutes:

\[
\begin{array}{ccc}
X & s & X \\
\downarrow x & \downarrow & \downarrow sx \\
\downarrow q(x) & \downarrow & \downarrow q(sx) \\
I^S & \longrightarrow & I^S
\end{array}
\]

That is, \( q \circ s = T_s \circ q \).

Define the \( s \)th coordinate function \( X_s: I^S \to I \) by \( X_s f = f(s) \). I will call \( f(s) \) the \( s \)th coordinate of \( f \). Note that \( X_s(qx) = i \) if and only if \( x \in s^{-1}A_i \), and that \( \{ f: X_s f = i \} = X_s^{-1}[i] \).

Hence \( q^{-1}(X_s^{-1}[i]) = s^{-1}A_i \). Let \( \mathcal{J}_S \) be the product \( \sigma \)-algebra on \( I^S \). \( \mathcal{J}_S \) is generated by \( \{ X_s^{-1}[i] : s \in S, i \in I \} \). \( T_s^{-1}(X_t^{-1}[i]) = \{ g: (T_s g)(t) = i \} = \{ g: g(ts) = i \} = X_t^{-1}(i) \). \( S \) is a semigroup so \( T_s \) is measurable. For a subset \( R \subset S \) define \( \mathcal{J}_R \) as the \( \sigma \)-algebra in \( I^S \) generated by \( \{ X_r^{-1}[i] : r \in R, i \in I \} \). Then \( \mathcal{J}_R \subset \mathcal{J}_S \). If \( R_1 \subset R_2 \subset S \) then \( \mathcal{J}_{R_1} \subset \mathcal{J}_{R_2} \).
Note that if h is a map of X into Y and if \( \mathcal{A} \) is a system of subsets of Y where \( \sigma(\mathcal{A}) \) is the smallest \( \sigma \)-algebra containing \( \mathcal{A} \), then \( h^{-1}(\sigma(\mathcal{A})) \) is a \( \sigma \)-algebra containing \( h^{-1}\mathcal{A} \) so that \( h^{-1}(\sigma(\mathcal{A})) > \sigma(h^{-1}\mathcal{A}) \). Also, \( \sigma(h^{-1}\mathcal{A}) > \{ h^{-1}B : B \subset Y \text{ and } h^{-1}B \in \sigma(h^{-1}\mathcal{A}) \} = h^{-1}\{ B : B \subset Y \text{ and } h^{-1}B \in \sigma(h^{-1}\mathcal{A}) \} > h^{-1}(\sigma(\mathcal{A})) \). The last containment holds because \( \{ B : h^{-1}B \in \sigma(h^{-1}\mathcal{A}) \} \) is a \( \sigma \)-algebra containing \( \mathcal{A} \), hence \( \{ B : h^{-1}B \in \sigma(h^{-1}\mathcal{A}) \} \subset \sigma(\mathcal{A}) \) and \( h^{-1}\{ B : h^{-1}B \in \sigma(h^{-1}\mathcal{A}) \} \supset h^{-1}(\sigma(\mathcal{A})) \). Thus if h is a map of \((X, \mathcal{F})\) into \((X, \mathcal{J})\) such that \( \sigma(\mathcal{A}) = \mathcal{J} \) and \( h^{-1}\mathcal{A} \subset \mathcal{J} \), then \( h^{-1}\mathcal{J} = h^{-1}(\sigma(\mathcal{A})) = \sigma(h^{-1}\mathcal{A}) \subset \mathcal{J} \); i.e., h is a measurable map of \((X, \mathcal{F})\) into \((X, \mathcal{J})\). Thus \( X_{\mathcal{J}} \), as defined above, is a measurable map of \((I^S, \mathcal{J}_S)\) into \((I, \sigma[\{ i : i \in I \}] \).

Next I shall show that q is a measurable measure preserving transformation of \((X, \mathcal{J}, m)\) into \((I^S, \mathcal{J}_R, m_R)\) where \( R \) is a subset of S and \( m_R(B) = m(q^{-1}B) \) for \( B \in \mathcal{J}_R \). To see that q is measurable it is sufficient to observe that \( q^{-1}\{ r^{-1}\{ i \} \} = \tau^{-1}A_i \) so that \( q^{-1}\mathcal{J}_R = \sigma[q^{-1}(r^{-1}\{ i \}) : r \in R, i \in I] \subset \mathcal{J} \). Now \( m_R \) defined above is a measure on \( \mathcal{J}_R \).
Next I shall show that $Q^*$, induced by $q$, is an isomorphism of $(\mathcal{E}_R, m_R)$ onto $(\mathcal{E}, m^*)$ if and only if $\{A_i : i \in I\}$ is an $R$-generator for $\mathcal{E}$. $Q^*B^* = (q^{-1}B)^*$ for $B \in B^* \in \mathcal{E}_R$ so $Q^*$ is a homomorphism of the measure algebra $(\mathcal{E}_R, m_R)$ into the measure algebra $(\mathcal{E}, m^*)$. Hence,

$$(\sigma[q^{-1}A_i : r \in R, i \in I])^* = (\sigma[q^{-1}(x^{-1}_r(i)) : r \in R, i \in I])^*$$

$$= (q^{-1}\sigma[x^{-1}_r(i) : r \in R, i \in I])^*$$

$$= Q^*E_R^*.$$ 

Now $Q^*$ is an isomorphism if and only if $Q^*$ is an onto map; i.e., $Q^*E_R^* = \mathcal{E}$. Hence $\{A_i : i \in I\}$ is an $R$-generator for $\mathcal{E}$ if and only if $Q^*$ is an isomorphism. Note that $\mathcal{E} \supseteq Q_*^{\mathcal{E}^*} \supseteq Q_*^{\mathcal{E}_R}$. Thus, if $\{A_i : i \in I\}$ is an $R$-generator then $\mathcal{E}^* = \mathcal{E}_R^*$. Now $\mathcal{E}_B$ is the smallest $\sigma$-algebra such that $\{x_s : s \in S\}$ are measurable, and $\mathcal{E}_R$ is the smallest $\sigma$-algebra for which $\{x_r : r \in R\}$ are measurable.

If $\{A_i : i \in I\}$ is an $R$-generator then $\mathcal{E}_S = \mathcal{E}_R(m)$; i.e., $\{x_s : s \in S\}$ are functions of $\{x_r : r \in R\}$. In particular, let $S$ be the cyclic group $\{T^j : j \text{ is an integer}\}$ where $T$ is a nonsingular invertible transformation and let $R = \{T^{-j} : j \geq 0\}$. If $\{A_i : i \in I\}$ is an $R$-generator then $\{x_{T^{-j}} : j \geq 1\}$ are functions of $\{x_{T^{-j}} : j \geq 0\}$; i.e., the process is deterministic. Note that in this case where $R$ is as above, $\{A_i : i \in I\}$ is an $R$-generator if and only if $\{A_i : i \in I\}$ is a strong generator.

The construction of $q$ may be interpreted as a coding procedure in the following way. To any $x \in X$ we assign $i$ as the $s$-th coordinate
of qx if \( x \in s^{-1}A_1 \). Now the measure space \((q(x), q(x) \cap \mathcal{E}_B, m_B)\) may be considered as a subspace of the product space \((\mathcal{E}', \mathcal{E}_B, m_B)\) with \(\{T_s : s \in S\}\) as a semigroup of shifts on \(\mathcal{E}'\), a representation of the semigroup action of \(S\). Note that the measure \(m_B\) is not in general a product measure.

Suppose that \(X\) is a product space \(J^{S'} = \{g : g\) is a mapping of \(S'\) into \(J\}\) where \(S'\) is a semigroup. Let \(T_s\) be a shift transformation of \(J^{S'}\) into \(J^{S'}\); i.e., \((T_s g)(t) = g(ts)\). Let \(Y_s\) be the \(s\)th coordinate mapping of \(J^{S'}\) into \(J\); i.e., \(Y_s g = g(s)\). Let \(\mathcal{J}\) be the product \(\sigma\)-algebra on \(J^{S'}\); i.e., the smallest \(\sigma\)-algebra containing \(\{Y_s^{-1}[j] : s \in S', j \in J\}\). Suppose that \(\{A_i : i \in I\}\) is an \(\mathcal{J}\)-measurable partition of \(J^{S'}\) where \(\mathcal{J}\) is generated by \(\{X_r^{-1}[j] : r \in R', j \in J\}\). Note that every \(A \in \mathcal{J}\) has the property that if \(g \in A\), if \(g \in J^{S'}\) and if \(Y_r g = Y_r g'\) for all \(r \in R'\) then \(g' \in A\) because \(\{A : g \in A\) and \(Y_r g = Y_r g'\) for all \(r\) implies that \(g \in A\)\) is a \(\sigma\)-algebra containing \(\{X_r^{-1}[j] : r \in R', j \in J\}\). The \(A_i\) to which \(g \in J^{S'}\) belongs is determined by the \(R\)th coordinates of \(g\). If there exists a set \(F\) such that \(S' = R'F^{-1}\) then \(\mathcal{J}\) is a \(\{T_p : p \in F\}^{-1}\)-exhaustive sub-\(\sigma\)-algebra of \(\mathcal{J}\). I shall show in chapter 1 that if \(S\) is left amenable and \(s \in S\) is invertible then the smallest group containing \(S\) is \(SS^{-1} \cup S \cup S^{-1}\). Now in the notation developed on p. 4, if \(X\) is \(\{g : g \in J^{S'}\}\), \(\mathcal{J}\) is the product \(\sigma\)-algebra on \(J^{S'}\) and \(S\) is \(\{T_s : T_s\) is a shift on \(J^{S'}\)\) then \(X_{T_s}(g) = X_{T_s}(fg) = f_g(T_s)\) where \(f_g(T_s) = i\) if and only if \(T_s g \in A_1\); i.e., if and only if \(g \in T_s^{-1}A_1\). If \(A_1 \in \mathcal{J}\) then the \(i \in I\).
such that $T_s g \in A_1$, is determined by the values $g(r s)$ of $g$ for $r \in R$; i.e., the $T_s^{th}$ coordinate of $q g$ is determined by the $R_s^{th}$ coordinates of $g \in J^S$. If $S = Z = \text{integers under addition}$ and $R = \text{nonnegative integers}$ then $S = R R^{-1}$ and $\{X_i = j\}_i \geq 0$ is a $[T_i: i \geq 0]^{-1}$-exhaustive sub-$\sigma$-algebra of $I^Z$. If $A_i \in \{X_i = j\}_i \geq 0$ then the $k^{th}$ coordinate of $q f$ is determined by the $k, k + 1, k + 2, \ldots$ coordinates of $f$.

As shall be seen in chapter 1, if $S$ is a left amenable semigroup which contains the identity then $G = SS^{-1}$. Cyclic semigroups and abelian semigroups are some examples of left amenable semigroups.

Let $T$ be a nonsingular invertible transformation on a separable measure space $(X, S, m)$. In [12], Krengel has shown that if $m(x) = 1$ and if there does not exist a $T$-invariant finite measure $p$ with $p \ll m$, then the system of sets $A$ for which $\{A, A^c\}$ is a strong generator is dense in every exhaustive sub-$\sigma$-algebra $J$ of $S$. Jones and Krengel([10], theorem 3.5) note that these sets may be constructed with orbits dense in $S$. Krengel ([11],[13]) showed also that if $T$ is conservative and measure preserving, if $m$ is a $\sigma$-finite infinite measure on $S$, if $J$ is an exhaustive sub-$\sigma$-algebra of $S$ such that $m$ restricted to $J$ is $\sigma$-finite, and if there does not exist any $J$-measurable $T$-invariant subsets of finite measure then the system of sets $A \in J$ with $m(A) < \infty$ which are strong generators is dense in the system of $J$-measurable sets of finite measure.
In the case that \( m \) is a finite invariant measure for \( T \), Krieger [15] has shown that if \( T \) is ergodic and has finite entropy \( h(T) \), and if \( J \) is an exhaustive sub-\( \sigma \)-algebra of \( J \), then there exists an \( J \)-measurable finite partition \( \{A_1, \ldots, A_k\} \) which is a generator for \( J \), where \( e^{h(T)} \leq k \leq e^{h(T)} + 1 \). If \( G \) is an ergodic freely acting hyperfinite group with a certain property C specified in [16], whose entropy \( h(G) \), is finite Krieger [16] showed that \( J \) has a generator \( \{A_1, \ldots, A_k\} \) in \( J \) such that \( k \leq e^{h(G)} + 1 \).

The results in this paper are for groups of measurable, nonsingular transformations on a separable measure space where there exists no finite invariant measure. They are analogous to the ones cited due to Krengel. Let \((X, J, \mu)\) be a separable measure space with \( \mu(X) = 1 \). Let \( S \) be a semigroup of invertible nonsingular transformations on \( X \). Let \( J \) be an \( S^{-1} \)-exhaustive sub-\( \sigma \)-algebra of \( J \) and assume that all \( s \) in \( S \) are \( J \)-measurable. If \( S \) is either left amenable or \( J \)-invertible, and if there does not exist any finite \( S \)-invariant measure \( \mu' \) on \( J \) with \( \mu' \ll \mu \), then the system of sets \( A \in J \) such that \( \{A, A^c\} \) is a size two \( S^{-1} \)-generator for \( J \) is dense in \( J \). These sets \( A \) may be constructed so that they have dense \( S^{-1} \)-orbits in \( J \). Note that the condition that \( S \) is invertible with respect to \( J \) is equivalent to \( J = J \). Krengel's first result is contained in the above since \( \{T_i^{-1}: i \geq 0\} \) is amenable.

For the second result assume that \((X, J, m)\) is a \( \sigma \)-finite
infinite separable measure space and that $X = \bigcup_{i=1}^{n} Z_i$ where the $Z_i$ are $S$-invariant, contain no nontrivial subinvariant sets in $\mathcal{J}$, have infinite measure and belong to $\mathcal{J}$, a $\sigma$-finite $S^{-1}$-exhaustive sub-$\sigma$-algebra of $\mathcal{J}$. As in the first result, $S$ is a semigroup of $\mathcal{J}$-invertible, $\mathcal{J}$-measurable transformations which preserve the measure $m$. Then the system of sets $A \cap \bigcup \{ Z_i : Z_i \text{ is nonatomic} \} \in \mathcal{J}$, $A \in \mathcal{J}$, such that $\{ A, A^c \}$ is an $S^{-1}$-generator for $\mathcal{J}$, is dense in the sets of finite measure in $\bigcup \{ Z_i : Z_i \text{ is nonatomic} \} \cap \mathcal{J}$. This is not an extension of Krengel's second result since his result allows for any number of infinite invariant sets; however, in the case that $T$ is conservative and ergodic on $X$, Krengel's result is contained in the above.

Lemmas, corollaries and theorems are numbered $x.y$ where $x$ indicates the chapter they appear in, and $y$ indicates the number of the lemma, corollary or theorem within the chapter. I have numbered certain statements consecutively within each chapter. In referring to a lemma, corollary or theorem I shall use lemma, theorem or corollary $x.y$ even when in chapter $x$. All references to numbered statements are within the same chapter and only the number of the statement is used.
I. WEAKLY WANDERING SETS

$S$ is a set of measurable nonsingular transformations on a measure space $(X, \mathcal{F}, p)$ with $p(X) = 1$. A set of positive measure, $W$, is called an $S$-weakly wandering set if there exists a sequence $(s_j)_{j=0}^\infty$ contained in $S$ such that $s_i^{-1}W \cap s_j^{-1}W = \emptyset$ for all non-negative integers $i \neq j$ where $s_0^{-1}W$ is defined to be $W$. I will call $(s_j)_{j=1}^\infty$ a weakly wandering sequence for $W$ and $(s_j^{-1}W)_{j=0}^\infty$ a weakly wandering sequence.

In this section I shall discuss the relationships between weakly wandering sets and the existence of a finite invariant measure. First I shall review a result from functional analysis that a convex set in a uniformly convex space has a unique element of smallest norm. Next, in proposition 1.1, I shall extend a theorem of Hajian and Ito [5] showing that the existence of $S^{-1}$-weakly wandering sets of measure arbitrarily close to 1 is equivalent to the nonexistence of a finite invariant measure which is absolutely continuous with respect to $p$. Following that, I prove related results for left amenable semigroups using methods of Krengel [13] and results due to Sucheston [17] and Granirer [4]; that is, for all $\varepsilon > 0$ there exists an $X' \in \mathcal{F}$ such that $p(X')$ is greater than $1 - \varepsilon$ and $\inf(p(sX')): s \in S = 0$ if
and only if there does not exist a finite invariant measure \( p_0 \ll p \) on \( \mathcal{J} \) where \( s \in \mathcal{S} \) is invertible on the \( \sigma \)-algebra \( \mathcal{J} \supset \mathcal{F} \). Furthermore, if there is no finite invariant measure on \( \mathcal{F} \), then for \( \varepsilon > 0 \) there exists an \( \mathcal{F} \)-weakly wandering set \( W \in \mathcal{F} \) such that \( p(W) > 1 - \varepsilon \).

A set \( C \) is called convex if for all \( 0 \leq t \leq 1 \) and for all \( c, d \in C \), \( tc + (1 - t)d \in C \). A Banach space \( X \) is uniformly convex if for all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for all \( x, y \in X \) with \( ||x|| \leq 1 \) and \( ||y|| \leq 1 \), \( ||x + y|| > 2 - \delta \) implies that \( ||x - y|| < \varepsilon \). Since in a Hilbert space, \( ||x + y||_2^2 + ||x - y||_2^2 = 2||x||_2^2 + 2||y||_2^2 \), it is the case that \( ||x - y||_2^2 < 4\delta \) when \( ||x + y||_2^2 > 2 - \delta \); thus \( L_2 \) is a uniformly convex space. Next I shall show that in an uniformly convex space every closed convex set \( H \) has a unique element of smallest norm (cf. Wilansky [18] p. 110). If \( 0 \in H \) then \( 0 \) is the unique element of smallest norm. If \( 0 \notin H \) then let \( a_n \in H \) such that

\[
\lim_{n \to \infty} ||a_n|| = \inf\{||h|| : h \in H\} = d > 0.
\]

Then \( 2||\frac{1}{2}(a_n + a_m)|| \geq 2d \) since

\[
\frac{1}{2}a_n + \frac{1}{2}a_m \in H.
\]

For all \( \delta > 0 \) there exists an \( N \) such that \( n, m \geq N \),

\[
||a_n|| \leq d + \delta, \quad ||a_m|| \leq d + \delta \quad \text{thus} \quad 2d + 2\delta \geq ||a_n|| + ||a_m|| \geq ||a_n + a_m|| \geq 2d.
\]

So \( 2d/(d + \delta) \leq ||(a_n + a_m)/(d + \delta)|| \leq 2. \) Since \( ||a_n/(d + \delta)|| \leq 1 \),

\[
||a_m/(d + \delta)|| \leq 1, \quad \text{uniform convexity now implies that} \quad ||a_n - a_m|| < \varepsilon
\]

when \( \delta \) is taken sufficiently small. So \( (a_n)_{n=1}^\infty \) is a Cauchy sequence and \( \lim_{n \to \infty} a_n = a \in H \) since \( H \) is closed. Thus \( ||a|| = d \). Now if there
exists an $a_1 \in H$ such that $||a_1|| = d$ then $\frac{1}{2}(a_1/d + a/d) = 1$,
thus $||a_1/d - a/d|| < \varepsilon$ for all $\varepsilon > 0$ and $a_1 = a$; therefore $a$ is
the unique element of smallest norm in $H$.

A linear operator $U$ on $L_2(X)$ is a mapping of $L_2$ into itself
so that $U(af + bg) = af + bg$ for all real $a, b$ and all $f, g$ in
$L_2$. The norm of $U$, $||U||$, is equal to $\sup\{||Uf||/||f|| : f \in L_2, ||f|| \neq 0\}$.
$U$ is a unitary operator if and only if for all $f \in L_2$, $||Uf|| = ||f||$
and $U$ is onto $L_2$.

Let $H$ be a closed convex subset of an $L_2$ space and assume
that $UH \subset H$, where $U$ is a linear operator of norm less than or
equal to 1. If $h \in H$ is the unique element of smallest norm in $H$
then $Uh \in H$ and $||Uh|| \leq ||h||$ imply that $Uh = h$; i.e., $h$ is a
fixed point for $U$.

In 1955, Y. N. Dowker [2] showed that for an invertible non-
singular transformation $T$, a necessary and sufficient condition that
there exist a finite $T$-invariant measure $p_0$ equivalent to $p$ is that
$\lim \inf p(T^nA) > 0$ as $n \to \infty$ for every measurable set $A$ such that
$p(A) > 0$. Hajian and Kakutani [6] proved that the existence of a
finite invariant measure is equivalent to the nonexistence of weakly
wandering sets. Hajian and Ito [5] extended these previous results to
groups $G$, showing that the following are equivalent: i) $\inf[p(gA);$
$g \in G] = 0$ for at least one set $A$ of positive measure; ii) $G$ has
no finite invariant measure equivalent to $p$; iii) there exists a $G$-weakly wandering set. In theorem 1.1, I shall use similar methods to prove:

**Theorem 1.1**: If $S$ is a semigroup of nonsingular invertible transformations on a finite measure space $(X, \mathcal{J}, p)$ with $p(X) = 1$ then the following are equivalent:

i) For all $\epsilon > 0$ there exists a $W \in \mathcal{J}$ such that $p(W) > 1 - \epsilon$ and $W$ is an $S$-weakly wandering set.

ii) There does not exist a nontrivial finite $S$-invariant measure on $\mathcal{J}$ absolutely continuous with respect to $p$.

iii) For all $\epsilon > 0$ there exists an $X' \in \mathcal{J}$ such that $p(X') > 1 - \epsilon$ and $\inf(p(s^{-1}X') : s \in S) = 0$.

**Proof**: i) implies ii): If there exist weakly wandering sets with measure arbitrarily close to 1, then any set of positive measure contains a weakly wandering set. Hence no set of positive measure can be the support of a finite $S$-invariant measure. So i) implies ii).

ii) implies iii): Following Hajian and Ito and Dowker I define $L^2(X, \mathcal{J}, p)$ operators $U_s$ such that

$$U_s r(x) = r(s^{-1}x) w_s^2$$

where $w_s$ is the Radon-Nikodym derivative of the measure $ps^{-1}$ with respect to the measure $p$. If $r(x)$ is measurable then so is $r(s^{-1}x) w_s^2$ for all $s \in S$. Let $1_B$ be the characteristic function of $B$; i.e., $1_B(x) = 1$ if $x \in B$ and $1_B(x) = 0$ for $x \notin B$. For
\[ A, B \in \mathcal{S}, \]
\[
\int_{A \cap s^{-1}B} l_B(sx) w(s^{-1}x) dp = \int_{A} w(s^{-1}x) dp = p(s(A \cap s^{-1}B)) = \int_{sA} l_B(x) dp.
\]

Note that \( s \) is \( \mathcal{S} \)-invertible implies that \( sA \in \mathcal{S} \), and that \( l_B(sx) \)
is \( \mathcal{S} \)-measurable since \( s^{-1}B \in \mathcal{S} \). Using standard methods of approximating \( L_1 \) functions by simple functions, it can be deduced that

\[
\int_{A} f(sx)w(s^{-1}x) dp = \int_{sA} f(x)dp \quad \text{for all } f \in L_1(X, \mathcal{S}, p).
\]

Let \( f(x) = w(s^{-1}x) \), then

\[
\int_{A} w(s^{-1}x) dp = \int_{sA} w(s^{-1}x) dp = \int_{A} w((ts)^{-1}x) dp.
\]

Since this holds for all \( A \in \mathcal{S} \), it follows that \( w(s^{-1}x) = w((ts)^{-1}x) \). Furthermore,

\[
U_s U_t r(x) = r(tsx)(w(s^{-1}x) w(s^{-1}x))^{1/2} = U_{ts} r(x).
\]

Also \( \|U_s r(x)\|^2 = \int_X r^2(sx) w(s^{-1}x) dp = \|r(x)\|^2 \). \( U_s \) is a unitary operator on \( L_2 \).

Let \( T = \{U_s : s \in \mathcal{S}\} \), where \( l(x) = 1 \), all \( x \in X \). Let \( T^* \)
closed convex hull of \( T \) in \( L_2 \). Since \( L_2 \) is a uniformly convex space there exists an unique element \( t_0 \) in \( T^* \) such that \( \|t_0\| = inf \{\|t\| : t \in T^*\} \). In addition, \( U_s T \subseteq T \); hence, \( U_s T^* \subseteq T^* \) and
$U_{s_t} = t_0$ for all $s \in S$. Define $m$ on $\mathcal{E}$ by

$$m(E) = \int_{E} t_0^2 \, dp.$$ 

Then

$$m(sE) = \int_{sE} t_0^2 \, dp = \int_{E} t_0^2 (s) \, w_{s^{-1}} dp = \int_{E} (U_{s_0})^2 \, dp = \int_{E} t_0^2 \, dp = m(E).$$

Hence $m$ is a finite $S$-invariant measure which is absolutely continuous with respect to $p$. Since $t_0$ is the strong limit of convex combinations of $U_{s_1}$, $\int_{E} t_0 \, dp \geq \inf \{ \int_{E} U_{s_1} \, dp : s \in S \}$. By the Cauchy-Schwartz inequality

$$m(E) \geq \int_{E} t_0^2 \, dp \geq (\int_{E} t_0 \, dp)^2 / p(E) \geq (\inf_{s \in S} \int_{E} U_{s_1} \, dp)^2 / p(E).$$

Since $m$ is a finite $S$-invariant measure, $m$ must be identically zero. So $0 = m(X) = \inf_{s \in S} \int_{X} U_{s_1} \, dp$ and there exists a sequence $U_{s_1}$ which converges pointwise a.e. to $O$. Egorov's theorem implies that for all $\epsilon > 0$ a set $X' \in \mathcal{E}$ can be found such that $p(X') > 1 - \epsilon$ and for every $x \in X'$ $U_{s_1}$ converges to 0 uniformly. Thus $p(s_1^{-1} X') = \int_{X'} (U_{s_1})^2 \, dp$ converges to 0 so that $\inf(p(s_1^{-1} X) : s \in S) = 0$ as required. iii) implies i): Let $W = X' - \bigcup_{i=1}^{\infty} s_1^{-1} X' \cup \bigcup_{i=2}^{\infty} \bigcup_{j=1}^{i-1} s_1^{-1} X'$, where $s_1$ are chosen so that $p(s_1^{-1} X') < \delta_1$, $\delta_1 < \epsilon / 2 \delta_1^{i+1}$ and $\delta_1$ is chosen such that $p(A) < \delta_1$ implies $p(s_1^{-1} X') < \epsilon / 12 \delta_1^{i+1}$ for $j = 1, 2, \ldots, i-1$. Thus $W \in \mathcal{E}$ is an $S$-weakly wandering set and
\( p(W) > p(X') - \varepsilon > 1 - 2\varepsilon \) (cf. Hajian and Kakutani [6]).

