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THE FOCAL-REGION FIELDS OF PARABOLOIDAL
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DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree of Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Dirk Edmond Baker, B.Sc., B.Sc.(Hons.), M.Sc.

* * * * *

The Ohio State University
1974

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INTRODUCTION

In recent years high-gain paraboloidal reflector antennas have found considerable application in fields such as world-wide radio and television communication, radar, space-probe tracking and radio astronomy. The paraboloidal reflector may be used either as a prime-focus instrument, in which case the incident energy is collected by a primary feed in the region of the focus, or as a paraboloid-subreflector combination, in which case the energy incident at the focus is redirected by the subreflector to be collected at some other location (e.g., in the Cassagrainian system the incident energy is redirected towards the vertex of the paraboloid by a hyperboloidal subreflector in the vicinity of the focus). The prime-focus paraboloid and feed-horn combination is simple and flexible; it is this configuration which will be investigated here.

The purpose of this introduction is to present some of the aspects of the use of prime-focus paraboloids in the field of antennas and to review the current literature pertaining to the subject. Although the study of the focusing properties of paraboloidal mirrors has formed a part of the theory of classical optics since the last century, it is not the aim of the present discussion to review this wealth of information. Rather, the techniques of classical optics will be mentioned and discussed with regard for their applicability to paraboloidal reflector
antennas. Further, terms such as "spillover power", "scalar diffraction theory", "induced current method", etc., which are in common usage in the antenna literature, will not be clarified in this introduction since it is felt that this would unnecessarily complicate the presentation at this stage. They will be described, however, in their proper context as they appear in the main body of the text of the following chapters.

A detailed discussion of the scope of this dissertation is deferred to the end of Chapter I which has the function of reviewing the theoretical aspects which apply to the subsequent analysis. This is felt to be desirable since at that stage the necessary clarification of antenna terms will have been accomplished, thereby facilitating the presentation.

The aperture efficiency (essentially the ratio of the power collected by the feed horn to the total power incident on the reflector) of prime-focus systems is usually below about 65%, which represents a substantial waste of reflector surface and loss of antenna gain. Perhaps the simplest way to increase the antenna gain is to increase the physical size of the reflector. This, however, may not be feasible either for mechanical reasons, such as excessive weight on the support structures and loss of agility in the steering system, or the high per unit-area cost of additional reflector surface. This does not solve the problem of wasted surface area and a more rewarding approach would be to increase the aperture efficiency. The problem then lies in finding the most efficient way of collecting the energy directed towards the feed
horn by the reflector. It should be borne in mind that the overall antenna performance does not depend solely on the aperture efficiency but is governed by a number of factors. The relationship between the dimensions of the primary-feed horn and the reflector focal length to diameter ratio, the aperture efficiency, the reradiated power, the spill-over power, the edge illumination of the reflector and the radiation pattern are of great importance.

When the antenna system is operating in the receive mode a uniform, linearly polarized plane wave is incident on the reflector from some direction, either on or off axis. This plane wave induces currents on the reflector surface which give rise to a particular field distribution in the focal region. In the transmit mode the primary feed transmits towards the reflector, either from an axial position or some scanned position off axis, and gives rise to the far-zone radiation fields.

These two situations are related by the reciprocity principle. It has been shown that the efficiency of the transfer of power between two antennas is related to the correlation between the fields which exist over a closed surface when the one antenna is transmitting and the other is terminated in a matched load \(1 - 3\). Thus, in the paraboloid-primary-feed-horn system, the efficiency of the power transmission may be obtained from the correlation between the focal-region fields produced by the incident plane wave and the fields existing in the feed-horn aperture when the horn is transmitting. A detailed knowledge of the fields produced in the focal region by the incident plane wave is thus essential to study the problem of achieving the optimum combination of
reflector and feed horn.

Some of the problems associated with obtaining the focal-region fields are well-known from classical optics. If the ratio of the focal length to the diameter of the paraboloid is large \((f/D > 2.0, \text{ say})\) the focal-region fields for the on-axis case may be obtained fairly simply. The technique is essentially to use scalar diffraction theory and to solve the Kirchhoff integral formula. The application of scalar diffraction theory is attractive since it can provide fairly simple analytical solutions for the principal components of the field distribution in the focal region. An excellent discussion of this procedure to find the fields diffracted by circular or rectangular apertures and the fields produced by waves converging to an axial focal point is given in the text by Born and Wolf [4, ch. 8]. In many applications it is useful to apply lateral displacement of the feed horn to achieve beam scanning. The degradation of the image produced by rays which impinge at an angle to the axis of a paraboloidal mirror forms part of the classical theory of aberrations of optical systems and is again discussed in detail in the text by Born and Wolf [4, chs. 5 and 9].

To achieve the solutions of classical optics approximations have to be made which, in essence, restrict the solutions so obtained to reflectors which have large focal ratios. The majority of reflectors in use at microwave frequencies have focal ratios in the range from 0.25 to 1.0 and this region is generally outside the range covered by the optical solutions. In addition optical systems are usually many orders of magnitude larger in aperture to wavelength ratio; they follow geo-
metrical optics behavior much more closely than antenna systems. Thus, it turns out that the techniques of classical optics are not readily applicable to antenna technology. These and other points are discussed in greater detail by Ruze [5]. From the above discussion it is clear that Airy's well-known equation $2J_1(u)/u$ [4, p. 396], for the amplitude distribution in the focal plane or a circular lens, is valid only for the large focal ratios common in optical systems.

We now turn our attention specifically to the paraboloid problem as regards antennas; a few authors have studied the problem in the transmit mode using infinitesimal dipole or tapered amplitude sources displaced either laterally or axially from the focus [5-7]. The analysis by Sandler [7] is the most extensive since he uses the current distribution method, retaining the vector character of the far-zone fields. These analyses give the character of the fields only for displacements along the principal axes and so are not suitable for studying the efficiency problem where the fields have to be specified over a surface.

In the receive mode the paraboloid has been studied almost exclusively for the case of axial incidence; the references cited below all deal with this configuration unless otherwise specified. A vector solution to the paraboloid problem is necessary to obtain a detailed description of the focal-region electromagnetic fields. Robieux and Tocquec [2] have obtained expressions for the vector components of the focal-region fields using the induced current method. Because of an inconsistency in their use of left- and right-handed coordinate systems their field expressions are in error. Kennaugh and Ott [8] obtained
a solution using a superposition of vector spherical waves (multipoles) centered at the focal point of the reflector but published only a brief communication in which the fields along the principal axes are shown graphically. Watson [9] has discussed the slightly more general case pertaining to the situation where the fields in the focal region are produced by a plane wave incident at a small angle to the axis. He paid particular attention to the exact distribution of the currents induced on the reflector surface, a generalization which leads to considerable analytical complication and which he concludes to be unnecessary as long as the focal-plane fields are not desired for displacements greater than a few tens of wavelengths from the focus.

Since 1963 Minnett and his collaborators have been involved in a program to improve the prime-focus feed system of the 210 ft diameter radio telescope of the Parkes Radio Observatory in Australia and they have produced a series of very interesting papers [10–16]. One of the papers [12] considered the problem of evaluating the focal-region fields produced by a linearly polarized wave incident normally on the aperture of a circular symmetric reflector. The induced current method was used to obtain a vector solution for the focal-region fields produced by a plane wave incident along the axis of a circular paraboloid. It was discovered that these focal-region fields could be expressed as an infinite set of hybrid waves propagating along the reflector axis. Further study showed that these hybrid waves are the natural modes of certain corrugated waveguide structures. As a result of this fact there is, in principle, a way of synthesizing the focal fields by appropriately
combining the hybrid waves in a corrugated waveguide. This is desirable since, by application of the formulas for the power transfer between two antennas, a perfect match between the focal-region fields and the fields in the horn aperture can be obtained if the horn fields are the complex conjugates of the focal fields produced by the incident wave. The radiation fields of these hybrid waves have inherent symmetry about the axis, a fact which may be exploited in polarization measurements [10], [11]. The application of corrugated waveguides as primary feeds for paraboloidal reflectors of large or small focal ratio is considered in references [13 - 16]. The topics introduced in this paragraph will be discussed in greater detail in a later chapter. Although the expressions presented by the above authors for the focal fields are suitable for the on-axis case, they are not useful for finding the focal-region fields produced by an off-axis plane wave.

Gniss and Ries [17] have also carried out a vector solution for obtaining the focal fields produced by an axial plane wave. They obtained results similar to those in [12] but did not discuss the problem of the feed at the focus in any detail. Their expressions for the focal fields are useful only for the on-axis case.

Rudge and Withers [18] have devised a novel way of obtaining the focal-region fields produced by an on-axis plane wave. Scalar diffraction theory was used to obtain the principal component of the electric field distribution in the focal plane of a paraboloidal reflector with a large focal ratio. In order to apply these results to a paraboloid of small focal ratio a "curvature-correction factor" was
developed to take into account the change in reflector curvature as the focal ratio changes. As may be expected, the results of the scalar analysis are not exact; however, their comparative simplicity makes them attractive to use in the study of the aperture-efficiency problem. This simplified analysis was used to derive a straightforward design procedure by which the dimensions of a rectangular primary feed, operating in the $TE_{10}$ mode, may be obtained for paraboloidal reflectors with either rectangular or circular cross section and having any focal ratio, so that the optimum solution for the paraboloid-primary-feed problem may be found. A technique for beam steering with a fixed paraboloidal reflector was also developed by using the results of the scalar analysis [19], [20].

Landry and Chassé [21] have described detailed measurements of the electric-field distribution produced in the focal region of a paraboloid of small focal ratio illuminated by a linearly polarized plane wave incident along its axis. In general their experimental results in the transverse planes agreed quite well with the theoretical predictions given in [12] and [17]. Along the paraboloid axis, however, there were discrepancies between the measured and predicted results, both in the positions of the nulls and in the amplitudes of the minor lobes of the primary component of the electric field. This disagreement between theory and measurement was resolved by Ingerson and Rusch [22] who carried out an analysis of the axial displacement without making the approximations commonly used in the phase expressions appearing in the integral formulas used to evaluate the focal fields.
The two panels to be discussed in this paragraph deal with the situation where a circular paraboloid is illuminated by a plane wave impinging at an angle to the axis of the paraboloid. Watson and Ghobarial [23] investigated the off-axis polarization characteristics of Cassegrainian and prime-focus paraboloids and concluded that the Cassegrainian system has superior cross polar isolation at off-axis incidence. Rusch and Ludwig [24] determined the scan-plane fields in the focal region of a beam-scanning paraboloid from physical optics. No analytical expressions for the fields were presented; however, their results were illustrated by a large number of graphs. In particular, they presented amplitude and phase contours of the focal-region electric field, for various angles of incidence, and contours for maximum scan gain as a function of focal ratio and illumination taper. Some of the results mentioned in this paragraph will be discussed in greater detail in a later chapter.

In summary of this review, we note that the optical analysis of the paraboloidal mirror, although attractive for its relative simplicity, is not, in general, readily applicable to the antenna problem. Further, the focal-region fields have been investigated primarily for the case of axial incidence by a plane wave on a circular paraboloidal reflector; in some cases these fields have been used to study the aperture efficiency of the prime-focus system. Off-axis incidence by a plane wave has, until recently, received very little attention. The paraboloidal reflector of rectangular cross section, although less common in practice than the paraboloidal reflector of circular cross section, has not been studied in any detail.
The foregoing does not represent all the work which has been done in the field of prime-focus paraboloids but does cover most of the significant work pertaining to the present study. Other pertinent references will be cited as the need arises.
CHAPTER I

GENERAL THEORY: THE SCATTERED FIELDS AND THEIR USE IN THE APERTURE EFFICIENCY PROBLEM

Before beginning our discussion of a perfectly conducting paraboloidal reflector illuminated by an incident plane wave, it is best to first consider the techniques which may be used to obtain the fields scattered by a body of arbitrary shape. For completeness' sake a brief review of some of the fundamentals of electromagnetic theory is given in Appendix A. The derivations of the integral formulas of scalar and vector diffraction theory are quite lengthy and the details of these derivations have been placed in Appendix B.

Section 1 discusses the scalar diffraction theory of Kirchhoff and Helmholtz. This is followed by a description of the vector Kirchhoff diffraction integrals in section 2. Here we also discuss two equivalent integral representations for the scattered fields and examine the physical optics approximation for the induced currents. In section 3 we investigate under what conditions the scalar and vector formulations are equivalent. Section 4 outlines how the fields scattered by a reflector may be used in conjunction with the fields existing in the aperture of a feed horn, when it is transmitting, to obtain the aperture efficiency of the system. This chapter concludes with a description of the objec-
tives and the scope of this dissertation.

1.1 Scalar diffraction theory of Kirchhoff and Helmholtz

The classical theory of geometrical optics is inadequate to describe certain of the phenomena resulting from the wavelength and phase properties of electromagnetic waves. These phenomena, known as "diffraction effects", are associated with the finite size of a scattering body or an aperture, edge effects, the non-zero wavelength of the source exciting the fields, etc. Deviations from a pure geometrical model of energy propagation occur in the neighborhood of shadow boundaries, in regions of geometrical shadowing or in regions where a large number of rays meet (i.e., in regions where the rays are focused).

The earliest attempts to explain the observed diffraction phenomena were developed from Huygens' construction by Fresnel who postulated that the secondary Huygens' wavelets mutually interfere. In optics this combination of Huygens' construction and the postulate of interference is known as the Huygens-Fresnel Principle. Helmholtz derived a rigorous diffraction formula for monochromatic waves by taking the scalar wave equation as the starting point. The theory for arbitrary time dependence was subsequently developed by Kirchhoff. A more detailed description of the historical development of scalar diffraction theory may be found in [4, ch. 8].

A solution to the scalar wave equation (also called the scalar Helmholtz equation)

$$\psi^2 \psi + k^2 \psi = 0$$

(1)
may be found with the aid of Green's second identity. This procedure is described in detail in Appendix B.1 where the solution $\Psi$ is found at any point $P$ within a multiply connected volume $V$ bounded by the surface $S$, representing the closed surfaces $S_1, S_2, ..., S_n$ and the outer bounding surface $E$ (see Fig. 72). It is assumed the $\Psi$ is a solution of Eq. (1) and that it is continuous and has continuous first derivatives within the volume $V$ and on the surfaces $S + E$. A vector surface normal $\hat{n}$ is directed into $V$. From Eq. (B.7) the value of $\Psi$ at any point $P$ within $V$ may be expressed in terms of $\Psi$ and $\partial \Psi/\partial n$ on $S + E$, thus

$$\Psi(P) = \frac{1}{4\pi} \int_{S + E} \left[ \psi \frac{\partial}{\partial n} \left( \frac{e^{-jkr}}{r} \right) - \frac{e^{-jkr}}{r} \frac{\partial \psi}{\partial n} \right] \, ds'$$

(2)

where $r$ is the distance from $P$ to the point of integration on the surface.

Equation (2) is more general than that normally required for problems related to diffraction by apertures. We specialize it to the situation where the volume $V$ is bounded by the single surface $E$ (see Fig. 1); for this situation Eq. (2) becomes

$$\Psi(P) = \frac{1}{4\pi} \int_{E} \left[ \psi \frac{\partial}{\partial n} \left( \frac{e^{-jkr}}{r} \right) - \frac{e^{-jkr}}{r} \frac{\partial \psi}{\partial n} \right] \, ds'$$

(3)

This equation is a general form of the Kirchhoff diffraction integral. Now, let the surface $E$ be an opaque screen separating a source outside $V$ from the observer within $V$. If an aperture $E_1$ is made in $E$ the incident field will penetrate into $V$. To find the field at $P$, the surface
\( \Sigma \) is divided into three portions: (1) the aperture \( \Sigma_1 \), (2) a portion \( \Sigma_2 \) on the non-illuminated side of the screen, and (3) a portion \( \Sigma_3 \) of a large sphere of radius \( R \) centered at \( P \) (see Fig. 2).
To apply Eq. (2) we need to know $\Psi$ and $\partial \Psi / \partial n$ over $\Sigma$; these values are usually not known but in order to obtain an approximate solution it is frequently assumed that

(a) on the surface of the screen $\Sigma_2$, $\Psi = 0$ and $\partial \Psi / \partial n = 0$.

(b) over the aperture surface $\Sigma_1$ the values of $\Psi$ and $\partial \Psi / \partial n$ are those which would exist if no screen were present.

When these approximations (known as the Kirchhoff approximations) are used in Eq. (2), the integral over $\Sigma_2$ vanishes by assumption (a) above and as $R$ becomes infinite the integration over the spherical sector $\Sigma_3$ vanishes as a result of the radiation condition (see the discussion in Appendix B.1) so that the approximation to the field at any point $P$ within $\Gamma$ is

$$
\Psi(p) = \frac{1}{4\pi} \int_{\Sigma_1} \left[ \Psi \frac{\partial}{\partial n} \left( \frac{\mathbf{e}}{r} \right) - \frac{\mathbf{e}}{r} \frac{\partial \Psi}{\partial n} \right] dS',
$$

(4)

Here the integration is carried out over the open surface $\Sigma_1$. Application of Eq. (4) to a spherical wave converging to an axial focal point and incident upon an aperture in an infinite plane screen has been illustrated by several authors [4, ch. 8], [25] and [26, ch. 3], for example.

In the above discussion we have assumed that the transverse dimensions of the aperture are sufficiently large in wavelengths that edge effects may be neglected. These edge contributions are generally significant within a few wavelengths of the edge; so the point of observa-
tion must not be too near the aperture if we are to apply Eq. (4). For the case of electromagnetic fields the polarization of the $E$ and $\mathbf{H}$ fields must be considered and $E$ and $\mathbf{H}$ must satisfy Maxwell's equations. The vector analog of Kirchhoff's scalar diffraction formula is discussed in the next section.

1.2 The vector Kirchhoff diffraction integrals

In the previous section we obtained a solution to the scalar Helmholtz equation; however, the scalar wave function does not always provide a useful description of the vector characteristics of the electromagnetic field. Each cartesian component of either $E$ or $\mathbf{H}$ is a scalar wave function, and thus Kirchhoff scalar formula Eq. (2) may be applied to it, but the various components are not independent wave functions since they must satisfy Maxwell's equations. A more complete vector formulation for the $E$ and $\mathbf{H}$ fields may be obtained from the vector wave equations for $E$ and $\mathbf{H}$ (Eqs. (A.8)).

The derivation of integral expressions for the $E$ and $\mathbf{H}$ fields is described in some detail in Appendix B.2 where pertinent references are cited. There we obtain two distinct sets of formulas which may be used to compute the electromagnetic field at an arbitrary point $P$ within a volume $V$ in terms of the field sources and the values of the field itself on the surfaces bounding the region. These two sets of formulas have been designated the Stratton-Chu formulation and the Franz formulation in a brief discussion of Kirchhoff theory by Taï [27]. The presentation in Appendix B.2 initially considers the electric and magnetic current sources $\mathbf{J}$ and $\mathbf{J}_m$ radiating in the presence of one or more scattering
obstacles. The region of interest is the multiply connected volume $V$ bounded by the finite closed surfaces $S_1, S_2, \ldots, S_n$ surrounding the scatterers and the "surface at infinity" $\Sigma$ (see Fig. 73). These results are then specialized to the situation where there are no primary sources within $V$, the only source being a source at infinity giving rise to an incident field impinging on a single scattering body enclosed by the surface $S$ (see Fig. 3).

![Incident wave diagram](image)

**Fig. 3.** Vector diffraction theory geometry.

The radiation condition at infinity requires that the integration over the outer bounding surface $\Sigma$ contribute nothing to the field in $V$ and the only inward travelling waves from $\Sigma$ must be due entirely to sources outside $\Sigma$. The relevant formulas for the scattered fields at any point $P$ in $V$ (cf. Eqs. (B,25) and (B,31)) are for convenience reproduced here.
Stratton-Chu formulation:

\[
\mathbf{E}_s(P) = \frac{1}{4\pi} \int_S \left[ -j\omega_0 \phi (\hat{n} \times \mathbf{n}) + (\hat{n} \times \mathbf{E}) \times \nabla' \phi + (\hat{n} \times \mathbf{E}) \nabla' \phi \right] dS' \quad (5a)
\]

\[
\mathbf{H}_s(P) = \frac{1}{4\pi} \int_S \left[ j\omega_0 \phi (\hat{n} \times \mathbf{n}) - (\hat{n} \times \mathbf{H}) \times \nabla' \phi - (\hat{n} \times \mathbf{H}) \nabla' \phi \right] dS' \quad (5b)
\]

Franz formulation:

\[
\mathbf{E}_s(P) = \frac{1}{4\pi} \int_S \left[ (\hat{n} \times \mathbf{E}) \times \nabla' \phi - j\omega_0 \phi (\hat{n} \times \mathbf{n}) + \frac{1}{j\omega_0} \left( (\hat{n} \times \mathbf{n}) \cdot \nabla' \right) \nabla' \phi \right] dS' \quad (6a)
\]

\[
\mathbf{H}_s(P) = \frac{1}{4\pi} \int_S \left[ (\hat{n} \times \mathbf{H}) \times \nabla' \phi + j\omega_0 \phi (\hat{n} \times \mathbf{n}) - \frac{1}{j\omega_0} \left( (\hat{n} \times \mathbf{n}) \cdot \nabla' \right) \nabla' \phi \right] dS' \quad (6b)
\]

where

\[
\phi = \frac{-jkr}{r} = \frac{-jk|\mathbf{R} - \mathbf{R}'|}{|\mathbf{R} - \mathbf{R}'|}
\]

\[
\hat{n} = \text{unit surface normal directed into } \mathbf{n}
\]

\[
\mathbf{E} = \text{total electric field} = \mathbf{E}_1 + \mathbf{E}_s
\]

\[
\mathbf{H} = \text{total magnetic field} = \mathbf{H}_1 + \mathbf{H}_s
\]

The operator \(\nabla'\) operates only on the coordinates of the source point.

It should be noted that the Stratton-Chu formulation requires that both tangential and normal components of the field be specified on the closed
surface \( S \), while the Franz formulation requires only the tangential components.

One further comment on the Stratton-Chu formulation is of importance. Equations (5) hold only if the vectors \( \mathbf{E} \) and \( \mathbf{H} \) are continuous and have continuous first derivatives on \( S \). This usually presents no difficulties provided \( S \) is a continuous, closed surface; however, when the integration is to be carried out over an open surface (e.g., in problems related to diffraction by apertures bounded by a conducting screen) it is necessary to postulate a distribution of line charges or currents along the boundary curve \( \Gamma \) to satisfy the field equations. A method of finding this line distribution of sources which is required to allow the electric and magnetic fields to satisfy Maxwell's equations has been proposed by Kottler (see Stratton [28, sect. 8.15]). He showed that if the contour integrals

\[
- \frac{1}{4 \pi j \omega \varepsilon_0} \oint_{\Gamma} \mathbf{\nabla} \phi (\hat{\mathbf{r}} \cdot \mathbf{H}) \, dl 
\]

(7a)

and

\[
\frac{1}{4 \pi j \mu_0} \oint_{\Gamma} \mathbf{\nabla} \phi (\hat{\mathbf{r}} \cdot \mathbf{E}) \, dl 
\]

(7b)

are added to Eqs. (5a) and (5b), respectively, compatibility with the field equations is achieved. The unit tangent vector \( \hat{\mathbf{r}} \) lies along the boundary curve and the sense of the line integral around \( \Gamma \) is such that the surface normal into \( V \) is on the left (These contour integrals vanish for a closed surface).
A modification such as the above is not required in the Franz formulation since the field contribution due to the line sources is already imbedded in the formulation. To see this we observe that since the integration is to be carried out over the open surface \( S_1 \) the following relationship (derived in [29, sect. 5.8]) can be applied

\[
\int_{S_1} \left[ (\hat{n} \times \hat{H}) \cdot \nabla' \phi \right] dS' = \int_{S_1} j_{0} \omega \epsilon_0 (\hat{n} \cdot E) \nabla' \phi \cdot dS' - \oint_{\Gamma} \nabla' \phi (\hat{t} \cdot \hat{n}) \cdot dl \quad (8)
\]

On substituting this equation into Eq. (6a) for \( \overline{E}_8 (P) \) we obtain an expression identical to Eq. (5a) of the Stratton-Chu formulation provided we add the line integral given in Eq. (7a) to Eq. (5a) and bear in mind that the integration is performed over the open surface \( S_1 \). For a more detailed discussion of the vector Kirchhoff equations and the associated boundary-line charge, the reader is referred to the paper by Sancer [30]. We conclude that for a closed surface the Franz and the Stratton-Chu formulations are equivalent, while for an open surface they are equivalent only if the contour integrals of Eqs. (7) are incorporated into the latter formulation.

The vector diffraction integrals presented above are frequently used to determine the scattering properties of a perfect electric-conducting body when it is illuminated by an incident wave. The boundary conditions on the perfectly conducting surface require that the tangential component of the electric field and the normal component of the magnetic field both be zero, i.e., \( \hat{n} \times \overline{E} = 0 \) and \( \hat{n} \cdot \hat{H} = 0 \). Under these conditions the general field expressions simplify considerably. The
currents exciting the scattered field are induced on the conducting surface by the incident wave which for the present purposes is taken to be a plane wave of known amplitude, phase and polarization everywhere in space. These induced currents may be obtained by the solution of an integral equation or they may be obtained by approximate methods. Both approaches require integration of the surface current distribution on the scatterer to find the scattered field. The technique of finding the scattered fields by integrating the induced surface currents is often referred to as the "induced-current method". The most frequently encountered approximation for the induced surface currents is the "physical-optics" approximation described below.

Consider the closed reflecting surface S illuminated by a plane wave. On the basis of geometrical optics there is a sharply defined shadow region behind the reflector in which the total field is zero.

*Fig. 4. On the physical-optics approximation.*
In Fig. 4 the closed curve $\Gamma$ divides the closed surface $S$ into the region $S_1$ illuminated by the incident wave and the region $S_2$ which is geometrically shadowed. Since the total field on the shadowed region $S_2$ is zero, the current distribution there is also zero. The current distribution over the illuminated region $S_1$ is obtained on the assumption that the currents induced on each element $dS'$ of surface are those that would be excited if $dS'$ formed part of an infinite tangent plane at that point. Thus the physical-optics approximation for the induced current, using the boundary condition Eq. (A.16a), is

$$J_s = \hat{n} \times \vec{H} = 2(\hat{n} \times \vec{H}_1) \quad \text{on } S_1 \quad (9a)$$

$$J_s = 0 \quad \text{on } S_2 \quad (9b)$$

where $\vec{H}_1$ is the incident magnetic field. The physical-optics approximation is valid provided the transverse dimensions of the reflector and the radii of curvature of the reflector and the incident wave front are large compared to the wavelength. Watson [9] has shown that in the focal-region of a paraboloidal reflector the contribution from secondary currents not included in Eq. (9b) is negligible compared with the field due to the primary current $2(\hat{n} \times \vec{H}_1)$. For the reflectors to be considered later, the conditions for applying the physical-optics approximation are satisfied and so this approximation will be used to obtain the final expressions for the scattered fields.

Since the fields on the non-illuminated side are taken to be zero the integrations are to be carried out over the open surface $S_1$ which means that the contour integrals of Eqs. (7) must be added to Eqs. (5).
With the aid of the boundary conditions Eqs. (A.16), as applied to reflection at a perfectly conducting tangent plane, all the field-related quantities in the diffraction integrals may be found (for details see [29, ch. 5]) as follows:

\[ \hat{n} \times \overline{E} = 0 \]  
(10)

\[ \hat{n} \times \overline{H} = 2(\hat{n} \times \overline{H}_4) \]  
(11)

\[ \hat{n} \cdot \overline{E} = 2(\hat{n} \cdot \overline{E}_4) \]  
(12)

\[ \hat{n} \cdot \overline{H} = 0 \]  
(13)

\[ \hat{t} \cdot \overline{E} = 0 \]  
(14)

\[ \hat{t} \cdot \overline{H} = 2(\hat{t} \cdot \overline{H}_4) \]  
(15)

Upon inserting Eqs. (10) through (15) into Eqs. (5) and (6) as required and adding the line integral terms to Eqs. (5) we obtain the final expressions for the scattered fields.

**Stratton-Chu formulation:**

\[ \overline{E}_s(P) = \frac{1}{2\pi} \int_{S_1} \left\{ -j\omega \varepsilon_0 (\hat{n} \times \overline{H}_4)\phi + (\hat{n} \cdot \overline{E}_4) \nabla^* \phi \right\} dS^, \]

\[ = -\frac{1}{2\pi j \omega \varepsilon_0} \oint_{\Gamma} \nabla^* \phi (\hat{t} \cdot \overline{H}_4) \, dl \]  
(16a)

\[ \overline{H}_s(P) = \frac{1}{2\pi} \int_{S_1} (\hat{n} \times \overline{H}_4) \times \nabla^* \phi \, dS^, \]  
(16b)
Franz formulation:

\[
E_b(P) = \frac{1}{2\pi} \int_{S_1} \left[ -j\omega_0 (\hat{n} \times \mathbf{H}_1) \phi + \frac{1}{j\omega_0} (\mathbf{E} \cdot \nabla') \nabla' \phi \right] dS', \quad (17a)
\]

\[
H_b(P) = \frac{1}{2\pi} \int_{S_1} (\hat{n} \times \mathbf{H}_1) \times \nabla' \phi \ dS'. \quad (17b)
\]

The Franz formulation appears conceptually simpler since it requires knowledge only of the induced surface current over the illuminated area, while the Stratton-Chu formulation requires the surface current over the illuminated area, the surface charges over the same area and a line distribution along the boundary curve \( \Gamma \). Both these formulations are examined in Chapter II where they are applied to the problem of finding the focal-region fields of a perfectly conducting paraboloidal reflector illuminated by an incident plane wave.

1.3 Comparison of the scalar and vector formulations

The relationship between the scalar diffraction integral (Eq. (3)) and the vector diffraction integrals (Eqs. (5) and (6)) for a source-free region may be established by deriving the former from Eq. (5a), for example. The necessary vector transformations are straightforward but lengthy and so we quote the desired results. For a closed surface \( S \), it may be shown [31, ch. 2] and [32, ch. 2] that Eq. (5a) is identical with
\[ \mathbf{E}_s'(P) = \frac{1}{4\pi} \int_S \left( \frac{\partial \phi}{\partial n} - \frac{\partial \mathbf{E}}{\partial n} \right) dS' \]

\[ + \frac{1}{4\pi} \int_S \left[ \mathbf{v}'(\hat{n} \cdot \mathbf{E}) - \hat{n}(\mathbf{v}' \cdot \mathbf{E}) \right] dS' \]

\[ - \frac{1}{4\pi} \int_S \left( (\mathbf{E} \cdot \mathbf{v}') \hat{n} + \mathbf{E} \times (\mathbf{v}' \times \hat{n}) \right) dS' \quad (18) \]

Further, for a closed surface the second and third surface integrals are zero \([31, \text{ch. 2}]\) and Eq. (18) becomes

\[ \mathbf{E}_s'(P) = \frac{1}{4\pi} \int_S \left( \frac{\partial \phi}{\partial n} - \frac{\partial \mathbf{E}}{\partial n} \right) dS' \quad (19) \]

This vector expression may now be written in terms of rectangular components, thus

\[ u_s'(P) = \frac{1}{4\pi} \int_S \left[ u \frac{\partial}{\partial n} \left( \frac{e^{-jkr}}{r} \right) - \frac{e^{-jkr}}{r} \frac{\partial u}{\partial n} \right] dS' \quad (20) \]

where \( u \) represents any one of the rectangular components of \( \mathbf{E} \) or \( \mathbf{H} \) and \( e^{-jkr} /r \) has been substituted for \( \phi \). This expression is the same as the scalar diffraction integral of Eq. (3) and so the scalar and vector formulations are identical provided the surface of integration is completely closed.

When the integrations are to be carried out over an open surface (as in the case of the physical-optics approximation) the line integral
term of Eq. (7a) must be added to Eq. (18). For an open surface the result now becomes [32, ch. 2]

\[ \mathbf{F}_s(P) = \frac{1}{4\pi} \int_{S_1} \left( \frac{\mathbf{E}}{\Delta n} - \phi \frac{\Delta \mathbf{E}}{\Delta n} \right) d\mathbf{s}' \]

\[ - \frac{1}{4\pi} \int_{S_1} \left( (\mathbf{E} \cdot \mathbf{n}) \mathbf{h} + \mathbf{E} \times (\mathbf{V}' \times \mathbf{n}) \right) d\mathbf{s}' \]

\[ - \frac{1}{4\pi} \int_{r} \phi (\mathbf{E} \times \mathbf{\hat{r}}) d\mathbf{l} - \frac{1}{4\pi j\omega} \int_{r} \mathbf{V}' \phi (\mathbf{\hat{r}} \cdot \mathbf{n}) d\mathbf{l} \quad (21) \]

The second surface integral of Eq. (18) has been transformed to the line integral involving the cross product \( \mathbf{E} \times \mathbf{n} \). Even though contributions from a part of \( S \) may be neglected (e.g., contributions from the shadowed region in the physical-optics approximation), Eq. (19) does not generally yield valid results for an open surface. At very large distances the scalar product contour integral in Eq. (21) gives a component which is entirely longitudinal (i.e., parallel to the radius vector \( \mathbf{r} \) connecting points on the surface of integration to the observation point) as a result of the \( V' \phi \) factor in the integrand, while the vector product contour integral may have a longitudinal component. The far-zone field must be entirely transverse, therefore the line integrals are required to cancel the longitudinal components of the surface integral.

There are, however, certain circumstances under which Eq. (19) does yield valid results for an open surface. A particular example concerns the class of problems related to plane apertures with the aperture
fields tangential to the aperture surface. Matters are greatly simplified since \( \hat{n} \) is constant for a plane aperture which means that the second surface integral in Eq. (21) is zero. Provided the fields are not required for wide angles from the axial direction (i.e., a direction perpendicular to the plane of the aperture), the results obtained by the scalar diffraction formula (Eq. (20)) will not differ significantly from those obtained by the more precise vector formulation. A more detailed discussion of the application of scalar diffraction theory to a plane aperture is given by [32, ch. 2] and [25]. The disparity between the results obtained by Eq. (19) and Eq. (6a), say, arises because Eq. (19) together with the analogous expression for \( \overline{H} \) does not satisfy Maxwell's equations, in the case of an open surface, while Eqs. (6) do. Care should be exercised in the application of the scalar diffraction formula to open surfaces, that is, ones with edges; the effects of the additional terms in Eq. (21) should be carefully evaluated before the field components are calculated by means of Eq. (20).

1.4 Aperture efficiency

In this section we develop a formula whereby the aperture efficiency of the paraboloid-feed-horn combination may be calculated in terms of the focal-region fields produced by a plane wave incident on the paraboloid aperture and the fields existing in the feed-horn aperture when the horn is transmitting.

The physical aperture \( A_p \) of an antenna is defined as the physical cross-sectional area of its aperture. If a paraboloid is illuminated
by an infinite plane wave, with power density $S_i$ watts per unit area, incident normally on the aperture, then a certain power, $P_r$ watts, will be delivered by the feed system to a load. The effective aperture $A_e$ is defined as the ratio of these two quantities [31] and [33]:

$$A_e = \frac{P_r}{S_i} = \frac{\text{power received}}{\text{incident power density}}$$

(22)

This definition incorporates any losses which may be caused by polarization and/or impedance mismatch.

The aperture efficiency $\epsilon_{ap}$ is defined as the ratio of the effective aperture to the physical aperture:

$$\epsilon_{ap} = \frac{A_e}{A_p} = \frac{P_r}{P_i}$$

(23)

where $P_i$ is the total power incident on the physical aperture of the paraboloid.

Kouyoumjian [34] and Richmond [35] have developed an expression for the voltage or current induced in one antenna by another in terms of the "reaction". Their result is of fundamental importance in finding the power delivered to a load by a receiving system and it will be briefly derived here:

Consider two harmonic sources of the same frequency, radiating in a medium ($\mu$, $\varepsilon$) which is linear and isotropic but not necessarily homogeneous. Let $\mathbf{J}_1$ and $\mathbf{J}_2$ represent the electric and magnetic current densities of source 1 and source 2 when radiating

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1 The derivation presented here parallels that used by Dr. J.H. Richmond in his course on Advanced Antenna Theory given in the Department of Electrical Engineering at the Ohio State University, Columbus, Ohio.
in the presence of the structure of source 2 and any scatterers which may be present. Similarly, the currents $\mathbf{J}_2$ and $\mathbf{J}_m^2$ set up the field $(E_2,H_2)$ when radiating in the same medium in the presence of source 1 and the scatterers. From Maxwell's equations (Eq.(A.7)) and a vector identity (Eq.(A.31)) we can show that

$$\nabla \cdot \left( \mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1 \right) = \left( \mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{E}_1 \cdot \mathbf{J}_2 \right) - \left( \mathbf{H}_2 \cdot \mathbf{J}_m^1 - \mathbf{H}_1 \cdot \mathbf{J}_m^2 \right)$$

(24)

which is the general form of the "point reciprocity theorem of Lorentz".

Now, let source 1 exist in the region $V_1$ bounded by the surface $S_1$ (source 2 is assumed to exist outside $S_1$), see Fig. 5. From the divergence theorem (Eq.(A.25))

$$\int_{S_1} \left( \mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1 \right) \cdot \hat{n} \, dS = \int_{V_1} \nabla \cdot \left( \mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1 \right) \, dv$$

(25)

where $\hat{n}$ is the outward surface normal to $S_1$. Together with Eq.(24) this leads to

$$\int_{S_1} \left( \mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1 \right) \cdot \hat{n} \, dS = \int_{V_1} \left( \mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_2 \cdot \mathbf{J}_m^1 \right) \, dv$$

(26)

where we have used the fact that $\mathbf{J}_2$ and $\mathbf{J}_m^2$ are zero in $V_1$. The integrals appearing in this equation do not generally represent power since no conjugates appear and they have been given the name "reaction" by Rumsey [36]. In deriving Eq.(26) no assumptions have been made on the nature of source 2 (it must be harmonic with the same frequency as source 1) or the medium outside region $V_1$ (except that it must be linear). Thus, Eq.(26) is valid even if source 2 is of infinite extent or an infinite distance away and the scatterers need not be finitely distributed.
SOURCE 2 (ANTENNA 2)

Fig. 5. Schematic diagram of two antennas and a scatterer, illustrating various surfaces of integration.

