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DISSENTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

by

Shu-Shen Sun, B.E.E., M.S.

The Ohio State University
1974

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CHAPTER I
INTRODUCTION

1.1 Historical Background

To develop efficient land locomotion machines to improve the quality of human life is among the most strongly felt desires of society. The wheeled vehicle was the first kind of land locomotion machine created by man. Since its creation, much time and effort has been expended in the further development of its speed and efficiency. Today, the man-made wheeled or tracked vehicle has proved its usefulness on a well constructed road or track. This situation has made most people believe that wheeled vehicles are the best design for land locomotion. However, for off-the-road locomotion, legged systems may prove to be faster, more economical, reliable, versatile, and adaptive to the changing environment. M. G. Bekker has made a comparative study on wheeled and legged locomotion systems. It is shown in his work that the operational speeds of man-made vehicles are of the order of a few miles per hour on rough terrain. A horse or antelope can traverse the same terrain at 40 mph or more [1].

In recent years, much work has been done in developing legged locomotion machines [2,3,4]. Legged locomotion studies, in general, have consisted of two different kinds of work: (i) designing the mechanism used to move the legged locomotion machine, and (ii) selecting a proper sequence of footfall patterns or gaits to control the
motion of the machine. Design of legged locomotion mechanisms is a very complex problem. It is not intended to study this problem here. The main work here is a theoretical study of gaits for legged locomotion systems.

The earliest studies of legged locomotion can be traced back to animal gait studies by zoologists and anatomists. Most of their work was done by observation. Muybridge and Hildebrand have done the most significant work [5,6].

The beginnings of a mathematical theory for legged locomotion systems have been introduced recently. The first work in this field was by Tomovic [7] who suggested a finite state approach to the theory of locomotion. McGhee later adopted this theory and considered each leg as a sequential machine with binary output [8]. He constructed a well-defined legged locomotion model based on the finite theory and a number of basic concepts; e.g., gait, gait matrix, duty factor, etc. McGhee and Jain used this model for regular gait definition, analysis, and enumeration, with very significant results [9].

1.2 Basic Objective of the Dissertation

In animal locomotion, a very complex control system is used to coordinate the legs to maintain a standing posture and stable locomotion [5,10]. The motivation of gait study is then to develop an efficient method to study and analyze different kinds of gaits. Hopefully, this study will yield some more knowledge about animal locomotion and will help in the eventual realization of efficient
control systems to coordinate the motion of legged locomotion machines.

As mentioned in the previous section, considerable work has been carried out relative to quadruped gait studies. The methods used for quadruped gait study, in general, cannot be used efficiently for studying more than four-legged locomotion systems. The objective of this dissertation is to develop new methods which can be used more efficiently for multi-legged gait study. The methods developed will be illustrated specifically for quadruped, hexapod, and eight-legged gaits.

1.3 Organization of the Dissertation

Chapter II is a summary of previous work done in the area of gait study. It includes work on gait representation methods, gait classification, gait enumeration, and static stability analysis.

Chapter III is a study of compatible gait and regular gait. Compatible gaits have the property that while one leg is lifting, before this leg touches the ground, no other legs in the same legged locomotion system will complete the action of lifting a leg and placing it back on the ground. Two basically different enumeration methods for compatible gaits are discussed in this chapter. Regular gaits are the class of gaits with the duty factor of all legs being the same. They constitute a proper subset of the set of all compatible gaits. They are enumerated also in this chapter. Some equivalence operations which help to reduce the number of equivalence classes are also presented.
In Chapter IV, symmetric gaits are studied. Such gaits have the property that legs on one side of the body move synchronously with legs on the other side of the body with a phase delay of exactly one-half a cycle. The permutation and complementation invariant properties possessed by symmetric gaits have been used for symmetric gait enumeration. Regular symmetric gait enumeration is also discussed in this chapter. This kind of gait has both the property of regular gait and symmetric gait. It has a smaller number of independent variables as compared with either regular gait or symmetric gait. This implies that this kind of gait is easy to generate and control. Static stability analysis is included in the study of this type of gait.

Chapter V summarizes the work done in this dissertation and poses some questions for further work in gait study.
CHAPTER II
SURVEY OF PREVIOUS WORK

2.1 Introduction

The purpose of this chapter is (i) to give an overview of the previous work which has been done on gait analysis; (ii) to present background for better understanding of the work in later chapters.

In general, work on gait analysis can be classified into four fields: 1) gait representation methods, 2) gait classification, 3) gait enumeration, and 4) static stability analysis. At the beginning of the following section, a brief history of previous gait studies is presented. Five different kinds of gait representation methods are examined following this. Gait classification work is summarized in Section 2.3. Section 2.4 deals with work on gait enumeration. Enumeration of regular quadruped gaits is the most complete work of this sort. A brief summary of it is given. Some equivalence relations which can be used to reduce the number of equivalence classes are also stated. In Section 2.5, the work on static stability of quadruped creeping gaits is discussed. The last section, Section 2.6, summarizes the contents of this chapter.

2.2 Gait Representation Methods

2.2.1 Introduction

Animal gaits have been studied by interested scientists for
quite a long time. Because of the fast motion of animals, a careful scientific investigation became possible only after the invention of the motion picture camera. The first comprehensive study of this type was made by E. Muybridge [5]. In the 1880's, Muybridge conducted experiments in which he triggered a battery of still cameras to take sequenced photographs of the motion of 25 kinds of animals. He noted that for each manner of moving, combinations of support by the several legs followed one another in a given sequence. He called this activity a "support sequence." Muybridge recognized several support sequences for quadrupeds.

The most significant work to date on quadruped gaits has been done by M. Hildebrand [6,11]. He has devised precise methods for describing and contrasting quadruped gaits and for interpreting the gaits used by a particular species relative to its physical characteristics and behavior. Hildebrand discovered a total of 164 gait "formulas" which are theoretically possible. Most of his work was based mainly on intuition, and lacked any substantial mathematical theory or analysis. Nevertheless, he was the first person to attempt theoretical analysis of duration of support patterns, variability of gaits, the interpretation of gaits and rate, and length and characteristics of the stride. In work similar to that of Hildebrand, D. Wilson has studied the gaits of six-legged and eight-legged animals [12,13].

The finite state approach to the theory of locomotion was first suggested by Tomovic [7] and later elaborated by Tomovic and
Karplus [14]. This was the first step towards establishing a mathematical theory for legged locomotion systems. The finite state model has been further developed by R. B. McGhee and his co-workers [8,9,15]. They utilized Tomovic's notion of a binary output state, i.e., on the ground pushing backward and in the air moving forward, to develop a mathematical structure for gait study. The following three definitions by McGhee are used to define the finite state model [8]:

Definition 1: A leg is a sequential machine $m$ with just two output states, 1 and 0. By convention, the state $m=0$ will be taken to represent the state of being in contact with a supporting surface while the state $m=1$ will correspond to a leg that is raised above the surface. These two states will also be referred to as the support phase and the transfer phase of a leg cycle.

Definition 2: A locomotion automaton, $M$, is an indexed set of legs $M = \{m_1, m_2, \ldots, m_k\}$.

Definition 3: A gait for a locomotion automaton composed of $k$ legs is a periodic sequence of binary $k$-tuples representing the successive states of the legs of the automaton. Within any complete cycle of a gait, the state of every leg must change exactly once from 0 to 1 and from 1 to 0.

In various gait studies, many different kinds of gait representation methods have been introduced. Some of the more useful ones are presented in the following paragraphs.
2.2.2 Footfall Formula

A footfall formula, as illustrated by Figure 2.1, is the most elementary gait representation method. The following statement describes this method [5]: "The animal is viewed from above as it moves from left to right. The feet are represented by circles. A blackened circle means that the foot is on the ground or in a weighted state. The cycle repeats after the eight support states shown."

![Figure 2.1. A Footfall Formula](image)

When a foot is on the ground, it is said to be in a support state. The transfer state is used to mean that a foot is swinging in the air. The supporting feet define the support pattern at any given moment. In Figure 2.1, there are eight different support patterns. Between two consecutive patterns, only one leg is seen to change from the support state to the transfer state or vice versa. Such a gait is said to be connected or non-singular [8]. A gait in
which more than one lifting or placing event occurs simultaneously is said to be singular [8].

2.2.3 Gait Diagram

A more clear representation of the gait of Figure 2.1 is shown in Figure 2.2. It is called a "gait diagram" [6]. Four horizontal rows of squares are assigned to the four feet. Vertical lines represent successive motion picture frames. Squares are lined-in if the respective foot is on the ground during the time interval represented.

In this diagram, the time duration has been introduced. The relative phase between legs is also shown in this graph.

![Figure 2.2: A Gait Diagram](image)

2.2.4 Gait Matrix and Gait Formula

The above two mentioned gait representation methods merely record certain aspects of animal motion. They are hard to use for systematic analysis. McGhee [8] has introduced a mathematical model
for gait study. The gait representation methods used for his model are called "gait matrix" and "gait formula". Their definitions are as follows:

Definition 4: A gait matrix G is a k-column matrix whose successive rows are binary k-tuples corresponding to the successive states of a particular gait of a k-legged locomotion automaton and whose total number of rows is equal to the length of one cycle of a gait sequence. Such a matrix will be said to represent the associated gait. A gait matrix is in row canonical form when it is written so that the first row begins with a 0 and the last row begins with a 1.

Definition 5: The duration vector t for an n-row gait matrix G is an n x 1 array whose entries specify the time duration of the corresponding rows of G. The cycle time τ for a given duration vector t is defined as

\[ \tau = \sum_{i=1}^{n} t_i, \quad t_i > 0. \]

Definition 6: A gait realization is a specific locomotion matrix G together with a particular duration vector t. Such a pair, denoted \( \{G, t\} \), will be said to realize the gait represented by G.

Definition 7: The 0 duration \( \tau_0 \) for a leg \( m_i \) is the total amount of time that \( m_i \) is in the 0 state during one cycle of a gait described by a specific gait realization \( \{G, t\} \). The duty factor \( \beta \) is the relative amount of time that a leg spends in the zero state; that is, \( \beta = \frac{\tau_0}{\tau} \).
Definition 8: The transition delay time \( d_i \) for a leg \( m_i \) is the time interval between the occurrence of the 1-to-0 transition of leg \( m_i \) and the following 1-to-0 transition of \( m_i \) within any cycle of a gait specified by a given gait realization. The relative phase of leg \( i \) is defined by

\[
\phi_i = \frac{d_i}{T}.
\]

Definition 9: A gait formula \( g \) for a particular mode of locomotion of a \( k \)-legged automaton is a point in a unit \((2k-1)\)-cube defined by

\[
g = (\beta_1, \beta_2, \ldots, \beta_k, \phi_2, \phi_3, \ldots, \phi_k).
\]

A gait formula will be said to implement a particular gait. Figures 2.3 and 2.4 are examples of the gait matrix and gait formula representation, respectively. The same animal gait as shown in Figure 2.1 and Figure 2.2 is used in these examples.

\[
G = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\tau = \begin{bmatrix}
1/10 \\
1/10 \\
1/10 \\
1/10 \\
1/10 \\
1/10 \\
1/10 \\
1/10
\end{bmatrix}
\]

Figure 2.3. Gait Matrix
### Figure 2.4. Gait Formula

| g = 5/8 |
| 5/8 |
| 5/8 |
| 5/8 |
| 5/8 |
| 5/8 |
| 1/2 |
| 3/4 |
| 1/4 |

\[ \tau = \frac{8}{10} \]

#### 2.2.5 Event Sequence Method

An **event sequence** is the placing and lifting sequence order of a legged locomotion system where **placing** of leg \( i \) refers to a 1-to-0 transition and **lifting** refers to a 0-to-1 transition [9,15]. It is convenient to present such a sequence on a circle to emphasize its periodic nature. The circumference of the circle can be taken to represent the time period of the gait. It is generally normalized to be 1. The quadruped example of Figure 2.3 is shown in event sequence form on Figure 2.5 where events 1, 2, 3, 4 are the placing events for legs 1, 2, 3, and 4, respectively, and 5, 6, 7, 8 are the corresponding lifting events.
Figure 2.5. Event Sequence Representation of a Gait

2.2.6 Some Further Properties of Gait

From the gait matrix and gait formula representation methods for gaits, some properties related to gait can be further defined. Those properties are useful for mathematical analysis of gait. The following definitions are taken from the work of McGhee [8].