Suppose \( S \) is a semigroup of nonsingular invertible transformations on a measure space \((X, \mathcal{E}, p)\). Replace \( s \) with \( s^{-1} \) and replace \( S^{-1} \) with \( S \) in i), ii) and iii) of theorem 1.1 to get i'), ii'), iii'). Then I will show that ii') is equivalent to ii), and the assumptions of theorem 1.1 hold for \( S^{-1} \); hence i'), ii') and iii') are equivalent by theorem 1.1. Thus i), ii), iii), i'), ii'), and iii') are all equivalent. If \( S \) is a semigroup of nonsingular invertible transformations, then \( s \in S \) is invertible and \( s^{-1} \) is also an invertible transformation and both \( s \) and \( s^{-1} \) are nonsingular so \( S^{-1} \) is a semigroup of nonsingular invertible transformations; i.e., the assumptions of theorem 1.1 hold for \( S^{-1} \) whenever they hold for \( S \). If \( p' \) is an \( S \)-invariant measure on \((X, \mathcal{E})\) then \( p'(s^{-1}A) = p'(A) \) for all \( A \in \mathcal{E} \). Since \( s \) is invertible, \( A \in \mathcal{E} \) implies \( sA \in \mathcal{E} \); hence for any \( A \in \mathcal{E} \), \( p'(sA) = p'(s^{-1}sA) = p'(A) \); i.e., \( p' \) is \( S^{-1} \) invariant. So ii) and ii') are equivalent since \( S \) and \( S^{-1} \) preserve the same measures. Note that corollary 1.1 to follow can also be modified in the same manner.

**Corollary 1.1:** Let \( S, X, \mathcal{E} \) be as in theorem 1.1; that is, \( S \) is a semigroup of nonsingular invertible transformations on a measurable space \((X, \mathcal{E})\). Let \( m \) be a \( \sigma \)-finite, infinite \( S \)-invariant measure on \( \mathcal{E} \). Then the following are equivalent:

1) For every \( A \in \mathcal{E} \) with \( m(A) < \infty \), and for every \( \varepsilon > 0 \) there exists an \( S \)-weakly wandering set \( W \) such that \( m(A-W) < \varepsilon \).
ii) There does not exist any nontrivial finite \(S\)-invariant measure on \(J\) absolutely continuous with respect to \(m\).

iii) For every pair of sets \(A, B \in J\) of finite measure and for every \(\varepsilon > 0\) there exists an \(s \in S\) such that
\[
m(A \cap s^{-1}B) = m(sA \cap B) < \varepsilon.
\]

Proof: i) implies ii): Let \(p\) be a finite \(S\)-invariant measure on \(J\) absolutely continuous with respect to \(m\). By i), for every \(A \in J\) with \(m(A)\) finite and for every \(\varepsilon > 0\), there exists an \(S\)-weakly wandering set \(W_{\varepsilon,A}\) such that \(m(A\setminus W_{\varepsilon,A}) < \varepsilon\). Now since \(p\) is a finite measure it must be that \(p(W_{\varepsilon,A}) = 0\). Now \(p \ll m\) so that
\[
\lim_{i \to \infty} p(A_i) = 0 \quad \text{implies that} \quad \lim_{i \to \infty} p(A_i) = 0.
\]
Furthermore,
\[
p(A) = p(A \setminus W_{\varepsilon,A}) + p(A \cap W_{\varepsilon,A})
\]
\[
= p(A \setminus W_{\varepsilon,A}) < \varepsilon \quad \text{where} \quad \lim_{\varepsilon \to 0} \varepsilon = 0.
\]
Consequently \(p(A) = 0\) for all \(A \in J\) with \(m(A)\) finite. But since \(m\) is \(\sigma\)-finite, \(X = \bigcup_{i=1}^{\infty} X_i\) where \(m(X_i)\) is finite. Hence
\[
p(X) = \sum_{i=1}^{\infty} p(X_i) = 0\] and \(p\) is the trivial measure on \(J\). ii) implies

i): For \(A \in J, m(A)\) finite, define \(P_{A,\delta}\) as follows
\[
P_{A,\delta}(c) = \frac{m(c \cap A)}{m(A)} (1-\delta) + \left( \sum_{i=1}^{\infty} \frac{m(c \cap V_i)}{2m(V_i)} \right) \delta
\]
where \(\{V_i\}_{i=1}^{\infty}\) is a \(\delta\)-measurable partition of \(X\setminus A\). By the above definition we have that \(P_{A,\delta}\) is equivalent to \(m\). Thus if \(p' \ll P_{A,\delta}\)
then \( p' \ll m \). Since ii) has been assumed there does not exist a \( p' \ll p_{A,B} \) where \( p' \) is a nontrivial finite \( S \)-invariant measure.

Let \( \varepsilon > 0 \) be given. Apply theorem 1.1 to \((X, \mathcal{A}, p_{A,B})\), \(S\) and find \( W \in \mathcal{J}\) such that \( W \) is an \( S \)-weakly wandering set and \( p_{A,B}(W) > 1 - \delta \). Choose \( \delta < \min\left(\frac{1}{2}, \frac{\varepsilon}{2m(A)}\right) \) so that \( 1 - \delta > \frac{1}{2}, \delta/(1 - \delta) < 2 - \frac{1}{2} < 2\varepsilon < \varepsilon/m(A), \) and \( p_{A,B}(W) = \frac{m(W \cap A)}{m(A)} (1 - \delta) + \left( \sum_{i=1}^{\infty} \frac{m(V_i \cap W)}{2^i m(V_i)} \right) \).

Thus

\[
m(W \cap A) = p_{A,B}(W) - \left( \sum_{i=1}^{\infty} \frac{m(V_i \cap W)}{2^i m(V_i)} \right) \frac{m(A)}{1 - \delta}
\]

\[
\geq \left( (1 - \delta) - \delta \right) \frac{m(A)}{1 - \delta} = (1 - \frac{\delta}{1 - \delta}) m(A)
\]

\[
\geq (1 - \frac{\varepsilon}{m(A)}) m(A) = m(A) - \varepsilon .
\]

Hence \( m(A - W) < \varepsilon \) as required.

i) implies iii): Let \( A, B, \varepsilon > 0 \) be given where \( m(A) \) and \( m(B) \) are finite. Then \( m(A \cup B) \) is finite and so by i) there exists a \( W \in \mathcal{J}\) such that \( m(A \cup B - W) < \varepsilon/2 \) where \( W \) is an \( S \)-weakly wandering set. Choose \( s \) such that \( s^{-1} W \cap W = \emptyset \); i.e., \( W \subset s^{-1} W^c \), then

\[
m(A \cap s^{-1}B) = m(A \cap W \cap s^{-1}B) + m(A \cap W^c \cap s^{-1}B)
\]

\[
\leq m(A \cap s^{-1}B \cap s^{-1}W^c) + m(A \cap W^c \cap s^{-1}B)
\]

\[
\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \text{because} \quad m(A \cap W^c) < \varepsilon/2 \quad \text{and}
\]

\( m(B \cap W^c) < \varepsilon/2 \) implies that \( m(s^{-1}(B \cap W^c)) < \varepsilon/2 \).
iii) implies i): Let $A$ and $\varepsilon > 0$ be given where $m(A)$ is finite.

Let $(V_i)_{i=1}^\infty$ be, as before, a $\mathcal{J}$-measurable partition of $X - A$. Notice that iii) of theorem 1.1 is satisfied since for any $\eta > 0$ there exists an $s \in S$ such that $m(A \cup \bigcup_{i=1}^n V_i) \leq \min \{m(V_i) : i = 1, 2, \ldots, n\}, m(A)$. Set $X' = A \cup \bigcup_{i=1}^n V_i$ where $n$ is chosen so that $\mathbb{P}_{A,s}(X') > 1 - (\eta/2)$. Now

$$\mathbb{P}_{A,s}(s^{-1}X') = \left(\frac{\sum_{i=1}^n m(s^{-1}X' \cap A)}{m(A)}\right)(1 - \varepsilon) + \sum_{i=1}^n \frac{m(s^{-1}X' \cap V_i)}{2m(V_i)}$$

$$\leq (\eta/2)(1 - \varepsilon) + (\eta/2)\delta + \eta/2 = \eta,$$

So iii) of theorem 1.1 holds. Hence theorem 1.1 i) holds; i.e., for every $\eta > 0$ there exists an $S$-weakly wandering set $W$ such that $\mathbb{P}_{A,s}(W) > 1 - \eta$. As in the proof of ii) implies i), $\eta$ sufficiently small implies that $m(W \cap A) > m(A) - \varepsilon$; i.e., i) of the corollary holds.

Let $S$ be a left amenable semigroup of nonsingular $\mathcal{J}$-invertible $\mathcal{J}$-measurable transformations on a probability space $(X, \mathcal{J}, p)$, where $\mathcal{J}$ is a separable exhaustive sub-$\sigma$-algebra of $\mathcal{J}$. In order to show that for such semigroups $S$ which do not preserve any finite measure $\mathbb{P}' \ll \mathbb{P}$ there exist sets $X'$ in $\mathcal{J}$ with measure arbitrarily close to 1 and whose $\inf\{p(sX') : s \in S\} = 0$, I need the following:

**Lemma 1.1:** Let $S$, $\mathcal{J}$, $(X, \mathcal{J}, p)$ be as in the above paragraph. Let $\mathcal{F}^S$ be a subalgebra (not necessarily a $\sigma$-algebra) of $\mathcal{J}$ with $\mathcal{G}(S) \subseteq \mathcal{F}^S$.
\( \mathcal{F}^g (G(S)) \) is the smallest group containing \( S \) and assume that \( \mathcal{F}^g \)
generates \( \mathcal{J} \). If \( \lambda \) is a \( G(S) \)-invariant finitely additive measure on \( \mathcal{F}^g \) and \( \lambda(X) = 1 \) then for all \( \epsilon > 0 \) there exists a \( U \in \mathcal{F}^g \)
such that \( \mu(U) < \epsilon \) and \( \lambda(U) > 1 - \epsilon \).

This lemma is a generalization of the following result due to
Krengel (lemma 5.1 in [13]) (note that I have interchanged \( \mathcal{F} \) and
\( \mathcal{J} \) in Krengel's result):

Let \( T \) be a nonsingular invertible transformation on a probability
space \((X, \mathcal{J}, \mu)\) such that no \( T \)-invariant probability measure \( \mu' \ll \mu \)
e xists on \( \mathcal{J} \). Let \( \mathcal{F} \) be a sub-\( \sigma \)-algebra of \( \mathcal{J} \) with \( T^{-1}\mathcal{F} \subseteq \mathcal{F} \) for
which \( \mathcal{F}^g = \bigcup_{k \geq 0} T^k \mathcal{F} \) generates \( \mathcal{J} \). Let \( \lambda \) be a \( T \)-invariant probabi-

Let \( \lambda \) on \( \mathcal{F}^g \) be defined by \( \lambda(T^k A) = \lambda(A) \) for \( k \geq 0 \) and \( A \in \mathcal{F} \).
Then there exists, for every \( \epsilon > 0 \), an integer \( k \geq 0 \) and a set
\( u \in T^k \mathcal{F} \) such that \( \mu(U) < \epsilon \) and \( \lambda(U) > 1 - \epsilon \).

In fact Krengel's proof of the above together with the following
observations prove lemma 1.1. First note that \( \mathcal{F}^g \) and not \( \mathcal{F} \)
is used in the proof. All that is needed is that \( \mathcal{F}^g \) is an algebra which
generates \( \mathcal{J} \) which is invariant under \( T^1 \) for all integers \( i \) and
that \( \lambda \) is a \( T \)-invariant finitely additive measure on \( \mathcal{F}^g \). Krengel's
proof, up to the point that \( P_k \) and \( P_\infty \) are defined, is valid also
for lemma 1.1. In fact this first part of his proof does not involve
\( T \). Krengel defines \( P_k = \bigcup_{|i|<k} T^i P_0 \) and \( P_\infty = \bigcup_{k=0} P_k \) where \( P_0 \in \mathcal{J} \).
with certain other characteristics developed in the first part. The
generalization of $P_e$ for lemma 1.1 would be $P_e = \bigcup_{g \in G(s)} gP_0$. However
if $G(s)$ is not countable then the remainder of the proof will not
follow. In order to get around this problem I shall essentially re-
duce $\bigcup_{g \in G(s)} gP_0$ to a countable union via the next lemma.

**Lemma 1.2:** Let $(X, \mathcal{S}, p)$ be a finite measure space with $p(X) = 1$.
Let $S$ be a set of $\mathcal{S}$ measurable nonsingular transformations on $X$.
Then for $P \in \mathcal{S}$ there exists a subset, $(s_i: i \text{ is an integer})$, of
$S$ such that

1) $m(s_i^{-1}P - \bigcup_{i=0}^{\infty} s_i^{-1}P) = 0$ for all $s \in S$.

2) If $S$ is a semigroup then $t^{-1}(\bigcup_{i=0}^{\infty} s_i^{-1}P) \subseteq \bigcup_{i=0}^{\infty} s_i^{-1}P(p)$
for all $t \in S$.

3) If $S$ is a group then $t^{-1}(\bigcup_{i=0}^{\infty} s_i^{-1}P) = \bigcup_{i=0}^{\infty} s_i^{-1}P(p)$ for all
t $\in S$.

**Proof:** Let $P$ be denoted by $s_0^{-1}P$. Let

$L_1 = \sup \{p(s_i^{-1}P - P): s \in S\}$.

Choose $s_1 \in S$ such that $p(s_1^{-1}P - P) \geq L_1/2$. Set

$L_2 = \sup \{p(s_1^{-1}P - \bigcup_{i=0}^{1} s_i^{-1}P): s \in S\}$.

Choose $s_2$ to be the element of $S$ for which $p(s_2^{-1}P - \bigcup_{i=0}^{1} s_i^{-1}P) \geq L_2/2$.

Continue choosing $L_i$ and $s_i$ in the above manner so that
\[(2) \quad 1 \geq p\left( \bigcup_{i=0}^{k-1} s_i^{-1}p \right) = \sum_{i=0}^{k-1} p(s_i^{-1}p - \bigcup_{j=0}^{i-1} s_j^{-1}p) \]

\[\geq p(p) + \sum_{i=1}^{\infty} (L_i/2).\]

Since for all natural numbers \( k \) and all \( s \in S \), \( I_k \geq p(s^{-1}p - \bigcup_{i=0}^{k-1} s_i^{-1}p) \), and since (2) implies that \( \lim_{k \to \infty} I_k = 0 \), I have that \( p(s^{-1}p - \bigcup_{i=0}^{\infty} s_i^{-1}p) = 0 \) for all \( s \in S \); i.e., I have shown (i).

If \( S \) is a semigroup and \( s, t \in S \), then \( t^{-1}s^{-1} \in S^{-1} \), and by what I have just shown, \( s^{-1}p \subset \bigcup_{i=0}^{\infty} s_i^{-1}p (p) \) for all \( s \in S \). Since \( \bigcup_{i=0}^{\infty} s_i^{-1}p \) is a countable union, for each \( t \in S \),

\[t^{-1}\left( \bigcup_{i=0}^{\infty} s_i^{-1}p \right) = \bigcup_{i=0}^{\infty} t^{-1}s_i^{-1}p \subset \bigcup_{i=0}^{\infty} s_i^{-1}p (p);\]

that is, (ii) is proven.

If \( S \) is a group then (ii) shows that

\[t^{-1}\left( \bigcup_{i=0}^{\infty} s_i^{-1}p \right) \subset \bigcup_{i=0}^{\infty} s_i^{-1}p = t \left( \bigcup_{i=0}^{\infty} s_i^{-1}p \right) = t^{-1}(\bigcup_{i=0}^{\infty} s_i^{-1}p) (p).\]

So (iii) is established and lemma 1.2 is proven.

Note that if \( m \) is a \( \sigma \)-finite measure on \((X, \mathcal{F})\) then a probability measure equivalent to \( m \) on \( X \) can be found. Since (i), (ii), (iii) hold up to sets of \( p \)-measure zero (i), (ii) and (iii) also hold up to sets of \( m \)-measure zero. Thus lemma 1.2 is also true for \( \sigma \)-finite measures.
Now then to complete the proof of lemma 1.1 via Krenkel's proof, it is sufficient to substitute $P_n^* = \bigcup_{i=0}^{\infty} g_i^{-1} P_0$ for $P_n^*$, and

$$P_k^* = \bigcup_{i=0}^{\infty} g_i^{-1} P_0$$

for $P_k^*$, where $g_i \in G(S)$ as selected by lemma 1.2 so that $g^{-1}(\bigcup_{i=0}^{\infty} g_i^{-1} P_0) = \bigcup_{i=0}^{\infty} g_i^{-1} P_0$. Then the remainder of Krenkel's proof proves lemma 1.1.

A semigroup $S$ is called left amenable if there exists a bounded functional $\lambda^*$ on the set $m(S)$ (which is the set of all bounded functions on $S$) with the following properties:

1) $\sup_{s \in S} (f(s)) \geq \lambda^*(f) \geq \inf_{s \in S} (f(s))$ for all $f \in m(S)$.

2) $\lambda^*(af + bg) = a\lambda^*(f) + b\lambda^*(g)$ for all $f, g \in m(S)$; $a, b$ real.

3) $\lambda^*(f(s)) = \lambda^*(f(ts))$ for any given $t \in S$.

If $\lambda^*$ satisfies iiiR $\lambda^*(f(s)) = \lambda^*(f(st))$ for $t \in S$ instead of iiiI, then $S$ is called right amenable. If there exists a $\lambda^*$ satisfying i), ii), iiiI), and iiiR, then $S$ is called amenable.

In order to use lemma 1.1 to get the desired result I need the following:

**Lemma 1.3:** If $S$ is a semigroup of 1-1 and onto $\mathcal{F}$-measurable transformations on a probability space $(X, \mathcal{F}, p)$, and if $G(S)$ is the smallest group containing $S$, where $S$ is left amenable, then:

1) $G(S) = SS^{-1} U S U S^{-1}$.

2) $S\mathcal{F}$ is an algebra.

3) $G(S) S\mathcal{F} = S\mathcal{F}$. 
iv) There exists a \( G(S) \)-invariant finitely additive measure \( \lambda \)
on \( S\bar{S} \) with \( \lambda(X) = 1 \).

Proof: If \( S \) is a left amenable semigroup, then it follows from
corollary 3.6 of I. Namoka [14] that for any finite subset \( E \) of \( S \)and any number \( k \), \( 0 < k < 1 \), there exists a finite subset \( A \) of \( S \)such that for each \( s \in E \), \( c(sA \cap A) \geq kc(A) \) where \( c(A) \) is thecardinality of \( A \). Now take \( E \) and \( k \) such that \( E = (s,t) \subset S \) and\( k \geq 2/3 \). Then there exists an \( A \) such that \( c(sA \cap A) \geq (2/3)c(A) \)and \( c(tA \cap A) \geq (2/3)c(A) \). Hence \( c(sA \cap tA) \geq (1/3)c(A) > 0 \). Thenfor \( s, t \in S \) there exist \( a, b \in S \) such that \( sa = tb \). That is,for all \( s, t \in S \) there exist \( a, b \in S \) such that \( t^{-1}s = ba^{-1} \). Thisimplies that for \( g = s_1s_2^{-1}s_3s_4^{-1} \cdots s_n \) there exist \( t_1, t_2 \in S \) with\( g = t_1t_2^{-1} \). Thus \( G(S) = SS^{-1} \cup S \cup S^{-1} \). Note that this is a resultabout left amenable semigroups where each element has an inverse; \( S \)need not be a semigroup of transformations.

To show ii) it must be shown that for any \( s \in S \) and any \( F \in J \),\((sF)^c \in S\bar{S} \), and that for any \( s_1, s_2 \in S \) and any \( F_1, F_2 \in J \),\( s_1F_1 \cup s_2F_2 \in S\bar{S} \). If \( s \in S \) and \( F \in J \), then \( (sF)^c = sF^c \in S\bar{S} \) since\( s \in S \) is 1-1 and onto, and \( F \in J \) implies \( F^c \in J \). Let \( s_1F_1 \cup s_2F_2 \)be given. Then there exist \( t_1 \) and \( t_2 \) in \( S \) such that \( s_2^{-1}s_1 = t_1t_2^{-1} \)so that

\[
\begin{align*}
s_1F_1 \cup s_2F_2 &= s_2(s_1^{-1}s_1F_1 \cup F_2) = s_2(t_1t_2^{-1}F_1 \cup F_2) = \\
&= s_2t_1(t_2^{-1}F_1 \cup t_1^{-1}F_2).
\end{align*}
\]
Since $s_2 t_1 \in S$ ($S$ is a semigroup) and $t_1, t_2 \in S$, $s_2 t_1(t_2^{-1} t_1 U t_1^{-1} t_2) \in S^2$.

Now $G(S)S^2 = G(S)F = (SS^{-1} U S U S^{-1})S = S^2$. Indeed, $S^{-1} S \subset S$ by the measurability of $s \in S$. Hence $SS^{-1} S = S$. Also, since $s \in S$ is 1-1 and onto, $S^{-1} S \subset S$ implies that $S \subset S^2$. Thus $S^2$ is a $G(S)$-invariant algebra and iii) has been proven.

Now I shall show iv): that there exists a finitely additive $G(S)$-invariant measure $\lambda$ on $S^2$ with $\lambda(\emptyset) = 1$. Since $S$ is left amenable there is a left $S$-invariant mean $\lambda^*$ on $m(S)$. Define $\lambda$ on $S^2$ as follows

$$\lambda(A) = \lambda^*(p(s^{-1} A)) .$$

Note that for a fixed $A \in S^2$, $f(s) = p(s^{-1} A)$ belongs to $m(S)$. Then

$$\lambda(A) = \lambda^*(p(s^{-1} A)) = \lambda^*(p((ts)^{-1} A)) = \lambda^*(p(s^{-1} t^{-1} A)) = \lambda(t^{-1} A)$$

for any $t \in S$ so that $\lambda$ is $S$-invariant on $S^2$. Extend $\lambda$ to $S^2$ by defining $\lambda$ on $S^2$:

(3) $\lambda(B) = \lambda(A)$ where $B = t A$ for some $A \in S^2$, and some $t \in S$.

First I must show that $\lambda$ is well defined. Suppose that $B = t A_1 = s A_2$.

Then $A_1 = t s^{-1} A_2 = r_1 r_2 A_2$. That is $r_1^{-1} A_1 = r_2^{-1} A_2 \in S$. Hence,

$$\lambda(A_1) = \lambda(r_1^{-1} A_1) = \lambda(r_2^{-1} A_2) = \lambda(A_2).$$

Consequently, $\lambda$ is well defined on $S^2$. By the above $\lambda$ is also $G(S)$-invariant since for any $g \in G(S)$, I have that $g = st^{-1}$ for some $s, t \in S$, so that for any $A \in S^2$,

$$\lambda(g A) = \lambda(st^{-1} A) = \lambda(A).$$

To show that $\lambda$ is finitely additive it is
sufficient to show that for $A_1, A_2 \in S\mathcal{F}$ with $A_1 \cap A_2 = \emptyset$, I have that $\lambda(A_1 \cup A_2) = \lambda(A_1) + \lambda(A_2)$. I have

$$\lambda(A_1 \cup A_2) = \lambda^*(p(s^{-1}(A_1 \cup A_2))) = \lambda^*(p(s^{-1}A_1) + p(s^{-1}A_2))$$

$$= \lambda^*(p(s^{-1}A_1)) + \lambda^*(p(s^{-1}A_2)) = \lambda(A_1) + \lambda(A_2)$$

as required.

Since $\sup p(s^{-1}x) = 1 = \inf (p(s^{-1}x))$, I have

$$\lambda(x) = \lambda^*(p(s^{-1}x)) = 1. \text{ Therefore lemma 1.3 is proven.}$$

I shall now use the preceding results to prove:

**Theorem 1.2:** Let $S$ be a left amenable semigroup of measurable non-singular invertible transformations on $(X, \mathcal{J}, p)$ where $p(x) = 1$. Let $\mathcal{F}$ be an $S^{-1}$-exhaustive sub-$\sigma$-algebra of $\mathcal{J}$, then the following are equivalent:

1) For all $\varepsilon > 0$ there exists a set $W \in \mathcal{J}$ such that $p(W) > 1 - \varepsilon$ and $W$ is an $S^{-1}$-weakly wandering set.

2) There does not exist any nontrivial finite $S^{-1}$-invariant measure on $\mathcal{J}$ which is absolutely continuous with respect to $p$.

3) For all $\varepsilon > 0$ there exists an $X' \in \mathcal{J}$ such that $p(X') > 1 - \varepsilon$ and

$$\inf (p(sX')) = 0.$$

**Proof:** Theorem 1.1 implies that 1) is equivalent to 2) and that 3) implies 1), so that theorem 1.2 will be proven if it can be shown that 2) implies 3). Lemma 1.3 implies that $S\mathcal{F}$ is a $\mathcal{G}(S)$-invariant
algebra with a finitely additive $G(s)$-invariant measure $\lambda$ on $S\mathcal{F}$ with $\lambda(X) = 1$. Applying lemma 1.1 with $\mathcal{F} = S\mathcal{F}$ to find that for all $\epsilon > 0$ there exists a $U \in S\mathcal{F}$ such that $p(U) < \epsilon$ and $\lambda(U) > 1 - \epsilon$. So $U = s_1F$ for some $s_1 \in S$ and some $F \in \mathcal{F}$. $\lambda(s_1F) = \lambda(F)$ since $\lambda$ is $G(s)$-invariant, and so $\lambda(F) > 1 - \epsilon$. Now sup $\{p(s^{-1}F)\} \geq \lambda(F) \geq 1 - \epsilon$. So there exists a $t \in S$ such that $p(t^{-1}F) \geq 1 - \epsilon$. Set $V = t^{-1}F$. Recall that if $\mathcal{F}$ is an $S^{-1}$-exhaustive sub-$\sigma$-algebra of $\mathcal{F}$, then $s \in S$ is $\mathcal{F}$-measurable and hence $V = t^{-1}F \in \mathcal{F}$ since $f \in \mathcal{F}$ and $t \in S$ is $\mathcal{F}$-measurable. Then $p(V) = p(t^{-1}F) \geq 1 - \epsilon$, and $p(s_1tV) = p(s_1F) = p(U) < \epsilon$. That is, for any $\epsilon > 0$, I can find a $V_{\epsilon} \in \mathcal{F}$ and a $t \in S$ such that $p(V_{\epsilon}) > 1 - \epsilon$ and $p(t_{\epsilon}V) < \epsilon$. Now find $V_{1} \in \mathcal{F}$ with $p(V_{1}) > 1 - (\epsilon/2^1)$ and $p(t_{1}V_{1}) < \epsilon/2^1$. Then $V_{1} \in \mathcal{F}$ and $\mathcal{F}$ is a $\sigma$-algebra so that $V_{1}^{*} = \bigcap_{i=1}^{\infty} V_{i} \in \mathcal{F}$. Also $p(\bigcap_{i=1}^{\infty} V_{i}) > 1 - \epsilon$ and $p(t_{1}(\bigcap_{i=1}^{\infty} V_{i})) < p(t_{1}V_{1}) < \epsilon/2^1$; i.e., $\inf_{s \in S} (p(sV_{1}^{*})) = 0$ as required.