In order to define antenna impedance, voltage and current, let us assume that the feed system of source 1 (or antenna 1) has a section of perfectly conducting waveguide or transmission line which will support only one propagating mode. We now choose a terminal surface $S_T$ at a point where only one mode exists and define the voltage and current at the terminals of antenna 1 when antenna 1 transmits as follows [37, ch. 8]:

$$\overline{E}_l^t = V_{11} \overline{e}$$  \hspace{1cm} (27a)

and on $S_T$

$$\overline{H}_l^t = I_{11} \overline{h}$$  \hspace{1cm} (27b)

where the superscript "t" indicates the tangential component of the field intensity on the terminal surface. The real vector-mode functions
are related by
\[ \overline{\mathbf{h}} = \mathbf{h} \times \overline{\mathbf{a}} \quad \text{and} \quad \overline{\mathbf{e}} \times \overline{\mathbf{h}} = (\overline{\mathbf{e}} \cdot \overline{\mathbf{e}}) \hat{\mathbf{n}} \quad (28) \]

and are normalized by letting
\[ \int_{S_T} \overline{\mathbf{e}} \cdot \overline{\mathbf{a}} \, dS = 1 \quad (29) \]

When antenna 2 transmits the voltage and current induced at the terminals of antenna 1 are similarly defined, thus
\[ \overline{E}_2^T = V_{12} \overline{a} \quad (30a) \]
and
\[ \overline{H}_2^T = I_{12} \overline{h} \quad (30b) \]

Since the right-hand side of Eq. (26) remains constant, provided the volume \( V_1 \) contains the sources, we can change the surface of integration from \( S_1 \) to \( S_m + S_T \). The surface \( S_m \) coincides with the perfectly conducting surface of the waveguide or transmission-line feed; the integral on the surface \( S_m \) vanishes because the tangential components of \( \overline{E}_1 \) and \( \overline{E}_2 \) vanish there. This leaves only the integration over the terminal surface \( S_T \). By substituting Eqs. (27) and (30) in the left-hand side of Eq. (26) and integrating over \( S_T \) we obtain
\[ V_{11} I_{12} - V_{12} I_{11} = \int_{V_1} \left( \overline{E}_2 \cdot \overline{J}_1 - \overline{H}_2 \cdot \overline{J}_{ml} \right) \, dv \quad (31) \]

Finally, from Eqs. (26) and (31), bearing in mind that the received cur-
rent $I_{12}$ is zero since the terminals are assumed to be open-circuited when receiving, we obtain the desired "reaction theorem":

$$V_{oc}^{12}_{11} = -\int_{S_1} \left( \overline{E_1} \times \overline{H_2} - \overline{E_2} \times \overline{H_1} \right) \cdot \hat{n} \, dS$$  \hspace{1cm} (32)

where $V_{oc}^{12}$ is the open-circuit voltage with antenna 1 receiving, $I_{11}$ is the current at the terminals when antenna 1 transmits and $S_1$ is any surface enclosing antenna 1. This equation forms the starting point for the derivation of an expression for the total power delivered to a load; we proceed as indicated below.

To find the terminal voltage for an arbitrary load impedance, rather than the voltage $V_{oc}^{12}$ induced at the open-circuited terminals, we may use Thevenin's theorem as shown in Fig. 6. The antenna impedance, defined at the load terminals, is $Z = R + jX$ and the load impedance is $Z_L = R_L + jX_L$. The received current $I_r$ with any load impedance $Z_L$ is

![Fig. 6 Equivalent circuits for Thevenin's theorem.](image-url)
given by

\[ I_r = \frac{-v_{12}^c}{z + z_L} \]  \hspace{1cm} (33)

and the received voltage across the load is

\[ V_r = -I_r z_L = \frac{v_{12}^c z_L}{z + z_L} \]  \hspace{1cm} (34)

The total power \( P_r \) delivered to the load is \( \frac{1}{2}|I_r|^2 \text{Re} z_L \); therefore

\[ P_r = \frac{R_L}{2|z + z_L|^2} \left| \int_{S_1} (\vec{E}_1 \times \vec{H}_1 - \vec{E}_2 \times \vec{H}_2) \cdot \vec{n} \ dS \right|^2 \]  \hspace{1cm} (35)

Now, divide throughout by \( P_1 = P_2 \), the power incident on the paraboloid, and multiply the numerator and the denominator of the right-hand side by \( \frac{1}{2} R \) to obtain

\[ \frac{P_r}{P_1} = \frac{R_L}{4|z + z_L|^2} \left| \int_{S_1} (\vec{E}_1 \times \vec{H}_1 - \vec{E}_2 \times \vec{H}_2) \cdot \vec{n} \ dS \right|^2 \]  \hspace{1cm} (36)

where we have set \( \frac{1}{2} |I_{11}|^2 \ = \ P_1 \), the total power transmitted by antenna 1. If the antennas are assumed to be lossless (i.e., zero resistive losses), then the total powers \( P_1 \) and \( P_2 \) can be expressed in terms of the corresponding fields (cf., Eq.(A,14)).
where $S_1$ and $S_2$ are two surfaces enclosing antennas 1 and 2, respectively. The surface $S_1$ need not correspond to the surface $S_1$ in Eq. (36).

When the load impedance is matched to the antenna impedance for maximum power transfer (i.e., $Z_L = Z^*$), we obtain the aperture efficiency $\varepsilon_{apm}$ for the matched condition as

$$\left( \frac{P_R}{P_L} \right)_{\text{matched}} = \varepsilon_{apm} = \frac{\left| \int_{S_1} (E_x H_y - E_y H_x) \cdot \hat{n} \, ds \right|^2}{16 P_1 P_2}$$

(39)

This expression agrees with that obtained by Hu [1] using a somewhat different procedure. From Eqs. (36) and (39) we may define an "impedance mismatching factor" $q$ as follows

$$q = \frac{4 \pi R R_L}{(R + R_L)^2 + (X + X_L)^2}$$

(40)

which is identical to the factor proposed by Tai [33]. The effect of any polarization mismatch is contained in the numerator of Eq. (39), for example.

The fields $(E_x H_y, E_y H_x)$ should be calculated with both antennas in place; this, however, is generally a difficult problem. As an approx-
imation we may make the following assumptions: the field \((\overrightarrow{E_1}, \overrightarrow{H_1})\) is the undisturbed waveguide field (i.e., in the absence of the reflector) and the field \((\overrightarrow{E_2}, \overrightarrow{H_2})\) exists in the absence of the waveguide. Essentially this means that we are neglecting interactions between the antennas. The closed surfaces \(S_1^f\) and \(S_2\) can be chosen such that they coincide with the perfectly conducting antenna surfaces and are closed by extending them across the plane apertures \(A_1\) and \(A_2\), respectively (see Fig. 5). \(S_1\) may also be set equal to \(A_1\) so that Eq. (39) becomes

\[
\epsilon_{apm} = \frac{1}{4} \left| \frac{\int_{A_1} (\overrightarrow{E_1} \times \overrightarrow{H_2} - \overrightarrow{E_2} \times \overrightarrow{H_1}) \cdot \hat{n}_1 \, dS}{\text{Re} \int_{A_1} (\overrightarrow{E_1} \times \overrightarrow{H_1^*}) \cdot \hat{n}_1 \, dS \cdot \text{Re} \int_{A_2} (\overrightarrow{E_2} \times \overrightarrow{H_2^*}) \cdot \hat{n}_2 \, dS} \right|^2
\]  

(41)

In this form the determination of the aperture efficiency requires a knowledge of the fields \((\overrightarrow{E_1}, \overrightarrow{H_1})\) and \(\overrightarrow{E_2, H_2}\) over a common surface and also the total powers \(P_1\) and \(P_2\) contained in the fields. When we apply this formula to the combination of a paraboloidal reflector and a primary feed-horn, the common surface may be taken as the aperture plane of the feed horn. The field \((\overrightarrow{E_2}, \overrightarrow{H_2})\) in the numerator of Eq. (41) is the focal-region field of the paraboloid, while \((\overrightarrow{E_1}, \overrightarrow{H_1})\) is the field existing in the horn aperture when it transmits. It should be noted that the field \((\overrightarrow{E_2}, \overrightarrow{H_2})\) in the numerator of Eq. (41) is not the same as the field \((\overrightarrow{E_2}, \overrightarrow{H_2})\) in the denominator.

For the matched condition \(\epsilon_{apm}\) represents the fraction of the power incident on the paraboloid which is actually delivered to the matched
load on reception. The factor \((1 - e_{apm})\), which is the fraction of the total power not absorbed by the feed system, consists of two components. The first component comprises that power, incident on the feed horn, which is not absorbed but reradiated back towards the reflector aperture. Midgley [3] has discussed the problem of reradiation by a receiving horn and concludes that to obtain complete absorption and zero reradiation the field existing in the feed-horn aperture must match the incident field in the conjugate sense. The second component is that power which is not intercepted by the horn aperture and thus constitutes the spillover power. Considered in the transmission mode, the spillover power corresponds to that power radiated by the primary feed but not intercepted by the reflector aperture, and the reradiated power is that which is incident on the reflector and reflected back to the feed horn.

The reradiated and spillover factors may be separated out by obtaining an expression for the transmission efficiency \(T_o\) of an "ideal horn", having the same aperture as the actual horn, which absorbs all the power incident on it. In the ideal case the fields \((\mathbf{E}_2, \mathbf{H}_2)\) would be matched across the aperture by their complex conjugates, that is, \(\mathbf{E}_1 = \mathbf{E}_2^*\) and \(\mathbf{H}_1 = -\mathbf{H}_2^*\). The minus sign arises because the direction of propagation is reversed on transmission. Eq. (39) now becomes

\[
T_o = \left\{ \text{Re} \int_{S_1} (\mathbf{E}_2 \times \mathbf{H}_2^*) \cdot \hat{n} \, dS \right\}^2
\]

(42)
where

\[ P_1 = \frac{1}{2} \text{Re} \int S_1 (E_1 \times \Pi_1^* \cdot \hat{n}) \, dS = \frac{1}{2} \text{Re} \int S_1 (\Pi_2^* \times E_2) \cdot \hat{n} \, dS \]  
(43)

Substituting for \( P_1 \) into Eq. (42) yields

\[ T_o = \frac{1}{2 \pi^2} \text{Re} \int S_1 (H_2^* \times E_2) \cdot \hat{n} \, dS \]  
(44)

where \( P_2 \) is given by Eq. (38). As before the surfaces \( S_1 \) and \( S_2 \) may be replaced by the apertures \( A_1 \) and \( A_2 \) (cf. Eq. (41)).

The reradiated power factor \( R \) is given by the difference between the transmission factors for the ideal horn and the actual horn, thus

\[ R = T_o - c_{apm} \]  
(45)

The spillover factor \( T_s \) represents that fraction of the incident power not intercepted by the horn and is given by

\[ T_s = 1 - T_o \]  
(46)

Note that \( c_{apm} + R + T_s = 1 \).

With the aid of formulas (41) through (46) we can investigate how the aperture efficiency \( c_{apm} \), the reradiated power factor \( R \) and the spillover power factor \( T_s \) depend on the paraboloid profile and \( f/D \) ratio as well as their dependence on the feed-horn aperture. In addition, the above factors may be evaluated for a particular horn-
paraboloid combination to obtain a measure of the performance of the system.

To conclude this section we derive a simplified form of Eq. (41) which is applicable in certain restricted circumstances. Note first of all that only the tangential components of the electric and magnetic fields contribute to the integrals in Eq. (41). If we assume that the tangential components of the $E$ and $H$ fields are orthogonal and that their magnitudes are related by the characteristic impedance of the medium, we may write

$$\overrightarrow{H}_1^t = \hat{n}_1 \times \overrightarrow{E}_1^t / Z_0$$  \hspace{1cm} (47)$$

$$\overrightarrow{H}_2^t = \hat{n}_2 \times \overrightarrow{E}_2^t / Z_0$$  \hspace{1cm} (48)$$

where the superscript "t" denotes tangential components on the aperture surface and $Z_0$ is the impedance of free space. The integrand in the numerator of Eq. (41) becomes

$$\frac{1}{Z_0} \left( \overrightarrow{E}_1^t \times (\hat{n}_2 \times \overrightarrow{E}_2^t) - \overrightarrow{E}_2^t \times (\hat{n}_1 \times \overrightarrow{E}_1^t) \right) \cdot \hat{n}_1$$

$$= \frac{1}{Z_0} \left( (\overrightarrow{E}_1^t \cdot \overrightarrow{E}_2^t) \hat{n}_2 - (\overrightarrow{E}_1^t \cdot \hat{n}_2) \overrightarrow{E}_2^t - (\overrightarrow{E}_2^t \cdot \overrightarrow{E}_1^t) \hat{n}_1 + (\overrightarrow{E}_2^t \cdot \hat{n}_1) \overrightarrow{E}_1^t \right) \cdot \hat{n}_1$$  \hspace{1cm} (49)$$

If we assume further that the antennas are facing directly towards each other (i.e., $\hat{n}_2 = -\hat{n}_1$) the second and fourth terms vanish and we obtain

$$\varepsilon_{apm} = \left| \frac{\int_{A_1} \overrightarrow{E}_1^t \cdot \overrightarrow{E}_2^t \, ds}{\int_{A_1} |\overrightarrow{E}_1^t|^2 \, ds \cdot \int_{A_2} |\overrightarrow{E}_2^t|^2 \, ds} \right|^2$$  \hspace{1cm} (50)$$
This simplified formula only requires a knowledge of the transverse electric field over the plane apertures; but it should be used with care since it applies only in the restricted circumstances outlined in its derivation.

The procedure proposed here to compute the aperture efficiency \( \text{(cf. Eq. (41))} \) differs from that normally used for reflector antennas. The standard procedure is to consider the transmit mode and to take the ratio of the power per unit solid angle radiated by the paraboloidal antenna and the power per unit solid angle radiated by an isotropic antenna to find the antenna gain, and hence the aperture efficiency \[ 31, \text{ sect. 3,22} \]. This procedure requires that the far-field radiation pattern of the feed horn be integrated over the surface of the reflector antenna. Clearly, if the feed horn is changed the integration over the reflector surface must be repeated for each new horn configuration. Although this method is suitable for the analysis of a particular feed horn, it could be very expensive if many different horns are to be evaluated to establish a design procedure since the numerical integrations have to be computed over a large surface (in square wavelengths). There is an obvious advantage in considering the receive mode; once the focal-region fields have been computed for a particular paraboloid by integrating over the reflector surface, they may be stored in the computer or on cards and used in Eq. (41) for any feed horn without need to integrate over the reflector surface again. The fields \( E_2, H_2 \) in Eq. (41) remain the same independent of the feed system; they are a property of the paraboloid. Successive determinations of the aperture efficiency
for different horns require integration over the horn apertures which are generally many orders of magnitude smaller than the reflector apertures; this should result in considerable savings in computer expenses.

1.5 Scope of the present investigation

Thus far we have reviewed the integral formulations of scalar and vector diffraction theory which may be used to describe diffraction phenomena. The physical optics approximation for the currents induced on the surface of a perfectly conducting scatterer has been discussed and its use in the vector diffraction integrals has been described. A formula has been derived whereby the aperture efficiency of a paraboloid-feed-horn combination may be calculated in terms of the field existing in the feed aperture when it transmits and the focal-region field produced by a plane wave incident on the reflector. In addition, formulas for the reradiated and spillover power factors have also been presented.

In Chapter II the vector diffraction integrals given in Chapter I are used to derive expressions for the $\mathbf{E}$ and $\mathbf{H}$ fields produced in the focal region of a truncated paraboloid of revolution with circular cross section when it is illuminated by a plane wave incident at an arbitrary angle to the reflector axis. The expressions for the fields involve integration over the surface of the reflector and some attention is given to the development of a double numerical integration scheme suitable for machine computation. Computer evaluation of double integrals, especially over large surfaces, can be costly. Chapter II
concludes by describing how one of the integrations in the general diffraction integrals may be carried out in closed form so that computation of the focal fields requires only a single numerical integration.

The expressions for the scattered field produced by a plane wave incident at an arbitrary angle to the reflector axis simplify considerably for the case of axial incidence; these simplified expressions are presented in Chapter III. Where possible and appropriate, expressions for the fields in the focal regions of paraboloids of large focal ratio \( f/D > 2.0 \), say) are obtained in closed form and compared with those appearing in the literature. The focal-region fields for paraboloids with \( f/D \) ratios of 0.25, 0.5, 1.0 and 2.0 are computed and presented in the form of graphs and contour maps. Reflectors commonly used at microwave frequencies have \( f/D \) ratios in the range of 0.25 to 1.0, while at \( f/D = 2.0 \) an interface between the results of the present study and those of classical optics can be obtained. An examination of the direction of energy flow in the focal region concludes this chapter.

Off-axis incidence by a plane wave is treated in Chapter IV where the results are again presented in the form of graphs and contour maps. The range of validity of the approximations which were made to make possible the reduction of the double integrals to single integrals is examined by direct comparison of some of the results obtained by each integration method.

In Chapter V some of the results obtained in Chapter III for axial incidence are used to examine the efficiency of aperture-type feeds in
the focal plane. Curves are presented showing the maximum aperture efficiency attainable by rectangular and circular apertures, the only losses being due to spillover. A straightforward design procedure is presented by means of which the dimensions of circular or rectangular waveguide feeds operating in the dominant $TE_{11}$ and $TE_{10}$ modes, respectively, may be determined so that maximum aperture efficiency may be achieved.
CHAPTER II

THE CIRCULAR PARABOLOIDAL REFLECTOR: DERIVATION OF THE
FIELD EQUATIONS FOR THE FOCAL REGION

This chapter is devoted to the study of the focal-region fields
produced by a plane wave incident at an arbitrary angle to the circular
aperture of an axially symmetric paraboloidal reflector. Section 1 de-
scribes the derivation of the integral expressions for the $\mathbf{E}$ and $\mathbf{H}$
fields in the focal region by using the vector representations for the
scattered fields presented in Chapter I. These expressions for the $\mathbf{E}$
and $\mathbf{H}$ fields involve integration over the reflector surface and certain
aspects of the double numerical integration techniques used to compute
the fields are discussed in section 2. The focal fields, as obtained
by the techniques of section 2, have to be used in the study of the apen-
ture efficiency of the horn-paraboloid combination. It soon becomes
apparent that the running times, and hence cost, of the computer evalua-
tion of the surface integrals for the fields prohibit the further compu-
tations required to investigate the aperture efficiency. This is so en-
ven for reflectors of moderate size (of the order of 50 wavelengths, say).

Section 3 describes how the double integration may be reduced to a
single integration by carrying out one of the integrations in closed form.
once suitable approximations and mathematical substitutions have been made in the integrands. This procedure shortens the computation time by an order of magnitude and makes possible the further computations required here. The techniques of section 2 are used to compute the focal-region fields both for axial and off-axis incidence by a plane wave; the results of these computations are presented in subsequent chapters.

2.1 The fields in the focal region of a circular paraboloidal reflector

Consider a linearly polarized plane wave incident at an arbitrary angle to the aperture of an axially symmetric paraboloidal reflector (Fig. 7). We wish to calculate the fields scattered by the reflector towards the focal region. For off-axis incidence the focal region may be somewhat loosely defined as the region of largest field strength in the image space. In the case of axial incidence the focal region is simply the region around the geometrical focus of the paraboloid. The origin of the various coordinate systems used in the analysis is taken to be at the focus of the paraboloid. A point M on the surface of the paraboloid is defined by the spherical coordinates \((R, \theta, \phi)\). For the present purposes it is convenient to use the angle \(\theta\) as measured from the negative z-axis, rather than the angle \(\theta'\) measured from the more usual positive z-axis, thus \(\theta = \pi - \theta'\). The observation point P in the focal region is defined by the cylindrical coordinates \((\rho', \phi', z')\). Finally,
Fig. 7 Paraboloid geometry and coordinate systems.
a point in the reflector aperture plane is defined by the cylindrical coordinates \((\rho, \phi, z)\). The scan angles, \(\eta\) and \(\gamma\), are difficult to visualize in Fig. 7 and, for the sake of clarity, they are shown in Fig. 8, where they are drawn with the apex of the paraboloid as origin. The \(\vec{E}\) and \(\vec{H}\) fields and the propagation vector \(\vec{k}\) are also shown.

![Diagram]

*Fig. 8 The scan angles and the incident fields.*

The procedure which follows assumes that we are dealing with a perfect mathematical paraboloid and does not take into account such factors as aperture blocking, surface deviations from the true paraboloidal shape or misalignment of the feed. A knowledge of the error-free antenna performance is generally required to estimate the additional effects of the above factors; methods by which this may be done appear in the literature \([31, \text{ch. 3}]\) and \([38]\).
As discussed in Chapter I, there are two equivalent integral representations (Eqs. (16) and (17)) which are commonly used to obtain the expressions for the scattered fields. During a preliminary study both representations were used to obtain the scattered fields. It turns out that the formulation using the edge current explicitly yields fewer terms in the surface integral; for this reason this latter representation has been used in the discussion which follows. For convenience Eqs. (16a) and (16b) for the $E$ and $H$ fields scattered by a perfectly conducting reflector are repeated here, thus

$$ E_s(p) = \frac{1}{2\pi} \int_{S_1} \left\{ -j\omega \epsilon_0 \left( \hat{n} \times \overline{H}_4 \right) \phi + \left( \hat{n} \cdot \overline{E} \right) \nabla' \phi \right\} dS' $$

$$ - \frac{1}{2\pi j\omega \epsilon_0} \oint_{\Gamma} \overline{\nabla}' \phi \left( \hat{\ell} \cdot \overline{H}_4 \right) d\ell $$ \hspace{1cm} (51)

and

$$ H_s(p) = \frac{1}{2\pi} \int_{S_1} \left( \hat{n} \times \overline{H}_4 \right) \times \nabla' \phi \hspace{1cm} dS' $$ \hspace{1cm} (52)

where $\phi = e^{-jkr}$ and $E_4$ and $H_4$ are the fields incident on the reflector. The integrations are carried out over the illuminated surface of the reflector and $\nabla'$ operates only on the source coordinates. As indicated in Fig. 7, $\hat{\ell}$ is a unit tangent vector along the boundary curve separating the illuminated and the shadowed regions. The sense of the line integral around the rim of the reflector is such that the outward normal to the surface $S_1$ is on the left. With this convention we may write $\hat{\ell}$ as

$$ \hat{\ell} = - \sin \phi \hat{x} + \cos \phi \hat{y} $$ \hspace{1cm} (53)
The \( \overrightarrow{E} \) and \( \overrightarrow{H} \) fields of the incident plane wave, resolved into rectangular components \((x,y,z)\), are given by

\[
\overrightarrow{E}_1 = E(-\cos\gamma, -\sin\gamma \sin\theta, \sin\gamma \cos\theta)
\]

\( \cdots \tag{54} \)

\[
\overrightarrow{H}_1 = \frac{E}{Z_0} (0, \cos\theta, \sin\theta)
\]

\( \cdots \tag{55} \)

where \( Z_0 = (\mu_0/\varepsilon_0)^{1/2} \) is the impedance of free space and the angles \( \theta \) and \( \gamma \) are as shown in Fig. 8. The propagation vector \( \overrightarrow{k} \) is given by

\[
\overrightarrow{k} = \hat{k} = -k(\sin\gamma, -\cos\gamma \sin\theta, \cos\gamma \cos\theta)
\]

\( \cdots \tag{56} \)

where \( k = \omega(\mu_0\varepsilon_0)^{1/2} \) and \( \hat{k} \) is a unit vector along \( \overrightarrow{k} \). For a plane wave, as above, the vectors \( \overrightarrow{E}, \overrightarrow{H} \) and \( \hat{k} \) are related as follows

\[
\overrightarrow{H} = \frac{\hat{k} \times \overrightarrow{E}}{Z_0}
\]

\( \cdots \tag{57} \)

Before proceeding with the derivation of the expressions for the \( \overrightarrow{E} \) and \( \overrightarrow{H} \) fields, we will list some of the properties of the paraboloidal surface. Most of these are well known in the \((R, \theta, \phi)\) spherical coordinate system [20, ch. 12]; however, in the present analysis there are advantages to be had in using the aperture-plane coordinates to express certain of the properties.

The fundamental equation of the paraboloid, in spherical coordinates, is

\[
R = 2f/(1 + \cos\theta) = f \sec^2(\theta/2)
\]

\( \cdots \tag{58} \)
where \( f \) is the focal length. In this coordinate system the unit normal to the surface is

\[
\hat{n} = -\cos(\theta/2) \hat{r} + \sin(\theta/2) \hat{\theta}
\]  

(59)

This may be expressed in rectangular coordinates as follows

\[
\hat{n} = \{-\sin(\theta/2) \cos\phi, -\sin(\theta/2) \sin\phi, \cos(\theta/2)\}
\]  

(60)

A differential element of the paraboloid surface may be expressed in the \( \rho \) and \( \phi \) coordinates of the aperture plane by

\[
dS = \rho \sec(\theta/2) \, d\rho \, d\phi
\]  

(61)

where \( \rho = R \sin\theta \). The \( \sec(\theta/2) \) factor takes into account the curvature of the reflector. A differential element of length on the rim of the paraboloid is given by

\[
dl = R_0 \sin\theta_0 \, d\phi = \rho_0 \, d\phi
\]  

(62)

where the subscript zero refers to the maximum values of \( \rho, R \) and \( \theta \).

One further relationship between \( \rho \) and \( \theta \) proves useful, namely

\[
\frac{\rho}{2f} = \tan(\theta/2) = \rho^*
\]  

(63)

where \( \rho^* \) may be considered as a normalized aperture-plane coordinate.

Returning now to Eqs. (51) and (52), we can express all the quantities appearing in them in terms of the coordinates of Fig. 7. Since the operator \( \bar{v} \) appearing in the integrands operates only on the coordinates of the source point \( M \) at \( \bar{r} \), we may write
\[ \mathbf{V'} = -\frac{2}{\partial x} \hat{r} \] (64)

hence
\[ \nabla' \phi = \frac{-jk r}{r} = (jk + \frac{1}{r}) \frac{e}{r} \hat{r} \] (65)

where \( \hat{r} \) is a unit vector from the source point \( M \) to the observation point \( P \). For distances \( r \) exceeding a few wavelengths, we may neglect the term in \( r^{-2} \) and write
\[ \nabla' \phi = jk \frac{e}{r} \hat{r} \] (66)

From Fig. 7 we find
\[ r'^2 = |\mathbf{r'} - \mathbf{R}|^2 = R^2 - 2R \rho' \sin \theta \cos(\phi - \phi') + 2R \rho' \cos \theta + \rho'^2 + z'^2 \] (67)

and
\[ \hat{r} = \frac{\mathbf{r}}{r} = \frac{1}{r} (\rho' \cos \phi' - R \sin \theta \cos \phi, \rho' \sin \phi' - R \sin \theta \sin \phi, z' + R \cos \theta) \] (68)

The incident plane wave is uniform in amplitude and phase; however, the \( \mathbf{E}_i \) and \( \mathbf{H}_i \) fields on the surface of the paraboloid do not have uniform phase across the surface since some parts of the wavefront (i.e., a plane normal to \( \hat{k} \)) have to travel farther before striking the reflector than others. To account for this difference in propagation length, we need to know the distance \( d \) from an arbitrary source point \( M \) on the reflector to the incident wavefront when it is at a suitable reference...
point. The derivation of an expression for \( d \) requires careful consideration of the relative geometries of the paraboloid and the propagation vector of the incident wave. This procedure is quite lengthy; and, in order to maintain continuity in the present discussion, this derivation is presented in Appendix C. Suffice, then, to quote the result given in Eq. (C.15), which is

\[
d = f\{2 - \sec^2(\theta/2)\} \cos \gamma \cos \eta \\
- R \sin \phi \cos \eta \sin \theta + R \sin \phi \sin \eta \cos \eta \sin \theta
\]

(69)

where \( R = f \sec^2(\theta/2) \). This distance \( d \) together with \( r \) gives the phase at the observation point as \((2\pi/\lambda)(r + d)\), where \( \lambda \) is the wavelength.

All the quantities appearing in Eqs. (51) and (52) have now been determined and on carrying out the operations indicated in Eq. (51), we obtain the following expression for the \( x \) component of the \( \vec{E} \) field at \( P \):
\[ E_x(P) = \frac{4kE}{2\pi} \int_0^{2\pi} \int_0^{D/2} \left[ (\sin(\theta/2)\sin\phi \sin \phi' + \cos(\theta/2)\cos \phi') + \frac{1}{r} \{ \rho' \cos \phi' - R \sin \theta \cos \phi \} \right. \\
\left. \{ \sin(\theta/2)\cos \phi \cos \gamma + \sin(\theta/2)\sin \phi \sin \gamma \sin \phi' + \sin \gamma \cos \phi \cos(\theta/2) \} \right] \]

\[ \exp\left\{-\frac{jk(r + d)}{r} \right\} \sec(\theta/2) \rho \, d\rho \, d\phi \]

\[ = \frac{E}{2\pi} R_o \sin \theta_o \cos \phi \int_0^{2\pi} (\rho' \cos \phi' - R_o \sin \theta_o \cos \phi') \cos \phi \frac{\exp\left\{-\frac{jk(r_o + d_o)}{r_o}\right\}}{r_o^2} \, d\phi \quad (70) \]

where \( r \) and \( d \) are given in Eqs. (67) and (69), and \( k = 2\pi/\lambda \).
We now express this equation, and similar ones for the remaining field components, in terms of the normalized aperture coordinate $\rho^*$ alluded to earlier. To do this we use the relation in Eq.(63) to convert all the terms in $\theta$ to terms in $\rho^*$. For convenience a list of the required substitutions is given in Appendix C. Before presenting the equations in their final form, we can make one observation concerning the terms involving $\rho'/r^2$ and $z'/r^2$ appearing in the amplitude factors of the integrands. For displacements $\rho'$ and $z'$ from the focus, small compared to the focal length, we have $\rho' \ll r^2$; hence the terms in $\rho'/r^2$ and $z'/r^2$ may be neglected without introducing any appreciable error. This approximation was checked for a paraboloid with a diameter of 34 wavelengths and the above observation was borne out.

A useful normalization factor $E_x(F)$ for the fields may be deduced from Eq.(70). Specifically, we consider the case of axial incidence by a plane wave and calculate the $x$ component of the $E$ field at the focus of a paraboloid of large focal ratio. In this situation $\gamma = n = \rho'$ $= z' = 0$; by neglecting the contribution from the term in $r^{-2}$ in the double integration and also the contribution from the line integration, both of which are negligible for large focal ratios and diameters of several wavelengths, we find

$$E_x(F) = \frac{jEm\pi^2}{4\pi\lambda} e^{-jk2\ell} = jEm\pi^2 (f/\lambda) e^{-jk2\ell}$$

(71)

The exact expression for the $x$ component of the $E$ field at the focus of a paraboloid of any focal ratio is given below; however $E_x(F)$ proves to be a useful short-hand notation.
The set of equations presented on the next few pages gives the expressions for the rectangular components of the $E$ and $H$ fields in the focal region of a paraboloid of any focal ratio which is illuminated by a plane wave impinging at an arbitrary angle to the axis of the paraboloid. The expressions are exact except for the approximations implicit in the use of the general equations (Eqs. (51) and (52)) and the two amplitude approximations made in the application of these equations. These latter approximations are concerned with the expression for $\nabla'\phi$ given by Eq. (66) and the terms in $\rho'/r^2$ and $z'/r^2$ as discussed above. Note that one of the $(2f/\lambda)$ terms appearing in Eq. (C.24) for $d$ has been absorbed into $E_x(F)$ multiplying the expressions for the fields.
\[ E_x(P) = E_x(F) \frac{(f/\lambda)}{\pi \sigma^2} \int_0^2 \int_0^{\phi^*} \left[ \sigma^* \sin \phi \sin \eta + \cos \eta - \frac{\sigma^* 2 (f/\lambda)}{r/\lambda} \{ \sigma^* \cos^2 \phi \cos \eta \\
+ \sigma^* \sin \phi \cos \phi \cos \gamma \sin \eta + \cos \phi \sin \gamma \cos \eta \} \right] \frac{\exp(-j(2\pi/\lambda)(r + d))}{(r/\lambda)} \sigma^* \rho^* \ d\rho^* \ d\phi \]

\[-jE_x(F) \frac{2\lambda (f/\lambda)^2 \sigma^* \cos \eta}{\pi^2 D(r_o/\lambda)^2} \int_0^{2\pi} \cos^2 \phi \exp(-j(2\pi/\lambda)(r_o + d_o)) \ d\phi \]

where \((r/\lambda)\) and \((d/\lambda)\) are given in Eqs. (C.25) and (C.24) as

\[ r/\lambda = \text{SQRT} \left\{ (f/\lambda)^2 (1 + \sigma^2)^2 - 4(f/\lambda)(\sigma^*/\lambda)\rho^* \cos(\phi - \phi^*) + 2(f/\lambda)(z^*/\lambda)(1 - \rho^2) \right\} \]

\[ + (\sigma^*/\lambda)^2 + (z^*/\lambda)^2 \] (73)

and

\[ d/\lambda = (f/\lambda) \{ (1 - \sigma^2) \cos \gamma \cos \eta - 2\sigma^* \cos \phi \sin \eta + 2\sigma^* \sin \phi \cos \gamma \sin \eta - 2 \} \] (74)

The expressions for \(r_o/\lambda\) and \(d_o/\lambda\) are identical to those for \(r/\lambda\) and \(d/\lambda\) except that the subscript zero is added to all the \(\sigma^*\) factors in Eqs. (73) and (74). The remaining field components are:
\[
E_y(P) = - E_x(F) \frac{(\varepsilon/\lambda)}{\pi \rho_0^2} \int_0^{2\pi} \int_0^{\rho_0} \left[ \rho^* \cos \phi \sin n + \frac{\rho^* 2(\varepsilon/\lambda)}{(r/\lambda)} \rho^* \sin \phi \cos \phi \cos y \right] \exp\left[-j(2\pi/\lambda)(r + d)\right] \rho^* \, d\rho^* \, d\phi \\
\]

\[
- jE_x(F) \frac{2\lambda(\varepsilon/\lambda)^2 \rho \phi \cos n}{\pi^2 D(r_0/\lambda)^2} \int_0^{2\pi} \sin \phi \cos \phi \exp\left[-j(2\pi/\lambda)(r_0 + d_0)\right] \, d\phi 
\]

(75)

\[
E_z(P) = E_x(F) \frac{(\varepsilon/\lambda)}{\pi \rho_0^2} \int_0^{2\pi} \int_0^{\rho_0} \left[ \rho^* \cos \phi \cos n + \frac{(1 - \rho^* 2)(\varepsilon/\lambda)}{(r/\lambda)} \rho^* \cos \phi \cos y \right] \exp\left[-j(2\pi/\lambda)(r + d)\right] \rho^* \, d\rho^* \, d\phi \\
\]

\[
+ jE_x(F) \frac{\lambda(\varepsilon/\lambda)^2 (1 - \rho_0^2) \cos n}{\pi^2 D(r_0/\lambda)^2} \int_0^{2\pi} \cos \phi \exp\left[-j(2\pi/\lambda)(r_0 + d_0)\right] \, d\phi 
\]

(76)
\[ H_x(P) = E_x(F) \frac{(\epsilon/\lambda)^2}{Z_0}\pi p_0^2 \int_0^{2\pi} \int_0^{\rho} \left[ (1 - \rho^2) \cos \phi \sin n - 2 \rho \sin \phi \cos \phi \cos n \right] \]

\[
\frac{\exp \{ -j(2\pi/\lambda)(r + d) \} \rho^2}{(r/\lambda)^2} \rho \, d\rho \, d\phi
\]

\[ H_y(P) = E_y(F) \frac{(\epsilon/\lambda)^2}{Z_0}\pi p_0^2 \int_0^{2\pi} \int_0^{\rho} \left[ 2\rho^2 \cos^2 \phi \cos n + (1 - \rho^2)(\rho \sin \phi \sin n + \cos n) \right] \]

\[
\frac{\exp \{ -j(2\pi/\lambda)(r + d) \} \rho}{(r/\lambda)^2} \rho \, d\rho \, d\phi
\]

\[ H_z(P) = 2E_z(F) \frac{(\epsilon/\lambda)^2}{Z_0}\pi p_0^2 \int_0^{2\pi} \int_0^{\rho} \left[ \rho \sin n + \sin \phi \cos n \right] \]

\[
\frac{\exp \{ -j(2\pi/\lambda)(r + d) \} \rho^2}{(r/\lambda)^2} \rho \, d\rho \, d\phi
\]
The expressions for $E_X(F)$, $r/\lambda$ and $d/\lambda$ appearing in the above equations for the $E$ and $H$ fields scattered towards the focal region are given by Eqs. (71), (73) and (74), respectively. As indicated earlier the expression for $E_X(P)$ given by Eq. (72) can be integrated directly for the case of axial incidence ($n = \gamma = 0$) when the observation point is at the focus ($o' = z' = 0$). The result of this integration may be used to normalize $E_X(P)$ to unity at the focus; the required normalization factor is

$$E_{xo}(F) = E_X(F) \left[ \frac{1}{(1 + o_0^*)} - \frac{\lambda}{\pi(1 + o_0^*)^2} \right]$$

where $o_0^* = (D/4f)$.

This concludes the derivation of the expressions for the focal-region fields and we are now in a position to apply these equations to the study of particular paraboloids. In general, the expressions for the fields cannot be integrated directly to yield closed-form solutions and we have to perform the integrations numerically on a digital computer. The numerical evaluation of integrals of the above type is the topic of the next section.

2.2 Numerical integration techniques

This section introduces some methods of single numerical integration and outlines some of the features. Double numerical integration is described in terms of repeated application of the principles of single numerical integration, a procedure which yields the so-called "product formulas" for double integration. Finally, we look at two particular
schemes for double numerical integration as applied to the reflector problem and examine their relative efficiency.

2.21 Single numerical integration

In antenna pattern calculations we generally encounter integrals of the form

\[ E(u) = \int_{a}^{b} f(x) e^{j\phi(u,x)} \, dx \]  

(81)

where \( f(x) \) is an amplitude function, \( \phi(u,x) \) is a phase function and \( u \) is a parameter related to the point of observation or the pattern angle. Numerical integration is normally discussed in terms of real functions; the integral in Eq. (81) can be reduced to the sum of a real part and an imaginary part by De Moivre's theorem, thus

\[ E(u) = \int_{a}^{b} f(x) \cos\phi(u,x) \, dx + j \int_{a}^{b} f(x) \sin\phi(u,x) \, dx \]

\[ = E_r(u) + jE_i(u) \]  

(82)

where \( E_r \) and \( E_i \) are both real functions. In this scheme the real and imaginary parts are evaluated separately and combined at the end to yield \( E(u) \). An alternative to this procedure is to write the integrand in Eq. (81) as a single complex function \( F(u,x) \) so that \( E(u) \) is now given by
and then to carry out all the necessary operations in complex arithmetic. The latter procedure is the more attractive when a computer capable of operating in the complex mode is available.

For the purposes of the present discussion we need only consider integrations of the type appearing in Eq. (82). The integrand may be written as a single real function, and, since \( u \) is held constant during any particular integration, it will not be written explicitly in subsequent expressions. The integral under consideration may be approximated by a finite summation as follows:

\[
\int_{a}^{b} g(x) \, dx \approx \sum_{i=1}^{n} w_i g(x_i)
\]

where the \( g(x_i) \) are samples of the integrand at the points \( x_i \) in the interval \((a, b)\) and the \( w_i \) are appropriate weight coefficients. When \( g(x) \) is a constant, the sum of the weights must equal \((b - a)\) for the formula to be exact.