Definition 10: A gait formula $g$ is **regular** if the duty factor of every leg is the same as that of every other leg. A gait represented by a gait matrix $G$ is **regularly realizable** if there exists a $t$ such that $\{G,t\}$ implies a regular $g$.

Definition 11: For a given gait formula $g$, any two legs of a locomotion automaton constitute a **symmetric pair** if the duty factor of either leg is the same as that of the other leg and the phase shift of one leg relative to the other is exactly equal to one half.

Definition 12: A gait formula $g$ for a 2K-legged automaton is **symmetric** if the legs of the automaton can be partitioned into $K$ symmetric pairs. A gait represented by a gait matrix $G$ is **symmetrically realizable** if there exists a $t$ such that $\{G,t\}$ implies a symmetric $g$. 

\[ E = 1 \ 6 \ 4 \ 7 \ 2 \ 5 \ 3 \ 8 \]
Definition 13: A gait formula $g$ is **singular** if there exists at least one component of $g$ such that for an arbitrarily small change in the value of this component, the gait matrix $G$ implied by $g$ is changed. A **singular gait** is a gait with a gait matrix $G$ such that there exists no duration vector $t$ with the property that $(G, t)$ implies a non-singular $g$.

Definition 14: A gait matrix is **connected** if every row differs in exactly one column from the row just above it and the row just below it. The first and last row must also differ in just one column.

Definition 15: A gait matrix is **Markov** if, when its rows are ordered from top to bottom, every binary $k$-tuple appearing as a row has a unique predecessor. The first row must be considered to follow the last row in applying this test.

2.3 Gait Enumeration

2.3.1 Introduction

As mentioned earlier, the work in this dissertation relates to connected gait, regular gait, symmetric gait, and regular symmetric gait. In order to make a complete study of any of the above mentioned classes of gait, it is necessary to know how many members or different event sequences are included in each class. This is the work of gait enumeration.

Hildebrand was the first research worker to attempt an enumeration of quadruped gaits [6,11]. In his work, all of the
quadruped gaits were shown on a plane. The two coordinates of the
plane are percentage stride interval that each hind foot is on the
ground and percentage of stride interval that the footfall of fore
foot follows the hind foot on the same side. A stride is one cycle
of motion. He stated that there are 104 possible symmetric quadruped
gaits including both connected and singular gaits. Forty-four of them
are regular symmetric, and 16 of these 44 are connected regular sym-
metric gaits. Hildebrand did not attempt to count any of the non-
symmetric gaits although he recognized their existence [11].

2.3.2 An Equivalence Operation for Gait Matrices

Hildebrand did not give a clear explanation of how his results
were obtained. A more clear and systematic method for gait enumera-
tion has been introduced by McGhee and Jain [9] and subsequently
developed further by Koozekanani and McGhee [15]. The possible number
of connected gaits has been shown to be (2K-1)! for a K-legged loco-
motion system [8]. In general, this number is too large to do an
efficient gait analysis. Some appropriate notion of gait equivalence
is therefore urgently needed to reduce the number of distinct gaits
to be analyzed. Two equivalence operations have been used in previous
work [9]. One involves certain permutations of rows and columns of a
gait matrix. The other involves complementation of a gait matrix.
The first of these two operations is used to reduce a given gait
matrix to canonical row and column form. Canonical row and column
form is defined as follows [9].
Definition 16: A connected gait matrix, $G$, is transformed to canonical row and column form by first cyclically permuting (rotating) the rows so that the first row begins with a 0 and the last row begins with a 1 and then permuting columns 2, 3, ..., $k$, so that the order of placing events in the event sequence becomes 2, 3, ..., $k$. The event sequence associated with the canonical row and column form of a connected gait matrix will be called the canonical event sequence.

In Figure 2.6a, a quadruped gait in canonical row form is given by its gait matrix and event sequence. Its canonical row and column form gait matrix and event sequence is shown in Figure 2.6b.

\[
\begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

E = 12546837

\[
\begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

E = 12536748

a) Canonical row form  

b) Canonical row and column form

Figure 2.6. Transformation of a Gait Matrix to Canonical Row and Column Form
Gait complementation is accomplished by changing all the 1's in a gait matrix to 0's and all 0's to 1's. For the event sequence representation, this is done by exchanging the placing and lifting event for each leg. The gait is then transformed to its canonical event sequence.

2.3.3 Regularly Realizable Gait Matrices

Some of the most advanced work done in gait analysis concerns regularly realizable gaits. Some of the more important results relative to this class of gaits were obtained by McGhee and Jain [9] and are summarized below.

Definition 17: A binary matrix \( \mathbf{G} \) is \textbf{regularly realizable} if and only if there exists at least one feasible solution to the following linear programming problem:

\[
\begin{align*}
\text{Objective function:} & \quad \tau = \sum_{i=1}^{n} t_i \\
\text{Inequality constraints:} & \quad t_i > 0, \ i=1,\ldots,n. \\
\text{Equality constraints:} & \quad \mathbf{G} \mathbf{t} = \mathbf{1} \quad \text{where } \mathbf{1} \text{ is a column vector of ones.}
\end{align*}
\]

Definition 18: A pair of columns, \( i \) and \( j \), of a \( k \)-column gait matrix \( \mathbf{G} \) are \textbf{compatible} if, when the event sequence associated with \( \mathbf{G} \) is presented on a circle, the arc from event \( i \) to event \( i+k \) does not overlap the arc from event \( j \) to event \( j+k \), and conversely, where \( i,j = 1,2,\ldots,k \).

Definition 19: An event sequence, \( \mathbf{E} \), for a \( k \)-column connected gait matrix, \( \mathbf{G} \), is \textbf{column comparable} if, when \( \mathbf{E} \) is presented on a
circle, the lifting events occur in the same order as the placing events.

In [9] the following theorems relating to the above definitions are proved:

Theorem 2-1: A necessary condition for the regular realizability of a gait matrix \( G \) is that every column of \( G \) be compatible with every other column of \( G \).

Theorem 2-2: Let \( G \) be an \( n \)-row, \( k \)-column binary matrix with the property that every column of \( G \) contains \( m \) ones. Such a \( G \) is always regularly realizable.

Theorem 2-3: The number of column comparable canonical event sequences for a \( k \)-column connected gait matrix is given by

\[
N = \frac{(2k-1)!}{((k-1)!)^2}.
\]

For a quadruped gait, \( k=4 \) and the number of column comparable gaits is 140. An examination of the canonical matrices shows that 82 of these 140 column comparable gaits are compatible gaits [9]. The linear programming problem has been solved for each of the 82 compatible gaits. The results show that there are 80 regularly realizable row and column canonical form quadruped gaits [9].

2.4 Static Stability Analysis

One question which arises from gait studies is how many of the theoretically possible gaits can provide footfall sequences which will keep a legged locomotion machine moving so that all of its phases are statically stable. McGhee and Frank have introduced a mathematical method for analysis of quadruped gait static stability [16]. This
work involves determining whether or not the vertical projection of the center of gravity of the machine onto the supporting surface is contained within a polygonal figure with vertices determined by the feet on the ground at that moment. The machine is said to be in a stable state if the above test is positive. For the six possible quadruped creeping gaits which always keep at least three feet on the ground at all times, three of them can have a stable state throughout the motion. However, one gait, the quadruped crawl, is shown to possess a greater degree of stability than the other two stable gaits and therefore constitutes a unique optimal gait from the point of view of static stability [16].

With respect to six-legged gait, Bessonov and Umnov have presented a paper on static stability analysis in which a computer program was used to measure stability margin of all the possible regular gaits to determine the range of stable gaits [17]. They reached the following three interesting conclusions: i) for hexapods statically stable gaits are possible for duty factors ranging from 0.5 to 1.0; ii) all optimal gaits are symmetric gaits; and iii) for a given duty factor, wave gaits have the maximum static stability margin. Wave gaits are a special group of gaits which were first studied by Wilson [12,13] for animals and by Sindall for vehicles [18]. This kind of gait has the property that the placing events on each side of the legged locomotion machine move from the rear to the front one after another in a wave motion.
2.5 Summary

This chapter begins with a short history of animal gait studies. Since mathematical analysis of gait is a newly developed field, some basic definitions and theorems taken from the literature are presented and consolidated.

Five different types of gait representation methods have been presented in this chapter. The gait matrix and gait event sequence are the two methods which can best be used for mathematical analysis of different properties of gait.

Quadruped gait analysis has been to date the most complete work on animal gait. Results obtained by previous investigators relating to quadruped compatible, regular, or regular symmetric gait enumeration are summarized in this chapter.

Static stability analysis is an important aspect of gait study. Two separate works in this area have been presented. One is on quadruped creeping gait stability analysis and the other is concerned with hexapod regular symmetric gait static stability analysis.

Hopefully, the material presented in this chapter has provided an adequate background for the reader to understand and appreciate the author's own contributions to the mathematical theory of locomotion. These contributions form the main part of this dissertation and are contained in the next two chapters.
CHAPTER III
COMPATIBLE GAITS AND REGULAR GAITS

3.1 Introduction

The work included in this chapter consists of the following three major parts:

(1) a general discussion of certain applications of group theory to gait enumeration,
(2) enumeration of compatible gaits,
(3) enumeration of regular gaits.

The purpose of the general discussion on gait enumeration is to try to find ways to reduce the set of all possible gaits to a small number of equivalence classes. This work in general involves two procedures. The first procedure is to specify the equivalence relations used to combine various patterns of gait. The next procedure is to count the number of equivalence classes. To simplify this work, only connected gait matrices will be considered. This is justified because it is known that only this class of matrices can actually be used in periodic motion by any machine or animal [16]. Moreover, because of this fact, no further consideration will be given to other types of matrices or to their corresponding singular gaits. Again, this is not a serious restriction since every singular gait can be represented as the limit of a non-singular gait [8].

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At the beginning of Section 3.2, a brief introduction of group theory and counting theory is presented. Five different permutation groups that are used to reduce the number of equivalence classes are discussed. Four of them preserve the following five properties of gait: i) connectivity, ii) Markov property, iii) compatibility, iv) regular realizability, and v) symmetric realizability.

In Section 3.3, an iterative computational method is developed in order to find all the possible canonical row and column forms of compatible gaits of k-legged locomotion systems. An additional property of gait, called interval invariants, is introduced to further reduce the number of equivalence classes of compatible gaits.

In Section 3.4, the regular gaits are enumerated by using a linear programming algorithm to test the equivalence classes deduced from Section 3.3. The results obtained are compared with previous results on quadruped regular gait.

The last section is a brief summary of the work in this chapter.

3.2 Permutation Groups for Partitioning of Event Sequences into Equivalence Classes

3.2.1 Introduction and Definitions

All the possible connected gait event sequences of a k-legged locomotion system can be considered as the arbitrary permutations among the 2k events of the k legs in a locomotion cycle. Of these 2k events, k of them are lifting events and the other k are placing events. Since arbitrary permutations among different objects form a symmetric
group [19,20], some understanding of group theory will be helpful to find out the different equivalence classes induced by such a group and certain of its subgroups. Group theory is discussed in many modern algebra textbooks [21]. However, to aid the reader, a brief summary of the definitions and theorems needed for equivalence class counting are presented in the next few pages. For a more detailed understanding of transformation groups, references [19,22] are helpful.

All of the following definitions are taken from these two sources.

Definition 20: A set is a collection of distinct elements.

Definition 21: An ordered pair is an ordered arrangement of two (not necessarily distinct) elements from a specified set $S$.

Definition 22: The cartesian product of two sets $S$ and $T$, denoted by $S \times T$, is the set of all ordered pairs $(x,y)$ in which $x$ is in $S$ and $y$ is in $T$.

Definition 23: A binary relation between two sets $S$ and $T$ is a subset of the ordered pairs in the cartesian product $S \times T$.

Definition 24: A set $S$ together with a binary operation $*$ on the set $S$ is said to form a group if the following conditions are satisfied:

1) the binary operation $*$ is closed

2) the binary operation $*$ is associative; that is, for any $a,b,c$ in $S$ $(a*b)*c = a*(b*c)$.

3) there is an element $e$ in $S$ which is such that $a*e = a$ for every $a$ in $S$. This element is called the identity element of the group.
4) for any element \( a \) in \( S \), there is another element in \( S \) denoted by \( a^{-1} \) and called the inverse of \( a \), which is such that \( a \cdot a^{-1} = e \).

