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Physics, nuclear

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POLARIZATION FROM SCATTERING POLARIZED SPIN-1 PARTICLES ON UNPOLARIZED SPIN-1 PARTICLES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Thomas Lindsay Terrall, B.S.

The Ohio State University
1974

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ACKNOWLEDGMENTS

I would not have completed this degree without the encouragement and aid in many forms given to me from my earliest years by my mother and late father. I became interested in physics in an excellent high school physics course taught by Mr. Merton Porter.

I believe also that this thesis would not have been completed without the encouragement, aid and advice of Dr. Seyler, who has been most patient and knowledgeable in guiding this work. Dr. Jossem, Chairman of the Ohio State Physics Department, and C. Eugene Maynard, Director of the Marion Regional Campus of Ohio State, have extended to me many considerations that have made this thesis possible.

Many persons have come to my aid in the waning moments of my third decade and my Ph.D. candidacy to enable me to complete the typing of the various drafts of the thesis; among them are C. N. Eschedor, Ms. Ann Peppard, Mrs. Collen Johnson and Ms. Donna Burkhart. Mrs. Eleanor Sapp has done the difficult typing of the text of the final draft rapidly and accurately and has been most helpful in the preparation of that draft.

Ms. Ann Davis has aided in the elimination of errors by rechecking the arithmetic of certain portions of the calculations.
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I. INTRODUCTION

This thesis concerns a theoretical non-relativistic description of the elastic scattering of a polarized beam of spin-1 particles by an unpolarized target of spin-1 particles. Expressions are derived for various measurable quantities such as differential cross section and vector and tensor polarization in terms of physically significant parameters. In deriving these equations only the well established principles of conservation of energy, linear and angular momentum, and parity together with time reversal invariance are assumed. In the special case of identical particles in beam and target the Pauli Exclusion Principle is also assumed.

Elastic scattering experiments, in which the polarizations of the beam particles and scattered particles are measured, provide a large amount of information about the nuclear interaction. The experimenter can measure, for various polarizations of the beam particles, the polarizations of the scattered particles as functions of energy and scattering angle. The expressions derived in this thesis present the scattered polarizations in terms of parameters which depend on the interaction between the particles.
These parameters are functions of energy and scattering angle.

The general motivation for this type of work where one expresses physical observables in terms of parameters is that as technology improves the variety and accuracy of data increases and one will be able to determine the values of the parameters as functions of energy. This spectroscopic information will eventually lead to the unravelling of the details of the interaction such as the spin dependence of the Hamiltonian.

This work is an extension of that of Seyler who treated the case of elastic scattering of polarized spin-1/2 particles by unpolarized spin-1 particles. Seyler's work corrected and generalized the earlier work on the same problem of Budianskii, Shih et al., and Goldberg who assumed that channel spin does not change in the interaction. Seyler's work as well as this thesis allows channel spin to change and also allows any state of polarization for the beam. We also consider the case of identical particles. Arvieux has also considered the spin-1/2 on spin-1 case but in the helicity representation and under the restriction that channel spin does not change. Lien has extended the work of Arvieux to the spin-1 on spin-1 case under the assumptions that channel spin and orbital angular momentum do not change. Meyer and Schiemenz at the University of Wisconsin Tandem Van de Graaf have recently measured vector and tensor observables
in the elastic scattering of deuterons on deuterons at 10 MEV. They did not meet with success in fitting Lien's expressions to their data and they felt Lien's assumptions were too restrictive.

The present derivation will now be considered in more detail. The state of the system before or after the scattering is represented in terms of the expectation values of 9 irreducible spherical tensor operators \( T^k_q \) of ranks \( k=0,1,2 \). The \( t^k_q \) are the expectation values of the operators \( T^k_q \) for the system before the scattering and the \( t'^k'_q' \) those for after the scattering. The change of the system in the scattering is represented by a 9x9 matrix, the transition matrix \( M \), whose elements are the set \( \{m_l\} \). The elements of \( M \) are functions of energy and scattering angle. We show the functional dependence of the \( t'^k'_q' \) on the \( t^k_q \) and the \( \{m_l\} \).

Time reversal invariance, conservation of parity and identity of the two particles impose linear relationships on the elements of the transition matrix. We develop these relationships.

The transition matrix elements are displayed in a form which reveals their dependence on scattering angle and represents their energy dependence in terms of the elements of the nuclear collision matrix. The nuclear collision matrix elements are parameterised in terms of the eigenphase shifts \( \delta^{j_1j_2}_l^s \) and the mixing parameters \( \epsilon^{j_1j_2}_l^s l's' \), both of which are energy dependent. The transition matrix is shown as a
linear expansion of spin operators. The coefficients, which depend on energy and scattering angle, are the elements of the set \( \{B_j\} \). This \( j \) is not the angular momentum quantum number. Expressions for the \( \{m_1\} \) as linear combinations of the \( \{B_j\} \) and vice versa are developed.

In addition to developing the expressions mentioned above certain theorems involving the products, sums and traces of the matrices appearing in this work are proved. These theorems reduced the calculational labor very greatly.
II. THEORY

This chapter contains a discussion of the theory on which our description of the scattering is based.

The Physical Problem

The physical arrangement of a scattering experiment is widely known and will not be discussed. Figure 1 and Figure 2 on page 6 show respectively the initial and final states of a classical scattering in the laboratory frame. Particle-1 is the incident or beam particle. Particle-2 is the target particle.

Coordinate Systems

We will use in this work two Cartesian frames that are at rest with respect to the center of mass of the system. The $xyz$ frame is a right handed frame whose $x$ and $z$ axis are in the plane of the scattering and whose $+z$ axis is along the initial momentum of particle-1. The $knp$ frame changes orientation with scattering angle and is defined below.

We define some unit vectors:

$k_1^i$ - in the direction of initial center of mass momentum of Particle-1,

$k_1^f$ - in the direction of final center of mass momentum of Particle-1,
FIGURE 1. CLASSICAL SCATTERING EXPERIMENT
INITIAL STATE
(LABORATORY FRAME)

FIGURE 2. CLASSICAL SCATTERING EXPERIMENT
FINAL STATE
(LABORATORY FRAME)
\( \hat{k}_2 \) - in the direction of initial center of mass momentum of Particle-2,
\( \hat{k}'_2 \) - in the direction of final center of mass momentum of Particle-2,

\[
\begin{align*}
\hat{p} &= (\hat{k}_1 + \hat{k}'_1) / |\hat{k}_1 + \hat{k}'_1|, \\
\hat{r} &= (\hat{k}'_1 - \hat{k}_1) / |\hat{k}'_1 - \hat{k}_1|, \\
\hat{n} &= \hat{p} \times \hat{r}.
\end{align*}
\]

We define also:

\[
\theta = \arccos (\hat{k}_1 \cdot \hat{k}'_1),
\]

which is the scattering angle in center of mass frame, and

\[
\epsilon = \arccos (\hat{p} \cdot \hat{k}_1).
\]

The vectors \( \hat{k}, \hat{n}, \hat{p} \) define a right-handed Cartesian frame which is illustrated in Figure 3 on page 8.

We note: \( \hat{k}_1 = -\hat{k}_2 \) and \( \hat{k}'_1 = -\hat{k}'_2 \).

The two angles \( \theta \) and \( \epsilon \) are related by \( \epsilon = \theta / 2 \).

If \( \hat{A} \) is a vector, then its components with respect of \( xyz \) frame are related to those with respect to \( knp \) frame by

\[
\begin{bmatrix}
A_k \\
A_n \\
A_p
\end{bmatrix} =
\begin{bmatrix}
\cos \theta/2 & 0 & -\sin \theta/2 \\
0 & 1 & 0 \\
\sin \theta/2 & 0 & \cos \theta/2
\end{bmatrix}
\begin{bmatrix}
A_x \\
A_y \\
A_z
\end{bmatrix}.
\]
FIGURE 3. CENTER OF MASS COORDINATE SYSTEMS
Symbols defined in one section retain that definition in subsequent sections unless noted otherwise.

We consider now the well known asymptotic forms of the wave functions describing spin-0 on spin-0 and spin-1 on spin-1 scattering experiments.

Form of the Asymptotic Wave Function for Scattering Spin-0 on Spin-0

We suppose first that a beam of spin-0 particles, which are all in the same eigenstate of linear momentum, is directed toward a target of spin-0 particles, and that there is no force between the two particles. The wave function $\psi_{PW}$ for the relative motion of the pair in the Schroedinger Representation is given by

$$\psi_{PW} = \sqrt{N} \ e^{i \mathbf{k} \cdot \mathbf{r}}$$

with $N$ - number of particles per unit volume of beam, and $\mathbf{k}$ - wave number for the relative motion of the two particles, and $\mathbf{r}$ - position vector of particle-1 with respect to particle-2.

We suppose now that the particles do exert forces on each other but under the restriction that the potential is a short range potential which depends only on the relative coordinates of the two particles. The asymptotic form of the wave function $\psi_{A}$ is given by

$$\psi_{A} = \sqrt{N} \ e^{i \mathbf{k} \cdot \mathbf{r}} + \sqrt{N} \ \frac{e^{i \mathbf{k} \cdot \mathbf{r}}}{r} f(\theta, \phi)$$
where \( f(\theta, \phi) \) is the scattering amplitude. Let us define the scattered wave \( \psi_{sc} \) with

\[
\psi_A = \psi_{pw} + \psi_{sc}. \tag{8}
\]

These two wave functions the plane wave \( \psi_{pw} \), which would be the system wave function if there were no interaction between the two particles, and the asymptotic wave \( \psi_A \), which is the actual wave function for the system, but in the limit of large \( r \), contain a description of the scattering experiment. The second term of \( \psi_{pw} + \psi_{sc} \) describes the scattered particles and it contains the information in which we are interested. Equation (6) is relative motion wave function, however apart from the mass of particle-2 this function is determinable from characteristics of the beam particles.

We will now briefly discuss the cross section in the case of spin-0 on spin-0.\(^8\)

The center of mass differential cross section \( \sigma(\theta, \phi) \) is defined by

\[
\text{(number of particles entering a detector at } (\theta, \phi) \text{ each sec.)} = \sigma(\theta, \phi) X \text{(plane wave probability current density)}
\]

that is,

\[
\frac{dN_d}{dt} = \sigma(\theta, \phi) j_{pw} \ d\Omega. \tag{9}
\]

The rate at which particles enter a detector of area \( dA \) that is perpendicular to \( \hat{k}_1 \) is given by
\[ \frac{dN_d}{dt} = j_{sc} \, dA. \]  

(10)

\( j_{sc} \) is the probability current density of the scattered wave at the detector.

If we compare equation (9) and equation (10) we find

\[ \sigma(\theta, \phi) \, j_{pw} \, d\Omega = j_{sc} \, dA. \]

Since

\[ d\Omega = dA/r^2 \]

we can write

\[ \sigma(\theta, \phi) = r^2 \frac{j_{sc}}{j_{pw}}. \]  

(11)

The probability current density \( \hat{\jmath} \) is defined by \( ^9 \)

\[ \hat{\jmath} = \text{Re} \left[ \psi^\dagger \left( \frac{\hbar}{im} \right) \nabla \psi \right]. \]  

(12)

\( m \) is the reduced mass of the system.

The plane wave probability current density is discovered by putting equation (6) in equation (12). The result is

\[ \hat{\jmath}_{pw} = \frac{\hbar k}{m} \, N \, \hat{k}_1 \]

and

\[ j_{pw} = \frac{\hbar k}{m} \, N. \]  

(13)

The probability current density for the scattered wave \( \hat{\jmath}_{sc} \) is from equations (12) and (7)
This gives

\[ r^2 j_{sc} = N \frac{\hbar k}{m} \left| f(\theta,\phi) \right|^2 + \frac{1}{r} \{ \} + \frac{1}{r^2} \{ \} . \]  

(14)

We drop the last two terms in this equation since the detector is located at large \( r \).

\[ r^2 j_{sc} = N \frac{\hbar k}{m} \left| f(\theta,\phi) \right|^2 . \]  

(14)

We insert \( r^2 j_{sc} \) from equation (14) and \( j_{PW} \) from equation (13) into equation (11) to find the differential cross section in terms of scattering amplitude. The result is

\[ \sigma(\theta,\phi) = \left| f(\theta,\phi) \right|^2 . \]  

(15)

Form of the Asymptotic Wave Function for Scattering Spin-1 on Spin-1

A beam of spin-1 particles, which are all in the same eigenstate of linear momentum, is directed toward a target of spin-1 particles. We assume for a moment that there is no force between the two particles. The wave function \( \psi_{PW} \) for the relative motion is given by

\[ \psi_{PW} = \sum_{s',v'} e^{i \mathbf{k} \cdot \mathbf{r}'} a_{s',v'} \chi_{s',v'} . \]  

(16)

This is the initial wave function for the system. \( \chi_{sv} \) is an eigenfunction of the total spin of the two particle system and of the z-component of total spin. \( s \) is the total spin quantum number, \( s = 0,1,2 \). \( v \) is the z-component
spin quantum number, $v = s, s - 1, \ldots, -s$. The $\chi_{sv}$ are column vectors in a 9 dimensional vector space. $\chi_{sv}$ has the symbol $|11 sv\rangle$ in the Dirac notation. This expression can represent any spin state for the system by choosing different values for the constants $a_{sv}$. The spin part of this initial wave function is given by $\psi_{pw}$,

$$\psi_{pw} = \sum_{s'v'} a_{s'v'} \chi_{s'v'}.$$  \hspace{1cm} (17)

We will now allow an interaction of the type allowed in the spin-0 case. Let us suppose that the sum of equation (16) reduces to a single term $a_{sv} \chi_{sv}$, that is, the system is in a eigenstate of total spin and z-component of spin. The asymptotic form of the wave function $\psi_A$ is then given by

$$\psi_A = a_{sv} e^{ikr} \chi_{sv} + a_{sv} e^{ikr} \sum_{s'v'} M_{s'v'sv} \chi_{s'v'}.$$  \hspace{1cm} (18)

This form allows that in the scattering the system may change from the state $sv$ to the state $s'v'$. The quantities $M(\theta \phi)$ give for each outgoing direction the amplitude of the outgoing spherical wave of spin state $s'v'$ that results from unit incident amplitude in spin state $sv$. A spin state described by a pair $sv$ is called a given channel.

If all of the terms of (16) are present in $\psi_{pw}$, then the asymptotic form of the wave function is given by
Let us define the scattered wave $V_{sc}$ by

$$\Psi_A = \sum_{s'v'} a_{s'v'} e^{ik'r} \chi_{s'v'}$$

$$+ \frac{e^{ikr}}{r} \sum_{s'v'} a_{s'v'} \sum_{s'v''} M(\theta, \phi) \chi_{s'v''}. \tag{19}$$

Let us define the scattered wave $\Psi_{sc}$ by

$$\Psi_A = \Psi_{pw} + \Psi_{sc}. \tag{20}$$

The scattered part of this wave $\Psi_{sc}$ describes the scattered particles. The spin part of the scattered portion of this wave function is given by $\Psi_{sc}$,

$$\Psi_{sc} = \sum_{s'v'} a_{s'v'} \sum_{s'v''} M(\theta, \phi) \chi_{s'v''}. \tag{21}$$

The asymptotic wave of equation (19) applies when all scattering events involve the same plane wave spin state, that given by equation (16). We wish to allow in this work for the possibility that in a given data run that different scattering events involve different plane wave spin states.

Figure 4 on the next page, shows symbolically the initial situations for a possible data run of 20 scattering events. The different arrows on particle-1 show different single particle spin states for that particle and not the system spin states we have discussed previously. The same is true of the arrows on particle-2. We notice that the target particles have random spin states while the beam particles are in certain spin states more often than in others. We have an unpolarized target and a partially polarized beam in
FIGURE 4. SYMBOLIC REPRESENTATION OF INITIAL SPIN STATES OF BEAM PARTICLES AND TARGET PARTICLES FOR A POSSIBLE DATA RUN
the case shown. The density matrix formalism which enables us to treat this problem is now discussed.

The Density Matrix

The density matrix formalism of von Neumann\textsuperscript{11} has been applied to spin state ensembles by several authors.\textsuperscript{12} This development follows Seyler.\textsuperscript{13}

Definition of the Density Matrix

We will discuss the density matrix in terms of the scattering of a spin-1 particle on a spin-1 particle. We have asserted that the scattering is represented by incident waves of the type

\[ \psi_{pw} = e^{i k \cdot r} \sum_{s'v'} a_{s'v'} \chi_{s'v'} \]  \hspace{1cm} (16)

together with scattered waves of the type

\[ \psi_{sc} = e^{i k r} \sum_{s'v'} (\sum_{s'v'} a_{s'v'} M(\theta) \chi_{s'v'}) \chi_{s'v'}' \] \hspace{1cm} (19)

Both of these wave functions are of the form

\[ \psi = F(r, \theta, \phi) \sum_{s'v'} b_{s'v'}^{(\theta, \phi)} \chi_{s'v'} \] \hspace{1cm} (22)

with a spin factor of

\[ \psi = \sum_{s'v'} b_{s'v'}^{(\theta, \phi)} \chi_{s'v'} \] \hspace{1cm} (23)

We will formulate the density matrix in terms of the function \( \psi \) given by equation (22) in order that the results
may be applied to either the \( \psi^p \) or the \( \psi^sc \) wave functions.

In a data run each scattering event may be described by one of the wave functions in the set \( \{ \psi \} \). \( W(\psi) \) is the probability of finding in a unit volume located by spherical coordinates \( (r, \theta, \phi) \) a particle in spin state \( \psi \).

Let the \( \psi \) be normalized as follows:

\[
\psi^\dagger \psi = \text{(number of particles in unit volume located at position (r,\theta,\phi) in spin state \( \psi \)).}
\]

\( \psi^\dagger \psi \) is the inner product in the 9 dimensional vector space.

The expectation value measured at position \( (r, \theta, \phi) \) for any operator \( \Omega \) that depends only on spin coordinates will be given by

\[
<\Omega> = \sum_{\psi} W(\psi) \frac{\psi^\dagger \Omega \psi}{\psi^\dagger \psi} .
\]  

(24)

The probability \( W(\psi) \) is given by

\[
W(\psi) = \frac{\text{(number of particles in unit volume at (r,\theta,\phi) in state \( \psi \))}}{\text{(number of particles in unit volume at (r,\theta,\phi) in all states \{\psi\}).}}
\]

(25)

It may be shown by direct substitution of equation (22) into equation (24) that if the density matrix is defined as

\[
\rho_{\psi^s,\psi'} = F(r, \theta, \phi) F^*(r, \theta, \phi) \sum_{\psi} b^*_{\psi^s,\psi'}(\psi) b^*_{\psi'}(\psi)
\]

(26)

or equivalently as

\[
\rho = \sum_{\psi} \psi \psi^\dagger ,
\]

(27)
then
\[ \langle \Omega \rangle = \frac{\text{Tr}[\rho \Omega]}{\sum_{\psi} \psi^\dagger \psi} \]  \hspace{1cm} (28)

**Properties of the Density Matrix**

We calculate the trace of \( \rho \).

\[ \text{Tr}[\rho] = \text{Tr}[\sum_{\psi} \psi \psi^\dagger] \]
\[ = \sum_{\psi} \text{Tr}[\psi \psi^\dagger] = \sum_{\psi} \psi^\dagger \psi. \]

Thus
\[ \text{Tr}[\rho] = \sum_{\psi} \psi^\dagger \psi. \]  \hspace{1cm} (29)

We combine equation (28) and equation (29) to find
\[ \langle \Omega \rangle = \frac{\text{Tr}[\rho \Omega]}{\text{Tr}[\rho]}. \]  \hspace{1cm} (30)

This is the expression which we will use to calculate expectation values of operators in terms of the density matrix which describes the system in the ensemble of states \( \{\psi\} \).

It is easy to show that \( \rho \) is Hermitean by using the definition of equation (27).

\[ \rho = \rho^\dagger. \]  \hspace{1cm} (31)

Additionally, \( \rho \) is an observable.\textsuperscript{14}

In Figure 4 we showed, for a possible data run, the ensemble of spin-1 single particle spin states of particle-1 and also those of particle-2. We could, knowing these ensembles, calculate the density matrices \( \rho_\text{B} \) and \( \rho_\text{T} \) for these two particles. It is useful to be able to derive the density
matrix of the system $\rho_{pw}$ from these two density matrices. The following result relates the density matrices $\rho_b$ and $\rho_t$, which describe two independent particles (such as beam particles and target particles before interaction) to the density matrix $\rho_{pw}$ for the system composed of the two particles.

The wave function for particle-1 expanded in terms of spin-1 spin wave functions $\chi_1^{\mu_1}$ is

$$\psi_b = \sum_{\mu_1} \alpha_{\mu_1} \chi_1^{\mu_1}$$

and that for particle-2 is

$$\psi_t = \sum_{\mu_2} \beta_{\mu_2} \chi_1^{\mu_2}$$

We form the density matrices for these two particles by using equation (26). We leave out the spatial part for reasons which will be discussed later.

$$\left(\rho_b\right)_{\mu_1,\mu_1} = \sum_{\psi_b} \alpha_{\mu_1} \psi_b \alpha_{\mu_1}^{*}\left(\psi_b\right)$$

and

$$\left(\rho_t\right)_{\mu_2,\mu_2} = \sum_{\psi_t} \beta_{\mu_2} \psi_t \beta_{\mu_2}^{*}\left(\psi_t\right)$$

The wave function for the system composed of particle-1 and particle-2 is

$$\psi_{pw} = \psi_b \otimes \psi_t = \sum_{\mu_1,\mu_2} \alpha_{\mu_1} \beta_{\mu_2} \chi_1^{\mu_1} \chi_1^{\mu_2}.$$
and its density matrix is from (26)

\[
\begin{aligned}
\langle \rho_{pw} \rangle _{\mu_1 \mu_2, \nu_1 \nu_2} &= \sum_{\psi_t} \alpha^\dagger_{\mu_1} (\psi_b) \beta_{\mu_2} (\psi_t) \cdot \alpha^\dagger_{\mu_1} (\psi_b) \beta^\dagger_{\mu_2} (\psi_t), \\
\langle x_1^\mu x_2^\mu \rangle &\text{ has the symbol } |1 \mu_1 \mu_2 \rangle \text{ in the Dirac notation.}
\end{aligned}
\]

Comparing equations (34), (35) and (37) we see

\[
\rho_{pw} = \rho_b \otimes \rho_t.
\]

This result gives the system density matrix represented in terms of the wave functions \( x_1^\mu x_1^\mu = |1 \mu_1 \mu_2 \rangle \). Since \( \rho_{pw} \) is an observable the following similarity transformation would give \( \rho_{pw} \) in the channel spin representation.

\[
\begin{aligned}
\left( \langle \rho_{pw} \rangle _{s',\nu',s\nu} \right) &= C^\dagger \left( \langle \rho_{pw} \rangle _{\mu_1 \mu_2, \nu_1 \nu_2} \right) C, \\
C &= \left( \langle 1 \mu_1 \mu_2 | s\nu \rangle \right).
\end{aligned}
\]

\( \langle 1 \mu_1 \mu_2 | s\nu \rangle \) is the inner product in the Dirac notation of the spin wave functions \( |s\nu \rangle \) and \( |1 \mu_1 \mu_2 \rangle \). It is a Clebsch-Gordon coefficient.

The ensemble of states \( \{ \psi_{pw} \} \), which depend on the preparation of the target and the beam, determines the plane wave density matrix \( \rho_{pw} \). The density matrix for the scattered particles \( \rho_{sc} \) is needed if we are to calculate the expectation values for operators for the scattered particles.
The ensemble \( \{\psi_{sc}\} \) is determined both by \( \{\psi_{pw}\} \) and by the Hamiltonian for the interaction. We wish to express \( \rho_{sc} \) in terms of the \( \rho_{pw} \) and a quantity which depends only on the Hamiltonian. The transition matrix is this quantity.

**Transition Matrix**

The transition matrix which relates \( \rho_{sc} \) to \( \rho_{pw} \) introduces into our formalism the effect of the detailed nature of the interaction on the results of the experiment.

**Definition of the Transition Matrix**

We have in \( \psi_{pw} \), given by equation (17), the spin part of the plane wave function and in \( \psi_{sc} \), given by equation (21), the spin part of the scattered portion of the asymptotic wave function. These wave functions, describing the system before and after the scattering, can be related in a matrix equation,

\[
\psi_{sc} = M\psi_{pw},
\]

(40)

where \( M \) is a 9x9 matrix with elements \( M_{s',s}\) and \( \psi_{pw} \) and \( \psi_{sc} \) are column vectors in 9-space. \( M \) is called the transition matrix.

The plane wave function of the system is given by

\[
\psi_{pw} = e^{ik \cdot \vec{r}} \psi_{pw}
\]

(41)

from equations (16) and (17).
The scattered portion of the asymptotic wave is given by

$$\psi_{sc} = \frac{e^{i k r}}{r} \psi_{sc}$$

from equations (19), (20) and (21).

The spatial part of these wave functions will not enter into the expectation values which we will calculate, so we can use density matrices defined only for the spin portion of the wave functions. We use:

$$\rho_{pw} = \sum_{\psi_{pw}} \psi_{pw}^* \psi_{pw}^\dagger$$

and

$$\rho_{sc} = \sum_{\psi_{sc}} \psi_{sc}^* \psi_{sc}^\dagger.$$  

It is a simple consequence of equations (40), (43) and (44) that

$$\rho_{sc} = M \rho_{pw} M^\dagger.$$ 

If we know the plane wave density matrix and the transition matrix we will be able to calculate expectation values for the scattered particles.

**Observable Quantities in Terms of the Transition Matrix**

It will be of interest to express the differential cross section in terms of the initial density matrix and the transition matrix. We take as our definition of differential cross section equation (11) with the probability current
densities $J_{sc}$ and $J_{pw}$ being those for all the states in the ensemble.

$$\sigma(\theta, \phi) = r^2 J_{sc} / J_{pw}.$$  \hspace{1cm} (46)

The plane wave probability current density $\mathbf{j}_{pw}$ for the state $\psi_{pw}$ is calculated below. We use equation (41) in equation (12).

$$\mathbf{j}_{pw} = \text{Re}[\psi_{pw}^\dagger \left(\frac{\hbar}{\text{Im}}\right) \mathbf{v} \psi_{pw}] = \text{Re}[\left(e^{-i \mathbf{k} \cdot \mathbf{r}} \psi_{pw}\right)^\dagger \left(\frac{\hbar}{\text{Im}}\right) \mathbf{v} \left(e^{i \mathbf{k} \cdot \mathbf{r}} \psi_{pw}\right)].$$

Since

$$\mathbf{v} \left(e^{i \mathbf{k} \cdot \mathbf{r}} \psi_{pw}\right) = i k e^{i \mathbf{k} \cdot \mathbf{r}} \psi_{pw} + e^{i \mathbf{k} \cdot \mathbf{r}} \mathbf{v} \psi_{pw}$$

and since $\psi_{pw}$ does not depend upon $(r, \theta, \phi)$ so $\mathbf{v} \psi_{pw} = 0$ we can write

$$\mathbf{j}_{pw} = \text{Re}[e^{-i \mathbf{k} \cdot \mathbf{r}} \psi_{pw}^\dagger \left(\frac{\hbar}{\text{Im}}\right) i k e^{i \mathbf{k} \cdot \mathbf{r}} \psi_{pw}]$$

$$= \text{Re}[\frac{\hbar k}{m} \psi_{pw}^\dagger \psi_{pw}] = \frac{\hbar k}{m} \psi_{pw}^\dagger \psi_{pw}.$$  \hspace{1cm} (47)

The current density for the ensemble may be expressed in terms of $\rho_{pw}$ by using equation (29).

$$\mathbf{j}_{pw} = \mathbf{\nabla} \cdot \mathbf{j}_{pw} = \frac{\hbar k}{m} \mathbf{\nabla} \cdot \psi_{pw}^\dagger \psi_{pw}$$

$$= \frac{\hbar k}{m} \text{Tr}[\rho_{pw}].$$  \hspace{1cm} (47)
The scattered probability current density \( \mathbf{j}_{sc} \) for the state \( \psi_{sc} \) is found by using equation (42) in equation (12).

\[
\mathbf{j}_{sc} = \text{Re}[\psi_{sc}^\dagger \frac{\hbar}{im} \nabla \psi_{sc}] = \text{Re}[\left(\frac{e^{ikr}}{r}\right)^\dagger \psi_{sc}^\dagger \left(\frac{\hbar}{im}\right) \nabla \left(\frac{e^{ikr}}{r}\right) \psi_{sc}].
\]

The gradient factor is

\[
\nabla \left(\frac{e^{ikr}}{r}\psi_{sc}\right) = \left(\frac{e^{ikr}}{r} i k + e^{ikr} \left(-\frac{1}{r^2}\right) \right) \hat{k}_{1} \psi_{sc} + \frac{e^{ikr}}{r} \nabla \psi_{sc}.
\]

\( \psi_{sc} \) depends on \( \theta \) and \( \phi \) and not on \( r \). The \( \nabla \psi_{sc} \) will thus involve terms in \( 1/r \) and this will lead to terms in \( 1/r^3 \) in \( \mathbf{j}_{sc} \). We will not need to keep these terms since we are interested in \( r^2 \mathbf{j}_{sc} \) in the asymptotic region.

\[
\mathbf{j}_{sc} = \text{Re}\left[\frac{e^{-ikr}}{r} \psi_{sc}^\dagger \frac{\hbar}{im} \frac{e^{ikr}}{r} (ik) \psi_{sc} \hat{k}_{1}\right]
\]

\[
= \text{Re}\left[\frac{\hbar k}{m} \psi_{sc}^\dagger \psi_{sc} \hat{k}_{1}\right]
\]

\[
= \frac{1}{r^2} \frac{\hbar k}{m} \psi_{sc}^\dagger \psi_{sc} \hat{k}_{1}.
\]

The current density for the ensemble \( \mathbf{j}_{sc} \) may be expressed in terms of \( \rho_{sc} \) by using equation (29).

\[
\mathbf{j}_{sc} = \sum_{\psi_{sc}} \mathbf{j}_{sc} = \sum_{\psi_{sc}} \frac{1}{r^2} \frac{\hbar k}{m} \psi_{sc}^\dagger \psi_{sc} \hat{k}_{1}
\]

\[
= \frac{1}{r^2} \frac{\hbar k}{m} \hat{k}_{1} \text{Tr}[\rho_{sc}]. \quad (48)
\]

We insert \( \mathbf{j}_{sc} \) and \( \mathbf{j}_{pw} \) in equation (46) to find the differential cross sections.
\[ \sigma(\theta, \phi) = \frac{\text{Tr}[\rho_{SC}]}{\text{Tr}[\rho_{PW}]} \]  

(49)

Equation (45) enables us to express this in the desired form

\[ \sigma(\theta, \phi) = \frac{\text{Tr}[M\rho_{PW}^* M^\dagger]}{\text{Tr}[\rho_{PW}]} \]  

(50)

The expectation value for the scattered particles of an operator such as the spin of particle-1 \( \vec{\mathbf{S}} \) is from equation (30) given by

\[ \vec{\mathbf{P}} = \langle \vec{\mathbf{S}} \rangle = \frac{\text{Tr}[\rho_{SC} \vec{\mathbf{S}}]}{\text{Tr}[\rho_{SC}]} \]  

(51)

Combining this with equation (45) we find

\[ \vec{\mathbf{P}} = \frac{\text{Tr}[M\rho_{PW}^* M^\dagger \vec{\mathbf{S}}]}{\text{Tr}[M\rho_{PW}^* M^\dagger]} \]  

(52)

We have expressed the observables such as differential cross section and polarization \( \vec{\mathbf{P}} \) in terms of the initial density matrix and the transition matrix. The initial density matrix will be determined for the case of an unpolarized target by the polarization state of the beam. The transition matrix which is determined by the exact nature of the interaction is thus the quantity of interest for predicting theoretically the observables. In order to express the observables in useful forms we will now study some of the properties of the transition matrix and we will find some different methods of expressing it.
Transition Matrix in Terms of the Nuclear Collision Matrix

The following expression from Lane and Thomas\textsuperscript{16} for the transition matrix in the channel spin representation reveals some of the structure of the transition matrix. The explicit dependence on the scattering angle is shown, and the energy dependence is, apart from the $1/k$ factor, concentrated in the nuclear collision matrix.

$$M_{S',\nu',_{S',\nu'}}^{(0)} = \sqrt{\pi} / k \left[ -C(0) \delta_{SS'} \delta_{\nu\nu'} + i \sum_{j',l',l} \sqrt{2l'+1} ight]$$

$$x |s\lambda\nu\rangle \langle s'\lambda'\nu' \nu-\nu' | j\nu\rangle \exp[i(\omega_\lambda + \omega_{\lambda'})]$$

$$x (\delta_{SS'} \delta_{\lambda\lambda'} - U^j_{S',L',S_L} y^{\nu-\nu'}_{\lambda'} (0,0))$$

with:

- $\lambda$ - quantum number for the orbital angular momentum of the relative motion, $\lambda = 1, 2, 3, 4, \ldots$,
- $j$ - quantum number for the total angular momentum, $j = \lambda + s$, $\lambda + s - 1, \ldots, |\lambda - s|$,
- $y^m_{\lambda}$ - normalized spherical harmonics of Condon and Shortley,\textsuperscript{17}
- $C(0)$ - Coulomb scattering amplitude,\textsuperscript{18}
- $\omega_\lambda$ - Coulomb phase shift,\textsuperscript{18}
- $U^j_{S',L',S_L}$ - element of the nuclear collision submatrix, $U^j_{S',L',S_L}$. 

The first term in this expression results from the long range Coulomb force, which is not polarizing. The second term results from the nuclear forces, assumed not to act beyond some certain distance. Our previous development which prohibited long range forces can be modified for Coulomb forces. Lane and Thomas have included this possibility and the results of this thesis will be valid when there are Coulomb forces.

We discuss next the modifications of M required by the situation of identical particles.

**Transition Matrix in Terms of the Nuclear Collision Matrix—Identical Particles Form**

Figure 5 on the next page illustrates the physics of the identical particle situation. The scattered wave function for case 1 is derivable from that of case 2 by the operation of interchange of particle coordinates. When particle 1 is identical with particle 2, these two cases are not distinguishable by any measurement and these wave functions are coherent. The Pauli Exclusion Principle requires that the wave functions of spin-1 particles be symmetric under the operation of particle exchange.

The plane wave from equation (41) is

\[ \psi_{ pw} = e^{ik \cdot r} \psi_{ pw} \]  \hspace{1cm} (41)

and the scattered wave from equation (42) is
FIGURE 5. SCATTERING OF IDENTICAL PARTICLES
\[ \psi_{SC} = e^{i kr}/r \psi_{SC}. \] (42)

Let us introduce the particle coordinates for particle-1 and particle-2 into our symbols as \( \psi_{SC}^{1,2}, \psi_{PW}^{1,2} \) and \( \chi_{SV}^{1,2} \). The symmetry of the spin wave functions under particle exchange is given by

\[ \chi_{SV}^{1,2} = (-1)^S \chi_{SV}^{2,1}. \] (54)

It is possible to form a plane wave \( \psi_{PW} \) which is symmetric in a particle exchange as

\[ \hat{\psi}_{PW}^{1,2} = \sum_{SV} \chi_{SV}^{1,2} \left[ e^{i \mathbf{k} \cdot \mathbf{r}} + (-1)^S e^{-i \mathbf{k} \cdot \mathbf{r}} \right]; \]

however the physical situation we are treating does not involve particles traveling in the -z direction so this is not relevant to our situation. We will form a scattered wave symmetric to particle exchange \( \hat{\psi}_{SC}^{1,2} \) as follows:

\[ \hat{\psi}_{SC}^{1,2} = \psi_{SC}^{1,2} + \psi_{SC}^{2,1}. \]

Since \( e^{i kr}/r \) does not change in a particle exchange we may write this as

\[ \hat{\psi}_{SC}^{1,2} = \frac{e^{i kr}}{r} [\psi_{SC}^{1,2} + \psi_{SC}^{2,1}]. \]

We define \( \hat{\psi}_{SC}^{1,2} \) by

\[ \hat{\psi}_{SC}^{1,2} = \psi_{SC}^{1,2} + \psi_{SC}^{2,1}. \]
then by inserting equation (21) into this last expression we find

$$\hat{\psi}_{sc}(1,2) = \sum_{su} \sum_{s'v'} M(\theta, \phi) \chi_{s'v'}(1,2)$$

$$+ \sum_{su} \sum_{s'v'} M(\pi-\theta, \phi+\pi) \chi_{s'v'}(2,1).$$

We use equation (54) to show that

$$\hat{\psi}_{sc}(1,2) = \sum_{su} \sum_{s'v'} [M(\theta, \phi) + (-1)^{s'v'v} M(\pi-\theta, \phi+\pi)] \chi(1,2)$$

(55)

We define the identical particle transition matrix $\hat{M}$ to have elements given by

$$\hat{M}(\theta, \phi) = [M(\theta, \phi) + (-1)^{s'v'} M(\pi-\theta, \phi+\pi)].$$

(56)

Equations (17), (55) and (56) lead to the identical particles form of equation (40)

$$\hat{\psi}_{sc} = \hat{M} \psi_{pw}.$$  

(57)

We insert equation (53) into equation (56) to find

$$\hat{M}(\theta, \phi) = \sqrt{\pi/k} \{ -C(\theta) + (-1)^{s'v'} C(\pi-\theta) \delta_{ss'} \delta_{vv'}$$

$$+ \frac{1}{j2l+1} \left< s'v' | jv \right> \langle s'v' | jv' \rangle \delta_{ss'} \delta_{ll'}$$

$$\times \exp[i(\omega_{s'} + \omega_{v'})] \left[ \delta_{ss'} \delta_{ll'} - \frac{uj}{s'v',s'v'} \right]$$

$$\times \left\{ Y_{s'}^{(v-v')}(\theta, \phi) + (-1)^{s'v'} Y_{s'}^{v-v'}(\pi-\theta, \phi+\pi) \right\} \}.$$
Using the parity of the spherical harmonics,\(^2\)

\[ Y_{\ell', v'}^{\nu'-\nu} (\pi-\theta, \phi+\pi) = (-1)^{\ell'} Y_{\ell', v'}^{\nu'-\nu} (\theta, \phi), \]

in this last equation one obtains

\begin{align*}
\hat{M} (\theta, \phi) &= \sqrt{\pi/k} \left[ -\left( C(\theta) + (-1)^{s'} C(\pi-\theta) \right) \delta_{ss'} \delta_{vv'} , \\
&\quad + i \sum_{j,l,l'} \sqrt{2l+1} \langle s'l'v'jv' | s'l'v'v'-v' | jv \rangle \langle jv | s'l'v'v'-v' | jv' \rangle \right] x \exp[i(\omega_l+\omega_{l'})] \left( \delta_{ss'} \delta_{ll'} - U^j_{s'l'sl} \right) x (1 + (-1)^{s'+l'}) Y_{\ell'}^{\nu'-\nu} (\theta, \phi) \right].
\end{align*}

(58)

This is the identical particle form of the transition matrix in terms of the nuclear collision matrix.

We can use equation (58) to deduce the effect of the conservation of parity restriction on the transition matrix and on the scattering.

Conservation of Parity Restriction

on Transition Matrix

The condition that parity be conserved leads to a relationship between elements of \(\hat{M}\). This relationship is

\[ \hat{M}_{s'-v', s-v} = (-1)^{s'-s} (-1)^{\nu'-\nu} \hat{M}_{s', v', s_v}. \]

(59)

The proof of this result is straightforward. Conservation of parity limits the sum of incident orbital angular momentum quantum number \(l\) and scattered orbital angular momentum quantum number \(l'\) to being an even number. We may
insert this into the calculation by rewriting the triple sum in $\hat{M}_{s'v'sv}$ as follows:

$$
\begin{align*}
\text{odd } & \text{ odd } \quad l+s \\
\sum_{j\ell'\ell} \ & \sum_{l'=1}^{l+s} \quad \sum_{j=|\ell-s|}^{l=1}
\end{align*}
$$

The expression for $\hat{M}_{s'\bar{v}',s-\bar{v}}$ may be written in the same manner. The properties of the C. G. Coefficients and the Spherical Harmonics under change of sign of the $z$-projection quantum number complete the proof. These are

$$
\begin{align*}
\langle s,l,-\bar{v},0|j-\bar{v}\rangle &= (-1)^{s+l+j} (-1)^{2\bar{v}} \langle s\ell\nu\sigma|j\nu\rangle, \\
\langle s'l'-\bar{v}',v'-\bar{v}|j-\bar{v}\rangle &= (-1)^{s'+l'+j'} (-1)^{2\bar{v}} \langle s'l'\nu'|v-\bar{v}'|j\nu\rangle, \\
(-1)^{\bar{v}'-\bar{v}} \chi_{\ell'}^{\nu'}(\nu,\sigma) &= \chi_{\ell}^{\nu}(\nu,\sigma).
\end{align*}
$$

The same proof can be used for the non-identical particles case starting with $M$ from equation (53).

Equation (59) implies that the parity conserving form of the transition matrix has only 41 independent elements. The elements of the transition matrix may be expressed in terms of the 41 members of the set $\{m_i\}$. These members are functions of energy and scattering angle. We show the transition matrix in terms of the $\{m_i\}$ in Table 1 on the next page.
**TABLE 1. TRANSITION MATRIX IN TERMS OF THE \{m_i\}**

<table>
<thead>
<tr>
<th>s'v'</th>
<th>22</th>
<th>21</th>
<th>20</th>
<th>2-1</th>
<th>2-2</th>
<th>11</th>
<th>10</th>
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</table>
Time Reversal Invariance Restriction on Transition Matrix

The transition matrix $M$ must be invariant under time reversal. This implies that the 41 elements of $M$ are not independent. A simple way to find the relationships between the elements of $M$ is to use the fact that the time reversal invariance restriction is simply expressed when the transition matrix is expressed in the helicity representation. This method follows Seyler.\textsuperscript{22}

1. The time reversal invariance condition on $M$ in the helicity representation, $M^h$, is\textsuperscript{23}

$$M^h(\theta)_{s'm'sm} = (-1)^{m'-m} M^h(\theta)_{s'm'sm}. \quad (60)$$

The relation between $M$ in the helicity representation and our $M$ is\textsuperscript{24}

$$M(\theta)_{s'v'sv} = \sum_{m'} d^s'_{v'm'}(\theta) M^h(\theta)_{s'm'sv}. \quad (61)$$

where $d^s'_{v'm'}(\theta)$ is an element of the reduced rotation matrix.