Integration formulas of the type given by Eq. (84), as well as many others, are discussed in great detail in the texts on numerical integration, for example [30 - 41]. Accordingly, only the essential features of four particular schemes are outlined below.
**Increment method.** Here the interval \((a, b)\) is divided into \(n\) equal subintervals, \(\Delta x = (b - a)/n\), and the \(x_i\) are taken at the center of each increment, \(x_i = (2i - 1)\Delta x/2\). The weights are equal to \(\Delta x\) since they must add up to give \((b - a)\) when \(g(x)\) is a constant, and Eq. (84) becomes

\[
\int_{a}^{b} g(x) \, dx \approx \Delta x \sum_{i=1}^{n} g(x_i)
\]

By taking \(n\) sufficiently large, or equivalently by making \(\Delta x\) sufficiently small, the integral can be calculated to any desired accuracy. Eq. (85) is exact for \(n\) as small as one when \(g(x)\) is a linear function of \(x\) (i.e., \(g(x) = Ax + B\)). For functions of higher degree the error introduced by approximating the integral depends on the second derivative of \(g(x)\).

**Trapezoidal rule.** The interval \((a, b)\) is divided into \((n - 1)\) equal subintervals, \(\Delta x = (b - a)/(n - 1)\) and the \(x_i\) are taken at the ends of the subintervals, \(x_i = a + (i - 1)\Delta x\). The required weights are \(W_1 = \Delta x/2\), \(W_2\) up to \(W_{n-1} = \Delta x\) and \(W_n = \Delta x/2\), so that Eq. (84) becomes

\[
\int_{a}^{b} g(x) \, dx \approx \frac{\Delta x}{2} [g(a) + g(b)] + \Delta x \sum_{i=2}^{n-1} g(x_i)
\]

The error again depends on the second derivative of \(g(x)\) and the trapezoidal rule is exact for \(n\) as small as two when \(g(x)\) is linear in \(x\).

For the same number of ordinates the increment method generally has less than one half the error of the trapezoidal rule.
**Simpson's rule.** This is perhaps the most frequently-used method for approximating integrals. The function \( g(x) \) is approximated by a series of short parabolic arcs, rather than the tangents or chords as in the increment and trapezoidal rules. To obtain the "compound" form of Simpson's rule, the interval \((a, b)\) is divided into an even number of subintervals, \( \Delta x = (b - a)/(n - 1) \), and the odd number of \( n \) ordinates are taken at the ends of each subinterval. With the notation \( g_i = g(x_i) \) the compound Simpson's rule may be written as

\[
\int_{a}^{b} g(x) \, dx \approx \frac{\Delta x}{3} \left[ g_1 + 4(g_2 + g_4 + \ldots + g_{n-1}) + 2(g_3 + g_5 + \ldots + g_{n-2}) + g_n \right]
\]

where \( g_1 = g(a) \) and \( g_n = g(b) \). The error is proportional to the fourth derivative of \( g(x) \) and so Simpson's rule is exact for all polynomials of degree three or less.

**Gaussian quadrature.** Gaussian quadrature differs from the schemes given above in that the \( n \) abscissae \( x_i \) are not equally spaced and so both the weights \( w_i \) and the nodes \( x_i \) must be determined. Schemes of this type are often referred to as quadratures of the highest degree of algebraic precision and can be made exact for polynomials of degree \( \leq 2n - 1 \). There are many forms of Gaussian quadrature, depending on the integrand, and they are discussed at length in the texts referenced earlier.

In the Gaussian scheme for integrals of the type given in Eq. (84) the upper and lower limits \((a, b)\) are usually taken as \((-1, 1)\). The procedure can be extended to limits other than \((-1, 1)\) by a change
of variables. The quadrature formula for this interval is

$$\int_{-1}^{1} g(x) \, dx \approx \sum_{i=1}^{n} W_i \, g(x_i)$$  \hspace{1cm} (88)$$

The n abscissae $x_i$ are the roots of the Legendre polynomials $P_n(x)$ and the weights $W_i$ are calculated from a formula involving $P_{n+1}(x)$ and the first derivative of $P_n(x)$. It is not common practice to calculate the $x_i$ and $W_i$ for a particular application since they are tabulated in the references cited above, for example Stroud and Secrest [41] give tables with $n$ up to 512. To obtain the quadrature formula for the interval $(a, b)$, we use the following change of variable

$$x = \frac{1}{2}( (b - a)t + (b + a))$$  \hspace{1cm} (89)$$

$$dx = \frac{1}{2}(b - a)dt$$  \hspace{1cm} (90)$$

and the quadrature formula becomes

$$\int_{a}^{b} g(x) \, dx \approx \frac{b - a}{2} \sum_{i=1}^{n} W_i \, g\left[ \frac{(b - a)t_i + (b + a)}{2} \right]$$  \hspace{1cm} (91)$$

where the $t_i$ are the roots of the Legendre polynomial $P_n(t)$.

It is difficult to compare the relative accuracies of the four schemes presented above since the error depends on the higher derivatives of the integrand. In fact the error consists of two parts: (1) a multiplicative constant which depends solely on the numerical rule employed and (2) a higher derivative evaluated at some intermediate point.
of the integration interval. The higher derivatives may be difficult to
obtain and, since the error terms of different schemes often require
different orders of differentiation, it may not be possible to compare
the different schemes directly. Allen [42] has carried out a study of
the above methods when applied to antenna pattern calculations to deter-
mine which method provides a given accuracy with the greatest economy.
He concludes that Gaussian quadrature has, in general, the highest de-
gree of precision and the lowest cost per integral to obtain a pre-
scribed accuracy. In addition, Gaussian quadrature has the advantage that
the error over most of the range of pattern angle which yields good ac-
curacy is smaller than that obtained by other methods.

2.22 Double numerical integration

The theory of integration formulas for one variable presented in
section 2.21 is well developed and is discussed in depth in the re-
ferences given there; however, the theory of multiple integration is not
nearly as complete. This is due in part to the greater complexity of
higher order spaces (two in the case of double integration) and to the
fact that the theory of orthogonal polynomials is more complicated in
more than one variable. In a recent text Stroud [43] discusses and tab-
ulates a class of non-product formulas which has been developed for and
is most efficient when applied to particular geometries in n-dimensional
space. It is his feeling that if a non-product formula for a particular
space is known, then it would usually be better to use that formula in-
stead of the product formula with the same number of points. The non-
product formulas will not be discussed here but they may warrant further study.

The method to be described below is perhaps the simplest to understand since it is based directly on the methods of single integration discussed earlier. Double numerical integration is performed by applying one of the methods of single numerical integration twice. While integration is carried out with respect to one variable, the other variable is held fixed. We are interested in integrals of the form

\[ E(u,v) = \int \int g(u,v,x,y) \, dx \, dy \quad (92) \]

where \( x \) and \( y \) are coordinates of the reflector surface, say, and \( u \) and \( v \) are related to the point of observation or the pattern angles. Eq. (92) may be written as

\[ E(u,v) = \int \int I(u,v,y) \, dy \quad (93) \]

where

\[ I(u,v,y) = \int g(u,v,x,y) \, dx \quad (94) \]

We now integrate Eq. (94) numerically \( N \) times for \( N \) fixed values of \( y \), the \( j \)th numerical integration is
\[ I(u,v,y_j) = \int_{a(y)}^{b(y)} g(u,v,x,y_j) \, dx \]

\[ \sum_{i=1}^{n} H_{ij} g(u,v,x_i,y_j) \quad (95) \]

where \( y_j \) is a particular value of \( y \) and the \( H_{ij} \) are \( N \) sets of weights with \( n \) weights in each set. Numerical integration of Eq. (93) then becomes

\[ E(u,v) = \sum_{j=1}^{N} W_j \, I(u,v,y_j) \quad (96) \]

where \( W_j \) are the weights of the \( N \)-point scheme used on the \( y \) integration. Combining Eqs. (95) and (96) yields

\[ E(u,v) \approx \sum_{j=1}^{N} \sum_{i=1}^{n} W_j \, H_{ij} \, g(u,v,x_i,y_j) \quad (97) \]

Note that the number of points in each of the single integration schemes need not be the same and that the total number of integration points is \( n \times N \). Eq. (97) forms the starting point for the discussion in the next section.
2.23 Case study of two double numerical integration schemes

In the light of Allen's conclusions regarding the higher efficiency of Gaussian quadrature as compared to the other schemes for single numerical integration; it was decided that two Gaussian schemes might be the best to use in Eq. (97). This yields the following double-Gaussian integration formula

$$\int_c^d \int_a^b g(u,v,x,y) \, dx \, dy$$

$$= \frac{(d - c)}{2} \sum_{j=1}^{N} w_j \left[ \frac{(b - a)}{2} \sum_{i=1}^{n} H_{ij} g\left[ \left( \frac{(b - a) s_i + (b + a)}{2} \right) \right] \right]$$

$$\left( \frac{(d - c) t_j + (c + d)}{2} \right)$$

$$\tag{98}$$

where $s_i$ and $t_j$ are the roots of the Legendre polynomials $P_n(s)$ and $P_N(t)$, respectively. Here $x$ and $y$ represent any two parameters defining a point on the surface of the reflector.

Even though the Gaussian quadrature scheme with the zeroes of the Legendre polynomials is the most efficient for a particular class of integrands in one variable, it does not necessarily follow that two
Gauss-Legendre schemes are the most efficient in two dimensions. The formulas given by Eq. (72) and Eqs. (75) through (79) for the field components require $\rho$ and $\phi$ integrations in a cylindrical coordinate system. When the wave is incident along the reflector axis there is symmetry in $\phi$, and it appears likely that an integration scheme with equally-spaced $\phi$ points might be more efficient than the Gauss-Legendre scheme which has unequally-spaced $\phi$ points. With this in mind, a second double numerical integration formula may be constructed by using a Gauss-Legendre scheme on the $\rho$ coordinate and a linear scheme (i.e., equal weights, and $\phi$ points at the center of equal subintervals) on the $\phi$ coordinate. This yields

$$
\int_a^b \int_c^d g(u,v,\rho,\phi) \, d\rho \, d\phi
$$

$$
= \Delta \phi \sum_{j=1}^{N} \left( \frac{b-a}{2} \right) \sum_{i=1}^{n} H_{ij} \left[ \left( \frac{(b-a)s_i + (b+a)}{2} \right), (2j-1) \Delta \phi/2 \right]_{oo}
$$

where $\Delta \phi = (d-c)/N = 2\pi/N$ for $d = 2\pi$ and $c = 0$. Stroud and Secrest [41, 1]

The author is indebted to Dr. G.W. Collins II of the Dept. of Astronomy who pointed this out to him and with whom he had several interesting discussions about numerical integration techniques.
give a result similar to this for one of the angular integrations over the surface of a sphere; the weights are $2\pi/N$ and the angular points $\theta_j$ may be any set of angles which are equally spaced at an interval of $2\pi/N$.

Before describing the results obtained in a preliminary study of the field integral for the principal component of the electric field Eq. (72), we digress briefly to consider some of the practical aspects of numerical integration. It is in order to ask: "Does the numerical integration yield the correct result?" This fundamental question is often difficult to answer. Obviously, the correct result to the general analysis is not available so the result obtained by numerical integration cannot be compared directly to it. Errors may be introduced into the numerical result either by an unintentional and sometimes well-concealed programming error or by the truncation error $E$ that arises from the fact that the sum is only approximately equal to the integral. Thus

$$\int_a^b g(x) \, dx = \sum_{i=1}^n W_i g(x_i) + E \quad \text{(100)}$$

where $E$ is the truncation error introduced by summing to $n$.

Let us assume that all programming errors have been eliminated and concentrate on the truncation error. Generally, if we have an integration rule which is convergent for the class of integrands $g(x)$, we can obtain the integral as accurately as desired by taking $n$ large enough. A simple test to see whether the integration has been carried out with a desired tolerance or accuracy $\varepsilon$ may be obtained by computing the integral
twice. Firstly, the integral is computed using \( n \) integration points, then this is repeated with a larger number \( m \) of integration points and these results are used to see whether \( c \) satisfies the following inequality

\[
\left| \sum_{i=1}^{n} W_i g(x_i) - \sum_{j=1}^{m} W_j g(x_j) \right| \leq c
\]

\[
\left| \sum_{i=1}^{n} W_i g(x_i) \right|
\]

The respective weights \( W_i \) and points \( x_i \) for the summations over \( n \) and \( m \) are in general different. All the parameters on which the integrand depends are held fixed during successive integrations.

The inequality in Eq. (101) may be monitored by eye on successive printouts or, alternately, the computer program may have a built-in facility to increase the number of points until the desired accuracy is reached or to terminate the computations once an upper limit for the number of functional evaluations is exceeded. This latter procedure leads us to what have been called "automatic integrators". Several automatic integrators, based on routines which have equally-spaced abscissae, have been developed[40, ch.6]. Basically the increment \( \Delta x \) is repeatedly halved until a desired accuracy is achieved; in this way the information obtained at the \( K \) th integration is not discarded but is used in forming the \((K+1)\) th integration. The Gaussian schemes are not readily applicable to automatic integrators since they do not have
equally-spaced abscissae; if the number of points is increased the previously computed values of the integrand go to waste.

If it is anticipated that the number of points required to yield the desired accuracy will vary widely (by an order of magnitude or more) over the range of parameters \( u \) and \( v \) in Eq. \((7)\), it would be best to use one of the automatic integrator schemes. The number of points will then be automatically adjusted so that the minimum number of points is used to yield the desired accuracy for the particular values of the parameters \( u \) and \( v \). However, if the number of points varies by only a small factor (two or so), it may be simpler to use that number of points yielding good accuracy at the extreme values of \( u \) and \( v \) to cover the entire range of \( u \) and \( v \). The results presented below follow the latter approach.

Some checks on the validity of the numerical results may be obtained if the integration can be carried out in closed form under certain special conditions, e.g., Eq. \((72)\) can be evaluated at the focus for axial incidence as shown in Eq. \((80)\). The values of the analytical integration may then be compared directly to the results of the numerical integrations. Agreement between the two evaluations in these special situations does not necessarily imply agreement throughout the range of parameters but it does instill a certain amount of confidence in the integration scheme. Further, the results of the numerical integration should be checked to see whether they are physically reasonable. Lastly, if measured data are available the numerical results should be compared with them, assuming that the measured data are reliable. The only response to the question posed at the beginning of this discussion
on p. 69 seems to be that there is no answer other than to build up confidence and experience and not to set excessively stringent tolerance (c) requirements.

We conclude this section with a presentation of some of the results obtained when the double numerical integration schemes given by Eqs. (98) and (99) were applied to Eq. (72). The procedure adopted here was as follows: (1) the number of $\phi$ points was set somewhat arbitrarily at 64 and the $\rho^*$ integrations were carried out successively for 4, 8 and 16 integration points, (2) the number of $\rho^*$ points was set at 16 and the $\phi$ integrations were carried out for 16, 32 and 64 points and (3) the results of the various integrations were compared for accuracy and computation time.

The above analysis was carried out for the case of axial incidence on a paraboloid with $f/D = 0.35$ and diameter $D = 69.68 \lambda$. The results are presented in Figs. 9 and 10. The patterns are obtained by repeatedly evaluating Eq. (72) for different values of the $\rho^*$ coordinate of the point of observation. The following values were used for the parameters of the point of observation: $\phi^* = \pi/2$, $z^* = 0$ and $\rho^*$ incremented from $-4\lambda$ to $4\lambda$ in steps of $0.1\lambda$. The values so obtained were plotted against $y'/\lambda$ to yield the principal component of the focal-region field along the $y$ axis. For axial incidence the $y$-axis patterns are symmetrical about the focus and so only the positive halves of the patterns are shown. Initial studies showed that the transverse pattern (i.e., on a line perpendicular to the reflector axis) was much more sensitive to the number of integration points than the axial pattern, therefore the
Fig. 9 $|E_x|$ versus $y'/\lambda$ for axial incidence showing the dependence on the number of points in the integration scheme ($\phi$ points fixed at 64).
Fig. 10 $|E_x|$ versus $y'/\lambda$ for axial incidence showing the dependence on the number of points in the integration scheme ($\phi^*$ points fixed at 16).
study concentrated on the transverse pattern. The solid curve in Figs. 9 and 10 may be compared with the measured pattern presented in [21, Fig. 4] for a paraboloid of similar dimensions. There is good agreement between the patterns both in the magnitude and position of the maxima and in the position of the minima. This affords a useful check on the integration schemes used above.

The interpretations of Figs. 9 and 10 are summarized in Table 1. We conclude that the double integration using the Gauss-Legendre scheme on $\rho^*$ and the linear scheme on $\phi$ is more efficient than the double Gauss-Legendre scheme (cf. items 7 and 1). In order to obtain the same pattern as the "master plot" (item 3), we need twice as many points with the double Gauss-Legendre scheme compared to the Gauss-linear scheme (cf. items 2, 3 and 8). At least 8 $\rho^*$ points and 64 $\phi$ points or 16 $\rho^*$ points and 32 $\phi$ points are needed to get an "accurate" pattern out to $2\lambda$, the latter being accurate to more than $3\lambda$.

The effect of off-axis incidence was investigated by considering a plane wave incident at an angle $\eta = 16^\circ$ and $\gamma = 0^\circ$ to the axis of a paraboloid with $f/D = 0.433$ and $D = 34.0\lambda$. The results of this study are shown in Fig. 11 and are summarized in Table 2. Only the Gauss-linear double integration curves are shown since this method was more efficient than the double Gauss-Legendre integration. Several interesting things happen to the pattern but they will not be discussed here as we are at present mainly concerned with the integration schemes; however, the effects of off-axis incidence are treated in detail in Chapter IV. From Fig. 11 and Table 2 we conclude that we require 16 $\rho^*$ points and 32 $\phi$...
TABLE 1
THE NUMBER OF INTEGRATION POINTS REQUIRED TO CALCULATE THE X COMPONENT OF THE FOCAL FIELD FOR AXIAL INCIDENCE

\( f/D = 0.35, \ D = 69.68\lambda, \ \eta = 0^\circ, \ \gamma = 0^\circ. \)

<table>
<thead>
<tr>
<th>No.</th>
<th>No. of ( \rho^* ) points</th>
<th>No. of ( \phi ) points</th>
<th>Total points</th>
<th>Time, sec.</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
<td>16</td>
<td>256</td>
<td>12.98\textsuperscript{a}</td>
<td>errors after 2.0(\lambda)</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
<td>32</td>
<td>512</td>
<td>18.76</td>
<td>same as 3</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>64</td>
<td>1024</td>
<td>30.59</td>
<td>&quot;master plot&quot;</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>64</td>
<td>512</td>
<td>18.00</td>
<td>errors after 2.4(\lambda)</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>16</td>
<td>128</td>
<td>10.25</td>
<td>errors after 1.7(\lambda)</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>64</td>
<td>256</td>
<td>12.54</td>
<td>errors after 1.0(\lambda)</td>
</tr>
<tr>
<td>7\textsuperscript{b}</td>
<td>16</td>
<td>16</td>
<td>256</td>
<td>14.35</td>
<td>errors after 1.6(\lambda)</td>
</tr>
<tr>
<td>8\textsuperscript{b}</td>
<td>16</td>
<td>64</td>
<td>1024</td>
<td>43.26</td>
<td>same as 3</td>
</tr>
</tbody>
</table>

Note: \textsuperscript{a} Times include a compilation time of about 7.1 sec.; all calculations are done on the IBM System 370 digital computer.  
\textsuperscript{b} Double Gauss-Legendre quadrature
points to obtain an accurate answer out to the third minimum on either side of the main-beam maximum. It should be noted that deviations between the results, obtained by using different total numbers of integration points, occur only on the axis side of the maximum while on the other side the patterns are identical in amplitude and phase. The solid curve in Fig. 11 is in good agreement with the contour map of the electric field presented by Rusch and Ludwig [24, Fig. 3] which shows a section of the main lobe.

The above analysis was repeated for the paraboloid with \( f/D = 0.35 \) and \( D = 69.68\lambda \) to see how many additional integration points would be needed for off-axis incidence as compared to axial incidence. The results of this study are not reproduced here because they are similar to those obtained for \( f/D = 0.433 \) except that the main beam deteriorates somewhat faster in the present case. Integration schemes with 16 \( \rho^* \) and 32 \( \phi \) points and 32 \( \rho^* \) and 64 \( \phi \) points yield identical results for an incidence angle \( \eta = 12^\circ \) out to a distance of 2\( \lambda \) on either side of the maximum. At \( \eta = 16^\circ \) small deviations in amplitude and phase are present on the axis side of the maximum. The above investigation leads to the conclusion that 32 \( \rho^* \) and 64 \( \phi \) points are necessary to obtain accurate results out to the third minimum on either side of the main-beam maximum for \( \eta = 16^\circ \) (i.e., four times the number of points for axial incidence).

In all the cases described above, we note that the truncation errors accumulate rapidly and become quite large when there are too few integration points. For a particular number of integration points
Fig. 11. $|E_x|$ versus $y'/\lambda$ for off-axis incidence showing the dependence on the number of points in the integration scheme.
**TABLE 2**

THE NUMBER OF INTEGRATION POINTS REQUIRED TO CALCULATE THE X COMPONENT OF THE FOCAL FIELD FOR OFF-AXIS INCIDENCE

\[ f/D = 0.433, \quad D = 34.0\lambda, \quad \eta = 16^\circ, \quad \gamma = 0^\circ. \]

<table>
<thead>
<tr>
<th>No.</th>
<th>No. of ( \rho^* ) points</th>
<th>No. of ( \phi ) points</th>
<th>Total points</th>
<th>Time, sec.</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>64</td>
<td>512</td>
<td>19.51(^a)</td>
<td>errors after first null on axis side</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>32</td>
<td>256</td>
<td>12.88</td>
<td>errors after first null on axis side</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>16</td>
<td>256</td>
<td>12.45</td>
<td>errors within 2( \lambda ) of maximum</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>32</td>
<td>512</td>
<td>18.95</td>
<td>same as 5</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>128</td>
<td>4096</td>
<td>100.62</td>
<td>&quot;master plot&quot;</td>
</tr>
</tbody>
</table>

Note: \(^a\) Times include a compilation time of about 7.1 sec.
there is a range over which the results are acceptable; we need fewer
integration points to obtain the main beam than we do for the side
lobes. In addition to the paraboloids discussed above several others with fo-
cal ratios between 0.25 and 1.0 and diameters between 16λ and 80λ were
investigated. The following rule of thumb for the number of integration
points required to give good accuracy for off-axis incidence at 16° and
field patterns out to the third minimum on either side of the maximum may
be obtained from the results of the above analysis. We require approx-
imately one point per wavelength along the radius of the reflector and
twice the total number of radial points around the circumference. This
is not a hard and fast rule since it also depends on the focal ratio but
it is a useful guide for establishing the approximate number of points
required to obtain useful answers.

The total number of integration points, and hence the computation
time, is proportional to the area (in square wavelengths) over which the
integration is to be carried out. Thus, if the diameter of the reflector
is increased by a factor of 10 the computation time and expense are
increased 100-fold. Antenna apertures of a 1000 or more wavelengths
in diameter are not uncommon and the field integrals over such large
surfaces may take a prohibitively long time. The results of the in-
tegration over the reflector surface must be used in a double integra-
tion over the feedhorn aperture to obtain a value for the aperture ef-
ficiency of the system. So, if both the reflector aperture and the
feedhorn aperture are increased in size, the cost of the computation
can increase very rapidly. Considerable savings in computation time
and money would be achieved if one of the integrations over the reflector surface could be carried out analytically. This has the advantage that the computation time would then be proportional to the aperture diameter (in wavelengths), rather than the square of the diameter, and the cost would increase linearly as the diameter is increased. The possibility of carrying out the \( \phi \) integration in Eq. (72) forms the topic of the next section. There we show that it is possible to carry out the \( \phi \) integration to yield Bessel functions provided we make suitable approximations and substitutions in the integrands.

2.3 The linear analysis for the fields in the focal region

As we have indicated in the previous section, evaluation of the field integrals by means of double numerical integration can be quite time-consuming and expensive, especially for large apertures (in square wavelengths). It would be advantageous to carry out one of the integrations in Eq. (70) analytically; this is precluded, however, by the presence of the square root in the expression for \( r \) (see Eq. (67)). In this section we show how the expression for \( r \) may be approximated by means of a binomial expansion and then establish the range of validity for the approximation subject to certain amplitude and phase criteria. Finally, this approximation is used to obtain expressions for the focal-region fields which involve only single numerical integration.

2.3.1 The linear approximation for displacements in the focal region

In the present analysis it is convenient to use the expression for \( r \) in the \((R, \theta, \phi)\) coordinate system, viz., Eq. (67), rather than
Eq. (73) in the \((p', \phi', z')\) system. Thus \(r\) is given by

\[
r = R \left[ 1 + \frac{1}{R^2} \left( -2Rp'\sin\theta \cos(\phi-\phi') + 2Rz' \cos\theta + p'^2 + z'^2 \right) \right]^{1/2}
\]

(102)

If the term in the square brackets is expanded binomially, and if terms of the order \((p'/R)^3\) and \((z'/R)^3\) are ignored, we obtain

\[
r = R + z'\cos\theta - p'\sin\theta \cos(\phi-\phi') + \frac{1}{2R} (p'^2 + z'^2) - \frac{1}{2R} (p' \sin\theta \cos(\phi-\phi') - z' \cos\theta)^2
\]

(103)

The analysis proceeds by making the following approximations (the Fraunhofer approximations)

\[
r \approx R + z'\cos\theta - p'\sin\theta \cos(\phi-\phi') \quad \text{(in phase expressions)} \quad (104a)
\]

\[
r \approx R \quad \text{(except in phase expressions)} \quad (104b)
\]

\[
\hat{r} \approx \hat{R} \quad (104c)
\]

This set of approximations leads to what is sometimes termed the "linear analysis" because the phase expression has been linearized. Limits on the displacements \(p'\) and \(z'\) may be obtained by requiring that their values are such that the following criteria are satisfied: (1) the maximum phase variation introduced by using Eq. (104a) rather than Eq. (102) must not exceed \(\pi/8\) (Rayleigh's far-field criterion) and (2) the maximum amplitude variation in \(r^{-1}\) in Eq. (70) should not exceed some value \(c\) when \(r^{-1}\) is set equal to \(R^{-1}\). It is convenient to examine the conditions for transverse displacement (i.e., \(z' = 0\)) and axial displacement (i.e., \(p' = 0\)) separately.
When \( z' = 0 \) the magnitude of the phase deviation \( \Delta \beta \) at some point \( p' \) is given by

\[
\Delta \beta = -\frac{2\pi}{\lambda} \frac{1}{2R} |(p'^2 - \rho'^2 \sin^2 \theta \cos^2(\phi - \phi'))|
\]

\[
= \frac{\pi \rho'^2}{\lambda R} \left| \left(1 - \frac{1}{2} \sin^2 \theta \right) - \frac{1}{2} \sin^2 \theta \cos^2(\phi - \phi') \right| \leq \pi/8 \quad (105)
\]

This leads to the condition that the maximum phase deviation will be less than \( \pi/8 \) provided \( \rho' \) is restricted so that:

\[
\frac{\rho}{\lambda} \leq 0.35 \sqrt{\frac{R}{\lambda}} \quad (106)
\]

for any value of \( \theta \) and \( (\phi - \phi') \). Since \( R \geq f \), we may write the maximum value of \( \rho'/\lambda \) as

\[
\frac{\rho'}{\lambda} \leq 0.35 \sqrt{\frac{f}{\lambda}} \quad \text{(107)}
\]

The radiation pattern of each element of surface \( dS \) in Eq. (70) is very broad so we may assume that the field amplitude contributed by each \( dS \) is uniform over the region of interest. The absolute value of the error \( \Delta A \) introduced for each \( dS \) by this assumption (cf., Eq. (104b)) is

\[
\Delta A = r |\frac{1}{r} - \frac{1}{R}|
\]

\[
= \rho' |\sin \theta \cos(\phi - \phi')| \leq \varepsilon
\]

or

\[
\rho'/\lambda \leq \varepsilon \left( \frac{f}{\lambda} \right) \frac{\sec^2(\theta/2)}{|\sin \theta \cos(\phi - \phi')|} \quad (108)
\]

where we have set \( R = f \sec^2(\theta/2) \). The maximum value of \( \cos(\phi - \phi') \) is
unity, while the factor in $\theta$ has a minimum at $\theta = 60^\circ$. Thus, the expression for the maximum value of $\rho'$ becomes

$$\frac{(\rho'/\lambda)}{\text{max A}} = 1.54 \frac{f/\lambda}{\epsilon}$$  \hspace{1cm} (109)$$

So, if we allow a maximum variation of 10% (i.e., $\epsilon = 0.1$) for the contribution to the total field by an element $dS$ on the surface, we find

$$\frac{(\rho'/\lambda)}{\text{max A}} = 0.15 \frac{f/\lambda}{\epsilon}$$  \hspace{1cm} (110)$$

The limits on $z'$ may be obtained by an analysis similar to the above. At a distance $z'$, with $\rho' = 0$, the phase error introduced by neglecting the second order terms in Eq. (103) will not exceed $\pi/8$ provided

$$\left|\frac{z'}{\lambda}\right| \leq 0.35 \frac{\sec(\theta/2)}{\sin\theta} \sqrt{\frac{f}{\lambda}}$$  \hspace{1cm} (111)$$

The $\theta$-dependent factor has a minimum at $\sin(\theta/2) = 1/\sqrt{3}$ (i.e., $\theta/2 = 35^\circ 14'$), which leads to

$$\left|\frac{z'}{\lambda}\right|_{\text{max Ph}} = 0.46 \frac{f}{\lambda}$$  \hspace{1cm} (112)$$

For an amplitude deviation of 10% we find

$$\left|\frac{z'}{\lambda}\right|_{\text{max A}} = 0.10 \frac{f}{\lambda}$$  \hspace{1cm} (113)$$

The above limits on the values of $\rho'$ and $z'$ in the linear approximation are intended only as a guide. If limits are desired for a particular problem, a separate estimate, based on the above procedures,
should be made. The need for this clearly arises in the case where the reflector under consideration subtends a half angle smaller than those quoted above and used to obtain the limits on \( \rho' \) and \( z' \). A comparison of Eqs. (107) and (110) shows that in the case of transverse displacement the phase restriction is generally more severe than the amplitude restriction provided the focal length of the reflector is greater than about 6 wavelengths. From Eqs. (112) and (113) we observe that the amplitude restriction on \( |z'| \) dominates the phase restriction for reflectors with focal lengths which are less than about 20 wavelengths.

2.32 The focal-region fields as obtained by

the linear approximation

The linear approximation for \( r \) in phase expressions as given in Eq. (104a) may be transformed into the \((\rho^*, \phi, z)\) coordinate system by means of the relations in Appendix C, thus

\[
\frac{r}{\lambda} = (\varepsilon/\lambda)(1 + \rho^* z) + (z'/\lambda) \frac{1 - \rho^* z}{1 + \rho^* z} = (\rho'/\lambda) \frac{2\rho^*}{1 + \rho^* z} \cos(\phi - \phi')
\]

(114)

The total phase \(-k(r + d)\) in Eq. (72) as obtained from Eqs. (114) and (74) for \( r \) and \( d \), respectively, is given by

\[
-(2\pi/\lambda)(r + d) = -2\pi \left[ (\varepsilon/\lambda)(\rho^* z - 1)(1 - \cos\gamma \cos\eta) + (z'/\lambda) \frac{1 - \rho^* z}{1 + \rho^* z} \\
- (\rho'/\lambda) \frac{2\rho^*}{1 + \rho^* z} \cos(\phi - \phi') - 2(\varepsilon/\lambda)\rho^* \cos\phi \sin\gamma \\
+ 2(\varepsilon/\lambda)\rho^* \sin\phi \cos\gamma \sin\eta \right]
\]

(115)
In amplitude expressions r as given by Eq. (104b) is

\[ \frac{r}{\lambda} = \frac{f}{\lambda}(1 + p^2) \]  \hspace{1cm} (116)

which is independent of \( \phi \).

At this point it is worthwhile to digress for a moment to examine the following integral representation for the Bessel function of the first kind of order \( n \) \([44, Eq. (9.1.21)]\):

\[ J_n(b) = \frac{1}{\pi} \int_0^\pi j b \cos \theta \cos(nt) \, d\theta \]  \hspace{1cm} (117)

from which it follows that

\[ J_n(b) = \frac{1}{2\pi} \int_0^{2\pi} j b \cos \theta \cos(nt) \, d\theta \]  \hspace{1cm} (118)

Two very useful relations may be obtained from this equation, they are:

\[ \int_0^{2\pi} \cos(n\beta) \, d\beta = 2\pi j^n \cos(n\xi) J_n(b) \]  \hspace{1cm} (119)

and

\[ \int_0^{2\pi} \sin(n\beta) \, d\beta = 2\pi j^n \sin(n\xi) J_n(b) \]  \hspace{1cm} (120)

The easiest way to check Eqs. (119) and (120) is to make the change of variable \( t = (\beta - \xi) \) and to observe that after some trigonometric manipulation Eq. (118) results in either case.
The form of Eqs. (119) and (120) suggests that it might be possible to carry out the \( \phi \) integration in Eq. (72), for example, if we use the phase as given in Eq. (115). Examination of this phase factor reveals that the last three terms depend on \( \phi \). The first of these terms is in a form similar to the phase factor in Eq. (119) or Eq. (120) and were it not for the last two terms the \( \phi \) integration could be carried out directly to yield Bessel functions. For axial incidence (i.e., \( \gamma = \eta = 0 \)) these last two terms disappear and we have circular symmetry in \( \phi \); however, for off-axis incidence the rotation symmetry appears to be lost.

The following substitution was devised to overcome the difficulty presented by the last two terms in the phase expression (Eq. (115)):

Let

\[
\Lambda \cos(\phi - \alpha) = (p'/\lambda) \frac{2p^*}{1 + p^*^2} \cos(\phi - \phi')
\]

\[
+ 2(f/\lambda)p^* \cos \psi \sin \gamma = 2(f/\lambda)p^* \sin \phi \cos \gamma \sin \eta \quad (121)
\]

where \( \Lambda \) and \( \alpha \) may be determined by means of the trigonometric relation

\[
\cos(C - D) = \cos C \cos D + \sin C \sin D \quad (122)
\]

This gives

\[
\tan \alpha = \frac{(p'/\lambda) \sin \phi' - (1 + p^*^2)(f/\lambda) \cos \gamma \sin \eta}{(p'/\lambda) \cos \phi' + (1 + p^*^2)(f/\lambda) \sin \eta} \quad (123)
\]

and
\[ \Lambda^2 = \left[ \frac{2\rho^*}{1 + \rho^*} \left( \rho^*/\lambda \right) \cos \phi^* + 2(\ell/\lambda)\rho^* \sin \gamma \right]^2 \]

\[ + \left[ \frac{2\rho^*}{1 + \rho^*} \left( \rho^*/\lambda \right) \sin \phi^* - 2(\ell/\lambda)\rho^* \cos \gamma \sin \gamma \right]^2 \]  

(124)

With the above substitution the equation for the total phase becomes

\[- (2\pi/\lambda)(r+d) = -2\pi \left[ (\ell/\lambda)(\rho^2 - 1)(1 - \cos \gamma \cos \eta) + (z^*/\lambda) \frac{1 - \rho^2}{1 + \rho^2} \right] \]

\[ + 2\pi \Lambda \cos(\phi - \alpha) \]  

(125a)

or

\[- (2\pi/\lambda)(r+d) = -\psi + 2\pi \Lambda \cos(\phi - \alpha) \]  

(125b)

where \( \psi \) is a convenient short-hand notation for the \( \phi \)-independent terms.

The \( \phi \) dependence is confined to the last term of Eq.(125b) and this term is of the same form as \( b \cos(\beta - \xi) \) in Eq.(119), for example.

An analysis similar to the above yields the total phase \(-(2\pi/\lambda) \times (r_0 + d_0)\) in the line integral of Eq.(72) in the form

\[- (2\pi/\lambda)(r_0+d_0) = -2\pi \left[ (\ell/\lambda)(\rho^2 - 1)(1 - \cos \gamma \cos \eta) + (z^*/\lambda) \frac{1 - \rho_0^2}{1 + \rho_0^2} \right] \]

\[ + 2\pi \Lambda_0 \cos(\phi - \alpha_0) \]  

(126a)

or

\[- (2\pi/\lambda)(r_0 + d_0) = -\psi_0 + 2\pi \Lambda_0 \cos(\phi - \alpha_0) \]  

(126b)
where

\[ A_0 \cos(\phi - \alpha_0) = \left(\frac{2\rho^*}{1 + \rho_0^*}\right) \cos(\phi - \phi') + 2(f/\lambda)\rho^* \cos \phi \sin y - 2(f/\lambda)\rho^* \sin \phi \cos y \sin n \] (127)

The parameters \(\alpha_0\) and \(A_0\) are given by

\[ \tan \alpha_0 = \frac{(\rho'/\lambda)\sin \phi' - (1 + \rho_0^*)(f/\lambda) \cos y \sin n}{(\rho'/\lambda)\cos \phi' + (1 + \rho_0^*)(f/\lambda) \sin y} \] (128)

and

\[ A_0^2 = \left[\frac{2\rho^*}{1 + \rho_0^*} (\rho'/\lambda) \cos \phi' + 2(f/\lambda)\rho^* \sin y\right]^2 + \left[\frac{2\rho^*}{1 + \rho_0^*} (\rho'/\lambda) \sin \phi' - 2(f/\lambda)\rho^* \cos y \sin n\right]^2 \] (129)

In amplitude expressions \(r_0\) becomes

\[ r_0/\lambda = (f/\lambda)(1 + \rho_0^2) \] (130)

Before we can carry out the integrations over \(\phi\) in Eq. (72) we require some further manipulations in the amplitude factors multiplying the exponential factor. The amplitude factors in the expressions for the Bessel functions are of the form \(\cos(n\beta)\) and \(\sin(n\beta)\), while in Eq. (72) there are product terms of the form \(\cos^2 \phi\) and \(\sin \phi \cos \phi\). The following trigonometric identities prove useful in reducing these product terms to the form required by the integral representation of the Bessel function:
\[ \sin x \cos x = \frac{1}{2} \sin 2x \]  

(131)

\[ \cos^2 x = \frac{1}{2} (1 + \cos 2x) \]  

(132)

\[ \sin^2 x = \frac{1}{2} (1 - \cos 2x) \]  

(133)

By employing the substitutions in Eqs. (125) through (133) the \( \vec{E} \) and \( \vec{H} \) fields may be represented by expressions requiring only single numerical integration. The procedure is illustrated below for the \( E_x \) component; thereafter the results for the remaining components are listed.
The integrations over $\phi$ may now be carried out term by term with the aid of Eqs. (119) and (120) to yield the final expression for $E_x(P)$, thus:
\[ E_x(P) = E_x(F) \frac{2}{\rho_0^2} \int_0^{\rho^*} \left[ \rho^* \sin \alpha \sin \theta_1 + \cos \theta_0 - \frac{\rho^*}{1 + \rho^*} \{ \rho^* \cos \gamma (\theta_0 - \cos 2\alpha \theta_2) \right. \\
\left. - \rho^* \sin 2\alpha \sin \gamma \sin \theta_2 + 2 \sin \alpha \sin \gamma \cos \theta_1 \} \right] \exp(-j\psi) \frac{\rho^*}{1 + \rho^*^2} \, d\rho^* \\
- jE_x(F) \frac{2\rho \rho_0 \cos \gamma}{\pi D(1 + \rho_0^2)^2} (\theta_0 - \cos 2\alpha \theta_2) \exp(-j\psi_0) \tag{135a} \]

The remaining components of the \( E \) and \( H \) fields are:
\[ E_y(P) = -E_x(F) \frac{2}{\rho_0^2} \int_0^{\rho_0^*} \left[ j\rho^* \cos \alpha \sin n J_1 + \frac{\rho^*}{1 + \rho^*} \{ -\rho^* \sin 2\alpha \cos \gamma J_2 \\
+ \rho^* \sin \gamma \sin (J_0 + \cos \alpha J_2) + 2j \sin \alpha \sin \gamma \cos n J_1 \} \exp(-j\gamma) \frac{\rho^*}{1 + \rho^*} d\rho^* \right. \\
+ \left. jE_x(F) \frac{2\lambda \rho^* \cos n}{\pi \eta (1 + \rho^*^2)^2} \sin 2\alpha_0 J_2 \exp(-j\gamma_0) \right] \]

\[ E_z(P) = jE_x(F) \frac{2}{\rho_0^2} \int_0^{\rho_0^*} \left[ \rho^* \cos \alpha \cos n J_1 + \frac{1 - \rho^*^2}{1 + \rho^*^2} \{ \rho^* \cos \alpha \cos \gamma J_1 \\
+ \rho^* \sin \alpha \sin \gamma \sin n J_1 - j \sin \gamma \cos n J_0 \} \exp(-j\gamma) \frac{\rho^*}{1 + \rho^*^2} d\rho^* \right. \\
+ \left. -E_x(F) \frac{2\lambda(1 - \rho^*^2) \cos n}{\pi \eta (1 + \rho^*^2)^2} \cos \alpha_0 J_1 \exp(-j\gamma_0) \right] \] (135b)

(135c)
\[ H_x(P) = \frac{2}{Z_0 \sigma^2} \int_0^{\rho^*} \left[ j(1 - \rho^2) \cos \alpha \sin n_1 + \rho \sin 2\alpha \cos n_2 \right] \exp(-j\eta) \frac{\rho^*}{(1 + \rho^2)^2} \, d\rho^* \quad (136a) \]

\[ H_y(P) = \frac{2}{Z_0 \sigma^2} \int_0^{\rho^*} \left[ \cos n - \rho^2 \cos 2\alpha \cos n_2 \right. \]
\[ + \left. j \rho^*(1 - \rho^2) \sin \alpha \sin n_1 \right] \exp(-j\eta) \frac{\rho^*}{(1 + \rho^2)^2} \, d\rho^* \quad (136b) \]

\[ H_z(P) = \frac{4}{Z_0 \sigma^2} \int_0^{\rho^*} \left[ \rho^* \sin n - j \sin \alpha \cos n_1 \right] \exp(-j\eta) \frac{\rho^*}{(1 + \rho^2)^2} \, d\rho^* \quad (136c) \]
where \( \rho_0^* = (d/4f) \) and \( Z_0 = (\nu_o/\nu_o) \). The phase factors \( \psi \) and \( \psi_o \) are given by

\[
\psi = 2\pi \left[ \left( f/\lambda \right) \left( \rho^* \rho^2 \right) - 1 \right] \left( 1 - \cos \psi \cos \eta \right) + \left( z'/\lambda \right) \frac{1 - \rho^2}{1 + \rho^2} \]
\]

and

\[
\psi_o = 2\pi \left[ \left( f/\lambda \right) \left( \rho_0^* \rho_0^2 \right) - 1 \right] \left( 1 - \cos \psi \cos \eta \right) + \left( z'/\lambda \right) \frac{1 - \rho_0^2}{1 + \rho_0^2} \]
\]

The argument of the Bessel function has for convenience been omitted; in the terms involving integration the argument is \((2\pi \lambda)\), while in the other terms the argument is \((2\pi \Lambda_o)\). The parameters \( \alpha, A, \alpha_o \) and \( \Lambda_o \) may be determined from Eqs. (123), (124), (128) and (129), respectively.