Definition 25: Let \( S \) be a finite set. A permutation of \( S \) is a one-to-one mapping of \( S \) onto itself. The notation \( \sigma = abcd \) is used for the permutation of the set \( \{a, b, c, d\} \) that maps \( a \) into \( b \), \( b \) into \( d \), \( c \) into \( c \), and \( d \) into \( a \).

Definition 26: Let \( \pi_1 \) and \( \pi_2 \) be the two permutations of a set \( S \). The composition of \( \pi_1 \) and \( \pi_2 \), denoted by \( \pi_1 \pi_2 \), or \( \pi_1 \cdot \pi_2 \) is the successive permutations of the set \( S \), first according to \( \pi_2 \) and then according to \( \pi_1 \).

Definition 27: Let \( G \) be a set of permutations of a set \( S \). Then \( G \) is said to be a permutation group of \( S \) if \( G \) and the binary operation of composition of permutations form a group.

Definition 28: The number of elements in a group \( G \) is called the order of \( G \). It is denoted by \( |G| \). If \( G \) is a permutation group on a set \( S \) and \( S \) contains \( s \) elements, then \( s \) is said to be the degree of \( G \). It is denoted by \( |S| \).

The cycle properties of elements of \( S \) operated on by a permutation group \( G \) provide very useful information for counting equivalence classes of functions under \( G \). The following definitions provide a formal description of cycles and of the cycle index polynomial of \( G \).

Definition 29: Let \( G \) be a permutation group on a set \( S \), and \( g \) be an element in \( G \). \( S \) can be split in a unique way into cycles;
that is, subsets of $S$ that are cyclically permuted by $g$. If $\ell$ is the
length of such a cycle, and if $s$ is any element of such a cycle, then
the cycle consists of $s$, $gs$, $g^2s$, ..., $g^{\ell-1}s$, where $g^2 = g \cdot g$, etc.

Definition 30: Let $G$ be a permutation group on a set $S$ and
$g$ be an element in $G$. If $S$ splits into $b_1$ cycles of length 1, $b_2$
cycles of length 2, etc., then the type of $g$ is $\{b_1, b_2, b_3, \ldots\}$. Fur­
thermore, the following relation holds with respect to these integers:

$$b_1 + 2b_2 + 3b_3 + \ldots = |S|$$

Definition 31: Let $G$ be a permutation group whose elements
are the permutations of $S$ and $|S| = m$. The cycle index polynomial
of $G$, $P_G$, is

$$P_G = |G|^{-1} \sum_{g \in G} x_1^{b_1} x_2^{b_2} \cdots x_m^{b_m}$$

where the $x_i$'s for $i=1,2,\ldots,m$ are dummy variables associated with
the cycle lengths and $\{b_1, b_2, \ldots, b_m\}$ is the type of $g$.

Before proceeding to count equivalence classes of gait,
the equivalence relation to be used is defined below.

Definition 32: Let $D$ and $R$ be two sets. Let $G$ be a permu­
tation group on $D$, and $H$ be a permutation group on $R$. A function $f_1$
from $D$ into $R$ is equivalent to another function $f_2$ if and only if
there is a permutation $\pi$ in $G$ and a permutation $\tau$ in $H$ such that

$$\tau f_1(d) = f_2[\pi(d)]$$

for all $d$ in $D$.

With the above equivalence relation, the number of equivalence
classes of functions from $D$ to $R$ can be found by using the generalized
Polya theorem [19, 22]. However, De Bruijn [20] has developed a simpler method to count the number of equivalence classes of functions in the case of one-to-one mappings and \(|D| = |R|\). De Bruijn's theorem will be the main tool used for gait counting in this section of this dissertation. A statement of this method is given by the following theorem.

**Theorem 3-1:** Let D and R be two sets, and \(|D| = |R|\). The number of equivalence classes of one-to-one mappings from D to R equals

\[
P_G \left( \frac{3}{3z_1}, \frac{3}{3z_2}, \ldots, \right) P_H(z_1, 2z_2, 3z_3, \ldots)
\]
evaluated at \(z_1 = z_2 = z_3 = \ldots = 0\), where \(G\) is the allowed permutation group on D, \(H\) is the allowed permutation group on R, \(P_G\) and \(P_H\) are the cycle index polynomials of \(G\) and \(H\), respectively, and the \(z_1\) are dummy variables substituted as indicated into \(P_G\) and \(P_H\).

The above discussion has provided a summary of those parts of group theory and counting theory which will be used in this dissertation for purposes of classification and enumeration of gait matrices. Before proceeding with this work, however, it is useful to define certain groups and subgroups of permutations which have been used by others or will be introduced here for this purpose. These groups are as follows:

1. The **full symmetric group**, \(S_n\): This group is the group of all permutations on \(n\) objects. Evidently it contains \(n!\) elements. That is, \(|S_n| = n!\)

2. The **row rotation group**, \(G_1\): This group, first introduced to gait analysis by McGhee [8], contains \(2k\) elements when
applied to k-column (2k rows) gait matrix. Each element merely rotates the rows of a gait matrix n steps where $0 \leq n < 2k$. Equivalently, this group may also be thought of as rotating event sequences.

3. The row and column canonical form group, $G_2$: This group contains all of $G_1$ plus arbitrary permutations of columns 2, 3, ..., $k$ together with all products (compositions) of these two types of permutations. It was first used by McGhee and Jain [9] to reduce the $(2k)!$ element set of all connected k-column gait matrices to a set of $N_2$ equivalence classes where [9]

$$N_2 = \frac{(2k-1)!}{(k-1)!} \quad (3-1)$$

The reduction is accomplished by transforming an arbitrary matrix to row and column canonical form.

4. The complementation and restricted row and column permutation group, $G_3$: This group consists of all elements of $G_2$ together with matrix complementation and all products to these two classes of operations. It was also used by McGhee and Jain in a study of quadruped gaits [9].

5. The relabelling group, $G_4$: This group includes all of $G_3$ together with all column rotations, and all compositions of these two classes of operations. It is first introduced to gait analysis in this dissertation and represents the
largest known group which preserves all of the five properties mentioned earlier in this chapter; namely, connectedness, Markov property, column compatibility, regular realizability, and symmetric realizability.

It should be noted that while \( G_1 \) through \( G_4 \) have been defined above in terms of operations on matrices, this is for convenience only. In fact, more fundamentally, these groups are applied to event sequences which correspond 1:1 to gait matrices and which have been defined in the previous chapter as total orderings of the set of integers \( I_{2k} = \{1, 2, \ldots, 2k\} \) for \( k \)-legged connected gait matrices.

While it has not been shown that the above sets of transformations \( G_1 \) through \( G_4 \) in fact constitute groups under composition, a little reflection will show that this is indeed the case and no proof will be presented here.

### 3.2.2 Equivalence Classes under Row Rotations

A connected gait matrix for \( k \)-legged locomotion is always a \( 2k \times k \) matrix \([8]\) in which each row represents a distinct phase in a locomotion cycle and where the top row is considered to be the successor to the bottom row. When a row rotation operation is applied to a gait matrix, the rows are simply moved down with those rows which are moved out of the matrix being moved to the top of the matrix. Row rotation does not change the footfall sequences of a gait. It represents the same gait, but at a different initial observation time. All gait matrices which can be matched by this kind of operation should obviously be included in one equivalence class.
For every gait matrix, there is a corresponding event sequence. Downward row rotation operations on a gait matrix correspond to clockwise rotation of the event sequence.

The problem of finding the number of equivalence classes of connected gaits under row rotations can be stated as follows: "In how many distinct ways can all of the 2k events in one cycle of k-legged locomotion be placed in 2k positions on a circle, if rotation of the circle is allowed?"

To use Theorem 3-1 to solve the above stated problem, the domain, D, is the set of integers \( I_{2k} = \{1, 2, 3, \ldots, 2k\} \). The one element group \( G = \{123\ldots2N\} \) is the only allowed permutation in D. The range, R, is the same as D. The allowed permutation group \( H \) on R is the 2k different rotations on the circle; i.e., the group \( C_1 \) as defined in the preceding Section 3.2.1. For a better understanding of this formulation, quadruped gaits will be used as an example.

The permutation group \( G \) for quadruped event sequences is \( G = \{1 2 3 4 5 6 7 8\} \). It contains 8 cycles of length 1. The cycle index polynomial of \( G \) is thus

\[
P_G = x_1^8.
\]  

(3-2)

The permutation group \( H \) is \( \{\pi_0, \pi_2, \pi_3, \ldots, \pi_r\} \), where

\[
\pi_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}, \quad \pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 \end{pmatrix}
\]

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By examining each of the $\pi_i$'s, the following results are obtained:

- $\pi_0$ contains 8 cycles of length 1.
- $\pi_1$ contains 1 cycle of length 8.
- $\pi_2$ contains 2 cycles of length 4.
- $\pi_3$ contains 1 cycle of length 8.
- $\pi_4$ contains 4 cycles of length 2.
- $\pi_5$ contains 1 cycle of length 8.
- $\pi_6$ contains 2 cycles of length 4.
- $\pi_7$ contains 1 cycle of length 8.

Thus, $P_H$ can be written as

$$P_H = \frac{1}{8} (x_1^8 + x_8^1 + x_4^2 + x_8^1 + x_2^1 + x_8^1 + x_4^2 + x_8^1)$$

$$= \frac{1}{8} (x_1^8 + x_2^4 + 2x_4^2 + 4x_8^1). \quad (3-3)$$

Since any connected gait is equivalent to a permutation of $I_{2k}$, and because permutations are 1:1 mappings of $I_{2k}$ onto itself, the number
of distinct functions under $H$ operating on $R$ is exactly the number, $N_1$, of distinct gaits under $G$. Thus,

$$N_1 = P_G \left( \frac{3}{z_1}, \frac{3}{z_2}, \ldots, \frac{3}{z_8} \right) P_H \left( z_1, 2z_2, \ldots, 2z_8 \right) \bigg|_{z_1 = 0, i = 1, 2, \ldots, 8}$$

$$= \frac{1}{8} \frac{8}{z_1} \left( z_1 + (2z_2)^4 + 2(4z_4)^2 + 4(8z_8) \right) \bigg|_{z_1 = 0, i = 1, 2, \ldots, 8}$$

$$= 7! = 5040.$$  \hspace{1cm} (3-4)

This result has also been obtained by more informal methods [8] in which each gait is transformed to row canonical form by rotating it until the event sequence begins with 1.

3.2.3 Equivalence under $G_2$

The permutation group $G_2$ discussed in the preceding Section 3.2.1 consists of the product of two permutation groups: arbitrary permutations of columns $2, 3, \ldots, k$ and row rotations of a gait matrix, where $k$ is the number of legs. Arbitrary permutation of columns $2, 3, \ldots, k$ of a gait matrix is equivalent to arbitrary permutation of placing events $2, 3, \ldots, k$ of a gait event sequence and subsequent permutation of the lifting events in the same order as their corresponding placing events. It was mentioned earlier that this is the operation used to transfer a gait to its canonical row and column form.
The following discussion will present a detailed procedure to find the total number of canonical row and column form connected gaits of k-legged locomotion systems. The quadruped case will again be used as the illustrative example.

Similar to the problem statement relative to $G_1$, the present problem can be posed as follows: "In how many ways can all of the totally ordered event sequences of k-legged gait be placed in $2k$ positions on a circle, if rotation of the circle is allowed and arbitrary permutation among events $2, 3, \ldots, k$ and simultaneous permutation of lifting events $k+2, k+3, \ldots, 2k$ according to the order of the placing events are also allowed?"

In this problem, the domain, $D$, is still the whole set of $2k$ gait events for k-legged locomotion. However, the allowed permutation in $D$ is the permutation group of all of the arbitrary permutations among placing events $2, 3, \ldots, k$ and simultaneous permutation of lifting events $k+2, k+3, \ldots, 2k$ as described above. The permutation group, $G$, for quadrupeds in this case is listed in Table 3.1. The cycle index product for each permutation is also listed.