We invert this last expression by multiplying by

$$\sum_{v'} d^s'_{v'm}(\theta) d^s'_{v'm'}(\theta) = \delta_{m'm'}.$$  We arrive at the equation

$$\sum_{v'} d^s'_{v'm}(\theta) M^h_{s'v'sv}(\theta) = M^h_{s'm'sv}(\theta). \quad (62)$$

Combining this last equation with equation (60) we find
\[
\sum_{s'v's'v} d_{s'v's'v}^S (\theta) M(\theta) = (-1)^{m-v} m_{sv}^h (\theta)_{svsm}.
\]

Equation (61) allows us to eliminate \( m_{sv}^h \) from this last equation. We find

\[
\sum_{s'v's'v} d_{s'v's'v}^S (\theta) M(\theta) = (-1)^{m-v} \sum_{s'v's'v} d_{s'v's'v}^S M(\theta).
\]

We change index \( m \) to \( v \) and index \( m' \) to \( v' \) to find the desired result.

\[
\sum_{s'v's'v} d_{s'v's'v}^S (\theta) M(\theta) = (-1)^{v'-v} \sum_{s'v's'v} d_{s'v's'v}^S M_{sv}^{sv's'm}.
\] (63)

This is a set of 81 equations, not all of which are independent, which relate the elements of \( M \). The independent equations are derived in Chapter III, page 69.

**Transition Matrix Represented in Terms of Spin Operators**

The spin dependence of \( M \) can be revealed by expanding it in terms of a linear combination of spin operators. The coefficients in the expansion will be functions of energy and scattering angle. This work is an extension to spin-1 of the spin-1/2 work of Wolfenstein and Ashkin.25

We need a complete set of 81 spin operators in the product space. We will work in the knp frame. We begin with the spin operators of particle-1 and particle-2 in product space together with the identity operator.
These operators are all Hermitean and all except \( I^9 \) have trace zero. We form all possible products of the operators in each set with themselves and each other to get

\[
\{I^9, E_k, E_n, E_p\} \quad \text{and} \quad \{I^9, S_k, S_n, S_p\}.
\]

We replace the operators \( E_\alpha E_\beta \) and \( S_\alpha S_\beta \) with

\[
E_{\alpha\beta} = \frac{1}{2} (E_\alpha E_\beta + E_\beta E_\alpha) - \frac{2}{3} I^9
\]

and

\[
S_{\alpha\beta} = \frac{1}{2} (S_\alpha S_\beta + S_\beta S_\alpha) - \frac{2}{3} I^9.
\]

for \( \alpha = k,n,p \) and \( \beta = k,n,p \). This makes each operator in the set be Hermitian and excepting \( I^9 \) have trace zero.

Since

\[
E_{nn} = -E_{kk} - E_{pp}
\]

and

\[
S_{nn} = -S_{kk} - S_{pp},
\]

(65)
we delete these dependent operators to get two sets of 9 operators; Set-1 is

\[ \{ l^9, \Sigma_k, \Sigma_n, \Sigma_p, \Sigma_{kk}, \Sigma_{kn}, \Sigma_{kp}, \Sigma_{np}, \Sigma_{pp} \} \]

and Set-2 is

\[ \{ l^9, S_k, S_n, S_p, S_{kk}, S_{kn}, S_{kp}, S_{np}, S_{pp} \} \].

The 81 possible products of these operators form a complete set. Let us refer to this set as Set 81.

If the following condition is obeyed for \( A_k \) and \( A_j \), any operators in our set of 81, then the set of operators will be complete.

\[ \text{Tr} [A_i A_j^\dagger] = (\text{const}) \delta_{ij}. \] (66)

This may be understood by an example in a two dimensional vector space. Suppose we have a set of 4 operators \( A, B, C, D \) which obey condition (66). We associate with each operator a 4 vector as follows:

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} = a
\]

\[
B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix} = b
\]

and similarly for \( C \) and \( D \).
\[
\text{Tr}[AB^\dagger] = (a_{11}b_{11}^* + a_{12}b_{12}^* + a_{21}b_{21}^* + a_{22}b_{22}^*).
\]
\[
b^\dagger a = (a_{11}b_{11}^* + a_{12}b_{12}^* + a_{21}b_{21}^* + a_{22}b_{22}^*).
\]
\[
\text{Tr}[AB^\dagger] = b^\dagger a.
\]
Thus the condition (66) for the operators is the same as the 4-vectors being orthogonal and hence linearly independent. The set of operators A,B,C,D will be a complete set.

We may use the trace theorems and properties on pages 91 through 94 to calculate the trace of the product of any operator in Set 81 with any other operator in that set. These operators do not satisfy the condition (66) because \(\Sigma_{kk}\) is not orthogonal to \(\Sigma_{pp}\). We can construct a set of operators which do satisfy condition (66) by replacing \(\Sigma_{kk}\) and \(\Sigma_{pp}\) with \(\Sigma_{kk} + \Sigma_{pp}\) and \(\Sigma_{kk} - \Sigma_{pp}\) in Set 1, and in Set 2 replacing \(S_{kk}\) and \(S_{pp}\) with \(S_{kk} + S_{pp}\) and \(S_{kk} - S_{pp}\). The set of 81 operators formed by the products of the operators in modified Set 1 with those in modified Set 2 are orthogonal. Thus the original set of operators is a linearly independent set and hence complete.

We do not need to include all 81 operators in our expansion but only those operators which are invariant to reflection and time reversal, because the transition matrix is invariant to reflection in space and time. We now identify the operators which have the required properties.
We see from the definitions (1) that the signs of the $k$ and $\hat{p}$ vectors change under space reflection and that of the $\hat{n}$ vector does not. The $\hat{p}$ and $\hat{n}$ vectors change sign under time reversal and $\hat{k}$ does not.

The spin operators $\hat{\pi}$ and $\hat{\sigma}$ are vector operators which have the same space and time reflection properties as $\hat{r} \times \hat{p}$, thus they change sign under time reversal and not under space reflection.

We now present some examples of determining the reflection and time reversal properties of the operators in Set 81.

<table>
<thead>
<tr>
<th>Direction Properties</th>
<th>Basic Operator Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Direction</strong></td>
<td><strong>Space</strong></td>
</tr>
<tr>
<td>$k$</td>
<td>-</td>
</tr>
<tr>
<td>$n$</td>
<td>+</td>
</tr>
<tr>
<td>$p$</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Specific Operators</th>
<th><strong>Direction</strong></th>
<th><strong>Operator</strong></th>
<th><strong>Space</strong></th>
<th><strong>Time</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>$l^9$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\Sigma_k$ or $S_k$</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$\Sigma_n$ or $S_n$</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$\Sigma_p$ or $S_p$</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

We see that $l^9$, $\Sigma_n$ and $S_n$ are invariant under both time and space reflection.
We see that $\Sigma_k S_k$ and $\Sigma_n S_n$ have the desired properties. The 25 acceptable operators are those appearing in Table 2 on the next page which presents the expansion of $M$ in terms of the spin operators. The coefficients $\{A_k\}$ are dependent on energy and angle of scattering.

Forty-one of the 81 operators have the correct space reflection property. This number agrees with the number of members in $\{m_i\}$. Forty-three of the 81 have the correct time reversal reflection property and as mentioned 25 have both the correct time and reflection property.

**Transition Matrix in Terms of Spin Operators--Identical Particles Cases**

The $M$ matrix must be invariant under the exchange of particles if the particles are identical. We now develop an expansion with that property.

We examine the particle exchange properties of some of the operators in (67).
TABLE 2
TRANSITION MATRIX EXPANDED IN TERMS OF SPIN OPERATORS

\[ M = A_1 \Sigma_n^9 + A_2 \Sigma_n + A_3 S_n + A_4 \Sigma_k S_k + A_5 \Sigma_n S_n + A_6 \Sigma_p \Sigma_p + A_7 \Sigma_{kk} + A_8 \Sigma_{pp} + A_9 S_{kk} + A_{10} S_{pp} + A_{11} \Sigma_k S_{kn} + A_{12} \Sigma_n S_{kk} + A_{13} \Sigma_n S_{pp} + A_{14} \Sigma_p S_{np} + A_{15} \Sigma_{kk} S_n + A_{16} \Sigma_{kk} S_{kk} + A_{17} \Sigma_{kk} S_{pp} + A_{18} \Sigma_{kn} S_k + A_{19} \Sigma_{kn} S_{kn} + A_{20} \Sigma_{kp} S_{kp} + A_{21} \Sigma_{np} S_p + A_{22} \Sigma_{np} S_{np} + A_{23} \Sigma_{pp} S_n + A_{24} \Sigma_{pp} S_{kk} + A_{25} \Sigma_{pp} S_{pp} \]

\( (67) \)
Under Particle exchange:

- $\Sigma_n$ goes to $\Sigma_n$
- $\Sigma_n$ goes to $\Sigma_n$
- $\Sigma_k S_p$ goes to $\Sigma_p S_k$
- $\Sigma_k S_k$ goes to $\Sigma_k S_k$
- $S_{kk}$ goes to $\Sigma_{kk}$
- $\Sigma_n + S_n$ goes to $\Sigma_n + S_n$
- $\Sigma_n - S_n$ goes to $S_n - \Sigma_n$
- $\Sigma_k S_{kn}$ goes to $\Sigma_k S_k$
- $\Sigma_k S_{kn} + \Sigma_k S_k$ goes to $\Sigma_k S_k + \Sigma_k S_{kn}$
- $\Sigma_k S_{kn} - \Sigma_k S_k$ goes to $\Sigma_k S_k - \Sigma_k S_{kn}$

We see that $\Sigma_k S_k$, $\Sigma_n + S_n$, and $\Sigma_k S_{kn} + \Sigma_k S_k$ remain unchanged in a particle exchange, while the other operators change.

Let us as an example consider a portion of equation (67)

$$A_2 \Sigma_n + A_3 S_n + A_4 \Sigma_k S_k + A_{11} \Sigma_k S_{kn} + A_{18} \Sigma_k S_{kn}.$$ 

We can rewrite this as

$$B_2 (\Sigma_n + S_n) + B_{18} (S_n - \Sigma_n) + B_3 \Sigma_k S_k$$

$$+ B_8 (\Sigma_k S_{kn} + \Sigma_k S_k) + B_{21} (\Sigma_k S_{kn} - \Sigma_k S_k)$$

if

$$B_2 + B_{18} = A_3, \quad B_2 - B_{18} = A_2, \quad B_8 + B_{21} = A_{11}$$

$$B_8 - B_{21} = A_{18}, \quad B_3 = A_4.$$
The portion of (67) in terms of the B coefficients will be invariant to particle exchange if $B_{18}$ and $B_{21}$ are zero.

We rewrite equation (67) in this fashion taking sums and differences of operators to get equation (68) of Table 3 on page 44. This expression is invariant to particle exchange if we set $B_{18}$ through $B_{25}$ equal to zero. The relationships between the A coefficients and the B coefficients are given in Table 4 on page 45.

**Spherical Polarization Observables**

We have seen that a knowledge of the transition matrix and the plane wave density matrix yields the expectation values of operators for the scattered particles. The beam density matrix determines the plane wave density matrix in the case of an unpolarized target. It is convenient to express the beam density matrix in terms of a complete set of spin operators. The Cartesian operators in Set-1 on page 37 which are complete in an order-3 vector space would serve. We will instead use a complete set of equivalent irreducible spherical tensor spin operators following Lakin. These operators hereinafter referred to as spherical tensor operators will also be used to describe the scattered particles. The advantage of this choice is that polarizations may be transformed between rotated frames by a linear transformation.
TABLE 3

TRANSITION MATRIX EXPANDED IN TERMS OF PARTICLE
SYMMETRIC AND ANTISYMOMETRIC SPIN OPERATORS

\[ M = B_1 \cdot 9 + B_2 (\Sigma_n + S_n) + B_3 (\Sigma_k S_k) \]
\[ + B_4 (\Sigma_n S_n) + B_5 (\Sigma_p S_p) + B_6 (\Sigma_{kk} + S_{kk}) \]
\[ + B_7 (\Sigma_{pp} + S_{pp}) + B_8 (\Sigma_{kn} + \Sigma_{kn} S_k) \]
\[ + B_9 (\Sigma_{nS_{kk}} + \Sigma_{kk} S_n) + B_{10} (\Sigma_{nS_{pp}} + \Sigma_{pp} S_n) \]
\[ + B_{11} (\Sigma_{pS_{np}} + \Sigma_{np} S_p) + B_{12} (\Sigma_{kk} S_{kk}) \]
\[ + B_{13} (\Sigma_{kk} S_{pp} + \Sigma_{pp} S_{kk}) + B_{14} (\Sigma_{kn} S_{kn}) \]
\[ + B_{15} (\Sigma_{kp} S_{kp}) + B_{16} (\Sigma_{np} S_{np}) \]
\[ + B_{17} (\Sigma_{pp} S_{pp}) + B_{18} (S_n - \Sigma_n) \]
\[ + B_{19} (S_{kk} - \Sigma_{kk}) + B_{20} (S_{pp} - \Sigma_{pp}) \]
\[ + B_{21} (\Sigma_{kS_{kn}} - \Sigma_{kn} S_k) + B_{22} (\Sigma_{nS_{kk}} - \Sigma_{kk} S_n) \]
\[ + B_{23} (\Sigma_{nS_{pp}} - \Sigma_{pp} S_n) + B_{24} (\Sigma_{pS_{np}} - \Sigma_{np} S_p) \]
\[ + B_{25} (\Sigma_{kk} S_{pp} - \Sigma_{pp} S_{kk}) \]

(68)
TABLE 4

RELATIONSHIPS BETWEEN COEFFICIENTS IN \{B_j\} AND IN \{A_k\}

\[
\begin{align*}
A_1 &= B_1 \\
A_2 &= B_2 - B_{18} \\
A_3 &= B_2 + B_{18} \\
A_4 &= B_3 \\
A_5 &= B_4 \\
A_6 &= B_5 \\
A_7 &= B_6 - B_{19} \\
A_8 &= B_7 - B_{20} \\
A_9 &= B_6 + B_{19} \\
A_{10} &= B_7 + B_{20} \\
A_{11} &= B_8 + B_{21} \\
A_{12} &= B_9 + B_{22} \\
A_{13} &= B_{10} + B_{23} \\
A_{14} &= B_{11} + B_{24} \\
A_{15} &= B_9 - B_{22} \\
A_{16} &= B_{12} \\
A_{17} &= B_{13} + B_{25} \\
A_{18} &= B_8 - B_{21} \\
A_{19} &= B_{14} \\
A_{20} &= B_{15} \\
A_{21} &= B_{11} - B_{24} \\
A_{22} &= B_{16} \\
A_{23} &= B_{10} - B_{23} \\
A_{24} &= B_{13} - B_{25} \\
A_{25} &= B_{17}
\end{align*}
\]
| \( B_{18} \) | 0 |
| \( B_{19} \) | 0 |
| \( B_{20} \) | 0 |
| \( B_{21} \) | 0 |
| \( B_{22} \) | 0 |
| \( B_{23} \) | 0 |
| \( B_{24} \) | 0 |
| \( B_{25} \) | 0 |

(70)
**Spherical Tensor Operators**

Spherical Tensor Operators may be formed from the Cartesian components of a vector operator. We begin with the spin operators of Particle-1 with respect to the xyz frame, $\Sigma_x$, $\Sigma_y$, $\Sigma_z$. Then

$$T_{l+1} = -\sqrt{3}/2 \ (\Sigma_x + i\Sigma_y),$$

$$T_{l0} = \sqrt{6}/2 \ (\Sigma_z), \quad (71)$$

and

$$T_{l-1} = \sqrt{3}/2 \ (\Sigma_x - i\Sigma_y)$$

are rank one irreducible spherical tensor operators.

$$T_{00} = 1^3 \quad (72)$$

is a rank zero operator. Rank two spherical operators may be formed from rank one operators by using the expression

$$T_{2q} = \sum_{q=-1}^{+1} \langle 1 \ l \ q \ Q-q|2Q \rangle T_{1q} \ T_{1Q-q},$$

Which give us the operators

$$T_{2+2} = \sqrt{3}/2 \ (\Sigma_x + i\Sigma_y)^2,$$

$$T_{2+1} = -\sqrt{3}/2 \ [(\Sigma_x+i\Sigma_y)\Sigma_z + \Sigma_z \ (\Sigma_x+i\Sigma_y)],$$

$$T_{20} = \sqrt{2} \ (3/2\Sigma_z^2-1),$$

$$T_{2-1} = \sqrt{3}/2 \ [(\Sigma_x-i\Sigma_y) \Sigma_z + \Sigma_z \ (\Sigma_x-i\Sigma_y)]$$

$$T_{2-2} = \sqrt{3}/2 \ (\Sigma_x - i\Sigma_y)^2. \quad (73)$$
These operators are not all Hermitean. A relationship that we will find helpful later is

\[ T_{KQ} = (-1)^Q T_{K-Q}^+ \]  \hspace{1cm} (74)

These operators form a complete set in 3-space. It may be verified that they satisfy the condition (66) by using the above definitions together with the trace properties of the spin operators shown on the pages following 91. Condition (66) in this case is

\[ \text{Tr} [T_{KQ} T_{K'}^{+} Q'] = 3 \delta_{KK'} \delta_{QQ'} \] \hspace{1cm} (75)

Beam Density Matrix in Terms of Spherical Operators

The beam density matrix \( \rho_b \) can be expressed as a linear combination of the \( T_{KQ} \) as

\[ \rho_b = C \sum_{KQ} \lambda_{KQ} T_{KQ}^+ \]

We can evaluate the constants \( C \) and \( \lambda_{KQ} \) by using the results (30), (74), and (75). This gives us:

\[ \rho_b = \frac{\text{Tr}[\rho_b]}{3} \sum_{KQ} \langle T_{KQ} \rangle_{\text{beam}} T_{KQ}^+ \]

We define the spherical polarizations of the beam \( t_{KQ} \) by

\[ t_{KQ} = \langle T_{KQ} \rangle_{\text{beam}} \] \hspace{1cm} (76)

Thus
\[
\rho_b = \frac{\text{Tr}[\rho_b]}{3} \sum_{KQ} t_{KQ} T_{KQ}^+.
\]  (77)

The specification of the beam density matrix requires 9 complex \(t_{KQ}\), however, since the density matrix is Hermitean and since its trace is determined this reduces to 8 independent real quantities.

We have seen in equation (38) that the density matrix for the two particle system is given by

\[
\rho_{pw} = \rho_b \otimes \rho_t
\]  (38)

since the target is unpolarized this becomes in our case

\[
\rho_{pw} = \rho_b \otimes 1^3.
\]  (78)

We find our plane wave density matrix \(\rho_{pw}\) in the product space by inserting (77) into (78).

\[
\rho_{pw} = \frac{\text{Tr}[\rho_b]}{3} \sum_{KQ} t_{KQ} [T_{KQ}^+ \otimes 1^3].
\]  (79)

The quantities \(T_{KQ}^+ \otimes 1^3\) in this expression would be expressed in terms of the spin eigenstates \(|l l_1 u_1 u_2\rangle\).

Since we wish to work in the channel spin representation we transform (79) to the channel spin representation by using \(C\) the unitary matrix whose elements are the Clebsch-Gordon Coefficients.

\[
C^+ \rho_{pw} C = \frac{\text{Tr}[\rho_b]}{3} \sum_{KQ} t_{KQ} C^+[T_{KQ}^+ \otimes 1^3]C.
\]
If we understand $\rho_{PW}$ and the $T_{KQ}$ to be in the channel spin representation then we can rewrite this last expression as

$$\rho_{PW} = \frac{\text{Tr}[\rho_D]}{3} \sum_{KQ} T_{KQ} T_{KQ}^\dagger. \quad (80)$$

If we let the $t_{KQ}$ be the expectation values of the $T_{KQ}$ in the product space then equation (80) becomes

$$\rho_{PW} = \frac{\text{Tr}[\rho_{PW}]}{9} \sum_{KQ} t_{KQ} T_{KQ}^\dagger. \quad (81)$$

We now present the scattered density matrix in terms of the spherical tensor operators by inserting equation (81) into equation (45).

$$\rho_{sc} = M \rho_{PW} M^\dagger = \frac{\text{Tr}[\rho_{PW}]}{9} \sum_{KQ} t_{KQ} M T_{KQ}^\dagger M^\dagger. \quad (82)$$

**Differential Cross Section**

We can find the polarized differential cross section $\sigma^P$ by using this last equation in equation (49). The result is

$$\sigma^P = \frac{\text{Tr}[\rho_{sc}]}{\text{Tr}[\rho_{PW}]} = \frac{1}{9} \sum_{KQ} t_{KQ} \text{Tr}[M T_{KQ}^\dagger M^\dagger]. \quad (83)$$

In the case of an unpolarized beam only $t_{00}$ is non-zero so the unpolarized differential cross section $\sigma^U$ is

$$\sigma^U = \frac{1}{9} t_{00} \text{Tr}[M M^\dagger] = \frac{1}{9} \text{Tr}[M M^\dagger]. \quad (84)$$
Polarizations of the Scattered Particles

The expectation values of the $T_{KQ}$ for the scattered particles are the scattered spherical polarizations $t'_{K'Q'}$. We can express the $t'_{K'Q'}$ in terms of the transition matrix and the beam spherical polarizations $t_{KQ}$ by using equations (51), (82), and (83).

$$t'_{K'Q'} \equiv \langle T_{KQ} \rangle_{\text{scatt}}$$

$$= \frac{1}{9} \frac{\text{Tr}[\rho_{pw}] \sum_{KQ} \text{tr}_{KQ} \text{Tr}[M_{T_{KQ}M^+T_{K'Q'}}]}{\sigma^P \text{Tr}[\rho_{pw}]}$$

or

$$= \frac{1}{9} \frac{\sum_{KQ} t_{KQ} \text{Tr}[M_{T_{KQ}M^+T_{K'Q'}}]}{\sigma^P} \quad (85)$$

This becomes for an unpolarized beam

$$t'_{K'Q'} = \frac{\sum_{KQ} t_{KQ} \text{Tr}[M_{T_{KQ}M^+T_{K'Q'}}]}{\sum_{KQ} \text{Tr}[M_{T_{KQ}M^+T_{00}}]} \quad (86)$$

$$t'_{K'Q'} = \frac{\text{Tr}[M_{T_{00}M^+T_{K'Q'}}]}{\text{Tr}[M M^+]} = \frac{\text{Tr}[M_{T_{K'Q'}}]}{\text{Tr}[M M^+]} \quad (87)$$
Equation (86) is the expression that we will use to give the scattered spherical polarizations in terms of the elements of the transition matrix and the beam spherical polarizations.

The $T_{KQ}$ operators and the polarizations were defined for the $xyz$ frame. If it is desired to know the spherical polarizations with respect to any other rotated frame, for example $x'y'z'$, the transformation is

$$
\hat{\mathbf{e}}_{KQ} = \sum_{Q'} t_{Q'Q} \hat{\mathbf{R}}_{Q'}^{Q} \hat{\mathbf{R}}_{Q} \quad (\alpha, \beta, \gamma)
$$

(88)

where $D_{Q'Q}^{K}(\alpha, \beta, \gamma)$ are the elements of Rose's rotation matrix.27 $\alpha, \beta, \gamma$ are the Euler angles of the rotation which takes $xyz$ into $x'y'z'$.

The Cartesian polarizations in any frame may be derived from the spherical polarizations in that frame by using the defining equations (71), (72), and (73).

**Phase Shift Analysis**

We have shown how the polarizations of the scattered particles depend on the transition matrix, which represents the effect of the interaction between the particles on the scattering. The elements of the transition matrix are functions of the energy and scattering angle. The Lane and Thomas expression, equation (53), shows explicitly the angle dependence of the transition matrix elements, the energy dependence being represented by the elements of the
nuclear collision matrix. In this section we express the nuclear collision matrix in terms of two types of physically significant parameters, the eigenphase shifts and the mixing parameters. It is thought that in the case of low energy only interactions involving small orbital angular momenta will occur and hence the nuclear collision matrix may be expressed in a simplified form with only the eigenphase parameters for small orbital angular momentum present in the expansion.

**Properties of the Nuclear Collision Matrix**

We mention as a preliminary to this parametrization some properties of the nuclear collision matrix.

The conservation of probability current implies that the nuclear collision matrix is unitary and time reversal invariance implies that it is symmetric.\(^28\)

We examine now how the rules for the addition of angular momenta and the conservation of angular momentum and parity restrict the form of the \(U^j\). We look at the various possibilities for the angular momentum quantum numbers \(j, \ell,\) and \(s\). \(s\), the total spin quantum number, can have values of 0, 1 or 2. \(\ell\), the orbital angular momentum quantum number, has values of 0, 1, 2, 3. \(j\), the total angular momentum quantum number, has values given by

\[
j = \ell + s, \ell + s - 1, \ell + s - 2, \ldots, |\ell - s|.\]
A given value of $j$ can occur in various ways; with an $s$ of 
**zero** when $l = j$, with an $s$ of **one** when $l = j$ or $l = j \pm 1$, 
with an $s$ of **two** when $l = j$ or $l = j \pm 1$ or $l = j \pm 2$. Each 
of the 81 positions in Table 6 on the next page represents 
a possible transition that conserves total angular momentum. The indices $l$ and $s$ are for plane wave states and $l'$ and $s'$ 
are for scattered states. Only transitions for which 
$l + l'$ is an even number will conserve parity. The 41 
transitions marked with a dot conserve total angular 
momentum and parity. Thus the conservation of parity 
gives $U^j$ its block form. The two blocks have opposite 
parity.

Let $U^{j\pi}$ be the submatrix of $U^j$ which has parity $\pi$. 
We will now indicate for some small values of $j$ the orders 
of the $U^{j\pi}$. Table 7 lists for the various $U^{j\pi}$ the possible 
values $s$ and $l$ (and also $s'$ and $l'$) can have for a total 
angular momentum of $j$. This leads to the order of that 
submatrix.

It is apparent from Tables 6 and 7 as long as $j \geq 2$ 
that if $j$ and $\pi$ are both even or both odd, we have a sub-
matrix of order 5 and if $j$ is even and $\pi$ is odd or $j$ is 
odd and $\pi$ is even, we have submatrix of order 4.

**Parameterizing $U^j$**

We will parameterize $U^{j\pi}$ by using the method of 
Seyler$^{29}$ who has generalized the work of Blatt and
<table>
<thead>
<tr>
<th>s' l'</th>
<th>0, j</th>
<th>1, j</th>
<th>2, j</th>
<th>2, j+2</th>
<th>2, j-2</th>
<th>1, j+1</th>
<th>1, j-1</th>
<th>2, j+1</th>
<th>2, j-1</th>
</tr>
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<tbody>
<tr>
<td>0, j</td>
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<td></td>
<td></td>
<td></td>
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<td></td>
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<td></td>
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<td>1, j</td>
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<td>2, j</td>
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TABLE 7
ORDERS OF THE SUBMATRICES $u^j\pi$

<table>
<thead>
<tr>
<th>$u^j\pi$</th>
<th>$u^0+$</th>
<th>$u^0-$</th>
<th>$u^1+$</th>
<th>$u^1-$</th>
<th>$u^2+$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
</tr>
<tr>
<td>0 0</td>
<td>1 1</td>
<td>1 0</td>
<td>0 1</td>
<td>2 0</td>
<td></td>
</tr>
<tr>
<td>2 2</td>
<td>1 2</td>
<td>1 1</td>
<td>0 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2 2</td>
<td>2 1</td>
<td>1 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2 3</td>
<td>2 2</td>
<td></td>
<td></td>
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</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2 4</td>
<td></td>
</tr>
<tr>
<td>order 2</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$u^j\pi$</th>
<th>$u^2-$</th>
<th>$u^3+$</th>
<th>$u^3-$</th>
<th>$u^4+$</th>
<th>$u^4-$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
</tr>
<tr>
<td>1 1</td>
<td>1 2</td>
<td>2 1</td>
<td>2 2</td>
<td>1 3</td>
<td></td>
</tr>
<tr>
<td>2 1</td>
<td>2 2</td>
<td>0 3</td>
<td>0 4</td>
<td>2 3</td>
<td></td>
</tr>
<tr>
<td>1 3</td>
<td>1 4</td>
<td>1 3</td>
<td>1 4</td>
<td>1 5</td>
<td></td>
</tr>
<tr>
<td>2 3</td>
<td>2 4</td>
<td>2 3</td>
<td>2 4</td>
<td>2 5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2 5</td>
<td>2 6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>order 4</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
We have seen that $U^j$ has the block form

$$
\begin{bmatrix}
U^j_{\pi_1} & 0 \\
0 & U^j_{\pi_2}
\end{bmatrix},
$$

with $\pi_1 = \pi_2 + 1$. It is easy to show that the unitarity and symmetry of $U^j$ imply the same properties for the $U^j_{\pi_1}$ and $U^j_{\pi_2}$.

The expression

$$(u^j)^\dagger \exp[2i(\delta^j)] (u^j)$$

is unitary and symmetric if $(\delta^j)$ is a real diagonal matrix and $(u^j)$ is a real orthogonal matrix. We can write then

$$U^j = (u^j)^\dagger \exp[2i(\delta^j)] (u^j).$$

(89)

The elements of the matrix $(\delta^j)$ are the eigenphase shifts.

We will now give explicit expressions for $U^j$ of different orders by using equation (89).

**1st Order**

$U^{0-}$ is the only $U^j$ submatrix with an order of one.

Its row is labeled 1 1.

$$U^{0-} = (u^{0-})^\dagger \exp[2i(\delta^{0-})] (u^{0-})$$

$$= \exp[2i(\delta^{0-})].$$

One parameter is required for the case of Order 1.
2nd Order

$U^{0+}$ is the only $U^{ij}_n$ submatrix with an order of two. It has row labels of 0 0 and 2 2 as seen in Table 7. The real diagonal matrix $\exp[2i(\delta^{0+})]$ is

$$\exp[2i(\delta^{0+})] = \begin{bmatrix} \exp[2i \delta^{0+}] & 0 \\ 0 & \exp[2i \delta^{22}] \end{bmatrix}$$

(90)

($u^{0+}$) being a real orthogonal matrix may be expressed in terms of a single real parameter $\epsilon^{0+}_{00,22}$, the mixing parameter, by

$$u^{0+} = \begin{bmatrix} \cos \epsilon^{0+}_{00,22} & \sin \epsilon^{0+}_{00,22} \\ -\sin \epsilon^{0+}_{00,22} & \cos \epsilon^{0+}_{00,22} \end{bmatrix}$$

(91)

Three parameters are required for the 2nd Order case. We find $U^{0+}$ in terms of the eigenphase shifts and the mixing parameters by inserting (90) and (91) into (89).

$$U^{0+} = \begin{bmatrix} U^{0+}_{00,00} & U^{0+}_{00,22} \\ U^{0+}_{22,00} & U^{0+}_{22,22} \end{bmatrix}$$

with
\[
\begin{align*}
U^{0+}_{00,00} &= \cos^2(\epsilon^{0+}_{00,22}) \exp[2i \delta^{0+}_{00}] + \sin^2(\epsilon^{0+}_{00,22})\exp[2i \delta^{0+}_{22}], \\
U^{0+}_{00,22} &= \sin(\epsilon^{0+}_{00,22})\cos(\epsilon^{0+}_{00,22})[\exp[2i \delta^{0+}_{00}] - \exp[2i \delta^{0+}_{22}]], \\
U^{0+}_{22,00} &= \sin(\epsilon^{0+}_{00,22})\cos(\epsilon^{0+}_{00,22})[\exp[2i \delta^{0+}_{00}] - \exp[2i \delta^{0+}_{22}]], \\
U^{0+}_{22,22} &= \sin^2(\epsilon^{0+}_{00,22})\exp[2i \delta^{0+}_{00}] + \cos^2(\epsilon^{0+}_{00,22})\exp[2i \delta^{0+}_{22}].
\end{align*}
\]

If the mixing parameter \(\epsilon^{0+}_{00,22}\) is zero then \(U^{0+}\) is diagonal and we see from equation (53) that the transition matrix will not involve any changes of channel spin due to interactions of total angular momentum zero and positive parity. In this case of zero mixing parameter the different eigenphase shifts \(\delta^{0+}_{00}\) give the phase shifts of the different partial waves of total angular momentum 0 and + parity which contribute to the scattering.

If the mixing parameter \(\epsilon^{0+}_{00,22}\) is not zero then \(U^{0+}\) is not diagonal and \(U^{0+}\) will contribute to a change of channel spin from 0 to 2 or 2 to 0 in the scattering. The mixing parameter determines for \(U^{0+}\) how much of \(\delta^{0+}_{00}\) and of \(\delta^{0+}_{22}\) occur in the diagonal elements and the size of the off diagonal elements. Also if \(\epsilon^{0+}_{00,22}\) is not zero then the \(\delta^{0+}_{00}\) is no longer the phase shift of a particular partial wave but instead the phase shift of a linear combination of different partial waves.
3rd Order

$U^{1+}$ is the only $U^{1+}$ submatrix with an order of three. It has row labels of 1 0, 1 2, 2 2. The real diagonal matrix \( \exp[2i(\delta^{1+})] \) is

\[
\exp[2i(\delta^{1+})] = \begin{bmatrix}
\exp[2i(\delta_{10}^{1+})] & 0 & 0 \\
0 & \exp[2i(\delta_{12}^{1+})] & 0 \\
0 & 0 & \exp[2i(\delta_{22}^{1+})]
\end{bmatrix}.
\]

A third Order real orthogonal matrix can in analogy to operators rotating 3-space Cartesian coordinate frames be written as

\[
(u^{1+}) = \begin{bmatrix}
\cos\gamma_{10,12} & \sin\gamma_{10,12} & 0 \\
-sin\gamma_{10,12} & \cos\gamma_{10,12} & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
x \begin{bmatrix}
\cos\gamma_{10,22} & 0 & \sin\gamma_{10,22} \\
0 & 1 & 0 \\
-sin\gamma_{10,22} & 0 & \cos\gamma_{10,22}
\end{bmatrix} x
\]

\[
x \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos\gamma_{12,22} & \sin\gamma_{12,22} \\
0 & -\sin\gamma_{12,22} & \cos\gamma_{12,22}
\end{bmatrix} .
\]
Thus six parameters are required for the third order submatrix.

Before we proceed to the Order-4 and Order-5 cases, let us make some definitions which will facilitate the writing of the results. We define:

\[
v(\alpha) = \begin{bmatrix}
\cos \alpha & \sin \alpha & 0 \\
-sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

\[
w(\beta) = \begin{bmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-sin \beta & 0 & \cos \beta
\end{bmatrix},
\]

and

\[
x(\gamma) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \gamma & \sin \gamma \\
0 & -\sin \gamma & \cos \gamma
\end{bmatrix}.
\]

The action of \(v(\alpha)\) on a matrix \((P_{ij})\) is shown below:

\[
v(\alpha)(P_{ij}) = \begin{bmatrix}
P_{11}\cos \alpha + P_{21}\sin \alpha, & P_{12}\cos \alpha + P_{22}\sin \alpha, & P_{13}\cos \alpha + P_{23}\sin \alpha \\
P_{11}\sin \alpha + P_{21}\cos \beta, & P_{12}\sin \alpha + P_{22}\cos \beta, & -P_{13}\sin \alpha + P_{23}\cos \beta \\
P_{31} & P_{32} & P_{33}
\end{bmatrix}.
\]
The result of multiplying by \( v(\alpha) \) is, for a given column, to mix rows 1 and 2 together. Row 3 is not affected. Similar results hold for \( w(\beta) \) and \( x(\alpha) \).

Rather than give examples of specific \( j \) and \( \pi \) for 4th and 5th orders we will write the expressions in a general notation by allowing the rows and columns of the nuclear collision matrix to be labeled 1, 2, ... instead of by \( s \) \( t \) pairs.

4th Order

The real diagonal matrix \( \exp[2i(\delta j^{\pi^2})] \) is

\[
\exp[2i(\delta j^{\pi^2})] = \begin{bmatrix}
\exp[2i(\delta_1^{\pi})] & 0 & 0 & 0 \\
0 & \exp[2i\delta_2^{\pi}] & 0 & 0 \\
0 & 0 & \exp[2i\delta_3^{\pi}] & 0 \\
0 & 0 & 0 & \exp[2i\delta_4^{\pi}]
\end{bmatrix}.
\]

The real orthogonal matrix \( (u_j^{\pi^2}) \) is expanded below:

\[
(u_j^{\pi^2}) = \begin{bmatrix}
& v(\epsilon_1^{\pi^2}) & 0 \\
0 & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
x \begin{bmatrix}
& w(\epsilon_1^{\pi^2}) & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Fourth order requires 10 parameters.

In order to simplify the writing of the 5th order submatrix let us give symbols to the matrices in the expansion of the 4th order submatrix.
(u^{j \pi_2}) = \psi_1(e^{j \pi_2}) \psi_2(e^{j \pi_2}) \psi_3(e^{j \pi_2})

= \psi_4(e^{j \pi_2}) \psi_5(e^{j \pi_2}) \psi_6(e^{j \pi_2}).

5th Order

The real diagonal matrix \( \exp[2i(\delta^{j \pi_1})] \) is

\[
\begin{bmatrix}
2i\delta_1^{j \pi_1} & 0 & 0 & 0 & 0 \\
0 & 2i\delta_2^{j \pi_1} & 0 & 0 & 0 \\
0 & 0 & 2i\delta_3^{j \pi_1} & 0 & 0 \\
0 & 0 & 0 & 2i\delta_4^{j \pi_1} & 0 \\
0 & 0 & 0 & 0 & 2i\delta_5^{j \pi_1}
\end{bmatrix}
\]

The real orthogonal matrix \( (u^{j \pi}) \) is

\[
(u^{j \pi_1}) = \begin{bmatrix}
\psi_1(e^{j \pi_1}) & 0 \\
\psi_2(e^{j \pi_1}) & 0 \\
\psi_3(e^{j \pi_1}) & 0 \\
\psi_4(e^{j \pi_1}) & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
x \begin{bmatrix}
  j \pi_1 \\
  \psi_5(\varepsilon_{24}) \\
  0 \\
  0 \ 0 \ 0 \ 0 \ 1
\end{bmatrix}
\quad
\begin{bmatrix}
  j \pi_1 \\
  \psi_6(\varepsilon_{34}) \\
  0 \\
  0 \ 0 \ 0 \ 0 \ 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
  j \pi_1 \\
  \cos(\varepsilon_{15}) \\
  0 \\
  0 \ 1 \ 0 \ 0 \ 0
\end{bmatrix}
\quad
\begin{bmatrix}
  j \pi_1 \\
  -\sin(\varepsilon_{15}) \\
  0 \\
  0 \ 0 \ 1 \ 0 \ 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \sin(\varepsilon_{15}) \\
  0 \\
  \cos(\varepsilon_{25}) \\
  0 \ 1 \ 0 \ 0 \ 0
\end{bmatrix}
\quad
\begin{bmatrix}
  \cos(\varepsilon_{25}) \\
  0 \\
  -\sin(\varepsilon_{25}) \\
  0 \ 0 \ 1 \ 0 \ 0
\end{bmatrix}
\]
The submatrix $U^{j\pi_1}$ requires 15 parameters.

Partial Waves

When the energy is low one expects an appreciable reaction to occur only for the system orbital angular momentum less than or equal to some small value. If $\ell_{\text{max}}$ is that maximum value of the orbital angular momentum quantum number then we expect $U^j$ to be the identity matrix for $j > \ell_{\text{max}} + 2$. It is of interest to calculate the number of mixing parameters and eigenphase shifts necessary
to represent the significant elements of the nuclear collision matrix for various small \( l_{\text{max}} \).

Let us consider the case of \( l_{\text{max}} = 1 \). We see from Table 7 that indices \( l=0 \) and/or \( l=1 \) occur in \( U^{0+}, U^{0-}, U^{1+}, U^{1-}, U^{2+}, U^{2-} \) and \( U^{3-} \). We expect that off diagonal elements whose column or row index involves an \( l \) greater than one will be zero and diagonal elements for \( l \) greater than one will be 1. Thus for \( U^{2-} \) we have the form

\[
U^{2-} = \begin{bmatrix}
a & c & 0 & 0 \\
c & b & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

For \( U^{2-} \) we need to parameterize a 2nd order unitary symmetric matrix. We have seen that this requires three parameters. The seven matrices listed above require respectively 1,1,1,6,1,3 and 1 parameters for a total of 14 parameters.

The number of parameters for various partial waves are listed in Table 8 on the next page.
<table>
<thead>
<tr>
<th>Partial Waves</th>
<th>$l_{\text{max}}$</th>
<th>Number of Parameters</th>
</tr>
</thead>
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<tr>
<td>S</td>
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</tr>
<tr>
<td>S+P</td>
<td>1</td>
<td>14</td>
</tr>
<tr>
<td>S+P+D</td>
<td>2</td>
<td>34</td>
</tr>
<tr>
<td>S+P+D+F</td>
<td>3</td>
<td>58</td>
</tr>
</tbody>
</table>
III. CALCULATION

We will explain in this chapter the development from the theory of the previous chapter of the explicit equations which form the description of the scattering.

**Derivation of the Time Reversal Invariance**

**Restrictions on the Elements of M**

The linear relationships between the \( \{m_i\} \) imposed by time reversal invariance are contained in equation (63) on page 35.

\[
\sum_{\nu} d_{\nu}^{s'} M_{s'\nu}^{s\nu} = (-1)^{\nu' - \nu} \sum_{\nu} d_{\nu}^{s} M_{s\nu}^{s\nu} \quad .
\]  

With \( s\nu \) taking values 22, 21, 20, 2-1, 2-2, 11, 10, 1-1, 00 and \( s'\nu' \) taking values 22, 21, 20, 2-1, 2-2, 11, 10, 1-1, 00. These 81 equations are not all independent. We develop now some theorems which enable us to identify some of the dependent equations and some of the identical equations. The elements of the reduced rotation matrix are given in Table 9 on the next page. We note that the elements of the reduced rotation matrix obey the following equation,

\[
d_{\nu}^{s'} M_{s'\nu} = (-1)^{\nu' + \nu} d_{-\nu}^{s'} M_{s\nu}^{s'\nu} \quad .
\]  

which is the same as + BCRS property. We will use the
TABLE 9
REDUCED ROTATION MATRIX

\[
\begin{array}{ccccccc}
\theta & x & y & z & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
22 & \omega_2 (\theta/2) & -(1/2)\sin \theta (1 + \cos \theta) & \sqrt{3/2} \sin \theta & (1/2)\sin \theta (1 + \cos \theta) & \sin^2 (\theta/2) & & \\
21 & (1/2)\sin \theta (1 + \cos \theta) & (1/2) (2 \cos \theta + 1) & -\sqrt{3/2} \sin \theta \sin \theta & (1/2) (1 + \cos \theta) & (1/2) \sin \theta (1 + \cos \theta) & & \\
20 & \sqrt{3/2} \sin^2 \theta & \sqrt{3/2} \sin \theta \cos \theta & (1/2) (1 + \cos \theta + 1) & -\sqrt{3/2} \sin \theta \sin \theta & \sqrt{3/2} \sin^2 \theta & & \\
2-1 & -(1/2)\sin \theta (1 + \cos \theta) & (1/2) (2 \cos \theta + 1) & \sqrt{3/2} \sin \theta \sin \theta & (1/2) (1 + \cos \theta) & -\sqrt{3/2} \sin \theta (1 + \cos \theta) & & \\
2-2 & \sin^2 (\theta/2) & -(1/2)\sin \theta (1 + \cos \theta) & \sqrt{3/2} \sin^2 \theta & (1/2) \sin \theta (1 + \cos \theta) & \sin^2 (\theta/2) & & \\
& 1 & 1 & \omega^2 (\theta/2) & -(1/2)\sin \theta & \sin \theta (1 + \cos \theta) & \\
& 10 & (1/2)\sin \theta & \cos \theta & -(1/2)\sin \theta & \\
& 1-1 & \sin^2 \theta/2 & (1/2)\sin \theta & \omega^2 (\theta/2) & \\
& 00 & & & & & & & & & & & \\
\end{array}
\]
notation that \((21, 10)\) denotes the equation resulting from evaluating equation (63) with \(sv = 21\) and \(s'v' = 10\).