We have shown how the \( \phi \) integration of the general equations for the \( \vec{E} \) and \( \vec{H} \) fields (Eqs. (72) and (75) through (79)) may be carried out analytically, yielding Bessel functions. The remaining integration in \( \rho^* \) must be evaluated numerically. Although more complicated functions (i.e., the Bessel functions) have been introduced in the integrands, it is to be anticipated that the computer time for the single numerical integration will be at least an order of magnitude less than that for the straightforward double integration. Before integrating the expressions for the fields as derived in this section, it is desirable to examine these expressions for the case of axial incidence by a plane wave. Simplified field expressions, valid for axial incidence, are presented in Chapter III.
CHAPTER III

THE FOCAL-REGION FIELDS FOR AXIAL INCIDENCE BY A PLANE WAVE

This chapter presents a detailed analysis of the focal-region fields produced by an axially incident plane wave. Equations (135) and (136) are specialized to the case of axial incidence and a convenient short-hand notation is introduced to express the field components in compact form. The relative amplitudes of the individual terms contributing to the field components are examined in the focal plane for various f/D ratios. Under certain conditions (depending on the aperture diameter and the f/D ratio) some terms are quite small and may be neglected, further simplifying the analysis.

The expressions for the components of the focal-region field are evaluated for four f/D ratios in the range from 0.25 to 2.00 and the results are presented in the form of graphs and contour maps. The presentation is facilitated by using a generalization of the dimensionless coordinates (related to $p'$ and $z'$) employed frequently in classical optics for systems with large f/D ratio. The large number of graphs and contour maps presented makes it possible to infer many of the characteristics of the focal-region fields produced by reflectors of focal ratios other than the specific cases considered here without carrying out a complete analysis for each particular case.
3.1 Simplified formulas for axial incidence

When the angles $\gamma$ and $\eta$ appearing in the general field expressions are set to zero, the parameters $\alpha$, $\Lambda$, $\alpha_o$, $\Lambda_o$, $\psi$ and $\psi_o$ given by Eqs. (123), (124), (128), (129), (137) and (138) simplify considerably. We find

$$\alpha = \phi'$$  \hspace{0.5cm} (139)

$$\Lambda = \frac{2\rho^*}{1 + \rho^*^2} \left(\alpha'/\lambda\right)$$  \hspace{0.5cm} (140)

$$\alpha_o = \phi'$$  \hspace{0.5cm} (141)

$$\Lambda_o = \frac{2\rho^*}{1 + \rho^*^2} \left(\alpha'/\lambda\right)$$  \hspace{0.5cm} (142)

$$\psi = 2\pi(z'/\lambda) \frac{1 - \rho^*^2}{1 + \rho^*^2}$$  \hspace{0.5cm} (143)

$$\psi_o = 2\pi(z'/\lambda) \frac{1 - \rho^*^2}{1 + \rho^*^2}$$  \hspace{0.5cm} (144)

Before using these relations, we introduce the following short-hand notations:

$$I_n(o',z') = \frac{2}{\rho^*} \int_0^{\rho^*} \frac{\rho^{n+1}}{(1 + \rho^*^2)^2} J_n(2\pi\lambda) \exp(-12\pi(z'/\lambda)\frac{1 - \rho^*^2}{1 + \rho^*^2}) d\rho^*$$  \hspace{0.5cm} (145)

and

$$I_n(o',z') = \frac{2\rho^*}{(1 + \rho^*^2)^2} J_n(2\pi\lambda_o) \exp(-12\pi(\frac{1 - \rho^*^2}{1 + \rho^*^2}))$$  \hspace{0.5cm} (146)
so that the equations for the field components \{Eqs.\,(135) and (136)\} may be written in the compact form:

\[
E_x(P) = E_x(\rho', \phi', z') = E_x(P) \left[ I_0 + \cos2\phi' \cdot I_2 - j\frac{\lambda}{\pi D} (I_0 - \cos2\phi' \cdot I_2) \right] \tag{147a}
\]

\[
E_y(P) = E_y(\rho', \phi', z') = E_x(P) \left[ \sin2\phi' \cdot I_2 + j\frac{\lambda}{\pi D} \sin2\phi' \cdot I_2 \right] \tag{147b}
\]

\[
E_z(P) = E_z(\rho', \phi', z') = jE_x(P) \left[ 2\cos\phi' \cdot I_1 + j\frac{\lambda}{\pi D} \cos\phi' \frac{1 - \rho'^2}{\rho^2} \cdot I_1 \right] \tag{147c}
\]

\[
H_x(P) = H_x(\rho', \phi', z') = \frac{E_x(P)}{Z_0} \sin2\phi' \cdot I_2 \tag{148a}
\]

\[
H_y(P) = H_y(\rho', \phi', z') = \frac{E_x(P)}{Z_0} (I_0 - \cos2\phi' \cdot I_2) \tag{148b}
\]

\[
H_z(P) = H_z(\rho', \phi', z') = j \frac{E_x(P)}{Z_0} 2\sin\phi' \cdot I_1 \tag{148c}
\]

It is of interest to examine the relative magnitudes of the \(I_n\) and the \(i_n\) appearing in the above equations. An investigation of this type forms part of the discussion in the next section where the fields are examined in the focal plane.

3.2 Fields in the focal plane

In general the \(I_n\) and the \(i_n\) are complex; however, in the focal plane \(z' = 0\) and these functions are real. Setting \(z' = 0\) in Eqs.\,(145) and \,(146) gives

\[
I_n(\rho', 0) = \frac{2}{\rho_{0z}} \int_{0}^{\rho_{0z}^*} \frac{\rho_{0z}^{*n+1}}{(1 + \rho_{0z}^*)^2} J_n \left(\frac{k\rho_{0z}^{*}}{1 + \rho_{0z}^*} \right) d\rho^* \tag{149}
\]
and

\[ i_n(\rho', 0) = \frac{2\rho_0^*}{(1 + \rho_0^* z^2)^2} J_n \left( \frac{k2\rho' \rho_0^*}{1 + \rho_0^* z^2} \right) \]  \hspace{1cm} (150) \]

where \( k = 2\pi/\lambda \) and \( \rho_0^* = (D/4f) \). At the focus \((\rho' = 0, z' = 0)\) these expressions may be integrated directly; the only non-zero functions are \( I_0 \) and \( i_o \) since \( J_n(0) = 0 \) unless \( n = 0 \). Thus,

\[ I_0(0, 0) = \frac{1}{1 + \rho_0^* z^2} \] \hspace{1cm} (151)

\[ i(0, 0) = \frac{2\rho_0^*}{(1 + \rho_0^* z^2)^2} \] \hspace{1cm} (152)

These expressions may be used to normalize the \( I_n \) and \( i_n \) to unity at the focus. The normalization factor \( E_{x0}(F) \) of Eq. (80) is obtained from Eq. (147a) with \( I_0 \) and \( i_0 \) as in Eqs. (151) and (152) and with \( I_n = i_n = 0 \) \((n \neq 0)\).

For large \( f/D \) ratios the \( I_n \) and the \( i_n \) assume very simple forms. If we consider reflectors with \( f/D \geq 2.0 \), which implies that the half angle \( \theta_o \) subtended at the focus by the reflector is less than about 14°, we can make the following approximation in Eq. (14a). With \( f/D = 2.0, \rho_0^* = (D/4f) = 1/8 \) and \( \rho_0^* z^2 = 1/64 \) so that \( 1 + \rho_0^* z^2 \approx 1 \) over the entire range of \( \rho_0^* \) to 1.6% or better. This approximation is the same as the usual small-angle approximation \( \sin \theta \approx \tan \theta \approx \theta \) one encounters in optics, for example. In fact, the analysis which follows may be carried out equally well by first transforming Eq. (14a) from \( \rho_0^* \) to \( \theta \) coordinates and making the small-angle approximation (noting that \( d\theta = 2d\rho_0^*/(1 + \rho_0^* z^2) \)). Setting \( 1 + \rho_0^* z^2 = 1 \) in Eq. (14a) yields
This equation may be integrated directly by using the following relation

\[
\int_{0}^{z} x^{n+1} J_p(x) \, dx = z^{p+1} J_{p+1}(z)
\]

Hence

\[
I_n(\rho', 0) = \frac{\rho_{\phi}^{n-1}}{k\rho'} J_n(2k\rho' \rho_{\phi})
\]

With \(1 + \rho_{\phi}^2 = 1\), Eq. (150) becomes

\[
i_n(\rho', 0) = 2\rho_{\phi} I_n(2k\rho' \rho_{\phi})
\]

We are initially interested in the relative amplitudes of the \(I_n\) and the \(i_n\) \((n = 0, 1, 2)\) for a particular \(f/D\) ratio; thereafter, it would be informative to see how the \(I_n\) and the \(i_n\) are influenced by changes in \(f/D\) ratio. In carrying out an analysis such as the above two problems immediately arise: firstly, what \(f/D\) values should be used in the study and, secondly, what parameter should be used as independent variable (i.e., \(\rho'\) or some function related to it)? It was decided to investigate the patterns for reflectors with \(f/D\) ratios of 0.25, 0.5, 1.0 and 2.0 since the range from 0.25 to 1.0 covers the \(f/D\) ratios of reflectors commonly used at microwave frequencies and at \(f/D = 2.0\) we should be able to compare the results with those of classical optics.
Fig. 12 Dependence of the magnitudes of the functions $I_n$ and $i_n$ on the normalized radius $V_0 (z' = 0)$ for $f/D = 0.25$. 
Fig. 13 Dependence of the magnitudes of the functions $I_n$ and $i_n$ on the normalized radius $V_0$ ($r' = 0$) for $F/D = 0.50$. 

\[ V_0 = k \cdot \text{amp} \cdot \sin(\theta_0) \]
Fig. 14 Dependence of the magnitudes of the functions $I_n$ and $i_n$ on the normalized radius $V_0 (z' = 0)$ for $F/D = 1.00$. 
Fig. 15 Dependence of the magnitudes of the functions $I_n$ and $i_n$ on the normalized radius $V_0$ ($z' = 0$) for $f/D = 2.00$. 

F/D=2.00

$[ \Delta ] - I_0$
$[ X ] - I_1$
$[ + ] - I_2$
The dimensionless parameter $V_0$, given by

$$V_0 = 2\pi A_0 = \frac{2kp'\rho^2}{1 + \rho^2} = kp' \sin \theta_0$$

(157)

was selected as independent variable since it accounts for the change in the scale of the patterns as the $f/D$ ratio changes. For large $f/D$ ratios, $V_0$ reduces to the parameter $v$ used extensively in optics (see [4, sect. 8.8], for example), that is

$$V_0 = 2kp'(\pi/4f) = \frac{2\pi}{\lambda} \left( \frac{\pi}{f} \right) p' = v \quad (f/D \geq 2.0)$$

(158)

where $a$ is the radius of the aperture.

The influence of $f/D$ ratio on the magnitudes of $I_n$ and $i_n$ can be seen quantitatively in Figs. 12 to 15; the results for $I_n$ were obtained by numerical integration of Eq. (149) using 32-point Gaussian quadrature. $I_0$ has a maximum at $V_0 = 0$ while $I_1$ and $I_2$ have nulls there. For $V_0 > 0$, $I_1$ and $I_2$ take on values which are comparable to those of $I_0$ for small $f/D$ ratios; while for large $f/D$ ratios $I_0$ dominates $I_1$ and $I_2$. The $i_n$ are all of about the same magnitude for a particular $f/D$ and their maximum values do not change appreciably as the $f/D$ ratio changes. At first sight the $i_n$ appear to be quite large in comparison with the $I_n$; it must be borne in mind, however, that the $i_n$ have to be modified by the factor $\lambda/(\pi D)$ in the expressions for the components of the $E$ field (see Eqn. (147)). In this study we will be concerned with reflectors which have diameters of $30 \lambda$ or more; with this value $\lambda/(\pi D) \approx -40$ dB. This means that the contribution to the $E$
field by the line integral in Eq. (51) will be negligible for moderate values of \( V_0 \) and provided the diameter of the reflector is large enough (\( \geq 30\lambda \), say). Clearly, the influence of the line integral is greatest in the regions where the \( I_n \) have their zeroes.

In what follows the line-integral term will be neglected so that the expressions for the electromagnetic field become

\[
E_x(P) = E_x(F)(I_0 + \cos 2\phi' \cdot I_2) \tag{159a}
\]

\[
E_y(P) = E_x(F) \sin 2\phi' \cdot I_2 \tag{159b}
\]

\[
E_z(P) = j E_x(F) 2\cos \phi' \cdot I_1 \tag{159c}
\]

\[
H_x(P) = \frac{E_x(F)}{Z_0} \sin 2\phi' \cdot I_2 \tag{160a}
\]

\[
H_y(P) = \frac{E_x(F)}{Z_0} (I_0 - \cos 2\phi' \cdot I_2) \tag{160b}
\]

\[
H_z(P) = j \frac{E_x(F)}{Z_0} 2\sin \phi' \cdot I_1 \tag{160c}
\]

For large \( f/D \) ratios these equations reduce further and yield very simple results for \( E_x \) and \( H_y \). The \( I_n \) become

\[
I_0(\rho', 0) = 2 \frac{J_1(k\rho'\theta_0)}{k\rho'\theta_0} \tag{161a}
\]

\[
I_1(\rho', 0) = \frac{J_2(k\rho'\theta_0)}{k\rho'\theta_0} \tag{161b}
\]

\[
I_2(\rho', 0) = \frac{\theta_0^2}{2} \frac{J_3(k\rho'\theta_0)}{k\rho'\theta_0} \approx 0 \tag{161c}
\]

where we have used \( \rho \delta = \tan(\theta_0/2) \approx \theta_0/2 \). On substituting these \( I_n \),
into Eqs. (159) and (160), we obtain the transverse components

\[
|E_x(P)| = |E_x(P)| \frac{2J_1(kp\theta_0)}{kp\theta_0} = 2 \frac{Ep\lambda}{4\pi} \frac{J_1(kp\theta_0)}{kp\theta_0} \quad \text{(162a)}
\]

\[
|H_y(P)| = \frac{|E_x(P)|}{Z_0} \quad \text{(162b)}
\]

Equation (162a) is Airy's formula for the amplitude distribution of the light in the focal plane of a lens or for the Fraunhofer diffraction pattern of a circular aperture. The amplitude distribution consists of concentric bright and dark rings with a bright spot at the center, the dark rings occurring at the circles where \(J_1(kp\theta_0) = 0\). There is no \(y\) component in the focal-plane \(E\) field, hence the polarization of the transverse field is parallel to that of the incident field.

For small \(f/D\) the amplitude distribution for \(E_x(P)\) distorts due to the presence of \(I_2\) in Eq. (159a) and the familiar Airy ring pattern takes on an elliptical contour. By using the results obtained earlier for the \(I_n\) (see Figs. 12 through 15), the field components can be readily calculated. The dependence of \(E_x, E_y\) and \(E_z\) on the normalized radius \(V_0\) is shown in Figs. 16 through 19 for various \(f/D\) ratios. Note that the fields have not been normalized to unity but the factor \(E_x(P)\) has been used for normalization. The cross-polarized component \(E_y\) and the longitudinal component \(E_z\) are negligible only for large \(f/D\) ratios.

In some applications, for example, the study of energy flow in the focal region, it is desirable to express the scattered fields in cylindrical coordinates; this may be done by means of the following trans-
Fig. 16 Dependence of the absolute value of the field intensity on the normalized radius $V_0$ ($z' = 0$) for $f/D = 0.25$. 
Fig. 17 Dependence of the absolute value of the field intensity on the normalized radius $V_o (z' = 0)$ for $f/D = 0.50$. 
Fig. 18 Dependence of the absolute value of the field intensity on the normalized radius $v_0 (z' = 0)$ for $f/D = 1.00$. 
Fig. 19 Dependence of the absolute value of the field intensity on the normalized radius $V_o (z' = 0)$ for $f/d = 2.00$. 

$F/D=2.00$

(+)-EX, PHIPRI=0 DEG

(-)-EX, PHIPRI=90 DEG

$O E G$

$-20$

$0$

$20$

$V_0 = k \cdot \rho \cdot \sin(\theta_0)$
Applying this transformation to Eqs. (159) and (160) the fields may be written in cylindrical coordinates, thus

\[
\begin{align*}
E_p &= E_x(F)(I_0 + I_2) \cos \phi' \\
E_\phi &= -E_x(F)(I_0 - I_2) \sin \phi' \\
E_z &= jE_x(F)I_1 \cdot 2 \cos \phi'
\end{align*}
\]

\[
\begin{align*}
H_p &= \frac{E_x(F)}{Z_0} (I_0 + I_2) \sin \phi' \\
H_\phi &= \frac{E_x(F)}{Z_0} (I_0 - I_2) \cos \phi' \\
H_z &= j\frac{E_x(F)}{Z_0}I_1 \cdot 2 \sin \phi'
\end{align*}
\]

These expressions enable us to make a very interesting observation about the \( \overline{E} \) and \( \overline{H} \) fields. The \( \overline{H} \) field is identical to the \( \overline{E} \) field except that the former is rotated by \( \pi/2 \) in \( \phi' \) with respect to the latter. To see this, replace \( \phi' \) by \( \xi - \pi/2 \) in the expressions for \( \overline{E} \), this leads to
expressions for $\vec{H}$ identical to those in Eqs. (165) except that $\phi'$ has been replaced by $\xi$ where $\xi = \phi' + \pi/2$ (the $1/Z_0$ factor must, of course, be taken into account). This result may also be deduced directly from Eqs. (159) and (160).

It is of interest to examine the focal-plane field distribution in detail; the configuration of the cross-polarized and longitudinal components is of considerable importance. Figures 20 through 23 show contours of the amplitude of the transverse electric field $E_z$ (i.e., transverse to the $z$ axis) as a function of $x'/\lambda$ and $y'/\lambda$ for various $f/D$ ratios, where $E_z$ is defined by:

$$E_z = \left( |E_x|^2 + |E_y|^2 \right)^{1/2}$$  \hspace{1cm} (166)

The normalized $\rho'$ coordinate $\nu_0$ is not used here but the field is plotted directly in terms of wavelengths along the $x$ and $y$ axes. The complete pattern may be obtained by successively reflecting the quadrant shown about the axes. For convenience the field has been normalized to unity by the normalization factor $E_{xo}(F) = E_x(F)/(1 + p_z^2)$. Note that the amplitude contours are not circular for small $f/D$ but they do approximate the circular Airy rings more and more closely as the $f/D$ ratio increases. In addition, the field around a particular ring is not uniform but depends on $\phi'$. For small $f/D$ ratios the secondary maxima show 2-cycle variation in $\phi'$ while the minima show 4-cycle variation. As the $f/D$ ratio increases the angular dependence of the field decreases until we reach the $\phi'$-independent Airy distribution of classical optics (cf. Eq. (113a)).

The phase of the principal component $E_x$ of the electric field in the fo-
Fig. 20 Equal-amplitude contours of the transverse electric field in the focal plane ($z' = 0$) for $f/d = 0.25$ (contour interval is 0.05).
Fig. 21 Equal-amplitude contours of the transverse electric field in the focal plane ($z' = 0$) for $f/D = 0.50$ (contour interval is 0.05).
Fig. 22  Equal-amplitude contours of the transverse electric field in the focal plane \((z' = 0)\) for \(f/D = 1.00\) (contour interval is 0.025).
Fig. 23 Equal-amplitude contours of the transverse electric field in the focal plane ($z' = 0$) for $f/D = 2.00$ (contour interval is 0.025).
cal plane, relative to the phase at the focus, changes by $180^\circ$ when the amplitude changes sign. As we move from the main beam through successive sidelobes, the phase increases discontinuously by $\pi$ radians on passing from one sidelobe to the next. For the transverse field $E_t$ the minima are not true zeroes; the minima are, however, zeroes on the axes (these positions are indicated by the dots).

The effect of the $E_y$ component of the field may be studied by plotting the direction of the transverse electric field, $\mathbf{E}_t = E_x \hat{x} + E_y \hat{y}$, as well as the relative amplitudes of the $E_y$ and $E_x$ components. To obtain a quantitative measure of the cross polarization, we may define a cross-polarization factor $X_P$ by

$$X_P = 20 \log_{10} \left| \frac{E_y}{E_x} \right|$$

Since the direction of $\mathbf{E}_t$ and $X_P$ are related quantities it is convenient to present these quantities in a composite figure. Figures 24 through 26 show plots of the direction of the transverse electric field and contours of the cross-polarization factor as functions of $x'/\lambda$ and $y'/\lambda$ for various $f/D$ ratios. The figure for $f/D = 2.00$ is very similar to that for $f/D = 1.00$ (cf. Fig. 26) and it is therefore not presented here. The transverse field is parallel to the x axis except on the circles where $E_x = 0$; in these regions the very small $E_y$ component dominates. The contours of $X_P$ are also similar to those in Fig. 26 except that the values are a further $10 \text{ dB}$ down for corresponding contours. For small $f/D$ ratios the polarization of the transverse electric field is not parallel to that of the incident field except where the $E_y$ component of the scattered field vanishes. This occurs (cf. Eq. (159b)) on the x and y axes ($\phi' = 0$, $\pi/2$).
Fig. 24 Direction of the transverse electric field (left) and equal-amplitude contours of the cross-polarization factor (right) in the focal plane ($z' = 0$) for $f/D = 0.25$ (contour interval is 10 dB).
Fig. 25 Direction of the transverse electric field (left) and equal-amplitude contours of the cross-polarization factor (right) in the focal plane ($z' = 0$) for $f/D = 0.50$ (contour interval is 10 dB).
Fig. 26 Direction of the transverse electric field (left) and equal-amplitude contours of the cross-
polarization factor (right) in the focal plane (z' = 0) for f/D = 1.00 (contour interval is 10 dB).
and on the circles where $I_2(\rho', 0) = 0$. For large $f/D$ ratios $I_2(\rho', 0) \approx 0$ and the scattered field is always parallel to the incident field. For small $f/D$ the cross-polarization factor becomes quite large (even in regions where $E_x$ is not very small) but $XP$ decreases quite rapidly as the $f/D$ ratio increases. In regions where $E_y \approx 0$, $XP \rightarrow \infty$; while in regions where $E_x \approx 0$, $XP \rightarrow 0$. For these singular regions $XP$ changes very rapidly and the contours tend to bunch together; not too much significance should be attached to the many closely-spaced contours in these transition regions but they are useful for indicating regions where $XP$ is either very large or very small.

The transverse electric and magnetic fields ($E_T, H_T$) are, in general, not orthogonal. To see under what conditions they are, we examine

$$E_T \cdot H_T = E_x H_x + E_y H_y = \frac{2E_x^2(f)}{Z_0} \sin 2\phi' \cdot I_0 \cdot I_2$$

(168)

from which we deduce that the transverse electric and magnetic fields are orthogonal only along the $x$ and $y$ axes ($\phi' = 0, \pi/2$) and on the sets of circles where $I_0(\rho', 0) = 0$ or $I_2(\rho', 0) = 0$. These are the same conditions under which the cross polarization is zero except that a further set of conditions $I_0(\rho', 0) = 0$ has been added. For large focal ratios $I_2(\rho', 0) \approx 0$ so that the transverse $E$ and $H$ fields are always orthogonal since they have only an $E_x$ and an $H_y$ component, respectively.

Figures 27 and 28 show the behavior of the longitudinal component $E_z$ in the focal plane ($z' = 0$) for $f/D = 0.25$ and $f/D = 1.00$, respectively. The contours are very similar and differ only in the maximum values attained and in the distances of the maxima and zeroes from the focus (in
Fig. 27 Equal-amplitude contours of the longitudinal component \( E_z \) of the electric field in the focal plane \( (z' = 0) \) for \( f/D = 0.25 \) (contour interval is 0.025).
Fig. 28 Equal-amplitude contours of the longitudinal component $E_z$ of the electric field in the focal plane ($z' = 0$) for $f/d = 1.00$ (contour interval is 0.005).
terms of wavelengths). The variation in the focal plane is simply that of \( I_1(\rho',0) \) rotated about the z axis and modified by the factor \( 2\cos\phi' \) in Eq. (159c). \( E_z \) is zero at the focus, on the y axis \( (\phi' = \pi/2) \) and on the circles \( I_1(\rho',0) = 0 \), while it has maxima (indicated by crosses) of alternate signs along the x axis \( (\phi' = 0) \) corresponding to the points where \( I_1(\rho',0) \) has its maxima. For \( f/D \) ratios equal to 0.25, 0.50, 1.00, and 2.00 the first maxima attain values of 0.640, 0.348, 0.178, and 0.090, respectively, relative to a maximum value of unity for the principal component \( E_x \) at the focus. In general the longitudinal component \( E_z \) is non-vanishing in the focal plane and attains values comparable to those of \( E_x \) in the focal plane.

To conclude this section some comments on the complete analysis of section 2.1 are in order. The results obtained by the approximate analysis were checked along the x and y axes and along the line \( \phi' = 45^\circ \) by numerically evaluating the double integration required by Eqs. (72), (75), and (76). In all cases the results obtained by the single numerical integration of the approximate analysis were in excellent agreement with those of the complete analysis; the only significant deviations occur in the deep nulls of the patterns where they are of relatively minor importance. It was found that the computation time for the linear analysis (single numerical integration) was less than that for the complete analysis (double numerical integration) by a factor of 10; herein lies the great advantage of the linear approximation. A useful check on the self-consistency of the integration schemes may be made by making the linear approximations in the integrands of the complete analysis (Eqs. (72), (75),
and (76)) but still carrying out the double numerical integration. When this was done it was found that the two sets of results were identical in all respects; this instills considerable confidence in the accuracy of the results presented in this section.

Finally, it is important to note that when the fields in Eqs. (164) and (165) are properly normalized they depend only on \( \rho \) (i.e., on the \( f/D \) ratio) and are independent of the precise values of either \( f \) or \( D \). In this sense the results presented here are universal and apply to any paraboloidal reflector with the corresponding \( f/D \) ratio. This is, however, no longer true when the complete analysis is used. In the next section we obtain a similar result for the field variation along the reflector axis.
3.3 Field variation along the reflector axis

Along the reflector axis \((\rho' = 0)\) the field distribution takes on a particularly simple form because the functions in Eqs. (145) and (146) may be integrated directly. The only non-vanishing functions for \(\rho' = 0\) are \(I_0\) and \(i_0\). Note that

\[
\frac{(z'/\lambda)}{1 + \rho'^2} = \frac{z'/\lambda - 2(z'/\lambda)}{1 + \rho'^2} \quad (169)
\]

and let

\[
\frac{\rho'^2}{1 + \rho'^2} = x
\]

(170)

This implies

\[
\frac{2\rho'd\rho'}{(1 + \rho'^2)^2} = dx
\]

(171)

so that we have an integrand of the form \(e^{j2kz'x}\) which may be integrated to yield

\[
I_0(0,z') = \frac{e^{-ja}}{1 + \rho'^2} \sin \left(\frac{kz'\rho'^2}{1 + \rho'^2}\right) \exp \left(\frac{jkz'\rho'^2}{1 + \rho'^2}\right) \quad (172)
\]

where we have used \((e^{-a} - e^a)/(2j) = \sin a\), and \(k = 2\pi/\lambda\).

Substitution of Eq. (172) for \(I_0\) and Eq. (146) for \(i_0\) (with \(\rho' = 0\), and using Eq. (169) in the exponential) in Eqs (147) and (148) gives the field variation along the axis as:
\[ E_x(r) = E_x(0,z') = E_x(r) e^{-j\frac{k_o z'}{k_o z'}} \left[ \frac{1}{1 + \rho o^2} \frac{\sin(k o z')}{k o z'} e^{j k_o z'} \right] \]

\[ H_y(r) = H_y(0,z') = \frac{E_x(r)}{z_o} e^{-j\frac{k z'}{k_o z'}} \left[ \frac{\cos(k o z')}{1 + \rho o^2} e^{j k_o z'} \right] \]

where \( k_o = \frac{k o_o^2}{1 + \rho o^2} \). The remaining components are all zero so that along the axis, the scattered field is parallel to the incident field. As before we may neglect the contribution from the line integral in Eq. (51) because of the \((\lambda / \pi D)\) factor, provided \( D/\lambda \) is large enough. When \( E_x \) is normalized to unity at the focus we obtain a \((\sin k_o z'/k_o z')\) amplitude variation for the field along the axis for all reflectors independent of the focal ratio. The range of validity of the linear approximation must be examined when applying Eq. (173) to a particular reflector. It is convenient to use a dimensionless coordinate \( U_o \) to plot the amplitude and phase variation along the axis. We choose

\[ U_o = k_o z' = \frac{k o_o^2 z'}{1 + \rho o^2} = k \sin^2(\theta_o/2) z' \]

For large focal ratio \((f/D)\) this reduces to

\[ U_o = k o_o^2 z' = \frac{\pi}{2 \lambda} \left( \frac{a}{f} \right)^2 z' = \bar{u}/4 \]
where \( u \) is the dimensionless parameter used in optics (see [4, sect. 8.8]) and \( a \) is the radius of the reflector.

The axial phase variation consists of two parts: (1) a purely linear phase change \((kz')\) of \(2\pi\) radians that occurs whenever the distance from the reflector is increased by a wavelength and (2) a phase change \((k_0z')\) which depends on the reflector shape (i.e., \(f/D\) ratio). In optics the quantity of interest is the "phase anomaly" which measures the deviation of the true total phase from that of a purely spherical wave converging to the focus in the half-space \(z' < 0\) and diverging from it in the half-space \(z' > 0\) [4, sect. 8.8]. We have

\[
\delta(\text{anomaly}) = \phi(\text{true phase}) - \tilde{\phi}(\text{spherical wave}) = k_0z' \quad \text{(177)}
\]

The amplitude variation and the phase anomaly along the \(z\) axis are shown by the curves labelled "approximate analysis" in Figs. 29 and 30. Discontinuities of \(\pi\) radians occur whenever \(\sin(k_0z')/k_0z'\) changes sign. Note that there is an additional phase shift of \(\pi\) radians as we pass through the focus. The expression for \(E_x(p)\) has been normalized to unity at the focus and may be written (neglecting the term proportional to \(\lambda/\pi\) in Eq. (173)) as

\[
E_{xo}(p) = \frac{\sin k_0z'}{k_0z'} e^{-jk_0z'} = Z_{olyo}(p) \quad \text{(178)}
\]

where
Expressed in this form the field variation along the axis is independent of the dimensions of the reflector and depends only on the $f/D$ ratio. The axial wavelength $\lambda_a$ is given by

$$\lambda_a = \lambda (1 + \rho^2) = \lambda \sec^2(\theta_o/2)$$

(180)

where $\lambda$ is the free-space wavelength. The phase velocity is $c \sec^2(\theta_o/2)$ which exceeds the free-space velocity $c = (\mu_0 \epsilon_0)^{-1/2}$. The amplitude has zeroes whenever $k_o z' = \pm m$, i.e., whenever

$$z'/\lambda = \pm \frac{m}{2} \frac{1 + \rho^2}{\rho_0^2} = \pm \frac{m}{2} \csc^2(\theta_o/2)$$

(181)

where $m$ is an integer, $m > 0$. The zeroes for paraboloids of small $f/D$ ratio are much more closely spaced than those for large $f/D$ ratio. For example, with $f/D = 0.25$ the zeroes occur at $z'/\lambda = \pm m$, while with $f/D = 2.00$ they occur at $z'/\lambda = 32.5 \text{ m}$. 

The complete analysis of section 2.1 gives further insight into the field variation along the reflector axis. For axial incidence ($\gamma = \eta = 0$) and for displacements along the axis ($\rho' = 0$) we may carry out the $\phi$ integration in Eq. (72) directly and the expression for the field along the axis (neglecting the line integral contribution) becomes:

$$E_{xo}(r) = \frac{2(1 + \rho^2)}{\rho_0^2} \left(\frac{f}{\lambda}\right) e^{-j k z'} \int \left[1 - \frac{\rho^2 (f/\lambda)}{(\tau/\lambda)}\right] \exp\left[-j (2\pi/\lambda)(r+d-z')\right] \frac{\rho^*}{(\tau/\lambda)} \, d\phi^*$$

(182)
Fig. 29 Comparison of the variation along the axis ($\theta' = 0$) of the amplitude (upper) and the phase anomaly (lower) of $E_x$ as computed by the approximate analysis (Eq. (178)) and by the complete analysis (Eq. (182), $D = 64.0 \lambda$, $f/n = 0.25$ and 0.50).
Fig. 30 Comparison of the variation along the axis ($\phi' = 0$) of the amplitude (upper) and the phase anomaly (lower) of $E_x$ as computed by the approximate analysis (Eq. (178)) and by the complete analysis (Eq. (182), $D = 64.0\lambda$, $f/D = 1.00$ and 2.00).
where

\[ \frac{r}{\lambda} = \left( \frac{f}{\lambda} \right)^2 (1 + \rho \ast^2) + 2 \left( \frac{f}{\lambda} \right) \left( \frac{z'}{\lambda} \right) (1 - \rho \ast^2) + \left( \frac{z'}{\lambda} \right)^2 \]  \tag{183}

and

\[ \frac{d}{\lambda} = - \left( \frac{f}{\lambda} \right) (1 + \rho \ast^2) \]  \tag{184}

The field has been normalized to unity at the focus by the factor \( E_{\infty}(F) = E_{\infty}(F)/(1 + \rho \ast^2) \), and a factor \( \exp(jkz' - jkz) \) has been included so that the phase anomaly may be conveniently computed. The field variation is now no longer independent of the focal length, and the integration must be carried out for each focal length of interest.

The amplitude variation and the phase anomaly have been computed for a reflector with a diameter of 64 wavelengths and \( |U_0| \leq 10 \). These curves are shown in Fig. 29 for \( f/D \) ratios of 0.25 and 0.50, while the curves for \( f/D \) ratios of 1.00 and 2.00 are presented in Fig. 30. The limits of the linear approximation (both amplitude and phase limits) as obtained from Eqs. (112) and (113) and expressed in \( U_0 \) units are shown by the vertical lines. The results obtained by the linear approximation are acceptable within the indicated limits; however, as \( |U_0| \) increases beyond these limits, the deviations become larger and larger. There is still a phase shift of \( \pi \) radians in the phase anomaly on passing through the focus.

The field distribution is not symmetrical about the focal plane and the minima are displaced from the positions predicted by the linear analysis. We find a large number of closely-spaced nulls as the reflector is approached, while as we move from the focus away from the reflector the positions of the nulls disperse. This effect is most evident in the curve
for $f/D = 2.00$ in Fig. 30. For large $f/D$ ratios the field maximum does not occur at the focus but occurs at a position slightly closer to the reflector. In particular, for $f/D = 2.00$ the maximum value is $+0.08$ dB and occurs at $U_o = -0.25$, which corresponds to $z'/\lambda = -2.59$. The results obtained for the amplitude variation along the axis by using the complete analysis and presented in Figs. 29 and 30 are in good agreement with measurements. The curve for $f/D = 0.25$ with $D = 64.0 \lambda$ is entirely compatible with the measured values presented by Landry and Chassé [21, Fig. 3] for a reflector $f/D = 0.35$ with $D = 69.68 \lambda$. Bachynski and Bekafi [45] have carried out measurements on a microwave lens with $f/D = 2.00$ and $D = 40.0 \lambda$ at $\lambda = 1.25$ cm; their results show all the features of the amplitude curve for $f/D = 2.00$ in Fig. 30, including the shift in the position of maximum intensity from the focus to a point slightly closer to the lens. This shift in the position of maximum intensity relative to the focal plane is not generally observed at optical frequencies because of the large aperture diameters (in wavelengths) commonly encountered there.