I will show that if $S$ is a left amenable semigroup of measurable nonsingular transformations on $(X, \mathcal{F}, p)$ with $p(X) = 1$ where $s \in S$ is 1-1 and onto, then there exists an $S$-weakly wandering set $W \in \mathcal{F}$ with measure arbitrarily close to 1 if and only if $S$ preserves no finite measure on $(X, \mathcal{F})$ which is absolutely continuous with respect to $p$. To do this I need the following result due to L. Sucheston [17]. Sucheston defines a pure charge $\lambda$ on $\mathcal{F}$ as a finite non-negative
finitely additive set function on \( S \) which does not dominate any nontrivial measure \( m \) on \( S \); i.e., if \( m \) is a measure and \( m \leq \lambda \) on \( S \) then \( m \) is identically zero. Sucheston proves ([17] theorem 3) that if \( \lambda \) is a charge on an algebra \( S \) then \( \lambda \) admits a unique decomposition \( \lambda = \lambda_m + \lambda_c \), where \( \lambda_c \) is a pure charge and \( \lambda_m \) is a measure. In addition \( \lambda_m(A) = \inf_C \lambda(A_1) \), where the infimum is taken over all at most countable partitions \( \{A_1\} \) of \( A \) in \( S \). Now I have \( \lambda_c(A) = \lambda(A) - \lambda_m(A) \). If \( \lambda \) is \( S \)-invariant and if \( s \in S \) is \( S \)-measurable, then \( \lambda_m \) is an \( S \)-invariant measure and \( \lambda_c \) is an \( S \)-invariant charge. Note that \( \lambda_m \) and \( \lambda_c \) are absolutely continuous with respect to \( \lambda \). I shall also need theorem 5 of [17] which states:

If \( \lambda \) is a pure charge and \( p \) is a measure on a \( \sigma \)-field \( S \) of subsets of \( X \), then for any \( \varepsilon > 0 \) there exists a set \( B \in S \) such that \( p(B) < \varepsilon \) and \( \lambda(B) = \lambda(X) \); or equivalently, for \( A = B^c \), I have that \( p(A) > 1 - \varepsilon \) and \( \lambda(A) = 0 \).

**Theorem 1.3:** Suppose \( S \) is a left amenable semigroup of 1-1, onto, measurable nonsingular transformations on \( (X, S, p) \), \( p(X) = 1 \). Then the following are equivalent:

1. For all \( \varepsilon > 0 \) there exists a set \( W \in S \) such that \( p(W) > 1 - \varepsilon \) and \( W \) is an \( S \)-weakly wandering set.

2. There does not exist any nontrivial finite \( S \)-invariant measure on \( S \) which is absolutely continuous with respect to \( p \).
iii) For all $\varepsilon > 0$ there exists an $X' \in \mathcal{F}$ such that $p(X') > 1 - \varepsilon$
and $\inf \{p(s^{-1}X') : s \in S\} = 0$.

Proof: i) implies ii): Same proof as i) implies ii) of theorem 1.1.

ii) implies iii): Define $\lambda(A) = \lambda^*(p(s^{-1}A))$ where $p(s^{-1}A)$ belongs
to $m(S)$, the set of bounded functionals on $S$ in the case that
$\lambda^*$ is a left $S$-invariant mean on $m(S)$. Then for $A \in \mathcal{F}$ and $t \in S$,
as in lemma 1.3, $\lambda(t^{-1}A) = \lambda(A)$ and $\lambda$ is finitely additive and
finite on $\mathcal{F}$ (cf. p. 24). That is $\lambda$ is an $S$-invariant charge on $\mathcal{F}$.
Applying theorem 3 of [17], $\lambda$ can be decomposed so that $\lambda = \lambda_m + \lambda_c$.
But since $\lambda_m$ is an $S$-invariant measure on $\mathcal{F}$ which is absolutely
continuous with respect to $p$, then $\lambda_m$ must be identically zero.
Hence, $\lambda = \lambda_c$ is a pure $S$-invariant charge on $\mathcal{F}$. By applying theorem
5 of [17] I find $X' \in \mathcal{F}$ such that $p(X') > 1 - (\varepsilon/2)$ and $\lambda(X') = 0$.

Then

$$0 = \lambda(X') = \lambda^*(p(s^{-1}X')) \geq \inf_{s \in S} (p(s^{-1}X'))$$

Hence, $\inf_{s \in S} (p(s^{-1}X')) = 0$ and ii) implies iii).

iii) implies i): Granirer ([14], theorem 2) has shown that if
$\inf \{p(s^{-1}C) : s \in S\} = 0$ where $s$ is nonsingular, then there exists
an $S$-weakly wandering set $W \subset C$ such that $p(W) > p(C) - (\varepsilon/2)$. For
the sake of completeness I shall reproduce his proof here with $X'$
substituted for $C$. Let $s_n \in S$ be such that $\lim p(s_n^{-1}X') = 0$. Let
$s_n$ satisfy $p(s_n^{-1}X') < \varepsilon_1$, and if $s_{n_1}, \ldots, s_{n_k}$ have been chosen,
let \( s_{n_k} \) be such that

\[
\begin{align*}
(4) \quad p(s_{n_k}^{-1} X') + \sum_{i=1}^{k-1} p((s_{n_k} \cdots s_{n_i})^{-1} X') < \varepsilon_k, \quad \text{where} \quad \sum_{k=1}^{\infty} \varepsilon_k = \varepsilon.
\end{align*}
\]

The choice in (4) is possible since \( s_{n_k} \cdots s_{n_1} \) is a nonsingular transformation with respect to \( p \), and so there exists a \( \delta > 0 \) such that if \( p(s_{n_k}^{-1} X') < \delta \), then \( p((s_{n_k} \cdots s_{n_i})^{-1} s_{n_k}^{-1} X') < \varepsilon_k/2 \), for \( i = 1, 2, \ldots, k - 1 \). If \( s_{n_k} \) is chosen so that \( p(s_{n_k}^{-1} X') < \min(\delta, \varepsilon_k/2) \), then (4) holds. Set

\[
D = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{k} (s_{n_k} \cdots s_{n_i})^{-1} X' \cap X' \in \mathcal{F}.
\]

Then \( p(D) \leq \sum_{k=1}^{\infty} \sum_{i=1}^{k} p((s_{n_k} \cdots s_{n_i})^{-1} X') < \sum_{k=1}^{\infty} \varepsilon_k = \varepsilon \). Let \( W = X' - D \).

Then \( p(W) = p(X') - p(D) > 1 - \varepsilon \). Moreover the sets \( s_{n_1}^{-1} W, (s_{n_2} s_{n_1})^{-1} W, \ldots, (s_{n_k} \cdots s_{n_1})^{-1} W, \ldots \) are pairwise disjoint. In fact if \( i < j \), then \( (s_{n_i} \cdots s_{n_1})^{-1} (X' - D) \subset (s_{n_i} \cdots s_{n_1})^{-1} (X' - D) \cap
\]
\[
(s_{n_1} \cdots s_{n_{i+1}} s_{n_i} \cdots s_{n_1})^{-1} (X' - D) \subset (s_{n_1} \cdots s_{n_{i+1}})^{-1} ((X' - D) \cap
\]
\[
(s_{n_j} \cdots s_{n_{i+1}})^{-1} (X' - D)) = \phi. \quad \text{So theorem 1.3 is complete.}
\]

**Corollary 1.2:** Let \( S, X, \mathcal{F} \) be as in theorem 1.3; that is, \( S \) is a left amenable semigroup of \( l_1 \)-onto, measurable nonsingular transformations on \( (X, \mathcal{F}, m) \). Let \( m \) be a \( \sigma \)-finite \( S \)-invariant measure on \( \mathcal{F} \). Then the following are equivalent:

1) For every \( A \in \mathcal{F} \) where \( m(A) \) is finite and for every \( \varepsilon > 0 \)
there exists an $S$-weakly wandering set $W$ such that $m(A - W) < \epsilon$.

ii) There does not exist any nontrivial finite $S$-invariant measure on $\mathcal{F}$ absolutely continuous with respect to $m$.

iii) For every pair of sets $A, B \in \mathcal{F}$ where both $A$ and $B$ have finite measure and for every $\epsilon > 0$ there exists an $s \in S$ such that $m(A \cap s^{-1}B) < \epsilon$.

Proof: The same as the proof of corollary 1.1 with applications of theorem 1.3 substituted for those of theorem 1.1.

Let $S$ be a subsemigroup of a group. Let $G(S)$ be the smallest group containing $S$. I will conclude this chapter with a brief discussion of the relationships between the amenability of $S$ and of $G(S)$. I will show that $S$ is amenable implies that $G(S)$ is amenable. The converse is not true.

Observe that if the subsemigroup $S$ is amenable then the subsemigroup $S'$ with $S' = S \cup \{\text{identity element of } G(S)\}$ is also amenable. Indeed, an invariant mean $\lambda^*$ on $m(S)$ is also an invariant mean on $m(S')$. Thus, without loss of generality, assume that $S$ contains the identity element.

A semigroup $S$ is said to have right cancellation if for $s_1 \in S$, $s_1s_2 = s_3s_2$ implies that $s_1 = s_3$. By I. Namioka (theorem 4.1 of [14]) the right amenability of a semigroup $S$ with right cancellation is equivalent to the existence of $k_0$, $0 < k_0 < 1$ such that for finitely
many $s_i, s_i'$, there exists a finite set $E \subseteq S$ with
\[ \frac{1}{n} \sum_{i=1}^{n} c(E s_i \cap E s_i') > k_0 c(E). \]
Since in the case I am considering $s_i$ has an inverse,
\[ \frac{1}{n} \sum_{i=1}^{n} c(E s_i s_i^{-1} \cap E) > k_0 c(E) \]
(where $c(E)$ = cardinality of $E$). Recall that I showed in lemma 1.3(1) that if $S$ is left amenable and contains the identity, then $G(S) = SS^{-1} \cup S \cup S^{-1} = SS^{-1}$; that is, for any $g_i, g_i' \in G(S)$, I have $g_i g_i^{-1} \in G(S)$. So there exist $u_i, v_i \in S$ such that $g_i g_i^{-1} = u_i v_i^{-1}$. Thus there exists a $k_0$,
\[ 0 < k_0 < 1, \]
and an $E \subseteq S$ depending on $u_i$ and $v_i$ such that
\[ \frac{1}{n} \sum_{i=1}^{n} c(E g_i g_i^{-1} \cap E) = \frac{1}{n} \sum_{i=1}^{n} c(E u_i v_i^{-1} \cap E) > k_0 c(E); \]
that is, $G(S)$ is right amenable and hence $G(S)$ is amenable. (For a proof of the equivalence of amenability and right amenability for groups see M. M. Day [1]. This paper delivered in 1968 surveys amenable and left amenable semigroups and discusses many properties which are implied by or equivalent to amenability. It contains an extensive bibliography on the subject.)

M. Hochster [9] has shown that a subsemigroup of an amenable group need not be amenable. Let $G$ be an amenable group which contains $S$, a subsemigroup of $G$, such that $S$ is not amenable. Then $G(S)$, the smallest group containing $S$, is contained in $G$. Since $G$ is amenable, every subgroup contained in $G$ is amenable. Hence $G(S)$ is amenable. Thus the condition that $S$ is amenable is stronger than the condition that $G(S)$ is amenable (see also Granirer [3], p. 108). According to
Hochster, G. Frey in his unpublished dissertation has shown that subsemigroups inherit amenability if and only if the group contains no copy of the free semigroup on two generators.
II. SIZE TWO GENERATORS

Let \((X, \mathcal{J}, p)\) be a finite measure space with \(p(X) = 1\). Let \(S\) be a semigroup of nonsingular invertible transformations on \(X\). In this section I shall show that if \(\mathcal{J}\) is a separable \(S^{-1}\) exhaustive sub-\(\sigma\)-algebra of \(\mathcal{J}\) and if \(S\) is either left amenable or \(\mathcal{J}\)-invertible, then the system of sets \(A\), such that \([A, A^0]\) is a size two \(S^{-1}\)-generator for \(\mathcal{J}\), is dense in \(\mathcal{J}\).

Lemma 2.1: Let \(S\) be a semigroup of \(\mathcal{J}\)-measurable nonsingular transformations on \(X\) which are one-to-one and onto. If for all \(\varepsilon > 0\), there exists a set \(X' \in \mathcal{J}\) with \(p(X') > 1 - \varepsilon\) and \(\inf\{p(sX') : s \in S\} = 0\), then for any decreasing sequence of positive numbers \(\varepsilon_k\), there exist transformations \(s_k\) in \(S\) and sets \(X_k \in s_k \mathcal{J}\) such that

1) \(p(x_k) > 1 - \varepsilon_k\),

ii) \(p(s_i s_k^{-1} x_k) < \varepsilon_k / 2^k\) for \(i < k\), and

iii) \(p(s_k (\cup_{i < k} s_i^{-1} x_i)) < \varepsilon_k\).

Proof: First I will show that I may find \(s_i \in S\) and \(x_i^n \in s_i \mathcal{J}\) for \(1 \leq i \leq n\) so that

1') \(p(x_i^n) > 1 - \varepsilon_i (1 - 2^{-n})\), \(1 \leq i \leq n\);

ii') \(p(s_j s_i^{-1} x_i^n) < 2^{-i} \varepsilon_i\), \(1 \leq j < i \leq n\);

iii') \(p(s_j (\cup_{i < j} s_i^{-1} x_i^n)) < \varepsilon_j\), \(1 \leq j \leq n\).
For $n = 1$, select $x_1^n$ so that $p(x_1^n) > 1 - \varepsilon_1(l - \frac{1}{2})$. So $i''$ is satisfied. Note that $ii''$) and $iii''$) are vacuously true for $n = 1$ since there does not exist any $j$, $1 \leq j < i \leq 1$, or any $i$ such that $1 \leq i < j \leq 1$. Next suppose that for $n < k$, $s_i \in S$,

$x_i^n \in s_i \mathcal{F}$ for $i$ such that $1 \leq i \leq n$ have been chosen so that $i''$, $ii''$ and $iii''$ hold. I shall now show that $s_k \in S$ and $x_k^n \in s_k \mathcal{F}$ for $i$ such that $1 \leq i \leq k$ may be chosen so that $i''$, $ii''$ and $iii''$ hold.

(1) Choose $\delta$ such that $\delta < 2^{-k} \varepsilon_k$ and $p(A) < \delta$ implies

\[ p(s_i A) < 2^{-k} \varepsilon_k \]  

for $1 \leq i \leq k$.

This is possible since \{s_i : 1 \leq i \leq k\} is a finite collection of non-singular transformations on $X$. Using the hypothesis that for all $\varepsilon > 0$ there exists a set $X' \in \mathcal{F}$ with $p(X') > 1 - \varepsilon$ and

\[ \inf\{p(sX') : s \in S\} = 0 \], choose $Y_k \in \mathcal{F}$ such that

(2) $p(Y_k) > 1 - \delta$ and $\inf\{p(sY_k) : s \in S\} = 0$.

Next select $s_k$ so that

(3) $p(s_k Y_k) < \delta$.

Note that $p(Y_k) > 1 - \delta$ implies $p(Y_k^c) < \delta$, so that by (1) and (2)

(4) $p(s_i Y_k) < \varepsilon_k 2^{-k}$ for $1 \leq i \leq k$.

Next define $x_k^n$ by
(5) \( x_k^k = y_k - s_k y_k \).

Since \( y_k \in \mathcal{F} \) and \( s_k \) is \( \mathcal{F} \)-measurable I have \( \frac{s_k}{y_k} x_k^k = s_k^{-1} y_k - y_k \in \mathcal{F} \); i.e.,

(6) \( x_k^k \in s_k \mathcal{F} \).

Also,

(7) \( s_k^{-1} x_k^k \in y_k^c \).

For \( 1 \leq i < k \), define \( x_i^k \) by

(8) \( x_i^k = x_i^{k-1} \cap s_1 y_k \subseteq x_i^{k-1} \) where \( s_1, x_i^{k-1} \in s_i \mathcal{F} \) for \( 1 \leq i \leq k - 1 \) have already been selected.

\( y_k \in \mathcal{F} \) implies that

(9) \( x_i^k \in s_i \mathcal{F} \) for \( 1 \leq i \leq k - 1 \).

Then also

(10) \( s_1^{-1} x_1^k = s_1^{-1} x_1^{k-1} \cap y_k \subseteq y_k \) for \( 1 \leq i \leq k - 1 \).

Now I will show that if i'), ii') and iii') hold for \( s_1 \in \mathcal{S} \) and \( x_i^n \in s_i \mathcal{F} \) where \( i \) is such that \( i \leq n \) and \( n \) is such that \( n < k \), then i'), ii') and iii') hold for \( s_k \) and \( x_k^k \) defined as above in (3), (5) and (8). If \( 1 \leq i \leq k - 1 \) then (8), (4) and the assumption that i') holds for \( x_i^{k-1} \) imply that
(11) \( p(x_i^k) > p(x_i^{k-1}) - p(s_i x_k^c) > 1 - \varepsilon_1 (1-2^{-(k-1)}) - \varepsilon_k 2^{-k} \)
\[
\geq 1 - \varepsilon_1 (1-2^{-k}) .
\]

The statements (5), (1), (2) and (3) imply that

(12) \( p(x_i^k) > p(Y_k) - p(B_k Y_k) > 1 - \varepsilon_1 (1-2^{-n}) \) for \( i \) such that \( 1 < i < n \), holds for \( n = k > 1 \).

If \( i \) and \( j \) are such that \( 1 < j < i < k - 1 \) I have that (8) and the assumption that \( ii' \) holds for \( x_i^{k-1} \) imply that

(13) \( p(s_i s_1^{-1} x_i^k) \leq p(s_i s_1^{-1} x_i^{k-1}) \leq \varepsilon_1 2^{-1} . \)

By (7) and (4) for \( j \) such that \( 1 < j < k \),

(14) \( p(s_j s_k^{-1} x_k^c) \leq p(s_j x_k^c) < 2^{-k} \varepsilon_k . \)

Now (13) and (14) show that \( iii' \), \( p(s_j s_1^{-1} x_i^k) < 2^{-1} \varepsilon_1 \) for \( i \) such that \( 1 < j < i < n \), holds for \( n = k \).

Statement (8) and the assumption that \( iii' \) holds for \( x_i^{k-1} \) and \( s_i \), where \( 1 < i < k - 1 \), imply that for \( i \) and \( j \) such that

\( 1 < i < j < k - 1 \), I have

(15) \[ p(s_j \cup s_i^{-1} x_i^k)) \leq p(s_j \cup s_i^{-1} x_i^{k-1}) < \varepsilon_j . \]

By (10), (1) and (3) for \( i \) such that \( 1 < i < k \),
(16) \( p(s_k \cup s^{-1}_k x') \leq p(s_k x_k) < \varepsilon_k / 2 \).

Now (15) and (16) imply that iii'), \( p(s_j \cup s^{-1}_j x^n) < \varepsilon_j \) for \( j \) such that \( 1 \leq j \leq n \), holds for \( n = k \).

So I have shown that \( s_i \) and \( x^n_k \in s_i \mathcal{F} \) may be selected so that iii'), ii') and iii') are satisfied.

Now set \( X_k = n x^{n_k} \), I shall show that this choice of \( X_k \) satisfies lemma 2.1. Since \( s_k \) is one-to-one and onto, \( s_k \mathcal{F} \) is a \( \sigma \)-algebra so that \( x^{n_k} \in s_k \mathcal{F} \) (by (6) and (9)) implies that \( X_k \in s_k \mathcal{F} \). Since for \( n \geq k \) I have \( x^{n_k} \supseteq x^{n+1} \) (by (8)) and since \( p(x) = 1 \), I have that
\[
p(X_k) = p(\cap x^{n_k}) = \lim_{n \to \infty} p(x^{n_k}) > \lim_{n \to \infty} (1 - \epsilon_k (1 - 2^{-n})) = 1 - \epsilon_k.
\]
That is i'), \( p(X_k) > 1 - \epsilon_k \), is proven. Now ii') implies that
\[
p(s_j s^{-1}_k x^n_k) < 2^{-k} \epsilon_k \]
for \( j \) and \( k \) such that \( 1 \leq j < k \leq n \) hence,
\[
p(s_i s^{-1}_k x^n_k) < \epsilon_k 2^{-k} \]
for \( i \) such that \( 1 \leq i \leq k \). Since in addition \( X_k \subseteq x^{n_k} \), the condition ii') holds. Now iii') similarly implies that iii) holds. Thus lemma 2.1 is proven.

Theorem 2.1: Let \( \mathcal{S} \) be a semigroup of nonsingular invertible transformations on a finite measure space \((X, \mathcal{J}, p)\) with \( p(x) = 1 \).

Assume that there does not exist a finite \( S \)-invariant measure \( p_0 \ll p \) on \( \mathcal{J} \). Let \( \mathcal{J} \) be a separable \( S^{-1} \) exhaustive sub-\( \sigma \)-algebra of \( \mathcal{J} \). If \( S \) is either left amenable or \( \mathcal{J} \)-invertible then the system of sets \( A \in \mathcal{J} \), such that \( \{A, A^c\} \) is a size two \( S^{-1} \) generator for \( \mathcal{J} \), is dense in \( \mathcal{J} \). There exist \( S^{-1} \)-generating sets \( A \) with
dense $S^{-1}$-orbits in $\mathcal{J}$ if $(X, J, \rho)$ is separable.

**Proof:** If $(F_i^1)_{i=1}^\infty$ is a dense generating set for $\mathcal{J}$ then by using the diagonal process:

\[
\begin{align*}
F_1 &\rightarrow F_2^1 \rightarrow F_3^1 \rightarrow F_4^1 \rightarrow \cdots \\
F_1^1 &\rightarrow F_2^1 \rightarrow F_3^1 \rightarrow F_4^1 \rightarrow \cdots \\
F_1^1 &\rightarrow F_2^1 \rightarrow F_3^1 \rightarrow F_4^1 \rightarrow \cdots \\
&\vdots \\
I \end{align*}
\]

I produce $(F_i^1)_{i=1}^\infty = (F_1^1, F_2^1, F_3^1, F_4^1, \ldots)$, which has the property that every $F_k$ appears infinitely often in the sequence $(F_i^1)_{i=1}^\infty$. $\mathcal{J}$ is separable so that it may be assumed that $(F_i^1)_{i=1}^\infty$, $F_i \in \mathcal{F}$, is a countable dense generating set for $\mathcal{J}$ and that every $F_i$ appears infinitely often.

If $\mathcal{S}$ is $\mathcal{J}$-invertible, apply theorem 1.1 (iii) with $\mathcal{J}$ replacing $\mathcal{J}$ to find an $X' \in \mathcal{J}$ with $\rho(X') > 1 - \epsilon$ and $\inf\{\rho(sX') : s \in \mathcal{S}\} = 0$.

If $\mathcal{J}$ is an $S^{-1}$-exhaustive sub-$\sigma$-algebra of $\mathcal{J}$ and if $\mathcal{S}$ is left amenable, then apply theorem 1.2 (iii) to find, again, an $X' \in \mathcal{J}$ with $\rho(X') > 1 - \epsilon$ and $\inf\{\rho(sX') : s \in \mathcal{S}\} = 0$. Let $F_0 \in \mathcal{J}$ and let $(s_k^0)_{k=0}^\infty$ be a decreasing sequence of positive numbers. Apply lemma 2.1 to find $s_k \in \mathcal{S}$ and sets $X_k \in s_k \mathcal{J}$, for $k \geq 0$ and $s_0$, taken to be the identity transformation. Let $A = \bigcup_{k=0}^\infty s_k^{-1} (F_k \cap X_k) \in \mathcal{J}$, where $(F_k)_{k=1}^\infty$ is the dense generating set of $\mathcal{J}$ constructed above.

Then $\rho(s_k A \cup F_k) \leq \rho(F_k - F_k \cap X_k) + \rho(\bigcup_{i<k} s_i s_i^{-1} X_k) + \rho(s_k A \cup F_k)$.
\[ \sum_{i=k}^{\infty} p(s_i^{-1}x_i) < 3\epsilon_k \] for a given decreasing sequence of positive numbers \((\epsilon_k)_{k=1}^{\infty}\). This shows that \((s_iA)_{k=1}^{\infty}\) is dense in \((F_i)_{i=1}^{\infty}\) since \(\epsilon_k\) decreases to zero. Thus \([A, A^0]\) is an \(S^{-1}\)-generator for \(\mathcal{J}\). Since \(\epsilon_0\) may be taken arbitrarily small these sets \(A\) are \(3\epsilon_0\) close in measure to any \(F_0 \in \mathcal{J}\); that is, that the system of sets \(A\) constructed above are dense in \(\mathcal{J}\).

If \(S\) is left amenable, the algebra \(S\mathcal{J}\) is dense in \(\mathcal{J}\). \(S\mathcal{J}\) being an algebra (lemma 1.3) and \(\mathcal{J}\) being an \(S^{-1}\)-exhaustive sub-\(\sigma\)-algebra of \(\mathcal{J}\) implies that \(\sigma(S\mathcal{J}) = \mathcal{J}\). If \(\mathcal{J}\) is separable let \(G_j\), \(j = 1, 2, \ldots\) be a countable dense set in \(\mathcal{J}\) such that every \(G_j\) appears infinitely often. There exists an \(H_j \in \mathcal{J}\) with
\[ p(t_j H_j \setminus G_j) < s_j \] and \(s_j A\) with
\[ p(t_j s_j A \setminus G_j) < 2s_j \]. If \(S\) is \(\mathcal{J}\)-invertible then \(\mathcal{J} = \mathcal{J}\). Therefore in both cases the \(S^{-1}\)-orbit of \(A\) is dense in \(\mathcal{J}\). Thus theorem 2.1 is proved.

Let \(G_+\) be a group of nonsingular invertible transformations on a finite separable measure space \((X, \mathcal{J}, \mu)\) with \(\mu(X) = 1\). The metric \(d(A, B) = \mu(A \Delta B)\) for \(A, B \in \mathcal{J}\) makes \(\mathcal{J}\) into a complete separable metric space. I shall now show that the system of \(G_+\)-generating sets with dense orbits in \(\mathcal{J}\) is a dense \(G_0\) set. Let \(G_j\) be, as before, a countable dense set in \(\mathcal{J}\) such that \(G_j\) appears infinitely often in the sequence. Let \((\epsilon_j)_{j=1}^{\infty}\) be a sequence of positive numbers which monotonically decreases to zero. Let \(A\) be a
Since $A$ has a dense $G_+$ orbit, there exist transformations $s_{j,A} G_+$ such that
\[ p(s_{j,A} A \Delta G_j) < \frac{1}{2} \varepsilon_j. \]
Note that these transformations $s_{j,A}$ may be different for different sets $A$, but that $G_j$ and $\varepsilon_j$ remain the same regardless of which $G_+$-generating set $A$ is used. For convenience I may denote any set $C$ by $s_{0,A} C$. Set $\Theta(i,k;A) = \{ B \in \mathcal{B} : p(B \Delta s_{i,A} A) < 1/n(i,k) \}$, where $n(i,k)$ is chosen so that $p(C) < 1/n(i,k)$ implies that $p(s_{j,A}^{-1} C) < \frac{1}{2} \varepsilon_j$ for fixed $i$ and for all $j$ such that $0 \leq j \leq k$. This is possible since all transformations in $G_+$ are nonsingular invertible and $p$ is a finite measure. Thus, $B \in \Theta(i, k; A)$ implies that
\[ p(s_{j,A}^{-1} B \Delta G_j) \leq p(s_{j,A}^{-1} (B \Delta s_{i,A} A)) + p(s_{j,A} A \Delta G_j) < \varepsilon_j \]
for fixed $i$ and for all $j$ such that $0 \leq j \leq k$. Now suppose $B \in \bigcup_{k=0}^{\infty} \bigcup_{i=0}^{A} \Theta(i,k;A)$. Then for each $k \geq 0$, there exists an $i_k$ and an $A_k$ such that $B \in \Theta(i_k, k; A_k)$. That is,
\[ p(s_{j,A}^{-1} B \Delta G_j) < \varepsilon_j \]
for fixed $i_k$ and for all $j$ with $0 \leq j \leq k$. Recall that $\lim \varepsilon_j = 0$ and $G_j$ appears infinitely often in the dense generating set $\{ G_i \}_{i=1}^\infty$. Hence
\[ \{ s_{j,A}^{-1} B \}_{i,j,A} \text{ is a dense generating set for } \mathcal{B} \text{ i.e., } \{ B ; B^c \} \]
is a $G_+$-generating set for $\mathcal{B}$ with a dense $G_+$ orbit. Now $\bigcap_{k=0}^{\infty} \bigcup_{i=0}^{A} \Theta(i,k;A)$ contains $\{ s_{i,A} A \}_{i=0}^\infty$, a dense orbit of $A$ in $\mathcal{B}$. 