Theorem 1. Equation \((sv, s'v')\) is identical with equation 
\((s-v, s'-v')\).

This is a consequence of the + BCRS of M and the property of reduced matrix of equation (92).

This theorem allows us to eliminate 22 equations from our consideration. We are left with 59 equations.

Theorem 2. Equation \((sv, s'v')\) is the same as equation 
\((sv', sv)\).

Proof. 1. Equation \((sv, s'v')\) is

\[
\sum_{v''} d^s_{v''v'}, M_{s'v''}, sv = (-1)^{v'-v} \sum_{v''} d^s_{v''v} M_{sv''}, s'v'.
\]

2. Equation \((s'v', sv)\) is

\[
\sum_{v''} d^s_{v''v} M_{sv''}, s'v' = (-1)^{v'-v'} \sum_{v''} d^s_{v''v'} M_{s'v''}, sv.'
\]

3. If we transfer \((-1)^{v'-v}\) in \((sv, s'v')\) to the other side, we will have equation \((s'v', sv)\).

Q.E.D.

This theorem reduces the equations of interest from 59 to 26, eliminating 33 equations.

It is easy to see from equation (63) that all equations \((sv, sv)\) are identities. This removes 6 more equations leaving 20 equations.
Theorem 3. All equations \((sv, s-v)\) are identities.

Proof. 1. Equation (63) for this case is

\[ \sum_v d^{sv}_{v''} M_{sv''},sv = (-1)^{v-v} \sum_v d^{sv}_{v''} M_{sv''},s-v \cdot \]

2. We insert the +BCRS of M and the property of equation (92) into this equation to get for the left hand side:

\[ \sum_v d^{sv}_{v''} (-1)^{v''+v} M_{s-v''},s-v (-1)^{s+s+v''+v} \]

3. This last sum may be rewritten and then substituted into the equation of step 1 to yield:

\[ \sum_v d^{sv}_{v''} M_{sv''},s-v = \sum_v d^{sv}_{v''} M_{sv''},s-v \]

which is an identity. Q.E.D.

This theorem eliminates 2 of the remaining equations to leave 18 equations.

It turns out by direct calculation that equations \((20,10)\) and \((10,00)\) are identities. This leaves 16 equations to be written. It is apparent from the form of equation (63) that these equations can be grouped into 6 groups. In a given group will occur only elements of M from one or at most two blocks of M. For example, Group 1 contains only elements from the \(s=2, s'=2\) block and Group 2 contains only elements from the \(s=2, s'=1\) block and the \(s=1, s'=2\) block.
Results

The 16 equations are presented in Appendix A on page 122. We have simplified these equations in certain instances by using sums and differences of elements of M as variables rather than the elements of M themselves. Group 1 consists of 4 equations in 13 unknowns. One of these equations does not involve any of the variables in the other 3 equations, so it is independent of them. Since the rank of the coefficient matrix of these 3 equations is 3 it is possible to solve for 3 of the unknowns in terms of the remaining 6 unknowns. The set of four equations of Group 1 are thus independent. Similar considerations for the other groups show that the 16 time reversal invariance equations are independent. The number of independent equations for each group is listed in Table 10 on the next page.

The form of M that conserves parity and is time reversal invariant has $41 - 16 = 25$ independent elements.

Formation of Spin Operator Expansion of the Transition Matrix

The expansion of the transition matrix in terms of a set of spin operators which have the correct rotation, reflection and time reversal properties is given in equation (68) on page 44. The identical particles form of this expansion is obtained by setting $B_{18}$ through $B_{25}$ equal to zero.
TABLE 10

NUMBER OF TIME REVERSAL INVARIANCE EQUATIONS

<table>
<thead>
<tr>
<th>Group</th>
<th>Blocks of $M$</th>
<th>Number of Independent Time Reversal Invariance Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$s=2, s'^{}=2$</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>$s=1, s'^{}=2$ and $s=2, s'^{}=1$</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>$s=1, s'^{}=1$</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>$s=0, s'^{}=2$ and $s=2, s'^{}=0$</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>$s=0, s'^{}=1$ and $s=1, s'^{}=0$</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>$s=0, s'^{}=0$</td>
<td>0</td>
</tr>
</tbody>
</table>

16
\[ M = B_1 l^9 + B_2 (\Sigma_n S_n) + B_3 (\Sigma_k S_k) + \]
\[ \cdots \cdots + B_{12} (\Sigma_{kk} S_{kk}) + \cdots \cdots \]
\[ + B_{24} (\Sigma_p S_{np} - \Sigma_{np} S_p) + B_{25} (\Sigma_{kk} S_{pp} - \Sigma_{pp} S_{kk}) \] 

(68)

We will now calculate explicit expressions for the spin operators in this expansion in order to make this expansion explicit except for \( \{ B_j (0, E) \} \). When we have the explicit-spin operators we will be able to express the \( \{ m_i \} \) in terms of the \( \{ B_j \} \).

We need to calculate the spin operators appearing in this expansion in the channel spin representation and with respect to the knp frame.

Formation of Spin Operators

\[ \Sigma_k \cdots S_p \]

\( \Sigma_k, \Sigma_n, \Sigma_p \) and \( S_k, S_n, S_p \) are the spin operators of Particle-1 and Particle-2 expressed with respect to the knp frame in the channel spin representation. We now discuss the formation of these operators.

The operators for a spin-1 particle expressed in the representation where the total spin squared and the z-component of spin are diagonal are given below.

\[ \Sigma_x^3 = S_x^3 = \frac{\sqrt{2}}{2} \hbar \times \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \] 

(93)
The superscript indicates that these operators are in single particle three dimensional space.

\[
\Sigma_z^3 = s_z^3 = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (94)
\]

\[
\Sigma_y^3 = s_y^3 = \frac{\sqrt{2}}{2} \hbar \begin{bmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{bmatrix}
\]

These operators are with respect to the xyz coordinate frame.

We let \( \hbar = 1 \) in this calculation.

We now construct the spin operators of Particle-1 and Particle-2 in the 2-particle system product space. These operators are derived by taking the tensor product of the single particle operators with \( \mathbf{1}^3 \), the three space identity. The bar indicates that these operators are represented in terms of eigenfunctions \( |1 \ 1 \ u_1 u_2 \rangle \), those of spin of particle-1 squared, z-component of spin of particle-1, spin of particle-2 squared, z-component of spin of particle-2.
\[ \hat{\Sigma}_x = \Sigma_x \otimes 1^3 = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & 1^3 & 0 \\ 1^3 & 0 & 1^3 \\ 0 & 1^3 & 0 \end{bmatrix} . \]

\[ \hat{\Sigma}_y = \Sigma_y \otimes 1^3 = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & -1^3 & 0 \\ 1^3 & 0 & 1^3 \\ 0 & 1^3 & 0 \end{bmatrix} . \]

\[ \hat{\Sigma}_z = \Sigma_z \otimes 1^3 = 1 \begin{bmatrix} 1^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1^3 \end{bmatrix} . \]

\[ \hat{S}_x = 1^3 \otimes S_x^3 = 1 \begin{bmatrix} S_x^3 & 0 & 0 \\ 0 & S_x^3 & 0 \\ 0 & 0 & S_x^3 \end{bmatrix} . \]

\[ \hat{S}_y = 1^3 \otimes S_y^3 = 1 \begin{bmatrix} S_y^3 & 0 & 0 \\ 0 & S_y^3 & 0 \\ 0 & 0 & S_y^3 \end{bmatrix} . \]

\[ \hat{S}_z = 1^3 \otimes S_z^3 = 1 \begin{bmatrix} S_z^3 & 0 & 0 \\ 0 & S_z^3 & 0 \\ 0 & 0 & S_z^3 \end{bmatrix} . \]
We now transform our spin operators into the channel spin representation where the total spin squared and the $z$-component of total spin of the two particle system are diagonal along with spin squared of particle-1 and of particle-2.

$\Sigma_x...S_z$ are in the channel spin representation. $C$ is the unitary matrix with elements $\langle 1 1 u_1 u_2 | s v 1 1 \rangle$, the Clebsch-Gordon Coefficients.

\[
\begin{align*}
\Sigma_x &= C^+ \Sigma_x^9 C, \\
\Sigma_y &= C^+ \Sigma_y^9 C, \\
\Sigma_z &= C^+ \Sigma_z^9 C,
\end{align*}
\]

The spin operators $\Sigma_x, \Sigma_y, \Sigma_z, S_x, S_y, S_z$ in the channel spin representation are shown in Appendix B on page 130.

The spin operators are vector operators and may be transformed from the $xyz$ frame to the $knp$ frame by using the vector transformation rule of equation (5). The $\Sigma_k...S_p$ are shown in Appendix C on page...

Properties of the $\Sigma_k...S_p$ Operators

We notice that the $\Sigma_k...S_p$ operators have a symmetry in each block denoted by a pair of indices, $s s'$. In each block a reflection through the center of the block yields
an element of the same magnitude. The sign alternates as one moves down one column after another. We say in this thesis that a matrix A, whose elements obey the equation

\[ A_{s', v'} = i_A (-1)^{s' + v'} A_{s', -v', s-v} \]

with \( i_A = i \), has plus or minus Block Centered Reflection Symmetry, \( ±BCRS \).

The operators \( E_k, S_k, E_p, S_p \) are symmetric and real and \( E_n, S_n \) are antisymmetric and pure imaginary. This is a reflection of their Hermiticity.

**Relationships between the Operators \( E_k \ldots S_p \)**

We mention some relationships between pairs of these operators:

1. \( \hat{E} \) and \( \hat{S} \).

\[ (\hat{E})_{s', v'}, s_v = (-1)^{s' + s} (\hat{S})_{s', v'}, s_v \]  \[ (97) \]

2. \( E_p \) and \( E_k \).

\[ (E_p)_{s', v'}, s_v = (-1)(-1)^{v'} (E_k)_{s', v'}, s_v \]  \[ (98) \]

if we interchange sines and cosines,

3. \( S_p \) and \( S_k \)

\[ (S_p)_{s', v'}, s_v = (-1)(-1)^{v'} (S_k)_{s', v'}, s_v \]  \[ (99) \]

if we interchange sines and cosines.
It is not surprising that these relationships exist given the method of formation of the operators as the tensor products of equations (95) and then the linear combinations of equation (5).

**Formation of the Operators Appearing in the Spin Operator Expansion of M**

The operators appearing in the expansion of $M$ of equation (68) may be formed by multiplication and addition operations on the $\Sigma_k \ldots S_p$ operators. Since these are order 9 matrices this is a time consuming task. It is desirable to reduce the work by proving theorems, about the products and sums, that are based on the symmetries and interrelationships of the $\Sigma_k \ldots S_p$. We discuss some of these theorems.

**Theorem 4.** If matrix $A$ is such that $A = aA^T$, $a=$constant, and matrix $B$ is such that $B = bB^T$, $b=$constant, then $AB = ab (BA)^T$.

This is a well known theorem which is used, for example, to show that $\Sigma_k \Sigma_k$ must be symmetric and to derive $\Sigma_k \Sigma_p$ from $\Sigma_p \Sigma_k$.

**Theorem 5.** If $A$ has Block Center Reflection Symmetry, of sign $i_A$,  

$$(A)_{s',v',s,v} = i_A (-1)^{s'+v'+s+v} (A)_{s'-v',s-v},$$
and B has Block Center Reflection Symmetry, of sign $i_B$,

$$(B)_{s'v',sv} = i_B(-1)^{s'v'+s+v} (B)_{s'-v',s-v},$$

then AB will have Block Center Reflection Symmetry, of sign $i_Ai_B$,

$$(AB)_{s'v',sv} = i_Ai_B(-1)^{s'v'+s+v} (AB)_{s'-v',s-v}.$$ 

Proof: 1. The product AB in terms of its elements is

$$(AB)_{s'v',sv} = \sum_{\tau f} (A)_{s'v',\tau f} (B)_{\tau f,sv}.$$ 

2. We can rewrite this by using the BCRS of A and B as

$$(AB)_{s'v',sv} = \sum_{\tau f} i_A(-1)^{s'+v'+\tau +f} (A)_{s'-v',\tau -f} \times i_B(-1)^{\tau +f+s+v} (B)_{\tau -f,s-v}$$

$$= i_Ai_B(-1)^{s'+v'+s+v} \sum_{\tau f} (-1)^{2\tau +2f} (A)_{s'-v',\tau -f} (B)_{\tau -f,s-v}.$$ 

3. $(-1)^{2\tau +2f} = +1$ and $\sum_{\tau f} C_{sv,\tau -f} = \sum_{\tau f} C_{sv,\tau f}$. 

4. Thus

$$(AB)_{s'v',sv} = i_Ai_B(-1)^{s'+s+v'+v} (AB)_{s'-v',s-v}.$$ 

All of the $S_{k...p}$ have BCRS property so this theorem will be applicable to all of our products of these operators. This is a very useful result since it relates 36 elements of the product to other elements of the product. In
addition the theorem gives 4 or 5 of the block centers as zero.

Theorem 6. If A and B have +BCRS then A + B will have +BCRS, and if A and B have -BCRS then A + B will have -BCRS.

The proofs of this Theorem and the remaining theorems in this section are simple being along the lines of the proof of Theorem 5.

This tells for instance that $\Sigma_{kn}$ which equals $1/2 \left(\Sigma_k \Sigma_n + \Sigma_n \Sigma_k\right)$ must have BCRS. As it turns out Theorems 5 and 6 insure that all the matrices that we have to calculate have +BCRS. Theorem 6 implies the $M$ itself must have +BCRS. This tells us nothing new about $M$ since +BCRS for $M$ is identical with the parity conservation restriction of equation (59).

Theorem 7. If both $A$ and $B$ are symmetric or antisymmetric and if the elements of $A$ are related to those of $B$ by

$$(A)_{s'v',sv} = (\pm 1) (-1)^{s'+s} (B)_{s'v',sv}$$

or by

$$(A)_{s'v',sv} = (\pm 1) (-1)^{v'+v} (B)_{s'v',sv}$$

then

$$(AB)_{s'v',sv} = (-1)^{s'+s} (AB)_{sv,s'v'}$$
This theorem together with the relationships of page 79 may be used to deduce a relationship between the elements symmetric about the main diagonal of $\Sigma_n S_n$ or $\Sigma_F E_k$. In the latter example it is necessary to interchange sines and cosines.

Theorem 8. If the elements of $A$ are related to those of $B$ by

$$ (A)_{s'v'},sv = I_{AB}(-1)^{s+s} (B)_{s'v'},sv \quad \text{and if the elements of $H$ are related to those of $D$ by} 
$$

$$ (H)_{s'v'},sv = I_{HD}(-1)^{s+s} (D)_{s'v'},sv \quad \text{then} 
$$

$$ (AH)_{s'v'},sv = I_{AB} I_{HD}(-1)^{s+s} (BD)_{s'v'},sv. 
$$

$I_{AB}$ and $I_{HD}$ are constants.

This theorem enables one to take shortcuts like deriving the elements of $\Sigma_k S_k$ from those of $S_k S_k$.

Theorem 9. Suppose the $A,B,H,D$ obey the postulates of Theorem 8 and additionally:

1. $A=\tau_A A^T$ and $B=\tau_A B^T$ with $\tau_A = \text{constant}$
2. $H=\tau_H B^T$ and $D=\tau_H D^T$ with $\tau_H = \text{constant}$, then

$$ (AH)_{s'v'},sv = I_{AB} I_{HD} \tau_A \tau_H (-1)^{s+s} (DB)_{sv,s'v'}. $$
This theorem enables one to derive $\Sigma_{n}S_{kk}$ from $\Sigma_{kk}S_{n}$, as an example.

**Theorem 10.** Suppose that $A,B,H,D$ obey the postulates of Theorem 8 and additionally have the type of modified main diagonal symmetry of

\[(A)_{sv',sv} = \tau_{A}(-1)^{s'+s} (A)_{sv',sv'} \]
\[(B)_{sv',sv} = \tau_{A}(-1)^{s'+s} (B)_{sv',sv'} \]
\[(H)_{sv',sv} = \tau_{H}(-1)^{s'+s} (H)_{sv',sv'} \]
\[(D)_{sv',sv} = \tau_{H}(-1)^{s'+s} (D)_{sv',sv'} ,\]

then

\[(AH)_{sv',sv} = I_{AB}I_{HD}I_{AT}I_{TH} (DB)_{sv',sv'} ,\]

that is

\[(AH) = I_{AB}I_{HD}I_{AT}I_{TH} (DB)^{T} .\]

**Theorem 11.** If $A$ and $B$ are related by

\[(A)_{sv',sv} = I_{AB}(-1)^{v'+v} (A)_{sv',sv} \]

and $H$ and $D$ are related by

\[(H)_{sv',sv} = I_{HD}(-1)^{v'+v} (D)_{sv',sv} ,\]

then

\[(AH)_{sv',sv} = I_{AB}I_{HD}(-1)^{v'+v} (BD)_{sv',sv} .\]

This theorem gives $\Sigma_{p}S_{p}$ from $\Sigma_{k}S_{k}$ with exchange of sines and cosines.

**Theorem 12.** If $A$ and $B$ are related by

\[(A)_{sv',sv} = (-1)^{s'+s} (B)_{sv',sv} \]
and if C and D are related by

$$(C)_{s',v',s_v} = (-1)^{s'+s} (D)_{s',v',s_v}$$

then

$$(AC+CA)_{s',v',s_v} = (-1)^{s'+s} (BD+DB)_{s',v',s_v},$$

This theorem is used to derive the $S_{is}$ operators from the $S_{is}$ operators.

The application of those theorems resulted in a great saving in time and offered at least one check on each operator.

Results

The explicit expressions for the spin operators in the expansion of $M$ are given in Appendix D on page 144.

**Derivation of the Elements of $M$ in Terms of the Spin Operator Coefficients**

The explicit expressions for the spin operators enable us to express the elements of the transition matrix, the $\{m_i\}$, in terms of the spin operator coefficients $\{B_j\}$. The explicit spin operators are inserted into equation (68) and the addition carried out element by element.

Results

The results are given in Appendix E on page 170.

It was observed that the coefficient of $\sin^n \cos^m$ in the expansion of $M_{s',v',s_v}$ in terms of $\{B_j\}$ was equal to or the negative of the coefficient of the $\sin^n \cos^m$ term in the
expansion of $M_{s'y'y'}$. This property was used to provide a check of the work.

The situation of identical particles requires that $B_{18}$ through $B_{25}$ be zero as shown in Table 5 on page 46. Since $B_{18}$ through $B_{25}$ appear only in the expressions for $m_{20}$...$m_{33}$ and $m_{40}$ and $m_{41}$ and since no other $B_j$ appear in these expressions these $m_i$ must be zero for the identical particles situation. This is shown in Table 11 on the next page. The identical particles form of the $M$ matrix is shown in Table 12 on page 88. We see that for this situation channel spin cannot change by one unit, but only by two units or zero units. We can understand this from quantum mechanics. If we discuss the scattering in terms of the time dependent Schroedinger equation we see that for a Hamiltonian which is symmetric to particle exchange, as the Hamiltonian for identical particles must be, that it is impossible for a symmetric initial state to turn into an antisymmetric final state or vice versa. We see from equation (54) that the spin part of the wave function is symmetric when $s = 0$ or 2 and antisymmetric when $s = 1$. Thus a change in channel spin by one unit would cause the wave function to change its symmetry and thus does not occur.

In the situation of identical particles the Nuclear Collision matrix will be simpler than for non-identical particles. The Nuclear Collision matrix for this situation is shown in Table 13 on page 89.
<table>
<thead>
<tr>
<th>$m_{20} = 0$</th>
<th>$m_{28} = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{21} = 0$</td>
<td>$m_{29} = 0$</td>
</tr>
<tr>
<td>$m_{22} = 0$</td>
<td>$m_{30} = 0$</td>
</tr>
<tr>
<td>$m_{23} = 0$</td>
<td>$m_{31} = 0$</td>
</tr>
<tr>
<td>$m_{24} = 0$</td>
<td>$m_{32} = 0$</td>
</tr>
<tr>
<td>$m_{25} = 0$</td>
<td>$m_{33} = 0$</td>
</tr>
<tr>
<td>$m_{26} = 0$</td>
<td>$m_{40} = 0$</td>
</tr>
<tr>
<td>$m_{27} = 0$</td>
<td>$m_{41} = 0$</td>
</tr>
</tbody>
</table>
### TABLE 12

**TRANSITION MATRIX IN TERMS OF ELEMENTS**

- IDENTICAL PARTICLES

<table>
<thead>
<tr>
<th>$s'v'$</th>
<th>21</th>
<th>20</th>
<th>2-1</th>
<th>2-2</th>
<th>11</th>
<th>10</th>
<th>1-1</th>
<th>00</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>$m_1$</td>
<td>$m_2$</td>
<td>$m_3$</td>
<td>$m_4$</td>
<td>$m_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>21</td>
<td>$m_6$</td>
<td>$m_7$</td>
<td>$m_8$</td>
<td>$m_9$</td>
<td>$m_{10}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>$m_{11}$</td>
<td>$m_{12}$</td>
<td>$m_{13}$</td>
<td>$-m_{12}$</td>
<td>$m_{11}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2-1</td>
<td>$-m_{10}$</td>
<td>$m_9$</td>
<td>$-m_8$</td>
<td>$m_7$</td>
<td>$-m_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2-2</td>
<td>$m_5$</td>
<td>$-m_4$</td>
<td>$m_3$</td>
<td>$-m_2$</td>
<td>$m_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$m_{14}$</td>
<td>$m_{15}$</td>
<td>$m_{16}$</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$m_{17}$</td>
<td>$m_{18}$</td>
<td>$-m_{17}$</td>
</tr>
<tr>
<td>1-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$m_{16}$</td>
<td>$-m_{15}$</td>
<td>$m_{14}$</td>
</tr>
<tr>
<td>00</td>
<td>$m_{37}$</td>
<td>$m_{38}$</td>
<td>$m_{39}$</td>
<td>$-m_{38}$</td>
<td>$m_{37}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
TABLE 13

THE PARITY CONSERVING ELEMENTS OF U^j
-IDENTICAL PARTICLES

<table>
<thead>
<tr>
<th>s',\kappa'</th>
<th>0, j</th>
<th>1, j</th>
<th>2, j</th>
<th>2, j+2</th>
<th>2, j-2</th>
<th>1, j+1</th>
<th>1, j-1</th>
<th>2, j+1</th>
<th>2, j-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, j</td>
<td>.</td>
<td>0</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>1, j</td>
<td>0</td>
<td>.</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>2, j</td>
<td>.</td>
<td>0</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>2, j+2</td>
<td>.</td>
<td>0</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>2, j-2</td>
<td>.</td>
<td>0</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>1, j+1</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>0</td>
<td>0</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>1, j-1</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>0</td>
<td>0</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>2, j+1</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
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<td>.</td>
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</tr>
<tr>
<td>2, j-1</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>


Derivation of the Spin Operator Coefficients
in Terms of the Elements of M

We will in this section express the spin operator coefficients \( \{B_j\} \) in terms of the elements of the transition matrix \( \{m_i\} \). It would be a lengthy task to invert the equations of Appendix E. This alternate method follows Seyler.\(^3\)

Explanation of the Method

The method will be explained by an example. Let us find \( B_3 \) in terms of the \( \{m_i\} \). The expansion of \( M \) in terms of the spin operators is

\[
M = B_{11}^9 + B_2(\Sigma_n S_n) + B_3 \Sigma_k S_k + \ldots + B_{25}(\Sigma_{kk} S_{pp} - \Sigma_{pp} S_{kk}).
\]

(68)

We multiply this from the right by \( \Sigma_k S_k \) and take the trace.

\[
\text{Tr}[M \Sigma_k S_k] = B_{11} \text{Tr}[\Sigma_k S_k] + B_2 \text{Tr}[\Sigma_n S_k S_k]
\]

\[
+ B_2 \text{Tr}[\Sigma_n S_k S_k] + B_3 \text{Tr}[\Sigma_k S_k S_k]
\]

(101)

\[
+ \ldots +
\]

\[
+ B_{25} \text{Tr}[\Sigma_{kk} S_{pp} S_k] - B_{25} \text{Tr}[\Sigma_{pp} S_{kk} S_k]
\]

We can evaluate the left hand side since we have explicit expressions for \( \Sigma_k S_k \) and for \( M \) in terms of \( \{m_i\} \). Since all of the traces on the right hand side are zero except
This method enables us to evaluate all \{B_j\} in terms of the \{m_i\}. In some cases two of the traces on the right are non-zero, it is necessary then to solve two or four simultaneous equations in the \(B\) coefficients.

This method requires that we know the trace of the product of any operation in (68) with any second operator in that equation.

**Traces of the Spin Operator Products—Properties and Theorems**

We will now develop results which enable us to calculate the needed traces of the spin operator products.

First we display the traces of the single particle spin operators. \(i, j, k, \ell\), are each any one of the directions \(k, n, p\).

**Property**

1. \(\text{Tr}[\Sigma_i] = 0\).

This is shown by direct calculation on the explicit operators of Appendix D.

2. \(\text{Tr}[\Sigma_i \Sigma_j] = 6\delta_{ij}\).

This is shown by direct calculation on the explicit operators of Appendix D.

3. \(\text{Tr} [\Sigma_{ij}] = 0\).
This is a straightforward consequence of the definition of $\Sigma_{ij}$ in equation (64) and 2.

4. $\text{Tr}[\Sigma_j \Sigma_k \Sigma_{\ell}] = 3i(\hat{j} \times \hat{k} \cdot \hat{\ell})$.

$\hat{j}, \hat{k}, \hat{l}$ are unit vectors in the direction of the directions $j, k, l$. This is based on an identity for the spin-1 operators from Seyler. The identity is

\[
(\hat{\Sigma} \cdot \hat{j}) (\hat{\Sigma} \cdot \hat{k}) (\hat{\Sigma} \cdot \hat{l}) = \sqrt{-1/4} \left[ (\hat{\Sigma} \cdot \hat{j} \times \hat{k}) (\hat{\Sigma} \cdot \hat{l}) + (\hat{\Sigma} \cdot \hat{l} \times \hat{j}) (\hat{\Sigma} \cdot \hat{k}) + (\hat{\Sigma} \cdot \hat{k} \times \hat{l}) (\hat{\Sigma} \cdot \hat{j}) \right] + 1/2 (\hat{j} \cdot \hat{k}) (\hat{\Sigma} \cdot \hat{l}) + 1/2 (\hat{l} \cdot \hat{k}) (\hat{\Sigma} \cdot \hat{j}). \tag{102}
\]

5. $\text{Tr}[\Sigma_j \Sigma_{\ell}] = \text{Tr}[\Sigma_{kj} \Sigma_j] = 0$.

This is a straightforward consequence of the definition of $\Sigma_{kj}$ and 4.

6. $\text{Tr}[\Sigma_j \Sigma_k \Sigma_{\ell} \Sigma_m] = 3(\delta_{jk}\delta_{\ell m} + \delta_{k\ell}\delta_{jm})$.

Equation (102) expresses a triple product of spin operators in terms of double products. It can be used in the proof of 6 to express a quadruple product in terms of a triple products and these in turn in terms of double products.

7. $\text{Tr}[\Sigma_{jk} \Sigma_{\ell m}] = (3/2\delta_{km}\delta_{j\ell} + 3/2\delta_{k\ell}\delta_{jm} - \delta_{jk}\delta_{\ell m})$.

This is a consequence of the definition of $\Sigma_{jk}$ and 6.
Identical expressions hold for the particle-2 spin operators.

The traces of operators which have as factors both particle-1 operators and particle-2 operators may be found using the following theorems.

8. **Theorem 13** If $E^3_1$ is an operator in particle-1 space and $S^3_2$ is an operator in particle-2 space and if $R^9_3$ is the tensor product of $E^3_1$ and $S^3_2$ i.e., $R^9_3 = E^3_1 \otimes S^3_2$, then $Tr[R^9_3] = Tr[E^3_1] \cdot Tr[S^3_2]$. This is a simple consequence of the definition of tensor product.

9. **Theorem 14** If $E^3_1$ and $E^3_3$ are any operators in particle-1 space and $S^3_2$ and $S^3_4$ are any operators in particle-2 space and $E^9_3$, $E^9_3$, $S^9_2$, $S^9_4$ are those operators in product space, then $E^9_3 E^9_3 S^9_2 S^9_4 = E^3_1 E^3_3 \otimes S^3_2 S^3_4$ and $E^9_3 E^9_3 S^9_2 S^9_4 = E^3_1 E^3_3 \otimes S^3_2 S^3_4$. The proof is shown in Appendix G on page 200.

10. **Theorem 15** If $U^3_3$ is an operator in particle-1 space or particle-2 space and $U^9_3$ is that operator in product space then $Tr[U^3_3] = Tr[U^9_3] / 3$. 
This is a simple consequence of Theorem 13.

We present as an example of the use of these Theorems the calculation of

\[ T = \text{Tr} [\Sigma_{kp}^9 S_{kp}^9 \Sigma_{kp}^9 S_{kp}^9]. \]

1. Theorem 14 tells us that

\[ T = \text{Tr} [\Sigma_{kp}^3 \Sigma_{kp}^3 \otimes S_{kp}^3 S_{kp}^3]. \]

2. Theorem 13 allows that

\[ T = \text{Tr} [\Sigma_{kp}^3 \Sigma_{kp}^3] \cdot \text{Tr} [S_{kp}^3 \cdot S_{kp}^3]. \]

3. Property 7 and Theorem 15 lead to

\[ T = \left[ \frac{3/2}{3} \right] \cdot \left[ \frac{3/2}{3} \right] = 1/4. \]

Results

The \( \{B_j\} \) expressed in terms of \( \{m_i\} \) are given in Appendix F on page 190.

We may check this work by showing that the insertion of the restrictions on the \( \{m_i\} \) of identical particles of equations (100) produces the correct restrictions on the \( \{B_j\} \), those of equations (70).

Number of Independent Parameters in the Transition Matrix

This is perhaps a good point to summarize the number of parameters required to specify the various forms of the transition matrix.
We consider first $M$ represented in terms of spin operators. There are 81 operators and hence 81 coefficients in the general expansion of a $9 \times 9$ matrix. We saw that only 41 of these operators were invariant to space reflection and of these only 25 were additionally time reversal invariant. The situation of identical particles required 8 of these operators to be discarded leaving 17 operators to represent an $M$ matrix which is space and time reflection invariant and particle exchange invariant.

We consider now $M$ represented in terms of its elements. A $9 \times 9$ matrix has 81 elements. We deduced that conservation of parity means that only 41 of these are independent. We saw that, when time reversal invariance is applied to the parity conserving form of $M$, that 16 independent equations result. Thus there are $41 - 16 = 25$ independent elements in the $M$ which conserves parity and is time reversal invariant. The situation of identical particles requires all the elements in blocks $s=1$, $s'=2$ and $s=2$, $s'=1$ and $s'=1$, $s=0$ and $s=1$, $s'=0$ to be zero. There are 16 independent elements in these blocks in the parity conserving form of $M$. There are 8 time reversal invariance equations involving elements in these blocks as shown in Appendix A. This leaves $16 - 8 = 8$ independent elements in these blocks in the parity conserving and time reversal invariant form of $M$. Thus the additional condition of identical particles reduces the number of independent elements to $25 - 8 = 17$. There are 17
independent elements in the form of $M$ appropriate to identical particles which conserves parity and is time reversal invariant. This agrees with the number of spin operators which are invariant to space and time reflection and particle exchange. Table 14 on the next page lists the numbers of independent elements for the various forms of the transition matrix.

**Expression of the Spherical Polarization Observables in Terms of Transition Matrix Elements**

One of the major results of this thesis is the expression of the polarization observables in terms of the beam polarizations and the matrix elements of the transition matrix. In order to do this for spherical polarizations we must express equation (86) on page 51 in terms of the $\{m_i\}$. This involves expressing the $81 \text{Tr}[M T_{KQ}^+ M^+ T_{K'Q'}]$ in terms of the $\{m_i\}$. The first step in doing this is to form the $9 T_{KQ}$.

**Formation of the Spherical Tensor Operators**

The Spherical Tensor Operators $T_{KQ}$ may be calculated by inserting the expressions for $\Sigma_X$, $\Sigma_Y$ and $\Sigma_Z$ from Appendix B into the defining equations for the $T_{KQ}$, equations (71), (72) and (73). For example:

$$T_{2+2} = \sqrt{3}/2 \ (\Sigma_X + i \Sigma_Y) \ (\Sigma_X + i \Sigma_Y).$$
TABLE 14
TRANSITION MATRIX--NUMBER OF INDEPENDENT ELEMENTS 
OF THE DIFFERENT FORMS

<table>
<thead>
<tr>
<th>Blocks</th>
<th>Parity Form</th>
<th>Parity and Time Reversal Form</th>
<th>Parity, Time Reversal and Identical Particles Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s=2, s'=2$</td>
<td>13</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>$s=1, s'=2$</td>
<td>14</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>and $s=2, s'=1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s=1, s'=1$</td>
<td>5</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$s=0, s'=2$</td>
<td>6</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>and $s=2, s'=0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s=0, s'=1$</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>and $s=1, s'=0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s=0, s'=0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>41</td>
<td>25</td>
<td>17</td>
</tr>
</tbody>
</table>
The method which we used avoided doing the matrix multiplications. We see from the last equation that

\[ T_{2+2} = \sqrt{3}/2 \left[ \Sigma_x^2 - \Sigma_y^2 + 2i \Sigma_{xy} \right] = \sqrt{3}/2 \left[ \Sigma_k^2(\theta=0) - \Sigma_n^2 + 2i \Sigma_{kn}(\theta=0) \right]. \]  \hspace{1cm} (102)

The other operators were calculated in a similar manner or by using equation (74). The method used for each is indicated below:

\[ T_{2+2} = \sqrt{3}/2 \left[ \Sigma_k^2(\theta=0) - \Sigma_n^2 + 2i \Sigma_{kn}(\theta=0) \right], \]

\[ T_{2+1} = -\sqrt{3} \left[ \Sigma_k^2(\theta=0) + i \Sigma_{kn}(\theta=0) \right], \]

\[ T_{20} = 3\sqrt{2}/2 \left[ \Sigma_{pp}(\theta=0) \right], \]

\[ T_{2-1} = (-1)^{-1} T_{2+1}^+ = \sqrt{3} \left[ \Sigma_k^2(\theta=0) - i \Sigma_{np}(\theta=0) \right], \]

\[ T_{2-2} = (-1)^{-2} T_{2+2}^+ = \sqrt{3}/2 \left[ \Sigma_k^2(\theta=0) - \Sigma_n^2 - 2i \Sigma_{kn}(\theta=0) \right], \]

\[ T_{1+1} = -\sqrt{3}/2 \left[ \Sigma_k^2(\theta=0) + i \Sigma_{n} \right], \]

\[ T_{10} = \sqrt{6}/2 \left[ \Sigma_{p}(\theta=0) \right], \]

\[ T_{1-1} = (-1)^{-1} T_{1+1}^+ = \sqrt{3}/2 \left[ \Sigma_k^2(\theta=0) - i \Sigma_{n} \right], \]

\[ T_{00} = 1^9. \]  \hspace{1cm} (103)

The explicit forms of the \( T_{KQ} \) are given in Appendix H on page 203.
Relationships between the $T_{KQ}$

The following relationship between the $T_{KQ}$ will be useful later. These relationships may be deduced in a simple manner from the BCRS of the $\Sigma_x$, $\Sigma_y$, $\Sigma_z$ or they may be verified by inspection of the explicit expressions for the $T_{KQ}$:

$$(T_{KQ})_{s',v',s''} = (-1)^{K+Q}(-1)^{s'+s+v'+v}(T_{K-Q})_{s'-v',s-v}. \tag{104}$$

This property reduces to BCRS for $T_{20}$ and $T_{10}$ and $T_{00}$.

Formation of the $[MT^\dagger_{KQ}M^\dagger_{K'Q'}]$:

The calculation of the 81 operators $[MT^\dagger_{KQ}M^\dagger_{K'Q'}]$ can be done most easily by first calculating the nine $MT^\dagger_{KQ}$ and the nine $M^\dagger_{K'Q'}$. We can avoid doing 6 of these 18 multiplications by using the result given below.

Theorem 16. If

$$(T_{KQ})_{s',v',s''} = (-1)^{K+Q}(-1)^{s'+s+v'+v}(T_{K-Q})_{s'-v',s-v},$$

then

$$(M^\dagger_{K'Q'})_{s',v',s''} = (-1)^{K'+Q'}(-1)^{s'+s+v'+v}(M^\dagger_{K'-Q'})_{s'-v',s-v}$$

and

$$(M T^\dagger_{KQ})_{s',v',s''} = (-1)^{K+Q}(-1)^{s'+s+v'+v}(M T^\dagger_{K-Q})_{s'-v',s-v}. \tag{105}$$
This is a simple consequence of the BCRS of the $M$ matrix and the properties of the $T_{KQ}$ mentioned in the equation (104). If $Q' = 0$ and $Q = 0$ these equations say $M^\dagger T_{K'Q'}$ and $MT_{KQ}^\dagger$ have BCRS.

**Relationship between Tr$[MT_{KQ}^\dagger M^\dagger T_{K'}Q']$**

and Tr$[MT_{K'-Q}T_{K'}^\dagger Q']$

We can avoid calculating all 81 of the Tr$[MT_{KQ}^\dagger M^\dagger T_{K'}Q']$ by using the following theorem:

**Theorem 17.** If

$$\begin{align*}
(M^\dagger T_{K}^\dagger Q_{K})_{s',v'},s_{v} &= (-1)^{K+Q} (-1)^{s'+v'} (-1)^{s+v} (MT_{K'-Q}^\dagger)^{s',v'}, s_{-v'}
\end{align*}$$

and

$$\begin{align*}
(M^\dagger T_{K'}^\dagger Q')_{s',v'},s_{v} &= (-1)^{K'+Q'} (-1)^{s'+v'} (-1)^{s+v} (M^\dagger T_{K'-Q'}^\dagger)^{s',v'}, s_{-v'}
\end{align*}$$

then

$$\text{Tr}[MT_{KQ}^\dagger M^\dagger T_{K'}Q'] = (-1)^{K+Q} (-1)^{K'+Q'} \text{Tr}[MT_{K'-Q}^\dagger M^\dagger T_{K'-Q'}].$$

This result which is based on the properties of the $MT_{KQ}^\dagger$ and the $M^\dagger T_{K'}^\dagger Q'$ developed in Theorem 16 reduces the number of traces that we must calculate from 81 to 41.

**Derivation of the Form of**

$[MT_{KQ}^\dagger M^\dagger T_{K'}Q']$

The calculation of a Tr$[MT_{KQ}^\dagger M^\dagger T_{K'}Q']$ is a time consuming task as inspection of one of them in Appendix I will
show. This work may be considerably reduced by fore-
knowledge of the general form of these quantities.

We now show that: either

\[ \text{Tr}[MT_K^* M^T K', Q'] = \sum_{\ell, k} P_{\ell k} [2\Re(m_k^* m^*)] \]

or

\[ \text{Tr}[MT_K^* M^T K', Q'] = \sum_{\ell, k} P_{\ell k} [2\Im(m_k^* m^*)], \]

with \( P_{\ell k} \) a real number in either case.

This result means that it is only necessary to calculate

terms of the traces for which \( \ell \leq k \), reducing the work by

nearly one-half.

The development of this result, which we now present,
is rather lengthy. We will use the BCRS property of \( M \) and

the fact that \( T_K Q \) can be expanded in a linear combination

of Hermitean Cartesian spin operators which are symmetric

or antisymmetric and have BCRS.

We begin by expressing the \( \text{Tr}[MT_K^* M^T K', Q'] \) in terms of

Cartesian Spin Operators.

Each of the operators \( T_K' Q' \) can be written in the

following form

\[ T_K' Q' = \sum_m a_m E_m, \]

with \( a_m = \pm 1 \) or \( a_m = \pm i \) and each \( E_m \) a Hermitean operator.
which is symmetric and real or antisymmetric and pure imaginary. The $E_m$ have BCRS. As an example we consider $T_{2+2}$:

$$T_{2+2} = \sqrt{3}/2 \left[ \Sigma_k^2(\theta=0) - \Sigma_n^2 + 2i \Sigma_{kn}(\theta=0) \right].$$

This can be rewritten as

$$T_{2+2} = E_1 + iE_2,$$

with

$$E_1 = \sqrt{3}/2 \left( \Sigma_k^2(\theta=0) - \Sigma_n^2 \right),$$

a real symmetric operator, and

$$E_2 = \sqrt{3} \left( \Sigma_{kn}(\theta=0) \right),$$

a pure-imaginary and antisymmetric operator. We can verify that any $T_{K'Q'}$ can be written in this form by inspecting equations (103). The symmetry properties and BCRS properties of the $E_m$ may be verified by reference to the explicit forms of the operators in Appendix D. The $T_{K'Q'}$ written in the form of equation (105) are shown in Table 15 on page 111.

We associate with each Cartesian operator $E$ a $\beta$ coefficient which has values $\pm 1$ as $E$ has $\pm$BCRS and a $\tau$ coefficient which has values $\pm 1$ as $E$ is symmetric and real or anti-symmetric and pure-imaginary.
We note by taking the Hermitean conjugate of equation (105) that for any $T_{KQ}$

$$T_{KQ}^+ = \Sigma b_n E_n$$  \hspace{1cm} (106)

with $b_n = \pm 1$ or $\pm i$.

We can use the expansions (105) and (106) to write our trace of interest as

$$\text{Tr}[M^T_{KQ} M^+_{K'Q'}] = \text{Tr}[M(b_n E_n) M^+ (\Sigma_m E_m)]$$

$$= \Sigma g_{m,n} \text{Tr}[M_{m} M^+_{m} E_{m}],$$  \hspace{1cm} (107)

with $g_{m,n} = \pm 1$ or $\pm i$.