From Table 3.1, the cycle index polynomial, $P_G$, of $G$ can be written by summing all cycle index products with the following result:

$$P_G = \frac{1}{6} \left( x_1^8 + 3x_1^4 x_2^2 + 2x_1^2 x_3^2 \right).$$  \hspace{1cm} (3-5)
<table>
<thead>
<tr>
<th>No.</th>
<th>Permutation</th>
<th>Cycle Index Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1 2 3 4 5 6 7 8)</td>
<td>(x^8)</td>
</tr>
<tr>
<td></td>
<td>(1 2 3 4 5 6 7 8)</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>(1 2 3 4 5 6 7 8)</td>
<td>(x_1^4 x_2)</td>
</tr>
<tr>
<td></td>
<td>(1 2 4 3 5 6 8 7)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(1 2 3 4 5 6 7 8)</td>
<td>(x_1^4 x_2)</td>
</tr>
<tr>
<td></td>
<td>(1 3 2 4 5 7 6 8)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(1 2 3 4 5 6 7 8)</td>
<td>(x_1^2 x_3)</td>
</tr>
<tr>
<td></td>
<td>(1 3 4 2 5 7 8 6)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(1 2 3 4 5 6 7 8)</td>
<td>(x_1^2 x_3)</td>
</tr>
<tr>
<td></td>
<td>(1 4 2 3 5 8 6 7)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(1 2 3 4 5 6 7 8)</td>
<td>(x_1^4 x_2)</td>
</tr>
<tr>
<td></td>
<td>(1 4 3 2 5 8 7 6)</td>
<td></td>
</tr>
</tbody>
</table>

The range, \(R\), the allowed permutation group, \(H\), in \(R\), and cycle index polynomial \(P_H\) are the same as in the problem statement for \(G_1\). Therefore according to Theorem 3-1, the number of equivalence classes of quadruped gait under \(G_2\) is 33.
This number has also been obtained previously by less general methods [9].

3.2.4 Matrix Complementation

It can be easily understood that by changing all the 1's in a gait matrix to 0's, and changing all the 0's to 1's, the gait matrix will be changed; but the connectivity, compatibility, realizability, symmetric realizability, and Markov properties of the original gait matrix are still preserved in the new gait matrix. This kind of operation is called gait matrix complementation. If the gait event sequence representation has been used, gait complementation can be accomplished by changing all the placing events to their corresponding lifting events and vice versa. In this section, the number of equivalence classes of connected gaits under the permutation of $G_2$ together with gait complementation are studied. Quadruped gaits are still used as the illustrative example.

For convenience, the symbol $G_3$ is used to represent the enlarged permutation group including complementation. Again row permutations will be applied in the range space $R$ so the permutation group $G$ in the present case is the composition of complementation and permutation of columns 2,3,...$k$. This group is listed in Table 3.2 for...
quadrupeds together with the cycle index product for each permutation.

Table 3.2: Allowed Column Permutations and Complementations under \( G_3 \) for Quadrupeds.

<table>
<thead>
<tr>
<th>No.</th>
<th>Permutation</th>
<th>Cycle Index Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 2 3 4 5 6 7 8</td>
<td>( x_1^8 )</td>
</tr>
<tr>
<td>2</td>
<td>1 2 3 4 5 6 7 8</td>
<td>( x_1^4 )</td>
</tr>
<tr>
<td>3</td>
<td>1 2 3 4 5 6 7 8</td>
<td>( x_1^2 )</td>
</tr>
<tr>
<td>4</td>
<td>1 2 3 4 5 6 7 8</td>
<td>( x_2^2 )</td>
</tr>
<tr>
<td>5</td>
<td>1 2 3 4 5 6 7 8</td>
<td>( x_1^4 )</td>
</tr>
<tr>
<td>6</td>
<td>1 2 3 4 5 6 7 8</td>
<td>( x_2^2 )</td>
</tr>
<tr>
<td>7</td>
<td>1 2 3 4 5 6 7 8</td>
<td>( x_1^2 )</td>
</tr>
<tr>
<td>8</td>
<td>1 2 3 4 5 6 7 8</td>
<td>( x_2^4 )</td>
</tr>
<tr>
<td>9</td>
<td>1 2 3 4 5 6 7 8</td>
<td>( x_1^2 )</td>
</tr>
<tr>
<td>10</td>
<td>1 2 3 4 5 6 7 8</td>
<td>( x_2^4 )</td>
</tr>
<tr>
<td>11</td>
<td>1 2 3 4 5 6 7 8</td>
<td>( x_2^4 )</td>
</tr>
<tr>
<td>12</td>
<td>1 2 3 4 5 6 7 8</td>
<td>( x_2^4 )</td>
</tr>
</tbody>
</table>
The cycle index polynomial in this case is thus

\[ p_G = \frac{1}{12} \left\{ x_1^8 + 2x_2x_6 + 2x_1^2x_3 + 3x_1^4x_2 + x_2^2x_4 + 3x_2^4 \right\} \]  \hspace{1cm} (3-7)

The polynomial \( p_H \) is the same as used in the last two sections, or

\[ p_H = \frac{1}{8} \left\{ x_1^8 + x_2^4 + 2x_4^2 + 4x_3 \right\}. \]

By using Theorem 3-1, the number of equivalence classes of connected gait under the permutations of \( G_3 \) is thus:

\[ N_3 = p_G \left( \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_4}, \ldots, \frac{\partial}{\partial z_8} \right) p_H(z_1, 2z_2, \ldots, 8z_8) \]

\[ = \frac{1}{12^8} \left\{ \left( \frac{z_1^8}{z_1} + 4 \frac{z_2^4}{z_2} + \ldots \right) \left\{ (z_1^8 + 16z_2^4 + \ldots) \right\} \right\} \]

\[ = \frac{1}{96} \left( 81 + 4 \cdot 16 \cdot 4! \right) \]

\[ = 436. \]  \hspace{1cm} (3-8)

This number is a new result in quadruped gait analysis.

3.2.5 Arbitrary Column Permutations

It has been noted earlier in this dissertation that arbitrary permutation of columns 2, 3, \ldots, k for k-legged gait matrix can preserve the connectivity, compatibility, regular realizability, symmetric realizability, and Markov properties of a gait. Similarly, arbitrary permutation of columns 1, 2, \ldots, k will also preserve all the above stated properties of a gait, but the number of equivalence classes should be further reduced. The permutation group \( G_4 \) discussed in this section consists of arbitrary column permutations and row rotations of a gait matrix, together with matrix complementation. In simpler terms, \( G_4 \) is
equivalent to the permutation group $G_3$ together with column rotations.

The permutation group $G$ for the current problem consists of $2(k!)$ permutations. For arbitrary $k$, considerable work is required to find the cycle index polynomial. No attempt will be made here to get a general result for this polynomial. However, quadruped gaits will once again be used as an example to show how to find the cycle index polynomial $P_G$.

The permutation group $G$ is given for quadrupeds in Table 3.3 together with the cycle index product for each permutation. From this table it can be seen that

$$P_G = \frac{1}{48} \left( x_1^8 + 13x_2^4 + 12x_4^2 + 8x_1^2x_3^2 + 8x_2x_6 + 6x_4^4x_2 \right) \quad (3-9)$$

Thus, using Theorem 3-1, the number of equivalence classes of connected quadruped gait matrices under $G_4$ can be calculated as

$$N_4 = P_G \left( \frac{3}{\partial z_1}, \frac{3}{\partial z_2}, \ldots, \frac{3}{\partial z_8} \right) \cdot P_H(z_1, 2z_2, 3z_3, \ldots, 8z_8)$$

$$= \frac{1}{48x8} \left( \frac{3^8}{\partial z_1^8} + 13 \frac{3^4}{\partial z_2^4} + 12 \frac{3^2}{\partial z_4^2} + \ldots \right) \cdot \left( z_1^8 + 32z_4^2 + 16z_2^4 + \ldots \right)$$

$$= \frac{1}{384} \left( 8! + 13 \cdot 16 \cdot 4! + 12 \cdot 32 \cdot 2! \right)$$

$$= 120. \quad (3-10)$$

The above number is also a new result in quadruped gait analysis. It is perhaps worthwhile to give another interpretation of its meaning as follows. In all of the work thus far in this dissertation,
Table 3.3. Column Permutations and Matrix Complementation under $G_4$

<table>
<thead>
<tr>
<th>No.</th>
<th>Permutations</th>
<th>Cycle Index</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 2 3 4 5 6 7 8</td>
<td>$x_1$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1 2 4 3 5 6 8 7</td>
<td>$4 \cdot x_2$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1 3 2 4 5 7 6 8</td>
<td>$4 \cdot x_2$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1 3 4 2 5 7 8 6</td>
<td>$4 \cdot x_2$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1 4 2 3 5 8 6 7</td>
<td>$x_2 \cdot x_3$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1 4 3 2 5 8 7 6</td>
<td>$x_2 \cdot x_3$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>2 1 3 4 6 5 7 8</td>
<td>$x_1 \cdot x_2$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2 1 4 3 6 5 8 7</td>
<td>$x_2^4$</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>2 3 1 4 6 7 5 8</td>
<td>$2 \cdot x_1 \cdot x_3$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>2 3 4 1 6 7 8 5</td>
<td>$x_4 \cdot x_2$</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>2 4 1 3 6 8 5 7</td>
<td>$x_4$</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>2 4 3 1 6 8 7 5</td>
<td>$2 \cdot x_2 \cdot x_3$</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>3 1 2 4 7 5 6 8</td>
<td>$2 \cdot x_2 \cdot x_3$</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>3 1 4 2 7 5 8 6</td>
<td>$x_4 \cdot x_2$</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>3 2 1 4 7 6 5 8</td>
<td>$4 \cdot x_1 \cdot x_2$</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>3 2 4 1 7 6 8 5</td>
<td>$2 \cdot x_1 \cdot x_3$</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>3 4 1 2 7 8 5 6</td>
<td>$x_2 \cdot x_4$</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>3 4 2 1 7 8 6 5</td>
<td>$x_2 \cdot x_4$</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>4 1 2 3 8 5 6 7</td>
<td>$x_4 \cdot x_2$</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>4 1 3 2 8 5 7 6</td>
<td>$x_4 \cdot x_2 \cdot x_3$</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>4 2 1 3 8 6 5 7</td>
<td>$x_2 \cdot x_1 \cdot x_3$</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>4 2 3 1 8 6 5 7</td>
<td>$x_2 \cdot x_1 \cdot x_3$</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>4 3 1 2 8 7 5 6</td>
<td>$x_4 \cdot x_2$</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>4 3 2 1 8 7 6 5</td>
<td>$x_4 \cdot x_2$</td>
<td></td>
</tr>
</tbody>
</table>

Note: In this table, the representation of permutations has been simplified by listing only the bottom row of each function, e.g., the entry above for $\pi_4$ means

$$\pi_4 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 4 & 2 & 5 & 7 & 8 & 6 \end{bmatrix}$$

38
certain conventions adopted from the work of McGhee and Frank [16] have been tacitly assumed. Specifically, a 0 entry in a gait matrix is taken to represent a leg in a support state while a 1 entry represents a transfer state. In addition, it has been implicit that leg 1 is the left front leg, leg 2 is the right front leg, etc. While these conventions will be adhered to in the balance of this dissertation, it must be realized that they are arbitrary. The operations included in $G_4$ permit exchange of the meaning of 1 and 0 (under matrix complementation) and arbitrary re-labelling of legs (column permutations). Thus the significance of Eq. (3-10) is that there are exactly 120 distinct connected quadruped gaits under arbitrary re-naming of legs and leg states.

3.2.6 All Permutations of 2k-Elements

There is one final permutation group which is of interest in gait analysis. This is the arbitrary permutations of the 2k events in a gait event sequence, or the full symmetric group, $S_{2k}$. It can easily be seen that this permutation group reduces the number of equivalence classes to 1. But, other than connectivity, all the other interesting properties associated with a gait matrix will be lost. This permutation group is therefore not of interest for further study.

3.2.7 Equivalence Classes of Hexapod Gait

The analysis which has been carried out above for quadrupeds can be repeated for six-legged gait. While no new principles are involved, a considerable amount of detailed calculation is needed.
These details are presented in Appendix I where it is shown that the number of connected hexapod gaits under $G_1$, $G_2$, $G_3$, and $G_4$ is 39916800, 332640, 166736, and 27974, respectively.