$G_+$-generating set with a dense $G_+$ orbit. Since $A$ has a dense $G_+$ orbit, there exist transformations $s_{j,A} G_+$ such that
\[ p(s_{j,A} A \Delta G_j) < \frac{1}{2} \varepsilon_j. \]
Note that these transformations $s_{j,A}$ may be different for different sets $A$, but that $G_j$ and $\varepsilon_j$ remain the same regardless of which $G_+$-generating set $A$ is used. For convenience I may denote any set $C$ by $s_{0,A} C$. Set $\Theta(i,k;A) = \{ B \in \mathcal{B} : p(B \Delta s_{i,A} A) < 1/n(i,k) \}$, where $n(i,k)$ is chosen so that $p(C) < 1/n(i,k)$ implies that $p(s_{j,A}^{-1} C) < \frac{1}{2} \varepsilon_j$ for fixed $i$ and for all $j$ such that $0 \leq j \leq k$. This is possible since all transformations in $G_+$ are nonsingular invertible and $p$ is a finite measure. Thus, $B \in \Theta(i, k; A)$ implies that
\[ p(s_{j,A}^{-1} B \Delta G_j) \leq p(s_{j,A}^{-1} (B \Delta s_{i,A} A)) + p(s_{j,A} A \Delta G_j) < \varepsilon_j \]
for fixed $i$ and for all $j$ such that $0 \leq j \leq k$. Now suppose
$B \in \bigcup_{k=0}^{\infty} \bigcup_{i=0}^{A} \Theta(i,k;A)$. Then for each $k \geq 0$, there exists an $i_k$ and an $A_k$ such that $B \in \Theta(i_k, k; A_k)$. That is,
\[ p(s_{j,A}^{-1} B \Delta G_j) < \varepsilon_j \]
for fixed $i_k$ and for all $j$ with $0 \leq j \leq k$. Recall that $\lim \varepsilon_j = 0$ and $G_j$ appears infinitely often in the dense generating set $\{ G_i \}_{i=1}^\infty$. Hence
\[ \{ s_{j,A}^{-1} B \}_{i,j,A} \text{ is a dense generating set for } \mathcal{B} \text{ i.e., } \{ B ; B^c \} \]
is a $G_+$-generating set for $\mathcal{B}$ with a dense $G_+$ orbit. Now $\bigcap_{k=0}^{\infty} \bigcup_{i=0}^{A} \Theta(i,k;A)$ contains $\{ s_{i,A} A \}_{i=0}^\infty$, a dense orbit of $A$ in $\mathcal{B}$.
Therefore \( \cap \bigcup_{k \in A} \mathcal{G}(i, k; A) \) is a dense \( G_0 \) set in \( \mathcal{J} \). Since

\[ s_{0,k} A = A \text{ the system of sets } \bigcap_{k \in A} \bigcup_{i \in A} \mathcal{G}(i, k; A) \text{ consists of all } \]

\( G_\perp \)-generating in \( \mathcal{J} \) sets with dense orbits.

Now suppose that \( m \) is a \( \sigma \)-finite infinite \( S \)-invariant measure on \((X, \mathcal{J})\) and that there exists no finite invariant measure \( p_0 \ll m \). For a probability measure, \( p \) equivalent to \( m \) apply theorem 2.1 and find \( A \) such that \( \{A, A^0\} \) is an \( S_{-1} \)-generator for \( \mathcal{J} \). The question arises as to whether this \( A \) has finite \( m \)-measure.

I will show in chapter 3 that given suitable conditions a size two \( S_{-1} \)-generator \( \{A, A^0\} \) may be found such that \( m(A) \) is finite.

Here however I would like to show that the \( A \) constructed in theorem 2.1 does not necessarily have finite measure.

Let \( \{G_i\}_{i=1}^\infty \) be an \( S_{-1} \)-generating set for \( \mathcal{J} \). If \( m(G_i \cap X_i) = \infty \) for any \( i \), then since \( m \) is \( S \)-invariant, \( m\left( \bigcup_{i=1}^\infty s_{-1}^{-1}(G_i \cap X_i) \right) = \infty \).

However, \( \{G_i\}_{i=1}^\infty \) may easily be chosen so that \( m(G_i) \) is finite for all \( i \). Even this choice will not necessarily produce an \( A \) of finite measure for theorem 2.1. Indeed if \( m(G_i) \) is finite then so is \( m\left( \bigcup_{j<i} s_{-1}^{-1}(G_j \cup G_i) \right) \); so for all \( \eta_i > 0 \) there exists a \( \delta_i > 0 \) such that \( p(A \cap ( \bigcup_{j<i} s_{-1}^{-1}(G_j \cup G_i)) < \delta_i \) implies \( m(A \cap ( \bigcup_{j<i} s_{-1}^{-1}(G_j \cup G_i)) < \eta_i \) (since \( p \) restricted to \( \bigcup_{j<i} s_{-1}(G_j \cup G_i) \)) \)
is equivalent to \( m \) restricted to that same set and on that set \( m \)
is a finite measure). Now

\[
m(\bigcup_{i=0}^{\infty} s_i^{-1}(g_i \cap W_i)) \geq \sum_{i=1}^{\infty} (m(s_i^{-1}g_i) - m(g_i \cap W_i) - m(s_i^{-1}(g_i \cap W_i))
\]

\( \bigcap_{j<i} s_j^{-1}g_j \) which is infinite if \( \sum_{i=1}^{\infty} \eta_i \) is finite since \( G_k \)
appears infinitely often in the sequence \( (g_i)_{i=1}^{\infty} \). So that if

\( e_1 \) is less than \( \delta_1 \) where \( \sum_{i=1}^{\infty} \eta_i \) is finite, then \( A \) will be a

set of infinite measure.
III. GENERATORS AND ERGODIC GROUPS

In this section I am concerned with a measure space \((X, \mathcal{J}, \mu)\) where \(\mu\) is a \(\sigma\)-finite infinite measure and \(\mathcal{S}\) is a semigroup of nonsingular \(\mathcal{J}\)-invertible, \(\mathcal{J}\)-measurable measure preserving transformations on \(X\). I will show that if \(X = \bigcup_{i=1}^{\infty} Z_i\) where \(Z_i\) is an \(S\)-\(\mu\)-invariant set of infinite measure in \(\mathcal{J}\) such that \(\mathcal{S}\) has no nontrivial \(\mu\)-subinvariant sets in \(Z_i\), and if \(\mathcal{F}\) is a separable \(S^{-1}\)-exhaustive sub-\(\sigma\)-algebra of \(\mathcal{J}\) such that \(\mu\) is \(\sigma\)-finite on \(\mathcal{F}\), then the system of sets \(H \in \mathcal{F}\), such that \(\{H, H^c\}\) is an \(S^{-1}\)-generator for \(\mathcal{J}\), has the property that the system of sets \(H \cap X_N\), where \(X_N\) is the nonatomic part of \((X, \mathcal{F})\), is dense in the nonatomic sets in \(\mathcal{F}\) of finite measure. First I need to establish a result about the way \(s \in \mathcal{S}\) transforms \(X\) by decomposing \(s\) into its strictly periodic pieces. I shall begin with a discussion of periodic and aperiodic sets for a nonsingular measurable transformation. The results are due to Jones and Krengel [10] and Helmberg and Simons [8] and are summarized as lemma 3.1.

I shall now give some definitions that I shall need in the sequel. Let \(\mathcal{S}\) be a set of nonsingular measurable transformations on \((X, \mathcal{J}, \mu)\) with \(\mu\) a \(\sigma\)-finite measure. A set \(F \in \mathcal{J}\) is \(S\)-invariant if \(F = s^{-1}F\) for all \(s \in \mathcal{S}\). \(S\) is ergodic if \(F\) an \(S\)-invariant set
implies that either $m(F) = 0$ or $m(F^c) = 0$. $F$ is an $m$-invariant set for $S$ if $F = s^{-1}F (m)$ for all $s \in S$. $S$ is $m$-ergodic if for all $m$-invariant sets $F$ I have $m(F) = 0$ or $m(F^c) = 0$. In a similar fashion define an $S$-subinvariant set as a set $F \in \mathcal{F}$ such that $s^{-1}F \subset F$ for all $s \in S$, and an $S$-m-subinvariant set as a set $F \in \mathcal{F}$ such that $s^{-1}F \subset F (m)$ for all $s \in S$. Call the system of sets $\{A_i : i \in I\}$ m-disjoint if $m(A_i \cap A_j) = 0$ for all $i, j \in I$ where $i \neq j$.

Let $s$ be a nonsingular transformation on $(X, \mathcal{F}, m)$. $W$ is called a wandering set if $s^{-i}W \cap s^{-j}W = \emptyset$ for all integers $i, j \geq 0$, where $i \neq j$. Let $\mathcal{A} = \{B: B = \bigsqcup_{i=1}^\infty W_i, W_i \in \mathcal{F}, W_i \text{ is a wandering set}\}$. Let $\mathcal{P}$ be a finite measure on $(X, \mathcal{F})$ equivalent to $m$. Let $B_i \in \mathcal{B}$ be such that $\lim_{i \to \infty} p(B_i) = \sup_{B \in \mathcal{B}} p(B)$. Then $\bigsqcup_{i=1}^\infty B_i \in \mathcal{F}$ and if $W$ is a wandering set and if $W \in \mathcal{F}$ then, $W \subset \bigsqcup_{i=1}^\infty B_i (p)$. Now, $\mathcal{P}$ is equivalent to $m$ so that $W \subset \bigsqcup_{i=1}^\infty B_i (m)$ also. Call $D = \bigsqcup_{i=1}^\infty B_i$, the dissipative part of $X$. Obviously $s^{-1}D \subset D (m)$. Call $C, C = D^c$, the conservative part of $X$; then $s^{-1}C \supset C (m)$. A transformation $s$ is periodic on $A$, $A \in \mathcal{F}$, if there exists a natural number $n$ such that $s^{-n}B \subset B (m)$ for all measurable sets $B \subset A$. The period of $s$ on $A$ is the least such natural number. The transformation $s$ is strictly periodic on $A$ with period $n, A \in \mathcal{F}$, if for every set $B$, such that, $B \subset A$ and $m(B) > 0$, $s$ is periodic with period $n$ on $B$. A transformation $s$ is aperiodic on $A, A \in \mathcal{F}$, if all measurable subsets on which $s$ is periodic have measure zero.
Since \( s \) is a not necessarily invertible transformation it may be useful to recall that \( ss^{-1} A \subset A \subset ss^{-1} A \), and that \( s(A \cap B) \subset sA \cap sB \). If \( s \) is nonsingular and \( \mathcal{J} \)-measurable, and if \( A, B \in \mathcal{J} \), then \( A \subset B \) \((m)\) implies that \( ss^{-1} A \subset ss^{-1} B \) \((m)\). If \( ss^{-1} A \in \mathcal{J} \), \( A \subset A \) \( \mathcal{J} \) with \( m(A) = 0 \), then \( 0 \leq m(ss^{-1} A) \leq m(s^{-1} A) = 0 \); i.e., \( m(ss^{-1} A) = 0 \).

Helmberg and Simons (lemma 1.3 in [8]) show that there exists a collection \( \{X_n : n = 0, 1, \ldots\} \) of \( m \)-disjoint sets such that
\[
\bigcup_{n=0}^{\infty} X_n = X(m),
\]
where \( s \) is strictly periodic with period \( n \) on \( X_n \) for \( n \geq 1 \) and aperiodic on \( X_0 \). Moreover, \( ss^{-1} X_n \supset X_n(m) \) for \( n \geq 1 \). Helmberg and Simons prove this result and also lemma 1.2 of their paper for \( m \) a finite measure. If \( p \) is a finite measure equivalent to \( m \), a \( \sigma \)-finite measure, then \( m \) and \( p \) have the same sets of zero measure, so that the statements of their lemmas 1.2 and 1.3 remain true if \( m \) is used rather than \( p \).

Now I will show that for any nonnegative integers \( i \) and \( j \) and for any \( n_1 \neq n_2 \), where \( n_1, n_2 > 0 \), \( m(s^{-1} X_{n_1} \cap s^{-1} X_{n_2}) = 0 \).

For \( k \geq 0 \) I have \( s^{-k} X_{n_2} \supset X_{n_2}(m) \), so \( s^{-k}(X_{n_1} \cap X_{n_2}) \supset s^{-k} X_{n_1} \cap X_{n_2}(m) \). Since \( m(X_{n_1} \cap X_{n_2}) = 0 \) implies \( m(s^{-k}(X_{n_1} \cap X_{n_2})) = 0 \), it must be that \( m(s^{-k} X_{n_1} \cap X_{n_2}) = 0 \). Then \( m(s^{-(k+j)} X_{n_1} \cap s^{-(k+j)} X_{n_2}) = 0 \) for any nonnegative integers \( j, k \). By the symmetry of \( n_1 \) and \( n_2 \) in the above argument, \( m(s^{-1} X_{n_1} \cap s^{-1} X_{n_2}) = 0 \) for \( i \) and \( j \geq 0 \), \( n_1, n_2 > 0 \), \( n_1 \neq n_2 \). Hence, also, \( m(\bigcup_{i=0}^{\infty} s^{-i} X_{n_1} \cap \bigcup_{i=0}^{\infty} s^{-i} X_{n_2}) = 0 \).

Since for \( n \geq 1 \), \( ss^{-1} X_n \supset X_n(m) \), so that \( ss^{-1}(\bigcup_{i=0}^{\infty} s^{-i} X_n) = \bigcup_{i=1}^{\infty} ss^{-i} X_n = \bigcup_{i=0}^{\infty} s^{-i} X_n(m) \). That is, \( \bigcup_{i=0}^{\infty} s^{-i} X_n \) is an \( s \)-\( m \)-invariant set for
n \geq 1$. Now if \(\{X_i: i \geq 0\}\) is an \(\mathcal{F}\)-measurable partition, then 
\[s^{-1}X_i: i \geq 0\] is also an \(\mathcal{F}\)-measurable partition of \(X\); hence 
\[m(s^{-1}X_0 \cap (\bigcup_{i=1}^{\infty} s^{-1}X_i)) = 0.\] But \((\bigcup_{i=1}^{\infty} s^{-1}X_i) \supseteq \bigcup_{i=1}^{\infty} X_i\) (m); hence, 
\[s^{-1}X_0 \subseteq (\bigcup_{i=1}^{\infty} s^{-1}X_i)^c \subseteq (\bigcup_{i=1}^{\infty} X_i)^c = X_0\ (m).\]

In lemma 1.2 Helmberg and Simons show that there exists 
\(Y_n \in \mathcal{F}, Y_n \subseteq X_n\), for \(n \geq 1\), such that \(\bigcup_{i=0}^{n-1} s^{-1}X_n \cap C = \bigcup_{i=0}^{n-1} s^{-1}Y_n \cap C\) (m) and \(s^{-1}Y_n \cap s^{-1}Y_n = \emptyset\) for \(i\) and \(j\) such that \(0 \leq i < j \leq n - 1\).

I will now establish that \(s^{-j}Y_n \supseteq s^{-i}Y_n\) (m) when \(i = j\) (n) \((i \neq j\) (n) means that \(j-i\) is a multiple of \(n\)), and \(0 \leq i \leq j\). I will also 
show that \(m(s^{-1}Y_n \cap s^{-j}Y_n) = 0\) when \(0 \leq i < j\) and \(i \neq j\) (n). Since 
s has strict period \(n\) on \(X_n\), for \(n \geq 1\), if in addition \(Y_n \subseteq X_n\) then 
\(s^{-j}Y_n \supseteq Y_n\ (m)\). For \(k_2 > k_1, s^{-k_2}Y_n \supseteq s^{-k_1}Y_n \supseteq Y_n\ (m)\) and
\[s^{-k_2-n}Y_n \supseteq s^{-k_2-n+1}Y_n \supseteq s^{-k_1-n}Y_n \supseteq Y_n\ (m).\] To see that \(m(s^{-1}Y_n \cap s^{-j}Y_n) = 0\) for \(0 \leq i < j\), \(j \neq i\) (n), observe that \(i = k_1n + r_1\) and \(j = k_2n + r_2\) 
with \(0 \leq r_1 < n\), \(0 \leq r_2 < n\), \(r_1 \neq r_2\) and \(k_1 \leq k_2\) imply 
\[s^{-i}Y_n \cap s^{-j}Y_n = s^{-k_1n-r_1}Y_n \cap s^{-k_2n-r_2}Y_n \subseteq s^{-k_2n-r_2}Y_n \cap s^{-k_2n-r_2}Y_n = \emptyset\ (m),\]
\[s^{-k_2n} (s^{-1}Y_n \cap s^{-2}Y_n) = \emptyset\ (m),\] because \(s^{-1}Y_n \cap s^{-2}Y_n = \emptyset\).

This decomposition above is the same for all equivalent, finite or \(\sigma\)-finite measures on \((X, \mathcal{F})\). I will use the notations \(X_n^\mathcal{F}\) (s) 
and \(Y_n^\mathcal{F}\) (s) where necessary to indicate the transformations and the 
\(\sigma\)-algebra with respect to which the decomposition is to be taken.
Where it will cause no confusion I may drop the explicit designation of either the $\sigma$-algebra or the transformation. I shall write $X_n^c$ for the complement of $X_n^T(s)$.

Next I will show, using the proof of theorems 1.11 of Jones and Krengels [10], that for a given natural number $m$, a given $\varepsilon > 0$ and a finite measure $p$ there exists a set $G_q \in \mathcal{F}$, $G_q \subseteq \bigcup_{i=0}^{\infty} s^{-i}(X_0 \cap C)$ such that

$$p\left(\bigcup_{i=0}^{\infty} s^{-i}(X_0 \cap C) - \bigcup_{j=1}^{m-l} s^{-j}G_q\right) < \varepsilon/2$$

and $s^{-1}G_q \cap s^{-j}G_q = \emptyset (p)$

for $0 \leq i < j \leq m-1$. It may be assumed that $p(X_0 \cap C) > 0$. Using Helmberg's and Simons' theorem 2.1, find $A$ such that $A \subseteq X_0 \cap C$,

$$p(A) > 0, \quad \bigcup_{i=0}^{\infty} s^{-i}A = \bigcup_{i=0}^{\infty} s^{-i}(X_0 \cap C) (p) \text{ and } A, s^{-1}_A, \ldots, s^{-r+1}_A$$

are pairwise disjoint. Choose $t \geq 1$, $t^{-1} < \varepsilon/2$ and let $r = mt$.

Let $G = \bigcup_{k=1}^{mk-1} (s^{-mk}A \cap (\bigcup_{i=0}^{\infty} s^{-i}A)^c)$. I have

1. $s^{-j}_G \cap s^{-i}_G = \emptyset$ for $0 \leq i < j \leq m-1$ and
2. $\bigcup_{i=0}^{\infty} s^{-i}A - \bigcup_{j=0}^{m-1} s^{-j}G \subseteq \bigcup_{i=0}^{\infty} s^{-i}A$. Since $s$ is conservative in $X_0 \cap C$, $\bigcup_{i=1}^{\infty} s^{-i}(X_0 \cap C) \supseteq X_0 \cap C (p)$. Thus $s(\bigcup_{i=1}^{\infty} s^{-i}A) = \bigcup_{i=1}^{\infty} s^{-i}A = \bigcup_{i=0}^{\infty} s^{-i}A (p)$. $\bigcup_{i=0}^{\infty} s^{-i}A$ are disjoint for $0 \leq h \leq t-1$ when $t^{-1} < \varepsilon/2$. So there exists a $q$ such that

$$p(\bigcup_{i=qm}^{(q+1)m-l} s^{-i}A) < \varepsilon/2.$$
Then \( p(\bigcup_{i=0}^{\infty} s^{-i}A - \bigcup_{j=0}^{m-1} s^{-j}G_0) = p(s^{-\infty}(\bigcup_{i=0}^{\infty} s^{-i}A - \bigcup_{j=0}^{m-1} s^{-j}G_0)) = \)
\[ p(s^{-\infty}(\bigcup_{i=0}^{\infty} s^{-i}A)) < \varepsilon/2 \quad \text{(see (2) and (3)).} \]
For \( G_q = s^{-\infty}G \in \mathcal{X} \)

I have that \( G_q \cap \bigcup_{i=0}^{\infty} s^{-i}(X_0 \cap C), p(\bigcup_{i=0}^{\infty} s^{-i}(X_0 \cap C) - \bigcup_{j=0}^{m-1} s^{-j}G_q) < \varepsilon/2 \)

\( s^{-1}G_q \cap s^{-j}G_q = \emptyset \) where \( 0 \leq i < j \leq m-1 \) (see (1)), as required.

Helmberg and Simons show that on \( D \), the dissipative part of \( X \), there exists a sequence of wandering sets \((W_n)_{n=1}^{\infty}\) such that

\[ \bigcup_{i=1}^{\infty} s^{-i}W_{n+1} \supset \bigcup_{i=1}^{\infty} s^{-1}W_n \text{ and } \bigcup_{i=1}^{\infty} (\bigcup_{i=1}^{\infty} s^{-1}W_n) = D. \]

So for all \( \varepsilon > 0 \)

there exists an \( n \) such that \( p(D - \bigcup_{i=1}^{\infty} s^{-1}W_n) < \varepsilon/2. \) Since \( W_n \)

is a wandering set \( W_n \subset D(p) \) and \( s^{-1}W_n \subset D(p). \) Let \( H = \bigcup_{i=0}^{\infty} s^{-km}W_n - \bigcup_{i=1}^{\infty} s^{-i}(X_0 \cap C) \subset X_0(p) \) and \( D \subset X_0(p). \) Then I have

\[ (4) \quad s^{-1}H \cap s^{-1}H = \emptyset \quad \text{for } 0 \leq i < j \leq m-1. \]

Let \( Y_0 = H \cup G_q. \) Then \( Y_0 \subset X_0(p). \) Since \( H \) and \( G_q \) are contained in disjoint s-invariant sets, so (1) and (4) imply

\[ s^{-i}Y_0 \cap s^{-j}Y_0 = \emptyset \quad 0 \leq i < j \leq m-1. \] Now,

\[ \bigcup_{i=0}^{\infty} s^{-i}X_0 = \bigcup_{i=0}^{\infty} s^{-i}(D \cup (X_0 \cap C)) = D \cup \bigcup_{i=0}^{\infty} s^{-i}(X_0 \cap C). \]

Also,

\[ \bigcup_{i=0}^{\infty} s^{-i}X_0 = \bigcup_{i=0}^{\infty} s^{-i}H \cup \bigcup_{i=0}^{m-1} s^{-i}G_q = \bigcup_{i=0}^{\infty} s^{-i}W_n \cup \bigcup_{i=0}^{\infty} s^{-i}(X_0 \cap C) \cup \bigcup_{i=0}^{m-1} s^{-i}G_q. \]
Now let \( m \) be a \( \sigma \)-finite measure of \((X, \mathcal{F})\) and let \( A \in \mathcal{F} \) with \( m(A) < \infty \). Since \( m \) is \( \sigma \)-finite I may find an \( \mathcal{F} \)-measurable partition 
\[(V_i : i = 2, 3, \ldots)\] of \( X - A \) with \( m(V_i) < \infty \). Define a measure \( \nu \) as follows:

\[
\nu(A) = \frac{m(A \cap V_i)}{2m(A)} + \sum_{i=1}^{\infty} \frac{m(A \cap V_i)}{2^i m(V_i)}.
\]

Note that \( \nu \) is a probability measure on \((X, \mathcal{F})\) and \( \nu \) is equivalent to \( m \). So the sets \( X_k \) of strict period \( k \) are the same for \( \nu \) and \( m \).

Now by what I have just shown, for any \( \delta > 0 \) and any \( n \geq 1 \), a \( y_0 \in \mathcal{F} \) may be found such that

\[
\nu(\bigcup_{i=0}^{n-1} s^{-i}y_0 \Delta \bigcup_{i=0}^{n-1} s^{-i}y_0) < \delta \quad \text{and} \quad s^{-i}y_0 \cap s^{-j}y_0 = \emptyset
\]

for \( i \) and \( j : 0 \leq i < j \leq n-1 \).

So

\[
\delta > \nu(\bigcup_{i=0}^{n-1} s^{-i}y_0 \Delta \bigcup_{i=0}^{n-1} s^{-i}y_0) \geq \frac{m((\bigcup_{i=0}^{n-1} s^{-i}y_0 \Delta \bigcup_{i=0}^{n-1} s^{-i}y_0) \cap A)}{2m(A)}.
\]
That is, $\delta < \varepsilon / 2m(A)$ implies that
\[ m((\bigcup_{i=0}^{n-1} s^{-i}X_0 - \bigcup_{i=0}^{n-1} s^{-i}Y_0) \cap A) < \varepsilon. \]

Now $s^{-1}X_0 \subseteq X_0 (m)$ implies that
\[ m((X_0 - \bigcup_{i=0}^{n-1} s^{-i}Y_0) \cap A) < \varepsilon \text{ and } s^{-i}Y_0 \cap s^{-j}Y_0 = \emptyset \]
for $0 \leq i < j \leq n-1$.

For future reference I summarize the preceding as:

**Lemma 3.1:** If $s$ is a nonsingular measurable transformation on $(X, \mathcal{F}, m)$ where $m$ is either finite or $\sigma$-finite then:

1) $X = \bigcup_{k=0}^{\infty} X_k(m)$ where $s$ has strict period $k$ on $X_k \in \mathcal{F}$, $k \geq 1$ and $s$ is aperiodic on $X_0$.

2) $s^{-1}X_k \supset X_k(m)$ for $k \geq 1$ and

3) $s^{-1}X_0 \subseteq X_0(m)$.

4) $\bigcup_{i=1}^{\infty} s^{-i}X_k$ is an $s$-m-invariant set for $k \geq 1$ and

\[ m((\bigcup_{i=0}^{n-1} s^{-i}X_k \cap \bigcup_{i=0}^{n-1} s^{-i}X_n) = 0 \text{ for } n \text{ and } k \text{ such that } n, k \geq 1, n \neq k. \]

5) For $k \geq 1$ there exists $Y_k \in \mathcal{F}$ such that

\[ \bigcup_{i=0}^{k-1} s^{-i}X_k \cap C = \bigcup_{i=0}^{k-1} s^{-i}Y_k \cap C \text{ and } \]
\[
\bigcup_{i=0}^{\infty} s^{-i}X_k = \bigcup_{i=0}^{\infty} s^{-j}X_k,
\]

\[
s^{-i}X_k \cap s^{-j}X_k = s^{-i}X_k (m) \text{ when } 0 \leq i \leq j \text{ and } i = j(k),
\]

and \[
s^{-i}X_k \cap s^{-j}X_k = \emptyset \text{ when } 0 \leq i < j \text{ and } i \neq j(k).
\]

v) For any \( \varepsilon > 0 \) and any \( n \geq 1 \) and for any set \( A \in \mathcal{F} \)
where \( m(A) \) is finite, there exists a \( Y_0 \in \mathcal{F} \), \( Y_0 \subset X_0(m) \)
such that \( s^{-i}Y_0 \cap s^{-j}Y_0 = \emptyset \) for \( i \) and \( j \) such that
\[
0 \leq i < j \leq n-1, \text{ and } m((X_0 - \bigcup_{i=0}^{n-1} s^{-i}Y_0) \cap A) < \varepsilon \text{ (or}
\]
equivalently, \( m(\bigcup_{i=0}^{n-1} s^{-i}Y_0 \cap A) > m(X_0 \cap A) - \varepsilon) \).