Some of the terms in this summation are zero, we can see this with the aid of the following lemmas.

Lemma 1. If operator $A$ has $-\text{BCRS}$, then

$$\text{Tr} A = 0.$$

This proof is obvious if in the trace sum we pair the elements of $A$ related by the BCRS symmetry.

Lemma 2. If $A$ has $+\text{BCRS}$ then $A^+$ has $+\text{BCRS}$.

The proof is a direct consequence of the definitions of BCRS and the tranjugate operation.

Since $M$ has $+\text{BCRS}$ then $M^+$ has $+\text{BCRS}$. $E_m$ and $E_n$ have BCRS so $ME_n M^+ E_m$ has BCRS by Theorem 5, and it is easy to show:
Lemma 3. \( ME_n M^\dagger E_m \) has \( \pm \)BCRS as \( \beta_n \beta_m \) is \( \pm 1 \).

These three lemmas allow us to conclude that in using the expression (107) we only need to include those terms in which \( E_n \) and \( E_m \) have the same sign BCRS, that is\n\[ \beta_n \beta_m = \pm 1. \]

One additional lemma:

Lemma 4. \( ME_n M^\dagger \) is Hermitean

Proof: \( (ME_n M^\dagger)^\dagger = (M^\dagger)^\dagger E_n^\dagger M^\dagger = ME_n M^\dagger. \) Q.E.D.

We consider now the form of one of the terms in equation (107). Let us write the term \( \text{Tr}[ME_n M^\dagger E_m] \) in terms of matrix elements.

\[
\text{Tr}[ME_n M^\dagger E_m] = \sum_{s'v'} \sum_{sv} (ME_n M^\dagger)_{s'v',sv}(E_m)_{sv,s'v'}. \quad (108)
\]

This double summation may be rewritten as

\[
\sum_{s'v'} \sum_{sv} \frac{1}{2} \delta_{ss'} \delta_{vv'} [(ME_n M^\dagger)_{s'v',sv}(E_m)_{sv,s'v'} + (ME_n M^\dagger)_{sv,s'v'} (E_m)_{s'v',sv}]. \quad (109)
\]

An example will show what has been done here.
\[ \text{Tr} \left[ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right] = a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22} \]

\[ = \frac{a_{11}b_{11} + a_{11}b_{11}}{2} + (a_{12}b_{21} + a_{21}b_{12}) + \frac{a_{22}b_{22} + a_{22}b_{22}}{2} \]

\[ = \sum_{\omega < \phi} \left( \frac{1}{2} \delta_{\omega \phi} (a_{\omega \phi}b_{\omega \phi} + a_{\phi \omega}b_{\phi \omega}) \right). \]

The example also explains the meaning of notation \( s'v' \leq sv \).

The expression (109) may be altered by using Lemma 4 and the fact that the parameter \( T_m \) reflects whether \( E_m \) is symmetric or anti-symmetric.

\[ \text{Tr}[M_{\mathbb{E}n}^M^+E_m] = \sum_{s'v'} \sum_{sv} (1/2) \delta_{s's} \delta_{v'v} \]

\[ \times \left( (M_{\mathbb{E}n}^M)^{s'v',sv} (E_m)_{sv,s'v'} + (M_{\mathbb{E}n}^M)^*_{s'v',sv} (E_m)_{sv,s'v'} (T_m)_{s'v'} \right). \]

This may be rearranged to give us an expression for the \( \text{Tr}[M_{\mathbb{E}n}^M^+E_m] \) term of \( \text{Tr}[M_{\mathbb{K}Q}^M^+T_{K',Q'}]. \)

\[ \text{Tr}[M_{\mathbb{E}n}^M^+E_m] \]

\[ = \sum_{s'v'} \sum_{sv} (1/2) \delta_{s's} \delta_{v'v} \]

\[ \times \left( (M_{\mathbb{E}n}^M)^*_{s'v',sv} (T_m)_{sv,s'v'} \right) (E_m)_{sv,s'v'}. \]

(110)
We will now consider the form of the terms in equation (110). Let us define \( H(s'v',sv) \) and \( G_{s'v',sv} \) as below for convenience in writing.

\[
G_{s'v',sv} = \left( (ME_nM^\dagger)_{s'v',sv} + \tau_m (ME_nM^\dagger)_{s'v',sv}^* \right). \tag{111}
\]

\[
H(s'v',sv) = G_{s'v',sv} (E_m)_{sv} sv'. \tag{112}
\]

Thus

\[
\text{Tr}[ME_nM^\dagger E_m] = \sum_{s'v'} \sum_{sv} (1/2) H(s'v',sv). \tag{112}
\]

Recalling the form of \( M \) shown in Table 1 on page 33 we see that the form of the matrix element \( (ME_nM^\dagger)_{s'v',sv} \) must be

\[
(ME_nM^\dagger)_{s'v',sv} = \sum_{\ell,k} h_{\ell k}(s'v',sv) m_\ell^* m_k. \tag{113}
\]

The coefficient \( h_{\ell k}(s'v',sv) \) will be real or pure-imaginary as \( E_n \) is real or pure-imaginary, which is as \( \tau_n = +1 \) or \( \tau_n = -1 \). From (113) we have also that

\[
(ME_nM^\dagger)^*_{s'v',sv} = \sum_{\ell,k} h_{\ell k}^*(s'v',sv) m_\ell m_k^*. \tag{114}
\]

Equations (113) and (114) give us an expression for \( G_{s'v',sv} \).
$G_{s', v', sv} = \left[ \sum_{\ell, k} h_{\ell k}(s', v', sv) m_\ell^* m_k + \tau_m \sum_{\ell, k} h_{\ell k}(s', v', sv) m_\ell^* m_k \right]$

\[= \sum_{\ell, k} \left[ h_{\ell k}(s', v', sv) \left( m_\ell^* m_k + \tau_m m_\ell^* m_k \right) \right]. \quad (115) \]

Also:

\[H(s', v', sv) = \sum_{\ell, k} \left[ h_{\ell k}(s', v', sv) \left( m_\ell^* m_k + \tau_m m_\ell^* m_k \right) (E_m)_{sv, s'v'} \right]. \quad (116)\]

Let us consider a simple case of equation (116).

\[H = (E_m) \sum_{\ell=1, 2} \sum_{k=1, 2} h_{\ell k} (m_\ell^* m_k \pm m_\ell^* m_k)

\[= (E_m) \left[ h_{11} (m_1^* m_1^* \pm m_1^* m_1^* ) + h_{12} (m_1^* m_2^* \pm m_1^* m_2^* )

+ h_{21} (m_2^* m_1^* \pm m_2^* m_1^* ) + h_{22} (m_2^* m_2^* \pm m_2^* m_2^* ) \right] \]

\[= (E_m) \left[ h_{11} (m_1^* m_1^* \pm m_1^* m_1^* ) + (h_{12} \pm h_{21}) (m_1^* m_2^* \pm m_1^* m_2^* )

+ h_{22} (m_2^* m_2^* \pm m_2^* m_2^* ) \right]. \]

We let

\[h'_{\ell k} = (h_{\ell k} \pm h_{k\ell}) \frac{1}{2} \delta_{\ell k} \]

then

\[H = (E_m) \left[ \sum_{k=1, 2} \sum_{\ell \leq k} h'_{\ell k} \left( m_\ell^* m_k \pm m_\ell^* m_k \right) \right]. \]
We rewrite equation (116) in this fashion. It is at this point that the time saving element of needing to sum only over the \( l \) values less than or equal to the \( k \) values enters our derivation.

\[
H(s',v',sv) = \sum_{k \geq l} (E_m)_{sv,s'v'} h^i_{kk}(s',v',sv) (m^*_km_k + \tau_m \tau_n m^*_km_k) \quad (117)
\]

with

\[
h^i_{kk}(s',v',sv) = (1/2)^\delta_{kk} [h^i_{kk}(s',v',sv) + \tau_m \tau_n h^i_{kk}(s',v',sv)].
\]

The form of the summand in equation (117) is discussed below in more detail.

We note first that if \( \tau_m \tau_n = +1 \),

\[
(m^*_km_k + \tau_m \tau_n m^*_km_k) = 2 \text{Re} \text{Re}(m^*_km_k),
\]

and if \( \tau_m \tau_n = -1 \),

\[
(m^*_km_k + \tau_m \tau_n m^*_km_k) = 2 \text{Im} \text{Im}(m^*_km_k).
\]

<table>
<thead>
<tr>
<th>( \tau_n )</th>
<th>( h^i_{kk}(s',v',sv) )</th>
<th>( \tau_m )</th>
<th>( (E_m)_{sv,s'v'} )</th>
<th>( \tau_m \tau_n )</th>
<th>( (m^<em>_km_k + \tau_m \tau_n m^</em>_km_k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>real</td>
<td>+</td>
<td>real</td>
<td>+</td>
<td>( 2 \text{Re}(m^*_km_k) )</td>
</tr>
<tr>
<td>+</td>
<td>real</td>
<td>-</td>
<td>imag</td>
<td>-</td>
<td>( 2 \text{Im}(m^*_km_k) )</td>
</tr>
<tr>
<td>-</td>
<td>imag</td>
<td>+</td>
<td>real</td>
<td>-</td>
<td>( 2 \text{Im}(m^*_km_k) )</td>
</tr>
<tr>
<td>-</td>
<td>imag</td>
<td>-</td>
<td>imag</td>
<td>+</td>
<td>( 2 \text{Re}(m^*_km_k) )</td>
</tr>
</tbody>
</table>
These four cases may be summarized: the summand is equal to

\[ c_{\lambda k} (s'v',sv) 2\text{Re} (m_k^* m_k^*) , \]

with \( c_{\lambda k} (s'v',sv) \) real, if \( \tau_m \tau_n = +1 \), and to

\[ c_{\lambda k} (s'v',sv) 2\text{Im} (m_k^* m_k^*) , \]

with \( c_{\lambda k} (s'v',sv) \) real, if \( \tau_m \tau_n = -1 \).

We can express \( H(s'v',sv) \) as expressed in equation (117) by using this last result as

\[ H(s'v',sv) = \sum_{k, \lambda} c_{\lambda k} (s'v',sv) 2\text{Re} (m_k^* m_k^*) , \]

when \( \tau_n \tau_m = +1 \), and as

\[ H(s'v',sv) = \sum_{k, \lambda} c_{\lambda k} (s'v',sv) 2\text{Im} (m_k^* m_k^*) , \]

when \( \tau_n \tau_m = -1 \). \( C_k \) is real in both of these equations.

We insert these last equations into equation (112) to find

\[ \text{Tr}[M E_n^t E_m] = \sum_{k, \lambda} 2\text{Re}(m_k^* m_k^*) \sum_{s'v',sv} (1/2) \delta_{s's'v'} \delta_{v'v} c_{\lambda k} (s'v',sv) \]

\[ = \sum_{k, \lambda} b_{\lambda k} 2\text{Re}(m_k^* m_k^*) , \]

(118)

when \( \tau_m \tau_m = +1 \), and
The quantities whose form we are attempting to derive are the $\text{Tr}[\mathbf{M}_{KQ}^T \mathbf{M}_{K^*Q'}^T]$, they are related to the $\text{Tr}[\mathbf{M}^T_{\text{E}_m}]$ by equation (107).

$$\text{Tr} [\mathbf{M}_{KQ}^T \mathbf{M}_{K^*Q'}^T] = \sum_{m,n} g_{m,n} \text{Tr} [\mathbf{M}^T_{\text{E}_m}] .$$  \hspace{1cm} (107)

Recall that if $\beta_m \beta_n = -1$ then

$$\text{Tr} [\mathbf{M}^T_{\text{E}_m}] = 0.$$

The spherical operators expanded in terms of Hermitean Cartesian operators which are symmetric or antisymmetric are shown in Table 15 on the next page. The Cartesian operators are arranged in columns according to the signs of their $\beta$ and $\tau$ coefficients. It is apparent from this table that for any $T_{KQ}$ the product $\beta \tau$ is the same for each Cartesian operator in its expansion. This means that for any spherical operators $T_{KQ}$ and $T_{K^*Q'}$ that all the $\mathbf{M}^T_{\text{E}_m}$ in the expansion (107) for which $\beta_m \beta_n = +1$ will also have $\tau_m \tau_n$ with either a plus sign or a minus sign and hence all the $\text{Tr}[\mathbf{M}^T_{\text{E}_m}]$ in that expansion will be either of the type

\begin{equation}
\text{Tr}[\mathbf{M}^T_{\text{E}_m}] = \sum_{\ell, k} 2 \text{Im}(m_\ell m_k^*), \quad \ell \leq k \end{equation}

when $\tau \tau' = -1$.  \hspace{1cm} (119)
<table>
<thead>
<tr>
<th>Operator</th>
<th>$\beta$</th>
<th>$\tau$</th>
<th>$\beta$</th>
<th>$\tau$</th>
<th>$\beta$</th>
<th>$\tau$</th>
<th>$\beta$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{2+2}$</td>
<td>$+1\left(\frac{\sqrt{3}}{2}(\Sigma_k^2-\Sigma_n^2)\right)$</td>
<td>+1 $[\sqrt{3} \Sigma_{kn}]$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
<td>$+1\left(\frac{3\sqrt{2}}{2} \Sigma_{pp}\right)$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{kn}\right)$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
</tr>
<tr>
<td>$T_{2+1}$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{kp}\right)$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
</tr>
<tr>
<td>$T_{2-1}$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
</tr>
<tr>
<td>$T_{2-2}$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
</tr>
<tr>
<td>$T_{1+1}$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
</tr>
<tr>
<td>$T_{10}$</td>
<td>$-1 \left(\sqrt{3} \Sigma_{np}\right)$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
</tr>
<tr>
<td>$T_{1-1}$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
<td>$+1\left[\sqrt{3} \Sigma_{np}\right]$</td>
</tr>
</tbody>
</table>
of equation (118) or of the type of equation (119). We therefore arrive at the desired result:

$$\text{Tr}[MT_{KQ}^+ M_{KQ}^T Q'] = \sum_{\ell, k} P_{\ell k} 2 \text{Re}(m^*_\ell m_k), \quad (120)$$

when \(\tau_m = +1\), and

$$\text{Tr}[MT_{KQ}^{*+} M_{KQ}^T Q'] = \sum_{\ell, k} P_{\ell k} 2 \text{Im}(m^*_\ell m_k), \quad (121)$$

when \(\tau_m = -1\). The \(P_{\ell k}\) are real in either case.

**Summary of Procedure**

We summarize the procedure for evaluating the 81

$$\text{Tr}[MT_{KQ}^{*+} M_{KQ}^T Q']$$

in terms of the \(\{m_\ell\}\). We have seen in Theorem 17 that we need to evaluate only 41 of these operators. One may determine most readily for a given pair of operators \(T_{KQ}\) and \(T_{K'Q'}\) whether the \(\text{Tr}[MT_{KQ}^{*+} M_{KQ}^T Q']\) is of the form of equation (120) or of equation (121) by checking the relative sign of a pair of terms \(m^*_\ell m_k\) and \(m^*_k m_\ell\). If they have the same sign it is equation (120) with the real part which applies and if they have the opposite sign it is equation (121) with the imaginary part. The quantity

$$\text{Tr}[MT_{KQ}^{*+} M_{KQ}^T Q']$$

is twice the real (or imaginary) part of the sum of those terms \(m^*_\ell m_k\) in the diagonal elements of the product of \(MT_{KQ}^{*+}\) and \(M_{KQ}^T Q'\), for which \(\ell < k\) together with the sum of those terms \(m^*_k m_\ell\) in the diagonal elements in this
product for which \( l = k \). There will be no terms \( m^*_k m^*_l \) in equation (121).

**Results**

The expressions for the 81 \( \text{Tr}[MT^+_K M^+_L T_{K'} Q'] \) in terms of the \( \{m_i\} \) are shown in Appendix I on page 213. In the situation of identical particles the number of non-zero terms in these traces is drastically reduced.
IV. CONCLUSION

Scattering experiments are conducted for the purpose of studying the nuclear interaction, experiments, in which polarizations are measured, can provide information on the spin dependence of the interaction. The information, from the experiment, about the nuclear interaction is contained in the measured values of the polarizations of the beam particles and the scattered particles. The results of this thesis will aid in the extraction of details of the interaction from the data of an experiment in which a polarized beam of spin-1 particles is scattered by an unpolarized target of spin-1 particles. The polarizations of the scattered particles $t'_{K'Q'}$ are related to the polarizations of the beam $t_{KQ}$ and to the transition matrix $M$ by the equation

$$
t'_{K'Q'} = \frac{\sum_{KQ} t_{KQ} \text{Tr}[M^\top_{KQ} M^\top_{K'Q'}]}{\sum_{KQ} t_{KQ} \text{Tr}[M^\top_{KQ} M^\top_{00}]} .
$$

(86)

The transition matrix represents the effect of the detailed nature of the interaction on the scattering. We provide in this thesis explicit expressions for the $\text{Tr}[M^\top_{KQ} M^\top_{KQ}]$ in terms of the elements of the transition matrix. A set of elements of $M$, $\{M_{s',v',sv}(\theta,E)\}$ which fit the measured
polarizations through equation (86) would contain the information available from the scattering experiment about the interaction.

In order to fit the data it is not necessary to search for 81 arbitrary functions of angle and energy, restrictions on the form of $M$ and on the form of the elements of $M$ are imposed by the laws of non-relativistic quantum mechanics as well as by the other restrictions of conservation of parity, time reversal invariance and the Pauli Exclusion Principle in the situation of identical particles in beam and target. We incorporate these restrictions into $M$.

The Lane and Thomas expression which results from applying the laws of quantum mechanics to the scattering of a particle with spin by another particle with spin gives the elements of the transition matrix explicitly in terms of scattering angle and the elements $U^j_{\ell's',\ell s}(E)$ of the nuclear collision matrix.

$$M_{s'v',sv}(\theta,E) = \sqrt{\pi/k} \left[ -C(\theta)\delta_{s's}\delta_{v'v} + i \sum_{j,\ell,\ell'} \frac{\sqrt{2\ell+1}}{j,\ell,\ell'} \right]$$

$$\times (s'v0|jv) (s'\ell'v' v-v'|jv)$$

$$\times \exp\{i(\omega_\ell + w_{\ell'})\} (\delta_{s's}\delta_{\ell\ell'} - U^j_{s',\ell',s'\ell})$$

$$\times Y^{v'v'}_{\ell'\ell}(\theta,0).$$

(53)
We derive in this thesis the form of $M$, which conserves parity, is time reversal invariant and obeys the Pauli Exclusion Principle.

The form of the Lane and Thomas result that must be used if the colliding particles are identical is $\hat{M}$.

$$\hat{M}_{s'v',sv}(\theta, \phi) = \sqrt{\pi/k} \left\{ \left\{ \delta_{s's'} \delta_{v,v} \right. \\
+ i \sum_{j, l, l'} \sqrt{2l+1} \left( s|v\rangle \langle j|v' \right) \left( s'\left| \begin{array}{c} v \\ \langle j|v' \langle l | s' \rangle \end{array} \right) \right. \\
\left. \times \left( 1 + (-1)^{s'+l} \right) \gamma_{l}^{v-v'}(\theta, \phi) \right\}. \tag{58}$$

The restriction that parity be conserved in the scattering gives $M$ and $\hat{M}$ the +BCRS symmetry property. This is shown in the equation

$$\hat{M}_{s'v',sv} = (-1)^{s'+s} (-1)^{v''v} \hat{M}_{s'v',sv} \tag{59}$$

The parity conserving form of the transition matrix has 41 independent elements $\{m_{j}(\theta, E)\}$ and is shown on page 33.

The restriction of time reversal invariance on $M$ is shown in the equation

$$\sum_{v''} d_{v''v',sv}^{s'} M_{s'v',sv} = (-1)^{v''v} \sum_{v''} d_{v''v}^{s} M_{sv''}, s'v' \tag{63}$$

This is a set of 16 independent equations in the elements of $M$. This equation relates elements in the block $s's'$ to
elements of the same block or relates elements in the block \( s' \) to those in block \( s, s' \). The restriction on the independence of the elements of \( M \) of time reversal invariance is shown on page 74.

The additional restriction that the Pauli Exclusion Principle implies in the situation of identical particles is that the elements of the blocks \( s'=2, s=1 \) and \( s'=1, s=2 \) and \( s'=0, s=1 \) and \( s'=1, s=0 \) are zero. This form of \( M \) is shown on page 88 and has 17 independent elements.

Our expressions for the \( \text{Tr}[M_{KQ}^T M_{K'Q'}^T] \) are in terms of the 41 elements of the set \( \{m_i\} \) which are the elements of the parity conserving form of \( M \).

One could attempt to find 41 functions of angle and energy \( \{m_i(\theta, E)\} \) to fit his experimental data through equation (86). An alternate approach is to express the elements of \( M \) in terms of parameters which depend only on the energy by using the Lane and Thomas result which contains the angle dependence of the elements of \( M \). The energy dependence is represented in the Lane and Thomas result by an infinite number of \( U^j \) which depend on energy. If the energy is low there will be an appreciable reaction only for orbital angular momenta less than or equal to some small value. Thus it will be necessary to include only a small number of the \( U^j \) in the expansion.

We parameterize the nuclear collision matrix by taking advantage of its form. The conservation of probability
current and parity together with time reversal invariance give $U^j$ the form shown on page 55 with the blocks being unitary, symmetric matrices. We show the parameterization of $U^j$ as an example of how the parameterization is done.

$$U^1_+ = (u^1_+)^\dagger \exp[2i(\delta^1_+)] (u^1_+).$$

$$\begin{bmatrix}
\cos(\epsilon^1_{10,12}) & \sin(\epsilon^1_{10,12}) & 0 \\
-sin(\epsilon^1_{10,12}) & \cos(\epsilon^1_{10,12}) & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\cos(\epsilon^1_{10,22}) & 0 & \sin(\epsilon^1_{10,22}) \\
0 & 1 & 0 \\
-sin(\epsilon^1_{10,22}) & 0 & \cos(\epsilon^1_{10,22})
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(\epsilon^1_{12,22}) & \sin(\epsilon^1_{12,22}) \\
0 & -\sin(\epsilon^1_{12,22}) & \cos(\epsilon^1_{12,22})
\end{bmatrix}
\exp[2i(\delta^1_+)]
\begin{bmatrix}
\exp[2i(\delta^1_{10})] & 0 & 0 \\
0 & \exp[2i(\delta^1_{12})] & 0 \\
0 & 0 & \exp[2i(\delta^1_{22})]
\end{bmatrix}.$$
The case shown required 6 parameters each a function of energy. If $j$ is greater than 1 then it will require 25 parameters to parameterize $U^j$ in this fashion. The numbers of parameters required to represent the nuclear collision matrix for various values of $l_{\text{max}}$, the maximum orbital angular momentum for which there is an appreciable interaction, is shown on page 68.

The values of the parameters reveal information about the interaction. If the mixing parameters are all zero then there is no change of channel spin by the system in the scattering and the $\delta_{j^n}^{i^n}$ give the phase shifts of the different waves of total spin $j$ and orbital angular momentum $l$ contributing to $M_{S\nu',S\nu}$. It can happen that some of the $\delta_{j^n}^{i^n}$ will be a zero giving no phase shift of that wave. It can also happen that an eigenphase shift will go through 90° as the energy goes through some value indicating a state of the compound nucleus at those values of energy, $j$, $l$ and $s$. The mixing parameters represent changes of orbital angular momentum and/or spin in the scattering. If some of the mixing parameters are non-zero then there can be a change of channel spin or orbital angular momentum by the system.

The form of $U^j$ for the situation of identical particles is shown on page 89.

A useful result is produced by expressing the transition matrix as a linear combination of spin operators which
have the same rotation, space reflection and time reflection properties as \( M \). This is

\[
M = B_1(\theta, E) \cdot l^g + B_2(\theta, E) \left( \Sigma_n + S_n \right) + B_3(\theta, E) \left( \Sigma_k S_k \right) + \ldots \ldots \ldots \\
\ldots + B_{25}(\theta, E) \left( \Sigma_{kk} S_{pp} - \Sigma_{pp} S_{kk} \right).
\]

The situation of identical particles requires that \( B_{18} \) through \( B_{25} \) be zero. The relationships between the \( \{m_i\} \) and the \( \{B_j\} \) as well as the inverse relationships are developed. Thus knowing the \( \{m_i\} \) the \( \{B_j\} \) can be found. It may turn out that some of the \( \{B_j\} \) are zero thus giving insight into the spin dependence of the Hamiltonian.

The results of this thesis can be used to describe the elastic scattering of a polarized spin-1 beam by an unpolarized spin-1 target and serve as a basis for acquiring knowledge of the nuclear interaction.

The expressions of this thesis relate the scattered polarizations to the beam polarizations and to the eigenphase shifts and mixing parameters. The next step in this work will be to develop a computer program that will for a set of experimental polarizations search for eigenphase shifts and mixing parameters to fit our expressions to the data.

This work could be extended in several directions. One possibility would be to allow the target as well as the beam
to be polarized. The density matrix of the target in \( \rho_{PW} = \rho_{b} \otimes \rho_{t} \) would then be something other than \( 1^{3} \). The target density matrix could be expressed in terms of the expectation values of the spherical spin operators 
\( \langle T_{KQ}^{*} \rangle_{\text{target}} = t_{K}^{*}m_{Q} \). The scattered polarizations would then depend on the \( t_{K}^{*}m_{Q} \) as well as the \( t_{KQ} \) for the beam.

The theory could be altered to describe the results of spin correlation experiments in which the polarizations of both particle-1 and particle-2 are measured simultaneously after the scattering. This would require the calculation of expectation values like \( \langle T_{KQ}^{1} T_{KQ}^{2} \rangle_{\text{scat}} \) rather than the \( \langle T_{KQ}^{*} \rangle_{\text{scat}} \) of the present work.

The extension of this work to higher spin systems such as spin-1 \( 1/2 \) on spin-1 would be possible; however, the manipulations of the \( 12 \times 12 \) matrices would present a formidable calculation.

The theory could be modified to allow for events other than elastic scattering. The modifications are of a more fundamental nature than those mentioned above.
APPENDIX A

TIME REVERSAL INVARIANCE RESTRICTIONS
ON THE ELEMENTS OF M

Abbreviations

$s = \sin \theta$
$c = \cos \theta$
Groupe 1 $s = 2, s' = 2$

\[
\begin{align*}
(m_1 + m_5) + (m_2 - m_4) + (m_6 + m_{10}) + (m_7 - m_9) + (m_{11}) + (m_{12}) + (m_3) + (m_8) + (m_{13}) \\
= 0 \\
(22,21) + (22,2-1)
\end{align*}
\]

\[
\begin{align*}
2sc & - (1 + c^2) & 2(1 - 2c^2) & -2(sc) & -4\sqrt{2}sc & -4\sqrt{3}s^2 \\
(22,20)
\end{align*}
\]

\[
\begin{align*}
4\sqrt{3}s^2 & - 4\sqrt{2}sc & 2(3c^2 - 1) & -2(1 + c^2) & -4(sc) & -4\sqrt{3}s^2 \\
(21,20)
\end{align*}
\]

\[
\begin{align*}
4\sqrt{3}s^2 & - 4\sqrt{2}sc & 2(3c^2 - 1) & -4sc & -4(1 - 2c^2) + 4\sqrt{2}sc
\end{align*}
\]
Groupe 1 (continued) \( s = 2, s' = 2 \)

\[
(m_1 - m_5) + (m_2 + m_4) + (m_6 - m_{10}) + (m_7 + m_9) = 0
\]

\[
(22, 21) - (22, 2-1)
\]

\[
s \quad -c \quad -c \quad -s
\]
Groupe 2 \( s = 1, s' = 2 \) and \( s = 2, s' = 1 \)

\[
\begin{align*}
(m_{30} + m_{28}) + (m_{33}) + (m_{27} - m_{31}) + (m_{22} - m_{20}) + (m_{25} - m_{23}) + (m_{21} - m_{24}) + (m_{32}) &= 0 \\
(21,11) + (21,1-1) \\
(c, \sqrt{2}s) - s &\quad c \\
(22,10) - (21,10) \\
-\sqrt{2}s &\quad 2c &\quad -\sqrt{2}s &\quad -2s &\quad -2c \\
(22,10) + (21,10) \\
\sqrt{2}s &\quad -2c &\quad -\sqrt{2}s &\quad -2c &\quad 2s \\
(22,11) - (22,1-1) \\
-c &\quad 2c &\quad s
\end{align*}
\]
Groupe 2 (continued) $s = 1, s' = 2$ and $s = 2, s' = 1$

\[
\frac{(m_{20}+m_{22})}{(m_{23}+m_{25})} + \frac{(m_{29})}{(m_{26})} + \frac{(m_{31}+m_{27})}{(m_{28}-m_{30})} = 0
\]

\[(20,11)\]

\[
2\sqrt[3]{3} s^2 - 2\sqrt[3]{2} sc 2 \quad (3c^2 - 1)
\]

\[(22,11) + (22,1-1)\]

\[
(1+c^2) 2(sc) 4\sqrt[3]{3} s^2 2
\]

\[(21,11) - (21,1-1)\]

\[
sc -(2c^2 - 1) -2\sqrt[3]{2} sc
\]
Groupe $3\ s = 1, s' = 1$

$$\frac{(m_{14})}{s} + \frac{(m_{17})}{s} + \frac{(m_{16})}{s} + \frac{(m_{15})}{s} + \frac{(m_{18})}{s} = 0.$$  

$(11, 10)$

$-\sqrt{2}s \quad 2c \quad \sqrt{2}s \quad 2c \quad \sqrt{2}s$
Groupe 4 \( s = 2, s' = 0 \) and \( s = 0, s' = 2 \)

\[
\begin{align*}
(m_{37}) + (m_{38}) + (m_{39}) + (m_{34}) + (m_{35}) + (m_{36}) &= 0 \\
(22,00) & \quad - (1+c^2) - 2sc - 2\sqrt[3]{8}s^2
\end{align*}
\]

\[
(21,00) & \quad -sc - (1-2c^2) - \frac{3}{2}sc
\]

\[
(20,00) & \quad 2 - 4\sqrt[3]{8}s^2 + \sqrt[3]{2}sc - (3c^2-1)
\]
Groupe 5 \( s = 1, s' = 0 \) and \( s = 0, s' = 1 \)

\[
\begin{align*}
\left( m_{40} \right) + \left( m_{47} \right) &= 0. \\
(11,00) \\

groupe 6 \ s = 0, \ s' = 0 \\
(00,00) \quad m_{19} = m_{19}
\end{align*}
\]
APPENDIX B

SINGLE PARTICLE SPIN OPERATORS

\( (\hat{L}_x, \hat{L}_y, \hat{L}_z, \hat{S}_x, \hat{S}_y, \hat{S}_z) \)
$$\Sigma_x = \frac{\sqrt{3}}{2} \cdot \begin{bmatrix} \ \end{bmatrix}$$

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<th>22</th>
<th>21</th>
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<tr>
<td>s'v'</td>
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\[ s_y = \frac{\sqrt{2}}{2} \cdot [ \] 

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0 & 0 & 0 & 0 & 0 & \frac{\varepsilon}{14r} & 0 & 0 & 0 & \frac{\varepsilon}{15r} \\
\end{bmatrix}
\cdot \tau = Z_S
\]
APPENDIX C

SINGLE PARTICLE SPIN OPERATORS

\( (\Sigma_k, \Sigma_n, \Sigma_p, S_k S_n, S_p) \)

**Abbreviations**

\[ s = \sin (\theta/2) \]

\[ c = \cos (\theta/2) \]
\[ \Sigma_k = \frac{\sqrt{2}}{24} \cdot [ \quad ] \]

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20 & 0 & 6/3s & 0 & 6/3s & 0 & 2/3s & 4/6c & -2/3s & 0 \\
2-1 & 0 & 0 & 6/3s & -6/2c & 6/2s & 0 & 6s & 6/2c & 0 \\
2-2 & 0 & 0 & 0 & 6/2s & -12/2c & 0 & 0 & 6/2s & 0 \\
11 & -6/2s & 6/2c & 2/3s & 0 & 0 & 6/2c & 6s & 0 & -4/6s \\
10 & 0 & -6s & 4/6c & 6s & 0 & 6s & 0 & 6s & 8/3c \\
1-1 & 0 & 0 & -2/3s & 6/2c & 6/2s & 0 & 6s & -6/2c & 4/6s \\
00 & 0 & 0 & 0 & 0 & 0 & -4/6s & 8/3c & 4/6s & 0 
\end{array} ]
\]
\[ S_k = \frac{\sqrt{2}}{24} \cdot \left[ \begin{array}{cccccccc} & & & & & & & \\ 22 & 21 & 20 & 2-1 & 2-2 & 11 & 10 & 1-1 & 00 \\ \hline 22 & -12/2a & 6/2c & 0 & 0 & 0 & 6/2c & 0 & 0 & 0 \\ 21 & 6/2c & -6/2a & 6/3c & 0 & 0 & 6/2c & 6c & 0 & 0 \\ 20 & 0 & 6/3c & 0 & 6/3c & 0 & -2/3c & 4/6a & 2/3c & 0 \\ 2-1 & 0 & 0 & 0 & 6/2c & 6/2s & 6/2c & 0 & -6c & 6/2c & 0 \\ 2-2 & 0 & 0 & 0 & 6/2c & 12/2s & 0 & 0 & -6/2c & 0 \\ 11 & 6/2c & 6/2s & -2/3c & 0 & 0 & -6/2s & 6c & 0 & 4/6c \\ 10 & 0 & 6c & 4/6s & -6c & 0 & 6c & 0 & 6c & 8/3s \\ 1-1 & 0 & 0 & 2/3c & 6/2s & -6/2c & 0 & 6c & 6/2s & -4/6c \\ 00 & 0 & 0 & 0 & 0 & 0 & 4/6c & 8/3s & -4/6c & 0 \end{array} \right] \]
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2-1 & 0 & 0 & 6/3s & -6/2c & 6/2s & 0 & -6s & -6/2c & 0 \\
2-2 & 0 & 0 & 0 & 6/2s & -12/2c & 0 & 0 & -6/2s & 0 \\
11 & 6/2s & -6/2c & -2/3s & 0 & 0 & 6/2c & 6s & 0 & 4/6s \\
10 & 0 & 6s & -4/6c & -6s & 0 & 6s & 0 & 6s & -8/3c \\
1-1 & 0 & 0 & 2/3s & -6/2c & -6/2s & 0 & 6s & -6/2c & -4/6s \\
00 & 0 & 0 & 0 & 0 & 0 & 4/6s & -8/3c & -4/6s & 0 \end{array} ] \]
APPENDIX D

SPIN OPERATORS IN THE $M$ EXPANSION

$(1^9, \Sigma_n s, \Sigma_k s_k, \ldots, \Sigma_{kk} s_{pp} - \Sigma_{pp} s_{kk})$

**Abbreviations**

$s = \sin \left( \theta/2 \right)$

c = \cos \left( \theta/2 \right)$
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APPENDIX E

ELEMENTS OF M IN TERMS OF SPIN OPERATOR COEFFICIENTS
({m_i} IN TERMS OF {B_j})

Abbreviations

\[ \cos = \cos \left( \theta/2 \right) \]
\[ \sin = \sin \left( \theta/2 \right) \]
\[ m_1 = \cos^\circ \left[ 8294B_1 + 8294B_3 + 55296B_6 - 27648B_7 ight. \\
\quad + 9216B_{12} - 9216B_{13} + 2304B_{17} \right] \\
+ \cos^2 \left[ \\
\quad - 8294B_3 + 8294B_5 - 8294B_6 + 8294B_7 \\
\quad - 27648B_{12} + 41472B_{13} + 20736B_{15} - 13824B_{17} \right] \\
+ \cos^4 \left[ \\
\quad + 20736B_{12} - 41472B_{13} - 20736B_{15} + 20736B_{17} \right] \\
\]

\[ m_2 = \cos^\circ \left[ -8294B_2 - 41472B_8 - 27648B_9 + 13824B_{10} \right] \\
\quad + \sin \cos \left[ -8294B_3 + 8294B_5 - 8294B_6 + 8294B_7 \\
\quad - 27648B_{12} + 41472B_{13} + 20736B_{15} - 13824B_{17} \right] \\
\quad + \cos^2 \left[ 41472B_8 + 41472B_9 - 41472B_{10} - 41472B_{11} \right] \\
\quad + \sin \cos^3 \left[ 41472B_{12} - 8294B_{13} - 41472B_{15} + 41472B_{17} \right] \]
Table continued.

\[ m_1 \text{ Coefficients In Terms Of } B_j \text{ Coefficients} \]

\[ m_3 = \cos^* \left[ -13824 \sqrt{6B_4} + 13824 \sqrt{6B_5} + 13824 \sqrt{6B_7} \\
+ 4608 \sqrt{6B_{13}} - 3456 \sqrt{6B_{14}} + 3456 \sqrt{6B_{15}} - 2304 \sqrt{6B_{17}} \right] \\
+ \cos^2 \left[ 13824 \sqrt{6B_3} - 13824 \sqrt{6B_5} \\
+ 13824 \sqrt{6B_6} - 13824 \sqrt{6B_7} + 18432 \sqrt{6B_{12}} - 41472 \sqrt{6B_{13}} \\
+ 3456 \sqrt{6B_{14}} - 20736 \sqrt{6B_{15}} - 3456 \sqrt{6B_{16}} + 23040 \sqrt{6B_{17}} \right] \\
+ i \sin \cos \left[ 27648 \sqrt{6B_8} + 27648 \sqrt{6B_9} - 27648 \sqrt{6B_{10}} \\
- 27648 \sqrt{6B_{11}} \right] + \cos^4 \left[ -20736 \sqrt{6B_{12}} + 41472 \sqrt{6B_{13}} \\
+ 20736 \sqrt{6B_{15}} - 20736 \sqrt{6B_{17}} \right] \]

\[ m_4 = i \cos^* \left[ -41472B_{10} - 41472B_{11} \right] \\
+ i \cos^2 \left[ -41472B_8 - 41472B_9 + 41472B_{10} + 41472B_{11} \right] \\
+ \sin \cos \left[ -41472B_{13} + 20736B_{14} - 20736B_{15} \\
- 20736B_{16} + 41472B_{17} \right] + \sin^3 \left[ -41472B_{12} \\
+ 82944B_{13} + 41472B_{15} - 41472B_{17} \right] \]
\[ m_5 = \cos^2 \left[ -20736B_{16} + 20736B_{17} \right] + \cos^4 \left[ 41472B_{13} - 20736B_{14} + 20736B_{15} + 20736B_{16} - 41472B_{17} \right] + \cos^4 \left[ 20736B_{12} - 41472B_{13} + 20736B_{15} + 20736B_{17} \right] \]

\[ m_6 = i \cos^4 \left[ 82944B_2 + 41472B_8 + 27648B_9 - 13824B_{10} \right] \]

\[ m_7 = \cos^4 \left[ 82944B_1 + 41472B_4 + 41472B_2 + 27648B_6 + 13824B_7 \right] - 18432B_{12} + 18432B_{13} + 10368B_{14} + 10368B_{15} - 4608B_{17} \]

\[ + \cos^2 \left[ 41472B_3 - 41472B_5 + 41472B_6 - 41472B_7 + 96768B_{12} - 16588B_{13} - 10368B_{14} - 82944B_{15} + 10368B_{16} + 69120B_{17} \right] \]

\[ + \cos^4 \left[ -82944B_{12} + 16588B_{13} + 82944B_{15} - 82944B_{17} \right] \]
Table continued.

$m_i$ Coefficients In Terms Of $B_j$ Coefficients

\[
m_i = 1 \cos^n \left[ -41472 \sqrt{6B_2} + 6912 \sqrt{6B_8} + 13824 \sqrt{6B_9} - 6912 \sqrt{6B_{11}} \right] \\
+ i \cos^2 \left[ -13824 \sqrt{6B_6} - 13824 \sqrt{6B_9} + 13824 \sqrt{6B_{10}} + 13824 \sqrt{6B_{11}} \right] + \sin \cos \left[ -13824 \sqrt{6B_3} + 13824 \sqrt{6B_5} - 13824 \sqrt{6B_6} + 13824 \sqrt{6B_7} + 23040 \sqrt{6B_{12}} - 41472 \sqrt{6B_{13}} - 3456 \sqrt{6B_{14}} - 20736 \sqrt{6B_{15}} + 3456 \sqrt{6B_{16}} + 18432 \sqrt{6B_{17}} \right] + \sin \cos^3 \left[ -41472 \sqrt{6B_{12}} + 82944 \sqrt{6B_{13}} + 41472 \sqrt{6B_{15}} - 41472 \sqrt{6B_{17}} \right] \\
\]

\[
m_9 = \cos^n \left[ -41472B_4 + 41472B_5 + 41472B_7 - 27648B_{13} + 10368B_{14} - 10368B_{15} + 13824B_{17} \right] + \cos^2 \left[ 41472B_3 - 41472B_5 + 41472B_6 - 41472B_7 - 69120B_{12} + 1655888B_{13} - 10368B_{14} + 82944B_{15} + 10368B_{16} - 96768B_{17} \right] + \cos^4 \left[ 82944B_{12} - 1655888B_{13} - 82944B_{15} + 82944B_{17} \right]
\]
Table continued.

\( m_1 \) Coefficients in Terms of \( B_j \) Coefficients

\[ m_{10} = i \cos^4 \left[-41472B_{10} - 41472B_{11}\right] + i \cos^2 \left[-41472B_8 \right. \\
\left. -41472B_9 + 41472B_{10} + 41472B_{11}\right] + \sin \cos \left[41472B_{13} \right. \\
\left. - 20736B_{14} + 20736B_{15} + 20736B_{16} - 41472B_{17}\right] + \sin \cos^3 \left[41472B_{12} - 82944B_{13} - 41472B_{15} + 41472B_{17}\right] \]

\[ m_{11} = \cos^8 \left[-13824 \sqrt{6B_4} + 13824 \sqrt{6B_5} + 13824 \sqrt{6B_7} \right. \\
\left. + 4608 \sqrt{6B_{13}} - 3456 \sqrt{6B_{14}} + 3456 \sqrt{6B_{15}} - 2304 \sqrt{6B_{17}}\right] + \cos^2 \left[13824 \sqrt{6B_3} - 13824 \sqrt{6B_5} + 13824 \sqrt{6B_6} \right. \\
\left. - 13824 \sqrt{6B_7} + 18432 \sqrt{6B_{12}} - 41472 \sqrt{6B_{13}} - 3456 \sqrt{6B_{16}} \right. \\
\left. + 3456 \sqrt{6B_{14}} - 20736 \sqrt{6B_{15}} + 23040 \sqrt{6B_{17}}\right] + i \sin \cos \left[-27648 \sqrt{6B_8} - 27648 \sqrt{6B_9} + 27648 \sqrt{6B_{10}} \right. \\
\left. + 27648 \sqrt{6B_{11}}\right] + \cos^4 \left[-20736 \sqrt{6B_{12}} + 41472 \sqrt{6B_{13}} + 20736 \sqrt{6B_{15}} - 20736 \sqrt{6B_{17}}\right] \]
Table continued.