At first sight it appears that the linear approximation gives very poor results for large $f/D$ ratios. It should be borne in mind, however, that the exact analysis predicts the first null on the reflector side of the focus at $U_o = -2.5$ corresponding to $z'/\lambda = -25.92$ (for $f/D = 2.00$ and $D = 64.0 \lambda$) which is about a quarter of the distance from the focus to the reflector; the linear analysis is not valid for such large displacements from the focus. For a reflector of similar diameter but with $f/D = 0.25$ the first null occurs at $z'/\lambda = -1.0$ which is only a small
fraction of the focal length \( f = 16.0 \lambda \). The above discussion emphasizes the fact that one should not apply the results of the approximate analysis to a reflector of arbitrary dimensions without first checking the range of validity of the approximations.

The results in section 3.2 for the field variation along the \( x \) and \( y \) axes and for the variation along the \( z \) axis presented above are combined in the next section to give an overall picture of the field variation in the \( x-z \) and \( y-z \) planes.

3.4 Field variation in the principal planes

To complete the investigation of the field in the focal region, we examine the field in the principal planes \( \phi' = 0 \) (the \( x-z \) plane) and \( \phi' = \pi/2 \) (the \( y-z \) plane). For axial incidence the \( x-z \) plane contains the electric field incident on the reflector and the \( y-z \) plane contains the incident magnetic field. Equations (159) and (160) show that the \( E_y \) and \( H_x \) components are always zero for \( \phi' = 0 \), \( \pi/2 \); while the \( E_z \) and \( H_z \) components are zero for \( \phi' = \pi/2 \) and for \( \phi' = 0 \), respectively. The transverse field has only an \( E_x \) component in the principal planes; Eq. (159a) for \( E_x \) has been evaluated and the results are presented in the form of contour maps. Figures 31 and 32 show equal-amplitude contours of \( E_x \) plotted in the normalized coordinates \((U_0, V_0)\) for various \( f/D \) ratios; field maxima and nulls along the axes are indicated by crosses and dots respectively. The results presented here also represent \( Z_0H_y \) but rotated through 90 degrees in \( \phi' \).

For large \( f/D \) ratios (see Fig. 31) the field patterns are almost identical in the \( \phi' = 0 \) and \( \phi' = \pi/2 \) planes. This result is in agreement with
Fig. 31 Equal-amplitude contours of the transverse field in the principal planes for $f/d = 1.00$ and 2.00 (contour interval is 0.05); * nulls, x maxima.
Fig. 32 Equal-amplitude contours of the transverse field in the principal planes for \( f/D = 0.25 \) and 0.50 (contour interval is 0.05); • nulls, x maxima.
that obtained by scalar diffraction theory in classical optics, in particular, the contours for \( f/D = 2.00 \) closely resemble those computed for a converging spherical wave diffracted at a circular aperture (see [4, sect. 8.8]). When the figure for \( f/D = 2.00 \) is rotated about the \( U_0 \) axis the maxima and minima on the \( V_0 \) axis generate the bright and dark Airy rings (cf. Fig. 23). As may be expected from the results obtained in section 3.2 for small \( f/D \) ratios, the rotational symmetry about the \( U_0 \) axis is lost for small \( f/D \) ratios. This is particularly evident for \( f/D = 0.25 \) where the differences between the \( \phi' = 0 \) and \( \phi' = \pi/2 \) planes are most noticeable.

This concludes our discussion of the fields produced in the focal region of a paraboloid illuminated by an axially incident plane wave.

We now turn our attention to the interesting problem of investigating the direction of the energy flow in the focal region.

### 3.5 Energy flow in the focal region

In order to investigate the energy flow in the region of the maximum scattered field, it is of interest to look at the time-averaged Poynting vector \( \langle \vec{P} \rangle \) given by

\[
\langle \vec{P} \rangle = \frac{1}{2} \Re (\vec{E} \times \vec{H}^*)
\]  

(185)

The average rate of energy flow in a given direction is given by the component of the time-averaged Poynting vector in that direction. A component of an arbitrary vector \( \vec{A} \) (representing \( \vec{E} \) or \( \vec{H} \)) may be expressed in terms of real and imaginary parts as \( A_a = A_{ar} + jA_{ai} \) where "a" may represent \((r, \phi, z)\) or \((x, y, z)\) components and the subscript \( r \) stands for...
the real part of \( \lambda_{a} \) and \( i \) for the imaginary part \((\lambda_{a1} \text{ and } \lambda_{a2} \text{ are both real quantities})\). Employing this separation into real and imaginary parts for \( \overline{E} \) and \( \overline{H} \) and the definition of \( \langle \overline{P} \rangle \), we may obtain general formulas for the \( \rho \), \( \phi \) and \( z \) components of \( \langle \overline{P} \rangle \) as follows:

\[
\langle P_{\rho} \rangle = \frac{1}{2} \left( E_{zr} H_{\rho r} + E_{z1} H_{\rho i} - H_{zr} E_{\rho r} - H_{z1} E_{\rho i} \right) \quad (186a)
\]

\[
\langle P_{\phi} \rangle = \frac{1}{2} \left( E_{zr} H_{\rho r} + E_{z1} H_{\rho i} - H_{zr} E_{\rho r} - H_{z1} E_{\rho i} \right) \quad (186b)
\]

\[
\langle P_{z} \rangle = \frac{1}{2} \left( E_{zr} H_{\rho r} + E_{z1} H_{\rho i} - H_{zr} E_{\rho r} - H_{z1} E_{\rho i} \right) \quad (186c)
\]

For the case of axial incidence these expressions can be written quite simply in terms of the \( I_{n} \). In general let \( I_{n} = I_{nr} + j I_{ni} \), where \( I_{nr} \) and \( I_{ni} \) are both real. Now, using Eqs. (164) and (165) for \( \overline{E} \) and \( \overline{H} \), we readily obtain the \( \rho \), \( \phi \) and \( z \) components of the Poynting vector, thus

\[
\langle P_{\rho} \rangle = \frac{|E_{x}(P)|^2}{Z_{o}} \left( I_{or} - I_{2r} \right) I_{11} - \left( I_{01} - I_{21} \right) I_{1r} \quad (187a)
\]

\[
\langle P_{\phi} \rangle = 0 \quad (187b)
\]

\[
\langle P_{z} \rangle = \frac{1}{2} \frac{|E_{x}(P)|^2}{Z_{o}} \left( (I_{or}^2 + I_{01}^2) - (I_{2r}^2 + I_{21}^2) \right) \quad (187c)
\]

Since the components are independent of the azimuthal angle \( \phi' \), the energy flow is symmetrical about the axis; further \( \langle \overline{P} \rangle \) has no \( \phi \) component so that the energy flow is directed parallel to the vector resultant of \( \langle P_{\rho} \rangle \hat{r} + \langle P_{z} \rangle \hat{z} \). In Eq. (187a), \( \text{Im} \) denotes the imaginary part.
In the focal plane \( z' = 0 \) and the \( I_n \) are purely real (see Eq. (149)), hence the energy flow is normal to the focal plane and is given by

\[
\langle P_z \rangle = \frac{\frac{1}{2} |E_r(P)|^2}{Z_0} (I_0^2 - I_2^2)
\]  

(188)

with \( \langle P_p \rangle = \langle P_\phi \rangle = 0 \). For convenience we may examine the quantity

\[
\langle P_z \rangle_0 = (1 + \rho_0^2)^2 (I_0^2 - I_2^2)
\]

(189)

which has been normalized to unity at the focus. We see that

\[
\langle P_z \rangle_0 > 0 \text{ if } |I_0(V_0,0)| > |I_2(V_0,0)|
\]

and

\[
\langle P_z \rangle_0 < 0 \text{ if } |I_0(V_0,0)| < |I_2(V_0,0)|
\]

which imply that there are certain regions in the focal plane where the energy flow is directed back towards the reflector (in the negative \( z \) direction).

Referring to Eqs. (164) we see that \( E_\phi = 0 \) when \( I_0 = I_2 \); and \( E_p = 0 \) when \( I_0 = -I_2 \). The radial dependence of \( |\langle P_z \rangle_0| \) is plotted as a function of \( V_0 \) (the normalized \( \rho' \) coordinate) for various \( f/D \) values in Figs. 33 and 34 where + and − in the lobes indicate energy flow away from and toward the reflector across the focal plane. For small \( f/D \) ratios the simple structure of alternate bright and dark rings (the Airy rings) predicted by Eqs. (162) is replaced by a more complex behavior caused by the presence of the \( I_2 \) term in the field expressions. Careful examination of Figs. 33 and 34 reveals that the position of the nulls corresponding to \( E_p = 0 \) remain fixed (in \( V_0 \) units) at the points where the classical
Fig. 33 Absolute magnitude of the Poynting vector in the focal plane 
\( z' = 0 \) as a function of \( V_0 \) for \( f/D = 1.00 \) and \( 2.00 \).
Fig. 34 Absolute magnitude of the Poynting vector in the focal plane ($z' = 0$) as a function of $V_o$ for $f/D = 0.25$ and 0.50.
Airy distribution has its zeroes; as the f/D ratio decreases from 2.00 to 0.25 a second null corresponding to \( E_\phi = 0 \) appears to the left of the fixed null and separates further and further from it. The reversal in the direction of energy flow occurs between these two nulls and can have a profound effect on the efficiency of aperture-type feeds in the focal region of a paraboloid of small f/D as discussed in Chapter V.

To obtain a more complete picture of the energy flow in the focal region and particularly in the region of the first null in the focal plane, the \( <P_z> \) and \( <P_\rho> \) components of the Poynting vector (Eqs. (187)) were used to obtain the direction of energy flow across the focal plane for various f/D ratios as shown in Figs. 35 and 36. It should be emphasized that the lines in the figures represent the direction of energy flow (indicated by arrows) and bear no relation to contours of equal energy density. For f/D = 2.00 the energy flow is almost entirely parallel to the axis of the reflector but the energy flow bypasses the null of the Airy pattern. This result is consistent with that of Farnell [46] who used scalar diffraction theory to compute the phase in the image space of a microwave lens with f/D = 2.1. The direction of the Poynting vector was found by assuming that it is normal to the phase front at each point in the field. His results further show that the energy flow bypasses all the zero rings in the Airy pattern as well as the nulls on the axis.

As the f/D ratio decreases the energy flow not only bypasses the null at \( E_\rho = 0 \) but there is an additional effect in that the energy circulates around the null at \( E_\phi = 0 \); this is most noticeable for f/D = 0.25.
Fig. 35 Direction of energy flow in the focal region for $f/D = 1.00$ and $2.00$. 
Fig. 36 Direction of energy flow in the focal region for $f/n = 0.25$ and $0.50$. 

$V_0 = k\rho \sin \theta_0$
Minnett and Thomas [12] have computed the lines of energy flow for the paraboloid of the Parkes radio telescope ($f/D = 0.4$); there is good agreement between their result and that obtained for $f/D = 0.50$ in Fig. 36. The results in Figs. 35 and 36 are very useful since they show the progressive change in the nature of the energy flow in the focal region as the $f/D$ ratio changes from large to small values.

In section 3.3 we found that the phase velocity along the reflector axis is $[c \sec^2(\theta_o/2)]$ which exceeds the free-space velocity $c$; it is of interest to investigate the velocity of energy propagation. The energy velocity $\overline{v_E}$ is defined by [26, sect. 1.6] and [47]:

$$\overline{v_E} = \frac{<P>}{<\Delta u>}$$

where $<P>$ is the time-averaged Poynting vector and $<\Delta u>$ is the time-averaged internal energy density. For a non-dispersive medium (i.e., $\mu$ and $\varepsilon$ are independent of the angular frequency $\omega$), such as free space in the present case, $<\Delta u>$ is simply the sum of the time-averaged magnetic and electric energy densities [47], thus

$$<\Delta u> = <w_m> + <w_e>$$

where

$$<w_m> = \frac{1}{4} \mu_0 \text{Re}(\overline{H} \cdot \overline{H^*})$$

and

$$<w_e> = \frac{1}{4} \varepsilon_0 \text{Re}(\overline{E} \cdot \overline{E^*})$$
For a dispersive medium an additional term, referred to as the "dispersion energy density", must be included on the right-hand side of Eq. (191).

From Eqs. (192) and (193), and using Eqs. (164) and (165) for $\mathbf{E}$ and $\mathbf{H}$, we find the magnetic and electric energy densities as follows:

\[ <w_m> = \frac{1}{4} \mu_0 \frac{|E_x(F)|^2}{Z_0^2} (|I_0|^2 + |I_2|^2 - 2 \cos 2\phi' \text{Re}(I_0 \cdot I_2^*) + 4|I_1|^2 \sin^2 \phi') \]

and

\[ <w_e> = \frac{1}{4} \epsilon_0 \frac{|E_x(F)|^2}{Z_0^2} (|I_0|^2 + |I_2|^2 + 2 \cos 2\phi' \text{Re}(I_0 \cdot I_2^*) + 4|I_1|^2 \cos^2 \phi') \]

where the $I_n$ are given by Eq. (145). The direction of $\mathbf{v}_E$ is given by the direction of $<\mathbf{P}>$ while the magnitude is given by the ratio of $|<\mathbf{P}>|$ and $<\Delta u>$. The magnitude of the time-averaged Poynting vector is

\[ |<\mathbf{P}>| = \frac{1}{2} \frac{|E_x(F)|^2}{Z_0^2} \left[ (|I_0|^2 - |I_2|^2)^2 + 4(\text{Im}(I_2 - I_0 I_1^*))^2 \right]^{1/2} \]

Thus

\[ |\mathbf{v}_E| = \frac{|<\mathbf{P}>|}{<\Delta u>} = c \frac{\left[ (|I_0|^2 - |I_2|^2)^2 + 4(\text{Im}(I_2 - I_0 I_1^*))^2 \right]^{1/2}}{(|I_0|^2 + 4|I_1|^2 + |I_2|^2)} \]

where we have used $Z_0 = \sqrt{\mu_0/\epsilon_0}$ and $c = 1/\sqrt{\mu_0 \epsilon_0}$. 
Along the reflector axis $p' = 0$ and $I_0$ is the only non-vanishing member of the $I_n$ (see Eq. (145)); Eq. (197) reduces to

$$|\bar{\nu}_E| = c$$

(198)

This means that the velocity of energy propagation is equal to the free-space velocity for all points along the reflector axis and is directed parallel to this axis. Away from the axis the behavior of the energy velocity is far more complicated as shown by Eq. (197); the direction is as shown in Figs. 35 and 36. We see that although the phase velocity ($c \sec^2(\theta_0/2)$) exceeds the free-space velocity $c$, the velocity of energy propagation $v_E$ is properly bounded above since it does not exceed $c$. For large $f/D$ ratios we found that the amplitudes of $I_1$ and $I_2$ are much smaller than $I_0$ (see Eqs. (155) and Fig. 15); neglecting $I_1$ and $I_2$ with respect to $I_0$, we obtain

$$|\bar{\nu}_E| = c$$

(199)

everywhere in the focal region. In addition, $\bar{\nu}_E$ is parallel to the $z$ axis. Referring to Fig. 35 for $f/D = 2.00$ we see that the direction of energy propagation is indeed parallel to the $z$ axis except in the region where $I_0$ has a null (near $V_0 = 3.8$); here the effect of $I_1$ and $I_2$ is most pronounced and they give rise to a non-zero $\rho$ component of $\bar{\nu}_E$.

To conclude this chapter we review again some of the results obtained by vector diffraction theory in references [12] and [17] which both considered axial incidence by a plane wave. Minnett and Thomas [12] investigated the field configuration in the focal plane of the Parkes radio
telescope (f/D = 0.4). They obtained results similar to those in Figs. 21 and 25 (f/D = 0.5) for the amplitude and the direction of the transverse component of the focal-plane field, respectively. Their investigation revealed a reversal of energy flow across the focal plane similar to that in Fig. 36 for f/D = 0.5. In addition, they considered the efficiency of optimum aperture-type feeds in the focal plane; this topic is discussed in detail in Chapter V. Gniss and Ries [17] examined reflectors with f/D = 0.25, 0.4 and 1.0 presenting curves similar to those in Figs. 16 through 19 for the E_x, E_y and E_z components. Since the standard optical coordinates (Eqs. (158) and (176)) were used, rather than the generalized dimensionless coordinates introduced in Eqs. (157) and (175), their results are coordinate-dependent in the sense that similar ranges of independent variable for different f/D ratios do not cover the same range of field pattern in the image space as is the case when the generalized coordinates are used. The present study shows the field configuration and energy flow for four different f/D ratios and allows the characteristics for other f/D ratios to be inferred. The cross-polarization factor defined by Eq. (167) permits direct comparison of the relative magnitudes of E_x and E_y at each point in the focal plane.

Neither of the studies mentioned in the above paragraph is able to investigate off-axis incidence. The equations derived in Chapter II are sufficiently general to enable us to consider paraboloidal reflectors illuminated by an off-axis plane wave; this interesting topic is taken up in the next chapter.
CHAPTER IV

THE FOCAL-REGION FIELDS FOR OFF-AXIS INCIDENCE BY A PLANE WAVE

This chapter investigates the field configuration produced in the focal region of a paraboloidal reflector illuminated by an off-axis plane wave. Initially, the principal component \( E_x \) of the focal field is evaluated by means of the linear analysis (Eq.(135a)) as discussed in section 2.3 and the field values so obtained are compared directly to those found from the complete analysis involving double integration (Eq.(72)). The double integration scheme is at least an order of magnitude more expensive, in terms of computer time, than the single integration scheme; thus the results are compared only for a limited range of parameters. Recall that the computation time for the double numerical integration scheme increases with the square of the aperture diameter while that of the single numerical integration scheme increases with the aperture diameter. The investigation is sufficiently detailed, however, to show that the approximate formulas involving single integration yield results in excellent agreement with those obtained by double integration but at considerably reduced cost. The great advantage of the single integration lies in this cost reduction; for very large reflectors computation of the scattered field by means of double integration may be completely prohibited by excessive computer expenses.
To facilitate the discussion we define two principal planes of scan with reference to the direction of the electric field vector $\vec{E}_1$ of a plane wave incident along the reflector axis (the scan angles $\gamma$ and $\eta$ are both zero here) as follows: (1) the E-plane is that plane containing the incident electric field vector $\vec{E}_1$ and the reflector axis, and (2) the H-plane is that plane containing the incident magnetic field vector $\vec{H}_1$ (this is normal to $\vec{E}_1$ and the direction of propagation $k$) and the reflector axis. Reference to Fig. 7 reveals that the E-plane is the x-z plane while the H-plane is the y-z plane. For a scan in the E-plane the propagation vector of the incident plane wave lies in the x-z plane, thus the scan angle $\gamma \neq 0$ while $\eta = 0$. The propagation vector lies in the y-z plane for an H-plane scan and the scan angles are $\gamma = 0$, $\eta \neq 0$. A third plane, the "transverse plane", is normal to both the E- and H-planes, i.e., it is parallel to the x-y plane but it need not pass through the origin (the reflector focus). The x and y axes are referred to as the "principal scan axes".

As for the case of axial incidence, field patterns are computed for $f/D$ ratios of 0.25, 0.5, 1.0 and 2.0 and the results are again presented in the form of graphs and contour maps. Since there are a large number of parameters which influence the field patterns (the $f/D$ ratio, the scan angles $\gamma$ and $\eta$, the reflector diameter, etc.) we are faced with the prospect of presenting an excessively large number of figures. The patterns presented are representative of the results obtained and are useful for inferring pattern characteristics for a wide range of parameters.
4.1 *Scan-plane fields along the principal axes in the focal region of a paraboloid illuminated by an off-axis plane wave*

In this section we examine how the position of the maximum field varies along the principal axes in the plane of scan as the incident plane wave changes orientation relative to the reflector axis. Take, for example, the case of an H-plane scan ($\gamma = 0$, $n \neq 0$). The incident plane wave arrives at an angle $\eta$ to the left of the reflector axis (looking along the axis away from the reflector, see Fig. 37); the maximum of the received field therefore moves to the right of the axis.

![Diagram](image)

**Fig. 37** Geometry for an H-plane scan.

A first approximation to the position of the maximum field along the $y$ axis may be found by considering reflection from a plane mirror rather
than the paraboloid, the maximum position is then predicted as
\((f/\lambda) \tan \eta\), where \(f\) is the focal length. This procedure predicts the
position of the maximum along the y axis reasonably well for large \(f/D\)
ratios; for small \(f/D\) ratios, however, the maximum occurs at a position
somewhat farther from the axis than \((f/\lambda) \tan \eta\). This topic is taken up
again in the discussion of the "beam deviation factor" (BDF) which is
a measure of the difference between the true position of the maximum and
that given by \((f/\lambda) \tan \eta\).

It is clear that a necessary first step in determining the positions
of the scanned maxima in the plane of scan is to establish the lateral
position. Since \(f\) varies by a factor of 8 for \(f/D\) ratios between 0.25
and 2.0 (for fixed diameter \(D\)) the position of the scanned maximum for
the same off-axis angle (\(\eta\), say) will vary approximately by this factor
over the above range of \(f/D\). This requires extensive scale changes to
make suitable plots and it proves advantageous to use coordinates re-
lated to the dimensionless parameter \(V_0\) introduced in section 3.2,
Eq. (157). We define

\[
X_0 = \frac{2kx' \rho_0}{1 + \rho_0^2} = kx' \sin \theta_0 \tag{200}
\]

\[
Y_0 = \frac{2ky' \rho_0}{1 + \rho_0^2} = ky' \sin \theta_0 \tag{201}
\]

\[
Z_0 = \frac{2kz' \rho_0}{1 + \rho_0^2} = kz' \sin \theta_0 \tag{202}
\]
where \( V_0 = \sqrt{X_0^2 + Y_0^2} \). The \( X_0, Y_0, Z_0 \) parameters are also referred to as "normalized coordinates". For later reference we tabulate the half-power beamwidths (HPBW's) of the principal component of the focal region field produced by an on-axis plane wave in the x-y plane. Table 3 shows the HPBW's along the principal x and y axes in terms of both the normalized coordinates \( (X_0, Y_0) \) and in wavelengths as obtained from Figs. 16 through 19.

**TABLE 3**

HALF-POWER BEAMWIDTHS OF \( E_x \) IN THE FOCAL PLANE FOR AXIAL INCIDENCE AS MEASURED ALONG THE x AND y AXES

<table>
<thead>
<tr>
<th>f/D</th>
<th>HPBW in normalized units</th>
<th>HPBW in wavelengths</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( X_0 )</td>
<td>( Y_0 )</td>
</tr>
<tr>
<td>0.25</td>
<td>3.20</td>
<td>2.46</td>
</tr>
<tr>
<td>0.50</td>
<td>3.20</td>
<td>2.96</td>
</tr>
<tr>
<td>1.00</td>
<td>3.20</td>
<td>3.20</td>
</tr>
<tr>
<td>2.00</td>
<td>3.20</td>
<td>3.20</td>
</tr>
</tbody>
</table>

We note that the HPBW's along the \( X_0 \) axis are constant independent of \( f/D \) ratio, while along the \( Y_0 \) axis the HPBW's for small \( f/D \) ratio are less than those along the \( X_0 \) axis but they have the same values for
large $f/D$ ratio. These nearly constant HPBW's along the $X_o$ and $Y_o$ axes are a very desirable feature of the normalized coordinates.

The principal component $E_X$ has been found from Eq. (135a) for $f/D = 0.25, 0.5, 1.00$ and $2.0$ with $D = 128\lambda$ and the results for line scans along the $Y_o$ axis ($X_o = Z_o = 0$) are shown in Figs. 38 to 41 for several values of the angle $\eta$ ($\gamma = 0$). The angles $\eta$ were chosen to correspond roughly to $0$, $5$ and $10$ HPBW's of scan along the $Y_o$ axis. The line integral contribution in Eq. (51) has been neglected in this, and in all subsequent analysis also, for the reasons discussed in section 3.2. The field has been normalized to unity at the focus by the factor $E_{X_0}(F) = E_X(F)/(1 + \rho_0^2)$. The results are presented in composite figures showing the phase of the $E_X$ component as well as $|E_X|$ in dB. The phase curves warrant a few words of explanation. The arctangent function used to find the phase gives principal values between $-\pi$ and $+\pi$ which means that when the phase increases to $180^\circ$, say, the value given by the arctangent function jumps down to $-180^\circ$ rather than increasing on towards $360^\circ$. Generally then, at the $\pm 180^\circ$ points where the phase curves show discontinuous jumps an integral number of $\pm 2\pi$ radians may be added to the curves. Except for the case of axial incidence the apparent phase jumps are not real and the phase increases or decreases smoothly. For comparison the results obtained by the exact analysis Eq. (72) are shown by the dashed curves in Figs. 38 and 40 with the angle $\eta$ corresponding to the maximum scan angle. At the intermediate scan angle and at $\eta = 0^\circ$ the amplitude and phase curves are almost identical for the linear analysis of Eq. (135a) and the exact analysis and
Fig. 38 Phase and amplitude scanned patterns of $E_x$ in the H-plane as a function of $y_o$ for various angles of incidence $\eta (\gamma = 0, f/d = 0.25)$. 
Fig. 39 Phase and amplitude scanned patterns of $E_x$ in the H-plane as a function of $Y_o$ for various angles of incidence $\eta(\gamma=0, f/D = 0.5)$.
Fig. 40 Phase and amplitude scanned patterns of $E_x$ in the H-plane as a function of $\eta_0$ for various angles of incidence $\eta(\gamma=0, f/D = 1.0)$. 
Fig. 41 Phase and amplitude scanned patterns of $E_x$ in the H-plane as a function of $y_0$ for various angles of incidence $\eta(\gamma=0, f/D = 2.0)$. 
are therefore not shown. The results obtained by the two methods show fair agreement in the $|E_x|$ curves out to about 10 HPBW's; large phase deviations are apparent, however, at this scan position. The phase curves do show similar variations and appear to be displaced by some constant amount.

Referring to the amplitude curves we note that the peak value of the field drops with scan, the beam broadens, the sidelobe on the far side of the peak from the axis (the coma lobe) increases; whereas the first sidelobe on the axis side decreases, changes sign, and merges with the main beam and second sidelobe causing additional beam broadening. This effect may be seen quite clearly in Fig. 39 for $\eta = 2.5$. Curves showing the scan loss and beamwidth increase for off-axis incidence will be presented later.

Since the focal field for axial incidence does not have equal HPBW's in the $E$- and $H$-planes, scan curves along the $X_0$ axis ($Y_0 = Z_0 = 0$) were also generated for $\gamma \neq 0$ and $\eta = 0$. It turns out that the curves for $f/D = 1.0$ and 2.0 are identical to those for the scans along the $Y_0$ axis in Figs. 40 and 41 and they are not repeated. Amplitude and phase curves for $f/D = 0.25$ and 0.5 are shown in Figs. 42 and 43, the dashed curve in Fig. 42 again being obtained by the exact analysis of Eq. (72). The $X_0$ scans exhibit the same features as the $Y_0$ scans except that the main beams are broader (as expected) and the sidelobes on the axis side of the peaks are generally much smaller in the present case. This is most evident for $f/D = 0.25$ in Fig. 42 where the sidelobes on the axis side are below $-60$ dB at the largest scan angle.
Fig. 42  Phase and amplitude scanned patterns of $E_x$ in the $E$-plane as a function of $X_0$ for various angles of incidence $\gamma$($\eta = 0$, $f/D = 0.25$).
Fig. 43 Phase and amplitude scanned patterns of $E_x$ in the E-plane as a function of $X_0$ for various angles of incidence $\gamma(n=0, f/d = 0.5)$. 
To examine the frequency dependence of the above results, phase and amplitude contours were generated at half the frequency considered there. The diameter is now 64$\lambda$ rather than 128$\lambda$ which means that to achieve approximately the same lateral displacement of the peak the scan angles $\eta$ must be about twice their values for the $D = 128\lambda$ case (since the focal length is now $f/2$). The curves produced for $D = 64\lambda$ and achieving the same lateral displacement as for the $D = 128\lambda$ case were found to be almost identical both in amplitude and phase to those shown in Figs. 38 to 43. The small discrepancies at large scan angles are thought to be caused by the fact that unless $n$ is small ($f/2\tan(2n)$) $\neq f\tan n$. Plotted in terms of HPBW's scanned the curves in Figs. 38 to 43 may be considered "universal" and taken to apply to reflectors of any diameter (provided $D$ is big enough) and the scan angles $\eta$ and $\gamma$ are chosen to yield scan maxima corresponding to the positions in the above figures. For convenience a scale suitable for conversion to HPBW's scanned has been inserted in these figures.

4.2 The position of the maximum field in the principal planes of scan for off-axis incidence

This section investigates the position of the maximum field in the principal planes of scan as a function of the $f/D$ ratio and the angle of off-axis incidence. This position of maximum field may be considered to be the position of best focus for a particular angle of incidence. In classical optics the surface of best focus in an optical system in the absence of astigmatism is given by the Petzval surface [48, ch. 3]. The radius of curvature of the Petzval surface for a single mirror is one
half the radius of curvature of that mirror [48, ch. 3]. In the field of microwave antennas the Petzval surface for a paraboloidal reflector has been derived by Ruze [5] as being another paraboloid of focal length "f/2" (i.e., one half the focal length of the reflector) tangent to the focal plane at the focus and described by the equation

\[(x/\lambda)^2 + (y/\lambda)^2 = 2(f/\lambda)(z/\lambda)\]  

(203)

where the coordinates are as in Fig. 37. This equation may also be expressed in terms of the normalized coordinates as follows:

\[X_o^2 + Y_o^2 = 2F_oZ_o\]  

(204)

where

\[F_o = \frac{2k\kappa_o^2}{1 + \rho_o^2} = kf \sin \theta_o\]  

(205)

The positions of the scanned maxima along the Y_o and X_o axes in Figs. 38 to 43 give the lateral position of the maximum field quite closely and these positions may be taken as the centers of rectangular grids of points to be used to establish the axial (i.e., parallel to the z axis) positions of the maxima in the scan plane. Phase and amplitude contours are plotted for three angles of incidence η in Fig. 44 for a scan in the H-plane, the paraboloid parameters being \(f/D = 0.5\) and \(D = 128\lambda\). The principal component \(E_x\) of the focal field is perpendicular to the plane of scan. For comparison the contours obtained by the linear analysis of Eq. (135a) and the exact analysis of
Fig. 44 Comparison of the phase and amplitude contours in the focal region plane-of-scan as obtained by the linear and exact analyses.
Eq. (72) are shown in the same figure. The contours for the linear analysis and the exact analysis are very similar; the positions of the field maxima, however, do not correspond in axial position for each of the scan angles but the maxima do show good agreement in their lateral positions. The Petzval surface given by Eq. (204), with \( X_0 = 0 \), is shown superimposed on each set of contours. It comes close to but does not pass through the scan maxima.

When amplitude and phase contours were generated for \( f/D = 1.0 \), it was found that the exact and linear analyses gave similar contours but that the axial positions of the field maxima differed by amounts greater than those shown in Fig. 44. In particular, for the case of axial incidence \((n = \gamma = 0)\), the linear analysis gives the position of the field maximum at the focus while the exact analysis gives the position at \( Z_0 = -0.4 \), i.e., the maximum occurs at a point closer to the reflector than the focus. This effect has been discussed in section 3.3 where we examined the field variation along the reflector axis. At this stage it was decided to investigate the effect of including some of the higher order terms in the binomial expansion for \( r \) given by Eq. (104a).

To do this we use \( r \) given by Eq. (104a) in Eq. (72) (the exact expression involving double integration) rather than \( r \) given by Eq. (102) and examine the effect of including the higher order terms appearing in Eq. (103). It was found that by including the terms proportional to \( z'^2 \) in the phase expressions of Eq. (72), the position of the field maximum for axial incidence is shifted to the position predicted by the exact analysis. In addition by including the term \( z'\cos\theta \) in amplitude expressions
the peak values were also found to be identical. Finally, by including the term \( p^2/(2R) \) in phase expressions, the phase difference between the exact and the linear analyses as presented in Figs. 38 to 43 became less than about \( \pm 20^\circ \) at the largest values of scan shown there. It was found that the terms involving \( \cos(\phi-\phi') \) make small changes to the phase but these changes are not nearly as dramatic as obtained by the other modifications described above.

The observations described in the preceding paragraph may be used to develop a "modified approximate analysis" as follows: In phase expressions \( r \) is given by

\[
r = R + z' \cos\theta - \rho' \sin\theta \cos(\phi-\phi') + \frac{1}{2R}(p^2 + z'^2 \sin^2\theta)
\]

(206)

while in amplitude expressions

\[
r = R + z' \cos\theta
\]

(207)

When these expressions are used in Eqs. (72) and (75) to (79) the \( \phi \) integration may still be carried out in closed form in a manner analogous to that described in section 2.32. The phase expressions \( \Psi \) and \( \Psi_0 \) are then modified by the addition of the term \( (2\pi/\lambda)(p^2 + z'^2 \sin^2\theta)/ (2R) \) as obtained from Eq. (206). There is no reason, a priori, for including these small "correction" terms in the linear analysis other than that by doing so, results corresponding more closely to those of the exact analysis may be achieved. Since the phase factor is now no
longer linear in \( o' \) and \( z' \) we will refer to the analysis involving Eqs. (206) and (207) as the "modified approximate analysis". It should be noted that the fields are still computed by single numerical integration and all the advantages of the linear analysis are retained.

The modified approximate analysis as described above was used to generate scan contours in the \( H \)-plane for the same parameters appearing in Fig. 44. The contours are almost identical to those for the exact analysis and the positions of the maxima for \( \eta = 2^o \) and \( 4^o \) correspond. Before presenting scan-plane contours for the remaining \( f/D \) ratios, some further comments on Fig. 44 are in order. The equiphase contours are separated by 180 electrical degrees and have been shifted by a constant amount for each angle of incidence so that the phase at the position of maximum field is zero. The phase contour through this maximum position is approximately parallel to the \( y \)-axis at each angle of incidence. For axial incidence (\( \eta = 0 \)) the equiphase contours corresponding to a phase change of \( 2\pi \) are separated by a distance of about 6.3 normalized units and this represents the wavelength of the scattered field along the axis. To convert \( Z_o = 6.3 \) to wavelengths we use Eq. (202) with \( 2\pi \sin \theta_o = 5.027 \) which gives a value of the wavelength of axial propagation as \( \lambda_a = 1.253\lambda \), where \( \lambda \) is the free-space wavelength. This value may be compared to that predicted for the axial wave by Eq. (180) in section 3.3. For \( f/D = 0.5 \), \( \theta_o = 53^\circ 13 \) so that \( \lambda_a = \lambda \sec^2(\theta_o/2) = 1.250\lambda \) and the results are in good agreement as they should be. For off-axis incidence the phase and amplitude contours disperse and the value of the field maximum decreases. At \( \eta = 2^o \) the maximum value is 0.93 (-0.63 dB) while at \( \eta = \)
4°, the maximum is 0.75 (-2.5 dB) compared to the value of unity at the focus for axial incidence.

Phase and amplitude contours were also generated for H-plane scans with \( f/D = 0.25, 1.0 \) and \( 2.0 \) (\( D = 128\lambda \)) by using the modified approximate analysis and the results for \( f/D = 0.25 \) and \( 1.0 \) are shown in Figs. 45 and 46. The contours for \( f/D = 2.0 \) are similar to those for \( f/D = 1.0 \) except that they are more nearly parallel to the z axis indicating that the field is very nearly constant over fairly large ranges of \( Z_0 \) for displacements parallel to the z axis. Note that for \( f/D = 1.0 \) the maximum occurs at \( Z_0 = -0.4 \) which is a distance of 0.14 wavelengths closer to the reflector than the position of the focus. Scans in the E-plane produce contours which are almost identical to those already presented for all the \( f/D \) ratios considered except for \( f/D = 0.25 \). The phase and amplitude contours for \( f/D = 0.25 \) are shown in Fig. 47 for the same parameters as in Fig. 45. The major difference between Figs. 45 and 47 for \( f/D = 0.25 \) lies in the difference between the E- and H-plane beamwidths as noted in Table 3, in the former case \( \eta = 3^\circ \) represents 5.41 HPBW's of scan while in the latter \( \gamma = -3^\circ \) represents 4.28 HPBW's of scan. By examining Eqs. (135) for a scan in the H-plane \((\phi' = \alpha = \pi/2, \text{here, and the scan angles are } \eta \neq 0, \gamma = 0)\) we observe that \( E_x \) is the only non-vanishing component of the electric field. This means that the electric vector is perpendicular to the plane of scan. For a scan in the E-plane \((\phi' = \alpha = 0, \text{here, and the scan angles are } \eta = 0, \gamma \neq 0)\) the electric field has both \( E_x \) and \( E_z \) components; the \( E_y \) component is still zero which implies that the electric vector lies in the plane of scan.
Region of a paraboloid away from reflector.

\[ y_0 = ky'y'\sin\theta_0 \]

H-plane scan,
\[ \frac{1}{D} = 0.25 \]
\[ D = 128\lambda \]

Phase and amplitude contours for an H-plane scan in the focal region of a paraboloid with \( f/D = 0.25 \).

\[ \eta = 0^\circ \]
\[ \eta = 3^\circ \]
\[ \eta = 6^\circ \]

Max. = 1.0

0.46

0.31
Fig. 46 Phase and amplitude contours for an H-plane scan in the focal region of a paraboloid with $f/D = 1.0$. 
Region of a paraboloid where $f/D = 0.25$.

Fig. 47. Phase and amplitude contours for an E-plane scan in the focal region of a paraboloid with $f/D = 0.25$.

$Z = k z \sin \theta$

$X = k x \sin \theta$

E-PLANE SCAN

$\frac{f}{D} = 0.25$

D = 128$

AWAY FROM REFLECTOR

PHASE
Plots of the scan maxima, without the corresponding amplitude and phase contours, are shown in Fig. 48 for $f/D = 0.25, 0.5, 1.0$ and $2.0$ for a scan in the E-plane. The diameter is still $128\lambda$ and the appropriate Petzval surfaces for the various focal lengths are also plotted. The positions of the scan maxima were found by both the modified approximate analysis and the exact analysis and the results are shown together for comparison. The agreement is quite good, particularly as regards the displaced position of the maximum from the focus towards the reflector for $f/D = 1.0$ and $2.0$. The plots of the scan maxima for an H-plane scan are similar to those in Fig. 48 and are not repeated.

It was observed that when the frequency is halved (i.e., the diameter is halved to $D = 64\lambda$) the plots of the position of maximum field are similar to those in Fig. 48, provided the angle of incidence $\gamma$ is chosen to yield equal numbers of HPBW's scanned in either case. There is, however, one major difference. For the $D = 64\lambda$ case the position of the maximum field for $f/D = 1.0$ and $2.0$ displaces axially towards the reflector by an amount approximately double that in Fig. 48 for these $f/D$ ratios. If this observation can be extrapolated to other frequencies we may conclude that if the diameter is successively doubled from $D = 128\lambda$, say, the position of the maximum for $f/D = 1.0$ and $2.0$ will move closer to the focus by successive halving of the displacement for $D = 128\lambda$.