**Lemma 3.2:** Let \( s \) be a nonsingular measurable transformation on
\((X, \mathcal{F}, m)\). If \((A_n)_{n=1}^{\infty}\) is a sequence of sets such that
\[
(5) \quad A_i \cap s^{-i}A_j = \emptyset (m) \text{ for } i, j \geq 1 \text{ and } i \neq j,
\]
then
\[
\bigcap_{n=0}^{\infty} s^{-n}(\bigcup_{j=1}^{k} A_j) = \bigcup_{j=1}^{k} \bigcap_{n=0}^{\infty} s^{-n}A_j (m).
\]

**Proof:** Obviously
\[
(6) \quad \bigcap_{n=0}^{\infty} s^{-n}(\bigcup_{j=1}^{k} A_j) \supseteq \bigcup_{j=1}^{k} \bigcap_{n=0}^{\infty} s^{-n}A_j .
\]

Set \( E = (\bigcup_{n=0}^{k} s^{-n}(\bigcup_{i,j \geq 1} (A_i \cap s^{-1}A_j)) = \bigcup_{n=0}^{k} \bigcup_{i,j \geq 1, i \neq j} (s^{-n}A_i \cap s^{-n-1}A_j). \)

[Image]
By (5) and the nonsingularity of $s$

\[(7) \quad m(E) = 0.\]

Next let $w \in \bigcap_{n=0}^{k} s^{-n}(\bigcup_{j=1}^{k} A_j) - E$. Then for all $n, 1 \leq n \leq k$ there exists a $j \geq 1$ such that $w \in s^{-n}A_j$. But $w \in A_{j_0} - E$ implies that $w \not\in s^{-1}A_j$ for any $j \neq j_0$. If $w \in s^{-n}A_j$, then $w \not\in s^{-n-1}A_j$ for any $j \neq j_0$, so that if $w \in \bigcap_{n=0}^{k} s^{-n}(\bigcup_{j=1}^{k} A_j) - E$ and $w \in A_{j_0}$, then $w \in s^{-n}A_{j_0}$ for all $n, 1 \leq n \leq k$; i.e.,

\[(8) \quad \bigcap_{n=0}^{k} s^{-n}(\bigcup_{j=1}^{k} A_j) - E \subseteq \bigcup_{j=1}^{k} \bigcap_{n=0}^{k} s^{-n}A_j .\]

Then (6), (7) and (8) imply the conclusion, that

$$\bigcap_{n=0}^{k} s^{-n}(\bigcup_{j=1}^{k} A_j) = \bigcup_{j=1}^{k} \bigcap_{n=0}^{k} s^{-n}A_j (m).$$

I will now make two observations to facilitate some later applications of lemma 3.2. First, if $\{Z_i : i = 1, 2, \ldots\}$ is a set, of $m$-disjoint $s$-$m$-invariant sets and if for each $i = 1, 2, \ldots,$ $A_i \subseteq Z_i$ then the condition $A_i \cap s^{-1}A_j = \emptyset (m)$ is satisfied and therefore $\bigcap_{n=0}^{k} s^{-n}(\bigcup_{j=1}^{k} A_j) = \bigcup_{j=1}^{k} \bigcap_{n=0}^{k} s^{-n}A_j (m)$. Secondly, if $s$ is a non-singular invertible transformation (both $s$ and $s^{-1}$ are nonsingular), i.e. (5), $A_i \cap s^{-1}A_j = \emptyset (m)$ for $i \neq j$, and if $k_1 \leq k_2$, then

$$\bigcap_{i=k_1}^{k_2} \bigcup_{j=1}^{k} A_j = s^{-k_2-k_1} \bigcap_{i=k_1}^{k_2} \bigcup_{j=1}^{i-k_1} A_j = s^{-k_2} \bigcap_{i=k_1}^{k} \bigcup_{j=1}^{i} \bigcup_{i=k_1}^{k} s^{-i} A_j.$$

**Lemma 3.3**: Assume that $(X, \mathcal{F}, m)$ is a $\sigma$-finite measure space, $s$ is a nonsingular measurable transformation on $(X, \mathcal{F}, m)$, $B \in \mathcal{F}$ and $m(B)$ is finite. Then
1) There exists \( B' \in \mathcal{F} \), \( B' \subseteq B \) with \( \frac{1}{2} \sum_{i=0}^{\infty} s_i B' = m(B') \geq \frac{1}{4} m(B \cap x_{\frac{1}{2}}^2) \), \( (x_{\frac{1}{2}}^2 = (x_{\frac{1}{2}}^1)') \).

ii) If \( \{ B'_i \}_{i=1}^{\infty} \) is a set of \( m \)-disjoint sets contained in \( B \) such that \( m(B'_i) \geq (1-r)m(B - \bigcup_{1 \leq j < \ell} B'_j) \) then \( m(\bigcup_{1 \leq j < \ell} B'_j) \geq (1-r^k)m(B) \).

**Proof**: i): Given \( B \in \mathcal{F} \) of finite measure, 
\[ e = m(X_0 \cap B - \bigcup_{j=2}^{\infty} \bigcup_{j=0}^{\infty} s^{-j}X_k)/3 \] and \( n=2 \), apply lemma 3.1 and find \( X_k \), \( Y_k \) in \( \mathcal{F} \), \( k \geq 0 \) such that (9) - (15) below hold:

(9) \( X = \bigcup_{k=0}^{\infty} X_k \);

(10) \( \bigcup_{j=0}^{\infty} s^{-j}X_k \) is an \( s-m \)-invariant set for \( k \geq 1 \);

(11) \( m(\bigcup_{i=0}^{\infty} s^{-i}X_{k_1} \cap \bigcup_{i=0}^{\infty} s^{-i}X_{k_2}) = 0 \) for \( k_1, k_2 \geq 1 \) and \( k_1 \neq k_2 \);

(12) \( s^{-i}Y_k \cap s^{-j}Y_k = \varnothing \) for \( k \geq 1 \) and \( j \geq i \geq 0 \) and \( i \neq j(k) \),  
\( s^{-i}Y_k \cap s^{-j}Y_k = s^{-i}Y_k(m) \) for \( k \geq 1 \), \( j \geq 1 \geq 0 \), \( i = j(k) \);

(13) \( m(\bigcup_{i=0}^{\infty} s^{-i}X_k - \bigcup_{i=0}^{\infty} s^{-i}Y_k) = 0 \) for \( k \geq 1 \);

(14) \( m(\bigcup_{j=0}^{1} s^{-i}Y_0 \cap (B - \bigcup_{j=0}^{2} \bigcup_{j=0}^{\infty} s^{-j}X_k)) \geq \frac{2}{3} m(X_0 \cap B - \bigcup_{j=0}^{2} \bigcup_{j=0}^{\infty} s^{-j}X_k) \);

(15) \( Y_0 \cap s^{-1}Y_0 = \varnothing \)(m).

Now for each \( k \geq 1 \) find an even number \( n_k \geq 1 \) such that 
\[ m(\bigcup_{j=0}^{n_k} s^{-j}X_k \cap B) \geq \frac{3}{4} m(\bigcup_{j=0}^{n_k} s^{-j}Y_k \cap B) = \frac{3}{4} m(\bigcup_{j=0}^{n_k} s^{-j}X_k \cap B). \]

This is possible since \( m(B) \) is finite. By (12) and (13)
\[ U[s^{-j}y_k: n_k k \leq j \leq n_k(k+1)-1] = U[s^{-j}y_k: 0 \leq j \leq n_k(k+1)-1] \]  

so that

\[ (16) \quad m((U[s^{-j}y_k: n_k k \leq j \leq n_k(k+1)-1]) \cap (\bigcup_{j=0}^{\infty} s^{-j}y_k \cap B)) \geq \frac{3}{4} m((\bigcup_{j=0}^{\infty} s^{-j}y_k \cap B). \]

Next define, for \( k \) an odd number and \( k \geq 1 \), \( B_k^0 \), \( B_k^1 \) and \( B_k^2 \) as follows:

Set \[ B_k^0 = U[s^{-j}y_k: j \text{ is even and } n_k k \leq j \leq (n_k+1)k-2] \cap B. \]

Since \( n_k k \) is even and \((n_k+1)k-2\) is odd,

\[ s^{-1}B_k^0 = U[s^{-(j+1)}y_k: j+1 \text{ is odd } n_k k+1 \leq j+1 \leq (n_k+1)k-1] \cap s^{-1}B. \]

Set \[ B_k^1 = U[s^{-j}y_k: j \text{ is odd and } n_k k \leq j \leq (n_k+1)k-2] \cap B. \]

Since \( n_k k \) is even and \((n_k+1)k-1\) is even,

\[ s^{-1}B_k^1 = U[s^{-(j+1)}y_k: j+1 \text{ is even } n_k k+2 \leq j+1 \leq (n_k+1)k-1] \cap s^{-1}B. \]

Set \[ B_k^2 = U[s^{-j}y_k: j \text{ is even and } n_k k+1 \leq j \leq (n_k+1)k-1] \cap B. \]

Since \( n_k k+1 \) is odd and \((n_k+1)k-1\) is even,

\[ s^{-1}B_k^2 = U[s^{-(j+1)}y_k: j+1 \text{ is odd } n_k k+3 \leq j+1 \leq (n_k+1)k] \cap s^{-1}B. \]

For \( k \geq 1 \) and \( k \) odd

\[ (17) \quad B_k^i \in \mathcal{F}, \quad B_k \subset B, \quad B_k^i \subset \bigcup_{j=0}^{\infty} s^{-j}y_k. \]

Now \( j+1 \) is odd (even) when \( j \) is even (odd), and \((n_k+1)k-2\) - \((n_k+1)k-1\), \((n_k+1)k-1\) - \((n_k+1)\), and \((n_k+1)k-1\) - \((n_k+2)\) are all less than \( k \); so that

\[ (18) \quad s^{-1}B_k^i \cap B_k^i = \emptyset (m) \text{ for } i = 0, 1, 2. \]
Since \( B_k^0 U B_k^1 U B_k^2 = U[s^{-j}y_k; n_k k \leq j \leq (n_k+1)k-1] \cap B \) there exists an \( i_k \in \{0, 1, 2\} \) such that
\[
(19) \quad m(B_k^i) \geq \left( \frac{1}{3} \right) \left( \frac{3}{4} \right) m\left( \bigcup_{j=0}^{\infty} s^{-j}X_k \cap B \right) \quad (cf. \ 16).
\]

For \( k \) even and \( k \geq 2 \) define \( B_k^0 \) and \( B_k^1 \) as follows:

Set \( B_k^0 = U \{ s^{-j}y_k; j \text{ is even and } n_k k \leq j \leq (n_k+1)k-1 \} \cap B \).

Now \( k \) is even implies that \( n_k k \) is even and \((n_k+1)k-1\) is odd, so
\[
B_k^0 = U \{ s^{-j}y_k; j \text{ is even and } n_k k \leq j \leq (n_k+1)k-2 \} \cap B, \quad \text{and}
\]
\[
s^{-1}B_k^0 = U \{ s^{-(j+1)}y_k; j+1 \text{ is odd and } n_k k+1 \leq j+1 \leq (n_k+1)k-2 \} \cap s^{-1}B.
\]

Set \( B_k^1 = U \{ s^{-j}y_k; j \text{ is odd and } n_k k \leq j \leq (n_k+1)k-1 \} \cap B \).

Now \( n_k k \) is even and \((n_k+1)k-1\) is odd so that
\[
B_k^1 = U \{ s^{-j}y_k; j \text{ is odd and } n_k k+1 \leq j \leq (n_k+1)k-1 \} \cap B, \quad \text{and}
\]
\[
s^{-1}B_k^1 = U \{ s^{-(j+1)}y_k; j+1 \text{ is even and } n_k k+2 \leq j+1 \leq (n_k+1)k \} \cap s^{-1}B.
\]

Now (17), (18) and (19) also hold for \( k \) such that \( k \geq 2 \) and \( k \) is even.

For \( k = 0 \) define \( B_0^0 \) and \( B_0^1 \) as follows:

Set \( B_0^0 = y_0 \cap B - \bigcup_{k=2}^{\infty} \bigcup_{j=0}^{\infty} s^{-j}X_k \), and
\[
B_0^1 = s^{-1}y_0 \cap B - \bigcup_{k=2}^{\infty} \bigcup_{j=0}^{\infty} s^{-j}X_k.
\]

So for \( k = 0 \) (17), (18) and (19) hold also, since \( y_0 \in \mathcal{F} \),
\[
y_0 \cap s^{-1}y_0 = \emptyset \quad (m), \quad \text{and} \quad \text{by (14)} \quad m\left( \bigcup_{j=0}^{\infty} s^{-j}y_0 \cap B \right) \geq \frac{2}{3} m\left( x_0 \cap B - \bigcup_{k=2}^{\infty} \bigcup_{j=0}^{\infty} s^{-j}X_k \right).
\]
Thus $B_k$ is defined for $k \geq 0$ and $k \neq 1$ so that (17), (18) and (19) hold.

Then

$$m\left(\bigcup_{k=0}^{\infty} B_k\right) \geq \left[\bigcup_{k=2}^{\infty} \frac{1}{4} m(B \cap \bigcup_{j=0}^{\infty} s^{-j}x_k) + \frac{1}{4} m(B \cap x_0 \cup \bigcup_{j=0}^{\infty} s^{-j}x_k) \right. \geq \frac{1}{4} m(B \cap \left.\bigcup_{k=2}^{\infty} \bigcup_{j=0}^{\infty} s^{-j}x_k \cup x_0\right) .$$

Hence

$$m\left(\bigcup_{k=0}^{\infty} B_k\right) \geq \frac{1}{4} m(B \cap x_1^C) .$$

Now apply lemma 3.2 with $A_k = \frac{1}{k}$ so that (5) is satisfied because of (10), (11) and (17).

Hence, $\frac{1}{i} s^{-i} \left(\bigcup_{k=0}^{\infty} B_k\right) = \bigcup_{k=0}^{\infty} \frac{1}{i} s^{-i} B_k = \phi (m)$ where the last equality is due to (18). Now set $B' = \bigcup_{k=0}^{\infty} B_k - s^{-1} \left(\bigcup_{k=0}^{\infty} B_k\right) \in \mathcal{F}$, then i) of the lemma is proven since $m(B') \geq \frac{1}{4} m(B \cap x_1^C)$ and

$$\frac{1}{i} s^{-i} B' = \phi .$$

Now I shall prove ii) by induction on $k$. By assumption in

ii) $m(B_1^{'k}) \geq (1-r) m(B - \bigcup_{j=1}^{\infty} B_1') ;$ hence

$$m(B_1^{'k}) \geq (1-r) m(B) .$$

Next assume that $m(\bigcup_{i=1}^{k} B_1') \geq (1-r^k) m(B)$ for $k < n$.

Then:

$$m(\bigcup_{i=1}^{k+1} B_1') = m(B_1^{'k+1}) + m(\bigcup_{i=1}^{k} B_1') \geq (1-r) m(B) - m(\bigcup_{i=1}^{k} B_1') + m(\bigcup_{i=1}^{k} B_1') = (1-r) m(B) + r (1-r^k) m(B) \geq (1-r) m(B) + r (1-r^k) m(B) .$$
= (1 - r^{k+1})m(B). The proof of lemma 3.3 is complete.

Lemma 3.4: If $S$ is a semigroup of $\mathcal{F}$-measurable nonsingular transformations with no nontrivial $m$-subinvariant sets, then for any $C \in \mathcal{F}$ with $m(C)$ finite, for any $A \in \mathcal{F}$ with $m(A) > 0$ and for all $\varepsilon > 0$ there exists $(s_i)_{i=1}^\infty$ in $S$ such that

1) $m\left( \bigcup_{i=1}^\infty s_i^{-1} A \Delta X \right) = 0$, and

2) $m(C - \bigcup_{i=1}^N s_i^{-1} A) < \varepsilon$ for some integer $N$.

Proof: By lemma 1.2 there exists a sequence $(s_i)_{i=1}^\infty \subset S$ such that for all $t \in S$, $t^{-1}(\bigcup_{i=1}^\infty s_i^{-1} A) \subset \bigcup_{i=1}^\infty s_i^{-1} A (m)$, so that $\bigcup_{i=1}^\infty s_i^{-1} A$ is an $S$-$m$-subinvariant set. $X = \bigcup_{i=1}^\infty s_i^{-1} A (m)$ since $m(A) > 0$ and $X$ has no nontrivial $S$-$m$-subinvariant sets. Observe that $m(C)$ finite implies that $\lim_{N \to \infty} m(C - \bigcup_{i=1}^N s_i^{-1} A) = m(C - \bigcup_{i=1}^\infty s_i^{-1} A) = 0$. Hence for $\varepsilon > 0$ there exists an $N$ such that ii) holds.

In lemma 3.4 above I have assumed that the semigroup $S$ has no nontrivial $m$-subinvariant sets. In the case that $S = \{ T^i : i \geq 0 \}$ and $T$ is a conservative transformation the assumption that $T$ has no nontrivial $m$-subinvariant sets is equivalent to the assumption that $T$ has no $m$-invariant sets; i.e., that $T$ is $m$-ergodic. Indeed, if $T^{-1}A \subset A (m)$, then $A - T^{-1}A$ differs from a wandering set $W$ by a set of measure zero. Hence $m(W) > 0$. But if $T$ is conservative then $m(W) = 0$ so $T^{-1}A = A (m)$. Also in the case that $S$ is a group the absence of nontrivial $m$-subinvariant
sets is equivalent to the absence of nontrivial $m$-invariant sets. Indeed, if $s^{-1}A \subseteq A(m)$ for all $s \in S$, then $s^{-1}A \subseteq A(m)$ and $sA \subseteq A(m)$. Hence the nonsingularity and invertibility of $s$ implies that $A = s^{-1}sA \subseteq s^{-1}A(m)$, so that $s^{-1}A \subseteq A \subseteq s^{-1}A(m)$; that is, $A = s^{-1}A(m)$.

Suppose $G$ is the smallest group containing $S$. Then the $m$-invariant (invariant) sets for $S$ and $G$ are the same because $s^{-1}\mathcal{F} = \mathcal{F}(m) \quad (s^{-1}\mathcal{F} = \mathcal{F})$ for all $s \in S$ implies that $\sum_{i=1}^{j_1} s_i^{-1} \mathcal{F} = \mathcal{F}(m)$ ($s_1^{-1} \mathcal{F} = \mathcal{F}$), $j_1$ an integer, and so $s_1^{-1} \ldots s_n^{-1} \mathcal{F} = \mathcal{F}(m)$ ($s_1^{-1} \ldots s_n^{-1} \mathcal{F} = \mathcal{F}$). Therefore $S$ and $G$ are either both $m$-ergodic (ergodic) or neither one is.

It is clear that if $S$ is $m$-ergodic then $S$ is ergodic. If the cardinality of $S$, $c(S)$, is at most countable then $S$ is ergodic implies that $S$ is $m$-ergodic as follows. If $c(S)$ is at most countable then $c(G)$ is at most countable since $g \in G$ implies that $g = s_1^{-1} \ldots s_n^{-1}$, $j_1 \in [1, -1]$. Suppose $\mathcal{F} \in \mathcal{F}$ is an $m$-invariant. Then $g^{-1}\mathcal{F} = \mathcal{F}(m)$ for all $g \in G$; hence, $\bigcup g^{-1}\mathcal{F} \in \mathcal{F}$ and is $G$-invariant. Also $\bigcup g^{-1}\mathcal{F} = \mathcal{F}(m)$. So if $S$ is ergodic then either $m(\bigcup g^{-1}\mathcal{F}) = 0$ or $m((\bigcup g^{-1}\mathcal{F})^c) = 0$, so either $m(\mathcal{F}) = 0$ or $m(\mathcal{F}^c) = 0$ and $S$ is $m$-ergodic.

The following example shows that, in general, ergodicity does not imply $m$-ergodicity. Let $G$ be a group of transformations $g_{ab}$
on the reals with Lebesgue measure such that \( g_{ab}(x) = x \) if \( x \neq a \) or \( b \), \( g_{ab}(a) = b \) and \( g_{ab}(b) = a \) where \( a \) and \( b \) are any real numbers. Then for any measurable set \( F \), \( \emptyset \neq F \subseteq \mathbb{R} \) there exist \( b \not\in F \), \( a \in F \) such that \( g_{ab}F \neq F \) and \( m(g_{ab}F \Delta F) = 0 \) for all \( a,b \). \( G \) is ergodic but not \( m \)-ergodic.

**Lemma 3.5** : Let \( S \) be a semigroup of \( \mathcal{F} \)-measurable nonsingular transformations on \((X, \mathcal{F}, m)\) with \( m \) a \( \sigma \)-finite infinite \( S \)-invariant measure. Then there exists a nontrivial finite measure \( p \), where \( p \ll m \) and \( p \) is an \( S \)-invariant measure if and only if there exists a set \( F \in \mathcal{F} \) such that \( m(F) \) is finite and \( F \) is an \( S \)-\( m \)-invariant set.

**Proof** : The existence of an \( S \)-\( m \)-invariant set \( F \in \mathcal{F} \) of finite measure implies that \( p(A) = m(A \cap F) \) is a finite \( S \)-invariant measure on \((X, \mathcal{F})\) which is absolutely continuous with respect to \( m \).

Next I will show that the existence of a nontrivial finite invariant measure \( p \) implies the existence of an \( S \)-\( m \)-invariant set. Let \( f(x) \) be the Radon-Nikodym derivative of \( p \) with respect to \( m \). Let \( A_i = \{ x : f(x) \geq \frac{1}{i} \} \) for \( i \geq 1 \). Now \( f(x) \) is \( \mathcal{F} \)-measurable implies \( A_i \in \mathcal{F} \). If \( m(A_i) = 0 \) for all \( i \), then \( m(\{ x : f(x) > 0 \}) = 0 \). Hence \( \int_X f(x) dm = 0 \); i.e., \( p \) would be identically zero. Since \( p \) is not identically zero there must be a \( k \geq 1 \) such that \( m(A_k) > 0 \). Choose \( A_k \) to be a set in the
sequence $A_1, A_2, \ldots, A_i, \ldots$ which has positive $m$-measure.
Thus $m(A_k) = m(\{x: f(x) \geq 1/k\}) > 0$. This choice of $A_k$ implies
that $f(x) < 1/k$ on $X - A_k$. Thus
\[
\int f(x) \, dm \leq (1/k)m(s^{-1}A_k - A_k),
\]
where equality holds if and only if $m(s^{-1}A_k - A_k) = 0$.
Now $\alpha > p(A_i) = \int_{A_i} f(x) \, dm \geq (1/i)m(A_i)$ implies that $m(A_i)$ is
finite for all $i \geq 1$.

(20) \[ (1/k)m(A_k - s^{-1}A_k) + p(A_k \cap s^{-1}A_k) \leq \int_{A_k - s^{-1}A_k} f(x) \, dm + \int_{A_k \cap s^{-1}A_k} f(x) \, dm = \int_{s^{-1}A_k} f(x) \, dm = p(s^{-1}A_k) = p(A_k) = p(A_k - s^{-1}A_k) \]
\[ + \int_{s^{-1}A_k - A_k} f(x) \, dm \leq p(A_k \cap s^{-1}A_k) + (1/k)m(s^{-1}A_k - A_k). \]
The last inequality is an equality if and only if $m(s^{-1}A_k - A_k) = 0$.

Observe that
\[
m(A_k - s^{-1}A_k) + m(A_k \cap s^{-1}A_k) = m(A_k) = m(s^{-1}A_k) = m(s^{-1}A_k - A_k) + m(A_k \cap s^{-1}A_k).
\]
So that $m(A_k - s^{-1}A_k) = m(s^{-1}A_k - A_k)$. Then we have that the first
and last terms of (20) are equal; therefore, $m(A_k \Delta s^{-1}A_k) = 0$,
and $A_k$ is an $m$-finite $S$-$m$-invariant set. This completes the proof.

For the remainder of this chapter I shall use the phrase
the usual conditions on \( S, F, G, X, Z_\lambda, m \) to mean the following:

i) \( S \) is a semigroup of measure preserving invertible transformations on \((X, \mathcal{J}, m)\) which are also \( \mathcal{J} \)-measurable.

ii) \( \mathcal{J} \) is a sub-\( \sigma \)-algebra of \( \mathcal{J} \).

iii) \((X, \mathcal{J}, m)\) is a \( \sigma \)-finite separable measure space.

iv) \( \bigcup_{i=1}^{\infty} Z_i = X \) where \( Z_i \in \mathcal{J} \), \( Z_i \) is \( S \)-\( m \)-invariant, the system \((Z_1)_{i=1}^{\infty}\) is \( m \)-disjoint, \( Z_i \) contains no nontrivial \( m \)-subinvariant sets in \( \mathcal{J} \), and \( m(Z_i) = \infty \).

I would like to make a few comments concerning some implications of the above conditions. First note that the condition that \( m \) is \( \sigma \)-finite on \( \mathcal{J} \) implies that we have \( m \) is \( \sigma \)-finite on \( \mathcal{J} \). Further, \( Z_i \cap \mathcal{J} \) is a \( \sigma \)-algebra on \( Z_i \) consisting of all sets \( A \in \mathcal{J} \) with \( A \subset Z_i \), and similarly \( Z_i \cap \mathcal{J} = \{ A \in \mathcal{J} : A \subset Z_i \} \) is a \( \sigma \)-algebra on \( Z_i \). Now \( Z_i \) is assumed to be \( S \)-\( m \)-invariant. Observe that \( s \in S \) restricted to \( Z_i \) is a \( Z_i \cap \mathcal{J} \)-measurable and \( Z_i \cap \mathcal{J} \)-invertible measure preserving transformation on \( Z_i \) with no \( m \)-subinvariant sets in \( Z_i \cap \mathcal{J} \) or \( Z_i \cap \mathcal{J} \). Thus lemma 3.4 may be applied so that for any \( \epsilon > 0 \), any \( C \in Z_i \cap \mathcal{J} \) with \( m(C) \) finite and any \( A \in Z_i \cap \mathcal{J} \) with \( m(A) > 0 \), there are sequences \((a^s_j)_{j=1}^{\infty} \subset \mathcal{J} \) and \((t^{-1}_j)_{j=1}^{\infty} \subset S^{-1} \), and positive integers \( N_1 \) and \( N_1 \) such that

...
\[ \bigcup_{j=1}^{N} s_j^{-1}A = Z_1(m) \quad \text{and} \quad m\left(C - \bigcup_{j=1}^{N} s_j^{-1}A\right) < \varepsilon, \]

and
\[ \bigcup_{j=1}^{N} t_j A = Z_1(m) \quad \text{and} \quad m\left(C - \bigcup_{j=1}^{N} t_j A\right) < \varepsilon. \]

Note that if \( A \in Z_1 \cap \mathcal{F} \) since \( s_j \) is \( Z_1 \cap \mathcal{F} \) measurable then
\[ \bigcup_{j=1}^{\infty} s_j^{-1}A \in Z_1 \cap \mathcal{F}. \] Since there are no \( S \)-invariant sets of finite measure in \( \mathcal{F} \) or \( \mathcal{J} \), (lemma 3.5) says there does not exist any nontrivial finite invariant measure which is absolutely continuous with respect to \( m \) on \( (X, \mathcal{F}) \) or \( (X, \mathcal{J}) \).