$m_1$ Coefficients In Terms Of $B_j$ Coefficients

$m_{12} = i \cos^2 \left(44472 \sqrt{6B_2} - 6912 \sqrt{6B_8} - 13824 \sqrt{6B_9} \right)
+ 6912 \sqrt{6B_{11}} \right] + i \cos \left[13824 \sqrt{6B_8} + 13824 \sqrt{6B_9} \right.
- 13824 \sqrt{6B_{10}} - 13824 \sqrt{6B_{11}} \right] + \sin \cos \left[- 13824 \sqrt{6B_3} + 13824 \sqrt{6B_5} - 138 \sqrt{6B_3} + 138 \sqrt{6B_5} - 13824 \sqrt{6B_6} + 13824 \sqrt{6B_7} + 23040 \sqrt{6B_{12}} - 44472 \sqrt{6B_{13}} = 3456 \sqrt{6B_{14}}
- 20736 \sqrt{6B_{15}} + 3456 \sqrt{6B_{16}} + 124432 \sqrt{6B_{17}} \right]
+ \sin \cos^3 \left[- 44472 \sqrt{6B_{12}} + 82944 \sqrt{6B_{13}} + 44472 \sqrt{6B_{15}} - 44472 \sqrt{6B_{17}} \right]

$m_{13} = \cos^4 \left[82944B_1 - 27648B_3 + 55296B_4 + 55296B_5 \right.
- 55296B_6 + 27648B_7 + 27648B_{12} - 27648B_{13} - 13824B_{14} - 13824B_{15} + 6912B_{16} + 13824B_{17} \right] + \cos^2 \left[82944B_3 \right.
- 82944B_5 + 82944B_6 - 82944B_7 - 13824B_{12} + 248832B_{13}
+ 20736B_{14} + 124416B_{15} - 20736B_{16} - 110592B_{17} \right]
+ \cos^4 \left[124416B_{12} - 248832B_{13} - 124416B_{15} + 124416B_{17} \right]
Table continued.

$m_1$ Coefficients In Terms Of $B_j$ Coefficients

\[ m_{14} = \cos^2 \left[ 8294 \cdot B_1 - 41472 B_4 - 41472 B_5 - 27G4B_6 ight. \]

\[ + 13824 B_7 - 18432 B_{12} + 18432 B_{13} - 10368 B_{14} \]

\[ - 184 B_{12} + 184 B_{13} - 103 B_{14} - 10368 B_{15} \]

\[ - 4608 B_{17} \] \[ + \cos^2 \left[ -41472 B_3 + 41472 B_5 + 41472 B_6 \right. \]

\[ - 41472 B_7 + 13824 B_{12} + 10368 B_{14} - 10368 B_{16} \]

\[ - 13824 B_{17} \]

\[ m_{15} = i \cos^2 \left[ -41472 \sqrt{2} B_2 + 20736 \sqrt{2} B_8 - 13824 \sqrt{2} B_9 \right. \]

\[ - 13824 \sqrt{2} B_{10} + 20736 \sqrt{2} B_{11} \] \[ + \sin \cos \left[ -41472 \sqrt{2} B_3 + 41472 \sqrt{2} B_5 + 41472 \sqrt{2} B_6 \right. \]

\[ - 41472 \sqrt{2} B_7 + 13824 \sqrt{2} B_{12} + 10368 \sqrt{2} B_{14} - 10368 \sqrt{2} B_{16} \]

\[ - 13824 \sqrt{2} B_{17} \]
Table continued.

$m_i$ Coefficients In Terms Of $B_j$ Coefficients

\[
m_{16} = \cos^2 \left( -41472B_4 + 41472B_5 - 41472B_7 + 27648B_{13} \right.
\]
\[
+ 10368B_{14} - 10368B_{15} - 13824B_{17} \right) + \cos^2 \left[ 41472B_3 \right.
\]
\[
- 41472B_5 - 41472B_6 + 41472B_7 - 13824B_{12} - 10368B_{14}
\]
\[
+ 10368B_{16} + 13824B_{17} \right]
\]

\[
m_{17} = i \cos \left( [41472 \sqrt{2B}_2 - 20736 \sqrt{2B}_8 + 13824 \sqrt{2B}_9 \right.
\]
\[
+ 13824 \sqrt{2B}_{10} - 20736 \sqrt{2B}_{11} \right)
\]
\[
+ \sin \cos \left( [-41472 \sqrt{2B}_3 + 41472 \sqrt{2B}_5 + 41472 \sqrt{2B}_6 \right.
\]
\[
- 41472 \sqrt{2B}_7 + 13824 \sqrt{2B}_{12} + 10368 \sqrt{2B}_{14} - 10368 \sqrt{2B}_{16}
\]
\[
- 13824 \sqrt{2B}_{17} \right]
\]

\[
m_{18} = \cos^2 \left( [82914B_1 - 82914B_3 + 55296B_6 - 27618B_7 + 9216B_{12} \right.
\]
\[
- 9216B_{13} - 20736B_{16} - 18432B_{17} \right) + \cos^2 \left[ 82914B_3 \right.
\]
\[
- 82914B_5 - 82914B_6 + 82914B_7 - 27618B_{12} - 20736B_{14}
\]
\[
+ 20736B_{16} + 27618B_{17} \right]
\]
Table continued.

\( m_1 \) Coefficients In Terms Of \( B_j \) Coefficients

\[
m_{19} = \cos \left[ 8294 B_1 - 55296 B_3 - 55296 B_4 - 55296 B_5 \\
+ 18432 B_{12} - 18432 B_{13} + 13824 B_{14} + 13824 B_{15} \\
+ 13824 B_{16} + 18432 B_{17} \right]
\]
Table continued.

\( m_1 \) Coefficients In Terms Of \( B_j \) Coefficients

\[ m_{20} = i \cos^2 \left[ -294^4 B_{19} + 41472 B_{21} + 2764 B_{22} - 1382^4 B_{23} \right] + \sin \cos \left[ -294^4 B_{19} + 294^4 B_{20} + 1382^4 B_{25} \right] + i \cos^2 \left[ 41472 B_{21} - 41472 B_{22} + 41472 B_{23} - 41472 B_{24} \right] \]

\[ m_{21} = \cos^2 \left[ 41472 \sqrt{2} B_{20} + 1382^4 \sqrt{2} B_{25} \right] + \cos^2 \left[ 41472 \sqrt{2} B_{19} - 41472 \sqrt{2} B_{20} - 2764 \sqrt{2} B_{25} \right] + i \sin \cos \left[ 41472 \sqrt{2} B_{21} - 41472 \sqrt{2} B_{24} \right] \]

\[ m_{22} = i \cos^2 \left[ -41472 B_{23} - 41472 B_{24} \right] + i \cos^2 \left[ -41472 B_{21} - 41472 B_{22} + 41472 B_{23} + 41472 B_{24} \right] + \sin \cos \left[ -41472 B_{25} \right] \]

\[ m_{23} = \cos^2 \left[ -294^4 B_{19} + 41472 B_{20} \right] + \cos^2 \left[ 12^4 16 B_{19} - 12^4 16 B_{20} \right] + i \sin \cos \left[ 41472 B_{21} - 294^4 B_{22} + 294^4 B_{23} - 41472 B_{24} \right] \]
Table continued.

$m_i$ Coefficients In Terms Of $B_j$ Coefficients

$m_{24} = i \cos^2 \left[ -\frac{41472}{2} \sqrt{B_{18}} + \frac{20736}{2} \sqrt{B_{21}} + \frac{13824}{2} \sqrt{B_{22}} + \frac{13824}{2} \sqrt{B_{23}} - \frac{20736}{2} \sqrt{B_{24}} \right] + \sin \cos \left[ \frac{41472}{2} \sqrt{B_{19}} - \frac{41472}{2} \sqrt{B_{20}} + \frac{27648}{2} \sqrt{B_{25}} \right] + i \cos^2 \left[ -\frac{41472}{2} \sqrt{B_{21}} + \frac{41472}{2} \sqrt{B_{24}} \right]$

$m_{25} = \cos^2 \left[ \frac{41472}{2} B_{20} - 27648 B_{25} \right] + \cos^2 \left[ \frac{41472}{2} B_{19} - \frac{41472}{2} B_{20} + \frac{55296}{2} B_{25} \right] + i \sin \cos \left[ -\frac{41472}{2} B_{21} - 82944 B_{22} + 82944 B_{23} + 41472 B_{24} \right]$

$m_{26} = i \cos^2 \left[ -\frac{13824}{2} \sqrt{B_{18}} - \frac{6912}{2} \sqrt{B_{21}} - \frac{23040}{2} \sqrt{B_{22}} + \frac{13824}{2} \sqrt{B_{23}} - \frac{6912}{2} \sqrt{B_{24}} \right] + \sin \cos \left[ \frac{41472}{2} \sqrt{B_{19}} - \frac{41472}{2} \sqrt{B_{20}} + \frac{13824}{2} \sqrt{B_{25}} \right] + i \cos^2 \left[ \frac{41472}{2} \sqrt{B_{22}} - \frac{41472}{2} \sqrt{B_{23}} \right]$
Table continued.

$m_1$ Coefficients In Terms Of $B_j$ Coefficients

\[ m_{27} = i \cos^* \left[ 82944B_{18} + 41472B_{21} - 27648B_{22} + 13824B_{23} \right. \]
\[ \left. + i \cos^2 \left[ - 41472B_{21} + 41472B_{22} - 41472B_{23} + 41472B_{24} \right] \right. \]
\[ \left. + \sin \cos \left[ - 82944B_{19} + 82944B_{20} + 13824B_{25} \right] \right. \]
\[ \left. + i \sin \cos \left[ - 82944B_{19} + 82944B_{20} + 13824B_{25} \right] \right. \]
\[ \left. + \sin \cos \left[ - 82944B_{19} + 82944B_{20} + 13824B_{25} \right] \right. \]
\[ + 27648B_{22} \]
\[ - 18432 \sqrt{6B_{23}} + 8912 \sqrt{6B_{24}} \]
\[ + 41472 \sqrt{6B_{22}} + 41472 \sqrt{6B_{23}} \]
\[ + 41472 \sqrt{6B_{19}} - 41472 \sqrt{6B_{20}} \]
\[ + 13824 \sqrt{6B_{25}} \]

\[ m_{29} = \cos^* \left[ 13824 \sqrt{6B_{18}} + 6912 \sqrt{6B_{21}} + 23040 \sqrt{6B_{22}} \right. \]
\[ - 18432 \sqrt{6B_{23}} + 6912 \sqrt{6B_{24}} \]
\[ + i \cos^2 \left[ - 41472 \sqrt{6B_{22}} + 41472 \sqrt{6B_{23}} \right] \]
\[ + \sin \cos \left[ 41472 \sqrt{6B_{19}} - 41472 \sqrt{6B_{20}} \right] \]
\[ + 13824 \sqrt{6B_{25}} \]

\[ m_{30} = \cos^* \left[ - 41472B_{20} + 27648B_{25} \right. \]
\[ + \cos^2 \left[ - 41472B_{19} + 41472B_{20} - 55296B_{25} \right. \]
\[ + \sin \cos \left[ - 41472B_{21} - 82944B_{22} + 82944B_{23} + 41472B_{24} \right. \]
\[ + \sin \cos \left[ - 41472B_{21} - 82944B_{22} + 82944B_{23} + 41472B_{24} \right. \]
Table continued.

\( m_i \) Coefficients In Terms Of \( B_j \) Coefficients

\[
m_{31} = i \cos \left[ 41472B_{23} + 41472B_{24} \right] + i \cos^2 \left[ 41472B_{21} + 41472B_{22} - 41472B_{23} - 41472B_{24} \right] + \sin \cos \left[- 41472B_{25} \right]
\]

\[
m_{32} = \cos \left[ 41472 \sqrt{B_{20}} + 13824 \sqrt{B_{25}} \right] + \cos^2 \left[ 41472 \sqrt{B_{19}} - 41472 \sqrt{B_{20}} - 27648 \sqrt{B_{25}} \right] + i \sin \cos \left[- 41472 \sqrt{B_{21}} + 41472 \sqrt{B_{24}} \right]
\]

\[
m_{33} = i \cos \left[ 41472 \sqrt{B_{18}} - 20736 \sqrt{B_{21}} - 13824 \sqrt{B_{22}} - 13824 \sqrt{B_{23}} + 20736 \sqrt{B_{24}} \right] + i \cos^2 \left[ 41472B_{21} - 41472 \sqrt{B_{24}} \right] \sin \cos \left[ 41472 \sqrt{B_{19}} - 41472 \sqrt{B_{20}} - 27648 \sqrt{B_{25}} \right]
\]
Table continued.

\[ m_1 \text{ Coefficients In Terms Of } B_j \text{ Coefficients} \]

\[ m_{34} = \cos^2 \left( 13824 \sqrt{3B_4} - 13824 \sqrt{3B_5} + 27648 \sqrt{3B_7} \right) \]
\[ + 9216 \sqrt{3B_{13}} + 3456 \sqrt{3B_{14}} - 3456 \sqrt{3B_{15}} - 4608 \sqrt{3B_{17}} \]
\[ + \cos^2 \left( -13824 \sqrt{3B_3} + 13824 \sqrt{3B_5} + 27648 \sqrt{3B_6} \right) \]
\[ - 27648 \sqrt{3B_7} \] \[ - 4608 \sqrt{3B_{12}} - 3456 \sqrt{3B_{14}} \]
\[ + 3456 \sqrt{3B_{16}} + 4608 \sqrt{3B_{17}} \] \[ + i \sin \cos \left( 13824 \sqrt{3B_8} \right) \]
\[ - 27648 \sqrt{3B_9} + 27648 \sqrt{3B_{10}} - 13824 \sqrt{3B_{11}} \]

\[ m_{35} = i \cos^2 \left( 13824 \sqrt{3B_8} - 27648 \sqrt{3B_9} + 27648 \sqrt{3B_{10}} \right) \]
\[ - 13824 \sqrt{3B_{11}} \] \[ + i \cos^2 \left( -27648 \sqrt{3B_8} + 55296 \sqrt{3B_9} \right) \]
\[ - 55296 \sqrt{3B_{10}} + 27648 \sqrt{3B_{11}} \] \[ + \sin \cos \left( -27648 \sqrt{3B_3} \right) \]
\[ + 27648 \sqrt{3B_5} + 55296 \sqrt{3B_6} - 55296 \sqrt{3B_7} - 9216 \sqrt{3B_{12}} \]
\[ - 6912 \sqrt{3B_{14}} + 6912 \sqrt{3B_{16}} + 9216 \sqrt{3B_{17}} \]
Table continued.

$m_i$ Coefficients In Terms Of $B_j$ Coefficients

\[
m_{36} = \cos^2 \left[ -27648 \sqrt{2B_3} + 13824 \sqrt{2B_4} + 13824 \sqrt{2B_5} \\
+ 55296 \sqrt{2B_6} - 27648 \sqrt{2B_7} - 9216 \sqrt{2B_12} + 9216 \sqrt{2B_13} \\
- 3456 \sqrt{2B_{14}} - 3456 \sqrt{2B_{15}} + 6912 \sqrt{2B_{16}} + 4608 \sqrt{2B_{17}} \right]
\]

\[
+ \cos^2 \left[ 41472 \sqrt{2B_3} - 41472 \sqrt{2B_5} - 82944 \sqrt{2B_6} \\
+ 82944 \sqrt{2B_7} + 13824 \sqrt{2B_{12}} + 10368 \sqrt{2B_{14}} - 10368 \sqrt{2B_{16}} - 13824 \sqrt{2B_{17}} \right] + i \sin \cos \left[ -41472 \sqrt{2B_8} + 82944 \sqrt{2B_9} \\
- 82944 \sqrt{2B_{10}} + 41472 \sqrt{2B_{11}} \right]
\]
Table continued.

\[ m_i \text{ Coefficients In Terms Of } B_j \text{ Coefficients} \]

\[ m_{37} = \cos^* \left[ 13824 \sqrt{3B_4} - 13824 \sqrt{3B_5} + 27648 \sqrt{3B_7} \right. \]
\[ + 9216 \sqrt{3B_{13}} + 3456 \sqrt{3B_{14}} - 3456 \sqrt{3B_{15}} - 4608 \sqrt{3B_{17}} \]
\[ + \cos^2 \left[ - 13824 \sqrt{3B_3} + 13824 \sqrt{3B_5} + 27648 \sqrt{3B_6} \right. \]
\[ - 27648 \sqrt{3B_7} - 4608 \sqrt{3B_{12}} - 3456 \sqrt{3B_{14}} + 3456 \sqrt{3B_{16}} \]
\[ + 4608 \sqrt{3B_{17}} \right] + i \sin \cos \left[ - 13824 \sqrt{3B_8} + 27648 \sqrt{3B_9} \right. \]
\[ - 27648 \sqrt{3B_{10}} + 13824 \sqrt{3B_{11}} \right] \]

\[ m_{38} = i \cos^* \left[ - 13824 \sqrt{3B_8} + 27648 \sqrt{3B_9} - 27648 \sqrt{3B_{10}} \right. \]
\[ + 13824 \sqrt{3B_{11}} \right] + i \cos^2 \left[ 27648 \sqrt{3B_8} - 55296 \sqrt{3B_9} \right. \]
\[ + 55296 \sqrt{3B_{10}} - 27648 \sqrt{3B_{11}} \right] + \sin \cos \left[ - 27648 \sqrt{3B_3} \right. \]
\[ + 27648 \sqrt{3B_5} + 55296 \sqrt{3B_6} - 55296 \sqrt{3B_7} - 9216 \sqrt{3B_{12}} \]
\[ - 6912 \sqrt{3B_{14}} + 6912 \sqrt{3B_{16}} + 9216 \sqrt{3B_{17}} \right] \]
Table continued.

$m_1$ Coefficients in Terms of $B_j$ Coefficients

$m_{39} = \cos^8 \left[ - 27648 \sqrt{2B_3} + 13824 \sqrt{2B_4} + 13824 \sqrt{2B_5} \
+ 55296 \sqrt{2B_6} - 27648 \sqrt{2B_7} - 9216 \sqrt{2B_{12}} + 9216 \sqrt{2B_{13}} \
- 3456 \sqrt{2B_{14}} - 3456 \sqrt{2B_{15}} + 6912 \sqrt{2B_{16}} + 4608 \sqrt{2B_{17}} \right] 
+ \cos^2 \left[ 41472 \sqrt{2B_3} - 41472 \sqrt{2B_5} - 82944 \sqrt{2B_6} \right. 
+ 82944 \sqrt{2B_7} + 13824 \sqrt{2B_{12}} + 10368 \sqrt{2B_{14}} 
- 10368 \sqrt{2B_{16}} - 13824 \sqrt{2B_{17}} \right] + i \sin \cos \left[ 41472 \sqrt{2B_8} \
- 82944 \sqrt{2B_9} + 82944 \sqrt{2B_{10}} - 41472 \sqrt{2B_{11}} \right]$
Table continued.

\( m_i \) Coefficients In Terms Of \( B_j \) Coefficients

\[
m_{40} = i \cos \theta \left[ -55296 \sqrt{3B_{18}} + 13824 \sqrt{3B_{21}} - 9216 \sqrt{3B_{22}} \\
- 9216 \sqrt{3B_{23}} + 13824 \sqrt{3B_{24}} \right]
\]
Table continued.

$m_i$ Coefficients In Terms Of $B_j$ Coefficients

$m_{41} = i \cos \left[ \frac{55296 \sqrt{3B_{18}} - 13824 \sqrt{3B_{21}} + 9216 \sqrt{3B_{22}}}{3} \right]$

$\quad + 9216 \sqrt{3B_{23}} - 13824 \sqrt{3B_{24}} \right]$
APPENDIX F

SPIN OPERATOR COEFFICIENTS IN TERMS OF THE ELEMENTS OF M

\( \{b_j \} \) IN TERMS OF \( \{m_i \} \)

Abbreviations

\[ s = \sin \left( \frac{\theta}{2} \right) \]
\[ c = \cos \left( \frac{\theta}{2} \right) \]
\[
B_1 = \frac{1}{9} (2m_1 + 2m_7 + m_{13} + 2m_{14} + m_{18} + m_{19})
\]

\[
B_2 = \frac{4}{12} [2m_2 - 2m_6 + \sqrt{6} m_8 - \sqrt{6} m_{12} + \sqrt{2} m_{15} - \sqrt{2} m_{17}]
\]

\[
B_3 = \frac{1}{24} [(-12c^2 + 12)m_1 + (-12sc)m_2 + 2/6c^2m_3 + (-12sc)m_6 \\
+ 6c^2m_7 + (-2/6sc)m_8 + 6c^2m_9 + 2/6c^2m_{11} + (-2/6sc)m_{12} \\
+ (6c^2 - 2)m_{13} + (-6c^2)m_{14} + (-6/2 sc)m_{15} + 6c^2m_{16} \\
+ (-6/2 sc)m_{17} + (6c^2 - 6)m_{18} + (4)m_{19} + (-2/3 c^2)m_{34} \\
+ (-1\sqrt{3} sc)m_{35} + (3/2 c^2 2/\sqrt{2})m_{36} + (-2/\sqrt{3} c^2)m_{37} \\
+ (-1\sqrt{3} sc)m_{38} + (3/2 c^2 - 2/\sqrt{2})m_{39}]
\]

\[
B_4 = \frac{1}{24} [+( -2/\sqrt{6})m_3 + (6)m_7 + (-6)m_9 + (-2/\sqrt{6})m_{11} + (4)m_{13} \\
+ (-6)m_{14} + (-6)m_{16} + (4)m_{19} + (2/3)m_{34} + 1/\sqrt{2} m_{36} + 2/\sqrt{3} m_{37} \\
+ \sqrt{2} m_{39}]
\]

\[
B_5 = \frac{1}{24} [(12c^2)m_1 + (12 sc)m_2 + (2/\sqrt{6} - 2/\sqrt{6} c^2)m_3 + (12sc)m_6 \\
+ (6 - 6c^2)m_7 + (2/\sqrt{6} sc)m_8 + (6 - 6c^2)m_9 + (2/\sqrt{6} - 2/\sqrt{6} c^2)m_{11} \\
+ (2/\sqrt{6} sc)m_{12} + (4 - 6c^2)m_{13} + (6c^2 - 6)m_{14} + (6/2 sc)m_{15} \\
+ (6 - 6c^2)m_{16} + (6/2 sc)m_{17} + (-6c^2)m_{18} + (-4)m_{19} \\
+ (2/\sqrt{3} c^2 - 2/\sqrt{3})m_{34} + (4/\sqrt{3} sc)m_{35} + (-3/2 c^2 + \sqrt{2})m_{36} \\
+ 2/\sqrt{3} (c^2 - 1)m_{37} + (4/\sqrt{3} sc)m_{38} + (-3\sqrt{2} c^2 + \sqrt{2})m_{39}]
\]
\[ B_6 = \frac{1}{18} \left\{ (-6c^2 + 6)m_1 + (-6sc)m_2 + (1c^2 + 1)\sqrt{6}m_3 + (-6sc)m_6 \\
+ (3c^2 - 3)m_7 + (-sc)\sqrt{6}m_8 + (+3c^2 + 3)m_9 + (c^2 + 1)\sqrt{6}m_{11} \\
+ (-sc)\sqrt{6}m_{12} + (+3c^2 - 3)m_{13} + (3c^2 - 3)m_{14} + (3sc)\sqrt{2}m_{15} \\
+ (+3c^2 - 3)m_{16} + (3sc)\sqrt{2}m_{17} + (-3c^2 + 3)m_{18} \\
+ (2c^2 + 2)\sqrt{3}m_{34} + (4sc)\sqrt{3}m_{35} + (-3c^2 + 3)\sqrt{2}m_{36} \\
+ (2c^2 + 2)\sqrt{3}m_{37} + (4sc)\sqrt{3}m_{38} + (-3c^2 + 3)\sqrt{2}m_{39} \right\} \]

\[ B_7 = \frac{1}{18} \left\{ (6c^2)m_1 + (6sc)m_2 + (-c^2 + 2)\sqrt{6}m_3 + (6sc)m_6 \\
+ (-3c^2)m_7 + (sc)\sqrt{6}m_8 + (-3c^2 + 6)m_9 + (-c^2 + 2)\sqrt{6}m_{11} \\
+ (sc)\sqrt{6}m_{12} + (-3c^2)m_{13} + (-3c^2)m_{14} + (3sc)\sqrt{2}m_{15} \\
+ (3c^2 - 6)m_{16} + (-3sc)\sqrt{2}m_{17} + (3c^2)m_{18} + (-2c^2 + 4)\sqrt{3}m_{34} \\
+ (-4sc)\sqrt{3}m_{35} + (3c^2)\sqrt{2}m_{36} + (-2c^2 + 4)\sqrt{3}m_{37} \\
+ (-4sc)\sqrt{3}m_{38} + (3c^2)\sqrt{2}m_{39} \right\} \]

\[ B_8 = \frac{1}{12} \left\{ (6 - 6c^2)m_2 + (-4\sqrt{6}sc)m_3 + 6c^2m_4 + (-6 + 6c^2)m_6 \\
+ (-\sqrt{6} + 2\sqrt{6}c^2)m_8 + (6c^2)m_{10} + (4\sqrt{6}sc)m_{11} \\
+ (1\sqrt{6} - 2\sqrt{6}c^2)m_{12} + (-3\sqrt{2})m_{15} + (3\sqrt{2})m_{17} + (-2\sqrt{3}sc)m_{34} \\
+ (-2\sqrt{3} + 4\sqrt{3}c^2)m_{35} + (3\sqrt{2}sc)m_{36} + (2\sqrt{3}sc)m_{37} \\
+ (2\sqrt{3} - 4\sqrt{3}c^2)m_{38} + (-3\sqrt{2}sc)m_{39} \right\} \]
\[ B_9 = \frac{1}{12} \left\{ (3 - 3c^2)m_2 + (-2 sc)\sqrt{6} m_3 + (3 + 3c^2)m_4 + (-3 + 3c^2)m_6 \\
+ (-2 + c^2)\sqrt{6} m_8 + (3 + 3c^2)m_{10} + (2sc)\sqrt{6} m_{11} + (2 - c^2)\sqrt{6} m_{12} \\
+ (3 \sqrt{2})m_{15} + (-3 \sqrt{2})m_{17} + (2sc)\sqrt{3} m_{34} + (2 - 4c^2)\sqrt{3} m_{35} \\
+ (-3 sc)\sqrt{2} m_{36} + (-2 sc)\sqrt{2} m_{37} + (-2 + 4c^2)\sqrt{3} m_{38} \\
+ (3 sc)\sqrt{2} m_{39} \right\} \]

\[ B_{10} = \frac{1}{12} \left\{ (3c^2)m_2 + (2 sc)\sqrt{6} m_3 + (6 - 3c^2)m_4 + (-3c^2)m_6 \\
+ (-1 - c^2)\sqrt{6} m_8 + (6 - 3c^2)m_{10} + (-2 sc)\sqrt{6} m_{11} \\
+ (1 + c^2)\sqrt{6} m_{12} + (3 \sqrt{2})m_{15} + (-3 \sqrt{2})m_{17} + (-2 \sqrt{3} sc)m_{34} \\
+ (-2 + 4c^2)\sqrt{3} m_{35} + (3 sc)\sqrt{2} m_{36} + (2 sc)\sqrt{3} m_{37} \\
+ (2 - 4c^2)\sqrt{3} m_{38} + (-3 sc)\sqrt{2} m_{39} \right\} \]

\[ B_{11} = \frac{1}{12} \left\{ (6c^2)m_2 + (4 \sqrt{6} sc)m_3 + (6 - 6c^2)m_4 + (-6c^2)m_6 \\
+ (1 - 2c^2)\sqrt{6} m_8 + (6 - 6c^2)m_{10} + (-4 sc)\sqrt{6} m_{11} \\
+ (-1 + 2c^2)\sqrt{6} m_{12} + (-3 \sqrt{2})m_{15} + (3 \sqrt{2})m_{17} + (2 sc)\sqrt{3} m_{34} \\
+ (2 - 4c^2)\sqrt{3} m_{35} + (-3 sc)\sqrt{2} m_{36} + (-2 sc)\sqrt{3} m_{37} \\
+ (-2 + 4c^2)\sqrt{3} m_{38} \right\} \]
\[ B_{12} = \frac{1}{6} \left[ (3 - 6c^2 + 3c^4) m_1 + (-6sc + 6sc^3) m_2 + (1 + 2c^2 - 3c^4) \sqrt{6} m_3 \right. \\
+ (-6sc - 6sc^3) m_4 + (3 + 6c^2 + 3c^4) m_5 + (-6sc + 6sc^3) m_6 \\
+ (6c + 6sc^3) m_{10} + (1 + 2c^2 - 3c^4) \sqrt{6} m_{11} + (6sc - 6sc^3) \sqrt{6} m_{12} \\
+ (5 - 12c^2 + 9c^4) m_{13} + (6 + 6c^2) m_{14} + (6sc) \sqrt{2} m_{15} \\
+ (6 - 6c^2) m_{16} + (6sc) \sqrt{2} m_{17} + (-6c^2) m_{18} + 4 m_{19} \\
+ (2 - 2c^2) \sqrt{3} m_{34} + (6sc) \sqrt{3} m_{35} + (-1 + 3c^2) \sqrt{2} m_{36} \\
+ (2 - 2c^2) \sqrt{3} m_{37} + (6sc) \sqrt{3} m_{38} + (-1 + 3c^2) \sqrt{2} m_{39} \]  \\

\[ B_{13} = \frac{1}{6} \left[ (c^2 - 3c^4) m_1 + (3sc - 6sc^3) m_2 \\
+ (1 - 3c^2 + 3c^4) \sqrt{6} m_3 + (-3sc + 6sc^3) m_4 \\
+ (6 + 3c^2 - 3c^4) m_5 + (3sc - 6sc^3) m_6 \\
+ (-12c^2 + 12c^4) m_7 + (-3sc + 6sc^3) \sqrt{6} m_8 \\
+ (6 + 12c^2 - 12c^4) m_9 + (3sc - 6sc^3) m_{10} \\
+ (1 - 3c^2 + 3c^4) \sqrt{6} m_{11} + (-3sc + 6sc^3) \sqrt{6} m_{12} \\
+ (1 + 9c^2 - 9c^4) m_{13} + 6 m_{16} + (-3)m_{18} \\
+ (2)m_{19} + 2 \sqrt{3} m_{34} + \sqrt{2} m_{36} + 2 \sqrt{3} m_{37} + \sqrt{2} m_{39} \]
\[ B_{14} = \frac{1}{6} \left\{ (-2 + 2c^2) \sqrt{6} m_3 + (12sc)m_4 + (-12c^2)m_5 + (6 - 6c^2)m_7 \\
+ (-2sc) \sqrt{6} m_8 + (6 - 6c^2)m_9 + (-12sc)m_{10} + (-2 + 2c^2) \sqrt{6} m_{11} \\
+ (-2sc) \sqrt{6} m_{12} + (-4 + 6c^2)m_{13} + (-6 + 6c^2)m_{14} + (6sc) \sqrt{2} m_{15} \\
+ (6 - 6c^2)m_{16} + (6sc) \sqrt{2} m_{17} + (-6c^2)m_{18} + 4m_{19} \\
+ (2 - 2c^2) \sqrt{3} m_{34} + (-4sc) \sqrt{3} m_{35} + (-1 + 3c^2) \sqrt{2} m_{36} \\
+ (2 - 2c^2) \sqrt{3} m_{37} + (-4 \sqrt{3}sc)m_{38} + (-1 + 3c^2) \sqrt{2} m_{39} \right\} \]

\[ B_{15} = \frac{1}{6} \left\{ (12c^2 - 12c^4)m_1 + (12sc - 24sc^3)m_2 + (2 + 12c^2 + 12c^4) \sqrt{2} m_3 \\
+ (-12sc + 24sc^3)m_4 + (12c^2 - 12c^4)m_5 + (12sc - 24sc^3)m_6 \\
+ (6 - 48c^2 + 48c^4)m_7 + (-12sc + 24sc^3) \sqrt{6} m_8 \\
+ (-6 + 48c^2 - 48c^4)m_9 + (12sc - 24sc^3)m_{10} \\
+ (2 + 12c^2 + 12c^4) \sqrt{6} m_{11} + (-12sc + 24sc^3) \sqrt{6} m_{12} \\
+ (-4 + 36c^2 - 36c^4)m_{13} + (-6)m_{14} + (-6)m_{16} + 4m_{19} \\
+ (-2 \sqrt{3})m_{34} + (-\sqrt{2})m_{36} + (-2 \sqrt{3})m_{37} + (-\sqrt{2})m_{39} \right\} \]
\[ B_{16} = \frac{1}{6} \left\{ (-2 \sqrt{6}c^2)m_3 + (-12sc)m_4 + (12c^2 - 12)m_5 + 6c^2m_7 \\
+ (2\sqrt{6}sc)m_8 + 6c^2m_9 + (12sc)m_{10} + (-2\sqrt{6}c^2)m_{11} + (2\sqrt{6}sc)m_{12} \\
+ (2 - 6c^2)m_{13} + (-6c^2)m_{14} + (-6\sqrt{2}sc)m_{15} + (6c^2)m_{16} \\
+ (-6\sqrt{2}sc)m_{17} + (6c^2 - 6)m_{18} + 4m_{19} + (2\sqrt{3}c^2)m_{34} \\
+ (4\sqrt{3}sc)m_{35} + (2 - 3c^2)\sqrt{2}m_{36} + 2\sqrt{3}c^2m_{37} + (4\sqrt{3}sc)m_{38} \\
+ (-3c^2 + 2)\sqrt{2}m_{39} \right\} \]

\[ B_{17} = \frac{1}{6} \left\{ (3c^4)m_1 + (6sc^3)m_2 + (4c^2 - 3c^4)\sqrt{6}m_3 + (12sc - 6sc^3)m_4 \\
+ (12 - 12c^2 + 3c^4)m_5 + (6sc^3)m_6 + (6c^2 - 12c^4)m_7 \\
+ (2sc - 6sc^3)\sqrt{6}m_8 + (-18c^2 + 12c^4)m_9 + (-12sc + 6sc^3)m_{10} \\
+ (4c^2 - 3c^4)\sqrt{6}m_{11} + (2sc - 6sc^3)\sqrt{6}m_{12} + (2 - 6c^2 + 9c^4)m_{13} \\
+ (-6c^2)m_{14} + (-6sc)\sqrt{2}m_{15} + 6c^2m_{16} + (-6sc)\sqrt{2}m_{17} \\
+ (-6 + 6c^2)m_{18} + 4m_{19} + (2c^2)\sqrt{3}m_{34} + 4sc\sqrt{3}m_{35} \\
+ (2 - 3c^2)\sqrt{2}m_{36} + 2c^2\sqrt{3}m_{37} + (4sc\sqrt{3})m_{38} \\
+ (2 - 3c^2)\sqrt{2}m_{39} \right\} \]

\[ B_{18} = \frac{1}{36} \left\{ + 6m_{20} + 3\sqrt{2}m_{24} + \sqrt{6}m_{26} + (-6)m_{27} + (-\sqrt{6})m_{29} \\
+ (-3\sqrt{2})m_{33} + 4\sqrt{3}m_{40} + (-4\sqrt{3})m_{41} \right\} \]
\[
\{ T_{\mu\nu}(\epsilon' \cdot \epsilon^*) + 0_{\mu\nu}(\epsilon' \cdot \epsilon^*) + \\
\epsilon_{\mu}(\epsilon' \cdot \epsilon - \epsilon -) + 0\epsilon(\epsilon' \cdot \epsilon -) + 6\epsilon_{\mu}(\epsilon' \cdot \epsilon -) + 6\epsilon_{\mu}(\epsilon' \cdot \epsilon -) + \\
\eta_{\mu}(\epsilon' \cdot \epsilon -) + \epsilon_{\mu}(\epsilon' \cdot \epsilon -) + \epsilon_{\mu}(\epsilon' \cdot \epsilon -) + 0\epsilon(\epsilon' \cdot \epsilon -) \} \frac{219}{7} = \frac{209}{7} \\
\{ \epsilon_{\mu}(\epsilon' \cdot \epsilon -) + \epsilon_{\mu}(\epsilon' \cdot \epsilon -) + 0\epsilon(\epsilon' \cdot \epsilon -) + 6\epsilon_{\mu}(\epsilon' \cdot \epsilon -) + \\
\eta_{\mu}(\epsilon' \cdot \epsilon -) + \epsilon_{\mu}(\epsilon' \cdot \epsilon -) + \epsilon_{\mu}(\epsilon' \cdot \epsilon -) + 0\epsilon(\epsilon' \cdot \epsilon -) \} \frac{219}{7} = \frac{209}{7} \\
\{ \epsilon_{\mu}(\epsilon' \cdot \epsilon -) + \epsilon_{\mu}(\epsilon' \cdot \epsilon -) + 0\epsilon(\epsilon' \cdot \epsilon -) + 6\epsilon_{\mu}(\epsilon' \cdot \epsilon -) + \\
\eta_{\mu}(\epsilon' \cdot \epsilon -) + \epsilon_{\mu}(\epsilon' \cdot \epsilon -) + \epsilon_{\mu}(\epsilon' \cdot \epsilon -) + 0\epsilon(\epsilon' \cdot \epsilon -) \} \frac{219}{7} = \frac{209}{7} \\
\} \frac{219}{7} = \frac{209}{7} 
\]
\[
B_{23} = \frac{1}{12} \left\{ (-3c^2)m_{20} + (6 - 3c^2)m_{22} + (-6sc)m_{23} + (-3\sqrt{2})m_{24} + (-6sc)m_{25} + (-1 + 3c^2)\sqrt{6}m_{26} + (3c^2)m_{27} + (6sc)m_{28} + (1 - 3c^2)\sqrt{6}m_{29} + (-6sc)m_{30} + (-6 + 3c^2)m_{31} + (3\sqrt{2})m_{33} + (2\sqrt{3})m_{40} + (-2\sqrt{3})m_{41} \right\}
\]

\[
B_{24} = \frac{1}{12} \left\{ 6c^2m_{20} + 6sc\sqrt{6}m_{21} + (6 - 6c^2)m_{22} + (6sc)m_{23} + (3 - 6c^2)\sqrt{2}m_{24} + (-6sc)m_{25} + \sqrt{6}m_{26} + (-6c^2)m_{27} + (-6sc)m_{28} + (-\sqrt{6}m)m_{29} + (-6sc)m_{30} + (-6 + 6c^2)m_{31} + (-6\sqrt{2}sc)m_{32} + (-3 + 6c^2)\sqrt{2}m_{33} + (-2\sqrt{3})m_{40} + (2\sqrt{3})m_{41} \right\}
\]
\[ B_{25} = \frac{1}{2} \left\{ \text{sc } m_{20} + (1 - 2c^2) \sqrt{2} m_{21} \right. \\
+ (-3 \text{sc}) m_{22} + (-2 \sqrt{2} \text{ sc}) m_{24} + (-2 + 4c^2) m_{25} \\
+ \text{sc } \sqrt{6} m_{26} + (\text{sc}) m_{27} + (\text{sc } \sqrt{6}) m_{29} \\
+ (2 - 4c^2) m_{30} + (-3 \text{sc}) m_{31} + (1 - 2c^2) \sqrt{2} m_{32} \\
+ (-2 \text{sc}) \sqrt{2} m_{33} \right\} \]
APPENDIX G

PROOF OF THEOREM 13

\[ \Sigma_1^3 s_2^2 \Sigma_3^3 s_4^3 = \Sigma_1^3 \otimes s_2^3 s_4^3 \]

and

\[ \Sigma_1^3 s_2^3 s_4^3 = \Sigma_1^3 \otimes s_2^3 s_4^3 \]
PROOF OF THEOREM 13

1. The definition of the tensor product operation relates $\Sigma^9$ with $\Sigma^3$ and $S^9$ with $S^3$ as follows:

\[
\begin{align*}
(S_1^9 \sigma_{m_1 m_2}, n_1 n_2) &= (S_1^3 \sigma_{m_1 n_1} \delta_{m_2 n_2}) \\
(S_3^9 \sigma_{v_1 v_2}, \theta_{1 \theta_2}) &= (S_3^3 \sigma_{v_1 \theta_1} \delta_{v_2 \theta_2}) \\
(S_2^9 \rho_{1 \rho_2}, \sigma_1 \sigma_2) &= \delta_{\rho_1 \sigma_1} (S_1^3 \rho_2 \sigma_2) \\
(S_4^9 \alpha_{1 \alpha_2}, \beta_1 \beta_2) &= \delta_{\alpha_1 \beta_1} (S_4^3 \alpha_2 \beta_2) .
\end{align*}
\]

2. We write the product $\Sigma_1^9 S_2^9 \Sigma_3^9 S_4^9$ in terms of elements as

\[
(S_1^9 \Sigma_2^9 \Sigma_3^9 S_4^9) m_{1 m_2}, \beta_1 \beta_2
\]

\[
= \Sigma (S_1^9 m_{1 m_2}, n_1 n_2) (S_2^9 n_{1 n_2} \sigma_1 \sigma_2) (S_3^9 \sigma_1 \sigma_2, \theta_1 \theta_2) (S_4^9 \theta_1 \theta_2, \beta_1 \beta_2) .
\]

3. We insert the equations of step 1 into this last expression and then sum over the indices occurring in the Kronecker delta functions to get

\[
(S_1^9 S_2^9 \Sigma_3^9 S_4^9) m_{1 m_2}, \beta_1 \beta_2
\]

\[
= \Sigma (S_1^3 \sigma_{1 \sigma_1} (S_3^3 \sigma_{1 \beta_1} (S_1^3 m_{2 \theta_2} \Sigma_3^3 \sigma_{1 \theta_2} (S_4^3 \theta_2 \beta_2)
\]

\[
= (S_1^3 \sigma_{1 \beta_1} (S_4^3 \sigma_{2 \beta_2}) (S_2^3 \theta_2 \beta_2)
\]

\[
= [(S_1^3 \Sigma_3^3 m_{1 \beta_1}] [(S_3^3 \sigma_{2 \beta_2}) m_{2 \beta_2}] .
\]
4. We write $\Sigma^3_1 \Sigma^3_3 \otimes S^3_2 S^3_4$ in terms of elements as

$$(\Sigma^3_1 \Sigma^3_3 \otimes S^3_2 S^3_4)_{m_1, m_2} = (\Sigma^3_1 \Sigma^3_3 \otimes S^3_2 S^3_4)_{m_1 m_2}.$$