3.3 Compatible Gaits

3.3.1 Introduction

While consideration of equivalence classes of gait under $G_4$ rather than individual gaits greatly simplifies many problems relating to gait analysis, there are still too many classes to allow for an exhaustive study when more than four legs are to be considered. Consequently, it is very desirable to find some condition for eliminating most gait equivalence classes from detailed analysis. One very natural requirement to impose is that of column compatibility since, with very few exceptions, animals use only compatible gaits [6,9,12,13]. One way to accomplish such a study is to examine each equivalence class of gait for column compatibility and to then discard all classes not possessing this property. McGhee and Jain did this for quadrupeds and found that out of the 840 row and column canonical forms, exactly 82 are column compatible classes [9]. It will be seen later in this section of this chapter that these 82 classes reduce to just 14 classes under $G_4$.

Unfortunately, if exhaustive examination of equivalence classes for compatibility is attempted for more than four-legged gait, the process becomes completely unwieldy due to the large number of classes which must be considered. Consequently, a more direct
method for determining equivalence classes of compatible gait is needed. In the following Section 3.3.2, a necessary condition due to McGhee and Jain [9] is used to develop an iterative method for generating all row and column canonical forms of \((k+1)\)-legged compatible gait matrices from the complete set of such forms for \(k\)-legged gait. Obviously, this step bypasses the classification problem described above.

The method of generation of gaits presented in the following Section 3.3.2 produces equivalence classes under \(G_2\). This is useful because it is known that each class of \(k\)-legged gait under \(G_2\) contains exactly \((k-1)!\) row canonical forms. However, it turns out that for six-legged gait, the number of such classes is 1648. This number is still too large to answer many interesting questions regarding the temporal characteristics of six-legged gait. Consequently, in Section 3.3.3, another method is developed which directly generates equivalence classes under \(G_4\). It is shown that there are just 145 classes of compatible six-legged gait under this group of transformations.

3.3.2 An Iterative Method for Generating All Canonical Row and Column Forms for Compatible Gait Matrices

In [9], McGhee and Jain show that every column compatible gait matrix must be column comparable. Theorem 2-3 presented in Chapter II of this dissertation states that the number of distinct row and column canonical forms of column comparable gait matrices is given by

\[
N = \frac{(2k-1)!}{[(k-1)!]^2}
\]  

(3-11)
For $k=4$, this number is 140 while for $k=6$ it is 2772. This is compared to the numbers 82 and 1648, respectively, for compatible gait. If $M_2$ is the number of compatible gaits under $G_2$, then for these two examples,
\[
\frac{N}{2} < M_2 < N
\] (3-12)

The iterative method to be developed in this section suggests that this may be a general result for any value of $k$, but no proof of this possibility has been found as yet.

Beginning first with biped gait, from [9], the total number of connected gaits (in row canonical form) is $3! = 6$. These are all shown in Figure 3.1. According to Eq. (3-11), these gaits should all be column comparable. Examination of each of the event sequences of Figure 3.1 shows that this is true. Further examination of the associated matrices shows that gaits 2, 3, 4, and 5 are compatible gaits while the other two are not. Thus, Eq. (3-12) is also satisfied for bipeds.

For tripeds, application of Eq. (3-12) yields
\[
15 < M_2 < 30
\] (3-13)

The column comparable gaits are thus 30 in number. They are all listed on Table 3.4 which also indicates which gaits are compatible. As can be seen, 18 of the gaits in this table are compatible, again satisfying Eq. (3-12).
<table>
<thead>
<tr>
<th>Gait 1</th>
<th>Gait 2</th>
<th>Gait 3</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
0 & 0 \\
1 & 0 \\
1 & 1 \\
1 & 0 \\
\end{bmatrix}
\] | \[
\begin{bmatrix}
0 & 1 \\
1 & 1 \\
1 & 0 \\
1 & 1 \\
\end{bmatrix}
\] | \[
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
1 & 0 \\
1 & 1 \\
\end{bmatrix}
\] |

<table>
<thead>
<tr>
<th>Gait 4</th>
<th>Gait 5</th>
<th>Gait 6</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
0 & 0 \\
0 & 1 \\
1 & 1 \\
1 & 0 \\
\end{bmatrix}
\] | \[
\begin{bmatrix}
0 & 0 \\
0 & 1 \\
0 & 0 \\
1 & 0 \\
\end{bmatrix}
\] | \[
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
0 & 1 \\
1 & 1 \\
\end{bmatrix}
\] |

Figure 3.1. Connected Biped Gait Matrices
### Table 3.4 Row and Column Canonical Forms for Comparable Triped Gaits.

<table>
<thead>
<tr>
<th>Equivalence Class No.</th>
<th>Event Sequence</th>
<th>Compatible</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 2 3 4 5 6</td>
<td>yes</td>
</tr>
<tr>
<td>2</td>
<td>1 2 3 5 6 4</td>
<td>no</td>
</tr>
<tr>
<td>3</td>
<td>1 2 3 6 4 5</td>
<td>no</td>
</tr>
<tr>
<td>4</td>
<td>1 2 4 3 5 6</td>
<td>yes</td>
</tr>
<tr>
<td>5</td>
<td>1 2 4 5 3 6</td>
<td>yes</td>
</tr>
<tr>
<td>6</td>
<td>1 2 4 5 6 3</td>
<td>no</td>
</tr>
<tr>
<td>7</td>
<td>1 2 5 3 6 4</td>
<td>no</td>
</tr>
<tr>
<td>8</td>
<td>1 2 5 6 3 4</td>
<td>no</td>
</tr>
<tr>
<td>9</td>
<td>1 2 5 6 4 3</td>
<td>no</td>
</tr>
<tr>
<td>10</td>
<td>1 2 6 3 4 5</td>
<td>yes</td>
</tr>
<tr>
<td>11</td>
<td>1 2 6 4 3 5</td>
<td>yes</td>
</tr>
<tr>
<td>12</td>
<td>1 2 6 4 5 3</td>
<td>yes</td>
</tr>
<tr>
<td>13</td>
<td>1 4 2 3 5 6</td>
<td>yes</td>
</tr>
<tr>
<td>14</td>
<td>1 4 2 5 3 6</td>
<td>yes</td>
</tr>
<tr>
<td>15</td>
<td>1 4 2 5 6 3</td>
<td>no</td>
</tr>
<tr>
<td>16</td>
<td>1 4 5 2 3 6</td>
<td>no</td>
</tr>
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<td>17</td>
<td>1 4 5 2 6 3</td>
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</tr>
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<td>18</td>
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</tr>
<tr>
<td>20</td>
<td>1 5 2 6 3 4</td>
<td>yes</td>
</tr>
<tr>
<td>21</td>
<td>1 5 2 6 4 3</td>
<td>yes</td>
</tr>
<tr>
<td>22</td>
<td>1 5 6 2 3 4</td>
<td>yes</td>
</tr>
<tr>
<td>23</td>
<td>1 5 6 2 4 3</td>
<td>yes</td>
</tr>
<tr>
<td>24</td>
<td>1 5 6 4 2 3</td>
<td>yes</td>
</tr>
<tr>
<td>25</td>
<td>1 6 2 3 4 5</td>
<td>yes</td>
</tr>
<tr>
<td>26</td>
<td>1 6 2 4 3 5</td>
<td>yes</td>
</tr>
<tr>
<td>27</td>
<td>1 6 2 4 5 3</td>
<td>yes</td>
</tr>
<tr>
<td>28</td>
<td>1 6 4 2 3 5</td>
<td>yes</td>
</tr>
<tr>
<td>29</td>
<td>1 6 4 2 5 3</td>
<td>yes</td>
</tr>
<tr>
<td>30</td>
<td>1 6 4 5 2 3</td>
<td>no</td>
</tr>
</tbody>
</table>

In order to develop the general iterative method for generation of \((k+1)\)-legged compatible gaits from \(k\)-legged gaits, it is useful to consider the relationship between triped and biped gaits. This comparison will be simplified by temporarily adopting a new
notation for event sequences. This notation is defined as follows: the placing events 1, 2, ..., k will be relabelled as P₁, P₂, ..., Pₖ, while the lifting events k+1, k+2, ..., 2k will be denoted L₁, L₂, ..., Lₖ. In this way, the meaning of a symbol will not change as k is increased.

Turning to Figure 3.2, all of the compatible biped gaits have been presented as event sequences on a circle using the P^, L^ notation introduced above. Beginning with the event sequence E = 1234 (Figure 3.2(a)), since in any row and column canonical form it must be that P₃ follows P₂, the former event can be placed in one of three locations: between P₂ and L₁, between L₁ and L₂, or between L₂ and P₁. Figure 3.3(a) through 3.3(i) all relate to these possibilities. In 3.3(a) through 3.3(c), the event P₃ has been placed between P₂ and L₁, while event L₃ is successively placed between L₂ and P₁, P₁ and P₂, and P₂ and P₃. Clearly, L₃ cannot be placed prior to L₂ or the gait would not be comparable. However, one more possibility does exist; namely, L₃ could be placed between P₃ and L₁ leading to the event sequence E = P₁P₂P₃L₃L₁L₂ (E = 123654 in the previous notation). Clearly, however, this event sequence is not column compatible even though it is column comparable. Consequently, this sequence does not appear in Figure 3.3.

Continuing to Figure 3.3(d) through 3.3(e), P₃ has been placed between L₁ and L₂. In this case, a column comparable gait results if L₃ is placed between L₂ and P₁, P₁ and P₂, or P₂ and L₁. Reference to Table 3.4 shows that these gaits are all compatible.
Figure 3.2. Compatible Biped Event Sequences
Figure 3.3. Compatible Triped Event Sequences
Finally, if $P_3$ is placed between $L_2$ and $P_1$, then a compatible gait results if $L_3$ is placed between $P_3$ and $P_1$, $P_1$ and $P_2$, or $P_2$ and $L_1$ yielding Figure 3.3(g) through 3.3(i). If $L_3$ is placed between $L_2$ and $P_3$, the resulting gait is comparable, but not compatible.

The above construction can be summarized in the following two rules.

Rule 1. Event $P_3$ can be placed in any interval between biped gait events so long as it follows $P_2$ and precedes $P_1$ when the placement circle is traversed in a clockwise direction.

Rule 2. Event $L_3$ can be placed in any interval following $L_2$ and preceding $L_1$ and the resulting gait will be column comparable. However, if such an interval includes $P_3$ at its right or left boundary, then while $L_3$ may be placed on either side of $P_3$, only one of these two possibilities leads to a compatible gait. The other is merely comparable.

It has been seen that application of the above Rules 1 and 2 to Figure 3.2(a) leads to 11 comparable tripod gaits of which 9 are also compatible. Turning now to Figure 3.2(b), in this case there are only two allowed placements for event $P_3$: between $P_2$ and $L_2$ or between $L_2$ and $P_1$. If $P_3$ is placed between $P_2$ and $L_2$, then $L_3$ can be placed between $L_2$ and $P_1$ or between $P_1$ and $L_1$, leading to Figure 3.3(j) and 3.3(k). Table 3.4 shows that these
two comparable gaits are both compatible. If $P_3$ is placed between $L_2$ and $P_1$ then $L_3$ can go on either side of $P_3$ and a comparable gait will result. In addition, $L_3$ can be placed between $P_1$ and $L_1$, leading to a total of 3 comparable gaits for this placing of $P_3$. However, according to Rule 2, only 2 of these gaits can be compatible. Both are listed as Figure 3.3(1) and 3.3(n). As can be seen, when $P_3$ is placed between $L_2$ and $P_1$, then a compatible gait results if $L_3$ is placed between $P_3$ and $P_1$, but does not result if $L_3$ falls between $L_2$ and $P_3$. Taken altogether, the above analysis shows that 5 comparable tripod gaits of which 4 are compatible can be generated from the biped gait of Figure 3.2(b).

Continuing the above analysis according to Rules 1 and 2 shows that Figure 3.2(c) also generates 5 comparable gaits of which 4 are compatible. Figure 3.2(d) produces just one comparable gait which is also compatible. Thus, altogether, application of Rules 1 and 2 to the set of all compatible biped gaits produces a total of 22 tripod comparable gaits of which 18 are compatible. The remaining 8 tripod comparable gaits which are not generated from Figure 3.2 are obtainable by application of Rules 1 and 2 to the two other biped gaits which are comparable, but not compatible. Since those biped gaits are not compatible, there is no possibility of the corresponding tripod gaits being compatible as indeed they are not.