A set \( A \in \mathcal{F}, \mathcal{F} \) a \( \sigma \)-algebra, is called an atom if for all sets
\( B \) contained in \( A \) such that \( B \in \mathcal{F} \), either \( m(A \Delta B) = 0 \) or \( A = B(m) \).

\( \mathcal{J} \) is called atomic if \( X = \bigcup A_a \) where \( A_a \) is an atom in \( \mathcal{J} \). \( \mathcal{J} \) is called nonatomic if \( \mathcal{J} \) has no atoms.

Lemma 3.6: Assume the usual conditions on \( S, \mathcal{F}, \mathcal{J}, X, Z_1, m \) (see p. 34). Then:

1) If \( Z_1 \cap \mathcal{J} \) contains one nontrivial atom then the
\( \sigma \)-algebra \( Z_1 \cap \mathcal{J} \) is atomic. Furthermore, any
nontrivial atom \( A \in Z_1 \cap \mathcal{J} \) is an \( S^{-1} \)-generating set for \( Z_1 \cap \mathcal{J} \), and \( \sigma(S(Z_1 \cap \mathcal{J})) = Z_1 \cap \mathcal{J}. \)

ii) \( Z_1 \cap \mathcal{J} \) is nonatomic if and only if \( \sigma(S(Z_1 \cap \mathcal{J})) \)
is nonatomic.

iii) For any \( A \in X \cap \mathcal{J} \) where \( m(A \cap Z_1) > 0 \) for all
\( Z_1 \subset X_n(m), X_n \) the nonatomic part of \( (X, \mathcal{J}, m) \),
there exists a sequence of positive numbers
and a sequence of disjoint sets
\[(A_k)_{k=1}^\infty, \quad A_k \subset A, \quad \text{which is an } S^{-1} \text{-generating set for } \sigma(S(X_\mu \cap \mathcal{J})) \] and whenever I have a sequence of sets \(B_k \in \mathcal{J}\) and \(m(A_k \Delta B_k) < \varepsilon_k\), I also have that \((B_k)_{k=1}^\infty\) is an \(S^{-1}\)-generating set for \(\sigma(S(X_\mu \Delta \mathcal{J}))\).

Proof: Note that ii) follows easily from i). Indeed, assume that i) has been proven. Then \(Z_1 \cap \mathcal{J} \in \sigma(S(Z_1 \cap \mathcal{J}))\) is either atomic or nonatomic. If \(\sigma(S(Z_1 \cap \mathcal{J}))\) is atomic then so is \(Z_1 \cap \mathcal{J}\), since the \(\mathcal{J}\)-measurability and \(\sigma\)-invertibility of \(s\) implies that \(Z_1 \cap \mathcal{J} \subset S(Z_1 \cap \mathcal{J})\). Hence \(Z_1 \cap \mathcal{J}\) is a sub-\(\sigma\)-algebra of \(\sigma(S(Z_1 \cap \mathcal{J}))\).

Assume that \(Z_1 \cap \mathcal{J}\) is atomic. Then i) implies that \(\sigma(S(Z_1 \cap \mathcal{J})) = Z_1 \cap \mathcal{J}\). Thus \(\sigma(S(Z_1 \cap \mathcal{J}))\) is atomic if and only if \(Z_1 \cap \mathcal{J}\) is atomic. Since both \(\sigma\)-algebras are either atomic or nonatomic ii) is established once i) is proven.

Next I shall prove i). I will begin by showing that any non-trivial atom \(A\) in \(Z_1 \cap \mathcal{J}\) is an \(S^{-1}\)-generating set for \(Z_1 \cap \mathcal{J}\), and any \(F \in Z_1 \cap \mathcal{J}\) has measure at least as big as that of \(A\). Let \(A\) be a nontrivial atom in \(Z_1 \cap \mathcal{J}\). Then there exist transformations \(\{t_j^{-1}\}_{j=1}^\infty \subset S^{-1}\) and \(\{s_j\}_{j=1}^\infty \subset S\) such that
\[(21) \quad \bigcup_{j=1}^\infty t_j A = Z_1 (m) \quad \text{and} \quad \bigcup_{j=1}^\infty s_j^{-1} A = Z_1 (m).\]

Let \(F \in Z_1 \cap \mathcal{J}\). Then \(F \cap \bigcup_{j=1}^\infty t_j A = F\). Hence there exists a subset \(\{t_{j_1}\}\) contained in \(\{t_j\}_{j=1}^\infty\) such that \(F \cap \bigcup_{1}^{j_1} t_j A = F(m)\).
and \( m(t^*_1 A \cap F) > 0 \) for each \( i \). Now \( 0 < m(t^*_1 A \cap F) = m(t^*_1 (A \cap t^*_1 F)) = m(A \cap t^*_1 F) \). But \( A \) is an atom, hence \( t^*_1 F \supset A \); i.e., \( F \supset t^*_1 A \). Thus \( m(F) \geq m(A) \) and \( F = \cup t^*_1 A \); i.e., any nontrivial atom \( A \) is an \( S^{-1} \)-generating set for \( Z_i \cap \mathcal{J} \). I shall now show that if \( A \) is a nontrivial atom then \( s^{-1} A \) is also an atom in \( Z_i \cap \mathcal{J} \). This result plus (21) obviously will imply that \( Z_i \cap \mathcal{J} \) is atomic if it contains one nontrivial atom. Indeed, for any \( B \in \mathcal{J} \) contained in \( s^{-1} A \), it is the case that \( m(B) \geq m(A) = m(s^{-1} A) \) or \( m(B) = 0 \); consequently, \( B = s^{-1} A(m) \) or \( B = \emptyset \), and \( s^{-1} A \) is an atom in \( Z_i \cap \mathcal{J} \). To see that \( Z_i \cap \mathcal{J} = \sigma(S(Z_i \cap \mathcal{J})) \), note that the \( \mathcal{J} \)-measurability and the \( \mathcal{J} \)-invertibility of \( s \in S \) imply that \( Z_i \cap \mathcal{J} \subset S(Z_i \cap \mathcal{J}) \). Further, \( F \in Z_i \cap \mathcal{J} \) implies that \( sF \subset \bigcup A_k \), where \( A_k \) are atoms in \( Z_i \cap \mathcal{J} \). Now \( sF \cap A_k = s(F \cap s^{-1} A_k) \), but \( A_k \) an atom implies that \( s^{-1} A_k \) is an atom; hence, \( F \supset s^{-1} A_k(m) \) or \( m(F \cap s^{-1} A_k) = 0 \). Thus \( F = \bigcup_{k} s^{-1} A_k \) for some subset \( \{A_k\} \) of \( \{A_k\} \). Therefore \( sF = \bigcup_{k} A_k \in \mathcal{J} \) and \( s^{-1} \in S^{-1} \) if \( \mathcal{J} \)-measurable. Hence \( s \in S \) is \( \mathcal{J} \)-invertible and \( S(Z_i \cap \mathcal{J}) \subset Z_i \cap \mathcal{J} \). Therefore \( Z_i \cap \mathcal{J} = \sigma(S(Z_i \cap \mathcal{J})) \). Now 1) is proven.

In order to prove 3) I shall first construct a generating set \( \{F_k\}_{k=1}^\infty \) for \( X_N \cap \mathcal{J} \) such that \( m(F_k) \) is finite, \( F_k \in Z_i \cap \mathcal{J} \) for some \( Z_i \subset X_N \), and for every \( k \), \( F_k \) appears infinitely often in the sequence \( \{F_k\}_{k=1}^\infty \). The separability and \( \sigma \)-finiteness of
$(X, J, m)$ implies that sequences $(F_{i,k})_{k=1}^\omega$ can be found such that $(F_{i,k})_{k=1}^\omega$ is a generating set for $Z_i \cap J$ and $F_{i,k}$ appears infinitely often in the sequence $(F_{i,k})_{k=1}^\omega$, and $m(F_{i,k})$ is finite. Next consider the array:

$$
\begin{array}{cccc}
F_{i_1,1} & F_{i_1,2} & F_{i_1,3} & \cdots & F_{i_1,k} & \cdots \\
F_{i_2,1} & F_{i_2,2} & F_{i_2,3} & \cdots & F_{i_2,k} & \cdots \\
F_{i_3,1} & F_{i_3,2} & F_{i_3,3} & \cdots & F_{i_3,k} & \cdots \\
\end{array}
$$

Let $i_1$ range over all indices such that $Z_i \subset X_N$. Then using the diagonalization process produce the sequence

$$(F_k)_{k=1}^\omega = (F_{i_1,1}, F_{i_1,2}, F_{i_2,1}, F_{i_3,1}, F_{i_2,2}, F_{i_1,3}, \cdots)$$

which is a generating set for $X_N \cap J$, where $m(F_k)$ is finite, and $F_k \in Z_i \cap J$ for a particular $i$, and $F_k$ appears infinitely often in the sequence $(F_k)_{k=1}^\omega$. Since $Z_i \cap J$ is nonatomic and $m(A \cap Z_i) > 0$, $A$ may be partitioned into sets $A_h \in Z_h \cap J$. Each $A_h$ may be further partitioned into sets $A_{h,i}$ so that $m(A_{h,i}) = 2^{-1} m(A_h)$ where $A_{h,i} \in Z_h \cap J \subset X_N \cap J$. Now fix $h$, and, for the moment, suppress the index $h$ on $A_h$; i.e., $A_h$ will be denoted by $A$, $A_{1,h}$ by $A_{1}'$ etc. Now $s_{i,k}$ may be found such that $m(F_i - \bigcup_{k=1}^\omega s_{i,k} A_{i}') < \delta_i$. Let

$$A_{i_1,1} \cap \cdots \cap A_{i_j,1} = A_{i_1}' \cap \cdots \cap A_{i_j,1}' \cap \bigcup_{k=1}^\omega s_{i,k} F_i$$

for $1 \leq i_1 < i_2 < \cdots < i_j \leq N_i$; $A_{i_1,1}', \ldots, A_{i_j,1} \in J$. These sets are disjoint and form a finite partition of a subset of $A_i'$. 
Reindex the sets in these partitions so that \( A_1 = A_{1,1}, A_2 = A_{1,2}, \) etc. Now if \( A_k = A_{i_1,1}, \ldots, i_j \), then take \( \varepsilon_j \) so that
\[
m(C) < \varepsilon_j \implies m(s_{i,j} C) < \frac{\varepsilon_j}{N_i^2} \quad \text{for} \quad j = 1, 2, \ldots, N_i.
\]
With each set \( A_{i_1,1}, \ldots, i_j = A_k \) associate a set of transformations \( X = \{s_{i_1,1}, \ldots, s_{i,j}\} \). Now if \( B_j \in X \) and \( m(A_j \Delta B_j) < \varepsilon_j \),
then \( m(s_{i,j} (A_j \Delta B_j)) < \frac{\varepsilon_j}{N_i^2} \), so that
\[
m(F_1 \Delta \bigcup_{i=1}^{N_i} s_{i,j} B_k) < 2\varepsilon_i.
\]
Now take \( \varepsilon_i \) decreasing to zero so that the condition that \( F_1 \) appears infinitely often in the sequence \( (F_k)_{k=1}^{\infty} \) implies that \( (R_k)_{k=1}^{\infty} \) is also an \( S^{-1} \)-generating set for \( \sigma(S(Z_h \cap 0)) \). Thus the combined partitions of \( A_h \), where \( h \) ranges over all indices such that \( Z_h \subset X_N \), is an \( S^{-1} \)-generating set for \( \sigma(S(X_N \cap 0)) \). Lemma 3.6 is complete.

Recall that corollary 1.1 says that if \( s \) is \( J \)-nonsingular invertible for all \( s \in S \), where \( S \) is a semigroup of transformations on \( X \), if \( m \) is a \( \sigma \)-finite infinite \( S \)-variant measure on \( \mathcal{F} \), and if there does not exist any finite \( S \)-invariant measure on \( \mathcal{F} \) which is absolutely continuous with respect to \( m \), then for any \( C \in \mathcal{F} \) where \( m(C) \) is finite there exists an \( S \)-weakly wandering set \( W \in \mathcal{F} \) such that \( m(W \cap C) > m(C) - \varepsilon \). Recall that corollary 1.2 says that if for all \( s \in S \), \( s \) is \( J \)-nonsingular invertible for some \( \sigma \)-algebra \( J \supset \mathcal{F} \), for all \( s \in S \) and if \( S \) is a left amenable semigroup of \( J \)-measurable transformations, if \( m \) is a \( \sigma \)-finite infinite \( S \)-invariant measure on \( \mathcal{F} \), and if there does not exist
any finite $S$-invariant measure on $\mathcal{F}$ which is absolutely continuous with respect to $m$, then for any $\varepsilon > 0$ and any $C \in \mathcal{F}$ where $m(C)$ is finite there exists an $S$-weakly wandering set $W \in \mathcal{F}$ such that $m(W \cap C) > m(C) - \varepsilon$. In addition to the usual conditions on $S, \mathcal{F}, \mathcal{J}, X, Z_\mathcal{J}, m$, assume that $S$ is either left amenable or $\mathcal{J}$-invertible, and that $X = \bigcup_{i=1}^{n} Z_i$. Then there exists an $S$-weakly wandering set $W \in \mathcal{F}$ such that $m(W \cap Z_i) > 0$ for each $i, 1 \leq i \leq n$. This will permit the construction of a size-3-$S^{-1}$-generator in $\mathcal{F}$ for $\sigma[S\mathcal{F}]$. Apply lemma 3.6 with $A = W$ and find a partition, $\{A_j\}_{j=1}^{\infty}$ of a subset of $W$ such that $\sigma(A_j; s \in S, j = 1, 2, \ldots) = \sigma[S\mathcal{F}]$. Then any set $E$ such that $\sigma(E)$ contains $\{A_j\}_{j=1}^{\infty}$ is also an $S^{-1}$-generating set for $\sigma[S\mathcal{F}]$. Set $B_1 = W \in \mathcal{F}$ and $B_2 = \bigcup_{j=1}^{\infty} s_j^{-1}A_j \in \mathcal{F}$, where $(s_j)_{j=1}^{\infty}$ is a weakly wandering sequence for $W$. That is, $s_j^{-1}W \cap s_j^{-2}W = \emptyset$ for $i \neq j$, and $W \cap s_j^{-1}W = \emptyset$ for $j = 1, 2, \ldots$. Thus if $A_j \subset W$ then $s_j^{-1}A_j \cap W = \emptyset$ for $j \neq k$ and $s_j^{-1}A_j \cap W = \emptyset$ for all $j = 1, 2, \ldots$. So $W \cap s_j(\bigcup_{i=1}^{\infty} s_j^{-1}A_i) = W \cap (A_j \cup \bigcup_{i \neq j} s_j^{-1}A_i) = (W \cap A_j) \cup (W \cap \bigcup_{i=1}^{\infty} s_j^{-1}A_i) = A_j \cup \bigcup_{i \neq j} s_j^{-1}W \cap s_j^{-1}A_i = A_j$. That is, $B_1 \cap s_jB_2 = A_j$. Furthermore, $\sigma[S\{B_1, B_2\}] \supseteq \{B_1, s_jB_2 = A_j \text{ for all } j = 1, 2, \ldots\}$. Then $\{B_1B_2, (B_1 \cup B_2)^c\} = \{W, \bigcup_{j=1}^{\infty} s_j^{-1}A_j, (W \cup \bigcup_{j=1}^{\infty} s_j^{-1}A_j)^c\} \subseteq \mathcal{F}$ is a size-3-$S^{-1}$-generator for $\sigma[S\mathcal{F}]$. 
Assume the usual conditions on $S, \mathcal{J}, J, X, Z_1, m$ (see p. 64).

I will prove the existence of an $S^{-1}$ generating set in two special cases (cf. lemma 3.9) which will enable me to prove the existence of size-2-$S^{-1}$-generators in the general case (theorem 3.1). In the first case there exists an $s \in S$ with $m(\bigcup_{k \neq 1, 2} X^J_k(s) \cap Z_1) > 0$ for each $i$, $i = 1, 2, \ldots$. In the second, $t^2 = \text{identity}$ for all $t \in S$. Lemmas 3.7 and 3.8 are technical lemmas which will allow me to show the existence of size-2-$S^{-1}$-generators in these two special cases. If $S$ is an $S^{-1}$-exhaustive sub-$\sigma$-algebra of $\mathcal{J}$ and if $X = Z_1$, then theorem 3.1 reduces to lemma 3.9.

**Lemma 3.7**: Let $S, \mathcal{J}, J, X, Z_1, m$ have the usual conditions (see p. 64). Let $s \in S$ and $\delta > 0$ be given. Assume that $A \in X_N \cap \mathcal{J}$ ($X_N$ is the nonatomic part of $(X, \mathcal{J})$), where $m(A)$ is finite. If $m(\bigcup_{k \neq 1, 2} X^J_k(s) \cap Z_1) > 0$ for each $i = 1, 2, \ldots$, then $L_1 \in \mathcal{J}$ and $A_1 \in \mathcal{J}$ may be chosen satisfying 1) - ix) below; if $m(\bigcup_{k \neq 1} X^J_k(s) \cap Z_1) > 0$ for each $i = 1, 2, \ldots$, then $L_1 \in \mathcal{J}$ and $A_1 \in \mathcal{J}$ may be chosen satisfying ii) - ix) below.

1) $L_1 \cap s^{-2} L_1 = \phi$.

2) $L_1 \cap s^{-1} L_1 = \phi$.

3) $s^4 L_1 \in \mathcal{J}$ for $i \leq 2$.

4) $m(L_1 \cap Z_1) > 0$ for each $i = 1, 2, \ldots$

5) $m(L_1 \cap X^J_1) < \delta$.
vi) \( L_1 \cap X_A \cap Z_1 \) is either empty or an atom (mod \( m \)) in \( \mathfrak{F} \).

vii) \( m(L_1) \) is finite if \( \sum m_i < \infty \), where \( m_i = \) measure of an atom in \( Z_1 \cap \mathfrak{F} \) when \( Z_1 \cap \mathfrak{F} \) is atomic,
and \( m_1 = 0 \) when \( Z_1 \subset X_N \cap \mathfrak{F} \).

viii) \( A_1 \cap A = \emptyset \).

ix) \( m(A \Delta A_1) < 48 \).

Proof: Apply lemma 3.1 to the space \( Z_1 \), and the \( \sigma \)-algebra \( Z_1 \cap \mathfrak{F} \) and the transformation \( s \) restricted to \( Z_1 \). Find
a \( y_{k_1}^i \in Z_1 \cap \mathfrak{F} \) with \( m(y_{k_1}^i) > 0 \) and \( s^{-h}y_{k_1}^i \cap s^{-j}y_{k_1}^i = \emptyset \) when
\( 0 \leq h < j \leq 2 \) for the case that \( m(\bigcup_{k=1}^{2} X_k^i \cap Z_1) > 0 \) and
\( k_1 \neq 1, 2 \). When \( m(\bigcup_{k=1}^{2} X_k^i \cap Z_1) = 0 \) pick \( y_2^i \in Z_1 \cap \mathfrak{F} \) such
that \( s^{-h}y_2^i \cap s^{-j}y_2^i = \emptyset \) for \( 0 \leq h < j \leq 1 \). Lemma 3.6 implies
that for each \( i \) either \( Z_1 \subset X_N(\mu) \) or \( Z_1 \subset X_A(\mu) \). If
\( Z_1 \subset X_N \) and if \( m(\bigcup_{k=1}^{2} X_k^i \cap Z_1) > 0 \) then, there exists an
\( L_0^i \subset y_{k_1}^i \subset Z_1 \) with \( k_1 \neq 1, 2 \), \( L_0^i \in \mathfrak{F} \) and \( 0 < m(L_0^i) < 2^{-1} \).

If \( Z_1 \subset X_N \), \( m(\bigcup_{k=1}^{2} X_k^i \cap Z_1) = 0 \) and \( m(\bigcup_{k=1}^{2} X_k^i \cap Z_1) > 0 \)
then there exists an \( L_0^i \subset y_{k_1}^i \subset Z_1 \) with \( k_1 \neq 1 \), \( L_0^i \in \mathfrak{F} \) and
\( 0 < m(L_0^i) < 2^{-1} \). For \( Z_1 \subset X_A \) there exists an \( L_0^i \subset y_{k_1}^i \)
equal to an atom of positive measure in \( Z_1 \cap \mathfrak{F} \) where \( k_1 \) is as
chosen before. Set

\[ L_1 = s^{-2}(\bigcup_{i=1}^{n} L_0^i) \in \mathfrak{F}. \]
Since \( L_0^i \) are contained in \( m \)-disjoint \( S \)-invariant sets \( Z_i \),
I can apply lemma 3.2 and find that because \( L_0^i \subseteq Y_k \) for \( k \neq 1 \),
\[
L_1 \cap s^{-1}L_2 = s^{-2}\left( \bigcup_{i=1}^{\infty} L_0^i \right) \cap s^{-1}(s^{-2}(\bigcup_{i=1}^{\infty} L_0^i)) = s^{-2}\left( \bigcup_{i=1}^{\infty} L_0^i \right) = \phi.
\]
If \( L_1 \cap s^{-1}L_2 = s^{-2}\left( \bigcup_{i=1}^{\infty} L_0^i \right) \cap s^{-1}(s^{-2}(\bigcup_{i=1}^{\infty} L_0^i)) = s^{-2}(\bigcup_{i=1}^{\infty} L_0^i) = \phi \).

If \( L_0^i \subseteq Y_k \) where \( k \neq 1,2 \), then by lemma 3.2,
\[
s^{-2}(L_1 \cap s^{-1}L_2) = s^{-2}\left( \bigcup_{i=1}^{\infty} L_0^i \right) \cap s^{-1}(s^{-2}(\bigcup_{i=1}^{\infty} L_0^i)) = s^{-2}(\bigcup_{i=1}^{\infty} L_0^i) = \phi.
\]
Hence i) and ii) are established. Note

Since \( s^{-2}L_0^i \subseteq Z_i \) and \( m(s^{-2}L_0^i) = m(L_0^i) > 0 \), I have iv);
that is, \( m(L_1 \cap Z_i) > 0 \) for all \( i \).

By lemma 3.6, either \( Z_i \subseteq X_N(m) \) or \( Z_i \subseteq X_A(m) \), and
\( X_N \cap X_A = \phi (m) \). Since \( m \) and \( Z_i \) are \( S \)-invariant,
\[
m(L_1 \cap X_N) = m(\bigcup_{i=1}^{\infty} L_0^i \cap X_N) = m(\bigcup_{i=1}^{\infty} L_0^i \cap Z_1 < X_N) < \delta.
\]
So, v) is satisfied.

As for vi): If \( Z_i \subseteq X_A(m) \), then \( L_0^i \) is an atom in \( Z_i \cap X \).
Hence by lemma 3.6, \( s^{-2}L_0^i \) is an atom in \( Z_i \cap X \) and so
\( L_1 \cap X_A \cap Z_i \) is empty or is an atom in \( Z_i \cap X \).

From the following observation I can deduce vii):
\[
m(L_1) = m(L_1 \cap X_N) + m(L_1 \cap X_A) < \delta + \sum_{i=1}^{\infty} m_1, \text{ where } m_1 \text{ is the measure of an atom in } Z_i \cap X \text{ if } Z_i \text{ is atomic in } X, \text{ and } m_1 = 0 \text{ if } Z_i \cap X \text{ is nonatomic}. \text{ If only finitely many } Z_i \text{'s are atomic then } \sum_{i=1}^{\infty} m_1 \text{ is finite since by lemma 3.6 every atom in } Z_i \cap X \text{ has finite measure.}
Set

\[ A_1 = A - \bigcup_{i=-2}^{1} s^i I_1. \]

Now \( s^i I_1 \in \mathcal{F} \) for \( i \leq 2 \) implies \( A_1 \in \mathcal{F} \), and obviously \( \text{viii) holds.} \)

The assumption that \( A \subset X_N \) implies \( m(A \cap A_1) = m(A \cap \bigcup_{i=-2}^{1} s^i I_1 \cap X_N) \)
\[ \leq 4m(I_1 \cap X_N) < 4 \varepsilon \] hence \( \text{ix) \ and \ lemma \ 3.7 \ is \ complete.} \)

**Lemma 3.8:** Assume the usual conditions on \( S, \mathcal{F}, \mathcal{J}, X, Z_1, m \) (see p. \( \Theta^4 \)).

Let \( s \in S, B \in \mathcal{F} \) with \( m(B) \) finite, \( k \geq 1 \) and \( \varepsilon > 0 \) be given.

Then for any \( k \geq 0 \) and any \( n \geq 1 \), sets \( B_0 \in \mathcal{F}, B_0 \subset B, B_n \subset B \) and distinct transformations \( r_n \in S \) can be found such that there exist

\( s_i \in S \) for which \( r_1 = s_1 \) and \( r_n = r_{n-1}s_n \) for \( n \geq 2 \), where:

1) \( B_0 \cap \bigcup_{i=0}^{k} s^{-1}(\bigcup_{n=1}^{\infty} r_n^{-1} B) = \emptyset. \)

2) \( m(B-B_0) < \varepsilon. \)

3) \( r_n^{-1} B_n \in \mathcal{F}. \)

4) \( r_n^{-1} B_n \cap \bigcup_{i=0}^{k} s^{-1}(B \cup \bigcup_{j \neq n} r_j^{-1} B) = \emptyset. \)

5) \( m(r_n^{-1} B - r_n^{-1} B_n) < 2^{-n \varepsilon}. \)

**Proof:** Given \( B, \) of finite measure, \( s \) a fixed element in \( S, \) and

\( \varepsilon > 0, \) apply corollary 1.1 and find an \( s_1 \in S \) such that

\[ m(\bigcup_{i=-k}^{k} s^i B \cap s_1^{-1} B) < \varepsilon/(2^k)(k+1). \]

Suppose \( r_1, \ldots, r_{n-1} \) have been found with \( r_n = r_{n-1}s_n, \) and
Then by applying corollary 1.1 again find an $s_B$ such that

\begin{equation}
\left(21\right) m\left(\bigcup_{i=-k}^{k} s^{-1}(B \cup \bigcup_{j<n} r^{-1}_n B) \cap \bigcap_{i=-k}^{k} r^{-1}_n B\right) < 5/(n^{2n+2})(k+1).
\end{equation}

Set $r_n = r_{n-1} s_n$. To see i) and ii) set

\begin{align*}
B_0 &= B - \bigcup_{i=0}^{k} s^{-1}(\bigcup_{n=1}^{\infty} r^{-1}_n B) \in \mathcal{F} \quad (B \in \mathcal{F}).
\end{align*}

Then

\begin{align*}
m(B - B_0) &= m(B \cap \bigcap_{i=-k}^{k} s^{-1}(\bigcup_{n=1}^{\infty} r^{-1}_n B)) \\
&< \sum_{n=1}^{\infty} (k+1)5/(n^{2n+2})(k+1) < 5.
\end{align*}

Since $m(s_B \cap r^{-1}_n B) < 5/(n^{2n+2})(k+1)$ and $m$ is preserved by $s$.