5. The right hand sides of step 4 and step 3 are identical so

$$(\Sigma^3_1 S^3_2 \Sigma^3_3 S^3_4)_{m_1 m_2} = (\Sigma^3_1 \Sigma^3_3 \otimes S^3_2 S^3_4)_{m_1 m_2}.$$

and

$$\Sigma^9_1 S^9_2 S^9_3 S^9_4 = \Sigma^3_1 \Sigma^3_3 \otimes S^3_2 S^3_4.$$

6. $\Sigma^9_1 S^9_2 S^9_3 S^9_4 = \Sigma^3_1 \Sigma^3_3 \otimes S^3_2 S^3_4$ is proved in the same fashion. Q.E.D.
APPENDIX H

SPHERICAL TENSOR OPERATORS

\((T_{KQ})\)
\[ T_{2+2} = \sqrt{3} \cdot \frac{1}{576} \cdot [ \begin{array}{cccccccc} 22 & 21 & 20 & 2-1 & 2-2 & 11 & 10 & 1-1 & 00 \\ 22 \vert & 0 & 0 & 96 \sqrt{6} & 0 & 0 & 0 & -288 \sqrt{2} & 0 & 192 \sqrt{3} \\ 21 \vert & 0 & 0 & 0 & 288 & 0 & 0 & 0 & -288 & 0 \\ 20 \vert & 0 & 0 & 0 & 0 & 96 \sqrt{6} & 0 & 0 & 0 & 0 \\ 2-1 \vert & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2-2 \vert & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 \vert & 0 & 0 & 0 & 288 & 0 & 0 & 0 & -288 & 0 \\ 10 \vert & 0 & 0 & 0 & 0 & 288 \sqrt{2} & 0 & 0 & 0 & 0 \\ 1-1 \vert & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 00 \vert & 0 & 0 & 0 & 0 & 192 \sqrt{3} & 0 & 0 & 0 & 0 \end{array} \]
\[ t_{2+1} = \frac{\sqrt{3}}{288} \cdot [ ] \]

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\[ t_{20} = \frac{\sqrt{2}}{192} \cdot [ ] \]

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$$T_{2-1} = \frac{\sqrt{3}}{288} \cdot [ \begin{array}{cccccccccc} 22 & 21 & 20 & 2-1 & 2-2 & 11 & 10 & 1-1 & 00 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +144 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +24\sqrt{6} & 0 & 0 & 0 & +72\sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & -24\sqrt{6} & 0 & 0 & 0 & +72\sqrt{2} & 0 & +96\sqrt{3} \\ 0 & 0 & 0 & -144 & 0 & 0 & 0 & -144 & 0 \\ -144 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +72\sqrt{2} & 0 & 0 & 0 & -72\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & +72\sqrt{6} & 0 & 0 & 0 & +72\sqrt{2} & 0 & 0 \\ 0 & 0 & -96\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \end{array} ]$$
\[ t_{2-2} = \frac{\sqrt{3}}{576} \cdot [ ] \]

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\[ T_{1+1} = \frac{\sqrt{6}}{48} \cdot [ ] \]

\[
\begin{array}{cccccccccc}
\text{su} & 22 & 21 & 20 & 2-1 & 2-2 & 11 & 10 & 1-1 & 00 \\
22| & 0 & -12 \sqrt{2} & 0 & 0 & 0 & +12 \sqrt{2} & 0 & 0 & 0 \\
21| & 0 & 0 & -12 \sqrt{3} & 0 & 0 & 0 & +12 & 0 & 0 \\
20| & 0 & 0 & 0 & -12 \sqrt{3} & 0 & 0 & 0 & +4 \sqrt{3} & 0 \\
2-1| & 0 & 0 & 0 & 0 & -12 \sqrt{2} & 0 & 0 & 0 & 0 \\
2-2| & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11| & 0 & 0 & -4 \sqrt{3} & 0 & 0 & 0 & -12 & 0 & +8 \sqrt{6} \\
10| & 0 & 0 & 0 & -12 & 0 & 0 & 0 & -12 & 0 \\
1-1| & 0 & 0 & 0 & 0 & -12 \sqrt{2} & 0 & 0 & 0 & 0 \\
00| & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 \sqrt{6} & 0 \\
\end{array}
\]
\[
\begin{bmatrix}
0 & 0 & 3/8 & 0 & 0 & 0 & 0 & 0 & 100 & 1
-1/2 & 0 & 0 & 0 & 0 & 9/6 & 0 & 0 & 10 & 11
3/8 & 0 & 0 & 0 & 0 & 0 & 6/6 & 0 & 0 & 20
0 & 0 & 0 & 5/6 & 0 & 0 & 0 & 6/6 & 0 & 21
0 & 0 & 0 & 0 & 6/6 & 0 & 0 & 5/6 & 0 & 22
0 & 0 & 0 & 0 & 0 & 5/6 & 0 & 0 & 0 & 22
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6/6 & 0 & 22
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6/6 & 22
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \cdot \frac{2\pi}{L} = 0 L
\]
\[ T_{1-1} = \frac{\sqrt{6}}{48} \cdot \begin{bmatrix} \nu \\ 22 \\ 21 \\ 20 \\ 2-1 \\ 2-2 \\ 11 \\ 10 \\ 1-1 \\ 00 \end{bmatrix} \]

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$$T_{00} = 1^* [ ]$$

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APPENDIX I

THE Tr[NTKQMTK'Q'] IN TERMS OF THE ELEMENTS
OF THE TRANSITION MATRIX \{m_\perp\}

Abbreviations

[\ell,k] = m_\ell m_k^*
\[
T \cdot [M T^+_2 M^+ T_{2+2}^+] = T \cdot [M T^+_2 M^+ T_{2-2}^+] = \frac{3}{24^2} \cdot 2 \text{ Re } [P] 
\]

\[
P = \left[12,20\right] \sqrt{6} (-48) + \left[11,21\right] \sqrt{3} 96 + \left[11,34\right] \sqrt{2} 96 + \left[8,27\right] \sqrt{6} (-48) + \left[24,27\right] \sqrt{2} (-144) + \left[7,7\right] 144/2 + \left[7,28\right] 144 + \left[23,28\right] 144 + \left[6,8\right] \sqrt{6} (-48) + \left[6,29\right] \sqrt{6} (-48) + \left[7,23\right] 144 + \left[7,14\right] 144 + \left[23,23\right] 144/2 + \left[6,24\right] \sqrt{2} (-144) + \left[6,15\right] \sqrt{2} (-144) + \left[6,35\right] (-96) + \left[6,40\right] \sqrt{3} (-96) + \left[3,11\right] 96 + \left[3,32\right] \sqrt{3} 96 + \left[3,37\right] 296 + \left[21,32\right] 288 + \left[21,37\right] \sqrt{6} 96 + \left[34,37\right] 192 + \left[2,12\right] \sqrt{6} (-48) + \left[2,33\right] \sqrt{2} (-144) + \left[2,38\right] \sqrt{3} (-96) + \left[20,33\right] \sqrt{2} (-144) + \left[20,38\right] \sqrt{3} (-96) + \left[1,13\right] 96 + \left[1,39\right] \sqrt{2} 96 + \left[2,26\right] \sqrt{6} (-48) + \left[2,17\right] \sqrt{2} (-144) + \left[2,41\right] \sqrt{3} (-96) + \left[20,26\right] \sqrt{6} (-48) + \left[20,41\right] \sqrt{3} (-96) + \left[1,18\right] 288 + \left[1,36\right] \sqrt{2} 96 + \left[1,19\right] 192 + \left[17,20\right] \sqrt{2} (-144) + \left[32,34\right] \sqrt{6} 96 + \left[15,27\right] \sqrt{2} (-144) + \left[28,28\right] 144/2 + \left[14,28\right] 144 + \left[27,29\right] \sqrt{6} (-48) + \left[14,23\right] 144 + \left[14,14\right] 144/2 + \left[27,35\right] \sqrt{3} (-96) + \left[27,40\right] \sqrt{3} (-96) \right]
\[
T_r \left[ M_{T_2+1}^+ H_{T_2+2}^+ \right] = -T_r \left[ M_{T_2+1}^+ H_{T_2-2}^+ \right] = \frac{3}{24^2} \cdot 2 \text{Re } [P]
\]

\[
P = [13,20] (-14^4) + [12,21] \sqrt{3} (48) + [11,22] \sqrt{6} 48 + [12,3^2] \sqrt{2} 96
+ [9,27] (-14^4) + [25,27] (-14^4) + [8,28] \sqrt{6} 24 + [24,28] \sqrt{2} (-72)
+ [7,8] \sqrt{6} 24 + [7,29] \sqrt{6} 24 + [23,29] \sqrt{6} 72 + [6,9] (-14^4)
+ [6,30] (-14^4) + [8,23] \sqrt{6} 72 + [8,23] \sqrt{6} 72 + [7,2^4] \sqrt{2} (-72)
+ [7,15] \sqrt{2} (-72) + [23,2^4] \sqrt{2} 72 + [6,25] (-14^4) + [6,16] (-14^4)
+ [7,35] \sqrt{3} (-96) + [7,40] \sqrt{3} (-96) + [4,11] \sqrt{6} 48 + [4,32] \sqrt{2} (14^4)
+ [4,37] \sqrt{3} 96 + [22,32] \sqrt{2} 144 + [22,37] \sqrt{3} 96 + [3,12] (-48)
+ [3,33] \sqrt{3} (-48) + [3,38] \sqrt{2} (-48) + [21,33] (14^4) + [21,38] \sqrt{6} 48
+ [34,38] 192 + [2,13] (-48) + [2,39] \sqrt{2} (-48)
+ [20,39] \sqrt{2} (-14^4) + [1,12] \sqrt{6} (-48) + [1,33] \sqrt{2} 144
+ [1,38] \sqrt{3} (-96) + [3,26] (-14^4) + [3,17] \sqrt{3} (-14^4)
+ [3,41] \sqrt{2} (-14^4) + [21,26] \sqrt{3} (-48) + [21,41] \sqrt{6} (-48)
+ [2,18] 144 + [1,26] \sqrt{6} 48 + [1,17] \sqrt{2} (-14^4) + [1,41] \sqrt{3} 96
+ [33,3^4] \sqrt{6} 96 + [16,27] (-14^4) + [15,28] \sqrt{2} (-72) + [28,29] \sqrt{6} 24
+ [14,29] \sqrt{6} 72 + [27,30] (-14^4) + [15,23] \sqrt{2} 72 + [14,2^4] \sqrt{2} 72
+ [14,15] \sqrt{2} 72 + [28,35] \sqrt{3} (-96) + [28,40] \sqrt{3} (-96)
\]
\[
T^\pm_F \left[ M T^\pm_{20} M T^\pm_{2+2} \right] = T^\pm_F \left[ M T^\pm_{20} M T^\pm_{2-2} \right] = \frac{\sqrt{6}}{24} \cdot 2 \Re \left[ P \right]
\]

\[
P = \left[ [12,20] \sqrt{6} \ 72 + [12,22] \sqrt{6} \ 72 + [13,34] \sqrt{3} \ 96 + [10,27] \left( -144 \right) \right.
+ \left[ 9,28 \right] \left( -72 \right) + \left[ 25,28 \right] \left( -216 \right) + \left[ 8,8 \right] 72 + \left[ 8,29 \right] 144 + \left[ 7,9 \right] \left( -72 \right)
+ \left[ 23,30 \right] 216 + \left[ 6,10 \right] \left( -144 \right) + \left[ 6,31 \right] \left( -144 \right) + \left[ 9,23 \right] 216
+ \left[ 9,34 \right] 216 + \left[ 24,24 \right] 72 + \left[ 7,25 \right] \left( -216 \right) + \left[ 7,16 \right] \left( -216 \right) + \left[ 23,25 \right] 72
+ \left[ 8,35 \right] \sqrt{2} \left( -144 \right) + \left[ 8,40 \right] \sqrt{2} \left( -144 \right) + \left[ 5,11 \right] \sqrt{6} \ 48 + \left[ 5,32 \right] \sqrt{2} \ 144
+ \left[ 5,37 \right] \sqrt{3} \ 96 + \left[ 4,12 \right] \sqrt{6} \ 24 + \left[ 4,33 \right] \sqrt{2} \ 72 + \left[ 4,38 \right] \sqrt{3} \ 48
+ \left[ 22,33 \right] \sqrt{2} \ 216 + \left[ 22,38 \right] \sqrt{3} \ 144 + \left[ 3,13 \right] \sqrt{6} \ \left( -48 \right) + \left[ 3,39 \right] \sqrt{3} \ \left( -96 \right)
+ \left[ 34,39 \right] \sqrt{6} \ 96 + \left[ 2,12 \right] \sqrt{6} \ \left( -24 \right) + \left[ 2,33 \right] \sqrt{2} \ 72 + \left[ 2,38 \right] \sqrt{3} \ \left( -48 \right)
+ \left[ 20,33 \right] \sqrt{2} \ \left( -216 \right) + \left[ 20,38 \right] \sqrt{3} \ \left( 144 \right) + \left[ 1,11 \right] \sqrt{6} \ 48 + \left[ 1,32 \right] \sqrt{2} \ \left( -144 \right)
+ \left[ 1,37 \right] \sqrt{3} \ 96 + \left[ 4,26 \right] \sqrt{6} \ \left( -72 \right) + \left[ 4,17 \right] \sqrt{2} \ \left( -216 \right) + \left[ 4,41 \right] \sqrt{3} \ \left( -144 \right)
+ \left[ 22,26 \right] \sqrt{6} \ \left( -24 \right) + \left[ 22,41 \right] \sqrt{3} \ \left( -48 \right) + \left[ 2,26 \right] \sqrt{6} \ 72 + \left[ 2,17 \right] \sqrt{2} \ \left( -216 \right)
+ \left[ 2,41 \right] \sqrt{3} \ \left( -144 \right) + \left[ 20,26 \right] \sqrt{6} \ \left( -24 \right) + \left[ 20,41 \right] \sqrt{3} \ \left( -48 \right) + \left[ 3,36 \right] \sqrt{3} \ 96
+ \left[ 3,19 \right] \sqrt{6} \ 96 + \left[ 17,20 \right] \sqrt{2} \ 72 + \left[ 18,21 \right] \sqrt{2} \ \left( -144 \right) + \left[ 17,22 \right] \sqrt{2} \ \left( -72 \right)
+ \left[ 16,38 \right] \left( -216 \right) + \left[ 29,29 \right] 72 + \left[ 28,30 \right] \left( -72 \right) + \left[ 14,30 \right] \left( 216 \right)
+ \left[ 27,31 \right] \left( -144 \right) + \left[ 16,23 \right] 72 + \left[ 15,24 \right] \left( 144 \right) + \left[ 15,15 \right] 72 + \left[ 14,25 \right] 72
+ \left[ 14,16 \right] 72 + \left[ 29,35 \right] \sqrt{2} \ \left( -144 \right) + \left[ 29,40 \right] \sqrt{2} \ \left( -144 \right)\right].\]
\[ T_2 [M T_2^+ N T_2^{-2}] = -T_2 [M T_2^+ N T_2^{-2}] = \frac{3}{24^2} \cdot 2 \Re [P] \]

\[ P = \left[ [11,20] \sqrt{6} (-48) + [12,21] 3\sqrt{48} + [13,22] \sqrt{144} + [12,34] \sqrt{2} (-96) \right. \]
\[ + [10,28] (-144) + [9,29] \sqrt{6} 24 + [25,29] \sqrt{6} (-72) + [8,9] \sqrt{6} 24 \]
\[ + [8,30] \sqrt{6} 24 + [24,30] \sqrt{2} 72 + [7,10] (-144) + [7,31] (-144) \]
\[ + [8,25] \sqrt{6} (-72) + [8,16] \sqrt{6} (-72) + [24,25] \sqrt{2} 72 + [9,35] \sqrt{3} (-96) \]
\[ + [9,40] \sqrt{3} (-96) + [5,12] \sqrt{6} 48 + [5,33] \sqrt{2} 144 + [5,38] \sqrt{3} 96 \]
\[ + [4,13] (-48) + [4,39] \sqrt{2} (-48) + [22,39] \sqrt{2} 144 + [3,12] 48 \]
\[ + [3,33] \sqrt{3} (-48) + [3,38] \sqrt{2} 48 + [21,33] (-144) + [21,38] \sqrt{6} 48 \]
\[ + [34,38] (-192) + [2,11] \sqrt{6} 48 + [2,32] \sqrt{2} (-144) + [2,37] \sqrt{3} 96 \]
\[ + [20,32] \sqrt{2} 144 + [20,37] \sqrt{3} (-96) + [5,26] \sqrt{6} (-48) + [5,17] \sqrt{2} (-144) \]
\[ + [5,41] \sqrt{3} (-96) + [4,18] (-144) + [3,26] 144 + [3,17] \sqrt{3} (-144) \]
\[ + [3,41] \sqrt{2} 144 + [21,26] \sqrt{3} (-48) + [21,41] \sqrt{6} (-48) + [4,36] \sqrt{2} 96 \]
\[ + [4,19] 192 + [17,21] 144 + [18,22] (-144) + [33,34] \sqrt{6} 96 \]
\[ + [16,29] \sqrt{6} (-72) + [29,30] \sqrt{6} 24 + [15,30] \sqrt{2} 72 + [28,31] (-144) \]
\[ + [14,31] 144 + [16,24] \sqrt{2} 72 + [15,25] \sqrt{2} 72 + [15,16] \sqrt{2} 72 \]
\[ + [30,35] \sqrt{3} (-96) + [30,40] \sqrt{3} (-96) \]
\[ T_r [M T_{2-2}^+ M^+ T_{2+2}] = T_r [M T_{2+2}^+ M^+ T_{2-2}] = \frac{3}{2^{1/2}} \cdot 2 \text{Re} [P] \]

\[ P = [11, 21] \sqrt{3} (-96) + [12, 22] \sqrt{6} (-48) + [11, 34] \sqrt{2} 96 \]
\[ + [10, 29] \sqrt{6} (-48) + [9, 9] 72 + [9, 30] 144 + [25, 30] (-144) \]
\[ + [8, 10] \sqrt{6} (-48) + [8, 31] \sqrt{6} (-48) + [24, 31] \sqrt{2} 144 + [10, 15] \sqrt{2} 144 \]
\[ + [9, 25] (-144) + [9, 16] (-144) + [25, 25] 72 + [10, 35] \sqrt{3} (-96) \]
\[ + [10, 40] \sqrt{3} (-96) + [5, 13] 96 + [5, 39] \sqrt{2} 96 + [4, 12] \sqrt{6} 48 \]
\[ + [4, 13] \sqrt{2} (-144) + [4, 38] \sqrt{3} 96 + [22, 33] \sqrt{2} 144 + [22, 38] \sqrt{3} (-96) \]
\[ + [3, 11] 96 + [3, 32] \sqrt{3} (-96) + [3, 37] \sqrt{2} 96 + [21, 32] 288 \]
\[ + [21, 37] \sqrt{6} (-96) + [34, 37] 192 + [5, 18] (-288) + [4, 26] \sqrt{6} 48 \]
\[ + [4, 17] \sqrt{2} (-144) + [4, 41] \sqrt{3} 96 + [22, 26] \sqrt{6} (-48) + [22, 41] \sqrt{3} (-96) \]
\[ + [5, 36] \sqrt{2} 96 + [5, 19] 192 + [17, 22] \sqrt{2} 144 + [32, 34] \sqrt{6} (-96) \]
\[ + [30, 30] 72 + [16, 30] (-144) + [29, 31] \sqrt{6} (-48) + [15, 31] \sqrt{2} 144 \]
\[ + [16, 25] 144 + [16, 16] 72 + [31, 35] \sqrt{3} (-96) + [31, 40] \sqrt{3} (-96) \]
\[ + [10, 24] \sqrt{2} 144 \]
\[ T_r \left[ M T_{1+1}^+ M^* T_{2-2} \right] = T_r \left[ M T_{1-1}^+ M^* T_{2-2} \right] = \frac{3\sqrt{2}}{24^2} \times 2 \ \text{Im} \ [P] \]

\[
P = \left[ \begin{array}{c}
[13,20] \sqrt{2} (-24) + [12,21] \sqrt{6} 24 + [11,22] \sqrt{3} (-48) + [26,34] (-96) \\
+ [9,27] \sqrt{2} (-72) + [25,27] \sqrt{2} (-72) + [8,28] \sqrt{3} 72 + [24,28] 72 \\
+ [7,29] \sqrt{3} (-72) + [7,29] \sqrt{3} (-72) + [23,29] \sqrt{3} (-24) + [6,9] \sqrt{2} 72 \\
+ [6,30] \sqrt{2} 72 + [8,31] \sqrt{3} 24 + [8,11] \sqrt{3} 24 + [7,24] (-72) \\
+ [7,15] (-72) + [23,24] 72 + [6,25] \sqrt{2} 72 + [6,16] \sqrt{2} 72 \\
+ [4,37] \sqrt{6} 48 + [22,32] 144 + [22,37] \sqrt{6} 48 + [3,12] \sqrt{2} (-72) \\
+ [3,33] \sqrt{6} (-72) + [3,38] (-144) + [21,33] \sqrt{2} (-72) + [21,38] \sqrt{3} (-48) \\
+ [1,38] \sqrt{6} 48 + [3,26] \sqrt{2} (-24) + [3,17] \sqrt{6} (-24) + [3,41] (-48) \\
+ [21,26] \sqrt{6} 24 + [21,41] \sqrt{3} 48 + [34,41] \sqrt{2} 96 + [2,18] \sqrt{2} 72 \\
+ [1,26] \sqrt{3} (-48) + [1,17] (+144) + [1,41] \sqrt{6} (-48) + [20,36] (-96) \\
+ [18,20] \sqrt{2} (72) + [17,21] \sqrt{2} (-72) + [17,34] \sqrt{3} (-96) \\
+ [16,27] \sqrt{2} (-72) + [15,28] 72 + [28,29] \sqrt{3} (-72) + [14,29] \sqrt{3} (-24) \\
+ [14,35] \sqrt{6} 48 + [14,40] \sqrt{6} 48 + [19,20] \sqrt{2} 96 \\
\end{array} \right]
\]
\[ T_{r} \left[ M T_{10}^{+} M^{+} T_{2-2} \right] = -T_{r} \left[ M T_{10}^{+} M^{+} T_{2-2} \right] = \frac{3}{24^2} \cdot 2 \text{ Im } [\mathbf{P}] \]

\[ \mathbf{P} = \left[ 12,20 \right] \sqrt{3} \cdot \frac{1}{8} + \left[ 13,21 \right] \cdot 96 + \left[ 12,22 \right] \sqrt{3} \cdot (-48) \]

+ \left[ 10,27 \right] \sqrt{2} \cdot (-144) + \left[ 9,28 \right] \sqrt{2} \cdot 72 + \left[ 25,28 \right] \sqrt{2} \cdot (-72)

+ \left[ 24,29 \right] \sqrt{6} \cdot 48 + \left[ 7,9 \right] \sqrt{2} \cdot (-72) + \left[ 7,30 \right] \sqrt{2} \cdot (-72) + \left[ 23,30 \right] \sqrt{2} \cdot (-72)

+ \left[ 6,10 \right] \sqrt{2} \cdot 144 + \left[ 6,31 \right] \sqrt{2} \cdot 144 + \left[ 9,23 \right] \sqrt{2} \cdot 72 + \left[ 9,14 \right] \sqrt{2} \cdot 72

+ \left[ 8,24 \right] \sqrt{6} \cdot (-48) + \left[ 8,15 \right] \sqrt{6} \cdot (-48) + \left[ 7,25 \right] \sqrt{2} \cdot (72)

+ \left[ 7,16 \right] \sqrt{2} \cdot 72 + \left[ 23,25 \right] \sqrt{2} \cdot 72 + \left[ 24,35 \right] \sqrt{3} \cdot 96 + \left[ 24,40 \right] \sqrt{3} \cdot 96

+ \left[ 5,11 \right] \sqrt{3} \cdot 96 + \left[ 5,32 \right] \cdot 288 + \left[ 5,37 \right] \sqrt{6} \cdot 96 + \left[ 4,12 \right] \sqrt{3} \cdot (-48)

+ \left[ 4,33 \right] \cdot (-144) + \left[ 4,38 \right] \sqrt{6} \cdot (-48) + \left[ 22,33 \right] \cdot 144 + \left[ 22,38 \right] \sqrt{6} \cdot 48

+ \left[ 21,39 \right] \sqrt{2} \cdot (-96) + \left[ 2,12 \right] \sqrt{3} \cdot (-48) + \left[ 2,33 \right] \cdot 144

+ \left[ 2,38 \right] \sqrt{6} \cdot (-48) + \left[ 20,33 \right] \cdot 144

+ \left[ 20,38 \right] \sqrt{6} \cdot (-48) + \left[ 1,11 \right] \sqrt{3} \cdot (-96) + \left[ 1,32 \right] \cdot 288 + \left[ 1,37 \right] \sqrt{6} \cdot (-96)

+ \left[ 4,26 \right] \sqrt{3} \cdot (-48) + \left[ 4,17 \right] \cdot (-144) + \left[ 4,41 \right] \sqrt{6} \cdot (-48) + \left[ 3,18 \right] \sqrt{3} \cdot 96

+ \left[ 2,26 \right] \sqrt{3} \cdot (-48) + \left[ 2,17 \right] \cdot 144 + \left[ 2,41 \right] \sqrt{6} \cdot (-48) + \left[ 20,26 \right] \sqrt{3} \cdot (-48)

+ \left[ 20,41 \right] \sqrt{6} \cdot (-48) + \left[ 21,36 \right] \sqrt{2} \cdot (-96) + \left[ 17,20 \right] \cdot (-144) + \left[ 17,22 \right] \cdot (-144)

+ \left[ 18,34 \right] \sqrt{6} \cdot (-96) + \left[ 16,28 \right] \sqrt{2} \cdot (-72) + \left[ 15,29 \right] \sqrt{6} \cdot (48)

+ \left[ 28,30 \right] \sqrt{2} \cdot (-72) + \left[ 14,30 \right] \sqrt{2} \cdot (-72) + \left[ 27,31 \right] \sqrt{2} \cdot (144)

+ \left[ 16,23 \right] \sqrt{2} \cdot (-72) + \left[ 14,25 \right] \sqrt{2} \cdot 72 + \left[ 14,16 \right] \sqrt{2} \cdot 72 + \left[ 15,35 \right] \sqrt{3} \cdot 96

+ \left[ 15,40 \right] \sqrt{3} \cdot 96 + \left[ 19,21 \right] \cdot 192 + \left[ 22,26 \right] \sqrt{3} \cdot 48 + \left[ 22,41 \right] \sqrt{6} \cdot 48]
\[ T_{\tau} \left[ M T_{1-1}^{+} M^{+} T_{2+2} \right] = \frac{3\sqrt{2}}{24} \cdot 2 \text{ Im } [P] \]

\[ P = \left[ [11,20] \sqrt{3} (-48) + [12,21] \sqrt{6} (-24) + [13,22] \sqrt{2} (-24) \right] + \left[ [26,34] (-96) + [10,28] \sqrt{2} (-72) + [9,29] \sqrt{3} 72 + [25,29] \sqrt{3} (-24) \right] + \left[ [8,9] \sqrt{3} (-72) + [8,30] \sqrt{3} (-72) + [24,30] 72 + [7,10] \sqrt{2} 72 \right] + \left[ [7,31] \sqrt{2} 72 + [23,31] \sqrt{2} (-72) + [10,14] \sqrt{2} 72 + [9,24] (-72) \right] + \left[ [9,15] (-72) + [8,25] \sqrt{3} 24 + [8,16] \sqrt{3} 24 + [24,25] 72 \right] + \left[ [25,35] \sqrt{6} 48 + [25,40] \sqrt{6} 48 + [5,12] \sqrt{3} 48 + [5,33] 144 \right] + \left[ [5,38] \sqrt{6} 48 + [4,13] \sqrt{2} (-72) + [4,39] (-144) + [22,39] 48 \right] + \left[ [3,12] \sqrt{2} (-72) + [3,33] \sqrt{6} 72 + [3,38] (-144) + [21,33] \sqrt{2} (-72) \right] + \left[ [21,38] \sqrt{3} (48) + [2,11] \sqrt{3} (-48) + [2,32] (144) \right] + \left[ [2,37] \sqrt{6} (-48) + [20,32] (-144) + [20,37] \sqrt{6} 48 + [5,26] \sqrt{3} (-48) \right] + \left[ [5,17] (-144) + [5,41] \sqrt{6} (-48) + [4,18] \sqrt{2} 72 + [3,26] \sqrt{2} (-24) \right] + \left[ [3,17] \sqrt{6} 24 + [3,41] (-48) + [21,26] \sqrt{6} (-24) + [21,41] \sqrt{3} (-48) \right] + \left[ [34,41] \sqrt{2} 96 + [22,36] (-96) + [17,21] \sqrt{2} (-72) \right] + \left[ [18,22] \sqrt{2} (-72) + [17,34] \sqrt{3} 96 + [16,29] \sqrt{3} (-24) \right] + \left[ [29,30] \sqrt{3} (-72) + [15,30] 72 + [28,31] \sqrt{2} 72 + [14,32] \sqrt{2} (-72) \right] + \left[ [16,24] (-72) + [15,25] 72 + [15,16] 72 + [16,35] \sqrt{6} 48 \right] + \left[ [16,40] \sqrt{6} 48 + [19,22] \sqrt{2} 96 + [10,23] \sqrt{2} 72 \right] \]
\[ T_{\tau} \left[ M_{T_{00} M_{T_{2+2}}}^{} \right] = T_{\tau} \left[ M_{T_{00} M_{T_{2-2}}}^{} \right] = \frac{\sqrt{3}}{24} \cdot 2 \Re \left[ P \right] \]

\[ P = \left[ [10,27] \right] (-288) + \left[ [9,28] \right] 288 + \left[ [8,8] \right] (-144) + \left[ [8,29] \right] (-288) + \left[ [7,9] \right] (288) + \left[ [7,30] \right] 288 + \left[ [6,10] \right] (-288) + \left[ [6,31] \right] (-288) + \left[ [24,24] \right] 144 + \left[ [23,25] \right] (-288) + \left[ [35,35] \right] (-144) + \left[ [35,40] \right] (-288) + \left[ [5,11] \right] \sqrt{6} \ 96 + \left[ [5,32] \right] \sqrt{2} (288) + \left[ [5,37] \right] \sqrt{3} 192 + \left[ [4,12] \right] \sqrt{6} (-96) + \left[ [4,33] \right] \sqrt{2} (-288) + \left[ [4,38] \right] \sqrt{3} (-192) + \left[ [3,13] \right] \sqrt{6} 96 + \left[ [3,39] \right] \sqrt{3} 192 + \left[ [2,12] \right] \sqrt{6} 96 + \left[ [2,33] \right] \sqrt{2} (-288) + \left[ [2,38] \right] \sqrt{3} 192 + \left[ [1,11] \right] \sqrt{6} 96 + \left[ [1,32] \right] \sqrt{2} (-288) + \left[ [1,37] \right] \sqrt{3} 192 + \left[ [22,26] \right] \sqrt{6} 96 + \left[ [22,41] \right] \sqrt{3} 192 + \left[ [20,26] \right] \sqrt{6} 96 + \left[ [20,41] \right] \sqrt{3} 192 + \left[ [34,36] \right] \sqrt{6} 96 + \left[ [17,20] \right] \sqrt{2} (-288) + \left[ [18,21] \right] \sqrt{2} (-288) + \left[ [17,22] \right] \sqrt{2} (288) + \left[ [29,29] \right] (-144) + \left[ [28,30] \right] 288 + \left[ [27,31] \right] (-288) + \left[ [16,23] \right] (-288) + \left[ [15,24] \right] (288) + \left[ [15,15] \right] 144 + \left[ [14,25] \right] (-288) + \left[ [14,16] \right] (-288) + \left[ [40,40] \right] (-144) + \left[ [19,34] \right] \sqrt{3} 192 \right] \]
\[ T_r \left[ M T_{2+2}^+ M^+ T_{2+1} \right] = -T_r \left[ M T_{2-2}^+ M^+ T_{2-1} \right] = \frac{3}{2} \cdot \text{Re} [P] \]

\[ P = \left[ [9,20] (1/4) + [10,21] \sqrt{2} (-1/4) + [10,34] \sqrt{3} (-96) + [13,27] (1/4) + [12,28] \sqrt{7} 72 + [26,28] \sqrt{6} 72 + [11,29] (1/4) \right] + \left[ [12,23] \sqrt{6} (24) + [12,14] \sqrt{6} 72 + [11,24] \sqrt{3} (-48) \right] + \left[ [11,15] \sqrt{3} (-1/4) + [11,35] \sqrt{2} (-48) + [11,40] \sqrt{2} (-1/4) \right] + \left[ [8,11] (-1/4) + [8,32] \sqrt{3} 48 + [8,37] \sqrt{2} 96 + [24,32] 144 \right] + \left[ [24,37] \sqrt{6} 96 + [35,37] 192 + [7,12] \sqrt{6} 24 + [7,33] \sqrt{2} (-72) \right] + \left[ [7,38] \sqrt{3} (-96) + [23,33] \sqrt{2} (-72) + [23,38] \sqrt{3} (-96) \right] + \left[ [6,13] (-1/4) + [6,39] \sqrt{2} 96 + [7,26] \sqrt{6} 24 + [7,17] \sqrt{2} (-72) \right] + \left[ [7,41] \sqrt{3} (-96) + [23,26] \sqrt{6} 24 + [23,41] \sqrt{3} (-96) + [6,18] (1/4) \right] + \left[ [6,36] \sqrt{2} (-1/4) + [6,19] (192) + [3,10] \sqrt{6} (-48) + [3,31] \sqrt{6} 48 \right] + \left[ [21,31] \sqrt{2} 144 + [2,9] (-1/4) + [2,30] 144 + [20,30] 144 \right] + \left[ [1,8] \sqrt{6} (-48) + [1,29] \sqrt{6} 48 + [2,25] 144 + [2,16] (-1/4) \right] + \left[ [20,25] 144 + [1,24] \sqrt{2} (1/4) + [1,15] \sqrt{2} (-1/4) + [1,35] \sqrt{3} (96) \right] + \left[ [1,40] \sqrt{3} 96 + [16,20] (-1/4) + [31,34] \sqrt{3} 96 + [18,27] (-1/4) \right] + \left[ [17,28] \sqrt{2} 72 + [17,23] \sqrt{2} (-72) + [32,35] \sqrt{6} 48 + [32,40] \sqrt{6} (-48) \right] + \left[ [29,32] \sqrt{3} (-48) + [15,32] (-1/4) + [28,33] \sqrt{2} 72 + [14,33] \sqrt{2} 72 \right] + \left[ [14,26] \sqrt{6} 72 + [14,17] \sqrt{2} 72 + [27,36] \sqrt{2} (-1/4) \right] \]
\[
T_r \left[ M^{T_{2+1}} M^{T_{2+1}} \right] = T_r \left[ M^{T_{2-1}} M^{T_{2-1}} \right] = \frac{3}{24^2} \cdot 2 \Pi [P]
\]

\[
P = \left[ [8,20] \sqrt{6} (72) + [9,21] \sqrt{2} (72) + [10,22] (\text{-}144) \right.
+ [9,34] \sqrt{3} 96 + [12,27] \sqrt{6} 72 + [26,27] \sqrt{6} (\text{-}72) + [13,28] 72


+ [13,14] 216 + [12,24] \sqrt{3} (\text{-}24) + [12,15] \sqrt{3} (\text{-}72) + [11,25] \sqrt{6} (\text{-}24)

+ [11,16]/6 (\text{-}72) + [12,35] \sqrt{2} (\text{-}48) + [12,40] \sqrt{2} (\text{-}144)

+ [9,11] \sqrt{6} (\text{-}24) + [9,32] \sqrt{2} 72 + [9,37] \sqrt{3} (96) + [25,32] 2\sqrt{2} 72

+ [25,37] \sqrt{3} 96 + [8,12] 24 + [8,23] \sqrt{3} (\text{-}24) + [8,38] \sqrt{2} (\text{-}48)


+ [7,39] \sqrt{2} (\text{-}48) + [23,39] \sqrt{2} (\text{-}144) + [6,12] \sqrt{2} 24 + [6,33] \sqrt{2} 72

+ [6,38] \sqrt{3} (\text{-}96) + [8,26] 72 + [8,17] \sqrt{3} (\text{-}72) + [8,41] \sqrt{2} (\text{-}144)

+ [24,26] \sqrt{3} 24 + [24,41] \sqrt{6} (\text{-}48) + [7,18] 72 + [6,26] \sqrt{6} (\text{-}24)

+ [6,17] \sqrt{2} (\text{-}72) + [6,41] \sqrt{3} 96 + [7,36] \sqrt{2} (\text{-}48) + [7,19] 192

+ [4,10] (\text{-}144) + [4,31] 144 + [22,31] 144 + [3,9] \sqrt{6} (\text{-}24)

+ [3,30] \sqrt{6} (24) + [21,30] \sqrt{2} (\text{-}72) + [2,8] \sqrt{6} (24) + [2,29] \sqrt{6} (\text{-}24)

+ [20,29] \sqrt{6} (\text{-}72) + [1,7] 144 + [1,28] (\text{-}144) + [3,25] \sqrt{6} 72

+ [3,16] \sqrt{6} (72) + [21,25] \sqrt{2} 72 + [2,24] \sqrt{2} 72 + [2,15] \sqrt{2} (\text{-}72)

+ [20,24] \sqrt{2} (\text{-}72) + [1,23] (\text{-}144) + [1,14] 144 + [2,35] \sqrt{3} (\text{-}96)

+ [2,40] \sqrt{3} 96 + [15,20] \sqrt{2} 72 + [16,21] \sqrt{2} (\text{-}72) + [30,34] \sqrt{3} (\text{-}96)

+ [17,27] \sqrt{2} 72 + [18,28] (\text{-}72) + [17,29] \sqrt{3} 72 + [18,23] (-72)

+ [17,24] (-72) + [33,35] \sqrt{6} (48) + [33,40] \sqrt{6} (\text{-}48)

+ [30,32] \sqrt{2} (\text{-}72) + [16,32] \sqrt{2} (\text{-}72) + [29,33] \sqrt{3} (24)

+ [15,33] (-72) + [27,33] \sqrt{2} (\text{-}72) + [15,26] \sqrt{3} 72 + [15,17] 72