Rules 1 and 2 can be generalized to produce the set of all $(k+1)$-legged compatible gaits from the set of all $k$-legged compatible
gaits by the following algorithm:

1. Select a k-legged compatible gait not previously considered.
2. Place $P_{k+1}$ in the first interval following $P_k$.
3. Consider all allowed placings of $L_{k+1}$ according to Rule 2. Note that a test for compatibility is required if $P_{k+1}$ is on either end of the interval containing $L_{k+1}$. Write down all compatible gaits generated by various placings of $L_{k+1}$.
4. Place $P_{k+1}$ in the next allowed interval according to Rule 1. If there is no such interval, go to step 5. Otherwise, go to Step 3.
5. If all k-legged compatible gaits have been exhausted, the procedure is finished. Otherwise, go to Step 1.

This algorithm has been programmed and executed on a digital computer to obtain the number, $M_2$, of compatible gaits under $G_2$ for $2 \leq k \leq 8$. The results are presented in Table 3.5 while the program is detailed in Appendix II.

3.3.3 Equivalence Classes of Quadruped Compatible Gaits under $G_4$

In the previous section, the row and column canonical form was used as a means for studying equivalence classes. As pointed out in Section 3.2.1, the row and column canonical form does not produce the set of equivalence classes which can group together the greatest number of gait event sequences without destroying the Markov, connectivity, compatibility, regular realizability, and symmetric
Table 3.5: Number of $k$-Legged Row and Column Canonical Form Gait Matrices Which Are Column Compatible or Comparable.

<table>
<thead>
<tr>
<th>Number of Legs, $k$</th>
<th>Number of Compatible Matrices, $M_2$</th>
<th>Number of Comparable Matrices, $N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>82</td>
<td>140</td>
</tr>
<tr>
<td>5</td>
<td>370</td>
<td>630</td>
</tr>
<tr>
<td>6</td>
<td>1648</td>
<td>2772</td>
</tr>
<tr>
<td>7</td>
<td>7252</td>
<td>12012</td>
</tr>
<tr>
<td>8</td>
<td>31582</td>
<td>51480</td>
</tr>
</tbody>
</table>

realizability properties possessed by gait event sequences. Gait matrix complementation and column rotation can be used to further reduce the number of equivalence classes and increase the number of members in each equivalence class. In [9], the 82 row and column canonical forms of compatible quadruped gait matrices are reduced to 45 equivalence classes by using matrix complementation. A hand computation has been carried out by the author to rotate the columns of the above mentioned 45 equivalence classes of quadruped compatible gaits. The number of equivalence classes turns out to be 14. These 14 equivalence classes are listed below in Table 3.6. In this table, the event sequence from each gait class which has the minimal value (when read as a number in base 9 or base 10 notation) is selected to represent that class [23].
The number of row canonical forms in each class is also listed in the table. It is interesting to note that this analysis revealed two typographical errors in [9]; namely, referring to Table I in this reference, for gait 13 the minimum duty factor should be 2/3 while for gait 24 the maximum duty factor should be 1/2.

3.3.4 Group Invariants for Compatible Quadruped Gaits under $G_4$

If an analysis of gait is carried out on the basis of equivalence classes, it then becomes important to be able to readily determine the class to which an arbitrary gait belongs. One way to do this is to define a set of transformations which reduces every member of a given class to the same canonical form. This has been done in Table 3.6 in the sense that each class is represented by its minimal member. Given an arbitrary gait event sequence, such a minimal form may be obtained in a systematic way as outlined in [23]. Another more direct approach is to try to find some set of numerical attributes of the elements of an equivalence class which is the same for all members of one class, but differs from class to class. Such attributes or features are called group invariants [19]. The following quantities provide examples of group invariants for gait matrices.

**Definition 33:** Let $N = \min(n_0, n_1)$, where $n_0$ is the number of rows of all 0's and $n_1$ is the number of rows of all 1's in a connected gait matrix.
Table 3.6: Equivalence Classes of Column Compatible Connected Quadruped Gait Matrices.

<table>
<thead>
<tr>
<th>Equivalence Class Number</th>
<th>Number of Members</th>
<th>Minimal Event Sequence</th>
<th>Maximum Duty Factor</th>
<th>Minimum Duty Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24</td>
<td>12345678</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>48</td>
<td>12354678</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>12356478</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>48</td>
<td>12356748</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>48</td>
<td>12358467</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>6</td>
<td>48</td>
<td>12365478</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>48</td>
<td>12365748</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>48</td>
<td>12365748</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>48</td>
<td>12365748</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>48</td>
<td>12385467</td>
<td>2/3</td>
<td>1/2</td>
</tr>
<tr>
<td>11</td>
<td>24</td>
<td>12385647</td>
<td>2/3</td>
<td>1/3</td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>12853467</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>13</td>
<td>12</td>
<td>15263478</td>
<td>1/4</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>12</td>
<td>17283546</td>
<td>3/4</td>
<td>1/2</td>
</tr>
</tbody>
</table>

Note: Class 1 and Class 11 are self-complementary classes. Class 12 is column compatible, but not regularly realizable. The duty factors given are for the minimal event sequence for each equivalence class. For classes which are not self-complementary, complementary gaits are realizable with complementary duty factor ranges [9].

Definition 34: Let \( M = \max(m_0, m_1) \), where \( m_0 \) is the number of rows of all 0's and \( m_1 \) is the number of rows of all 1's in a connected gait matrix.

Definition 35: Let \( L = \min(l_0, l_1) \), where \( l_0 \) is the minimum number of 0's in any column and \( l_1 \) is the minimum number of 1's in any column of a connected gait matrix.

Definition 36: Let \( K = \max(k_0, k_1) \), where \( k_0 \) is the number of rows with three 0's in a connected gait matrix, and \( k_1 \) is the number of rows with three 1's in the same gait matrix.
Evidently, all of these quantities are invariant under $G_4$ so it is reasonable to hope that all or some subset of them will be sufficient to identify the class of a given compatible gait. Note that compatibility is easily determined by inspection of the columns of a given matrix and that the above invariants are to be used only for those matrices which pass such a compatibility test.

In Table 3.7, the 14 equivalent classes of quadruped compatible gaits are listed, together with their associated $K$, $L$, $M$ and $N$. From this table, it can be seen that $K$, $L$, $M$, and $N$ are different for different gaits. These four numbers are thus sufficient to determine the class of a given compatible quadruped gait. It also can be easily checked that no two of these four numbers are sufficient for this purpose. However, still further examination of the table shows that knowledge of $K$, $L$, and $M$ is adequate to identify the class of a compatible quadruped gait. No other subset of these invariants constitutes such a complete set of invariants [19].

It would be interesting to attempt to extend this type of invariants to encompass compatible gaits for more than four legs. However, this has not been done in this dissertation. Instead, in the next section, a new type of invariant which is still easier to compute is introduced. This invariant is used to define a new type of canonical form analogous to the row and column canonical form, but which is applicable under $G_4$ rather than just $G_2$. 
3.3.5 Interval Canonical Form for a Gait Matrix

The minimum number of equivalence classes for quadruped compatible gaits has been worked out by hand calculation and presented in the preceding Section 3.3.3. The same work could be extended to hexapod compatible gaits. However, since there are 1648 row and column forms of hexapod compatible gait, this would be very burdensome for hand computation. A new method which uses the interval between placing events and lifting events as invariants to identify different gait classes has been developed to find the number of equivalence classes of compatible gaits under $G_4$. This method can be generally used for any number of legs. The hexapod compatible gaits are used as an illustrative example.

The following steps describe how a gait matrix may be transformed to its interval canonical form.

**Step 1.** From a given gait matrix $G$, derive the row and column canonical form event sequence for $G$ and put it on a circle.

**Step 2.** Starting with event 1, compute the intervals between one placing event and the next placing event. Write down the sequence of values of these intervals following the canonical placing order, i.e., $1, 2, 3, \ldots, k$. This sequence is called the placing interval index, $I_0$.

**Step 3.** Perform all possible rotations of the numbers in $I_0$. The sequence which has the minimum value under such rotations is called the canonical placing interval index. It is symbolized by $I_1$.

**Step 4.** Complement the gait matrix, and repeat Steps 1, 2, and 3.
Table 3.7: Group Invariants for Compatible Quadruped Gaits under $G_4$.

<table>
<thead>
<tr>
<th>Equivalence Class No.</th>
<th>Minimal Event Sequence</th>
<th>Group Invariants</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12345678</td>
<td>2 4 1 16</td>
</tr>
<tr>
<td>2</td>
<td>12354678</td>
<td>2 3 1 14</td>
</tr>
<tr>
<td>3</td>
<td>12356478</td>
<td>1 2 1 12</td>
</tr>
<tr>
<td>4</td>
<td>12356748</td>
<td>3 1 2 10</td>
</tr>
<tr>
<td>5</td>
<td>12385467</td>
<td>2 2 1 12</td>
</tr>
<tr>
<td>6</td>
<td>12536478</td>
<td>4 2 1 10</td>
</tr>
<tr>
<td>7</td>
<td>12536748</td>
<td>4 1 2 8</td>
</tr>
<tr>
<td>8</td>
<td>12563478</td>
<td>4 2 2 8</td>
</tr>
<tr>
<td>9</td>
<td>12563748</td>
<td>4 1 3 6</td>
</tr>
<tr>
<td>10</td>
<td>12835467</td>
<td>3 3 0 14</td>
</tr>
<tr>
<td>11</td>
<td>12835647</td>
<td>2 4 0 16</td>
</tr>
<tr>
<td>12</td>
<td>12853467</td>
<td>2 3 0 16</td>
</tr>
<tr>
<td>13</td>
<td>15263748</td>
<td>4 1 4 4</td>
</tr>
<tr>
<td>14</td>
<td>17283546</td>
<td>4 3 0 12</td>
</tr>
</tbody>
</table>

The newly obtained interval index is called $I_2$.

Step 5. The canonical interval sequence, $I(G)$, is defined as $I(G) = \min(I_1, I_2)$. The interval canonical form for $G$ is obtained by taking the form corresponding to $I(G)$ and rotating it to row canonical form. The corresponding event sequence will be called $E^*$.

The following example is used to illustrate the above procedures.

The gait event sequence is a compatible hexapod gait.
Step 1: $G$ is defined by the event sequence

$$E_G = 1-12-3-11-2-6-10-7-9-5-4-8$$

Thus, if $G'$ is the row and column canonical form for $G$,

$$G = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix} \rightarrow G' = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
\end{bmatrix}$$

The event sequence for $G'$ is evidently

$$E_G = 1-10-2-11-3-4-12-7-8-5-6-9$$

Presenting this sequence on a circle produces the following diagram:

Event sequence for $G'$
Step 2: \begin{align*} I_o &= 2-2-1-4-1-2 \\
\end{align*}

Step 3: By rotating \( I_o \) right two positions:

\[ I_1 = 1-2-2-2-1-4 \]

Step 4: If \( H \) is the row canonical form of the complement of \( G \), and if \( H' \) is the row and column canonical form for \( H \), then from Step 1:

\[
H = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
H' = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

\[ E_{H'} = 1-2-11-12-3-7-4-8-5-9-10-6 \]
The placing index for $H'$ is evidently:

$$I = 1-3-2-2-3-1$$

Rotation of $I$ one step right produces:

$$I_2 = 1-1-3-2-2-3$$

Step 5: $I(G) = \min\{I_1, I_2\} = I_2 = 1-1-3-2-2-3$

Thus, since $I_2 < I_1$, if $P$ is the interval canonical form for $G$, then $P$ is obtained by rotating $H'$ right one to correspond to $I_2$ and then rotating the result down one to restore it to row canonical form; i.e.,

$$P = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix} \quad (3-14)$$

Thus, in summary, $I(G) = 1-1-3-2-2-3$ \hspace{1cm} (3-15)

$$E^* = E_p = 1-2-3-12-7-4-8-5-9-6-10-11 \quad (3-16)$$
This example should make clear the existence of a unique canonical interval sequence and a unique interval canonical form for any connected gait matrix. These forms partition the set of all k-column gait matrices into equivalence classes under $G^4$.

Corresponding to the canonical placing interval index $I$ associated with a given gait, is a unique canonical lifting interval index, $L$. $L$ is computed by starting at event 2 and counting intervals between subsequent lifting events. This sequence is then rotated to obtain a minimal value when the sequence is read as a number. Thus, for example, if the canonical interval sequence found in the preceding example is considered, namely $I(G) = 1-1-3-1-3-3$, then reference to the circle showing the event sequence for $G'$ reveals that $L(G) = 1-2-1-4-1-3$. From this discussion, evidently $L(G)$ is precisely $I_1(\overline{G})$, where $\overline{G}$ is the complement of a gait matrix $G$.