The contours shown in Fig. 44 are in good agreement with the results presented by Rusch and Ludwig [34] for a reflector with $f/D = 0.433$ and $D = 34.0\lambda$. Their analysis retains the phase in its exact form; they do
Fig. 48: Comparison of the maximum field locus obtained by approximate analysis and the exact analysis.

1. $f/D = 0.25$
2. $f/D = 0.5$
3. $f/D = 1.0$
4. $f/D = 2.0$

E-Plane Scan
$D = 128\lambda$

--- PETZVAL SURFACE

FOCUS

--- MODIFIED APPROXIMATE ANALYSIS
--- EXACT ANALYSIS

HPBW

0 1 2 3
not present their field expressions, but it is clear that the scattered field in the H-plane is found by double integration of the physical optics currents over the reflector surface. They also present plots of the maximum field locus for $f/D = 0.433$ and $0.604$ ($D = 34.0\lambda$) which are similar to those for $f/D = 0.5$ in Fig. 48. Since the diameter considered there is quite small ($D = 34.0\lambda$) the displaced position of the maximum field for axial incidence is evident at $f/D = 0.604$. Because of the limitations imposed by available computer time, their analysis concentrates almost entirely on an aperture diameter of $34.0\lambda$; although a limited study at twice the frequency ($D = 64.0\lambda$) was carried out. It is felt that the approximate analysis presented in this chapter is suitable for studying large apertures that are beyond the reach of the exact analysis because the former is not nearly as severely restricted by computer expenses as the latter. There are, of course, small differences between the results obtained by the exact and approximate analyses, but this represents the trade-off between being able to produce usable results at moderate expense and not being able to obtain results at all.

To conclude this section we look briefly at some of the features of off-axis incidence which may be deduced fairly simply from the results already presented. Consider first the beam deviation factor (BDP). In the transmit mode the BDP is defined as the ratio of the secondary pattern maximum to the angle of the displaced feed in the transverse plane and it depends upon the reflector $f/D$ ratio as well.
as the aperture illumination [50]. In general the BDF is less than unity and approaches unity for large f/D ratio. A definition consistent with that of the transmit mode may be stated for the receive mode as follows: the BDF is the ratio of the angle of incidence of the plane wave to the angle of the displaced maximum of the focal-region field, that is,

$$\text{BDF} = \frac{\text{angle of incidence}}{\text{angle of displaced maximum}} \quad (208)$$

Fig. 49 shows the BDF plotted as a function of the f/D ratio. The BDF is not independent of the number of HPBW's scanned by the maximum field position. The solid curve in Fig. 49 shows the BDF for displacement of the field maximum by 2 HPBW's; the vertical bars indicate the increase in BDF when the main-beam maximum is displaced by 8 HPBW's.

From Figs. 38 to 43 it is clear that the maximum value of the focal field decreases with off-axis incidence. When plotted against the number of HPBW's that the maximum is displaced from the focus (rather than the angle of incidence of the plane wave), the scan loss depends only on the f/D ratio and is almost independent of whether the scan is in the E- or H-plane. Figure 50 shows the scan loss in dB for various f/D ratios plotted against the number of HPBW's through which the maximum is displaced. For f/D = 1.0 and 2.0 the scan loss is negligible out to 10 HPBW's, while for f/D = 0.25 the scan loss is greater than 10 dB at 10 HPBW's of scan indicating that reflectors of very small f/D ratio are unsuitable for beam-scanning applications requiring large displacement of the field maximum.
Fig. 49 Beam deviation factor as a function of f/D for a paraboloid.
Fig. 50  Scan loss in dB as a function of the displaced position in HPBW's of the field maximum.
As the field maximum moves away from the reflector axis, the main beam broadens as described in section 4.1. Convenient measures of the beam-broadening are given by plotting the half-power and tenth-power beamwidths at the scanned position of the maximum against the HPBW for axial incidence (zero beamwidths scanned). Figure 51 shows the results for an E-plane scan. Note that the 3 dB and 10 dB beamwidths for axial incidence are independent of the f/D ratio being 3.2 and 5.4 normalized units, respectively. The curves for a scan in the H-plane are similar except that the 3 dB and 10 dB beamwidths for f/D = 0.25 and 0.5 for axial incidence are displaced downwards (as are the complete curves) to the points indicated by the dots and crosses for f/D = 0.25 and 0.5, respectively.

In the next section we examine the field configuration in the transverse plane parallel to the x-y plane but passing through the position of a scanned maximum in the plane of scan.

4.3 Field variation in transverse planes through the scan maxima

In section 3.2 we examined the field configuration produced in the focal plane (the transverse plane through the focus) by an axially incident plane wave. From the results already presented in this chapter (e.g., Figs. 44 to 47) it is clear that the field patterns produced in a transverse plane by an off-axis plane wave can differ markedly from those obtained in the on-axis case. It is the purpose of this section to present some typical patterns which illustrate the changes in the field configuration as the incident wave moves off axis. The scanned
Fig. 51 Half- and tenth-power beamwidths at the position of the scanned maximum.
patterns depend on the $f/D$ ratio and the angles of incidence $\theta$ and $\phi$, among other things. Since there are three field components ($E_x, E_y, E_z$) to be computed for each $f/D$ ratio at a particular scan angle the number of figures required to illustrate the results in detail is quite large and this makes for a cumbersome presentation. For the purpose of illustrating some typical results (rather than detailed results) we look at a reflector with $f/D = 0.5$ and $D = 128\lambda$ for a particular scan angle and compare the results with those obtained for axial incidence. In addition, we show the field configuration of the principal component $E_x$ at $f/D$ ratios of 0.25 and 1.0.

In the figures which follow, the $x$ and $y$ axes are expressed in terms of the normalized coordinates $X_0$ and $Y_0$ defined by Eqs. (200) and (201), similarly, the displacement of the maximum field parallel to the $z$ axis in the plane of scan is given in terms of $Z_0$ defined by Eq. (202). The normalized coordinates prove to be very useful since all patterns (independent of $f/D$) may be plotted on the same scales. If patterns are plotted on scales expressed in wavelengths each $f/D$ requires a new scale along the $x$ and $y$ axes to display similar portions of the field patterns. An example of this difference in scaling requirements for wavelength scales may be seen in Figs. 20 to 23 where the scales change by a factor of about 4 as the $f/D$ changes from 0.25 to 2.0 while the field patterns are shown out to the second sidelobe in each case. Figures 16 to 19 indicate that the normalized $V_0$ scale used there is the same for all $f/D$ while displaying similar regions of the field patterns.

Figure 52 shows the magnitude of the field distribution of the prin-
Fig. 52 Equal-amplitude contours of $E_x$ in the transverse plane through the scan maximum in the $H$-plane for off-axis incidence ($f/D = 0.5$, $\eta = 4^\circ$).
principal component $E_x$ in the transverse plane passing through the scan maximum at $z_0 = 0.2$ for an $H$-plane scan with the angle of incidence $\eta = 4^\circ$.

The position of the main-beam maximum has been scanned 8.72 HPBW's off axis. For comparison we show the configuration of $E_x$ produced in the focal plane by an axially incident plane wave (Fig. 53). The portion of the field distribution shown in Fig. 53 for $E_x$ is about the same as that shown in Fig. 21 for $E_z$ (cf. Eq. (166)). For axial incidence the pattern is symmetrical with respect to the $x$ and $y$ axes as described in section 3.2. The symmetry with respect to the $x$ axis is lost for off-axis incidence ($H$-plane scan); the pattern, however, retains the symmetry with respect to the $H$-plane and the complete pattern may be found by reflecting the portion shown in Fig. 52 about the $H$-plane. For off-axis incidence the contours of the axis side of the maximum disperse and distort while on the other side of the maximum the structure of the coma lobe in the transverse plane becomes evident. The phase of the $E_x$ component in Fig. 53 remains constant ($= 0$) in the main lobe (enclosed by the inner dashed ellipse) and changes to $\pi$ radians in the first sidelobe. In Fig. 52 the phase of $E_x$ does not vary by more than $\pi/8$ radians within the region enclosed by the 0.04 contour of the main lobe. Outside the 0.04 contour the phase changes continuously as the coma lobe (enclosed by the dashed curve) is approached; in the coma lobe the phase differs by about $\pi$ radians from that in the central portion around the field maximum.

Figures 54 and 55 show the field configuration of $E_x$ for $f/D = 0.25$ with $\eta = 3^\circ$, corresponding to 5.37 HPBW's of scan, and $f/D = 1.0$ with $\eta = 4^\circ$, corresponding to 8.59 HPBW's of scan, respectively. The on-axis
Fig. 53 Equal-amplitude contours of $E_x$ in the focal plane for axial incidence ($f/d = 0.5$).
Fig. 54 Equal-amplitude contours of $E_o$ in the transverse plane through the scan maximum in the H-plane for off-axis incidence ($f/D = 0.25$, $\eta = 3^\circ$).
counterparts of these figures are shown in Figs. 20 and 22; although Figs. 20 and 22 are slightly different from $E_x$, since they show $E_z$ (cf. Eq. (166)), they do give a qualitative picture of the on-axis field, particularly for $f/D = 1.0$ where $E_y$ is small. We observe that the coma lobe in Fig. 54 has a completely different character to those in Figs. 52 and 55. For $f/D = 0.25$ the coma lobe has split into two peaks and the coma-lobe maximum does not occur on the $y$ axis as it does for the other two $f/D$ ratios. This bifurcation of the coma lobe is typical and also occurs at sufficiently large angles of scan for $f/D = 0.5$. The phase in the main lobe is constant to better than $\pi/8$ within the 0.02 and 0.12 contours for $f/D = 0.25$ and 1.0, respectively. The field patterns presented here confirm the observation made earlier that the field deteriorates much more rapidly with small $f/D$ ratios than with large $f/D$ ratios as the field maximum is scanned off axis. Figures 52, 54 and 55 are similar to those presented in Born and Wolf [4, Figs. 9.6 and 9.7] for primary coma, in the absence of astigmatism, and were obtained for paraboloidal reflectors of large $f/D$ ratio. It is difficult to compare the figures quantitatively since the notations used in this dissertation and in reference [4] are not simply related and cannot be readily transformed one into the other. The figures in [4] do, however, show the same general features as those shown here.

For axial incidence $E_y$ is simply the function $I_2$ shown in Fig. 13 modified by a $\sin 2\phi'$ factor as given by Eq. (159b). Figure 56 shows equal-amplitude contours of $E_y$ for axial incidence on a reflector with $f/D = 0.5$. The shape of the $E_y$ contours for axial incidence is the same
Fig. 55 Equal-amplitude contours of $E_x$ in the transverse plane through the scan maximum in the $H$-plane for off-axis incidence ($f/D = 1.0$, $\eta = 4^\circ$).
independent of the f/D ratio; the maximum values and null positions depend on the f/D ratio since I₂ does. Equal-amplitude contours of E_y computed for the same conditions as in Fig. 52 are shown in Fig. 57. The contours for off-axis incidence distort and the reflection symmetry in the E-plane evident in Fig. 56 is lost. The two "lobes" to the left of the x axis in Fig. 56 reduce in amplitude while the "main lobe" to the right remains about the same and the minor lobe in the top right-hand corner increases in amplitude as the angle of incidence increases from zero. This redistribution of amplitude among the lobes of the E_y pattern is typical of the patterns for f/D = 0.25 and 1.0 also. The null regions in Fig. 56 are replaced by regions where E_y is a minimum except along the y axis where E_y still has a null (see Fig. 57). As in the case of E_x the changes in the pattern shape for E_y are much more severe for f/D = 0.25 than for f/D = 1.0 (see Figs. 54 and 55). In the former case the contours are almost completely distorted while in the latter a high degree of symmetry is still evident.

As in section 3.2 we may obtain a quantitative measure of the relative sizes of E_x and E_y by plotting the cross-polarization factor X_P defined in Eq.(167). Contours of the cross-polarization factor for off-axis incidence are shown in Fig. 58 for the same parameters as in Fig. 52. The on-axis counterpart of Fig. 58 is shown in Fig. 25 where we observed that the cross-polarization factor tends to \( \pm \) on the circles where E_y = 0 and tends to \( + \) on the ellipses where E_x = 0 (see Eq.(167)). The reflection symmetry in the E-plane, for axial incidence, is lost in the off-axis case as is to be expected since both E_x and E_y lose their E-plane symmetry. The right-hand half of Fig. 58 retains some of the features of
Fig. 56 Equal-amplitude contours of $E_y$ in the focal plane for axial incidence ($f/D = 0.5$).
Fig. 57 Equal-amplitude contours of $E_y$ in the transverse plane through the scan maximum in the H-plane for off-axis incidence ($f/D = 0.5$, $\eta = 4^\circ$).
Fig. 25 except that the circles of large negative $\chi P$ are not present in the former figure because the null region of $E_y$ has been replaced by a region of minimum $E_y$. The regions of large positive $\chi P$ still follow the curves where $E_x$ has its nulls. With the parameters in Figs. 54 and 55, the cross-polarization factor for $f/D = 0.25$ shows greater distortion while $\chi P$ for $f/D = 1.0$ shows less distortion than is evident in Fig. 58 for $f/D = 0.5$.

The investigation into the off-axis characteristics of the scattered field concludes with a look at the $E_z$ component. Figure 59 shows equal-amplitude contours of $E_z$ for axial incidence by a plane wave. This figure is similar to Figs. 27 and 28 but is plotted in normalized coordinates in the present case. For off-axis incidence (using the same parameters as in Fig. 52) the contours in Fig. 59 distort and the amplitude of the inner main lobe decreases while that of the outer minor lobe increases (see Fig. 60). As before the distortion of the contours is more apparent at $f/D = 0.25$ than at $f/D = 1.0$, Fig. 60 for $f/D = 0.5$ being intermediate between these two cases.

The results presented in this section are, to the best of the author's knowledge, unique. The only comparable results are those obtained for large $f/D$ ratios in classical optics as shown in the text by Born and Wolf [4], for example. Since scalar diffraction theory is used to obtain the results of classical optics information is obtained only for the principal component $E_x$ of the electric field as the main-beam maximum is scanned off axis. In the present analysis vector diffraction theory is used to compute the electric field and this enables us to examine the $E_y$
Fig. 58 Equal-amplitude contours of the cross-polarization factor in the transverse plane through the scan maximum in the H-plane for off-axis incidence ($f/D = 0.5$, $\eta = 4^\circ$).
Fig. 59 Equal-amplitude contours of $E_x$ in the focal plane for axial incidence ($f/D = 0.5$).
and $E_z$ components as well. The analysis reveals that the $E_y$ and $E_z$
components suffer distortions for off-axis incidence which are no less
interesting than those observed for the principal $E_x$ component. The
cross-polarization factor $X_P$ enables us to compare directly the relative
magnitudes of the $E_x$ and $E_y$ components. The use of normalized coor-
dinates should be reemphasized since this makes it possible to present
field patterns for different $f/D$ ratios and different angles of incidence
without having to resort to tedious scale changes which would be required
for plots made in terms of wavelengths.

In Chapter V we look again at the fields produced for axial incidence
by a plane wave and use these fields to examine the efficiency of
aperture-type feeds in the focal plane. The investigation suggests a
straightforward design procedure which may be used to achieve maximum
aperture efficiency for a paraboloid-primary-feed combination. The feeds
considered are dominant-mode circular and rectangular waveguides.
Fig. 60 Equal-amplitude contours of $E_z$ in the transverse plane through the scan maximum in the H-plane for off-axis incidence ($f/D = 0.5$, $\eta = 4^\circ$).
CHAPTER V

EFFICIENCY OF APERTURE-TYPE FEEDS

In this chapter the focal-region fields produced by a plane wave incident along the axis of the paraboloid are applied to the study of the aperture efficiency of paraboloid-feed-horn combinations. Some of the results obtained in Chapter III for axial incidence by a plane wave are used in conjunction with the formulas derived in section 1.4 to investigate the dependence of the aperture efficiency on the primary-feed dimensions and the reflector f/D ratio.

Rectangular and circular waveguides are often used as primary feeds for paraboloidal reflector antennas. The design procedures to obtain an efficient combination of reflector and feed are based to a great extent on empirical guides [29, ch. 12], [49, ch. 12]. For example, to achieve maximum gain the usual procedure is to design the feed horn so that its radiation pattern provides an illumination taper approximately -10dB at the reflector edges relative to the illumination at the center, while for good side-lobe performance the illumination taper should be about -20dB [49, ch. 12].

The overall performance of the reflector-primary-feed combination depends on the relationship between the dimensions of the primary feed and the reflector f/D ratio, the aperture efficiency, the reradiated pow-

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er, the spillover power, the relative edge illumination and the radiation pattern when the system is transmitting. It is the purpose of this chapter to establish a straightforward design procedure whereby the primary-feed dimensions of circular and rectangular waveguide feeds (operating in the dominant TE_{11} and TE_{10} modes, respectively) may be determined for reflectors of arbitrary f/D ratio. The design criterion will be taken to be that choice of primary-feed dimensions which will provide the maximum transmission of power to a matched load when the reflector is illuminated by a uniform plane wave. Stated in another way this means that the aperture efficiency defined by Eq. (41) is to be maximized by a suitable choice of primary-feed dimensions.

Section 1 investigates what fraction of the total power incident on the reflector can be absorbed by circular or rectangular apertures in the focal plane, the only losses being due to spillover (i.e., power incident on the reflector but not intercepted by the aperture in the focal plane). The aperture efficiency for a circular waveguide operating in the dominant TE_{11} mode is studied in section 2 as a function of reflector f/D ratio and feed radius. In section 3 a similar analysis is carried out for a rectangular waveguide operating in the dominant TE_{10} mode. Finally, the results of sections 2 and 3 are used in section 4 to establish the design procedure to achieve maximum aperture efficiency.
5.1 Power absorbed by circular and rectangular apertures in the focal plane

It is of interest to examine the fraction of the total power incident on the reflector which can be absorbed by an aperture of prescribed dimensions in the focal plane. At present the aperture represents a surface through which the incident energy flows, subsequently it will become the aperture of an actual feed horn in the focal plane. The total power $P_1$ incident on the reflector aperture $A_2$ ($=\pi d^2/4$) is

$$P_1 = \frac{|E_4|^2}{8\varepsilon_0} \pi d^2$$

which may be written as

$$P_1 = \frac{\pi |E_x(r)|^2}{2\varepsilon_0 \delta^2 k^2}$$

by using the definitions of $E_x(r)$ (Eq. (71)) and $\delta$ (Eq. (63)) and where $k = 2\pi/\lambda$. The power $P_r$ absorbed by an aperture $A_1$ in the focal plane is

$$P_r = \frac{1}{2} \text{Re} \int_{A_1} (\overline{E}_2 \times \overline{H}_2) \cdot \hat{n} \, dS$$

where $\overline{E}_2$ and $\overline{H}_2$ are the fields scattered by the reflector and $\hat{n}$ points in the direction of energy flow (parallel to $\hat{z}$ for a planar aperture in the focal plane). The ratio of the power absorbed by the aperture to the total power incident on the reflector is

$$\frac{P_r}{P_1} = \frac{1}{2P_1} \text{Re} \int_{A_1} (E_2 \times H_2) \cdot \hat{n} \, dS$$
Comparison of this equation with Eq. (44) for the transmission efficiency of an "ideal horn", i.e., one absorbing all the power incident on it, reveals that the two expressions are identical ($P_1 = P_2$ here). Thus, for an exact conjugate match the feed absorbs all the energy incident on its aperture and in this sense the feed will be optimum for the given aperture dimension. We now investigate the optimum transmission efficiency $T_0$ which can be attained by circular and rectangular apertures as the aperture dimensions are varied.

Consider a circular aperture of radius "a" in the focal plane ($z' = 0$) and centered on the focus; the power received by this aperture is

$$P_r = \int_{0}^{2\pi} \int_{0}^{a} \langle P_z \rangle \rho \, d\rho \, d\phi$$

(213)

where $\langle P_z \rangle$ is the $z$ component of the time-averaged Poynting vector given by Eq. (188). Upon introducing the normalized coordinate $V_0$ into Eq. (213) and carrying out the $\phi'$ integration, we obtain the aperture efficiency of an optimum aperture-type feed of circular cross section as

$$T_0 = \frac{P_r}{P_1} = \frac{1}{2} \left( 1 + \rho_0^2 \right)^2 \int_{0}^{V_{oa}} \left( I_0^2(V_0) - I_2^2(V_0) \right) V_0 \, dV_0$$

(214)

where

$$V_0 = \frac{2k_0'p_0}{1 + \rho_0^2} = k_0' \sin \theta_0$$

(215)
and

\[ V_{oa} = \frac{2ka\theta_o}{1 + \rho^2} = ka \sin \theta_o \]  

(216)

For large \( f/D \) ratios (\( \geq 2.0 \), say) \( I_0 \) and \( I_2 \) are given by Eqs.(161) so that Eq.(214) becomes

\[ T_o = 2 \int_0^{\frac{ka\theta_o}{V_o}} \frac{J_2^2(V_o)}{V_o} dV_o \]  

(217)

where we have used \( ka \sin \theta_o \approx ka\theta_o \) and \( 1 + \rho^2 \approx 1 \). This equation may be integrated directly by using the relation

\[ \frac{J_2^2(x)}{x} = -\frac{1}{2} \frac{d}{dx} \{ J_0^2(x) + J_1^2(x) \} \]  

(218)

and remembering that \( J_0(0) = 1, J_1(0) = 0 \) to yield

\[ T_o = 1 - J_0^2(ka\theta_o) - J_1^2(ka\theta_o) \]  

(219)

This is identical to Rayleigh's formula for the fraction of the total energy falling within a radius "a" centered at the focus of a circular lens [4, sect. 8.5].

Eq.(214) has been evaluated numerically for various \( f/D \) ratios and the efficiency curves are shown in Fig.61 as functions of \( V_{oa} \). The curve for \( f/D = 2.0 \) (found from Eq.(219)) corresponds to the optical result; for small \( f/D \) ratios the efficiency is much less than and increases more slowly with \( V_{oa} \) than predicted by Eq.(219). The reason for this may be seen in Figs. 33 and 34 which show that for small \( f/D \) ratios some of the energy in the main lobe is redistributed into the minor lobes. The regions in which the slope of the efficiency curves
Fig. 61 Aperture efficiency $T_o$ of a paraboloid with focal ratio $f/D$ and optimum aperture-type feed of radius $v_{oa}$. 

$V_{oa} = k \alpha \sin \theta_o$
becomes negative correspond to the regions marked by "-" in Figs. 33 and 34 where the direction of energy flow is reversed. The dashed lines in Fig. 61 coincide with the end points of the energy reversal zones defined by $E_p = 0$ as discussed in section 3.5. Even under conditions of total absorption by an ideal horn (the only losses being due to spill-over) the aperture efficiency for small $f/D$ ratios is markedly reduced by the reversal of energy flow across the focal plane. This requires that the aperture size in wavelengths must be increased for small $f/D$ to achieve a given efficiency. Upon referring to Figs. 20 through 24 we see that the dimensions of the field distribution in the focal plane decrease with decreasing $f/D$ and this tends to offset the increase in diameter required to achieve a given efficiency. Fig. 62 shows contours of equal aperture efficiency as functions of the reflector $f/D$ ratio and the radius $a/\lambda$ of an ideal feed. The horizontal scale is also given in terms of $\theta_0$, the half angle subtended by the reflector, and this serves as a useful conversion between $f/D$ and $\theta_0$ (see Eq. (63)). The radius to achieve a given efficiency is a minimum in the range of $f/D$ ratios between 0.4 and 0.75. The fluctuations on the contours result from the energy reversals or from the fact that the efficiency remains constant over certain small regions at particular $f/D$ ratios (see Fig. 61).

To obtain the aperture efficiency of an optimum aperture-type feed of rectangular cross section we proceed as for the circular aperture; there is, however, one important difference. The area of a rectangular aperture can be controlled independently by two parameters, the width $2W$ and the height $2H$, while the circular aperture requires only a single parameter, the radius $a$. The power received by a rectangular
Fig. 62 Contours of equal aperture efficiency $T_0$ for an ideal feed with radius $a/\lambda$ and a paraboloid with focal ratio $f/D$. 
aperture of width $2W$ and height $2H$, centered on the focus, is

$$P = \int_{-W}^{W} \int_{-H}^{H} <P_x> \, dx' \, dy'$$

(220)

where $<P_x>$ is given by Eq.(188) as before. We now introduce the following normalized coordinates into Eq.(220):

$$X_o = \frac{2kx'\rho_o^*}{1 + \rho_o^*} = kx' \sin \theta_o$$

(221)

and

$$Y_o = \frac{2ky'\rho_o^*}{1 + \rho_o^*} = ky' \sin \theta_o$$

(222)

so that $V_o = \sqrt{X_o^2 + Y_o^2}$. The aperture efficiency of an optimum aperture-type feed of rectangular cross section may now be written as

$$T_o = \frac{P_o}{P_1} = \frac{(1 + \rho_o^*2)^2}{4\pi} \int_{-W_o}^{W_o} \int_{-H_0}^{H_0} (I^2_o(V_o) - I^2_o(V_o)) \, dX_o \, dY_o$$

(223)

where

$$H_o = \frac{2kH_o \theta}{1 + \rho_o^*} = kH \sin \theta_o$$

(224)

and

$$W_o = \frac{2kW_o \theta}{1 + \rho_o^*} = kW \sin \theta_o$$

(225)

Since $W_o$ and $H_o$ can vary independently, we are faced with the problem of establishing some criterion for a relationship between them so that we need only vary a single parameter. The ideal situation would
be to set one of the parameters to its optimum value (i.e., that value giving maximum efficiency) while the other is varied. Unfortunately the right-hand side of Eq. (223) represents quite a complicated function because the integrand itself is the result of squaring an integration over the reflector surface and there seems to be no simple way of establishing the optimum values of $W_0$ and $H_0$ in advance.

We will subsequently investigate the aperture efficiency of a dominant-mode TE$_{10}$ rectangular waveguide and it seems desirable to relate $W_0$ and $H_0$ by examining this mode. The coordinate system as well as the aperture electric-field distribution for a TE$_{10}$-mode rectangular waveguide is shown in Fig. 63. In addition, we show the field distribution for a TE$_{11}$ circular waveguide; the expressions for the aperture fields will be presented in subsequent sections where they are used to examine the efficiency of dominant-mode circular and rectangular waveguides.

![Fig. 63 Rectangular and circular aperture geometries.](image-url)
In the rectangular aperture the electric field is polarized in the x direction so that the x-z plane is the E-plane of the system while the y-z plane is the H-plane. The field radiated by a square waveguide operating in the TE\(_{10}\) mode does not have equal E- and H-plane beamwidths, the E-plane beamwidth being the narrower. Because we are to use the rectangular waveguide with a reflector of circular cross section it seems desirable to choose \(W_0\) and \(H_0\) so that the waveguide field has equal E- and H-plane beamwidths when transmitting. To broaden the beam in the E-plane the height of the waveguide aperture has to be reduced; to achieve equal half-power beamwidths the relation between \(W_0\) and \(H_0\) is [50, ch. 1]

\[
W_0 = 1.364 H_0
\]  

An alternative choice could be \(W_0 = 1.5H_0\) which gives equal beamwidths between the first nulls for the E- and H-plane patterns. In the discussion of the efficiency of rectangular apertures Eq. (226) is used throughout to relate \(W_0\) and \(H_0\).

The results of numerical computation of Eq. (223) are shown in Fig. 64 where the optimum efficiency curves are plotted as a function of \(W_0\). The curves are similar to those in Fig. 61 for a circular aperture but the effect of the energy reversals is not as marked in the present case. Contours of equal aperture efficiency were not generated in this case; because of the similarity between the curves in Figs. 61 and 64 the efficiency contours for the rectangular aperture will be very much like those in Fig. 62 but the bumps will be smoothed out to a great extent.
Fig. 64 Aperture efficiency $T_o$ of a paraboloid with focal ratio $f/D$ and optimum aperture-type feed of half-width $W_o$. 

$$W_o = k W \sin \theta_o$$
The aperture efficiencies of optimum aperture-type feeds obtained in this section (Figs. 61 and 64) are used in the next two sections to obtain the spillover factor $T_a$ defined by Eq. (46). There we also examine the maximum aperture efficiency that can be achieved by circular and rectangular waveguide feeds operating in the dominant $\text{TE}_{11}$ and $\text{TE}_{10}$ modes, respectively, as well as the reradiated power factor $R$ (Eq. (45)).

5.2 Aperture efficiency for circular waveguides operating in the $\text{TE}_{11}$ mode

In order to apply Eq. (41) to find the aperture efficiency $e_{\text{apm}}$ for the matched-load condition, we require the field $(E_1, H_1)$ existing in the waveguide aperture when it transmits and the field $(E_2, H_2)$ scattered toward the focal region when the reflector is illuminated by an incident plane wave. Consider a waveguide of circular cross section with radius $a$ in the focal plane of the reflector and centered at the focus. From the numerator of Eq. (41) it is apparent that only the transverse components of the field in the waveguide aperture contribute to the integrand. The transverse electric field components of the dominant $\text{TE}_{11}$ mode are given by Silver [29, sect. 7.13] as follows:

$$E_{\phi 1} = j\omega \sigma \kappa_{\text{TE}_{11}} \frac{J_1(K_{\text{11}0'})}{K_{\text{11}0'}} \cos \phi'$$

(227a)

and

$$E_{\rho 1} = -j\omega \sigma \kappa_{\text{TE}_{11}} J_1'(K_{\text{11}0'}) \sin \phi'$$

(227b)

The aperture is assumed to be perfectly matched to free space here so that reflections at the aperture are neglected.
For the region far enough away from cutoff the magnetic field components are

\[ H_{01} = \frac{-j\omega \mu_0 k_{1a}}{Z_0} J_1(\frac{k_{11}'}{Z_0} \phi') \quad (227c) \]

and

\[ H_{\phi 1} = \frac{-j\omega \mu_0 k_{1a}}{Z_0} \frac{J_1(\frac{k_{11}'}{Z_0} \phi')}{k_{11}'} \quad (227d) \]

where

- \( J_1 \) = first-order Bessel function of the first kind
- \( J'_1 \) = first derivative of \( J_1 \) with respect to its argument
- \( k_{11a} \) = first root of \( J'_1 = 1.841 \)
- \( \mu_0 \) = permeability of free space
- \( Z_0 \) = impedance of free space

The coordinate system and the aperture electric-field distribution are as shown in Fig. 63.

The focal-region fields incident on the waveguide aperture are given by Eqs.(164) and (165) which are repeated here for convenience, thus

\[ E_{\phi 2} = E_x(F)(I_0 + I_2) \cos \phi' \quad (228a) \]

\[ E_{\phi 2} = -E_x(F)(I_0 - I_2) \sin \phi' \quad (228b) \]

\[ H_{\phi 2} = \frac{E_x(F)}{Z_0} (I_0 + I_2) \sin \phi' \quad (228c) \]
\[ H_{\phi 2} = \frac{E_x(F)}{Z_0} (I_o - I_2) \cos \phi' \]  

(228d)

where \( I_o \) and \( I_2 \) are given by Eq. (149).

The total power incident on the paraboloidal reflector is given by Eq. (210) and the total power radiated by the waveguide is given by Silver [29, sect. 10,3] as:

\[ P_1 = \frac{\pi \kappa a^2}{4} \left( K_{11}^2 - 1 \right) J_1^2(K_{11}a) \]  

(229)

where \( k \) is the free-space propagation constant.

Upon carrying out the desired vector operations the numerator \( N \) of Eq. (41) becomes

\[
N = \left| \frac{2\mu a E_x(F)}{Z_0} \int_0^{2\pi} \int_0^a \left[ \frac{I_o}{K_{11} \rho'} \left( \cos^2 \phi' + J_1'(K_{11} \rho') \sin^2 \phi' \right) \right] \rho' \ d\rho' \ d\phi' \right|^2
\]

(230)

At this stage it proves advantageous to introduce the following recurrence relations for the Bessel functions:

\[ J_1'(x) = \frac{1}{2} \{ J_0(x) - J_2(x) \} \]  

(231)

and

\[ \frac{J_1(x)}{J_0(x)} = \frac{1}{2} \{ J_0(x) + J_2(x) \} \]  

(232)

Equation (230) now becomes
\[ N = \frac{\omega^2 \mu_0^2 k_1^2 \left| E_\infty (r) \right|^2}{Z_0^2} \left( \int_0^{2\pi} \int_0^a I_0 \left[ J_0 (k_{11} r') - J_2 (k_{11} r') \cos 2\phi' \right] r' \, dr' \, d\phi' \right)^2 \]  

(233)

Since \( I_0, J_0 \) and \( J_2 \) are all independent of \( \phi' \), the \( \phi' \) integration may be carried out directly -- the \( \cos 2\phi' \) term integrates to zero while the other yields \( 2\pi \). Substituting for \( \rho_1 \) and \( \rho_2 (= \rho_1) \) in Eq. (41) yields

\[ \varepsilon_{apm} = \frac{2\omega \mu_0 a^2 k}{Z_0 a^2 (1 - \frac{1}{k_{11} a^2}) J_0^2 (k_{11} a)} \left( \int_0^a I_0 J_0 (k_{11} r') \rho' \, dr' \right)^2 \]  

(234)

Finally, introducing the normalized variables \( V_o \) and \( V_{oa} \) of Eqs. (215) and (216) we obtain

\[ \varepsilon_{apm} = \frac{(1 + \rho^2)^2}{2V_o^2 (1 - \frac{1}{k_{11} a^2}) \frac{J_0^2 (k_{11} a)}{V_o} \frac{V_o}{V_{oa}}} \left( \int_0^{V_{oa}} I_0 (V_o) J_o (1.841 \frac{V_o}{V_{oa}}) \, dV_o \right)^2 \]  

(235)

where we have used \( k_{11} = 1.841/a \) in the argument of the \( J_0 \) Bessel function.

Equation (235) relates the aperture efficiency \( \varepsilon_{apm} \) to the radius of the primary feed and the angular dimension of the reflector. The spillover power factors may be obtained as described in section 1.4 (see Eqs. (45) and (46)). The problem of studying the aperture efficiency is essentially that of studying the right-hand side of Eq. (235) as a function of the primary feed radius and the reflector \( f/D \) ratio. Equation (235) has been investigated for \( f/D \) ratios of 0.25, 0.32, 0.40,
0.50, 0.75, 1.00 and 2.00 with $V_{oa}$ values in the range $0 \leq V_{oa} \leq 5$.

Figure 65 shows the variation of the aperture efficiency for the extreme values of $f/D$ considered, namely 0.25 and 2.00. The curves for the remaining $f/D$ ratios are similar, showing a gradual increase in the peak value of the aperture efficiency, denoted by $\varepsilon_{apM}$, and a slow shift in the position of the maximum to higher $V_{oa}$ values as the $f/D$ ratio increases. The relative magnitudes of the spillover and reradiated power factors are also shown. As the aperture radius ($V_{oa}$) increases from zero the aperture efficiency increases, the spillover power factor decreases while the reradiated power factor remains at a relatively low value. Beyond a certain point the reradiated power factor starts to increase fairly rapidly and so the aperture efficiency decreases thus giving rise to the maximum $\varepsilon_{apM}$. The results of the study of the relevant power factors are summarized in Table 4 where the values of the maximum aperture efficiency $\varepsilon_{apM}$, the reradiated and spillover power factors are shown with the values of $V_{oa}$ corresponding to the maximum $\varepsilon_{apM}$ for various $f/D$ ratios. It was observed that when the maximum aperture efficiency values in Table 4 are plotted against $\sin^2 \theta_o$ ($\theta_o$ being the half angle subtended by the reflector at its focus) $\varepsilon_{apM}$ increases linearly with decreasing $\sin^2 \theta_o$ (i.e., increasing $f/D$) from a value of $\theta_o$ corresponding to $f/D \leq 0.46$. The spillover power factor decreases linearly with decreasing $\sin^2 \theta_o$ (for $f/D$ greater than 0.46) while the reradiated power factor remains almost constant. This enables us to extrapolate the curves to find the limiting values as $f/D$ tends to infinity. These values are included in Table 4 and are denoted by
Fig. 65 Aperture efficiency $\varepsilon_{\text{apm}}$, spillover power factor $T_s$ and reradiated power factor $R$ for paraboloids with $f/D = 0.25, 2.00$ and dominant TE$_{11}$-mode circular waveguide feeds.
TABLE 4

POWER FACTORS FOR PARABOLOIDAL REFLECTORS OF
VARIOUS f/D RATIOS FED BY CIRCULAR WAVEGUIDE

<table>
<thead>
<tr>
<th>f/D</th>
<th>Maximum aperture efficiency e_{app}</th>
<th>Spillover factor T_a</th>
<th>Reradiated factor R</th>
<th>Position of weak V_{on} = ka sin θ_o</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.547</td>
<td>0.418</td>
<td>0.035</td>
<td>2.61</td>
</tr>
<tr>
<td>0.32</td>
<td>0.610</td>
<td>0.340</td>
<td>0.050</td>
<td>2.75</td>
</tr>
<tr>
<td>0.40</td>
<td>0.655</td>
<td>0.282</td>
<td>0.063</td>
<td>2.88</td>
</tr>
<tr>
<td>0.50</td>
<td>0.684</td>
<td>0.249</td>
<td>0.067</td>
<td>3.00</td>
</tr>
<tr>
<td>0.75</td>
<td>0.722</td>
<td>0.209</td>
<td>0.069</td>
<td>3.06</td>
</tr>
<tr>
<td>1.00</td>
<td>0.736</td>
<td>0.195</td>
<td>0.069</td>
<td>3.10</td>
</tr>
<tr>
<td>2.00</td>
<td>0.751</td>
<td>0.180</td>
<td>0.069</td>
<td>3.12</td>
</tr>
<tr>
<td>∞</td>
<td>0.760</td>
<td>0.171</td>
<td>0.069</td>
<td>3.12</td>
</tr>
</tbody>
</table>

"∞" in the f/D column.

Further discussion of the TE_{11}-mode circular waveguide feed is deferred to section 5.4 where the results in Table 4 are used to establish a straightforward design procedure for TE_{11}-mode circular waveguides to be used as primary feeds for paraboloidal reflector antennas to achieve maximum aperture efficiency. In the next section we examine the aperture efficiency of TE_{10}-mode rectangular waveguides in the focal plane of a paraboloidal reflector.
5.3 Aperture efficiency for rectangular waveguides operating in the TE\textsubscript{10} mode

Consider a waveguide of rectangular cross section with aperture dimensions 2W and 2H centered at the focus as shown in Fig. 63. The transverse components of the dominant TE\textsubscript{10} mode in the waveguide aperture are given by Silver [29, sect. 7.11] as follows (the aperture is assumed to be perfectly matched to free space):

\begin{align*}
E_{x1} &= A \cos\left(\frac{\pi y}{2W}\right) \\
E_{y1} &= 0 \\
H_{x1} &= 0 \\
H_{y1} &= -\frac{A}{Z_0} \cos\left(\frac{\pi y}{2W}\right)
\end{align*}

where A is a constant.