+ [14,18] (72) + [28,36] \sqrt{2} (\text{-}144) \]
\[ T_r \left[ M T_{20}^+ M_t^{+ T_{2+1}} \right] = -T_r \left[ M T_{20}^+ M_t^{+ T_{2-1}} \right] = \frac{\sqrt{6}}{2^{4^2}} \cdot 2 \text{ Re } [P] \]

\[ P = \left[ [7,20] (216) + [9,22] (216) + [8,34] \sqrt{2} (-144) \right. \]

\[ + [11,27] \sqrt{6} (-72) + [12,28] \sqrt{6} (36) + [26,28] \sqrt{6} (-108) \]

\[ + [13,29] \sqrt{6} 72 + [12,30] \sqrt{6} (-36) + [26,30] \sqrt{6} (108) \]

\[ + [11,31] \sqrt{6} (-72) + [12,23] \sqrt{6} (-36) + [12,14] \sqrt{6} (-108) \]

\[ + [12,25] \sqrt{6} (-36) + [12,16] \sqrt{6} (-108) + [13,35] \sqrt{3} (-48) \]

\[ + [13,40] \sqrt{4} (-144) + [10,11] \sqrt{6} (-24) + [10,32] \sqrt{2} 72 \]

\[ + [10,37] \sqrt{3} 96 + [9,12] \sqrt{6} (-12) + [9,33] \sqrt{2} 36 + [9,38] \sqrt{3} 48 \]

\[ + [25,33] \sqrt{2} 108 + [25,38] \sqrt{3} 144 + [8,13] \sqrt{6} 24 + [8,39] \sqrt{3} (-96) \]

\[ + [35,39] \sqrt{6} 96 + [7,12] \sqrt{6} 12 + [7,33] 36 + [7,38] \sqrt{3} (-48) \]

\[ + [23,33] \sqrt{2} (-108) + [23,38] \sqrt{3} 144 + [6,11] \sqrt{6} (-24) \]

\[ + [6,32] \sqrt{2} (-72) + [6,37] \sqrt{3} 96 + [9,26] \sqrt{6} 36 + [9,17] \sqrt{2} (-108) \]

\[ + [9,41] \sqrt{3} (-144) + [25,26] \sqrt{6} 12 + [25,41] \sqrt{3} (-48) \]

\[ + [7,26] \sqrt{6} (-36) + [7,17] \sqrt{2} (-108) + [7,41] \sqrt{3} (144) \]

\[ + [23,26] \sqrt{6} 12 + [23,41] \sqrt{3} (-48) + [8,36] \sqrt{3} (-48) + [8,19] \sqrt{6} 96 \]

\[ + [5,10] (-144) + [5,31] 144 + [4,9] 72 + [4,30] (-72) \]


\[ + [20,28] (216) + [1,6] (-144) + [1,27] 144 + [4,25] 216 \]


\[ + [2,14] (216) + [20,23] 72 + [3,35] \sqrt{2} (-144) + [3,40] \sqrt{2} 144 \]

\[ + [14,20] (-72) + [15,21] 144 + [16,22] (-72) + [29,34] \sqrt{2} 144 \]

\[ + [17,28] \sqrt{2} 108 + [17,30] \sqrt{2} (108) + [17,23] \sqrt{2} 36 + [18,24] \sqrt{2} (-72) \]

\[ + [17,25] \sqrt{2} (-36) + [31,32] \sqrt{2} (-72) + [30,33] \sqrt{2} (-36) \]

\[ + [16,33] \sqrt{2} (-108) + [28,33] \sqrt{2} (-36) + [14,33] \sqrt{2} (108) \]

\[ + [27,32] \sqrt{2} (72) + [16,26] \sqrt{6} 36 + [16,17] \sqrt{2} (36) + [15,18] \sqrt{2} (72) \]

\[ + [14,26] \sqrt{6} 36 + [14,17] \sqrt{2} (-36) + [29,36] \sqrt{3} (-144) \]
\[ T_r \left[ M T_{2-1} \right] = T_r \left[ M T_{2+1} \right] = \frac{3}{2} \cdot 2 \Re \left[ \mathbf{P} \right] \]

\[ \mathbf{P} = \left[ [6,20] (144) + [7,21] \sqrt{2} (-72) + [8,22] \sqrt{6} (-72) \right] + [7,31] \sqrt{3} 96 + [11,28] \sqrt{6} (-72) + [12,29] (-72) + [26,29] (-216) + [13,30] 72 + [12,31] \sqrt{6} (-72) + [26,31] \sqrt{6} (72) + [11,23] \sqrt{6} (24) + [11,14] \sqrt{6} (72) + [12,24] \sqrt{3} (-24) + [12,15] \sqrt{3} (-72) + [13,25] (-72) + [13,16] (-216) + [12,35] \sqrt{2} 48 + [12,40] \sqrt{2} (144) + [10,12] \sqrt{6} (-24) + [10,33] \sqrt{2} 72 + [10,38] \sqrt{3} (96) + [9,13] 24 + [9,39] \sqrt{2} (-48) + [25,39] \sqrt{2} 144 + [8,12] (-24) + [8,33] \sqrt{3} (-24) + [8,38] \sqrt{2} 48 + [24,33] (-72) + [24,38] \sqrt{6} 48 + [35,38] (-192) + [7,11] \sqrt{6} (-24) + [7,32] \sqrt{2} (-72) + [7,37] \sqrt{3} 96 + [23,32] \sqrt{2} 72 + [23,37] \sqrt{3} (-96) + [10,26] \sqrt{6} 24 + [10,17] \sqrt{2} (-72) + [10,41] \sqrt{3} (-96) + [9,18] (-72) + [8,26] (-72) + [8,17] \sqrt{3} (-72) + [8,41] \sqrt{2} (144) + [24,26] \sqrt{3} 24 + [24,41] \sqrt{6} (-48) + [9,36] \sqrt{2} (-48) + [9,19] (192) + [5,9] 144 + [5,30] (-144) + [4,8] \sqrt{6} (24) + [4,29] \sqrt{6} (-24) + [22,29] \sqrt{6} 72 + [3,7] \sqrt{6} (-24) + [3,28] \sqrt{6} 24 + [21,28] \sqrt{2} (72) + [2,6] (-144) + [2,27] (144) + [20,27] (-144) + [5,25] 144 + [5,16] (-144) + [4,24] \sqrt{2} (-72) + [4,15] \sqrt{2} (72) + [22,24] \sqrt{2} (-72) + [3,23] \sqrt{6} (-72) + [3,14] \sqrt{6} 72 + [21,23] \sqrt{2} 72 + [4,35] \sqrt{3} (-96) + [4,40] \sqrt{3} 96 + [14,21] \sqrt{2} (-72) + [28,34] \sqrt{3} (-96) + [17,29] \sqrt{3} 72 + [18,30] 72 + [17,31] \sqrt{2} 72 + [17,24] 72 + [18,25] (-72) + [33,35] \sqrt{6} 48 + [33,40] \sqrt{6} (-48) + [31,33] \sqrt{2} (-72) + [29,33] \sqrt{3} 24 + [15,33] 72 + [28,32] \sqrt{2} 72 + [14,32] \sqrt{2} (-72) + [16,18] 72 + [15,26] \sqrt{3} 72 + [15,17] (-72) + [30,36] \sqrt{2} (-144) + [15,22] \sqrt{2} 72 \]
\[ T_r \left[ M \, T_{2-2}^+ \, M^+ T_{2+1} \right] =-T_r \left[ M \, T_{2-2}^+ \, M^+ T_{2-1} \right] = \frac{3}{2^4} \cdot 2 \, \text{Re} \, [P] \]

\[ P = [[[6,21]] \cdot (144) + [7,22] \cdot (144) + [6,34] \cdot \sqrt{3} \cdot (-96) + [11,29] \cdot (-144) + [12,30] \cdot \sqrt{6} \cdot (-72) + [26,30] \cdot \sqrt{6} \cdot (-72) + [13,31] \cdot (-144) + [11,24] \cdot \sqrt{3} \cdot 48 + [11,15] \cdot \sqrt{3} \cdot 144 + [12,25] \cdot \sqrt{6} \cdot 24 + [12,16] \cdot \sqrt{6} \cdot 72 + [11,35] \cdot \sqrt{2} \cdot (-48) + [11,40] \cdot \sqrt{2} \cdot (-144) + [10,13] \cdot \sqrt{6} \cdot (-8) + [10,39] \cdot \sqrt{2} \cdot 192 + [9,12] \cdot \sqrt{6} \cdot (-24) + [9,33] \cdot \sqrt{2} \cdot (-72) + [9,28] \cdot \sqrt{3} \cdot 96 + [25,33] \cdot \sqrt{2} \cdot 72 + [25,38] \cdot \sqrt{3} \cdot (-96) + [8,11] \cdot (-48) + [8,32] \cdot \sqrt{3} \cdot (-48) + [8,37] \cdot \sqrt{2} \cdot 96 + [24,32] \cdot 72 + [24,37] \cdot \sqrt{6} \cdot (-96) + [35,37] \cdot 192 + [10,18] \cdot (-144) + [9,26] \cdot \sqrt{6} \cdot (-24) + [9,17] \cdot \sqrt{2} \cdot (-72) + [9,41] \cdot \sqrt{3} \cdot (96) + [25,26] \cdot \sqrt{6} \cdot 24 + [25,41] \cdot \sqrt{3} \cdot (-96) + [10,36] \cdot \sqrt{2} \cdot (-48) + [10,19] \cdot 192 + [5,8] \cdot \sqrt{6} \cdot (-48) + [5,29] \cdot \sqrt{6} \cdot 48 + [4,7] \cdot (-144) + [4,28] \cdot (144) + [22,28] \cdot (-144) + [3,6] \cdot \sqrt{6} \cdot (-48) + [3,27] \cdot \sqrt{6} \cdot 48 + [21,27] \cdot \sqrt{2} \cdot (-144) + [5,24] \cdot \sqrt{2} \cdot (-144) + [5,15] \cdot \sqrt{2} \cdot (144) + [4,23] \cdot (-144) + [4,14] \cdot 144 + [22,23] \cdot (144) + [5,35] \cdot \sqrt{3} \cdot (-96) + [5,40] \cdot \sqrt{3} \cdot 96 + [14,22] \cdot (-144) + [27,34] \cdot \sqrt{3} \cdot 96 + [17,30] \cdot \sqrt{2} \cdot 72 + [18,31] \cdot 144 + [17,25] \cdot \sqrt{2} \cdot 72 + [32,35] \cdot \sqrt{6} \cdot (-48) + [32,40] \cdot \sqrt{6} \cdot 48 + [30,33] \cdot \sqrt{2} \cdot 72 + [16,33] \cdot \sqrt{2} \cdot (-72) + [29,32] \cdot \sqrt{3} \cdot 48 + [15,32] \cdot (-144) + [16,26] \cdot \sqrt{6} \cdot 72 + [16,17] \cdot \sqrt{2} \cdot (-72) + [31,36] \cdot \sqrt{2} \cdot (-144) \]
\[
T_\tau \left[ M T_{1+1}^+ M^T_{2+1} \right] = T_\tau \left[ M T_{1-1}^+ M^T_{2-1} \right] = \frac{3\sqrt{2}}{24^2} \cdot 2 \text{Im} [P]
\]

\[
P = \left[ [8,20] \sqrt{3} \, 2^4 + [9,21] \, 72 + [10,22] \sqrt{2} \, 72 + [25,34] \sqrt{6} \, 48 \right.
\]
\[
+ [12,27] \sqrt{3} \, 72 + [26,27] \sqrt{3} \, (-72) + [13,28] \sqrt{2} \, (108) \]
\[
+ [12,29] \sqrt{2} \, (-108) + [26,29] \sqrt{2} \, (-36) + [11,30] \sqrt{3} \, 72 \]
\[
+ [13,23] \sqrt{2} \, 12 + [13,14] \sqrt{2} \, 36 + [12,24] \sqrt{6} \, (-12) + [12,15] \sqrt{6} \, (-36) \]
\[
\[
+ [26,40] \, 144 + [9,11] \sqrt{3} \, (-24) + [9,32] \, (72) + [9,37] \sqrt{6} \, 48 \]
\[
+ [25,32] \, 72 + [25,37] \sqrt{6} \, 48 + [8,12] \sqrt{2} \, 36 + [8,33] \sqrt{6} \, (-36) \]
\[
+ [8,38] \, (-144) + [24,33] \sqrt{2} \, (-36) + [24,38] \sqrt{3} \, (-48) \]
\[
+ [7,13] \sqrt{2} \, (-36) + [7,39] \, (144) + [23,39] \, 48 + [6,12] \sqrt{3} \, (-24) \]
\[
+ [6,33] \, (-72) + [6,38] \sqrt{6} \, 48 + [8,26] \sqrt{2} \, 12 + [8,17] \sqrt{6} \, (-12) \]
\[
+ [8,41] \, (-48) + [24,26] \sqrt{6} \, (-12) + [24,41] \sqrt{3} \, 48 \]
\[
+ [35,41] \sqrt{2} \, 96 + [7,18] \sqrt{2} \, 36 + [6,26] \sqrt{3} \, 24 + [6,17] \, 72 \]
\[
+ [6,41] \sqrt{6} \, (-48) + [23,36] \, 48 + [4,10] \sqrt{2} \, (-72) + [4,31] \sqrt{2} \, 72 \]
\[
+ [22,31] \sqrt{2} \, 72 + [3,9] \sqrt{3} \, (-72) + [3,30] \sqrt{3} \, 72 + [21,30] \, 72 \]
\[
+ [2,8] \sqrt{3} \, (-72) + [2,29] \sqrt{3} \, 72 + [20,29] \sqrt{3} \, 24 + [1,7] \sqrt{2} \, (-72) \]
\[
+ [1,28] \sqrt{2} \, (72) + [3,25] \sqrt{3} \, 24 + [3,16] \sqrt{3} \, (-24) + [21,25] \, (-72) \]
\[
+ [20,24] \, (-72) + [2,24] \, 72 + [2,15] \, (-72) + [1,23] \sqrt{2} \, 72 \]
\[
+ [1,14] \sqrt{2} \, (-72) + [20,35] \sqrt{6} \, 48 + [20,40] \sqrt{6} \, (-48) + [15,20] \, (-72) \]
\[
+ [16,21] \, (-72) + [16,34] \sqrt{6} \, (-48) + [17,27] \, 72 + [18,28] \sqrt{2} \, 36 \]
\[
+ [17,29] \sqrt{6} \, (-12) + [18,23] \sqrt{2} \, (36) + [17,24] \sqrt{2} \, (-36) \]
\[
+ [17,35] \sqrt{3} \, (-48) + [17,40] \sqrt{3} \, 48 + [30,32] \, (-72) + \]
\[
+ [16,32] \, (-72) + [29,33] \sqrt{6} \, 36 + [15,33] \sqrt{2} \, 36 + [27,33] \, (+72) \]
\[
+ [15,26] \sqrt{6} \, (-36) + [15,17] \sqrt{2} \, (-36) + [14,18] \sqrt{2} \, 36 + [14,36] \, 144 \]
\[
+ [19,23] \sqrt{2} \, 64 \]
\[
T_{[M T_{10}^{+} M^{+} T_{241}^{+}]} = T_{[M T_{10}^{+} M^{+} T_{2-1}^{+}]} = \frac{3}{2^{4\frac{1}{2}}} \cdot 2 \quad \text{Im} [P]
\]

\[
P = [[7,20] \sqrt{2} (-72) + [8,21] \sqrt{6} (-48) + [9,22] \sqrt{2} (-72) + [24,34] \sqrt{3} (-96) + [11,27] \sqrt{3} (-144) + [12,28] \sqrt{3} (-72) + [26,28] \sqrt{3} (-72) + [12,30] \sqrt{3} (-72) + [26,30] \sqrt{3} (-72)]
\]

\[
+ [11,31] \sqrt{3} 144 + [12,23] \sqrt{3} (-24) + [12,14] \sqrt{3} (-72) + [13,24] (-48) + [13,15] (-144) + [12,25] \sqrt{3} 24 + [12,16] \sqrt{3} 72 + [10,11] \sqrt{3} (-48) + [10,32] 144 + [10,37] \sqrt{6} 96 + [9,12] \sqrt{3} 24 + [9,33] (-72) + [9,38] \sqrt{6} (-48) + [25,33] 72 + [25,38] \sqrt{6} 48 + [24,39] \sqrt{2} (-96) + [7,12] \sqrt{3} 24 + [7,33] 72 + [7,38] \sqrt{6} (-48) + [23,33] 72 + [23,38] \sqrt{6} (-48) + [6,11] \sqrt{3} 48 + [6,32] 144 + [6,37] \sqrt{6} (-96) + [9,26] \sqrt{3} 24 + [9,17] (-72) + [9,41] \sqrt{6} (-48) + [25,26] \sqrt{3} (-24) + [25,41] \sqrt{6} 48 + [8,18] \sqrt{3} 48 + [7,26] \sqrt{3} 24 + [7,17] 72 + [7,41] \sqrt{6} (-48) + [23,26] \sqrt{3} 24 + [23,41] \sqrt{6} (-48) + [24,36] \sqrt{2} 48 + [5,10] \sqrt{2} (-144) + [5,31] \sqrt{2} (144) + [4,9] \sqrt{2} (-72) + [4,30] \sqrt{2} 72 + [22,30] \sqrt{2} (-72) + [21,29] \sqrt{6} (-48) + [2,7] \sqrt{2} (72) + [2,28] \sqrt{2} (-72) + [20,28] \sqrt{2} (-72) + [1,6] \sqrt{2} (144) + [1,27] \sqrt{2} (-144) + [4,25] \sqrt{2} 72 + [4,16] \sqrt{2} (-72) + [22,25] \sqrt{2} (-72) + [3,24] \sqrt{6} 48 + [3,15] \sqrt{6} (-48) + [2,23] \sqrt{2} 72 + [2,14] \sqrt{2} (-72) + [20,23] \sqrt{2} 72 + [21,35] \sqrt{3} (96) + [21,40] \sqrt{3} (-96) + [16,22] \sqrt{2} (-72) + [15,34] \sqrt{3} 96 + [17,28] 72 + [18,29] \sqrt{3} 48 + [17,30] (-72) + [17,23] (-72) + [17,25] (-72) + [18,35] \sqrt{6} (-48) + [18,40] \sqrt{6} 48 + [31,32] (-144) + [30,33] 72 + [16,33] (-72) + [28,33] (-72) + [14,33] (-72) + [27,32] (-144) + [16,26] \sqrt{3} (-72) + [16,17] (-72) + [14,26] \sqrt{3} 72 + [14,17] (-72) + [15,36] \sqrt{2} 144 + [19,24] 192 + [14,20] \sqrt{2} 72]
\[ T_x \left[ M T_{1-1}^+ M T_{2+1}^+ \right] = T_x \left[ M T_{1+1}^+ M T_{2-1}^+ \right] = \frac{3\sqrt{2}}{24} \cdot \text{Im } [P] \]

\[
\begin{align*}
\text{P} &= \left[ [6, 20] \sqrt{2} \right] (72) + [7, 21] \sqrt{3} 24 + [23, 34] \sqrt{6} 48 \\
&\quad + [11, 28] \sqrt{3} (-72) + [12, 29] \sqrt{2} (-108) + [26, 29] \sqrt{2} (-36) \\
&\quad + [13, 30] \sqrt{2} (-108) + [12, 31] \sqrt{3} 72 + [26, 31] \sqrt{3} (-72) \\
&\quad + [13, 25] \sqrt{2} 12 + [13, 16] \sqrt{2} (+36) + [26, 35] \sqrt{4} 8 + [26, 40] \sqrt{144} \\
&\quad + [10, 12] \sqrt{3} (-24) + [10, 33] 72 + [10, 38] \sqrt{6} 48 + [9, 13] \sqrt{2} 36 \\
&\quad + [9, 39] (-144) + [25, 39] \sqrt{4} 8 + [8, 12] \sqrt{2} 36 + [8, 33] \sqrt{6} 36 \\
&\quad + [8, 38] (-144) + [24, 38] \sqrt{3} 48 + [7, 11] \sqrt{3} 24 + [7, 32] 72 \\
&\quad + [7, 37] \sqrt{6} (-48) + [23, 32] (-72) + [23, 37] \sqrt{6} 48 + [10, 26] \sqrt{3} 24 \\
&\quad + [10, 17] (-72) + [10, 41] \sqrt{6} (-48) + [9, 18] \sqrt{2} 36 + [8, 26] \sqrt{2} 12 \\
&\quad + [8, 17] \sqrt{6} 12 + [8, 41] (-48) + [24, 26] \sqrt{6} 12 + [24, 41] \sqrt{3} (-48) \\
&\quad + [35, 41] \sqrt{2} 96 + [25, 36] \sqrt{4} 48 + [5, 9] \sqrt{2} 72 + [5, 30] \sqrt{2} (-72) \\
&\quad + [4, 8] \sqrt{3} (72) + [4, 29] \sqrt{3} (-72) + [22, 29] \sqrt{3} 24 + [3, 7] \sqrt{3} 72 \\
&\quad + [3, 28] \sqrt{3} (-72) + [21, 28] 72 + [2, 6] \sqrt{2} 72 + [2, 27] \sqrt{2} (-72) \\
&\quad + [20, 27] \sqrt{2} 72 + [5, 25] \sqrt{2} 72 + [5, 16] \sqrt{2} (-72) + [4, 24] 72 \\
&\quad + [4, 15] (-72) + [22, 24] 72 + [3, 23] \sqrt{3} 24 + [3, 14] \sqrt{3} (-24) \\
&\quad + [21, 23] 72 + [22, 35] \sqrt{6} 48 + [22, 40] \sqrt{6} (-48) + [14, 21] 72 \\
&\quad + [15, 22] 72 + [14, 34] \sqrt{6} (-48) + [17, 29] \sqrt{6} 12 + [18, 30] \sqrt{2} 36 \\
&\quad + [17, 31] (-72) + [17, 24] \sqrt{2} (-36) + [18, 25] \sqrt{2} (-36) \\
&\quad + [17, 35] \sqrt{3} 48 + [17, 40] \sqrt{3} (-48) + [31, 33] (-72) + [29, 33] \sqrt{6} (-36) \\
&\quad + [15, 33] \sqrt{2} 36 + [28, 32] (-72) + [14, 32] 72 + [16, 18] \sqrt{2} (-36) \\
&\quad + [15, 26] \sqrt{6} (+36) + [15, 17] \sqrt{2} (-36) + [16, 36] \sqrt{144} \\
&\quad + [19, 25] \sqrt{2} 96 + [24, 33] \sqrt{2} (-36) 
\end{align*}
\]
\[ T_r [M T_{00}^+ M_{T_{2+1}}] = T_r [M T_{00}^+ M_{T_{2-1}}] = \frac{\sqrt{3}}{24^2} \cdot 2 \text{Re} [\mathcal{P}] \]

\[ \mathcal{P} = [[[11,27]] \sqrt{6} (-144) + [12,28] \sqrt{6} (-144) + [13,29] \sqrt{6} (-144) \]
\[ + [12,30] \sqrt{6} (144) + [11,31] \sqrt{6} (-144) + [36,40] \sqrt{6} (-144) \]
\[ + [10,11] 6 (-48) + [10,32] \sqrt{2} (144) + [10,37] \sqrt{3} (192) \]
\[ + [9,12] \sqrt{6} (48) + [9,33] \sqrt{2} (-144) + [9,38] \sqrt{3} (-192) \]
\[ + [8,13] \sqrt{6} (-48) + [8,39] \sqrt{3} (192) + [7,12] \sqrt{6} (-48) \]
\[ + [7,33] \sqrt{2} (-144) + [7,38] \sqrt{3} (192) + [6,11] \sqrt{6} (-48) \]
\[ + [6,32] \sqrt{2} (-144) + [6,37] \sqrt{3} (192) + [25,26] \sqrt{6} (-48) \]
\[ + [25,41] \sqrt{3} 192 + [23,26] \sqrt{6} (-48) + [23,41] \sqrt{3} (192) \]
\[ + [35,36] \sqrt{6} (-48) + [5,10] (-288) + [5,31] (288) + [4,9] (-288) \]
\[ + [21,24] (-288) + [20,23] (-288) + [34,35] (-288) + [34,40] (288) \]
\[ + [14,20] 288 + [15,21] 288 + [16,22] (288) + [17,23] \sqrt{2} (-144) \]
\[ + [18,24] \sqrt{2} (-144) + [17,25] \sqrt{2} (144) + [31,32] \sqrt{2} (-144) \]
\[ + [30,33] \sqrt{2} (144) + [28,33] \sqrt{2} (144) + [27,32] \sqrt{2} 144 \]
\[ + [16,26] \sqrt{6} (-144) + [16,17] \sqrt{2} (-144) + [15,18] \sqrt{2} (144) \]
\[ + [14,26] \sqrt{6} (-144) + [14,17] \sqrt{2} (144) + [19,35] \sqrt{3} 192 ] \]
\[
T_x [M T_{2+2}^+ N_{20}^+] = T_x [M T_{2-2}^+ N_{20}^+] = \frac{\sqrt{6}}{24^2} \cdot 2 \text{Re} [P]
\]

\[
\]
\[
+ [24,27] /2 (-216) + [9,28] 216 + [25,28] (-216) + [10,29] /6 72
\]
\[
\]
\[
\]
\[
+ [12,12] 144/2 + [12,38] /2 (-144) + [26,38] /2 (-144)
\]
\[
\]
\[
+ [26,26] 144/2 + [26,41] /2 (-144) + [11,36] /3 (-96)
\]
\[
\]
\[
+ [7,9] (-72) + [7,30] 216 + [6,8] /6 (-24) + [6,29] /6 (72)
\]
\[
\]
\[
+ [6,15] /2 (-216) + [6,35] /3 (-48) + [6,40] /3 (144) + [3,5] /6 48
\]
\[
+ [2,4] 144 + [1,3] /6 8 + [2,22] (-144) + [20,22] (-144)
\]
\[
+ [1,21] /2 (-144) + [1,34] /3 96 + [15,27] /2 72 + [16,28] 72
\]
\[
+ [16,23] (-216) + [31,35] /3 144 + [31,40] /3 (-48)
\]
\[
+ [18,32] /2 (-144) + [33,33] 144/2 + [17,33] 144 + [17,17] 144/2
\]
\[
+ [29,31] /6 (-24) + [15,31] /2 (-72) + [28,30] (-72) + [14,30] (-72)
\]
\[
+ [27,29] /6 (-24) + [14,25] (-216) + [14,16] 72 + [27,35] /3 (144)
\]
\[
+ [27,40] /3 (-48) + [23,30] 216
\]
\[ T \left[ M \frac{T^+}{2+1} M^+ T_{20} \right] = T \left[ M \frac{T^+}{2-1} M^+ T_{20} \right] = \frac{\sqrt{6}}{24^2} \cdot 2 \text{ Re } [P] \]

\[ P = \left[ [3,20] \sqrt{6} (-72) + [4,21] \sqrt{2} (-72) + [5,22] \frac{144}{\sqrt{6}} + [4,34] \sqrt{3} (-96) \right. \\
+ [7,27] (-216) + [23,27] (216) + [8,28] \sqrt{6} (-36) + [24,28] \sqrt{2} (-108) \\
+ [9,29] \sqrt{6} 36 + [25,29] \sqrt{6} (-108) + [10,30] (216) + [8,23] \sqrt{6} 36 \\
+ [8,14] \sqrt{6} (-108) + [9,24] \sqrt{2} 36 + [9,15] \sqrt{2} (-108) + [10,25] (-72) \\
+ [10,16] 216 + [9,35] \sqrt{3} 48 + [9,40] \sqrt{3} (-144) + [12,37] \sqrt{2} (-144) \\
+ [26,37] \sqrt{2} 144 + [13,38] \sqrt{3} (-48) + [36,38] \sqrt{6} 96 + [12,13] \sqrt{6} 24 \\
+ [12,39] \sqrt{3} (-48) + [26,39] \sqrt{3} (-144) + [11,12] 144 \\
+ [11,38] \sqrt{2} (-144) + [13,26] \sqrt{6} 72 + [13,41] \sqrt{3} (-144) \\
+ [11,26] (-144) + [11,41] \sqrt{2} (144) + [12,36] \sqrt{3} (-96) \\
+ [12,19] \sqrt{6} 96 + [9,10] (-72) + [9,31] (216) + [25,31] (216) \\
+ [8,9] \sqrt{6} (-12) + [8,30] \sqrt{6} 36 + [24,30] \sqrt{2} (-108) + [7,8] \sqrt{6} 12 \\
+ [7,29] \sqrt{6} (-36) + [23,29] \sqrt{6} (-108) + [6,7] 72 + [6,28] (-216) \\
+ [7,15] \sqrt{2} (-108) + [23,24] \sqrt{2} (-36) + [6,23] (-72) + [6,14] (216) \\
+ [7,35] \sqrt{3} (-148) + [7,40] \sqrt{3} (144) + [4,5] 144 + [3,4] \sqrt{6} 24 \\
+ [2,3] \sqrt{6} (-24) + [1,2] (-144) + [3,22] \sqrt{6} (-72) + [21,22] \sqrt{2} (-72) \\
+ [2,21] \sqrt{2} (-72) + [20,21] \sqrt{2} 72 + [1,20] 144 + [2,34] \sqrt{3} (96) \\
+ [14,27] (-72) + [15,28] \sqrt{2} 36 + [16,29] \sqrt{6} 36 + [15,23] \sqrt{2} (108) \\
+ [16,24] \sqrt{2} (-108) + [30,35] \sqrt{3} (-144) + [30,40] \sqrt{3} 48 \\
+ [17,32] 144 + [18,33] \sqrt{2} (-72) + [32,33] (-144) + [17,18] \sqrt{2} 72 \\
+ [30,31] (-72) + [29,30] \sqrt{6} (-12) + [15,30] \sqrt{2} 36 + [28,29] \sqrt{6} 12 \\
+ [14,29] \sqrt{6} 36 + [27,28] 72 + [15,25] \sqrt{2} (-108) + [15,16] \sqrt{2} 36 \\
+ [14,21] \sqrt{2} (108) + [14,15] \sqrt{2} (-36) + [28,35] \sqrt{3} 144 \\
+ [28,40] \sqrt{3} (-48) + [16,31] (-72)]
\[ T_r \left[ M T_{20}^+ M^+ T_{20} \right] = T_r \left[ M T_{20}^+ M^+ T_{20} \right] = \frac{2}{2t^2} \cdot 2 \text{Re} \left[ \mathbf{P} \right] \]

\[ \mathbf{P} = \left[ \left[ 13,13 \right] 72 + \left[ 13,39 \right] \sqrt{2} \left( -144 \right) + \left[ 36,39 \right] 288 + \left[ 12,12 \right] 72 \right. \]
\[ + \left[ 12,38 \right] \sqrt{2} \left( -144 \right) + \left[ 26,38 \right] \sqrt{2} \left( 432 \right) + \left[ 11,11 \right] \left( -144 \right) \]
\[ + \left[ 11,37 \right] \sqrt{2} 288 + \left[ 12,26 \right] \left( -432 \right) + \left[ 12,41 \right] \sqrt{2} \left( 432 \right) + \left[ 26,26 \right] \left( 72 \right) \]
\[ + \left[ 26,41 \right] \sqrt{2} \left( -144 \right) + \left[ 13,36 \right] \sqrt{2} \left( -144 \right) + \left[ 13,19 \right] 288 + \left[ 10,10 \right] \left( -72 \right) \]
\[ + \left[ 10,31 \right] \left( 432 \right) + \left[ 9,9 \right] \left( 36 \right) + \left[ 9,30 \right] \left( -216 \right) + \left[ 25,30 \right] \left( -648 \right) \]
\[ + \left[ 8,8 \right] 72 + \left[ 8,29 \right] \left( -432 \right) + \left[ 7,7 \right] 36 + \left[ 7,28 \right] \left( -216 \right) \]
\[ + \left[ 23,28 \right] 684 + \left[ 6,6 \right] \left( -72 \right) + \left[ 6,27 \right] \left( 432 \right) + \left[ 9,25 \right] \left( 216 \right) \]
\[ + \left[ 9,16 \right] \left( -648 \right) + \left[ 25,25 \right] 36 + \left[ 24,24 \right] \left( -72 \right) + \left[ 7,23 \right] \left( -216 \right) \]
\[ + \left[ 7,14 \right] \left( 648 \right) + \left[ 23,23 \right] 36 + \left[ 8,35 \right] \sqrt{2} \left( -144 \right) + \left[ 8,40 \right] \sqrt{2} \left( 432 \right) \]
\[ + \left[ 5,5 \right] 144 + \left[ 4,4 \right] \left( -72 \right) + \left[ 3,3 \right] \left( -144 \right) + \left[ 2,2 \right] \left( -72 \right) + \left[ 1,1 \right] \left( 144 \right) \]
\[ + \left[ 2,22 \right] \left( -432 \right) + \left[ 22,22 \right] \left( -72 \right) + \left[ 21,21 \right] 144 + \left[ 2,20 \right] 432 \]
\[ + \left[ 20,20 \right] \left( -72 \right) + \left[ 3,34 \right] \sqrt{2} 288 + \left[ 27,27 \right] \left( -72 \right) + \left[ 28,28 \right] 36 \]
\[ + \left[ 14,28 \right] \left( -216 \right) + \left[ 29,29 \right] 72 + \left[ 30,30 \right] 36 + \left[ 16,30 \right] 216 \]
\[ + \left[ 31,31 \right] \left( -72 \right) + \left[ 14,23 \right] \left( -216 \right) + \left[ 14,14 \right] 36 + \left[ 15,24 \right] 432 \]
\[ + \left[ 15,15 \right] \left( -72 \right) + \left[ 16,25 \right] \left( -216 \right) + \left[ 16,16 \right] 36 + \left[ 29,35 \right] \sqrt{2} \left( 432 \right) \]
\[ + \left[ 29,40 \right] \sqrt{2} \left( -144 \right) + \left[ 32,32 \right] 144 + \left[ 33,33 \right] \left( -72 \right) + \left[ 17,33 \right] 432 \]
\[ + \left[ 18,18 \right] 72 + \left[ 17,17 \right] \left( -72 \right) \]
$$T_r \left[ M \ T_{11}^* \ M^+ \ T_{20} \right] = T_r \left[ M \ T_{1-1}^* \ M^+ \ T_{20} \right] = \frac{2 \sqrt{3}}{24} \cdot 2 \text{ Im} \left[ P \right]$$

$$P = \left[ [3,20] \sqrt{3} (-24) + [4,21] (-72) + [5,22] \sqrt{2} (-72) \right. + [22,3^4] \sqrt{6} (-48) + [7,27] \sqrt{2} (-108) + [23,27] \sqrt{2} (108) + [8,28] \sqrt{3} (-108) + [2^4,28] (108) + [9,29] \sqrt{3} (-108) + [25,29] \sqrt{3} 36 + [10,30] \sqrt{2} (-108) + [8,23] \sqrt{3} 12 + [8,1^4] \sqrt{3} (-36) + [9,2^4] (36) + [9,1^5] (-108) + [10,2^5] \sqrt{2} (36) + [10,1^6] \sqrt{2} (-108) + [25,3^5] \sqrt{6} (2^4) + [2^5,4^0] \sqrt{6} (-72) + [12,3^7] (-1^44) + [26,3^7] (1^44) + [13,3^8] \sqrt{6} (-72) + [12,1^3] \sqrt{3} (-72) + [12,3^9] \sqrt{6} 72 + [26,3^9] \sqrt{6} 2^4 + [11,1^2] \sqrt{2} (-72) + [11,3^8] 1^44 + [13,2^6] \sqrt{3} (2^4) + [13,1^4] \sqrt{6} (-2^4) + [3^6,4^1] \sqrt{3} 96 + [11,2^6] \sqrt{2} 72 + [11,1^4] (-1^44) + [26,3^6] \sqrt{6} 4^8 + [9,1^0] \sqrt{2} (-36) + [9,3^1] \sqrt{2} (108) + [25,3^1] \sqrt{2} (108) + [8,9] \sqrt{3} (-36) + [8,3^0] \sqrt{3} 108 + [2^4,3^0] 108 + [7,8] \sqrt{3} (-36) + [7,2^9] \sqrt{3} 108 + [23,2^9] \sqrt{3} (36) + [6,7] \sqrt{2} (-36) + [6,2^8] \sqrt{2} (108) + [8,2^5] \sqrt{3} 12 + [8,1^6] \sqrt{3} (-36) + [2^4,2^5] (-36) + [7,2^4] (36) + [7,1^5] (-108) + [23,2^4] (-36) + [6,2^3] \sqrt{2} 36 + [6,1^4] \sqrt{2} (-108) + [23,3^5] \sqrt{6} 2^4 + [2^3,4^0] \sqrt{6} (-72) + [4,5] \sqrt{2} 72 + [3,1^4] \sqrt{3} 72 + [2,3] \sqrt{3} 72 + [1,1^2] \sqrt{2} 72 + [3,2^2] \sqrt{3} (-2^4) + [21,2^2] 72 + [2,2^1] (-72) + [20,2^1] 72 + [1,2^0] \sqrt{2} (-72) + [20,3^4] \sqrt{6} (-4^8) + [1^4,2^7] \sqrt{2} (-36) + [15,2^8] (-36) + [16,2^9] \sqrt{3} (-1^2) + [15,2^3] (-108) + [16,2^4] (-108) + [16,3^5] \sqrt{6} (-72) + [16,4^0] \sqrt{6} 2^4 + [18,3^3] 72 + [32,3^3] \sqrt{2} 72 + [17,1^8] 72 + [17,3^2] \sqrt{2} 72 + [30,3^1] \sqrt{2} (-36) + [16,3^1] \sqrt{2} (-36) + [29,3^0] \sqrt{3} (-36) + [15,3^0] (-36) + [28,2^9] \sqrt{3} (-36) + [1^4,2^9] \sqrt{3} (-1^2) + [27,2^8] \sqrt{2} (-36) + [15,2^5] (108) + [15,1^6] (-36) + [1^4,2^4] (108) + [1^4,1^5] (-36) + [1^4,3^5] \sqrt{6} (-72) + [1^4,4^0] \sqrt{6} (2^4) + [19,2^6] \sqrt{3} 96 ]$$
\[ T^\frac{\text{r}}{\text{r}} \left[ M^\frac{\text{T}_0^0 \text{M}^+^\text{T}_0^0} \right] = T^\frac{\text{r}}{\text{r}} \left[ M^\frac{\text{T}_0^0 \text{M}^+^\text{T}_0^0} \right] = \frac{\sqrt{2}}{24^2} \cdot 2 \text{Re} \left[ \mathbf{P} \right] \]

\[
\[ T_\tau [M T_{2+2}^t M_{1+1}^t] = T_\tau [M T_{2-2}^t M_{1-1}^t] = \frac{3 \sqrt{2}}{2} \cdot 2 \quad \text{Im} \{P\} \]

\[ P = [[9,20]/2 (-72) + [10,21] (-144) + [10,34] /6 (-48) \]
\[ + [13,27]/2 (-24) + [12,28]/3 24 + [26,28]/3 24 + [11,29]/2 (-24) \]
\[ + [12,23]/3 (72) + [12,14]/3 24 + [11,24]/6 (-72) + [11,15]/6 (-24) \]
\[ + [11,35] (-144) + [11,40] (-48) + [8,11]/2 72 + [8,32]/6 24 \]
\[ + [24,32]/2 72 + [7,12]/3 (-72) + [7,33]/2 (-72) + [23,23]/2 (-72) \]
\[ + [6,13]/2 72 + [7,26]/3 (-72) + [7,17]/2 (-72) + [23,26]/3 (-72) \]
\[ + [6,18]/2 (72) + [6,36] 144 + [3,10]/3 48 + [3,31]/3 (-48) \]
\[ + [21,31]/4 (-144) + [2,9]/2 72 + [2,30]/2 (-72) + [20,30]/2 (-72) \]
\[ + [1,8]/3 48 + [1,29]/3 (-48) + [2,25]/2 (-72) + [2,16]/2 72 \]
\[ + [20,25]/2 (-72) + [1,24]/2 (-144) + [1,15]/4 144 + [1,35]/6 48 \]
\[ + [1,40]/6 (-48) + [16,20]/2 (-72) + [31,34]/6 48 + [18,27]/2 72 \]
\[ + [17,28]/(-72) + [17,23]/2 + [32,35]/3 (-48) + [32,40]/3 48 \]
\[ + [29,32]/6 (-24) + [29,37]/2 (-96) + [15,32]/2 (-72) \]
\[ + [15,37]/3 (-96) + [28,33]/2 + [28,38]/6 48 + [14,33]/72 \]
\[ + [14,38]/6 48 + [27,39]/(-96) + [28,41]/6 48 + [14,26]/3 (-24) \]
\[ + [14,17]/2 + [14,41]/6 48 + [27,36]/48 + [19,27]/2 96 \]
\[ + [37,40]/2 96 \]
\[ \mathcal{T} \frac{M}{2+1} \frac{M^t T_{1+1}}{\mathcal{T}} = -\mathcal{T} \frac{M}{2-1} \frac{M^t T_{1-1}}{\mathcal{T}} = \frac{3\sqrt{2}}{24} \cdot 2 \text{Im} [P] \]

\[ P = [[8,20] \sqrt{3} 72 + [9,21] 72 + [10,22] \sqrt{2} (-72) + [9,34] \sqrt{6} (48) \]
+ [12,27] \sqrt{3} 24 + [26,27] \sqrt{3} (-24) + [13,28] \sqrt{2} 12 + [12,29] \sqrt{2} 12
+ [26,29] \sqrt{2} 36 + [11,30] \sqrt{3} (-24) + [13,23] \sqrt{2} 108 + [13,14] \sqrt{2} 36
+ [12,24] \sqrt{6} (-36) + [12,15] \sqrt{6} (-12) + [11,25] \sqrt{3} (-72)
+ [11,16] \sqrt{3} (-24) + [12,35] (-144) + [12,40] (-48) + [9,11] \sqrt{3} 72
+ [9,32] 72 + [25,32] 72 + [8,12] \sqrt{2} (-36) + [8,33] \sqrt{6} (-12)
+ [24,33] 36 + [7,13] \sqrt{2} (-36) + [6,12] \sqrt{3} (-72) + [6,33] 72
+ [8,26] \sqrt{2} (-108) + [8,17] \sqrt{6} (-36) + [24,26] \sqrt{6} (-36)
+ [7,18] \sqrt{2} (+36) + [6,26] \sqrt{3} 72 + [6,17] (-72) + [7,36] (144)
+ [4,10] \sqrt{2} (72) + [4,31] \sqrt{2} (-72) +
+ [22,31] \sqrt{2} (-72) + [3,9] \sqrt{3} 24 + [3,30] \sqrt{3} (-24) + [21,30] 72
+ [2,8] \sqrt{3} (-24) + [2,29] \sqrt{3} 24 + [20,29] \sqrt{3} 72 + [1,7] \sqrt{2} (-72)
+ [1,28] \sqrt{2} 72 + [3,25] \sqrt{3} (-72) + [3,16] \sqrt{3} 72 + [21,25] (-72)
+ [1,14] \sqrt{2} (-72) + [2,35] \sqrt{6} 48 + [2,40] \sqrt{6} (-48) + [15,20] 72
+ [33,35] \sqrt{3} (-48) + [33,40] \sqrt{3} (+48) + [30,32] (-72)
+ [30,37] \sqrt{6} (-48) + [16,32] (-72) + [16,37] \sqrt{6} (-48) + [29,33] \sqrt{6} 12
+ [29,38] 48 + [15,33] \sqrt{2} (-36) + [15,38] \sqrt{3} (-48) + [28,39] 48
+ [14,39] 144 + [27,33] (-72) + [27,38] \sqrt{6} 48 + [29,41] 144
+ [15,26] \sqrt{6} (-12) + [15,17] \sqrt{2} 36 + [15,41] \sqrt{3} 48 + [14,18] \sqrt{2} 36
+ [27,41] \sqrt{6} (-48) + [28,36] 48 + [19,28] \sqrt{2} 96 + [38,40] \sqrt{2} 96
+ [16,21] (-72) + [30,34] \sqrt{6} (-48) + [17,27] (-72) + [18,28] \sqrt{2} 36
+ [17,29] \sqrt{6} (-36) + [18,23] \sqrt{2} 36 + [17,24] \sqrt{2} 36]
\[ T_r \left[ M T_{20}^{+} M^{+} T_{1+1}^{-1} \right] = T_r \left[ M T_{20}^{+} M^{+} T_{1-1}^{-1} \right] = \frac{2\sqrt{3}}{24} \cdot 2 \quad \text{Im} [P] \]

\[ P = \left[ [7,20] \sqrt{2} (-108) + [9,22] \sqrt{2} (108) + [8,34] (-144) \right. \]
\[ + [11,27] \sqrt{3} (-24) + [12,28] \sqrt{3} (12) + [26,28] \sqrt{3} (-36) \]
\[ + [13,29] \sqrt{3} (24) + [12,30] \sqrt{3} (-12) + [26,30] \sqrt{3} 36 \]
\[ + [11,31] \sqrt{3} (-24) + [12,23] \sqrt{3} (-108) + [12,14] \sqrt{3} (-36) \]
\[ + [12,25] \sqrt{3} (-108) + [12,16] \sqrt{3} (-36) + [13,35] \sqrt{6} (-72) \]
\[ + [13,40] \sqrt{6} (-24) + [10,11] \sqrt{3} 72 + [10,32] 72 + [9,12] \sqrt{3} 36 \]
\[ + [9,33] 36 + [25,33] (108) + [8,13] \sqrt{3} (-72) + [7,12] \sqrt{3} (-36) \]
\[ + [7,33] 36 + [23,33] (-108) + [6,11] \sqrt{3} (72) + [6,32] (-72) \]
\[ + [9,26] \sqrt{3} (-108) + [9,17] (-108) + [25,26] \sqrt{3} (-36) \]
\[ + [7,26] \sqrt{3} (108) + [7,17] (-108) + [23,26] \sqrt{3} (-36) + [8,36] \sqrt{6} 72 \]
\[ + [5,10] \sqrt{2} 72 + [5,31] \sqrt{2} (-72) + [4,9] \sqrt{2} (-36) + [4,30] \sqrt{2} 36 \]
\[ + [22,30] \sqrt{2} (108) + [3,8] \sqrt{2} (-72) + [3,29] \sqrt{2} 72 + [2,7] \sqrt{2} (-36) \]
\[ + [2,28] \sqrt{2} 36 + [20,28] \sqrt{2} (-108) + [1,6] \sqrt{2} 72 + [1,27] \sqrt{2} (-72) \]
\[ + [4,25] \sqrt{2} (-108) + [4,16] \sqrt{2} (108) + [22,25] \sqrt{2} (-36) \]
\[ + [21,24] \sqrt{2} (72) + [2,23] \sqrt{2} (108) + [2,14] \sqrt{2} (-108) \]
\[ + [20,23] \sqrt{2} (-36) + [3,35] \sqrt{4} + [3,40] (-144) + [14,20] \sqrt{2} (-36) \]
\[ + [15,21] \sqrt{2} 72 + [16,22] \sqrt{2} (-36) + [29,34] \sqrt{4} + [17,28] (-108) \]
\[ + [31,32] (-72) + [31,37] \sqrt{6} (-48) + [30,33] (-36) + [30,38] \sqrt{6} (-24) \]
\[ + [16,33] (-108) + [16,38] \sqrt{6} (-72) + [29,39] \sqrt{6} h8 + [28,33] (-36) \]
\[ + [28,38] \sqrt{6} 24 + [14,33] (108) + [14,38] \sqrt{6} (-72) + [27,32] 72 \]
\[ + [27,37] \sqrt{6} (-48) + [30,41] \sqrt{6} 72 + [16,26] \sqrt{3} (-12) + [16,17] 36 \]
\[ + [16,41] \sqrt{6} 2h + [15,18] 72 + [28,41] \sqrt{6} (-72) + [14,26] \sqrt{3} (-12) \]
\[ + [4,17] (-36) + [14,41] \sqrt{6} 24 + [29,36] \sqrt{6} (24) + [19,29] \sqrt{3} 96 \]
\[ + [39,40] \sqrt{3} 96 \]
\[ T_R \begin{bmatrix} M T_{2-1}^i & N T_{1+1}^j \end{bmatrix} = -T_R \begin{bmatrix} M T_{2+1}^j & N T_{1-1}^j \end{bmatrix} = \frac{\sqrt{2}}{24} \cdot 2 \quad \text{Im} [P] \]