While $I(G)$ uniquely determines $L(G)$, in general the $k$ lifting events can be placed in a total of $k!$ arbitrary ways in the locations allocated to lifting events. However, an exception occurs in the case of column comparable gaits. Column comparability demands that the lifting sequence be in the same order as the placing sequence. Thus, if a gait is in interval canonical form, the lifting sequence can only be a rotation of the sequence $k+1, k+2, \ldots, 2k$. To make this notion more precise, define a reference orientation of the lifting events as that interval canonical form in which the first lifting event following event 1 is event $k+1$. The shift index, $i$,
of any other interval canonical form of comparable gait possessing
the same value for $I(G)$ is the amount of clockwise rotation of lifting
events needed to bring the reference orientation into agreement with
the given gait. As an example, consider the interval canonical form
given by Eq. (3-16). Evidently this is a comparable gait. Since
event 7 is the second placing event in this sequence rather than
the first, $i=1$ in this case.

In general, the shift index is uniquely defined for a given
gait. But in case $I_1 = I_2$, some special considerations are necessary.
Since $I(G) = \min(I_1, I_2)$, this quantity is not changed by gait matrix
complementation. Providing that the two shift indices $i_1$ and $i_2$ are
also equal, such gaits are evidently self-complementary under $G_4$,
and $I_1$ and $I_2$ consequently lead to the same value for $E^*$. Thus either
$I_1$, $i_1$, or $I_2$, $i_2$ can be used as the interval canonical form for such
a gait. However, in the general case, since $I_1$ and $I_2$ have switched
their placing and lifting states the two shift indices may be expected
to differ. One way to uniquely define the shift index in this case is
to choose the value associated with $I_1$. This will be done in what
follows.

The above statements have given some basic ideas concerning
the use of interval canonical form to represent gait equivalence
classes. These concepts have been used to find the hexapod compatible
gait equivalence classes by means of a hand calculation. The following
paragraph presents an algorithm for this procedure. It turns out that
the 1648 connected row and column canonical form hexapod gaits can be further grouped into 148 canonical interval forms. These 148 canonical forms are all listed in Table 3.8. It should be noted that the use of the shift index notation allows for a very compact presentation of these results. Since the gait defined by Eq. (3-14) is obviously compatible, it should appear in Table 3.8 under $i, i = 113223, 1$. Inspection of the table reveals that this entry does in fact appear in row 29.

The algorithm used to obtain Table 3.8 proceeds according to the following steps:

1. Write down the smallest six-digit number using only the integers 1, 2, ..., 7 and such that the digits sum to twelve. Evidently this number is:
   
   $I = 111117$

   Consider this number to be the canonical interval sequence for the placing events of a six-legged gait and derive the corresponding complementary interval sequence. Evidently
   
   $L = 111117$.

2. Consider all 6 possible values for the shift index, $i$, and determine which lead to a compatible gait for the given pair $I, L$. Write down these values for $i$.

3. Determine the next smallest six-digit number satisfying the constraints of step 1 and check to see if this number
appears as a previous value of I or L or as a rotation of some such previous value. If no, continue. If yes, repeat this step.

4. Compute L and list the pair I,L. Determine all values of shift index i which lead to a compatible gait and add these to the table. If I = 222222, the procedure is finished. Otherwise, go to step 3.

3.4 Regular Gait

Regular gaits are defined to be those gaits which have the property that the duty factor of all legs is the same. There is no known general formula or iterative method to count regular gaits. Instead, a given gait event sequence must be tested to see whether it is qualified to be used as a regular gait; i.e., whether or not it is regularly realizable. Two methods have been used to accomplish such tests. The first method [8] involves solving a quadratic programming problem. The second method [9] involves solving a linear programming problem. Since a linear programming problem is much easier to handle, it is adopted here to test for regular realizability. The particular linear programming problem to be solved is defined in Chapter II in Definition 17.

The total possible number of gaits is, in general, too large to be used to determine regular realizability. In finding the quadruped regular gaits, McGhee and Jain [9] have used the necessary condition of compatibility. They tested the 82 row and column
Table 3.8: Interval Invariants and Shift Indices for all Equivalence Classes of Compatible Gait under $G_4$.

<table>
<thead>
<tr>
<th>No.</th>
<th>Canonical Interval Sequence, $I$</th>
<th>Complementary Interval Sequence, $L$</th>
<th>Complementary Shift Indices, $i$</th>
<th>Duty Factor Range</th>
<th>Number of Members</th>
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canonical forms of quadruped compatible gaits and found that 80 of them are also regularly realizable. In the preceding Section 3.3.3, the equivalence classes associated with these gaits are all listed together with their allowable duty factor ranges. This range is also obtained from the solution of the associated linear programming problem [9].

In the previous section, it was shown that there are 148 six-legged compatible gait equivalence classes under $G_4$. To find the six-legged regularly realizable gaits, it is only necessary to test these 148 equivalence classes. The work has been done by a standard linear programming routine on a PDP-9 computer. The result is that there are 135 equivalence classes of six-legged regularly realizable gaits. These are all indicated in Table 3.8 along with their duty factor ranges. Inspection of row 29 of this table shows that the gait defined by Eq. (3-14) is regularly realizable with a duty factor in the range $\frac{1}{3} \leq \beta \leq \frac{1}{2}$.

3.5 Summary

In this chapter, four different kinds of permutation groups have been discussed with regard to their ability to reduce the number of equivalence classes of connected gait event sequences. Event sequence permutations under these four groups are able to preserve the connectivity, compatibility, regular realizability, symmetric realizability, and Markov properties of a gait. Three of these four permutation groups have been used in previous work on gait
enumeration [8,9]. The other one is a new permutation group developed in this chapter. By using this new permutation group, called the relabelling group, $G_4$, the 5040 possible row canonical form connected quadruped gaits have been grouped into 120 equivalence classes.

Compatibility is a necessary condition for a gait to be regularly realizable. A general iterative method has been developed in this chapter to count the number of row and column canonical forms of compatible gait for any number of legs. Following this work, the concept of interval invariants is introduced. These invariants have been used to develop a procedure to use the interval canonical form of gait matrices to find the number of equivalence classes of hexapod compatible gait. Some other types of invariants relating to quadruped compatible gaits have also been discussed.

The newly derived results on hexapod compatible gaits have been used to extend existing results on regular gait enumeration. Out of the 145 interval canonical forms of hexapod compatible gait, 135 are found to be also regularly realizable. It is in addition shown that the 80 previously known row and column canonical form quadruped regular gaits can be reduced to 13 equivalence classes if the interval canonical form or some other canonical form under $G_4$ is used.
4.1 Introduction

The purpose of this chapter is to introduce some methods for symmetric gait study. The first work is on the enumeration problem. A general enumeration formula for symmetric gait is developed. Next, the subsets of such gaits which are also regular are studied, the so-called regular symmetric gaits [8]. Numerical results are given for quadruped and hexapod gaits. The stability problem is discussed next. A detailed investigation has been carried out on the stability margin of six-legged regular symmetric gaits. The optimal gait is defined to be one with the largest stability margin. A CRT display simulation program is also given which permits a visual study of gait stability. From the stability study, it is shown that the optimal regular symmetric gaits have wave gait footfall sequences. A general discussion of wave gaits is then presented. Finally, a comparison is given of the observed animal gaits and the theoretically studied regular symmetric gaits.

4.2 Symmetric Gaits

4.2.1 Mathematical Symmetry and Physical Symmetry

Referring to Definitions 11 and 12 in Chapter II, within the set of all connected gait matrices, there is a subset having the
special property of symmetric realizability. For such a matrix, a
duration vector can be found such that the legs in each symmetric
pair have a relative phase of exactly one-half a cycle. These gaits
are called symmetric gaits. Symmetric gaits are preferred above all
others by animals with the few exceptions occurring mainly in the
higher species [6,10,23]. Due to the evident importance of this
class of gaits, this section of this chapter is devoted to a study
of their general properties.

Let \((A,B)\) be a symmetric pair of legs in a legged locomotion
system; the placing events are \(a, b\), and \(\bar{a}, \bar{b}\) are the lifting events.
From the definition of a symmetric pair, the following relations hold:

\[ \phi_a = \phi_b + 0.5 \pmod{1} \]  \hspace{1cm} (4-1)

\[ \phi_{\bar{a}} = \phi_{\bar{b}} + 0.5 \pmod{1} \]  \hspace{1cm} (4-2)

where the notation \((\text{mod } 1)\) means the residue of a number after
dividing by 1; i.e., the fractional part of the number. In these
equations, \(a\) and \(b\) could refer to any two columns in a gait matrix
constituting a symmetric pair. This, therefore, a mathematical
concept. However, in zoology and physiology, the term "symmetrical"
also has a geometrical meaning. Specifically, in discussions of
animal gaits, the symmetric pair cannot be an arbitrary set of two
legs. Rather, such a pair must consist of a given leg together with
its adjacent leg on the other side of the body [6,12,13]. For
quadruped animals, this means the observed symmetric pairs must be
front leg and hind leg pairs. This kind of symmetry will be called
physical symmetry to distinguish it from the previous definition of
symmetric gait which will henceforth be referred to as mathematical
symmetry.

4.2.2 Enumeration of Physically Symmetric Gaits

The permutation groups discussed in the last chapter preserve
the mathematical symmetry of a gait. However, physical symmetry of
a gait is a geometrical property which can be destroyed by arbitrary
column permutations. In the remainder of this chapter, only physical
symmetry will be studied and any reference to symmetry or symmetric
gait can be taken to mean this type of symmetry. In the ensuing
analysis, two permutation groups which preserve the physical symmetry
of gait event sequences are introduced and used to enumerate the
physical symmetric gaits in row canonical form. The following two
theorems define these permutations:

Theorem 4-1: Let P and Q be two columns of a gait matrix, G,
with an assigned duration vector t, such that legs P and Q consti­
ture a symmetric pair and let R and S be another pair of legs with
phase variables defined by: \( \phi_r = \phi_p \), \( \phi_{r'} = \phi_q \), \( \phi_s = \phi_q \), \( \phi_{s'} = \phi_p \).
Then R and S are again a symmetric pair.
Proof: From Eq. (4-1) and (4-2)
\[
\phi_p = \phi_q + \frac{1}{2} \pmod{1} \quad (4-3)
\]
\[
\phi_{p'} = \phi_{q'} + \frac{1}{2} \pmod{1} \quad (4-4)
\]
Thus, \[ \phi_{\tau} = \phi_{p} = \phi_{q} + 0.5 = \phi_{s} + 0.5 \quad \text{(mod 1)} \] (4-5)

and

\[ \phi_{\tau} = \phi_{q} = \phi_{p} + 0.5 = \phi_{s} + 0.5 \quad \text{(mod 1)} \] (4-6)

This proves the theorem.

The next class of transformations is best explained through the introduction of yet another gait representation called a gait phase diagram. This diagram is just a presentation of an event sequence on a circle in which the angular position of event \( i \) is given by

\[ \theta_{i} = 2\pi \phi_{i} \quad , \quad i = 1, 2, \ldots, k \] (4-7)

\[ \theta_{i} = 2\pi [\phi_{i-k} + \beta_{i-k} \quad \text{(mod 1)}] \quad , \quad i = k+1, \ldots, 2k \] (4-8)

Figure 2.5 is in fact a gait phase diagram for the gait of Figure 2.2.

Gait phase diagrams for physically symmetric gaits have the special property that the two placing events of a symmetric pair lie on opposite ends of a diameter. The same is true for the two lifting events of such a pair. In terms of gait phase diagram, Theorem 4-1 simply states that any two events lying on opposite ends of a diameter can be exchanged without destroying physical symmetry. The next theorem relates to exchanges of two pairs of such events.

Theorem 4-2: Let \( a_{1} \) and \( a_{2} \) be two events associated with the two ends of a diameter \( A \) of a physically symmetric gait and let \( b_{1} \) and \( b_{2} \) be the corresponding events associated with another diameter \( B \). Then the two diameters may be exchanged along with their events without destroying gait symmetry.
Proof: Evidently, the phase separation of .5 which is required for both the lifting and placing events of a physical symmetric gait is not affected by exchange of diameters. Therefore, physical symmetry is preserved by this type of event permutation.