Next set

\begin{align*}
r_{n-1} B_n &= r^{-1}_n B - \bigcup_{i=0}^{k} s^{-1}(B \cup \bigcup_{j \neq n} r^{-1}_j B) \in \mathcal{F} \quad \text{since } B \in \mathcal{F}.
\end{align*}

So iii) and iv) are established. I have

\begin{align*}
m(r^{-1}_n B - r^{-1}_n B_n) &= m(r^{-1}_n B \cap \bigcap_{i=0}^{k} s^{-1}(B \cup \bigcup_{j \neq n} r^{-1}_j B)) \\
&= m(r^{-1}_n B \cap \bigcap_{i=0}^{k} s^{-1}(B \cup \bigcup_{j \neq n} r^{-1}_j B)) \\
&+ \sum_{j>n} m(r^{-1}_n B \cap \bigcup_{i=0}^{k} s^{-1} r^{-1}_j B)) \\
&\leq \left[5/(n^{2n+2})(k+1)\right] + \sum_{j>n} \left[(k+1)5/(j^{2j+2})(k+1)\right] < 5/2^n \\
&\quad \text{(see 21)}.
\end{align*}
That is I have proven v).

If \( \mathcal{B} \) is chosen less than \( m(\mathcal{B}) \) then (21) implies that for 
\( j < n, \) the \( m(x_j^{-1}B \cap x_n^{-1}B) < \mathcal{B}/(k+1)n^{2n+2} < m(\mathcal{B}). \) Hence 
\( x_j^{-1}B \neq x_n^{-1}B(m) \) and all \( x_j \) are distinct. Lemma 3.8 is complete.

Before I begin the last lemma I would like to point out a few things. If \( t^2 = \text{identity} \) for all \( t \in \mathcal{S} \) then the semigroup \( \mathcal{S} \) is actually 
an abelian group since \( t^{-1} = t \) for all \( t \in \mathcal{S} \); hence, \( t_1 t_2 = (t_1 t_2)^{-1} = 
\) \( t_2^{-1} t_1^{-1} = t_2 t_1, \) and \( t \in \mathcal{S} \) is \( 3 \)-invertible. Thus \( \sigma(\tau \mathcal{S}) = \sigma(\tau \mathcal{F}: s \in \mathcal{S}, \mathcal{F} \in \mathcal{J}) = \mathcal{J}. \) 
\( \mathcal{S} \) is an abelian group implies that \( s^{-1} t^{-1} A = t^{-1} A \) if and only if \( t^{-1} s^{-1} A = t^{-1} A; \) that is, if and only if \( s^{-1} A = A. \) So if \( B \in \mathcal{J} \) and \( B \subseteq \mathcal{J}(s), \) then 
\( s^{-1} t^{-1} B = t^{-1} B. \) Hence \( t^{-1} \mathcal{J}(s) = \mathcal{J}(s)(m) \) and similarly, \( t \mathcal{J}(s) = \mathcal{J}(s)(m). \)

Hence \( t^{-1} \mathcal{J}(s) = \mathcal{J}(s)(m). \) In the case that \( Z_1 \) has no nontrivial \( \mathcal{S} \)-sub-invariant sets, and \( m(\mathcal{J}(s) \cap Z_1) > 0, \) then \( \mathcal{J}(s) \supseteq Z_1(m). \) In 
particular if \( t^2 = \text{identity} \) for all \( t \in \mathcal{S}, \) then \( Z_1 \subseteq \mathcal{J}(s) \) or 
\( Z_1 \subseteq \mathcal{J}(s). \) Furthermore, if \( m(\bigcup_{k=1}^\infty \mathcal{J}(s) \cap Z_1) > 0 \) for each \( i, \)
then \( Z_1 \subseteq \mathcal{J}(s) \) for each \( i. \)

Using the notation of lemma 3.7 suppose that \( \sum_{i=1}^\infty m_i = \infty, \) where 
m\( _1 \) equals the measure of an atom in \( Z_1 \cap \mathcal{J} \) when \( Z_1 \) is atomic and 
equal to zero when \( Z_1 \) is nonatomic in \( \mathcal{J}. \) Define \( m' \) on \((X, \mathcal{J})\) as 
follows:
$m'(B) = m(B) \quad \text{if} \quad B \subseteq X_N$;

$m'(B) = m(B)/2^i m_i \quad \text{if} \quad B \subseteq Z_i \cap X_A$ ($X_A$ is the atomic part of $(X, \mathcal{F})$).

Then

$$m'(s^{-1}B) = m(s^{-1}B \cap X_N) + \sum_{i: Z_i \subseteq X_A} \frac{m(s^{-1}B \cap Z_i)}{2^i m_i}$$

$$= m(s^{-1}(B \cap X_N)) + \sum_{i: Z_i \subseteq X_A} \frac{m(s^{-1}(B \cap Z_i))}{2^i m_i}$$

(since $X_N$, $Z_i$ are $S$-invariant)

$$= m(B \cap X_N) + \sum_{i: Z_i \subseteq X_A} \frac{m(B \cap Z_i)}{2^i m_i} = m'(B).$$

So $m'$ is also an $S$-invariant measure on $\mathcal{F}$ and $m$ is equivalent to $m'$. If $m$ is $\sigma$-finite then $m'$ is also $\sigma$-finite. If $m_i$ is equal to the $m'$-measure of an atom in $Z_i$ when $Z_i \subseteq X_A$, and equal to zero when $Z_i \subseteq X_N$, then $\sum_{i=1}^{\infty} m_i = \sum_{i=1}^{\infty} \frac{m_i}{2^i} \leq \sum_{i=1}^{\infty} 2^{-i} < \infty$.

Now if $A \subseteq X_N$, then $m'(H \cap X_N \Delta A) = m(H \cap X_N \Delta A)$. So for the lemma 3.9 I can, without loss of generality, assume that $\sum_{i=1}^{\infty} m_i$ is finite, and hence the set $L_1$ as found in lemma 3.7 is a set of finite measure.

If $s$ is 1-1 and onto then

$$A \cap \bigcup_{i} s^{-1}B = \bigcup_{i} (A \cap s^{-1}B) = \bigcup_{i} (A \cap s^{-1}B) = \bigcup_{i} s^{-1}(s^{-1}A \cap B).$$

Now
So $A \cap \bigcup_{i=2}^{k_2} s^{-1}B = \phi$ if and only if $\bigcup_{i=2}^{k_2} (s^{-1}A \cap B) = \phi$. Thus Lemma 3.7

viii), $(A_1 \cap \bigcup_{i=1}^{k_1} s^{-1}L_1 = \phi)$ if and only if $\bigcup_{i=1}^{k_1} s^{-1}A_1 \cap L_1 = \phi$. Also

Lemma 3.81), $B_0 \cap \bigcup_{i=0}^{k} s^{-1}(B \cup r_n^{-1}B) = \phi$, holds if and only if

$\bigcup_{i=0}^{k} s^{-1}B_0 \cap (B \cup s^{-1}(B \cup r_j^{-1}B)) = \phi$. Lemma 3.8 iv), $r_n^{-1}B_n \cap \bigcup_{j=0}^{k} s^{-1}(B \cup r_j^{-1}B) = \phi,

is equivalent to $\bigcup_{j=0}^{k} s^{-1}r_n^{-1}B_n \cap (B \cup s^{-1}r_j^{-1}B) = \phi$.

Since $r_n^{-1} = s_n^{-1}r_{n-1}^{-1} = s_{n-1}^{-1} ... s_{n-1}^{-1} r_{n-1}^{-1}$ in Lemma 3.8, if $r_{n-1}^{-1} A \in \mathcal{I}$ then $r_n^{-1} A \in \mathcal{I}$.

**Lemma 3.9:** Assume the usual conditions on $S, \mathcal{I}, \mathcal{J}, X, Z_1, m$ (see p. 64). Let $A \in \bigcap_{i=1}^{k} X_{i} \cap \mathcal{I}$ with finite measure, $s$ such that $m(\bigcup s_i^{-1}(X_1 \cap Z_1)) > 0$ for $i = 1, 2, \ldots$, and $s > 0$ be given. Assume either (I):

$m(\bigcup_{k=1}^{2} s_i^{-1}(X_1 \cap Z_1)) > 0$ for all $i$; or (II): $t^2 =$ identity for all $t \in S$. Then sets $L \in \mathcal{I}, A^* \in \mathcal{I}, W_j$ with $W_j \subset L$, and transformations $t_j \in S$ can be found which have the following properties:

1) $m(L \cap X_i) < \delta$.

ii) $m(A^* \cap \bigcup_{i=1}^{k} s_i^{-1}L) = 0$.

iii) $m(L \cap s^{-1}L) = 0$. 

m(L \cap s^{-2}L) = 0 \text{ if (I) holds.}

L = s^{-2}L \text{ if (II) holds.}

iv) \( m(A \triangle A^*) < 58. \)

v) \( \{W_j\}^\infty_{j=1}, \) a partition of a subset of \( L, \) is an \( S^{-1} \)-generating system for \( \sigma(S^3), \) \( \{t^{-1}_jW_j\}^\infty_{j=1} \subset \mathcal{F}. \)

If (II) holds: \( L = \bigcup W_j \) and \( W_j \in \mathcal{F}. \)

vi) \( \bigcup_{i=0}^4 \bigcup_{j=1}^2 s^{-1}(A^* \cup (\bigcup s^{-1}L)) \cup \bigcup_{i=0}^4 t^{-1}_j(A^* \cup (\bigcup s^{-1}L)) \cap t^{-1}_nW_n = \emptyset, n = 1, 2, \ldots \)

vii) \( \bigcup_{i=0}^4 s^{iA^*} \cap (\bigcup t^{-1}_j(A^* \cup (\bigcup s^{-1}L)) = \emptyset. \)

viii) \( \bigcup_{i=0}^4 \bigcup_{j=1}^\infty t^{-1}_jW_j - \bigcup_{j=1}^\infty t^{-1}_jW_j = x_1^m(s), W_j \subset W_j. \)

If (II) holds then \( \bigcup_{i=0}^4 \bigcup_{j=1}^\infty t^{-1}_jW_j = \emptyset. \)

If \( H = A^* \cup \bigcup \bigcup_{i=0}^4 s^{-1}L \cup s^{-1}_jW_j \) then \( H \in \mathcal{F}, \) \( m(H \cap X_N \triangle A) < 88 \) and \( \sigma(S^3 \cap A) = \sigma(S^3); \) i.e., \( H \) is an \( S^{-1} \)-generating set for \( \sigma(S^3). \)

Proof: Note that by the comments prior to this lemma if (II) holds then \( Z_i \subset C_2^m(s) \) for each \( i = 1, 2, \ldots \) Hence, the second part of viii) holds if the first part is established because in this case \( m(x_1^m(s)) = 0. \) Apply lemma 3.7 to \( A \) and find \( L_1 \) and \( A_1 \in \mathcal{F} \) where \( A_1 \subset A \) and \( m(L_1) < \infty. \) In addition I have the following:

\[ s^iL_1 \in \mathcal{F} \text{ for } i \leq 2. \]

\[ m(L_1 \cap Z_i) > 0 \text{ for } i = 1, 2, \ldots. \]

\[ m(L_1 \cap X_n) < 8. \]
\[ L_1 \cap X \cap Z_1 \text{ is either an atom in } Z_1 \cap Y \cap X \text{ or is empty.} \]
\[ A_1 \cap \bigcup_{i=-2}^{1} s^i L_1 = \emptyset. \]
\[ m(A \Delta A_1) < 45. \]
\[ m(L_1) < \infty. \]
\[ L_1 \cap s^{-1} L_1 = \emptyset. \]
\[ L_1 \cap s^{-2} L_1 = \emptyset \text{ if (I) holds.} \]
\[ s^{-2} L_1 = L_1 \text{ if (II) holds.} \]

I will construct \( L \in \mathfrak{F} \) with \( L \subseteq L_1 \). Hence i), ii) and iii) will be automatically satisfied if \( A^* \subseteq A_1 \).

Next apply lemma 3.8 to \( B = A_1 \cup \bigcup_{i=0}^{2} \bigcup_{s^i L_1} \in \mathfrak{F} \) and find transformations \( (x_k)_{k=1}^{\infty} \) sets \( B_0 \in \mathfrak{F} \) with \( B_0 \subseteq B \) and \( m(B-B_0) < \delta \), and

sets \( B_n' \) with \( x_k B_n' \in \mathfrak{F} \) for \( k \geq n \) and \( m(B-B_n') < 2^{-2n} \) such that

\[ \bigcup_{i=0}^{4} s^{-1} (B \cup (\bigcup_{j \neq n} x_j^{-1} B) \cap x_j^{-1} B) \cap x_n^{-1} B_n' = \emptyset \text{ for all } n = 1, 2, \ldots, \]

and

\[ \bigcup_{i=0}^{4} s^{-1} B_0 \cap \bigcup_{n=1}^{\infty} x_n^{-1} B = \emptyset. \]

Now set \( A^* = B_0 \cap A_1 \subseteq A \). Then \( A^* \subseteq B_0 \) and \( m(A \Delta A^*) = m(A \Delta A_1) \)
\[ + m(A_1 \Delta A^*) < 55. \text{ Hence iv) is satisfied. Now } A^* \subseteq B_0 \text{ and} \]
\[ A^* \cup \bigcup_{i=0}^{2} s_i L \subseteq B \text{ so for } t_j = x_f(j), B_j = B^j(j) \text{ where } x_f(j) \text{ is a} \]
subsequence of \( r_j \) to be defined. Since \( r_j \) are distinct, \( t_j \) are also distinct, then for \( W_k \subset B_k \) and \( L \subset L_1 \), I have
\[
\bigcup_{i=0}^4 s^i((A^* U \bigcup s^iL) U \bigcup t_{j=k}^{-1}(A^* U \bigcup s^iL)) \cap t_k^{-1} W_k = \emptyset \quad \text{(vi)}, \quad \text{and}
\]
\[
\bigcup_{i=0}^4 s^iA^* \cap \bigcup_{k=1}^2 t_k^{-1}(A^* U \bigcup s^iL) = \emptyset \quad \text{(vii)}.
\]

It will suffice, then, to find \( L \subset L_1 \), \( W_j \subset L \) and \( W_j \subset B_j \), and distinct elements \( t_j \) of \( (r_j)^\infty_{j=1} \) such that (v) and (viii) are satisfied.

Since \( m(L_1 \cap Z_1) > 0 \) for \( i = 1, 2, \ldots \), I can apply lemma 3.6 to \( L_1 \) to produce a sequence of sets \( (W_j)^\infty_{j=1} \) and a sequence of numbers \( (\epsilon_j)^\infty_{j=1} \) with the following properties: If \( j \) is such that \( W_j \subset X_A \) the \( \epsilon_j = 0 \); and if \( j \) is such that \( W_j \subset X_N \), then \( \epsilon_j \) is positive and the sequences \( (W_j) \) and \( (\epsilon_j) \) are such that \( V_j \in J \) and
\[
M(W_j \Delta V_j) < \epsilon_j \quad \text{implies that} \quad (V_j)^\infty_{j=1} \quad \text{is an} \quad s^{-1} \quad \text{generating system for} \quad \sigma(S\beta).
\]
Lemma 3.6 implies that \( W_j \subset Z_{i_j}(m) \) for some \( i_j \) and for each \( i_j: Z_{i_j} \subset X_N(m) \) or \( Z_{i_j} \subset X_A(m) \) and \( X_N \cap X_A = \emptyset \) (m). So that \( W_j \subset X_N(m) \) or \( W_j \subset X_A(m) \).

Now I shall construct \( (W_j)^\infty_{j=1} \). By the construction of \( (W_j)^\infty_{j=1} \) in lemma 3.6 and the construction of \( L_1 \) in lemma 3.7, if \( W_j \subset Z_{i_j} \subset X_A \) then \( W_j \) is an atom in \( Z_{i_j} \cap 3 \). If \( W_j \subset Z_{i_j} \subset X_A \) then find
\[ r_{k_1} \in (x_j)_{j=1}^m \text{ such that } m(3 - R_{k_1}') < 2^{-k_1} < m_{1_1} = \text{the measure of an atom in } Z_{1_1} \cap \mathcal{J} = m(W_1'). \text{ Thus } m(W_1' \cap R_{k_1}') > 0 \text{ implies } W_1 \subset R_{k_1}'(m). \]

In this case set:

\[(2^4) \quad W_1 = W_1', \]
\[t_1 = r_{k_1}, \text{ and} \]
\[R_1 = R_{k_1}'. \]

Now \( W_1 \) is an atom in \( \mathcal{J} \) implies that \( t^{-1}W_1 \) and \( s^{-1}t^{-1}W_1 \) are atoms in \( \mathcal{J} \). Hence \( t^{-1}W_1 \subset X_1^J(s)(m) \) or \( t^{-1}W_1 \subset X_1^C(s)(m) \). If \( t^{-1}W_1 \subset X_1^J(s) \), then \( s^{-1}t^{-1}W_1 = t^{-1}W_1 \subset X_1^J(s) \), and \( 1 \bigcap s^i(t^{-1}W_1) = t^{-1}W_1 \subset X_1^J(s) \). If \( t^{-1}W_1 \subset X_1^C(s) \), then \( s^{-1}t^{-1}W_1 \cap t^{-1}W_1 = \emptyset (m) \), so that \( 1 \bigcap s^i(t^{-1}W_1) = \emptyset (m) \). So

\[ 1 \bigcap s^i(t^{-1}W_1) = t^{-1}W_0 \subset X_1^J(s). \]

If \( W_1' \subset Z_{1_1} \subset X_N \), the nonatomic part of \( (X, \mathcal{J}) \), then find \( x_1 \) such that \( 2^{-k_1} < m(W_1')/8 \). Apply lemma 3.3(1) to \( x_{k_1}^{-1}W_1' \) and find \( x_{k_1}^{-1}W_1'' \in \mathcal{J} \) such that

\[ 1 \bigcap s^i(x_{k_1}^{-1}W_1'') = \emptyset, \text{ and} \]

\[ m(x_{k_1}^{-1}W_1') \geq 1/4m(x_{k_1}^{-1}W_1' \cap X_1^C(s)); \text{ moreover, if (II) holds,} \]
\[ m(r_{k_1}^{-1}w''_1) \geq \frac{1}{4} m(r_{k_1}^{-1}w'_1) . \]

Set
\[ W_1 = (w''_1 \cup (w'_1 \cap r_{k_1}x_1^*(s))) \cap R_{k_1}^i , \]
\[ t_1 = r_{k_1} , \text{ and} \]
\[ B_1 = R_{k_1}^i . \]

Then
\[ W'_1 \supset W_1 , \]
\[ m(W'_1) \geq m(W_1)/8 , \text{ and} \]
\[ \bigcap_{i=0}^{i=1} s^i(t_1^{-1}w_1) = t_1^{-1}w_1 \subset x_1^*(s) . \]

Next find \( k_2 > k_1 \) such that \( 2^{-k_2} < m(W'_1 - W_1)/8 \). As above apply lemma 3.3(i) to \( r_{k_2}^{-1}(w'_1 - w_1) \) and find \( r_{k_2}^{-1}w''_2 \in \mathcal{Y} \) such that
\[ \bigcap_{i=0}^{i=1} s^i(r_{k_2}^{-1}w''_2) = \emptyset , \text{ and} \]
\[ m(r_{k_2}^{-1}w''_2) \geq \frac{1}{4} m(r_{k_2}^{-1}w'_1 \cap x_1^*(s)) . \]

If (II) holds, \( m(r_{k_2}^{-1}w''_2) \geq \frac{1}{4} m(r_{k_2}^{-1}(w'_1 - w_1)) . \)

Set:
\[ W_2 = (w''_2 \cup ((w'_1 - w_1) \cap r_{k_2}x_1^*(s))) \cap R_{k_2}^i , \]
\[ t_2 = r_{k_2} , \text{ and} \]
\[ B_2 = R_{k_2}^i . \]
Then:

(27) \[ w_1 - w_1 \supset w_2 \]

\[ \text{m}(w_2) \geq \text{m}(w_1 - w_1)/8, \quad \text{and} \]

\[ \frac{1}{\text{m}(w_2)} \sum_{i=0}^{n} t_{2}^{-1}(w_2) = t_{2}^{-1}w_2 \subset x_1(s) \]

Continue in this way: At the \( n \)th step find \( k_n > k_{n-1} \) such that

\[ 2^{-k_n} < \text{m}(w_1 - \bigcup_{j<n} w_j)/8; \]

apply lemma 3.3(1) to \( x_{k_n}^{-1}(w_1 - \bigcup_{j<n} w_j) \) and

find \( x_{k_n}^{-1}w'' \in \mathcal{F} \) such that

\[ x_{k_n}^{-1}w'' \subset x_{k_n}^{-1}(w_1 - \bigcup_{j<n} w_j), \]

\[ \frac{1}{\text{m}(x_{k_n}^{-1}w'')} = \phi, \quad \text{and} \]

\[ \text{m}(x_{k_n}^{-1}w'') \geq \frac{1}{4} \text{m}(x_{k_n}^{-1}(w_1 - \bigcup_{j<n} w_j) \cap x_1(s)); \quad \text{moreover, if (II)} \]

holds, \( \text{m}(x_{k_n}^{-1}w'') \geq \frac{1}{4} \text{m}(x_{k_n}^{-1}(w_1 - \bigcup_{j<n} w_j)) \).

Set:

(28) \[ w_n = (w'' \cup ((w_1 - \bigcup_{j<n} w_j) \cap x_1(s))) \cap B_{k_n} \]

\[ t_n = x_{k_n}, \quad \text{and} \]

\[ B_n = B_{k_n} \]

Then
(29) \( W'_1 = \bigcup_{j=1}^k W_j \supseteq W_n, \)
\[ m(W'_n) \geq m(W'_1 - \bigcup_{j=1}^k W_j) / 8, \]
and
\[ \bigcap_{i=0}^1 \left( t_n^{-1} W_n \right) - t_n^{-1} W_n^o \subset \chi_1^0(s). \]

Now \((W'_1)_{i=1}^\infty\) is a system of disjoint sets contained in \(W'_1\),
a set of finite measure, such that \( m(W'_i) \geq (1 - 7/8)m(W'_1 - \bigcup_{j=1}^k W_j) \).
Therefore the conditions of lemma 3.3 ii) are satisfied.
Hence,
\[ m\left( \bigcup_{i=1}^k W_i \right) \geq (1 - (7/8)^k)m(W'_1). \]
Choose \(n_1\) such that \( (7/8)^n m(W'_1) < \varepsilon_1 \); i.e., choose \(n_1 > \frac{\log(e_1/m(W'_1))}{\log (7/8)} \).

Then
\[ m\left( \bigcup_{i=n_1}^\infty W_i \right) \geq (1 - (7/8)^{n_1})m(W'_1) \text{ and } W_1 \subseteq W'_1, \]
so that
\[ m(W'_1 - \bigcup_{i=n_1}^\infty W_i) \leq (7/8)^{n_1} m(W'_1) < \varepsilon_1. \]

Now (30) shows that \(\{W'_j\}_{j=1}^{n_1}\) may be substituted for \(W'_1\) in \(\{W'_j\}_{j=1}^\infty\).
The new class of sets will still be an \(S^1\)-generating system for \(\sigma[S3]\).
Repeat this process for each \(W'_j\), \(j = 2, \ldots\), so that a \(W_j\) is obtained such that \(\{W_j\}_{j=1}^\infty\) is an \(S^1\)-generating system for \(\sigma[S3]\), where \(\Sigma n_i < j \leq \Sigma n_i\), and the following hold:
\[ W_j \subseteq W'_1 - \bigcup_{i=1}^{n_1} W_i, \]
\[ W_j \subseteq W'_1 - \bigcup_{i=1}^{n_1} W_i, \]
\[ i = \Sigma_{n_i} + 1. \]
\[ m\left( \bigcup_{k=1}^{n_1} W_j \right) > m(W_k^1) - \varepsilon_k, \]
\[ j = n_1 + 1 \]

\[ W_j \subseteq B_j = B^i_f(j), \]
\[ t_j = r_f(j) \quad \text{where } r(j_1) \neq r(j_2) \text{ if } j_1 \neq j_2, \]
\[ \bigcap_{i=0}^{1} s_i^1(t_j^{-1} W_j) = t_j^{-1} W_j \subseteq x_1^1(s), \text{ and} \]

\[ t_j^{-1} W_j \in \mathcal{F}. \]

Now \( s^{-1}(t_1^{-1} W_1) \cap t_1^{-1} W_1 = \emptyset \) by (22) since \( t_1^{-1} W_1 \subseteq r_1^{-1} F^i_f(j) \)

and \( t_1^{-1} W_1 \subseteq x_1^{-1}(i) B \). Apply lemma 3.2 and find that

\[ \bigcap_{i=0}^{1} s_i^1(\bigcup_{j=1}^{\infty} t_j^{-1} W_j) = \bigcup_{j=1}^{n_1} \bigcap_{i=0}^{1} s_i^1(t_j^{-1} W_j) = \bigcup_{j=1}^{n_1} t_j^{-1} W_j \subseteq x_1^1(s) \]

where \( W_j^0 \subseteq W_j \). So VIII) is established.

If (I) holds set \( L = L_1 \) and if (II) holds set \( L = \bigcup_{j=1}^{n_1} W_j \).

Note that if (II) holds then \( t \in S \) is \( \mathcal{F} \)-invertible, hence \( W_j \in \mathcal{F} \).

So that in either case \( L \in \mathcal{F} \). Thus v) is satisfied. Note that

\[ H = A^* \cup L \cup s^{-1} L \cup s^{-1} \left( \bigcup_{j=1}^{\infty} t_j^{-1} W_j \right) \in \mathcal{F}. \]

Now in both cases

\[ m(H \cap X_N \Delta A) = m(H \cap X_N \setminus A) + m(A \setminus H \cap X_N) \]
\[ \leq 3m(L) + m(A \Delta A^*) \]
\[ \leq 35 + 55 = 85, \]
because \( W_j \subseteq L \) and \( m \) is \( S \)-invariant.