\[ P = \begin{bmatrix} [6.20] \sqrt{2} 72 + [7.21] (-72) + [8.22] \sqrt{3} (-72) + [7.34] \sqrt{6} 148 \\
+ [11.28] \sqrt{3} (-24) + [12.29] \sqrt{2} (-12) + [26.29] \sqrt{2} (-36) \\
+ [13.30] \sqrt{2} (12) + [12.31] \sqrt{3} (-24) + [26.31] \sqrt{3} 24 + [11.32] \sqrt{3} 72 \\
+ [13.25] \sqrt{2} (-108) + [13.35] \sqrt{2} (-36) + [12.35] (144) + [12.40] 48 \\
+ [8.33] \sqrt{6} (-12) + [24,33] \sqrt{2} (-36) + [7.14] \sqrt{3} 72 + [7.32] (-72) \\
+ [23.22] 72 + [10.26] \sqrt{3} (-72) + [10.17] (-72) + [9.15] \sqrt{2} (-36) \\
+ [8.26] \sqrt{2} (+108) + [8.17] \sqrt{6} (-36) + [24,26] \sqrt{6} (-36) + [9.36] 144 \\
+ [5.9] \sqrt{2} (-72) + [5.30] \sqrt{2} 72 + [4.8] \sqrt{3} (-24) + [4.29] \sqrt{3} 24 \\
+ [22.29] \sqrt{3} (-72) + [3.7] \sqrt{3} 24 + [3.28] \sqrt{3} (-24) + [21.28] (-72) \\
+ [2.6] \sqrt{2} (72) + [2.27] \sqrt{2} (-72) + [20.27] \sqrt{2} (+72) + [5.25] \sqrt{2} (-72) \\
+ [3.14] \sqrt{3} (-72) + [21.23] (-72) + [4.35] \sqrt{6} 48 + [4.40] \sqrt{6} (-48) \\
+ [14,21] (-72) + [15.22] 72 + [28,34] \sqrt{6} (-48) + [17.29] \sqrt{6} (-36) \\
+ [18.30] \sqrt{2} (-36) + [17.31] (-72) + [17.24] \sqrt{2} (-36) + [18.25] \sqrt{2} 36 \\
+ [33,35] \sqrt{3} (-48) + [33,40] \sqrt{3} (+48) + [31,33] (-72) \\
+ [31,38] \sqrt{6} (-48) + [30.39] 48 + [16,39] (-144) + [29.33] \sqrt{6} 12 \\
+ [29,38] (-48) + [15.33] \sqrt{2} 36 + [15.38] \sqrt{3} (-48) + [28,32] 72 \\
+ [28,37] \sqrt{6} (-48) + [14,32] (-72) + [14.37] \sqrt{6} 48 + [31,41] \sqrt{6} 48 \\
+ [16,18] \sqrt{2} 36 + [29,44] (-144) + [15,26] \sqrt{6} (-12) \\
+ [15,17] \sqrt{2} (-36) + [15,41] \sqrt{3} 48 + [30,36] 48 + [19,30] \sqrt{2} 96 \\
+ [38,40] \sqrt{2} (-96) \]
\[ T_r [M T_{2-2}^+ M_{1+1}^+] = T_r [M T_{2+2}^+ M_{1-1}^+] = \frac{3 \sqrt{2}}{2^4} \cdot 2 \cdot \text{Im}[P] \]

\[ P = [[6,21] 96 + [7,22] \sqrt{2} 72 + [6,31] \sqrt{6} (-48) + [11,29] \sqrt{2} (-24) \]
\[ + [12,30] \sqrt{3} (-24) + [26,30] \sqrt{3} (-24) + [13,31] \sqrt{2} (-24) \]
\[ + [11,35] (-144) + [11,40] (-48) + [10,13] \sqrt{2} 72 + [9,12] \sqrt{3} 72 \]
\[ + [9,33] (-72) + [25,33] 72 + [8,11] \sqrt{2} 72 + [8,32] \sqrt{6} (-24) \]
\[ + [24,32] \sqrt{2} 72 + [10,18] \sqrt{2} (-72) + [9,26] \sqrt{3} 72 + [9,17] (-72) \]
\[ + [25,26] \sqrt{3} (-72) + [10,36] 144 + [5,8] \sqrt{3} 48 + [5,29] \sqrt{3} (-48) \]
\[ + [4,7] \sqrt{2} 72 + [4,28] \sqrt{2} (-72) + [22,28] \sqrt{2} 72 + [3,6] \sqrt{3} 48 \]
\[ + [4,23] \sqrt{2} 72 + [4,14] \sqrt{2} (-72) + [22,23] \sqrt{2} (-72) + [5,35] \sqrt{6} 48 \]
\[ + [5,40] \sqrt{6} (-48) + [14,22] \sqrt{2} (-72) + [27,34] \sqrt{6} 48 + [17,30] (-72) \]
\[ + [18,31] \sqrt{2} (-72) + [17,25] (-72) + [32,35] \sqrt{3} 48 + [32,40] \sqrt{3} (-48) \]
\[ + [31,39] (-96) + [30,33] 72 + [30,38] \sqrt{6} (-48) + [16,33] (-72) \]
\[ + [16,38] \sqrt{6} 48 + [29,32] \sqrt{6} 24 + [29,37] (-96) + [15,32] \sqrt{2} (-72) \]
\[ + [15,37] \sqrt{3} 96 + [30,41] \sqrt{6} (-48) + [16,26] \sqrt{3} (-24) + [16,17] (-72) \]
\[ + [16,41] \sqrt{6} 48 + [31,36] 48 + [9,31] \sqrt{2} 96 + [37,40] \sqrt{2} 96 ] \]
\[ T_\tau [ M^{T_{l+1}} M^{T_{l+1}}] = T_\tau [ M^{T_{l-1}} M^{T_{l-1}}] = \frac{6}{24^2} \cdot 2 \text{ Re } [\mathbf{p}] \]

\[ \mathbf{p} = \left[ [8,20] \right) \sqrt{6} \ 12 + [9,21] \sqrt{2} \ 36 + [10,22] \ 72 + [25,34] \sqrt{3} \ 48 \]

\[ + [12,27] \sqrt{6} \ 12 + [26,27] \sqrt{6} \ (-12) + [13,28] \ 36 + [12,29] \ (-36) \]

\[ + [26,29] \ (-12) + [11,30] \sqrt{6} \ 12 + [13,23] \ 36 + [13,14] \ 12 \]

\[ + [12,24] \sqrt{3} \ (-36) + [12,15] \sqrt{3} \ (-12) + [11,25] \sqrt{6} \ (36) \]

\[ + [11,16] \sqrt{6} \ (12) + [26,35] \sqrt{2} \ 72 + [26,40] \sqrt{2} \ 24 + [9,11] \sqrt{6} \ 36 \]

\[ + [9,32] \sqrt{2} \ 36 + [25,32] \sqrt{2} \ 36 + [8,12] \ (-108) + [8,33] \sqrt{3} \ (-36) \]

\[ + [24,33] \ (-36) + [7,13] \ (108) + [6,12] \sqrt{6} \ 36 + [6,33] \sqrt{2} \ (-36) \]

\[ + [8,26] \ (-36) + [8,17] \sqrt{3} \ (-12) + [24,26] \sqrt{3} \ 36 + [7,18] \ 36 \]

\[ + [6,26] \sqrt{6} \ (-36) + [6,17] \sqrt{2} \ 36 + [23,36] \sqrt{2} \ (-72) + [4,10] \ 72 \]

\[ + [4,31] \ (-72) + [22,31] \ (-72) + [3,9] \sqrt{6} \ 36 + [3,30] \sqrt{6} \ (-36) \]

\[ + [21,30] \sqrt{2} \ (-36) + [2,8] \sqrt{6} \ 36 + [2,29] \sqrt{6} \ (-36) + [20,29] \sqrt{6} \ (-12) \]

\[ + [1,7] \ 72 + [1,28] \ (-72) + [3,25] \sqrt{6} \ (-12) + [3,16] \sqrt{6} \ 12 \]

\[ + [21,25] \sqrt{2} \ 36 + [2,24] \sqrt{2} \ (-36) + [2,15] \sqrt{2} \ 36 + [20,24] \sqrt{2} \ (36) \]

\[ + [1,23] \ (-72) + [1,14] \ 72 + [20,35] \sqrt{3} \ (-48) + [20,40] \sqrt{3} \ 48 \]

\[ + [15,20] \sqrt{2} \ (-36) + [16,21] \sqrt{2} \ (-36) + [16,34] \sqrt{3} \ (-48) \]

\[ + [17,27] \sqrt{2} \ (-36) + [18,28] \ (-36) + [17,29] \sqrt{3} \ 12 + [18,23] \ (-36) \]

\[ + [17,24] \ 36 + [17,35] \sqrt{6} \ 24 + [17,40] \sqrt{6} \ (-24) + [30,32] \sqrt{2} \ (-36) \]

\[ + [30,37] \sqrt{3} \ (-48) + [16,32] \sqrt{2} \ (-36) + [16,37] \sqrt{3} \ (-48) \]

\[ + [29,33] \sqrt{3} \ 36 + [29,38] \sqrt{2} \ 72 + [15,33] \ 36 + [15,38] \sqrt{6} \ 24 \]

\[ + [28,39] \sqrt{2} \ (-72) + [14,39] \sqrt{2} \ (-24) + [27,33] \sqrt{2} \ 36 \]

\[ + [27,38] \sqrt{3} \ (-18) + [29,41] \sqrt{2} \ 24 + [15,26] \sqrt{3} \ 12 + [15,17] \ (-36) \]

\[ + [15,41] \sqrt{6} \ (-24) + [40,41] \sqrt{6} \ (-96) + [14,18] \ 36 + [27,41] \sqrt{3} \ 48 \]

\[ + [14,36] \sqrt{2} \ (-24) + [14,19] \ 96 \]
\[
T_{r} \left[ M^{+}_{10} M^{+}_{11} \right] = -T_{r} \left[ M^{+}_{10} M^{+}_{1-1} \right] = \frac{3}{24^2} \cdot 2 \text{Re} \left[ \mathbf{p} \right]
\]

\[
\mathbf{p} = \left[ [7,20] (-72) + [8,21] \sqrt{3} (-48) + [9,22] (-72) + [24,34] \sqrt{6} (-48) \right.
\]
\[+ [11,27] \sqrt{6} (-24) + [12,28] \sqrt{6} (-12) + [26,28] \sqrt{6} (-12) \]
\[+ [12,30] \sqrt{6} (-12) + [26,30] \sqrt{6} (-12) + [11,31] \sqrt{6} (24) \]
\[+ [12,23] \sqrt{6} (-36) + [12,14] \sqrt{6} (-12) + [13,24] \sqrt{2} (-72) \]
\[+ [13,15] \sqrt{2} (-24) + [12,25] \sqrt{6} (36) + [12,16] \sqrt{6} (36) + [10,11] \sqrt{6} (36) \]
\[+ [10,32] \sqrt{2} 72 + [9,12] \sqrt{6} (-36) + [9,33] \sqrt{2} (-36) \]
\[+ [25,33] \sqrt{2} 36 + [7,12] \sqrt{6} (-36) + [7,33] \sqrt{2} 36 + [23,33] \sqrt{2} 36 \]
\[+ [6,11] \sqrt{6} (-72) + [6,32] \sqrt{2} 72 + [9,26] \sqrt{6} (-36) + [9,17] \sqrt{2} (-36) \]
\[+ [25,26] \sqrt{6} 36 + [8,18] \sqrt{6} 24 + [7,26] \sqrt{6} 36 + [7,17] \sqrt{2} 36 \]
\[+ [23,26] \sqrt{6} (-36) + [24,36] (-144) + [5,10] 144 + [5,31] (-144) \]
\[+ [4,16] 72 + [22,25] 72 + [3,24] \sqrt{3} (-144) \]
\[+ [3,15] \sqrt{3} 48 + [2,23] (-72) + [2,14] (+72) + [20,23] (-72) \]
\[+ [21,35] \sqrt{6} (-144) + [21,40] \sqrt{6} 48 + [14,20] 72 + [16,22] (-72) \]
\[+ [15,31] \sqrt{6} 48 + [17,28] \sqrt{2} (-36) + [18,29] \sqrt{6} (-24) + [17,30] \sqrt{2} 36 \]
\[+ [17,23] \sqrt{2} 36 + [17,25] \sqrt{2} 36 + [18,35] \sqrt{3} 48 + [18,40] \sqrt{3} (-48) \]
\[+ [31,32] \sqrt{2} (-72) + [31,37] \sqrt{3} (-96) + [30,33] \sqrt{2} 36 \]
\[+ [30,38] \sqrt{3} 48 + [16,33] \sqrt{2} (-36) + [16,38] \sqrt{3} (-148) + [15,39] 96 \]
\[+ [28,33] \sqrt{2} (-36) + [28,38] \sqrt{3} 48 + [14,33] \sqrt{2} (-36) \]
\[+ [14,38] \sqrt{3} 48 + [27,32] \sqrt{2} (-72) + [27,37] \sqrt{3} 96 + [30,41] \sqrt{3} 48 \]
\[+ [16,26] \sqrt{6} 12 + [16,17] \sqrt{2} (-36) + [16,41] \sqrt{3} (-148) \]
\[+ [28,41] \sqrt{3} 48 + [14,26] \sqrt{6} (-12) + [14,17] \sqrt{2} (-36) \]
\[+ [14,41] \sqrt{3} 48 + [15,36] (-148) + [15,19] \sqrt{2} 96 \]
\[ T_\tau \left[ \mathbf{M} T_{1-1}^{\tau} \right] = T_\tau \left[ \mathbf{M} T_{1+1}^{\tau} \right] = \frac{6}{2^{4/2}} \cdot 2 \text{Re} \left[ T \right] \]

\[ P = \left[ (16,20) \right] 72 + \left[ (7,21) \right] \sqrt{2} \cdot 36 + \left[ (8,22) \right] \sqrt{6} \cdot (12) + \left[ (23,34) \right] \sqrt{3} \cdot 48 + \left[ (11,28) \right] \sqrt{6} \cdot (-12) + \left[ (12,29) \right] \sqrt{-36} + \left[ (26,29) \right] \sqrt{-12} + \left[ (13,30) \right] \sqrt{-36} + \left[ (12,31) \right] \sqrt{6} \cdot 12 + \left[ (26,31) \right] \sqrt{-12} + \left[ (11,23) \right] \sqrt{6} \cdot 12 + \left[ (12,24) \right] \sqrt{3} \cdot 36 + \left[ (12,15) \right] \sqrt{3} \cdot 12 + \left[ (13,25) \right] 36 + \left[ (13,16) \right] 12 + \left[ (26,35) \right] \sqrt{2} \cdot 72 + \left[ (26,40) \right] \sqrt{2} \cdot 24 + \left[ (10,12) \right] \sqrt{6} \cdot 36 + \left[ (10,33) \right] \sqrt{2} \cdot 36 + \left[ (9,13) \right] (-108) + \left[ (8,12) \right] (-108) + \left[ (8,33) \right] \sqrt{3} \cdot 36 + \left[ (24,33) \right] \sqrt{-36} + \left[ (7,11) \right] \sqrt{6} \cdot (-36) + \left[ (7,32) \right] \sqrt{2} \cdot 36 + \left[ (23,32) \right] \sqrt{2} \cdot (-36) + \left[ (10,26) \right] \sqrt{6} \cdot (-36) + \left[ (10,17) \right] \sqrt{2} \cdot (-36) + \left[ (9,18) \right] 36 + \left[ (8,26) \right] \sqrt{6} \cdot 36 + \left[ (24,26) \right] \sqrt{3} \cdot (-36) + \left[ (25,36) \right] \sqrt{2} \cdot (-72) + \left[ (5,9) \right] \sqrt{2} \cdot (-72) + \left[ (4,8) \right] \sqrt{6} \cdot (-36) + \left[ (4,29) \right] \sqrt{6} \cdot 36 + \left[ (22,29) \right] \sqrt{6} \cdot (-12) + \left[ (3,7) \right] \sqrt{6} \cdot (-36) + \left[ (3,28) \right] \sqrt{6} \cdot 36 + \left[ (21,28) \right] \sqrt{2} \cdot (-36) + \left[ (2,6) \right] \sqrt{2} \cdot (-72) + \left[ (2,27) \right] 72 + \left[ (20,27) \right] (-72) + \left[ (5,25) \right] (-72) + \left[ (5,16) \right] 72 + \left[ (4,24) \right] \sqrt{2} \cdot (-36) + \left[ (4,15) \right] \sqrt{2} \cdot 36 + \left[ (22,24) \right] \sqrt{2} \cdot (-36) + \left[ (3,23) \right] \sqrt{6} \cdot (-12) + \left[ (3,14) \right] \sqrt{6} \cdot (12) + \left[ (21,23) \right] \sqrt{2} \cdot (-36) + \left[ (22,35) \right] \sqrt{3} \cdot (-48) + \left[ (22,40) \right] \sqrt{3} \cdot 48 + \left[ (14,21) \right] \sqrt{2} \cdot 36 + \left[ (15,22) \right] \sqrt{2} \cdot 36 + \left[ (14,34) \right] \sqrt{3} \cdot (-12) + \left[ (17,29) \right] \sqrt{3} \cdot (-12) + \left[ (18,30) \right] \sqrt{2} \cdot (-36) + \left[ (17,31) \right] \sqrt{2} \cdot 36 + \left[ (17,24) \right] 36 + \left[ (18,25) \right] 36 + \left[ (17,35) \right] \sqrt{6} \cdot (-24) + \left[ (17,40) \right] \sqrt{6} \cdot 24 + \left[ (31,33) \right] \sqrt{2} \cdot (-36) + \left[ (31,38) \right] \sqrt{3} \cdot (-48) + \left[ (30,39) \right] \sqrt{2} \cdot 72 + \left[ (16,39) \right] \sqrt{2} \cdot (-24) + \left[ (29,33) \right] \sqrt{3} \cdot (-36) + \left[ (29,38) \right] \sqrt{2} \cdot 72 + \left[ (15,33) \right] 36 + \left[ (15,38) \right] \sqrt{6} \cdot (-24) + \left[ (28,32) \right] \sqrt{2} \cdot (-36) + \left[ (28,37) \right] \sqrt{3} \cdot 48 + \left[ (14,32) \right] \sqrt{2} \cdot 36 + \left[ (14,37) \right] \sqrt{3} \cdot (-48) + \left[ (31,41) \right] \sqrt{3} \cdot 48 + \left[ (16,18) \right] \sqrt{6} \cdot (-36) + \left[ (29,11) \right] \sqrt{2} \cdot 24 + \left[ (15,26) \right] \sqrt{3} \cdot (-12) + \left[ (15,17) \right] \sqrt{6} \cdot (-12) + \left[ (15,41) \right] \sqrt{2} \cdot (24) + \left[ (40,41) \right] \sqrt{2} \cdot (-96) + \left[ (16,36) \right] \sqrt{2} \cdot (-24) + \left[ (16,19) \right] 96 + \left[ (5,30) \right] 72 \right] \]
\[ T_r [M T^0 \; M^+ T_{1+1}] = T_r [M T^0 \; M^+ T_{1-1}] = \frac{\sqrt{6}}{24^2} \cdot 2 \text{ Im } [P] \]

\[ P = \left[ [11,27] \sqrt{3} (-48) + [12,28] \sqrt{3} (-48) + [13,29] \sqrt{3} (-48) \\
+ [12,30] \sqrt{3} (144) + [11,31] \sqrt{3} (-48) + [36,40] \sqrt{3} (-48) \\
+ [10,11] \sqrt{3} (144) + [10,32] (144) + [9,12] \sqrt{3} (-144) + [9,33] (-144) \\
+ [8,13] \sqrt{3} (144) + [7,12] \sqrt{3} (144) + [7,33] (-144) + [6,11] \sqrt{3} (144) \\
+ [6,32] (-144) + [25,26] \sqrt{3} (144) + [23,26] \sqrt{3} (144) + [35,36] \sqrt{3} (144) \\
+ [5,10] \sqrt{2} (144) + [5,31] \sqrt{2} (144) + [4,9] \sqrt{2} (144) \\
+ [4,30] \sqrt{2} (-144) + [3,8] \sqrt{2} (144) + [3,29] \sqrt{2} (-144) + [2,7] \sqrt{2} (144) \\
+ [2,28] \sqrt{2} (-144) + [1,6] \sqrt{2} (144) + [1,27] \sqrt{2} (-144) \\
+ [22,25] \sqrt{2} (144) + [21,24] \sqrt{2} (144) + [20,23] \sqrt{2} (144) \\
+ [34,35] \sqrt{2} (144) + [34,40] \sqrt{2} (-144) + [14,20] \sqrt{2} (144) \\
+ [15,21] \sqrt{2} (144) + [16,22] \sqrt{2} (144) + [17,23] (144) + [18,24] (144) \\
+ [17,25] (-144) + [31,32] (-144) + [31,37] \sqrt{6} (-96) + [30,33] (144) \\
+ [30,38] \sqrt{6} (96) + [29,39] \sqrt{6} (-96) + [28,33] (144) + [28,38] \sqrt{6} (-96) \\
+ [27,32] (144) + [27,37] \sqrt{6} (-96) + [16,26] \sqrt{3} (48) + [16,17] (-144) \\
+ [16,41] \sqrt{6} (-96) + [15,18] (144) + [14,26] \sqrt{3} (48) + [14,17] (144) \\
+ [14,41] \sqrt{6} (-96) + [19,40] \sqrt{6} (96) \]
\[ T \cdot [M T_{22}^+ M_{10}^+] = -T \cdot [M T_{2-2}^+ M_{10}^+] = \frac{3}{24^2} \cdot 2 \text{Im} \{P\} \]

\[ P = [(4,20) \sqrt{2} 144 + [5,21] 288 + [5,34] \sqrt{6} 96 + [8,27] \sqrt{3} 48 + [24,27] (-144) + [9,28] \sqrt{2} 72 + [25,28] \sqrt{2} (-72) + [10,29] \sqrt{3} 48 + [9,23] \sqrt{2} 72 + [9,14] \sqrt{2} 72 + [10,24] 144 + [10,15] 144 + [10,35] \sqrt{6} 48 + [10,40] \sqrt{6} 48 + [13,32] 96 + [12,33] \sqrt{6} (-48) + [26,33] \sqrt{6} (-48) + [12,17] \sqrt{6} (-48) + [11,18] \sqrt{3} 96 + [8,10] \sqrt{3} (-48) + [8,31] \sqrt{3} (-96) + [24,31] (-144) + [7,9] \sqrt{2} (-72) + [7,30] \sqrt{2} (-72) + [23,30] \sqrt{2} (-72) + [6,8] \sqrt{3} (-48) + [6,29] \sqrt{3} (-48) + [7,25] \sqrt{2} 72 + [7,16] \sqrt{2} 72 + [23,25] \sqrt{2} 72 + [6,24] 144 + [6,15] 144 + [6,35] \sqrt{6} (-48) + [6,40] \sqrt{6} (-48) + [3,5] \sqrt{3} (-96) + [2,4] \sqrt{2} (-144) + [1,3] \sqrt{3} (-96) + [2,22] \sqrt{2} 144 + [20,22] \sqrt{2} 144 + [1,21] 288 + [1,34] \sqrt{6} (-96) + [15,27] (-144) + [16,28] \sqrt{2} (-72) + [16,23] \sqrt{2} (-72) + [31,35] \sqrt{6} 48 + [31,40] \sqrt{6} 48 + [18,37] \sqrt{6} (-96) + [33,38] \sqrt{3} 96 + [17,38] \sqrt{3} 96 + [32,39] \sqrt{2} (-96) + [33,41] \sqrt{3} 96 + [17,26] \sqrt{6} 48 + [17,41] \sqrt{3} 96 + [32,36] \sqrt{2} (-96) + [29,31] \sqrt{3} (-48) + [15,31] (-144) + [28,30] \sqrt{2} (-72) + [14,30] \sqrt{2} (-72) + [27,29] \sqrt{3} (-48) + [14,25] \sqrt{2} (72) + [14,16] \sqrt{2} 72 + [27,35] \sqrt{6} (-48) + [27,40] \sqrt{6} (-48) + [19,32] 192 ] \]
\[ T_r \left[ M T_{21}^+ M^+ T_{10} \right] = T_r \left[ M T_{21}^+ M^+ T_{10} \right] = \frac{3}{244} \cdot 2 \text{ Im } [P] \]

\[ P = \left[ [3,20] \sqrt{3} (-144) + [4,21] (-144) + [5,22] /2 144 \right. \]

\[ + [6,33] \sqrt{6} (-96) + [7,27] \sqrt{2} (-72) + [23,27] \sqrt{2} 72 + [8,28] \sqrt{3} (-24) \]

\[ + [24,28] (-72) + [9,29] \sqrt{3} 24 + [25,29] \sqrt{3} (-72) + [10,30] \sqrt{2} 72 \]

\[ + [8,23] \sqrt{3} (-72) + [8,14] \sqrt{3} (-72) + [9,24] (-72) + [9,15] (-72) \]

\[ + [10,25] \sqrt{2} 72 + [10,16] \sqrt{2} 72 + [9,35] \sqrt{6} (-48) + [9,40] \sqrt{6} (-48) \]

\[ + [12,32] \sqrt{6} (-48) + [26,32] \sqrt{6} 48 + [13,33] (-48) + [11,33] \sqrt{6} 48 \]

\[ + [13,17] (-144) + [12,18] \sqrt{3} 48 + [11,17] \sqrt{6} (-48) + [9,10] \sqrt{2} (-72) \]

\[ + [9,31] \sqrt{2} (-72) + [25,31] \sqrt{2} (-72) + [8,9] \sqrt{3} (-24) \]

\[ + [8,30] \sqrt{3} (-24) + [24,30] 72 + [7,8] \sqrt{3} 24 + [7,29] \sqrt{3} 24 \]

\[ + [23,29] \sqrt{2} 72 + [6,7] \sqrt{2} 72 + [6,28] \sqrt{2} 72 + [8,25] \sqrt{3} 72 \]


\[ + [6,23] \sqrt{2} (-72) + [6,14] \sqrt{2} (-72) + [7,35] \sqrt{6} (-48) \]

\[ + [7,40] \sqrt{6} (-48) + [4,5] \sqrt{2} (-144) + [3,4] \sqrt{3} (-48) + [2,3] \sqrt{3} 48 \]

\[ + [1,2] \sqrt{2} 144 + [3,22] \sqrt{3} 144 + [21,22] 144 + [2,21] 144 \]

\[ + [20,21] (-144) + [1,20] \sqrt{2} (-144) + [2,31] \sqrt{6} (-96) \]

\[ + [14,27] \sqrt{2} 72 + [15,28] (-72) + [16,29] \sqrt{3} (-72) + [15,23] 72 \]

\[ + [16,24] (-72) + [30,35] \sqrt{6} (-48) + [30,40] \sqrt{6} (-48) \]

\[ + [33,37] \sqrt{3} (-96) + [17,37] \sqrt{3} 96 + [18,38] \sqrt{6} (-48) \]

\[ + [33,39] \sqrt{2} 48 + [17,39] \sqrt{2} 144 + [32,38] \sqrt{3} 96 + [18,26] \sqrt{3} 48 \]

\[ + [18,41] \sqrt{6} 48 + [32,41] \sqrt{3} (-96) + [33,36] \sqrt{2} (-96) \]

\[ + [30,31] \sqrt{2} (-72) + [16,31] \sqrt{2} (-72) + [29,30] \sqrt{3} (-24) \]

\[ + [15,30] 72 + [28,29] \sqrt{3} 24 + [14,29] \sqrt{3} 72 + [27,28] \sqrt{2} 72 \]


\[ + [28,35] \sqrt{6} (-48) + [28,40] \sqrt{6} (-48) + [19,33] 192 ] \]
\[ T_\tau [M T_{11}^\dagger M^T T_{10}^\dagger] = -T_\tau [M T_{11}^\dagger M^T T_{10}^\dagger] = \frac{3\sqrt{2}}{24^2} \cdot 2 \Re [P] \]

\[ P = \left[ \begin{array}{c}
[1,2] (-1^4) & + [3,22] / 6 & 2^4 & + [21,22] / 2 (-72) & + [2,2^1] / 2 & 72 \\
[16,35] / 3 & (-48) & + [16,40] / 3 & (-48) & + [33,37] / 6 (-48) \\
[17,37] / 6 & 48 & + [18,38] / 3 & 48 & + [33,39] (-1^4) & + [17,39] (-48) \\
[32,38] / 6 & (-48) & + [18,2^6] / 6 & (-2^4) & + [18,4^1] / 3 (-48) \\
[28,29] / 6 (-36) & + [1^4,2^9] / 6 & (-12) & + [27,28] (-72) \\
[1^4,15] / 2 (-36) & + [1^4,35] / 3 & 48 & + [1^4,4^0] / 3 & 48 \end{array} \right] \]
\[ T_T[M^T_{10} M^T_{10}] = T_T[M^T_{10} M^T_{10}] = \frac{-3}{2t^2} \cdot 2 \Re \left[ \rho \right] \]

\[ P = \left[ [1,1]288 + [2,2] \frac{144}{14} + [4,4] (-\frac{144}{144}) + [5,5] (-\frac{288}{144}) + [2,20] 288 + [20,20] \frac{144}{14} + [3,21] \frac{\sqrt{3}}{192} + [4,22]] 288 + [22,22] \left(-\frac{144}{144}\right) \right. \]
\[ + [21,31] \sqrt{6}(192) + [6,6] \frac{144}{144} + [6,27] 288 + [7,7] 72 + [7,28] \frac{144}{144} \]
\[ + [23,28] \frac{144}{144} + [24,29] \sqrt{3} 96 + [9,9] (-72) + [9,30] \left(-\frac{144}{144}\right) \]
\[ + [25,30] \frac{144}{144} + [10,10] \left(-\frac{144}{144}\right) + [10,31] \left(-\frac{288}{144}\right) + [7,23] \frac{144}{144} \]
\[ + [23,23] 72 + [8,24] \sqrt{3} 96 + [8,15] \sqrt{3} 96 + [9,25] \frac{144}{144} + [9,16] \frac{144}{144} \]
\[ + [25,25] (-72) + [24,35] \sqrt{6} 96 + [24,40] \sqrt{6} 96 + [11,32] \sqrt{3} 96 \]
\[ + [12,33] \sqrt{3} 48 + [26,33] \sqrt{3} 48 + [12,33] \sqrt{3} 48 + [26,33] \sqrt{3} 48 \]
\[ + [11,32] \sqrt{3} 96 + [12,17] \sqrt{3} 48 + [13,18] 96 + [12,17] \sqrt{3} 48 \]
\[ + [27,27] \frac{144}{144} + [28,28] 72 + [14,28] \frac{144}{144} + [14,29] \sqrt{3} 96 \]
\[ + [30,30] (-72) + [16,30] \frac{144}{144} + [31,31] \left(-\frac{144}{144}\right) + [14,23] \frac{144}{144} \]
\[ + [14,14] 72 + [16,25] \left(-\frac{144}{144}\right) + [16,16] (-72) + [15,35] \sqrt{6} 96 \]
\[ + [15,40] \sqrt{6} 96 + [32,37] \sqrt{6} 96 + [33,38] \sqrt{6} 48 + [17,38] \sqrt{6} 48 \]
\[ + [18,39] \sqrt{2} 96 + [33,38] \sqrt{6} 48 + [17,38] \sqrt{6} 48 + [32,37] \sqrt{6} 96 \]
\[ + [33,41] \sqrt{6} 48 + [17,26] \sqrt{3} 48 + [17,41] \sqrt{6} 48 + [33,41] \sqrt{6} 48 \]
\[ + [17,26] \sqrt{3} 48 + [17,41] \sqrt{6} 48 + [18,36] \sqrt{2} 96 + [18,19] 192 \]
\[ + [7,14] \frac{144}{144} \]
\[ T^*_x \{ M^* T_{22}^+ M^*_2 T_{00} \} = T^*_x \{ M^* T_{2-2}^+ M^*_0 T_{00} \} = \frac{\sqrt{3}}{24^2} \cdot 2 \, \text{Re} \, [P] \]

\[ P = \left[ [4, 20] \frac{288}{2} + [5, 21] \frac{\sqrt{2}}{2} 288 + [5, 34] \frac{\sqrt{3}}{3} 192 + [9, 23] \frac{238}{2} \right. \]

\[ + [10, 24] \frac{\sqrt{2}}{2} 288 + [10, 35] \frac{\sqrt{3}}{3} 192 + [12, 12] \frac{(-144)}{2} + [11, 13] \frac{\sqrt{6}}{6} 96 \]

\[ + [12, 26] \frac{(-288)}{2} + [26, 26] \frac{(-144)}{2} + [11, 36] \frac{\sqrt{3}}{3} 192 + [8, 10] \frac{\sqrt{6}}{6} 96 \]

\[ + [7, 9] \frac{288}{2} + [6, 8] \frac{\sqrt{6}}{6} 96 + [7, 25] \frac{(-288)}{2} + [23, 25] \frac{(-288)}{2} \]

\[ + [6, 24] \frac{\sqrt{2}}{2} \frac{(-288)}{2} + [6, 35] \frac{\sqrt{3}}{3} 192 + [3, 5] \frac{\sqrt{6}}{6} 96 + [2, 4] \frac{288}{2} \]

\[ + [1, 3] \frac{\sqrt{6}}{6} 144 + [2, 22] \frac{(-288)}{2} + [20, 22] \frac{(-288)}{2} + [1, 21] \frac{\sqrt{2}}{2} \frac{(-288)}{2} \]

\[ + [1, 34] \frac{\sqrt{3}}{3} 192 + [15, 27] \frac{\sqrt{2}}{2} \frac{(-288)}{2} + [16, 28] \frac{(-288)}{2} \]

\[ + [31, 40] \frac{\sqrt{3}}{3} 192 + [18, 32] \frac{\sqrt{2}}{2} \frac{(-288)}{2} + [33, 33] \frac{(-144)}{2} + [17, 33] \frac{288}{2} \]

\[ + [17, 17] \frac{(-144)}{2} + [29, 31] \frac{\sqrt{6}}{6} 96 + [15, 31] \frac{\sqrt{2}}{2} 288 + [28, 30] \frac{288}{2} \]

\[ + [14, 30] \frac{288}{2} + [27, 29] \frac{\sqrt{6}}{6} 96 + [14, 16] \frac{(-288)}{2} + [27, 40] \frac{\sqrt{3}}{3} 192 \]

\[ + [19, 37] \frac{\sqrt{3}}{3} 192 + [38, 38] \frac{(-144)}{2} + [37, 39] \frac{\sqrt{6}}{6} 96 + [38, 41] \frac{(-288)}{2} \]

\[ + [41, 41] \frac{(-144)}{2} \]
\[
T_x \left[ M^+ T_{2+1} T_{00}^+ \right] = -T_x \left[ M^+ T_{2-1} T_{00}^+ \right] = \frac{\sqrt{3}}{24} \cdot 2 \text{Re} \left[ P \right]
\]

\[
P = \left[ [3,20] \sqrt{6} (-144) + [4,21] \sqrt{2} (-144) + [5,22] 288 + [4,34] \sqrt{3} (-192) + [8,23] \sqrt{6} (-144) + [9,24] \sqrt{2} (-144) + [10,25] 288 + [9,35] \sqrt{3} (-192) + [12,13] \sqrt{6} (-144) + [11,12] (-288) + [13,26] \sqrt{6} (-144) + [11,26] (288) + [12,36] \sqrt{3} (192) + [9,10] (288) + [8,9] \sqrt{6} 48 + [7,8] \sqrt{6} (-48) + [6,7] (-288) + [23,24] \sqrt{2} 144 + [6,23] 288 + [7,35] \sqrt{3} (192) + [4,5] 288 + [3,4] \sqrt{6} (48) + [2,3] \sqrt{6} (-48) + [1,2] (-288) + [3,22] \sqrt{6} (-144) + [21,22] \sqrt{2} (-144) + [21,21] \sqrt{2} (-144) + [20,21] \sqrt{2} (144) + [1,20] 288 + [2,34] \sqrt{3} 192 + [14,27] 288 + [15,28] \sqrt{2} (-144) + [16,29] \sqrt{6} (-144) + [30,40] \sqrt{3} (-192) + [17,32] 288 + [18,33] \sqrt{2} (-144) + [32,33] (-288) + [17,18] \sqrt{2} 144 + [30,31] 288 + [16,31] (288) + [29,30] \sqrt{6} 48 + [15,30] \sqrt{2} (-144) + [28,29] \sqrt{6} (-48) + [14,29] \sqrt{6} (-144) + [27,28] (-288) + [15,16] \sqrt{2} (-144) + [14,15] \sqrt{2} (144) + [28,40] \sqrt{3} 192 + [19,38] \sqrt{3} 192 + [38,39] \sqrt{6} (-144) + [37,38] (-288) + [39,41] \sqrt{6} (-144) + [37,41] 288 \right]
\]
\[
T_x [M T_{20}^+ M^+ T_{00}] = T_x [M T_{20}^+ M^+ T_{00}] = \sqrt{n} / 2 \cdot 2 \text{Re} [P]
\]

\[
P = \{(1,1) 288 + (2,2) (-144) + (3,3) (-288) + (4,4) (-144) +
+ (5,5) 288 + (2,20) 864 + (20,20) (-144) + (21,21) (864) +
+ (4,22) (-864) + (22,22) (-144) + (3,34) \sqrt{2} 288 + (6,6) 288 +
+ (7,7) (-144) + (8,8) (-288) + (9,9) (-144) + (10,10) 288 +
+ (7,23) (864) + (23,23) (-144) + (24,24) 288 + (9,25) (-864) +
+ (25,25) (-144) + (8,35) \sqrt{2} (576) + (11,11) 144 + (12,12) (-72) +
+ (13,13) (-144) + (12,12) (-72) + (11,11) 144 + (12,26) 432 +
+ (26,26) (-72) + (12,26) 432 + (26,26) (-72) + (13,36) \sqrt{2} 288 +
+ (27,27) 288 + (28,28) (-144) + (14,28) (864) + (29,29) (-288) +
+ (30,30) (-144) + (16,30) (-864) + (31,31) 288 + (14,14) (-144) +
+ (15,15) 288 + (16,16) (-144) + (29,40) \sqrt{2} (576) + (32,32) 144 +
+ (33,33) (-72) + (17,33) 432 + (33,33) (-72) + (17,33) 432 +
+ (32,32) (144) + (17,17) (-72) + (18,18) 144 + (17,17) (-72) +
+ (37,37) 144 + (38,38) (-72) + (39,39) (-144) +
+ (19,39) \sqrt{2} 288 + (38,38) (-72) + (37,37) 144 + (38,41) 432 +
+ (38,41) 432 + (41,41) (-72) + (41,41) (-72)\}
\[
T_r \left[ M T_{1+1}^+ M^T T_{00} \right] = T_r \left[ M T_{1-1}^+ M^T T_{00} \right] = \frac{\sqrt{6}}{24^2} \cdot 2 \text{ Im } [P]
\]

\[
P = [ [3,20] \sqrt{3} (-48) + [4,21] (-144) + [5,22] /2 (-144)
+ [22,34] \sqrt{6} (-96) + [8,23] \sqrt{3} (-48) + [9,24] (-144)
+ [10,25] /2 (-144) + [25,35] \sqrt{6} (-96) + [12,13] \sqrt{3} 144
+ [11,12] \sqrt{2} 144 + [13,26] \sqrt{3} (-48) + [11,26] \sqrt{2} (-144)
+ [26,36] \sqrt{6} (-96) + [9,10] \sqrt{2} 144 + [8,9] \sqrt{3} 144
+ [7,8] \sqrt{3} 144 + [6,7] \sqrt{2} 144 + [8,25] \sqrt{3} (-48) + [24,25] 144
+ [7,24] (-144) + [23,24] 144 + [6,23] \sqrt{2} (-144)
+ [23,35] \sqrt{6} (-96) + [4,5] \sqrt{2} 144 + [3,4] \sqrt{3} 144 + [2,3] \sqrt{3} 144
+ [1,2] \sqrt{2} 144 + [3,22] \sqrt{3} (-48) + [21,22] 144 + [2,21] (-144)
+ [20,21] 144 + [1,20] \sqrt{2} (-144) + [20,34] \sqrt{6} (-96) + [14,27] \sqrt{2} 144
+ [15,28] 144 + [16,29] \sqrt{3} 144 + [16,40] \sqrt{6} (-96) + [17,32] \sqrt{2} 144
+ [18,33] 144 + [32,33] \sqrt{2} 144 + [17,18] 144 + [30,31] \sqrt{2} 144
+ [16,31] \sqrt{2} 144 + [29,30] \sqrt{3} 144 + [15,30] 144 + [28,29] \sqrt{3} 144
+ [14,29] \sqrt{3} 144 + [27,28] \sqrt{2} 144 + [15,16] 144 + [14,15] 144
+ [14,40] \sqrt{6} (-96) + [38,39] \sqrt{3} 144 + [37,38] \sqrt{2} 144
+ [39,41] \sqrt{3} (-144) + [19,41] \sqrt{6} 96 + [37,41] \sqrt{2} (-144)]
\]
\[ T_x \left[ M T^*_00 \quad M^*_T00 \right] = T_x \left[ M^*_00 \quad M^*_T00 \right] = \frac{1}{2 \pi^2} \quad 2 \Re \left[ \mathbf{P} \right] \]

\[ \mathbf{P} = [[1,1] \ 576 + [2,2] \ 576 + [3,3] \ 576 + [4,4] \ 576 + [5,5] \ 576 + [6,6] \ 576 \]


LIST OF REFERENCES


Swanson, D. R., Phys. Rev. 89 (1953), 740.

Oehme, R., Phys. Rev. 98 (1955), 147, 216.


20. Ibid., p. 19.


32. Ibid., 270.