In rectangular coordinates the transverse components of the focal-region field produced by an incident plane wave are given by Eqs. (159) and (160) as

\begin{align*}
E_{x2} &= E_x(F)(I_0 + \cos 2\phi' \cdot I_2) \\
E_{y2} &= E_y(F) \sin 2\phi' \cdot I_2 \\
H_{x2} &= \frac{E_x(F)}{Z_0} \sin 2\phi' \cdot I_2 \\
H_{y2} &= \frac{E_y(F)}{Z_0} (I_0 - \cos 2\phi' \cdot I_2)
\end{align*}

where \(I_0\) and \(I_2\) are given by Eq. (149).
The total power incident on the reflector is given by Eq. (210) and the total power radiated by the wave guide is

\[ P_1 = \frac{1}{2} \text{Re} \int_{A_1} (\mathbf{E}_2 \times \mathbf{H}_2^*) \cdot (\mathbf{E}_2 \mathbf{H}_2^*) \, dS = \frac{A^2 W H}{Z_0} \quad (238) \]

Upon substituting for \((\mathbf{E}_1, \mathbf{H}_1)\) and \((\mathbf{E}_2, \mathbf{H}_2)\) the numerator \(N\) or Eq. (41) becomes

\[ N = \left| \frac{2AF_x(F)}{Z_0} \int_{-W}^{W} \int_{-H}^{H} I_0 \cos\left(\frac{\pi y}{2W}\right) \, dx \, dy \right|^2 \quad (239) \]

Dividing this equation by \(P_1P_2\) as required by Eq. (41) and introducing the normalized variables of Eqs. (221) through (225) we obtain the expression for the aperture efficiency as

\[ \varepsilon_{apm} = \left(\frac{1 + \delta^2}{8\pi W_0 H_0}\right)^2 \left| \int_{-W_0}^{W_0} \int_{-H_0}^{H_0} I_0(V_0) \cos\left(\frac{\pi y_0}{2N_0}\right) \, dx_0 \, dy_0 \right|^2 \quad (240) \]

where \(V_0 = \sqrt{x_0^2 + y_0^2}\).

In order to evaluate Eq. (240) numerically we use \(H_0 = W_0/1.364\) to give equal E- and H-plane half-power beamwidths as discussed in section 5.1. The aperture efficiency may now be studied as a function of the width of the rectangular guide (maintaining the constant relation between \(W_0\) and \(H_0\)) and the reflector \(f/D\) ratio. As for the circular guide, \(f/D\) ratios of 0.25, 0.32, 0.40, 0.50, 0.75, 1.00 and 2.00 were investigated for \(W_0\) values in the range 0 \(\leq W_0 \leq 5\). The curves for aperture
efficiency, spillover power factor and reradiated factor are shown in Fig. 66 for \( f/D = 0.25, 2.00 \). The curves for the other \( f/D \) ratios are similar, the maximum value of the aperture efficiency \( \varepsilon_{\text{apM}} \) increases with increasing \( f/D \) ratio and at the same time the position of the maximum shifts to higher \( W_0 \) values. The curves are remarkably similar to those for the \( \text{TE}_{11} \)-mode circular guide (see Fig. 65), the most obvious difference is a slightly smaller reradiated power factor at peak efficiency for the \( \text{TE}_{10} \)-mode rectangular guide. The values of the maximum aperture efficiency, the spillover factor and the reradiated power factor at the position \( W_0 \) of \( \varepsilon_{\text{apM}} \) are shown in Table 5 for the \( f/D \) ratios considered.

**TABLE 5**

POWER FACTORS FOR PARABOLOIDAL REFLECTORS OF VARIOUS \( f/D \) RATIOS FED BY RECTANGULAR WAVEGUIDE

<table>
<thead>
<tr>
<th>( f/D )</th>
<th>Maximum aperture efficiency ( \varepsilon_{\text{apM}} )</th>
<th>Spillover factor ( T_s )</th>
<th>Reradiated factor ( R )</th>
<th>Position of peak ( W_0 = kW \sin \theta_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.555</td>
<td>0.436</td>
<td>0.009</td>
<td>2.85</td>
</tr>
<tr>
<td>0.32</td>
<td>0.618</td>
<td>0.358</td>
<td>0.024</td>
<td>2.93</td>
</tr>
<tr>
<td>0.40</td>
<td>0.664</td>
<td>0.300</td>
<td>0.036</td>
<td>3.08</td>
</tr>
<tr>
<td>0.50</td>
<td>0.692</td>
<td>0.265</td>
<td>0.043</td>
<td>3.15</td>
</tr>
<tr>
<td>0.75</td>
<td>0.729</td>
<td>0.225</td>
<td>0.046</td>
<td>3.25</td>
</tr>
<tr>
<td>1.00</td>
<td>0.745</td>
<td>0.209</td>
<td>0.046</td>
<td>3.32</td>
</tr>
<tr>
<td>2.00</td>
<td>0.760</td>
<td>0.194</td>
<td>0.046</td>
<td>3.35</td>
</tr>
<tr>
<td>---</td>
<td>0.768</td>
<td>0.186</td>
<td>0.046</td>
<td>3.35</td>
</tr>
</tbody>
</table>
Fig. 66 Aperture efficiency $\varepsilon_\text{apm}$, spillover power factor $T_S$ and reradiated power factor $R$ for paraboloids with $f/D = 0.25$, 2.00 and dominant $TE_{10}$-mode rectangular waveguide feeds.
5.4 Design procedure

All the information required to establish the aperture dimensions of rectangular and circular dominant-mode feeds so that maximum aperture efficiency is obtained when the feeds are used with paraboloidal reflectors of arbitrary $f/D$ ratio is contained in Tables 4 and 5. Before presenting the set of design curves derived from these tables, we develop expressions for the relative edge illumination $\Delta$ corresponding to the feed dimensions providing maximum aperture efficiency.

Although the edge illumination produced by the feed radiation characteristics is not used directly in the design procedure, further insight into the design problem may be gained by examining this quantity.

Consider first of all the circular waveguide operating in the dominant $\text{TE}_{11}$ mode. The polar and azimuthal component radiation patterns of the $\text{TE}_{11}$ mode, respectively, are given by Silver [29, sect. 10.2] as follows:

$$ E_\theta = -\frac{\omega u}{2R} \left(1 + \beta_{11} \cos^2 \theta/k\right) J_1(K_{11} a) \left[ \frac{J_1(k a \sin \theta)}{\sin \theta} \right] \cos \phi \ e^{-j kR} \tag{241} $$

and

$$ E_\phi = \frac{-k a u}{2R} \left(\beta_{11}/k + \cos \theta\right) J_1(K_{11} a) \left[ \frac{\sin \phi}{1-(k \sin \theta/K_{11})^2} \right] \sin \phi \ e^{-j kR} \tag{242} $$

where the symbols have the same meanings as in Eqs. (227) and where $\Delta$

---

$^2$ The aperture is assumed to be perfectly matched to free space.
\[ \beta_{11} = \sqrt{k^2 - k_{11}^2} \]

and

\[ R = \text{distance from center of aperture to observation point} \]

\[ = f \sec^2(\theta/2) \text{ for a point on the paraboloid.} \]

For large apertures \( \beta_{11}/k \approx 1 \) but this approximation is not used here since at \( f/D = 0.25 \) the aperture radius is about 0.4 wavelengths.

At the reflector edges \( (\theta = \theta_o) \), in the horizontal and vertical planes, the amplitude of the field distribution, normalized to unity at the apex of the reflector, is given by

\[
\Delta T E_{11}^0 = \left[ \frac{2 J_1(V_{oa})}{V_{oa}} \right] \left( \frac{1 + \beta_{11}\cos\theta_o/k}{1 + \beta_{11}/k} \right) \frac{1}{\sec^2(\theta_o/2)}
\]

(E-plane, \( \phi = 0 \) ) \hspace{1cm} (243)

\[
\Delta T E_{11}^\phi = \left[ \frac{2J_1(V_{oa})}{1 - (V_{oa}/1.841)^2} \right] \left( \frac{\beta_{11}/k + \cos\theta_o}{1 + \beta_{11}/k} \right) \frac{1}{\sec^2(\theta_o/2)}
\]

(H-plane, \( \phi = \pi/2 \) ) \hspace{1cm} (244)

If \( V_{oa} \) is now chosen to maintain a specific illumination condition (e.g., those particular values in Table 4 which give maximum aperture efficiency) the edge illumination \( \Delta \) will vary only as a function of the half angle \( \theta_o \). The \( 1/\sec^2(\theta_o/2) \) factor in the denominator is called the "space attenuation factor" [49, ch. 12] and takes into account the difference in propagation path from the focus to the apex and the focus to the reflector edge. For small \( f/D \) this factor \( \approx 0 \) dB, however for \( f/D = 0.25 \) the space attenuation is \(-6 \) dB. Figure 67 shows \( \Delta \) plotted as
Fig. 67 Reflector edge-field illumination factor $A$ for maximum aperture efficiency with feeds of rectangular and circular cross section.
a function of $\sin \theta_0$ (the conversion from $\sin \theta_0$ to $f/D$ ratio is conveniently obtained from Fig. 68) for the values of $V_{oa}$ in Table 4. Also shown are the edge illumination factors for the rectangular $\text{TE}_{10}$ mode which are derived in the next paragraph.

Expressions for the polar and azimuthal patterns of the $\text{TE}_{10}$ rectangular guide are again given by Silver [29, sect. 10.3] and by a similar procedure to that in the last paragraph we obtain the relative edge illumination factors as follows:

$$\Delta \text{TE}_1^{\theta} = \left[\frac{\sin H_0}{H_0}\right] \left(\frac{1 + \beta_{10} \cos \theta_0 / k}{1 + \beta_{10} / k}\right) \frac{1}{\sec^2(\theta_0/2)}$$

(E-plane, $\phi = 0$) \hspace{1cm} (245)

and

$$\Delta \text{TE}_1^{\phi} = \left[\frac{\pi^2}{4 \cdot (\pi^2/4 - W_0^2)}\right] \left(\frac{\cos \theta_0 + \beta_{10} / k}{1 + \beta_{10} / k}\right) \frac{1}{\sec^2(\theta_0/2)}$$

(H-plane, $\phi = \pi/2$) \hspace{1cm} (246)

where

$$W_0 = 1.364 H_0 = kW \sin \theta_0$$

and

$$\beta_{10} = \sqrt{k^2 - (\pi/2W)^2}$$

Again we choose $W_0$ (and so also $H_0$) to maintain maximum aperture efficiency by using the values for $W_0$ in Table 5.
Fig. 68 Variation of $\sin \theta_o$ with f/D ratio for a paraboloidal reflector.
The values of $A$ for the rectangular waveguide are shown in Fig. 67 together with those for the circular waveguide.

Examination of Fig. 67 reveals that the edge illumination corresponding to maximum aperture efficiency is a function of the reflector f/D ratio. For the rectangular $TE_{10}$ mode the edge illumination for maximum aperture efficiency is about $-12$ dB for $\sin \theta_0$ values up to 0.6, corresponding to f/D ratios greater than 0.75. This is not much different from the $-10$ dB empirical guide recommended by Jasik [49, ch. 12]. For smaller f/D ratios the maximum aperture efficiency criterion indicates edge illumination levels of less than $-15$ dB. The situation is complicated somewhat for the circular $TE_{11}$-mode waveguide because of the difference between the E- and H-plane beamwidths. The relative edge taper in the H-plane does not reach the $-10$ dB level until $\sin \theta_0 \leq 0.7$, corresponding to $f/D \leq 0.6$. If a circular waveguide is to be used as a primary feed careful attention should be paid to the sidelobe levels to ensure that they are acceptable for the particular application.

It was observed that when the normalized aperture radius ($V_{oa}$) and halfwidth ($W_o$) parameters corresponding to maximum aperture efficiency were converted to wavelengths, the dimensions giving maximum aperture efficiency increase almost linearly with increasing f/D ratio, see Fig. 69. This figure is a useful design curve since the feed aperture dimension to give maximum aperture efficiency may be read directly from it for f/D ratios between 0.25 and 2.0. A final set of design curves, obtained from Tables 4 and 5 to yield the value of the maximum aperture efficiency, the spillover and reradiated power factors is shown in Fig. 70.
Fig. 69 Dimensions in wavelengths of dominant-mode TE_{11} circular and TE_{10} rectangular waveguide feeds giving maximum aperture efficiency as a function of f/D ratio.
Fig. 70 Maximum aperture efficiency $\epsilon_{apM}$, spillover power factor $T_S$ and reradiated power factor $R$ for paraboloids with focal ratio $f/D$ and dominant-mode $TE_{11}$ circular or $TE_{10}$ rectangular waveguide feeds.
The design procedure to obtain maximum aperture efficiency is as follows: Given the \( f/D \) ratio of the reflector (or given \( \theta_0 \), the half angle, convert to \( f/D \) by Fig. 68) find the radius or halfwidth (in wavelengths) of a \( TE_{11} \) circular waveguide feed or a \( TE_{10} \) rectangular waveguide feed, respectively, from Fig. 69. The values of the aperture efficiency, spillover and reradiated power factors are then read from Fig. 70. Finally, using \( \sin \theta_0 \) (convert \( f/D \) to \( \sin \theta_0 \) by Fig. 68) the relative edge illumination is found from Fig. 67. An estimate of the level of the first sidelobe of the paraboloid far-field radiation pattern, when the feed transmits, may be found from the curves in Jasik [49, ch. 12].

Up till the present the discussion has been limited to rectangular and circular waveguide feeds. If the dimensions of the desired waveguide feed are sufficiently large to allow free propagation of more than one mode, the problem of controlling the modes arises; it is difficult to excite a large-sized waveguide so that only a single mode is generated. A convenient solution to obtain the required large aperture with a single-mode-field can be achieved by a gradual transition produced by flaring the terminal section of the waveguide to form an electromagnetic horn. The rectangular waveguide is flared into a pyramidal horn while the circular guide is flared into a conical horn; the required feed dimensions now being those in the flared-horn aperture. The flare angle of the horn should be kept as small as practicable. A flare angle of less than \( 10^\circ \) is desirable in that more rapid flaring may introduce phase variations or mode changes which are not allowed for in
Further details of the theoretical and experimental performance of horns with small flare angle may be found in references [29], [49], [51] and [52].

To conclude this chapter we look briefly at some other feed configurations which have been suggested as primary feeds for paraboloids. Potter [53] has devised a conical horn which utilizes the dominant $TE_{11}$ mode and the higher-order $TM_{11}$ mode in cylindrical waveguide to achieve equal beamwidths in all planes when the horn transmits. This represents a substantial improvement over the dominant-mode $TE_{11}$ cylindrical guide which does not have equal E- and H-plane beamwidths. The use of cylindrical corrugated waveguides and corrugated conical horns as feeds for reflector antennas has been considered by many authors [10] - [16], [54] and [55], among others. Thomas [16] has shown that by suitably combining a sufficient number of hybrid modes propagating in a corrugated waveguide with circumferential corrugations it is possible, in principle, to achieve aperture efficiencies in excess of 90%. The hybrid mode of nth order consists of the two characteristic modes $TE_{1n}$ and $TM_{1n}$ of a smooth circular waveguide. The problem of the realization of a feed with more than two hybrid modes has not been solved and requires the development of suitable mode generators. For a 2-hybrid-mode feed the aperture efficiency is found to be 83% for the paraboloid of the Parkes radio telescope ($f/D \approx 0.41$); this represents a substantial improvement over the 66% for a dominant-mode $TE_{11}$ waveguide feed (see Fig. 70). Corrugated rectangular waveguides and corrugated pyramidal horns have also been considered in the literature, for example [56].
All the feed configurations mentioned in the previous paragraph are considerably more complex than the simple $TE_{11}$ circular and $TE_{10}$ rectangular dominant-mode waveguide feeds considered in the design procedure described earlier. They present the possibility of substantially increasing the aperture efficiency of large reflector antennas and warrant further study. Investigation of the properties of and the aperture efficiencies attainable by these more complex horns are, however, beyond the scope of the present study.
CHAPTER VI
SUMMARY AND CONCLUSIONS

The goal of this dissertation has been to investigate the fields produced in the focal region of a paraboloidal reflector, with circular cross section, when it is illuminated by a plane wave incident at an arbitrary angle to the reflector axis. The motivation for studying the field for arbitrary incidence angles, rather than the more restricted case of axial incidence, arises since lateral displacement of the feed is frequently employed to achieve a beam-scanning capability. The study of the fields produced in the image space of paraboloidal reflectors of large f/D ratio (≥ 2,0, say) forms part of the theory of diffraction in classical optics. The scalar diffraction theory used there gives very little information about the detailed configuration of the electromagnetic field in the focal region. A vector solution, requiring integration of the induced physical optics currents over the reflector surface, was therefore obtained in the present study and has been used to examine the field configuration in the focal region in detail. The vector solution presented herein is valid for paraboloidal reflectors of arbitrary f/D ratio and it is shown that the image structure for small f/D ratio differs significantly from that deduced by scalar analysis in classical optics. The field expressions obtained by vector diffraction theory
reduce to those of classical optics when approximations valid for large $f/D$ ratios are made.

In many applications the paraboloidal reflector is used primarily in the receive mode, radio astronomy being a pertinent example in the author's experience, and the question as to the configuration of the received field naturally arises. Throughout this dissertation the paraboloidal reflector is studied in the receive mode for plane wave illumination. Many of the results presented in this study for the receive mode may be interpreted in the transmit mode by application of the reciprocity principle [24], [26]. As described at the end of section 1.4, the received fields in the focal region are a property of the paraboloid geometry and they remain the same independent of the particular feed system used to absorb the received energy. This property of the received fields is a highly desirable feature since the fields may be calculated once-and-for-all and then used in subsequent applications.

The vector solution for the fields requires double numerical integration over the reflector surface (see Eq. (70), for example). A brief study of two double numerical integration schemes revealed that a product formula employing Gaussian quadrature (with the zeroes of the Legendre polynomials) on the radial coordinate $\rho$ and the increment method on the azimuthal coordinate $\phi$ is more efficient, in terms of computer time, than a product formula involving Gauss-Legendre quadrature on both coordinates. By means of suitable approximations in the integrands of the general vector formulas, the $\phi$ integration was carried out in closed form to yield Bessel functions in $\rho$ (see Eq. (135a), for
example); the remaining integration was then carried out numerically by means of Gauss-Legendre quadrature to yield the results in this dissertation. The net computer time is at least an order of magnitude less than that required for the straightforward two-dimensional integration. The cost of double numerical integration increases with the square of the reflector aperture diameter, while the cost of single integration increases linearly with the diameter. Thus, the advantages of single integration over double integration in terms of computer expense becomes greater the larger the reflector diameter. The reduction in computer expenses achieved by carrying out the \( \phi \) integration in closed form cannot be overemphasized; without the techniques outlined in section 2.3 to reduce the double integration to a single integration most of the results presented in this dissertation could not have been obtained. Evaluation of the double integration to generate the numerous field patterns presented in the text is almost completely prohibited by excessive computer costs.

The general field expressions for the focal fields produced by a plane wave incident at an arbitrary angle to the reflector axis were reduced to comparatively simple forms for the case of axial incidence. These simplified field expressions were then used to compute field patterns for paraboloids with \( f/D = 0.25, 0.5, 1.0 \) and 2.0. The range from 0.25 to 1.0 covers the \( f/D \) ratios of reflectors commonly used at microwave frequencies and at \( f/D = 2.0 \) a suitable interface between the results of the present study and those of classical optics was achieved by showing that the general results of the vector theory reduce to the
well-known classical optics results at \( f/D = 2.0 \). A generalization of the dimensionless coordinates used in classical optics for the point of observation was made to facilitate the graphical presentation of the results. These generalized coordinates permit the use of constant scale factors along the coordinate axes, independent of the \( f/D \) ratio under consideration. The line integral contribution to the scattered field appearing in the general vector diffraction formula Eq. (51) was seen to be more than 40 dB below the surface integral contribution for reflectors with diameters of 30 wavelengths or more and could for this reason be neglected for large aperture diameters.

The analysis revealed that the electric and magnetic fields have identical field configurations in the focal plane except that the latter is rotated by \( 90^\circ \) in \( \phi' \) with respect to the former (see discussion on p. 112). Further, it was observed that the electric and magnetic fields are generally not orthogonal, the conditions for orthogonality being given by Eq. (168). It was also determined that for small \( f/D \) ratios the polarization of the transverse field in the focal region is not parallel to that of the incident field except where the \( E_y \) component (the cross-polarized component) vanishes, these regions being defined by Eq. (159b). The effect of the \( E_y \) component was clearly illustrated by plotting the direction of the transverse component of the focal field as well as the cross-polarization factor defined by Eq. (167). An examination of the field variation along the reflector axis revealed that the effective wavelength for propagation along the reflector axis is greater than the free space wavelength (see Eq. (180)). The exact analysis
revealed that the field variation along the reflector axis is not symmetrical about the focus and that the position of the field maximum is displaced from the focus towards the reflector. Since a vector formulation had been used to compute the fields, it proved to be a reasonably simple matter to investigate the energy flow in the focal region by looking at the Poynting vector. This study revealed that at small \( f/D \) ratio vortices of energy circulate about the positions where the \( \phi \) component of the electric field vanishes. This effect becomes less evident for larger \( f/D \) and at \( f/D = 2.0 \) there is no circulation of energy but the energy flow bypasses the nulls in the focal plane (see Figs. 35 and 36).

The field configuration produced in the focal region of a paraboloidal reflector illuminated by an off-axis plane wave was investigated in some detail. Initially, the field distribution along the principal scan axes (the \( x \) and \( y \) axes) was examined for \( E \)- and \( H \)-plane scans, respectively. The results obtained by the linear analysis of section 2.3 were compared directly with those of the exact analysis and it was shown that the two methods give results in close agreement for the field amplitudes out to lateral displacements of maximum field, from the reflector axis, corresponding to 10 half-power beamwidths of scan. At these scan angles the phase obtained by the two methods showed significant differences although the phase curves have the same general shape. It was observed that the field patterns are the same for reflectors with the same \( f/D \) ratio provided the angles of off-axis incidence \((\eta, \gamma)\) are chosen so that the lateral displacements of the field maximum from the
reflector axis correspond to equal numbers of HPBW's scanned as the reflector dimensions change. Next, the positions of the scanned maxima in the E- and H-planes were examined and it was shown that the maximum field locus deviates significantly from the Petzval surface. Finally, the field configuration in transverse planes (planes parallel to the x-y plane) through the positions of maximum field in the H-plane was investigated. Since vector diffraction theory was used, it was possible to examine not only the principal component $E_x$ of the scattered field but all the cross and longitudinal components $E_y$ and $E_z$. Studies using scalar diffraction theory are limited to the principal component $E_x$; the vector treatment reveals that the $E_y$ and $E_z$ components experience beam degradations no less interesting than those found for $E_x$.

Some of the results obtained for axial incidence were used to examine the efficiency of aperture-type feeds of circular and rectangular cross-section in the focal plane. It was demonstrated that the efficiency of an ideal feed, that is, one absorbing all the power incident on it and delivering this power to a matched load, is significantly influenced by the reversal of the direction of energy flow across the focal plane. This reversal in the direction of energy flow causes the aperture efficiency which may be achieved by paraboloidal reflectors of small $f/D$ ratio to be less than that achieved by reflectors of large $f/D$ ratio. The results of the aperture efficiency calculations were used to present a straightforward design procedure by means of which the dimensions of circular or rectangular waveguide feeds operating in the dominant $TE_{11}$ or $TE_{10}$ modes, respectively, may be determined so that max-
imum aperture efficiency is achieved. The relative levels of the absorbed, spillover and reradiated components of the total power intercept- ed by the given reflector are predicted.

The results presented in this dissertation allow the characteristics of the focal-region fields to be inferred for reflectors of any f/D ratio for illumination by a plane wave incident at an arbitrary angle to the reflector axis. The reversal of energy flow across the focal plane has been shown to decrease the maximum aperture efficiency attainable by reflectors with small f/D ratio. Suggested future work includes further investigation of the efficiency of aperture-type feeds and should include feeds more complex than the simple dominant-mode circular and rectangular feeds discussed herein. The decrease in aperture efficiency for displaced feeds should also be investigated. Finally, a more de- tailed analysis of the progressive degradation of the field configura- tion in the focal region, considering $E_x$, $E_y$ and $E_z$ components, should be undertaken to gain further insight into the effects produced by an off-axis plane wave incident on reflectors of arbitrary f/D ratio.
APPENDIX A

FUNDAMENTALS OF ELECTROMAGNETIC THEORY

A.1 Maxwell's equations and associated relationships

At each instant of time the state of an electromagnetic system may be described by Maxwell's equations. In differential form these equations are

\[ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad (A.1a) \]

\[ \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \quad (A.1b) \]

\[ \nabla \cdot \vec{D} = 0 \quad (A.1c) \]

\[ \nabla \cdot \vec{B} = \rho \quad (A.1d) \]

where \(\vec{D}\) and \(\vec{B}\) are the electric and magnetic flux density vectors, respectively, \(\vec{E}\) and \(\vec{H}\) are the electric and magnetic field intensity vectors, respectively, \(\vec{J}\) is the electric current density vector and \(\rho\) is the electric charge density\(^1\).

\(^1\)The author uses a vector notation for the nabla operator when the result of the operation is a vector (e.g., \(\text{grad } \phi = \nabla \phi\)), or if the operation is performed on a vector (e.g., \(\text{div } \vec{A} = \nabla \cdot \vec{A}\)) but if the result is a scalar, vector notation is not used (e.g., Laplacian \(\phi = \nabla^2 \phi\)).
In time-harmonic field problems it is desirable to eliminate the time dependence of the field vectors, the charge density, and the current density to obtain a set of equations which are functions only of position. To do this we assume a time dependence of the form $e^{j\omega t}$, where $\omega$ is the angular frequency, and represent the electric field, for example, by

$$\mathbf{E}(\mathbf{r},t) = \text{Re}\{\mathbf{E}(\mathbf{r})e^{j\omega t}\} \quad (A.2)$$

$\text{Re}\{\mathbf{E}(\mathbf{r})e^{j\omega t}\}$ denotes taking the real part of a complex number and $\mathbf{E}(\mathbf{r})$ is, in general, a complex function of position but independent of time. We may represent the other quantities in the set of equations (A.1) in a similar manner to produce the time-harmonic form of Maxwell's equations

$$\nabla \times \mathbf{E} + j\omega \mathbf{B} = 0 \quad (A.3a)$$
$$\nabla \times \mathbf{H} - j\omega \mathbf{D} = \mathbf{J} \quad (A.3b)$$
$$\nabla \cdot \mathbf{B} = 0 \quad (A.3c)$$
$$\nabla \cdot \mathbf{D} = \rho \quad (A.3d)$$

The field quantities in Eqs. (A.3) are related by the constitutive parameters of the medium. In free space

$$\mathbf{B} = \mu_0 \mathbf{H} \quad (A.4a)$$
$$\mathbf{D} = \varepsilon_0 \mathbf{E} \quad (A.4b)$$

where $\mu_0$ and $\varepsilon_0$ are the permeability and the permittivity of free space, respectively.
The equation of continuity which relates the electric current and charge densities may be deduced from Eqs. (A.3b) and (A.3d), thus

\[ \nabla \cdot \mathbf{J} + j \omega \rho = 0 \]  \hspace{1cm} (A.5)

Although no magnetic currents or isolated magnetic charges are known to exist in nature, it is often expedient to introduce fictitious magnetic charges and currents. In some field problems it is convenient to replace the actual sources by a set of equivalent sources on the boundary surface of the region of interest. The fictitious magnetic charges and currents facilitate the analysis of field discontinuities at the boundary surface. The magnetic charge density \( \rho_m \) and the magnetic current density \( \mathbf{J}_m \) are related by an equation of continuity similar to Eq. (A.5)

\[ \nabla \cdot \mathbf{J}_m + j \omega \rho_m = 0 \] \hspace{1cm} (A.6)

If we introduce these magnetic quantities into the field equations (A.3), we obtain the generalized set of Maxwell's equations, given by

\[ \nabla \times \mathbf{E} + j\omega \mathbf{B} = -\mathbf{J}_m \] \hspace{1cm} (A.7a)

\[ \nabla \times \mathbf{H} - j\omega \mathbf{D} = \mathbf{J} \] \hspace{1cm} (A.7b)

\[ \nabla \cdot \mathbf{B} = \rho_m \] \hspace{1cm} (A.7c)

\[ \nabla \cdot \mathbf{D} = \rho \] \hspace{1cm} (A.7d)

A more detailed discussion of magnetic charges and currents is given by Stratton [28, sect. 8.14] and Silver [29, sect. 3.1].
By using the constitutive relations (A.4) and carrying out some vector operations Eqs. (A.7) may be manipulated to yield the wave equations satisfied by electromagnetic fields. For linear, homogeneous media the permittivity and the permeability are constants independent of position. We obtain the inhomogeneous vector wave equations for the space dependence of the fields in the form

\[ \nabla \times \nabla \times \vec{E} - k^2 \vec{E} = -j\omega \mu_0 \vec{J} - \nabla \times \vec{J}_m \]  
(A.8a)

\[ \nabla \times \nabla \times \vec{H} - k^2 \vec{H} = -j\omega \mu_0 \vec{J}_m + \nabla \times \vec{J} \]  
(A.8b)

where \( k^2 = \omega^2 \mu_0 \varepsilon_0 \). These equations form the starting point for the derivation of the vector diffraction integrals in Appendix B.

In a source-free region, the sources \( \vec{J}, \vec{J}_m, \rho \) and \( \rho_m \) are zero and, with the help of the vector identity (A.36), Eqs. (A.8) reduce to the pair of homogeneous equations (often called the vector Helmholtz equations)

\[ \nabla^2 \vec{E} + k^2 \vec{E} = 0 \]  
(A.9a)

\[ \nabla^2 \vec{H} + k^2 \vec{H} = 0 \]  
(A.9b)

Eqs. (A.9) imply that each rectangular component \( E_x, E_y, \ldots, H_z \) satisfies the scalar Helmholtz equation

\[ \nabla^2 \psi + k^2 \psi = 0 \]  
(A.10)

This equation is used in Appendix B to derive the scalar diffraction integral of Kirchoff and Helmholtz.
A.2 The Poynting vector and energy flow

The Poynting vector is defined by

\[ \mathbf{P} = \mathbf{E} \times \mathbf{H} \]  
\[(A.11)\]

and represents the instantaneous rate of energy flow out through a unit area with its normal in the direction of \( \mathbf{E} \times \mathbf{H} \). For a time-harmonic field the instantaneous Poynting vector is given by

\[ \mathbf{P} = \text{Re}(\mathbf{E}(r)e^{j\omega t}) \times \text{Re}(\mathbf{H}(r)e^{j\omega t}) \]  
\[(A.12)\]

The time-averaged Poynting vector \( <\mathbf{P}> \) is

\[ <\mathbf{P}> = \frac{1}{T} \int_{0}^{T} \mathbf{P} \, dt = \frac{1}{2} \text{Re}(\mathbf{E}(r) \times \mathbf{H}(r)^*) \]  
\[(A.13)\]

where \( \mathbf{H}(r)^* \) is the complex conjugate of \( \mathbf{H}(r) \). Finally, the total power \( P \) crossing a surface \( S \) is given by

\[ P = \int_{S} <\mathbf{P}> \cdot d\mathbf{S} \]  
\[(A.14)\]

This expression proves useful when we consider the total power absorbed by an aperture in the focal region of a paraboloidal reflector.

A.3 Boundary conditions

The boundary conditions on \( \mathbf{E}, \mathbf{D}, \mathbf{B} \) and \( \mathbf{H} \) at the interface between two dissimilar regions are determined by applications of Maxwell's equations in integral form at the surface between them. Fig. 71 shows two regions separated by a boundary surface \( S; \mathbf{n}_1 \) and \( \mathbf{n}_2 \) are unit normal vectors at
an arbitrary point on $S$, directed into regions 1 and 2, respectively.

![Fig. 71 Boundary between two regions.](image)

At each point on the surface $S$ the fields in regions 1 and 2, distinguished by corresponding subscripts, must satisfy the following conditions:

\[
\hat{n}_1 \times \vec{H}_1 + \hat{n}_2 \times \vec{H}_2 = \vec{J}_s \quad \text{(A.15a)}
\]

\[
\hat{n}_1 \times \vec{E}_1 + \hat{n}_2 \times \vec{E}_2 = -\vec{J}_{ms} \quad \text{(A.15b)}
\]

\[
\hat{n}_1 \cdot \vec{B}_1 + \hat{n}_2 \cdot \vec{B}_2 = \rho_{ms} \quad \text{(A.15c)}
\]

\[
\hat{n}_1 \cdot \vec{D}_1 + \hat{n}_2 \cdot \vec{D}_2 = \rho_s \quad \text{(A.15d)}
\]

where $\vec{J}_s$, $\vec{J}_{ms}$, $\rho_s$, and $\rho_{ms}$ are the surface densities of electric current, magnetic current, electric charge, and magnetic charge on $S$.

A case of special interest arises when the fields vanish in one of the regions. This occurs, for example, when one of the media is a perfect electrical conductor. The boundary conditions (A.15) now reduce to

\[
\hat{n} \times \vec{H} = \vec{J}_s \quad \text{(A.16a)}
\]

\[
\hat{n} \times \vec{E} = 0 \quad \text{(A.16b)}
\]
where $\hat{n}$ is the outward unit normal from the perfect conductor. These boundary conditions prove useful for evaluating the surface currents and charges induced by a plane wave incident on a perfectly conducting paraboloidal reflector.

When we study electromagnetic fields in unbounded regions, we have to consider the behavior of the fields at infinity. In addition to the usual boundary conditions on finite boundaries, it is also necessary to specify a boundary condition or "radiation condition" at infinity. Sommerfeld [58] has stated the radiation condition for the three dimensional just scalar field $\psi$ (for e time dependence) in the form

$$\lim_{r \to \infty} |r\psi| < K \quad (A.17a)$$

$$\lim_{r \to \infty} r(jk\psi + \frac{\partial \psi}{\partial r}) = 0 \quad (A.17b)$$

where $K$ is a finite constant. The condition (A.17a) may be dropped since it is satisfied whenever (A.17b) holds. When the wave function satisfies Eq. (A.17b), there are no sources at infinity and the wave function represents outward travelling waves only. The importance of the radiation condition will be emphasized in Appendix B where it is used in the general solution of the scalar Helmholtz equation (Eq. (A.10)).

The radiation condition may also be stated for electromagnetic fields rather than scalar fields. In this case the mathematical statement of
the radiation condition corresponding to (A.17) is (see Jones [59 , ch. 1 ]).

\[
\lim_{r \to \infty} |r\mathbf{E}| < K \quad (A.18a)
\]

\[
\lim_{r \to \infty} |r\mathbf{H}| < K \quad (A.18b)
\]

\[
\lim_{r \to \infty} r \left[ (\hat{r} \times \mathbf{H}) + \left( \frac{\varepsilon_0}{\mu_0} \right)^{1/2} \mathbf{E} \right] = 0 \quad (A.18c)
\]

\[
\lim_{r \to \infty} r \left[ \left( \frac{\varepsilon_0}{\mu_0} \right)^{1/2} (\hat{r} \times \mathbf{E}) - \mathbf{H} \right] = 0 \quad (A.18d)
\]

where $\hat{r}$ is the outward unit normal to a sphere of large radius centered in the vicinity of the sources. These conditions essentially require that, at very great distances, the field solutions represent an outward propagating wave and not an inward propagating wave. In addition, $\mathbf{E}$ and $\mathbf{H}$ must be perpendicular to each other and to $\hat{r}$. This may be seen by taking the dot product of $\hat{r}$ with Eq. (A.18c) and rearranging Eq. (A.18d). These radiation conditions are used explicitly in the derivation of the vector diffraction integrals in Appendix D.

A.4 Some useful formulas and identities

In this section we list some formulas and vector identities which prove useful in the derivation of certain equations and their subsequent manipulation.

Dirac delta function. In an orthogonal curvilinear coordinate system with coordinate variables $u_1$, $u_2$, $u_3$ and with metric coefficients $h_1$, $h_2$, $h_3$ the three-dimensional delta function is given by
\[
\frac{\delta(u_1-u_1')\delta(u_2-u_2')\delta(u_3-u_3')}{h_1h_2h_3} = \delta(\vec{r} - \vec{r}')
\]  
\hspace{1cm} (A.10)

where \(\delta(\vec{r} - \vec{r}')\) is a convenient short-hand notation. The Dirac delta function is defined by the functional properties

\[
\delta(\vec{r} - \vec{r}') = 0 \hspace{1cm} \text{when } \vec{r} \neq \vec{r}'
\]  
\hspace{1cm} (A.20)

\[
\int_V \delta(\vec{r} - \vec{r}') \, dv' = \begin{cases} 1 & \text{when } \vec{r} = \vec{r}' \text{ is in } V \\ 0 & \text{when } \vec{r} = \vec{r}' \text{ is not in } V \end{cases}
\]  
\hspace{1cm} (A.21)

\[
\int_V F(\vec{r}')\delta(\vec{r} - \vec{r}') \, dv' = \begin{cases} F(\vec{r}) & \text{when } \vec{r} = \vec{r}' \text{ is in } V \\ 0 & \text{when } \vec{r} = \vec{r}' \text{ is not in } V \end{cases}
\]  
\hspace{1cm} (A.22)

Green's identities. Consider a volume \(V\) bounded by a closed surface \(S\) having the unit vector \(\hat{n}\) as the outward surface normal. Let \(\phi\) and \(\psi\) be two scalar functions of position which together with their first and second derivatives are continuous throughout \(V\) and on the surface \(S\), then

\[
\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) \, dv' = \int_S \phi \nabla \psi \cdot \hat{n}' \, ds'
\]  
\hspace{1cm} (A.23)

which is Green's first identity. By interchanging the roles of \(\phi\) and \(\psi\) in Eq. (A.23) and subtracting this new equation from Eq. (A.23) we obtain Green's second identity (also known as Green's theorem)
\[
\int_{V} (\phi \nabla^{2} \psi - \nabla \cdot \nabla \phi \cdot \nabla \cdot \mathbf{v}) \, dv = \int_{S} (\phi \nabla^{2} \psi - \nabla \cdot \nabla \phi) \cdot \mathbf{n} \, dS. \tag{A,24}
\]

**Gauss' theorem.** For any vector function of position \( \mathbf{A} \) with continuous first derivatives throughout a volume \( V \) and over the enclosing surface \( S \), Gauss' theorem states

\[
\int_{V} \nabla \cdot \mathbf{A} \, dv = \int_{S} \mathbf{A} \cdot \mathbf{n} \, dS. \tag{A,25}
\]

This result is also known as the divergence theorem.

**Stokes' theorem.** Stokes' theorem states that for any vector function of position with continuous first derivatives on an open surface \( S \) bounded by a contour \( C \),

\[
\oint_{C} \mathbf{A} \cdot d\mathbf{l} \quad \text{or} \quad \int_{S} (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, dS = \oint_{C} \mathbf{A} \cdot d\mathbf{l}. \tag{A,26}
\]

The direction of the line integral and that of \( \mathbf{n} \) follow the right-hand screw rule.

**Vector Stokes' theorem.** Another version of Stokes' theorem is given by

\[
\int_{V} (\nabla \times \mathbf{A}) \, dv = \int_{S} (\mathbf{n} \times \mathbf{A}) \, dS. \tag{A,27}
\]

where \( \mathbf{n} \) is the outward normal to the closed surface \( S \) bounding \( V \).
Duality of Maxwell's equations. The pair of equations (A.7a) and (A.7b) are duals of each other. A systematic interchange of symbols changes the first equation into the second, and vice versa. Similarly Eqs. (A.7c) and (A.7d) are duals. If we make use of the constitutive relations Eqs. (A.4), we can draw up a table showing the dual quantities (see Table 6).