To show that \( \sigma(SH) = \sigma(S^3) \), it will suffice to show that

\[ \sigma(SH) = \{ W_j \}_{j=1}^{\infty} \]

Consider the following equalities:

\[
H = A^* \cup L \cup s^{-1}L \cup s^{-1}( \bigcup_{j=1}^{\infty} t_j^{-1}W_j ) ;
\]

\[
sH = sA^* \cup sL \cup L \cup ( \bigcup_{j=1}^{\infty} t_j^{-1}W_j ) ; \quad \text{and}
\]

\[
s_H^2 = s^2A^* \cup s^2L \cup sL \cup s( \bigcup_{j=1}^{\infty} t_j^{-1}W_j ) .
\]

Also:

\[
A^* \cap (sA^* \cup sL \cup L \cup ( \bigcup_{j=1}^{\infty} t_j^{-1}W_j ) ) = \emptyset \ (\text{ii, vii}).
\]

\[
A^* \cap (sL \cup L \cup ( \bigcup_{j=1}^{\infty} t_j^{-1}W_j ) ) = \emptyset \ (\text{ii, iii, vi}) \text{ if (I) holds;}
\]

\[
= sL \ (\text{ii, iii, vi}) \text{ if (II) holds.}
\]

\[
s^{-1}( \bigcup_{j=1}^{\infty} t_j^{-1}W_j ) \cap (sA^* \cup sL \cup L ) = s^{-1}( ( \bigcup_{j=1}^{\infty} t_j^{-1}W_j ) \cap (s^2A^* \cup s^2L \cup sL ) ) = \emptyset \ (\text{vi, vii}).
\]

\[
s^{-1}( \bigcup_{j=1}^{\infty} t_j^{-1}W_j ) \cap \bigcup_{j=1}^{\infty} t_j^{-1}W_j = \bigcup_{j=1}^{\infty} t_j^{-1}W_j \subseteq x_1^0 \text{ if (I) holds (viii);} \]

\[
= \emptyset \text{ if (II) holds (viii).}
\]

So that:

\[
H \cap sH = (A^* \cap sA^*) \cup L \cup \bigcup_{j=1}^{\infty} t_j^{-1}W_j \quad \text{if (I) holds;}
\]

\[
= (A^* \cap sA^*) \cup L \cup sL \quad \text{if (II) holds.}
\]

Also if (I) holds:

\[
A^* \cap sA^* \cap (s^2L \cup sL \cup s( \bigcup_{j=1}^{\infty} t_j^{-1}W_j ) = \emptyset \ (m) \ (\text{ii, viii}).
\]
\[ I \cap (s^2A \cup s^2L \cup sL \cup s(\bigcup_{j=1}^{\infty} t_j^{-1}W_j)) = \emptyset \text{ (i)} \text{, (ii), (vi)} . \]

\[ \bigcup_{j=1}^{\infty} t_j^{-1}W_j \cap (s^2A^* \cup sL \cup s^2L) = \emptyset \text{ (vii), (vi)} . \]

\[ \bigcup_{j=1}^{\infty} t_j^{-1}W_j \cap s(\bigcup_{j=1}^{\infty} t_j^{-1}W_j) = \bigcup_{j=1}^{\infty} t_j^{-1}W_j \text{ (viii)} . \]

So that if (I) holds

\[ H \cap sH - s^2H = ((A^* \cap sA^*) - s^2A^*) \cup L \text{, and} \]

\[ sH - \bigcup_{i=0}^{1} s^i(H \cap sH - s^2H) = sA^* \cup sL \cup L \cup (\bigcup_{j=1}^{\infty} t_j^{-1}W_j) - \]

\[ \bigcup_{i=0}^{1} s^i((A^* \cap sA^* - s^2A^*) \cup L) = (sA^* - \bigcup_{i=0}^{1} (A^* \cap sA^* - s^2A^*)) \cup \bigcup_{j=1}^{\infty} t_j^{-1}W_j , \]

because

\[ \bigcup_{j=1}^{\infty} t_j^{-1}W_j \cap \bigcup_{i=0}^{1} s^i(A^* \cup L) = \emptyset \text{ (vi), (vii)} . \]

\[ sA^* \cap \bigcup_{i=0}^{1} s^iL = s(A^* \cap \bigcup_{i=0}^{1} s^iL) = \emptyset \text{ (ii).} \]

\[ t_k(sH - \bigcup_{i=0}^{1} s^i(H \cap sH - s^2H) \cap (H \cap sH - s^2H) = \]

\[ = (t_k((sA^* - \bigcup_{i=0}^{1} s^i(A^* \cap sA^* - s^2A^*)) \cup \bigcup_{j \neq k} t_j^{-1}W_j) \cup W_k) \cap ((A^* \cap sA^* - s^2A^*) \cup L) = \]

\[ = W_k , \]

because \( t_k(sA^* \cap (A^* \cup L) = \emptyset \text{ (vii)} \) and \( t_k(\bigcup_{j \neq k} t_j^{-1}W_j) \cap (A^* \cup L) = \)

\[ t_k(\bigcup_{j \neq k} t_j^{-1}(A^* \cup L)) = \emptyset \text{ (vi)} \) and \( W_k \subseteq L \text{ (v)} . \)
If (II) holds

\[ H \cap sH = (A* \cap sA*) \cup L \cup sL. \]

Hence

\[ (34) \quad H - H \cap sH = A* - (A* \cap sA*) \cup \bigcup_{j=1}^{\infty} t_j^{-1}w_j. \]

Then

\[ t_k(H - H \cap sH) \cap (H \cap sH) = \]

\[ (t_k(A*-sA*)) \cup \bigcup_{j \neq k} t_j^{-1}w_j \cup w_k) \cap ((A* \cap sA*) \cup L \cup sL) = \]

\[ = w_k, \]

because:

\[ t_k(A*-sA*) \cap ((A* \cap sA*) \cup L \cup sL) = \]

\[ t_k((A*-sA*) \cap (t_k^{-1}(A* \cap sA*) \cup t_k^{-1}sL)) = \]

\[ = \emptyset \quad (vii), \]

\[ t_k t_j^{-1}w_j \cap ((A* \cap sA*) \cup (L \cup sL)) = \]

\[ t_k(t_j^{-1}w_j \cap t_k^{-1}((A* \cap sA*) \cup (L \cup sL))) = \]

\[ = \emptyset \text{ for } j \neq k \quad (vi, vii), \text{ and} \]

\[ W_k \subseteq L \quad (v). \]

So lemma 3.9 is proven.
Theorem 3.1: Assume the usual conditions hold on $S, \mathcal{F}, \mathcal{J}, X, Z_1, m$:

1) $S$ is a semigroup of measure preserving invertible transformations on $(X, \mathcal{J}, m)$ which are also $\mathcal{J}$-measurable.

2) $\mathcal{J}$ is a sub-$\sigma$-algebra of $\mathcal{A}$.

3) $(X, \mathcal{J}, m)$ is a $\sigma$-finite, infinite, separable measure space.

4) $X = \bigcup_{i=1}^{\infty} Z_i$ where $Z_i \in \mathcal{J}, (Z_i)$ are $m$-disjoint, $Z_i$ is $S$-invariant, $Z_i$ contains no nontrivial $S$-invariant $\mathcal{J}$-measurable sets and $m(Z_i)$ is infinite.

In addition assume that $\mathcal{J}$ is an $S^{-1}$-exhaustive sub-$\sigma$-algebra of $\mathcal{J}$.

Let $X_N$ be the nonatomic part of $(X, \mathcal{J})$. Then the system of sets $\mathcal{H} \in \mathcal{J}$ such that $\{H, H^c\}$ is an $S^{-1}$-generator for $\mathcal{J}$ has the property that the system $X \cap X_N$ is dense in the sets of $X_N \cap \mathcal{J}$ which have finite measure.

Proof: Set $X^{(1)} = \bigcup [Z_i: there exists an $s \in S$ such that $m(\bigcup_{k=1,2}^\infty x_k^\mathcal{J}(s) \cap Z_i) > 0 \}, and set $X^{(2)} = X - X^{(1)}$. Then for each $i, Z_i \subset X^{(1)}(m)$ or $Z_i \subset X^{(2)}(m)$. For all $s \in S, s^2$ = identity on $X^{(2)}$. For each $Z_i \subset X^{(1)}$ there exists an $s \in S$ such that $m(\bigcup_{k=1,2}^\infty x_k^\mathcal{J}(s) \cap Z_i) > 0$.

Let $A \in X_N \cap \mathcal{J}$ with $m(A)$ finite and $s > 0$ be given. I
shall construct $H_* \in X^{(1)} \cap \mathfrak{F}$ and $H_* \in X^{(2)} \cap \mathfrak{F}$ such that
\[
m(H_1 \cap X_N \Delta A \cap X^{(1)}) < \frac{1}{2} \varepsilon , \quad \text{and} \quad (H_2 \cap X_N \Delta A \cap X^{(2)}) < \frac{1}{2} \varepsilon ,
\]
\[\sigma(H_1) = \sigma(S(X^{(1)} \cap \mathfrak{F})), \quad \text{and} \quad \sigma(H_2) = \sigma(S(X^{(2)} \cap \mathfrak{F})).\]

Choose $s_1 \in S$ and find all $Z_j$ such that $m(\bigcup_{k=1,2} X^{(k)}(s_1) \cap Z_j) > 0$.
Relabel these $Z_j$ as $Z_{1,i}$, $i = 1, 2, \ldots$. Apply lemma 3.9 with $A$ replaced by $A \cap \bigcup_i Z_{1,i}$ and $X = \bigcup_i Z_{1,i}$, and $s = s_1$, and $\delta = \varepsilon/2^2 \cdot 7$. I can then find $L_1$, $A_1^*$, $W_{1,j}$, $t_{1,j}$, $H_1$ such that
\[
H_1 = A_1^* \cup \bigcup_{i=0}^{1} L_1 \cup \bigcup_{j=1}^{1} W_{1,j} \cup \bigcup_{i=1}^{1} Z_{1,i} \cap \mathfrak{F},
\]
\[\{H_1, H_1^c\} \text{ is an } S^{-1} \text{-generator for } \sigma(S(\bigcup_i Z_{1,i} \cap \mathfrak{F})).\]

$m(H_1 \cap X_N \Delta A \cap \bigcup_i Z_{1,i}) < \varepsilon/2^2$.

Next select $s_2 \in S$ and find all $Z_j \in \{Z_{1,i}\}_{i=1}^{\infty}$ such that $m(\bigcup_{k=1,2} X^{(k)}(s_2) \cap Z_j) > 0$. Relabel these $Z_j$ as $Z_{2,i}$, $i = 1, 2, \ldots$.

Apply lemma 3.9 with $A$ replaced by $A \cap \bigcup_i Z_{2,i}$, and $X = \bigcup_i Z_{2,i}$, and $s = s_2$, and $\delta = \varepsilon/2^3 \cdot 7$. Thus $L_2$, $A_2^*$, $W_{2,j}$, $t_{2,j}$, $H_2$ such that
\[
H_2 = A_2^* \cup \bigcup_{i=0}^{1} L_2 \cup \bigcup_{j=1}^{1} W_{2,j} \cup \bigcup_{i=1}^{1} Z_{2,i} \cap \mathfrak{F},
\]
\[\{H_2, H_2^c\} \text{ is an } S^{-1} \text{-generator for } \sigma(S(\bigcup_i Z_{2,i} \cap \mathfrak{F})).\]

$m(H_2 \cap X_N \Delta A \cap \bigcup_i Z_{2,i}) < \varepsilon/2^3$.

Continue producing $H_1$s in this manner. Since for every $Z_j \subset X^{(1)}$
there exists an \( s \in S \) such that \( \text{m}(\bigcup_{k \neq 1, 2} x_k^3(s) \cap Z_j) > 0 \), eventually the set consisting of all \( Z_j \) contained in \( X(1) \) will be exhausted.

Set \( H_1 = \bigcup_i H_i \).

Choose \( s_1 \in S \) and find all \( Z_j \subset X(2) \) such that \( \text{m}(\bigcup_{k \neq 1} x_k^3(s) \cap Z_j) > 0 \).

As noted previously this is equivalent to \( Z_j \subset X_2^3(s) \). Relabel these \( Z_j \) as \( Z_{1,i} \), \( i = 1, 2, \ldots \). Apply lemma 3.9 with \( A \) replaced by \( A \cap \bigcup_{i=1}^{\infty} Z_{1,i} \), \( X = \bigcup_{i=1}^{\infty} Z_{1,i} \), \( s = s_1 \), and \( \varepsilon = \varepsilon/2^2 \).

Thus find \( L_i', A_i', W_i,j, t_i,j, H_i' \) such that

\[
H_i' = A_i' \cup \bigcup_{i=0}^{\infty} s_{1,i+1} \cup \bigcup_{j=0}^{\infty} t_{i,j+1} \cup W_{i,j} \in \bigcup_{i=1}^{\infty} Z_{1,i} \cap F,
\]

\( \{H_i', H_i'\} \) is an \( S^{-1} \)-generator for \( \sigma(s(\bigcup_{i=1}^{\infty} Z_{1,i} \cap F)) \), and

\( \text{m}(H_i' \cap X N A \cap \bigcup_{i=1}^{\infty} Z_{1,i}) < \varepsilon/2^2 \).

Continue in this manner producing \( H_i' \)s. Since for every \( Z_j \subset X(2) \) there exists an \( s \in S \) such that \( \text{m}(\bigcup_{k \neq 1} x_k^3(s) \cap Z_j) > 0 \), and since \( Z_j \) has no nontrivial \( S \)-invariant sets, eventually the set of \( Z_1 \) contained in \( X(2) \) will be exhausted.

Set \( H_2 = \bigcup_i H_i' \).

Now if \( H = H_1 \cup H_2 \), then \( H \in F \) and \( \text{m}(H \cap X N A \cup \bigcup_{i=1}^{\infty} Z_{1,i}) < \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon \).

So to complete the proof it will be sufficient to show that \( \sigma(H \cup F) \subset \sigma(SF) \).

Note, the assumption that \( F \) is an \( S^{-1} \)-exhaustive sub-\( \sigma \)-algebra of \( \mathcal{J} \) implies that \( \sigma(SF) \cap \mathcal{J} \). To show that \( \sigma(H \cup F) = \sigma(SF) \), it is sufficient to show that
\( \sigma(\mathcal{SH}) \supset \{ H_1, H'_1 \} \) for all \( i \). Consider \( \bigcap_{j=0}^{1} s_{jH}^1 = \bigcap_{j=0}^{1} s_{H_1}^j \cup \bigcap_{j=0}^{1} s_{(H - H_1)}^j \). Now \( s_1^2 \) is identity on \( H - H_1 \) because all \( Z_j \) such that \( s_1^2 \neq \text{identity on some part of} \ Z_j \) are members of \( Z_{1,i} \). Hence \( B_2 = \bigcap_{j=0}^{1} s_{jH}^1(H - H_1) \) is an \( s_1^{-}\)-invariant set. By (33)

\[
\bigcup_{i=1}^{1} Z_{1,i} = \bigcap_{j=0}^{1} s_{H_1}^j \cap \bigcap_{j=0}^{1} s_{H_1}^j = \bigcap_{j=0}^{1} s_{H_1}^j \cap \bigcap_{j=0}^{1} s_{H_1}^j \subseteq \mathcal{L}_1,
\]

and \( m(L_1 \cap Z_{1,i}) > 0 \) for each \( i = 1, 2, \ldots \). By lemma 3.4 there exists a sequence of \( (x_k)_{k=1}^{\infty} \) such that \( \bigcup_{k=1}^{\infty} x_k \left( \bigcap_{j=0}^{1} s_{H_1}^j \cap \bigcap_{j=0}^{1} s_{H_1}^j \right) = \bigcup_{i=1}^{1} Z_{1,i} \).

Hence \( \left( \bigcup_{k=1}^{\infty} x_k \left( \bigcap_{j=0}^{1} s_{H_1}^j \cap \bigcap_{j=0}^{1} s_{H_1}^j \right) \right) \cap H = H_1 \).

Similarly \( H_2 \) may be separated from \( H - H_1 \), etc. Separating off all \( H_1 \) produces the system of sets \( \{ H_1, H_2, \ldots, H_{H_1} \} \). By (34) and because of the fact that \( s_1^j \) is the identity on \( H_2 - H_1 \) and \( H_1 \), \( (H - H_1 = H_1) \),

\[
\bigcup_{i=1}^{1} Z_{1,i} \supset H_1 - \bigcap_{j=0}^{1} s_{H_2}^j = H_1 - \bigcap_{j=0}^{1} s_{H_1}^j \cup_{j=0}^{1} s_{H_1}^j \cup_{j=0}^{1} s_{H_1}^j \subseteq \mathcal{W}_{1,j}.
\]

and \( m(Z_{1,i} \cap \bigcup_{j=0}^{1} s_{H_1}^j \cup_{j=0}^{1} s_{H_1}^j ) > 0 \) for each \( i \). Hence, as before,

for some \( (x_k)_{k=1}^{\infty} \), \( \bigcup_{k=1}^{\infty} x_k (H_2 - \bigcap_{j=0}^{1} s_{H_1}^j ) \cap H_2 = H_1 \), and \( H_1 \)

may be separated for all \( i \). Thus I have shown that \( \sigma(\mathcal{SH}) = \sigma(\mathcal{S3}) \)
as required and hence theorem 3.1 is proven.

Corollary 3.1, to follow, shows that the requirement in theorem 2.1 that \( S \) be left amenable or \( \mathcal{S} \)-invertible can be dropped if there
exists a σ-finite invariant measure an \( J \) equivalent to \( p \), and if 
there are no \( S \)-m-invariant sets in \( J \). Although the density 
statement about the system of size-2-\( S^{-1} \)-generators is not necessarily 
true the existence of a size-2-\( S^{-1} \)-generator for \( J \) in \( F \) is established.

**Corollary 3.1**: Let \( S \) be a semigroup of nonsingular invertible 
transformations on a finite separable measure space \((X, \mathcal{J}, p)\) 
with \( p(x) = 1 \). Let \( F \) be an \( S^{-1} \)-exhaustive sub-\( \sigma \)-algebra of \( J \).

If there exists an \( F \)-σ-finite \( S \)-invariant measure \( m \gg p \) on \( J \), 
where \( X = \bigcup_{i=1}^{\infty} Z_i \) and \( Z_i \in \mathcal{J} \) is an \( S \)-m-invariant set of infinite 
m-measure which contains no nontrivial \( S \)-m-subinvariant sets in \( J \), 
then there exists a size-2-\( S^{-1} \)-generator \([H, H^c]\) in \( F \) for \( J \).

**Proof**: Apply theorem 3.1 and find \( H \in \mathcal{F} \) which is an \( S^{-1} \)-generating 
set for \( J(m) \). Now \( p \ll m \) implies that \( H \) is an \( S^{-1} \)-generating 
set for \( J(p) \).

Theorem 3.1 also has the following as its corollaries.

**Corollary 3.2**: Let \( T \) be a conservative ergodic measure preserving 
transformation on \((X, \mathcal{J}, m)\), an infinite measure space. Let \((X, \mathcal{J}, m)\) 
be a σ-finite infinite separable measure space where \( \mathcal{J} \) is an ex­ 
haustive sub-\( \sigma \)-algebra of \( J \). Then the system of sets \( A \) such that 
\( [A, A^c] \) is a strong generator is dense in the sets of finite 
measure in \( \mathcal{J} \).

**Proof**: Let \([T^i : i \geq 0] = S\). \( T \) conservative and ergodic implies 
that \( S \) has no nontrivial subinvariant sets and that \( X \) is nonatomic.
in both $\mathcal{I}$ and $\mathcal{J}$. $\mathcal{J}$ an exhaustive sub-$\sigma$-algebra of $\mathcal{I}$ is equivalent to $\mathcal{J}$ an $S^{-1}$-exhaustive sub-$\sigma$-algebra of $\mathcal{I}$. Also $\{A, A^c\}$ an $S^{-1}$-generator for $\mathcal{J}$ is equivalent to $\{A, A^c\}$ a strong generator. Hence theorem 3.1 implies that the system of sets $A \in \mathcal{I}$ such that $\{A, A^c\}$ is a strong generator is dense in the sets of finite measure in $\mathcal{I}$.

**Corollary 3.3:** Let $S$ be an $m$-ergodic group of invertible measure preserving transformations on $(X, \mathcal{I}, m)$, an infinite measure space. Let $(X, \mathcal{I}, m)$ be a $\sigma$-finite infinite separable measure space where $\mathcal{I}$ is an $S^{-1}$-exhaustive sub-$\sigma$-algebra of $\mathcal{I}$. Then either $(X, \mathcal{I})$ is atomic and for every atom $A \in \mathcal{I}$ $\{A, A^c\}$ is an $S^{-1}$-generator for $\mathcal{I}$; or $(X, \mathcal{I})$ is nonatomic and the system of sets $A \in \mathcal{I}$ such that $\{A, A^c\}$ is an $S^{-1}$-generator for $\mathcal{I}$ is dense in the sets of finite measure in $\mathcal{I}$.

**Proof:** If $S$ is a group and if $S$ is $m$-ergodic then $S$ has no non-trivial sub-invariant sets. Therefore $X = Z_1$ in the notation of theorem 3.1. Then by lemma 3.6, $(X, \mathcal{I})$ is either atomic or nonatomic. If $(X, \mathcal{I})$ is atomic then by lemma 3.6 we have that every atom in $\mathcal{I}$ is an $S^{-1}$-generating set for $\mathcal{I}$. If $(X, \mathcal{I})$ is nonatomic then apply theorem 3.1 to get the desired result.

**Corollary 3.4:** Let $S$ be an $m$-ergodic semigroup of conservative invertible measure preserving transformations on $(X, \mathcal{I}, m)$, an infinite measure space. Let $(X, \mathcal{I}, m)$ be a $\sigma$-finite infinite
separable measure space where $\mathcal{J}$ is an $\mathcal{S}^{-1}$-exhaustive sub-$\mathcal{C}$-algebra of $\mathfrak{M}$. Then either $(X, \mathcal{J})$ is atomic and for every atom $A \in \mathcal{J}$, $(A, A^c)$ is an $\mathcal{S}^{-1}$-generator for $\mathfrak{M}$ or $(X, \mathcal{J})$ is nonatomic and the system of sets $A \in \mathcal{J}$ such that $(A, A^c)$ is an $\mathcal{S}^{-1}$-generator for $\mathfrak{M}$ is dense in the sets of finite measure in $\mathcal{J}$.

Proof: If $s \in S$ is conservative and if $sA \subseteq A(m)$ then, as before (p.60), $sA = A(m)$. Hence if $A$ is an $S$-m-subinvariant set, then $A$ is an $S$-m-invariant set. Thus if $S$ is also $m$-ergodic, $S$ has no nontrivial subinvariant sets. The rest follows as in corollary 3.3.

The following example shows that it is not sufficient simply to require that $S$ be $m$-ergodic and drop the requirement that $S$ has no nontrivial subinvariant sets.

Let $X$ be the real line with Lebesque measure. $X$ is a nonatomic $\sigma$-finite infinite separable measure space. Let $t_2: x \mapsto x+1$. Let $t_2'$ be any ergodic measure preserving transformation on $[0,1)$ and set $t_2 = t_2'$ on $[0,1)$ and equal to the identity elsewhere. I will show that $S$, the smallest semigroup containing $t_1$ and $t_2$, is $m$-ergodic, $S$ has nontrivial subinvariant sets, and there is no set of finite measure which is an $\mathcal{S}^{-1}$-generating set for $X$. Note also that since $t_1$ is dissipative and $t_2$ conservative, $S$ contains both conservative and dissipative transformations.

First I will show that there exists nontrivial subinvariant sets for $S$. Any $s \in S$ has the form $s = t_1^{i_1}t_2^{i_2} \cdots t_1^{i_{2n+1}}$ where $n$ and $i_k$ are nonnegative integers. So $s^{-1} = t_1^{-i_{2n+1}}t_2^{-i_2} \cdots t_2^{-i_2}t_1^{-i_1}$. Then for any integer $i$ it is the case that $(\neq, i)$ is an invariant set.
since \( s^{-1}(-\infty, 1) = (-\infty, 1 - \sum_{k=0}^{n} i_{2k+1}) \).

Now I will show that \( S \) is \( m \)-ergodic on \( X \); i.e., that if \( s^{-1}A = A(m) \) for all \( s \in S \), then either \( A = X(m) \) or \( A = \emptyset \). Assume \( s^{-1}A = A(m) \) for all \( s \in S \). Then \( t_{-1}^{-1}(A \cap [i, i+1]) = A \cap [i-1, i)(m) \), which implies that for all integers \( i \) and \( j \), \( A \cap [i, i+1] = t_{-1}^{-1}(A \cap [j, j+1]) \)(m). Thus \( m(A \cap [0, 1]) > 0 \) or \( m(A) = 0 \). Since \( t_{2} \) is conservative and ergodic on \([0,1]\), the alternative, \( m(A \cap [0, 1]) > 0 \), along with the invariance of \( A \), imply that \( A \supset \bigcup_{n=0}^{\infty} t_{2}^{-n}(A \cap [0, 1]) = [0, 1)(m) \). Thus either \( A = \emptyset(m) \) or \( A = X(m) \).

If, for a given set \( A \), there exists sets \( C \) and \( D \) such that \( sA \cap C \subset D \) or \( sA \cap C \subset D \) for every \( s \in S \) then for all \( B \in \sigma(sA) : s \in S \) either \( B \cap C \subset D \) or \( B \cap C \subset D \). Indeed, let \( \Theta = \{ B \in \sigma(sA) : B \cap C \subset D \text{ or } B \cap C \subset D \} \). Then \( \Theta \) is obviously closed under complementation. If \( B_{i} \in \Theta \) and for all \( i \), \( B_{i} \cap C \subset D \), then \( \bigcup_{i=1}^{\infty} B_{i} \cap C \subset D \). If there exists an \( i \) such that \( B_{i} \cap C \subset D \), then \( \bigcup_{i=1}^{\infty} B_{i} \cap C \subset D \). Hence \( \Theta \) is a \( \sigma \)-algebra and \( \Theta \supset \{ sA \} \), so that \( \Theta = \sigma(sA) \). Now \( sA \cap [k, k+1] \subset \bigcup_{j=0}^{\infty} t_{j}^{i}A \cap [k, k+1] \) for \( k \leq -1 \). Indeed, for \( s = t_{1}^{i} \ldots t_{2}^{i_{2n}} t_{1}^{i_{2n+1}} \) where \( i = \sum_{k=0}^{n} i_{2k+1} \), \( i_{2k+1} \) is a nonnegative integer and \( k \leq -1 \), I
have $A \cap (k, k+1) = s(A \cap t_{1}^{-1}(k,k+1)) = t_{1}^{-1}(A \cap t_{1}^{-1}(k,k+1)) = t_{1}^{-1}A \cap (k,k+1) \subset \bigcup_{j=0}^{\infty} t_{1}^{j}A \cap (k,k+1)$. Thus by the above, $B \in \sigma(A)$ implies that either $B \cap (k,k+1) \subset \bigcup_{j=0}^{\infty} t_{1}^{j}A \cap (k,k+1)$ or $B^{c} \cap (k,k+1) \subset \bigcup_{j=0}^{\infty} t_{1}^{j}A \cap (k,k+1)$. Now

$$m(\bigcup_{j=0}^{\infty} t_{1}^{j}A \cap (k,k+1)) \leq \sum_{j=0}^{\infty} m(t_{1}^{j}A \cap (k,k+1))$$

$$= \sum_{j=0}^{\infty} m(A \cap (k-j, k+1-j))$$

$$= m(A \cap (-\infty, k+1)) = 0$$

Since $\lim_{k \to -\infty} m(A \cap (-\infty, k+1)) = 0$, for a fixed $\varepsilon > 0$ there exists $k$ such that for every $B \in \sigma(A)$ either $m(B \cap (k,k+1)) < \varepsilon$ or $m(B \cap (k,k+1)) > 1 - \varepsilon$. There exists a Lebesgue set $C$ of positive measure strictly contained in $(\bigcup_{j=0}^{\infty} t_{1}^{j}A)^{c} \cap (k,k+1)$ since, for the latter set, $m(\bigcup_{j=0}^{\infty} t_{1}^{j}A)^{c} \cap (k,k+1) \geq 1 - m(\bigcup_{j=0}^{\infty} t_{1}^{j}A \cap (k,k+1)) \geq 1 - \varepsilon$.

However, for all $B \in \sigma(A)$, either $B \cap (k,k+1) \supset (\bigcup_{j=0}^{\infty} t_{1}^{j}A)^{c} \cap (k,k+1)$ or $B \cap (k,k+1) \subset (\bigcup_{j=0}^{\infty} t_{1}^{j}A)^{c} \cap (k,k+1)$. Hence, the fact that $C \cap (k,k+1)$ neither contains $(\bigcup_{j=0}^{\infty} t_{1}^{j}A)^{c} \cap (k,k+1)$ nor is contained in $(\bigcup_{j=0}^{\infty} t_{1}^{j}A) \cap (k,k+1)$ implies that $C \not\in \sigma(A)$. Thus $\sigma(A)$ is
not the same as the $\sigma$-algebra of Lebesgue sets up to sets of measure zero, and $\{A, A^c\}$ is not an $S^{-1}$-generator.
BIBLIOGRAPHY