**TABLE 6**

**DUAL QUANTITIES DERIVED FROM MAXWELL'S EQUATIONS**

<table>
<thead>
<tr>
<th>Electric Field</th>
<th>Magnetic Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vec{E} )</td>
<td>( \vec{H} )</td>
</tr>
<tr>
<td>( \vec{H} )</td>
<td>( -\vec{E} )</td>
</tr>
<tr>
<td>( \vec{J} )</td>
<td>( \vec{J}_m )</td>
</tr>
<tr>
<td>( \vec{J}_m )</td>
<td>( -\vec{J} )</td>
</tr>
<tr>
<td>( \rho )</td>
<td>( \rho_m )</td>
</tr>
<tr>
<td>( \rho_m )</td>
<td>( \rho )</td>
</tr>
<tr>
<td>( \varepsilon_0 )</td>
<td>( \mu_0 )</td>
</tr>
<tr>
<td>( \mu_0 )</td>
<td>( \varepsilon_0 )</td>
</tr>
</tbody>
</table>

Once we have obtained the solution for the \( \vec{E} \) field, say, we can obtain the corresponding solution for the \( \vec{H} \) field by interchanging all symbols as indicated in Table 6.
Vector identities. In the following formulae scalar functions of position are denoted by $\phi$ and $\psi$ and vector functions by $\vec{A}$, $\vec{B}$, etc.

\[ \nabla(\phi \psi) = \psi \nabla \phi + \phi \nabla \psi \]  
(A.28)

\[ \nabla \cdot (\psi \vec{A}) = \vec{A} \cdot \nabla \psi + \psi \nabla \cdot \vec{A} \]  
(A.29)

\[ \nabla \times (\psi \vec{A}) = \nabla \psi \times \vec{A} + \psi \nabla \times \vec{A} \]  
(A.30)

\[ \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B} \]  
(A.31)

\[ \nabla \times (\vec{A} \times \vec{B}) = \vec{A} \nabla \cdot \vec{B} - \vec{B} \nabla \cdot \vec{A} + (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} \]  
(A.32)

\[ \nabla \cdot \nabla \phi = \nabla^2 \phi \]  
(A.33)

\[ \nabla \cdot (\nabla \times \vec{A}) = 0 \]  
(A.34)

\[ \nabla \times (\nabla \phi) = 0 \]  
(A.35)

\[ \nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \]  
(A.36)

\[ \vec{A} \times (\vec{B} \times \vec{C}) = -(\vec{B} \times \vec{C}) \times \vec{A} = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \]  
(A.37)
APPENDIX B

THE INTEGRAL FORMULAS OF SCALAR AND VECTOR DIFFRACTION THEORY

B.1 The scalar diffraction integral

Green's second identity may be used to obtain an expression for the scalar wave function \( \psi \) at any point in the volume \( V \) shown in Fig. 72. If there are no sources in \( V \), we use the homogeneous scalar wave equation (A.10) in the derivation; however, in the most general case there will be a primary source (or sources), \( f(r') \), of the scalar field in \( V \). To analyze this situation we use the inhomogeneous wave equation in the form

\[
\nabla^2 \psi + k^2 \psi = f(r') \text{ in } V
\]

(B.1)

Fig. 72 Volume and surface configuration for the field integral derivation.
Note that \( \vec{R} \) is a fixed position vector which locates the point \( P \) at which the field \( \psi(\vec{R}) = \psi(P) \) is to be found. The points in the region occupied by \( f(\vec{r'}) \) and the points on the surfaces \( S_1, S_2, ..., S_n \), \( \Sigma \) are described by the variable position vector \( \vec{r}' \). The region of interest is \( V \) enclosed by the outer bounding surface \( \Sigma \) and the surface \( S \), composed of the closed surfaces \( S_1, S_2, ..., S_n \). The unit normal vector \( \hat{n} \) is directed into \( V \) (the outward normal \( \hat{n}' = -\hat{n} \)).

Green's second identity Eq. (A.24) may be written in the form

\[
\int_V \left\{ \phi(\vec{r}') \nabla'^2 \psi(\vec{r}') - \psi(\vec{r}') \nabla'^2 \phi(\vec{r}') \right\} \, dv' = -\int_{S + \Sigma} \left( \phi(\vec{r}') \nabla' \psi(\vec{r}') - \psi(\vec{r}') \nabla' \phi(\vec{r}') \right) \cdot \hat{n} \, dS' \quad (B.2)
\]

where the surface integration is carried out over all the boundary surfaces. The operator \( \nabla' \) operates only on the source coordinates (functions of \( \vec{r}' \)) and \( \nabla' = -\nabla \).

An equation for the field function may be obtained by putting \( \phi(\vec{r}') \) equal to the free-space Green's function. In the three-dimensional case

\[
\phi(\vec{r}') = \frac{-jkr}{4\pi r}
\]

(B.3)

where \( r = |\vec{R} - \vec{r}'| \) and \( \phi(\vec{r}') \) satisfies the following equation

\[
\nabla'^2 \phi = \nabla^2 \phi = -\delta(|\vec{R} - \vec{r}'|) - k^2 \phi
\]

(B.4)

When the Green's function is introduced into Eq. (B.2) the volume integral becomes
The inhomogeneous wave equation (B.1) has been used to introduce the source function \( f(\mathbf{r}') \). The volume integration for the second term may be carried out to yield \( \psi(\mathbf{r}) \). After some rearranging Eq. (B.2) becomes

\[
\psi(\mathbf{r}) = -\int_{V} \frac{f(\mathbf{r}') e^{-jkr}}{4\pi r} d\mathbf{r}'
\]

\[
= \int_{S+\Sigma} \left[ \left( \mathbf{\nabla}' \psi(\mathbf{r}') \right) \frac{e^{-jkr}}{4\pi r} - \psi(\mathbf{r}') \frac{\mathbf{n} \cdot \mathbf{\nabla}'}{4\pi r} \right] \cdot \mathbf{n} dS'
\]

(B.6)

This is a general solution to Eq. (B.1) where boundary effects are included. Sources within \( V \) contribute to the wave function \( \psi \) through the volume integral.

Equation (B.6) proves to be particularly useful in situations where there are no sources in \( V \). If we note that \( \mathbf{n} \cdot \mathbf{\nabla} = -\partial / \partial n \), which implies \( \mathbf{n} \cdot \mathbf{\nabla}' = \partial / \partial n \), we obtain

\[
\psi(\mathbf{r}) = \frac{1}{4\pi} \int_{S+\Sigma} \left[ \psi \left( \frac{\partial}{\partial n} \left( \frac{e^{-jkr}}{r} \right) \right) - \frac{e^{-jkr}}{r} \frac{\partial \psi}{\partial n} \right] dS'
\]

(B.7)

where \( r \) is the distance from the observation point \( p \) to the source point on the surface. Equation (B.7) is a general form of the diffraction formula of Kirchhoff and Helmholtz as used in classical optics.
We will be interested in the exterior problem where $\Sigma$ is the surface at infinity. An application of the radiation conditions (A.17) shows that the integration over $\Sigma$ contributes nothing to the right-hand side of Eq. (B.7). The integration over $\Sigma$ expresses the sum of all waves travelling inwards from $\Sigma$ and this contribution must vanish as $\Sigma$ recedes to infinity. If we denote this inward travelling contribution by $\psi_{\text{in}}$, we have from Eq. (B.7) that:

$$\psi_{\text{in}} = \frac{1}{4\pi} \int_{\Sigma} \left[ \frac{\partial}{\partial n} \left( \frac{e^{jk\rho}}{r} \right) - \frac{e^{-jk\rho}}{r} \frac{2\psi}{\partial n} \right] dS'$$  \hspace{1cm} (B.8)

By allowing the surface $\Sigma$ to approach a sphere of infinite radius, we may set $\partial/\partial n = -\partial/\partial r$. For convenience the surface increment $dS'$ is expressed in terms of an element of solid angle $d\Omega$; thus $dS' = r^2 d\Omega$. The first term in the integrand becomes

$$-\psi \frac{\partial}{\partial r} \left( \frac{e^{-jk\rho}}{r} \right) = \left\{ jk\psi \frac{e^{-jk\rho}}{r} + \psi \frac{e^{-jk\rho}}{r^2} \right\}$$  \hspace{1cm} (B.9)

Finally, Eq. (B.8) becomes

$$4\pi \psi_{\text{in}} = \int_{\Sigma} r \left( \frac{\partial \psi}{\partial r} + jk\psi \right) e^{-jk\rho} d\Omega + \int_{\Sigma} \psi e^{-jk\rho} d\Omega$$  \hspace{1cm} (B.10)

where the integration over $d\Omega$ extends over the finite domain $4\pi$.

In order to exclude sources at infinity, which would produce inward travelling waves, the right-hand side of Eq. (B.10) must vanish for large $r$. The second integral vanishes as $r \to \infty$ provided $\psi$ is regular at infinity as required by Eq. (A.17a). For the first integral to vanish it is sufficient for the integrand to vanish, this requires that
\[
\lim_{r \to \infty} r \left( \frac{\partial \psi}{\partial r} + jkr \right) = 0 \quad (B.11)
\]

which is precisely the radiation condition expressed in Eq. (A.17b). We conclude that, if the scalar function \( \psi \) satisfies the radiation condition at infinity, there are no sources at infinity and the wave function represents outward travelling waves only.

In problems related to diffraction by apertures in screens, the volume \( V \) is often taken as being bounded only by the outer surface \( \Sigma \) (i.e., there are no scatterers in \( V \)) and a somewhat different procedure to the above is used to let the surface \( \Sigma \) recede to infinity. The surface \( \Sigma \) is divided into three parts and various assumptions are made about the values of \( \psi \) and \( \partial \psi / \partial n \) on the sub-surfaces comprising \( \Sigma \). A procedure of this kind leads to the "Kirchhoff approximation" for diffraction by an aperture in an opaque screen as discussed in section 1.1.

### B.2 The vector diffraction integrals

In this section we seek a vector analog of Kirchhoff's scalar diffraction formula. The electric and magnetic fields must satisfy Maxwell's equations as well as the vector wave equations for \( \overline{E} \) and \( \overline{H} \) (Eqs. (A.8)) which are repeated here for convenience, thus

\[
\nabla \times \nabla \times \overline{E} - k^2 \overline{E} = -j\omega \mu_0 \overline{J} - \nabla \times \overline{J}_m \quad (B.12a)
\]

\[
\nabla \times \nabla \times \overline{H} - k^2 \overline{H} = -j\omega \epsilon_0 \overline{J}_m + \nabla \times \overline{J} \quad (B.12b)
\]

These wave equations may be solved for the electromagnetic field by
making use of the vector analog of Green's second identity: The region of interest is $V$ enclosed by the outer bounding surface $\Sigma$ and the surface $S$, composed of the closed surfaces $S_1, S_2, \ldots, S_n$ (see Fig. 73). If $\overrightarrow{F}$ and $\overrightarrow{G}$ are two vector functions of position, each continuous and having continuous first and second derivatives everywhere within $V$ and on the boundary surfaces, then

$$\int_V \left( \overrightarrow{F} \cdot \nabla \times \nabla \times \overrightarrow{G} - \overrightarrow{G} \cdot \nabla \times \nabla \times \overrightarrow{F} \right) \, dv'$$

$$= -\int_{S + \Sigma} \left( \overrightarrow{G} \times \nabla \times \overrightarrow{F} - \overrightarrow{F} \times \nabla \times \overrightarrow{G} \right) \cdot \hat{n} \, ds' \quad (B.13)$$

where $\hat{n}$ is a unit vector normal to a boundary surface, directed into $V$. With the aid of this theorem one can express the electromagnetic field at an arbitrary point $P$ in the volume $V$ in terms of the field sources within this volume and the values of the field itself over the boundaries of the region.

![Diagram](Fig. 73 Notation for Green's theorem)
The analysis proceeds as follows: $\vec{v}$ is set equal to the field component of interest ($E$ or $H$) and the vector function of position $\vec{G}$ is chosen to have a singularity at $P$. Since the function $\vec{G}$ in the Green's function must be continuous and differentiable, $P$ must be excluded from the domain of integration. This is done by surrounding $P$ by a small sphere $S_o$ of radius $r_o$ and considering that portion $V'$ of $V$ which is bounded by $S + E + S_o$. In the limit as $S_o$ shrinks to zero a term expressing the field vector at $P$ is obtained.

Various choices for $\vec{G}$ appear in the literature and these choices are dependent upon the differential equation which $\vec{G}$ is required to satisfy. Two common choices for $\vec{G}$ are a solution of the vector Helmholtz equation (Eq. (A.9a))

$$\nabla' \times \nabla' \times \vec{G} - \nabla' (\nabla' \cdot \vec{G}) - k^2 \vec{G} = \nabla' \times \nabla' \times \vec{G} + k^2 \vec{G} = 0 \quad \text{(B,14)}$$

$$\vec{G} = -\frac{jkr}{r} \vec{a} = \phi \vec{a} \quad (r \neq 0) \quad \text{(B,15)}$$

and a solution to the homogeneous wave equation, derived from Eqs. (B,12) with $\vec{J}$ and $\vec{J}_m$ set to zero,

$$\nabla' \times \nabla' \times \vec{G} - k^2 \vec{G} = 0 \quad \text{(B,16)}$$

$$\vec{G} = \nabla' \times \left( -\frac{jkr}{r} \vec{a} = \nabla' \times \phi \vec{a} \right) \quad (r \neq 0) \quad \text{(B,17)}$$

where $r$ is the distance from $P$ to any point in the region $V$ and $\vec{a}$ is an arbitrary but otherwise constant vector. As before, the prime on the nabla operator indicates operations in source coordinates. The $\vec{G}$'s given by Eqs. (B,15) and (B,17) are referred to as "free-space" vector Green's
functions, i.e., they are fields of sources radiating into unbounded space. It is possible to make other choices for \( \mathbf{G} \) such that they satisfy boundary conditions on \( S \) (both \( \mathbf{F} \) and \( \mathbf{G} \) will be required to satisfy the radiation condition at infinity); Harrington [37] and Tai [60] have outlined a number of alternative choices for \( \mathbf{G} \). However, in the following we consider only the free-space Green's functions of Eqs. (B.15) and (B.17) so that the boundary conditions must be imposed for each particular problem as the need arises.

Stratton and Chu [28, sect. 8.14] have investigated the relationship between the field vectors and their sources for Green's functions of the type in Eq. (B.15); their analysis is described in detail by Silver [29, sect. 3.8]. Green's functions of the type in Eq. (B.17) were first investigated by Franz [61] who derived his formulas for the fields by the method of the dyadic Green's function. Levine and Schwinger [62] also

\[ \nabla \times \nabla \times \mathbf{F}(\mathbf{r}, \mathbf{r}') = k^2 \mathbf{F}(\mathbf{r}, \mathbf{r}') = \nabla \times \{ \delta(\mathbf{r} - \mathbf{r}') \mathbf{I} \} \]

where \( r = |\mathbf{r} - \mathbf{r}'| \) and \( \mathbf{I} \) is the unit dyad. The differential equation for the dyadic Green's function used by Levine and Schwinger is given by

\[ \nabla \times \nabla \times \mathbf{F}(\mathbf{r}, \mathbf{r}') - k^2 \mathbf{F}(\mathbf{r}, \mathbf{r}') = \mathbf{I} \delta(\mathbf{r} - \mathbf{r}') \]

where \( r \) and \( \mathbf{I} \) are as above. If either of these differential equations is inserted into Eq. (B.13) rather than Eq. (B.15) or Eq. (B.17), the limiting procedure usually required to find the field at \( \mathbf{r} = \mathbf{r}' \) is circumvented. This method has been illustrated in the derivation of the scalar diffraction integral of Kirchhoff and Helmholtz in Section B.1.
used this method but presented their analysis in a different way. A derivation of the Franz formula which parallels that of the Stratton-Chu formula may be found in the book by Mentzer [53]. All the above analyses require extensive vector manipulations; therefore the derivations will only be outlined.

The derivation of the Stratton-Chu formula presented here closely follows that given by Silver while the derivation of the Franz formula is described in the manner given by Mentzer rather than by the method of the dyadic Green's function as used by Franz. The desired representation of the $\mathbf{E}$ field at a point $P$ in $V$ is now obtained by putting $\mathbf{F} = \mathbf{E}$ and inserting Eq. (B.15) or Eq. (B.17) and Eq. (B.12a) into Eq. (B.13) and carrying out the limiting procedure to find $\mathbf{E}$ at $P$. If $P$ lies on $S$, then a different limiting procedure is required to find $\mathbf{E}$ at $P$ on $S$ as described by Kouyoumjian [34].

**Stratton-Chu formulation.** On carrying out the substitutions indicated above and after extensive vector manipulations Eq. (B.13) becomes

$$\mathbf{a} \cdot \int_{V'} \left( j \omega \mu_0 \mathbf{J} + \mathbf{J}_m \times \nabla \phi - \rho / \varepsilon_0 \nabla \phi \right) \, dv'$$

$$= \mathbf{a} \cdot \int_{S + \Sigma + S_0} \left[ -j \omega \mu_0 \mathbf{h} \times \mathbf{H} + (\mathbf{h} \times \mathbf{E}) \times \nabla \phi + (\mathbf{h} \cdot \mathbf{E}) \nabla \phi \right] \, dS' \quad (B.18)$$

Now since $\mathbf{a}$ is an arbitrary vector and since Eq. (B.18) must hold for all $\mathbf{a}$, the integrals themselves must be equal. That is,
\[ \int_{S_0} \left[ -j\omega \mu_0 \phi (\hat{n} \times \hat{h}) + (\hat{n} \times \vec{E}) \times \nabla' \phi + (\hat{n} \cdot \vec{E}) \nabla' \phi \right] dS' \]

\[ = \int_{V'} \left( j\omega \mu_0 \phi \vec{J} + \vec{J}_m \times \nabla' \phi - \rho / \varepsilon_0 \nabla' \phi \right) dv' \]

\[ - \int_{S + \Sigma} \left[ -j\omega \mu_0 \phi (\hat{n} \times \hat{h}) + (\hat{n} \times \vec{E}) \times \nabla' \phi + (\hat{n} \cdot \vec{E}) \nabla' \phi \right] dS' \]  \hspace{1cm} (B.19)

where for convenience the surface integral over \( S_0 \) has been split off.

It can be shown [29 sect. 3.8] that

\[ \lim_{r_0 \to 0} \int_{S_0} \left[ \right] dS' = -4\pi \vec{E}(P) \]  \hspace{1cm} (B.20)

In this limit \( V' \) includes all of \( V \) and we obtain finally

\[ \vec{E}(P) = - \frac{1}{4\pi} \int_{V} \left( j\omega \mu_0 \phi \vec{J} + \vec{J}_m \times \nabla' \phi - \rho / \varepsilon_0 \nabla' \phi \right) dv' \]

\[ + \frac{1}{4\pi} \int_{S + \Sigma} \left[ -j\omega \mu_0 \phi (\hat{n} \times \hat{h}) + (\hat{n} \times \vec{E}) \times \nabla' \phi + (\hat{n} \cdot \vec{E}) \nabla' \phi \right] dS' \]  \hspace{1cm} (B.21a)

The analysis for \( \vec{H} \) proceeds as above, or the expression for \( \vec{H} \) may be written directly from Eq. (B.21a) by using the duality of Maxwell's equations (see Table 6), hence
\[
\tilde{H}(P) = -\frac{1}{4\pi} \int_V \left( j \omega \epsilon_0 \phi \widetilde{\nabla} \cdot \mathbf{m} - \mathbf{j} \times \nabla' \phi - \rho_m / \mu_o \nabla' \phi \right) \, dv'
\]

\[
+ \frac{1}{4\pi} \int_{S + \Sigma} \left[ j \omega \epsilon_0 \phi (\mathbf{\hat{n}} \times E) + (\mathbf{\hat{n}} \times \mathbf{\nabla}' \phi) \times \mathbf{\hat{n}} + (\mathbf{\hat{n}} \cdot \mathbf{\nabla}' \phi) \right] \, ds'
\]  

(B.21b)

A situation of great interest occurs when the region \( V \) is unbounded and the sources within \( V \) are confined to a region of finite extent. The only boundary surface is \( \Sigma \) which we take as a sphere of very large radius \( R \) centered on \( P \) so that \( \mathbf{\hat{r}} = -\mathbf{\hat{n}} \). With this value for \( \mathbf{\hat{n}} \) and using \( \nabla' = -\mathbf{\hat{r}} \, \partial / \partial \mathbf{\hat{r}} \), the surface integral of Eq. (B.21a) becomes

\[
\frac{1}{4\pi} \int_{\Sigma} \left[ -j \omega \epsilon_0 (\mathbf{\hat{n}} \times \mathbf{\nabla}) - (\mathbf{\hat{n}} \times \mathbf{E}) \times \mathbf{\nabla}' \phi + (\mathbf{\hat{n}} \cdot \mathbf{E}) \nabla' \phi \right] \, ds'
\]

\[
= \frac{1}{4\pi} \int_{\Sigma} \left[ j \omega \epsilon_o (\mathbf{\hat{r}} \times \mathbf{E}) - \left( j k + \frac{1}{R} \right) \left( \mathbf{\hat{r}} \times (\mathbf{\hat{r}} \times \mathbf{E}) - (\mathbf{\hat{r}} \cdot \mathbf{E}) \mathbf{\hat{r}} \right) \right] \frac{-j k R}{R} \, ds'
\]

\[
= \frac{1}{4\pi} \int_{\Sigma} \left[ j \omega \epsilon_o \mathbf{R} \left( \mathbf{\hat{r}} \times \mathbf{E} \right) + (\epsilon_o / \mu_o) \frac{1}{2} \mathbf{E} \right] e^{-j k R} \, d\Omega
\]

\[
+ \frac{1}{4\pi} \int_{\Sigma} \frac{1}{E} e^{-j k R} \, d\Omega
\]  

(B.22)

where \( \nabla' \phi \) has been evaluated as in Eq. (B.9), \( ds = R^2 \, d\Omega \), and the integration over the solid angle extends over \( 4\pi \). We have used the fact that \( (\mathbf{\hat{r}} \times \mathbf{E}) \times \mathbf{\hat{r}} + (\mathbf{\hat{r}} \cdot \mathbf{E}) \mathbf{\hat{r}} = \mathbf{E} \). Now as \( R \to \infty \) the second integral van-
ishes provided $\vec{E}$ is regular at infinity as required by the condition (A.18a). The first integral vanishes provided the integrand satisfies the radiation condition as expressed by Eq. (A.18c). Thus a contribution to the field from the integral over $\Sigma$ can arise only from sources outside $\Sigma$. This contribution (if it exists) will be referred to as an "incident field" and we subsequently write Eq. (B.21a) as

$$
\vec{E}(P) = \vec{E}_1(P) - \frac{1}{4\pi}\int_{V}(j\omega \mu_0 \vec{J} + \vec{J}_m \times \nabla'\phi - \rho/c_0 \nabla'\phi) \, dv'
$$

$$
+ \frac{1}{4\pi}\int_{S}[-j\omega \mu_0 \phi (\hat{n} \times \vec{H}) + (\hat{n} \times \vec{E}) \times \nabla'\phi + (\hat{n} \cdot \vec{E})\nabla'\phi] \, ds' \quad (B.23)
$$

with an analogous expression for $\vec{H}(P)$.

In the present analysis we are interested in determining the scattered fields ($\vec{E}_s(P)$ or $\vec{H}_s(P)$) outside the volume $V_s$ containing the scatterer, where

$$
\vec{E}(P) = \vec{E}_1(P) + \vec{E}_s(P) \quad (B.24a)
$$

$$
\vec{H}(P) = \vec{H}_1(P) + \vec{H}_s(P) \quad (B.24b)
$$

So, for a source-free region with an incident wave impinging on the scatterer, we may write the scattered fields as

$$
\vec{E}_s(P) = \frac{1}{4\pi}\int_{S}[-j\omega \mu_0 \phi (\hat{n} \times \vec{H}) + (\hat{n} \times \vec{E}) \times \nabla'\phi + (\hat{n} \cdot \vec{E})\nabla'\phi] \, ds' \quad (B.25a)
$$

and
\[ \overline{H}_s(p) = \frac{1}{4\pi} \int_S \left[ j\omega \mu_0 (\hat{n} \times \overline{E}) + (\hat{n} \times \overline{H}) \times \overline{V}' \phi + (\hat{n} \times \overline{H}) \overline{V}' \phi \right] dS' \quad (B.25b) \]

where \( S \) now represents the single closed surface bounding \( V_s \). Equations (B.25) express the scattered electromagnetic field in terms of the tangential components as well as the normal components of the \( \overline{E} \) and \( \overline{H} \) fields on the closed surface.

**Franz formulation.** On carrying out an analysis similar to the above procedure for the Stratton-Chu formulation but using \( \overline{\mathcal{E}} = \overline{V}' \times \phi a \) in Eq. (B.13) we find, for a source-free region,

\[ \int_{S + \Sigma + \Sigma_0} \left( \overline{\mathcal{E}} \times (\overline{V}' \times \overline{F}) - \overline{F} \times (\overline{V}' \times \overline{\mathcal{E}}) \right) \cdot \hat{n} dS' = 0 \quad (B.26) \]

where \( \overline{F} \) represents either \( \overline{E} \) or \( \overline{H} \). It can be shown [63, ch. 2] that

\[ \lim_{r_0 \to 0} \int_{\Sigma_0} \left( \overline{\mathcal{E}} \times (\overline{V}' \times \overline{F}) - \overline{F} \times (\overline{V}' \times \overline{\mathcal{E}}) \right) \cdot \hat{n} dS' = 4\pi a \overline{V} \times \overline{F}(p) \quad (B.27) \]

where the operator \( \overline{\nabla} \) on the right-hand side operates only on the coordinates of the observation point. Thus Eq. (B.26) becomes

\[ 4\pi a \overline{V} \times \overline{F}(p) = \int_{S + \Sigma} \left( \overline{\mathcal{E}} \times (\overline{V}' \times \overline{F}) - \overline{F} \times (\overline{V}' \times \overline{\mathcal{E}}) \right) \cdot \hat{n} dS' \quad (B.28) \]

and after some manipulation we obtain...
This equation may be expressed in the form presented by Franz by substituting $\vec{E}$ and $\vec{H}$ for $\vec{F}$ in turn and by using the source-free Maxwell's equations to relate $\vec{E}$ and $\vec{H}$. The operator identity $\vec{\nabla} \times \vec{\nabla} \times = \vec{\nabla} \nabla - \nabla^2$ and the fact that, since $\vec{\nabla}$ operates only on the observation coordinates, $\vec{\nabla} \cdot (\hat{n} \times \vec{F}) = \vec{\nabla} \cdot (\hat{n} \times \vec{F})$ prove useful in this endeavor (see Eqs. (A.36) and (A.29)). An argument similar to that for the Stratton-Chu formulation shows that when the radiation condition is imposed at infinity, the contribution from the integral over $\Sigma$ must come entirely from sources outside $\Sigma$.

Thus the scattered fields are

$$4\pi \vec{E}_{s} (P) = \vec{\nabla} \times \int \left( \hat{n} \times \vec{E} \right) \phi \, dS', \quad \frac{1}{\omega \epsilon_0} \nabla \times \nabla \times \int \left( \hat{n} \times \vec{H} \right) \phi \, dS' \quad (B.30a)$$

and

$$4\pi \vec{H}_{s} (P) = \vec{\nabla} \times \int \left( \hat{n} \times \vec{H} \right) \phi \, dS' + \frac{1}{\omega \mu_0} \nabla \times \nabla \times \int \left( \hat{n} \times \vec{E} \right) \phi \, dS' \quad (B.30b)$$

where $S$ now represents the closed surface bounding the scattering volume $V_s$. Eqs. (B.30) express the scattered electromagnetic field in terms of the tangential components of $\vec{E}$ and $\vec{H}$; the normal components are not required. If there are sources within the original volume $V$, the source
terms in Eqs. (B.12) must be retained; an analysis of this type, using
the method of the dyadic Green's function, is presented by Sancer[30].

Eqs. (B.30) are not in a form suitable for numerical calculations; it
would be advantageous to eliminate all the curl operations so that
computation of the fields requires only numerical integration. To do
this we make use of the vector identities Eqs. (A.30) and (A.37) and
bear in mind that the operator \( \nabla \) operates only on the coordinates of
the point of observation. The first term in Eq. (B.30a) becomes

\[
\nabla \times \int_S (\hat{n} \times E) \, dS' = \int_S \nabla \phi \times (\hat{n} \times E) \, dS' = \int_S (\hat{n} \times E) \times \nabla' \phi \, dS'
\]

while the second term is

\[
\frac{-j}{\omega e_0} \nabla \times \nabla \times \int_S (\hat{n} \times H) \phi \, dS' = \frac{-j}{\omega e_0} \nabla \times \int_S \nabla \phi \times (\hat{n} \times H) \, dS'
\]

We now have an expression of the form \( \nabla \times (\hat{n} \times H) \) and from Eq. (A.32)
we obtain

\[
\nabla \times ((\nabla \phi) \times (\hat{n} \times H)) = \nabla \phi \nabla \times (\hat{n} \times H) = (\hat{n} \times H) \nabla \phi \cdot \nabla \phi
\]

\[
+ (\hat{n} \times H) \cdot \nabla \nabla \phi = \nabla \phi \cdot \nabla (\hat{n} \times H)
\]

The first and last terms vanish since \( \nabla \) operates on the observation co-
dinates and the second term becomes \( k^2(\hat{n} \times H) \phi \) since \( \nabla^2 \phi + k^2 \phi = 0 \),
where \( k^2 = \omega^2 \mu \epsilon_0 \). The second integral in Eq. (B.30a) may now be written as
\[
\int_\mathbf{S} \left[ -j\omega\mu_0 \phi (\hat{n} \times \mathbf{H}) + \frac{1}{j\omega\epsilon_0} \{ (\hat{n} \times \mathbf{H}) \cdot \mathbf{\nabla} \} \phi \right] d\mathbf{s}
\]

so that the scattered \( \mathbf{E} \) field is

\[
\mathbf{E}_s (p) = \frac{1}{4\pi} \int_\mathbf{S} \left[ \{ (\hat{n} \times \mathbf{E}) \times \nabla' \phi \} - j\omega\mu_0 \phi (\hat{n} \times \mathbf{H}) \right. \\
\left. + \frac{1}{j\omega\epsilon_0} \{ (\hat{n} \times \mathbf{H}) \cdot \nabla' \} \cdot \nabla' \phi \right] d\mathbf{s} \quad \text{(B.31a)}
\]

where we have used the fact that \( \nabla' = -\nabla \). A procedure similar to the above may be used on Eq. (B.30b) or the \( \mathbf{H} \) field may be obtained directly from Eq. (B.31a) by the duality of Maxwell's equations, thus

\[
\mathbf{H}_s (p) = \frac{1}{4\pi} \int_\mathbf{S} \left[ \{ (\hat{n} \times \mathbf{H}) \times \nabla' \phi \} + j\omega\epsilon_0 \phi (\hat{n} \times \mathbf{E}) \right. \\
\left. - \frac{1}{j\omega\mu_0} \{ (\hat{n} \times \mathbf{E}) \cdot \nabla' \} \cdot \nabla' \phi \right] d\mathbf{s} \quad \text{(B.31b)}
\]

Schelkunoff [64] and [65] has obtained a result identical to that of Eqs. (B.31) by using the magnetic and electric vector potentials together with the field equivalence theorem. A further derivation of Eqs. (B.31) is presented by Rusch and Potter [31] who use a test dipole at the point of observation; application of the reciprocity principle then yields the desired fields.
APPENDIX C

THE PHASE OF THE SCATTERED FIELDS AND
THE NORMALIZED APERTURE COORDINATES

C.1 The distance from a source point to the incident wavefront

From the geometry of Fig. 7 the phase angle at the observation point \( P : (\rho', \phi', z') \) associated with a ray reflected from the source point \( M : (R, \theta, \phi) \) is given by

\[
\delta = \frac{2\pi}{\lambda} (d + r)
\]  

(C.1)

where \( r \) is the distance from the source point to the observation point and \( d \) is the distance from the source point to the incident wavefront, as measured along the wave normal. For axial incidence \( \delta \) assumes the particularly simple form

\[
\delta = \frac{2\pi}{\lambda} R \cos \theta
\]  

(C.2)

where \( R \cos \theta \) is the distance from the focal plane (i.e., a plane through the focus, normal to the axis of the paraboloid) to the source point. However, for oblique incidence the situation is somewhat more complicated as outlined below.

In order to clarify the geometry Fig. 74 shows the rotation of the wavefront in two special situations, namely those when one of the angles
(η or γ) is non-zero while the other is set to zero. Firstly, the front
of the plane wave is rotated about an axis parallel to the x axis; the
degree of angular rotation is η, being the angle between the z axis and
the direction of propagation. Secondly the wavefront has been rotated
about an axis parallel to the y axis; here the angular off-axis displacement is γ, again being the angle between the z axis and the direction
of propagation.

![Diagram of rotations of the incident plane wave](image)

**Fig. 74** Rotations of the incident plane wave.

For these special cases we may write

\[ \cos \eta = \hat{k}_1 \cdot (-\hat{z}) \]  \( \text{(C.3)} \)

\[ \cos \gamma = \hat{k}_2 \cdot (-\hat{z}) \]  \( \text{(C.4)} \)

where \( \hat{k}_1 \) and \( \hat{k}_2 \) are the appropriate unit propagation vectors depending
on whether \( \gamma = 0 \) or \( \eta = 0 \).
If the wave is incident at an arbitrary angle both angles must be considered to define the direction of propagation completely. To find the phase of the $\mathbf{E}$ and $\mathbf{H}$ fields incident upon the reflector, we make use of Fig. 75 which shows a two-dimensional representation of a somewhat hard-to-visualize three-dimensional situation. The point $M$ is an arbitrary source point on the reflector, $M'$ is the axial projection of $M$ onto the aperture plane, and $Q$ is the projection of the apex $A$ onto the aperture plane.

The distance from $M$ to the wavefront of the plane wave passing through $M'$ is $MM''$, as measured along the wave normal. The distance from $Q$, lying on the axis in the aperture plane, to the wavefront is $QQ'$, also
measured along $\hat{k}$. Finally, we require the distance $QQ''$ from $Q$ to a point on a plane wave passing through the focus so that this plane may be used as phase reference.

One more representation of the paraboloid proves useful, namely

$$R = f + z$$

where $f$ is the focal length and $z$ is the paraxial distance from a fixed plane, normal to the axis, to the paraboloid. By using (C.5) we find

$$\Delta = MM' = z_o - z = f\left(\sec^2(\theta_o/2) - \sec^2(\theta/2)\right)$$

which is a useful relation for finding the distance from a point on the paraboloid to the aperture plane.

To facilitate the discussion the coordinates of the points $M$, $M'$ and $Q$ are listed below. Thus

$$M : (R \sin\theta \cos\phi, R \sin\theta \sin\phi, -R \cos\theta)$$

$$M' : (R \sin\theta \cos\phi, R \sin\theta \sin\phi, -R \cos\theta + \Delta)$$

$$Q : (0, 0, -R \cos\theta + \Delta)$$

Recall that $\hat{k}$ as given by Eq. (56) is

$$\hat{k} = - (\sin\gamma, -\cos\gamma \sin\eta, \cos\gamma \cos\eta)$$

The distances, measured along $\hat{k}$, required to find $d$ are obtained as follows

$$M'H = M'H' \cdot \hat{k} = \Delta \cos\gamma \cos\eta$$
Finally, the expression for the distance $d$ from an arbitrary source point $M$ to a point on the wavefront of a plane wave passing through the focus takes the form

$$d = Q''Q - Q'Q + M'M$$  \hspace{1cm} (C.14)

Substitution of the values given above yields

$$d = f\{2 - \sec^2(\theta/2)\} \cos \gamma \cos \eta - R \sin \theta \cos \phi \sin \gamma + R \sin \theta \sin \phi \cos \gamma \sin \eta$$  \hspace{1cm} (C.15)

where $R = f \sec^2(\theta/2)$. For the case of axial incidence this yields

$$d = f\{2 - \sec^2(\theta/2)\} = f - z = R \cos \theta$$  \hspace{1cm} (C.16)

as expected. This expression for $d$, together with that for $r$, Eq. (67), will be converted to $\rho*$ and $\phi$ coordinates in the next section.

C.2 The normalized aperture coordinates

The relation

$$\rho* = \tan(\theta/2) = (\rho/2f)$$  \hspace{1cm} (C.17)

forms the basis for the discussion which follows. The change of variables
required to convert the field integrals to integrals over the $\rho^*$ and $\phi$ coordinates is accomplished by means of the following equalities derived from Eq. (C.17). These are

\[
\sec^2(\theta/2) = 1 + \tan^2(\theta/2) = 1 + \rho^*^2 
\] (C.18)

\[
\sin\theta = \frac{2 \tan(\theta/2)}{\sec^2(\theta/2)} = \frac{2\rho^*}{1 + \rho^*^2} 
\] (C.19)

\[
\cos\theta = 1 - 2\sin^2(\theta/2) = \frac{1 - \rho^*^2}{1 + \rho^*^2} 
\] (C.20)

\[
R = f \sec^2(\theta/2) = f(1 + \rho^*^2) 
\] (C.21)

\[
R \sin\theta = 2f\rho^* 
\] (C.22)

\[
R \cos\theta = f(1 - \rho^*^2) 
\] (C.23)

With the above substitutions the expression for $d$ given in (C.15) becomes

\[
d/\lambda = f/\lambda (1 - \rho^*^2) \cos\gamma \cos\eta - 2\rho^* \cos\phi \sin\gamma \\
+ 2\rho^* \sin\phi \cos\gamma \sin\eta - 2) + (2f/\lambda) 
\] (C.24)

Note that the distance $d$ has been expressed in wavelengths and that a term equal to \((2f/\lambda) - (2f/\lambda)\) has been added to $d$. This latter procedure makes no difference to $d$ but helps to define a suitable normalization factor for the fields. The same procedure is used to express the distance $r$ given by Eq. (67) as
\[ \frac{r}{\lambda} = \sqrt{\left(\frac{\epsilon}{\lambda}\right)^2 \left(1 + \rho^2\right)^2 - 4\left(\frac{\epsilon}{\lambda}\right)\left(\rho'\right)\rho \cos(\phi - \phi')} \]

\[ + 2\left(\frac{\epsilon}{\lambda}\right)\left(\frac{z'}{\lambda}\right)\left(1 - \rho^2\right) + \left(\rho'\right)^2 + \left(\frac{z'}{\lambda}\right)^2 } \]

where \( r \) has also been expressed in wavelengths.

We are now in a position to obtain the final expressions for the components of the focal-region \( \overline{E} \) and \( \overline{H} \) fields expressed in the normalized aperture coordinates.
REFERENCES


47. R.G. Kouyoumjian, "On the field energy in dispersive media," and "Some special notes on group velocity and energy velocity," Unpublished lecture notes for the course on Advanced Electromagnetic Theory given in the Department of Electrical Engineering at the Ohio State University, Columbus, Ohio.