Theorem 4-1 and 4-2 provide the basis for enumeration of the number of event sequences which can be realized as physically symmetric gaits. To obtain this number, let $d_1$ be the diameter whose ends are labelled with events 1 and 2, $d_2$ be the diameter with ends 3 and 4, etc. Thus, in general, $d_j$ relates to the pair of events $(2j-1, 2j)$. The reference orientation of the diameter is defined as the event placement such that the event $2j-1$ precedes the event $2j$ in traversing the gait phase diagram beginning at event 1. The complementary orientation is denoted $\overline{d}_j$ and for $j \geq 2$ is obtained by exchanging the events on the two ends of $d_j$ as permitted by Theorem 4-1. Complementation is not defined for $j = 1$. These definitions permit the statement and proof of the next theorem.

Theorem 4-3: The number of distinct row canonical forms of event sequences for physically symmetric gaits is given by

$$M = \frac{(k-1)!}{2}$$

where $k$ is the (even) number of legs.

Proof: Evidently the event sequence for any $k$-legged symmetric gait can be described as a sequence of diameters, $D = d_1 d_2 \ldots d_k$, where each $d_j$ is either a diameter $d_j$ or a complementary diameter $\overline{d}_j$. 

73
Each diameter $d_1$ must appear exactly once in such a sequence in either of its two possible forms (except for $d_1$, for which no complement is defined) and the order of appearance is arbitrary. There are therefore $k!2^{k-1}$ distinct diameter sequences. Any such sequence is put into canonical row form by one of $k$ rotations. The number of canonical row event sequences is thus

$$M = \frac{k!2^{k-1}}{k} = (k-1)^2.$$  

This proves the theorem.

4.2.3 Tabulation of Symmetric Gaits

The proof of Theorem 4-3 is constructive. Consequently, it is possible to tabulate the symmetric gaits by making use of diameter sequences. This has been done by hand calculation for quadrupeds. The results are presented in Table 4.1. These 48 gaits are presumably the non-singular members of the 104-element set of all symmetric quadruped gaits studied by Hildebrand [11].

An alternative method of studying symmetric gaits is to recognize that a $k$-legged symmetric gait is specified by a reduced gait formula,

$$g = (\beta_1, \beta_3, \ldots, \beta_{k-1}, \phi_3, \phi_5, \ldots, \phi_{k-1}) \quad (4-10)$$

In general, for $2K$-legged gait, this formula involves $2K-1$ parameters rather than the $4K-1$ associated with an arbitrary gait. That is, only the odd numbered components of the gait formula are independent variables for symmetric gait. In the particular case $k=4$, Eq. (4-10)
Table 4.1: Symmetric Quadruped Gaits

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reduces to the three independent variables

\[ g = (\beta_1, \beta_2, \phi_3) \]  \hspace{1cm} (4-11)

Thus, each symmetric quadruped gait can be related to a point in the unit 3-cube. It is known that, in such circumstances, a set of separating planes can be found which divides this cube into regions, in this case 48 in number, with each such region corresponding to one of the event sequences of Table 4.1. This is essentially the method used by Hildebrand [11] and this analysis will not be repeated here. A similar method will be used in the next section of this chapter, however, to study regular symmetric gait.

4.3 Regular Symmetric Gait

4.3.1 General Properties of Regular Symmetric Gaits

The following are two general properties obtained from studies of regular gaits and symmetric gaits [8,9,16]:

(1) For regular gaits, the duty factor of all the legs is the same. Thus, only one duty factor parameter is required for all legs of a regular gait.

(2) For 2K-legged symmetric gait, there are K symmetric pairs. The relative phase of the two legs of a symmetric pair is exactly equal to one-half of a cycle. This implies the motion of the legs on one side are synchronized with the legs on the other side, but with a phase difference of half a cycle. This reduces the parameters used to specify the relative phase of a 2K-legged...
symmetric gait to \((K-1)\) instead of \((2K-1)\) for the general case.

The 2K-legged regular symmetric gaits have both the above stated properties. They need only \(K\) parameters to define a 2K-legged regular symmetric gait formula, rather than the 2K-1 required by an arbitrary symmetric gait. Specifically the general gait formula for 2K-legged regular symmetric gaits is

\[
g = (\beta, \phi_1, \phi_3, \ldots, \phi_{k-1})
\]

where \(\phi_i\)'s are the relative phase lag of the placing event of leg \(i\) to that of leg 1, \(\beta\) is the duty factor of all the legs, and \(k = 2K\).

4.3.2 Enumeration Methods

Thus far in this dissertation, the enumeration of regular gaits and symmetric gaits has been discussed. Therefore, there are at least two methods that could be used for regular symmetric gait enumeration. They are:

Method 1: Test all the regular gaits to find out how many of them are also symmetric.

Method 2: Test all the symmetric gaits to see how many of them are also regularly realizable.

For the following reason, these two methods are not used in what follows.

The number of possible regular gaits or symmetric gaits is quite large for more than four-legged locomotion. The above mentioned two methods need a great deal of storage to keep the data. The test procedure is also very tedious. As mentioned earlier in this chapter,
there are only K independent parameters required to specify a 2K-legged regular symmetric gait. By giving values to these parameters, a regular symmetric gait event sequence is obtained. The values of \( \beta \) and all of the \( \phi_i \)'s have ranges varying between 0 and 1. It is then possible to generate all the regular symmetric event sequences by first quantizing the values between 0 and 1 in all the coordinates of \( \beta \) and \( \phi_i \)'s and then testing event sequences at the generated sample points.

In general, this method is not sufficient to give a definite solution. The quantization value is hard to determine so that the sample points will cover all the possible cases. But if the distribution of the different gaits is known, there might be a way to find satisfactory quantization values. The distribution of the gaits in the unit cube is determined by separation planes [6,11]. For 2K-legged regular symmetric gaits, the separation planes are of the following two forms:

\[
\phi_i \equiv \phi_j \pmod{1} \text{ for } i=1,\ldots,2K, \ j=1,2,\ldots,i-1,i+1,\ldots,2K, \quad (4-13)
\]

\[
\phi_i \equiv \beta \pmod{1} \text{ for } i=1,\ldots,2K. \quad (4-14)
\]

These equations were solved for quadrupeds by Hildebrand [6,11]. His results are summarized by the diagram shown on Figure 4.1. This figure also shows the event sequences for each region of this diagram together with their minimal canonical forms. As can be seen, only 3 of the 13 equivalence classes of regular quadruped gait are
<table>
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<th>Gait No.</th>
<th>Event Sequence</th>
<th>Minimal Canonical Event Sequence</th>
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<tr>
<td>16</td>
<td>1 3 6 8 2 4 5 7</td>
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</tr>
</tbody>
</table>

Figure 4.1. Regular Symmetric Quadruped Gaits
also symmetric. Referring to Table 3.6, these 3 are equivalence classes 8, 13, and 14. Each of these classes is indicated by a different style of cross-hatching on Figure 4.1.

While Hildebrand's results were obtained by intuitive processes, the analysis of six-legged regular symmetric gait is sufficiently complicated to demand the more formal treatment implied by Eq. (4-13) and (4-14). The following is a step-by-step determination of six-legged regular symmetric gaits by means of separating planes. It is hoped that this work will help to clarify the general method.

The three independent variables for six-legged regular symmetric gaits are $\phi_3$, $\phi_5$, and $\beta$. The following are the 12 placing and lifting events for the six legs. $1,2,\ldots,6$ are placing events, and $7,8,\ldots,12$ are lifting events.

<table>
<thead>
<tr>
<th>placing events</th>
<th>lifting events</th>
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<tbody>
<tr>
<td>$\phi_1 = 0 \pmod{1}$</td>
<td>$\phi_7 = \beta \pmod{4}$</td>
</tr>
<tr>
<td>$\phi_2 = 0.5 \pmod{1}$</td>
<td>$\phi_8 = \phi_7 + 0.5 \pmod{4}$</td>
</tr>
<tr>
<td>$\phi_3 = \phi_3 \pmod{1}$</td>
<td>$\phi_9 = \phi_3 + \beta \pmod{1}$</td>
</tr>
<tr>
<td>$\phi_4 = \phi_3 + 0.5 \pmod{1}$</td>
<td>$\phi_{10} = \phi_9 + 0.5 \pmod{1}$</td>
</tr>
<tr>
<td>$\phi_5 = \phi_5 \pmod{1}$</td>
<td>$\phi_{11} = \phi_5 + \beta \pmod{1}$</td>
</tr>
<tr>
<td>$\phi_6 = \phi_5 + 0.5 \pmod{1}$</td>
<td>$\phi_{12} = \phi_{11} + 0.5 \pmod{1}$</td>
</tr>
</tbody>
</table>

The separation planes are determined by using Eq. (4-13) and (4-14). That is, by finding all those places in the unit three-cube where two or more of the above events occur simultaneously.
The complete derivation of these conditions is very lengthy and will not be repeated here. Instead, it is illustrated by the following three examples:

Example 1: Find the separation plane determined by

\[ \phi_3 = \phi_4 \pmod{1}. \]

Solution: \( \phi_3 = \phi_3 + 0.5 \pmod{1} \)

\[ 0 = 0.5 \pmod{1} \]

The above equation cannot be satisfied. This shows that no separation plane can be found under \( \phi_3 = \phi_4 \pmod{1} \).

Example 2: Find the separation plane determined by

\[ \phi_4 = \beta \pmod{1}. \]

Solution: \( \phi_4 = \beta \pmod{1} \)

Since \( 0 \leq \phi_4 < 1 \), there is one separation plane,

\[ \phi_4 - \beta = 0, \text{ under } \phi_4 = \beta \pmod{1}. \]

Example 3: Find the separation plane determined by

\[ \phi_4 = \phi_{11} \pmod{1}. \]

Solution: \( \phi_4 = \phi_{11} \pmod{1} \)

\[ \phi_3 + 0.5 = \phi_5 + \beta \pmod{1} \]

\[ \phi_3 - \phi_5 - \beta = -0.5 \pmod{1} \]

This implies:

\[ \phi_3 - \phi_5 - \beta = 0.5 \]

\[ \phi_3 - \phi_5 - \beta = -0.5 \]

\[ \phi_3 - \phi_5 - \beta = -1.5. \]

Thus there are three separation planes under \( \phi_4 = \phi_{11} \pmod{1} \).
Completion of all calculations shows that there are 34 different separation planes for six-legged regular symmetric gaits. They are listed in Table 4.2.

All the six-legged regular symmetric gaits are distributed in a three-dimensional space of $\phi_3$, $\phi_5$, and $\beta$. In order to see the relation between the separation planes and the gait distribution, 10 graphs are presented in Appendix III. These ten graphs are sections of the gait space along the $\beta$ coordinate with a 0.10 unit separation.

From the ten graphs in Appendix III, it can be seen that 10 sample points for each $\phi_3$, $\phi_5$, and $\beta$ between 0 and 1 will be enough to reveal all the possible regular symmetric gaits. A computer program has been written to generate all of these gaits. The detailed program is included in Appendix III. Figure 4.1 exhibits the quadruped gait event sequences and equivalence classes thus obtained. Table 4.3 presents the same information for hexapod gaits.
Table 4.2: Separation Planes for Six-Legged Regular Symmetric Gaits

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<th>(\phi_5)</th>
<th>(\beta)</th>
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<td>(\phi_5 - \beta) = 0</td>
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Table 4.3: Equivalence Class Numbers and Shift Indices for Hexapod Regular Symmetric Gaits

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<th>Event Sequence</th>
<th>Interval Invariant Equivalence Class</th>
<th>Shift Index</th>
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| 256 | 1 10 7 6 12 3 2 9 8 5 11 4 | 41 | 1
| 257 | 1 4 7 10 5 11 2 3 8 9 6 12 | 41 | 0
| 258 | 1 4 7 13 6 12 2 3 8 9 5 11 | 41 | 1
| 259 | 1 4 7 5 10 11 2 3 8 6 9 12 | 41 | 0
| 260 | 1 12 4 7 10 5 2 11 3 8 9 6 | 41 | 0
| 261 | 1 4 7 6 10 12 2 3 8 5 9 11 | 41 | 0
| 262 | 1 11 4 7 10 6 2 12 3 8 9 5 | 41 | 0
| 263 | 1 5 7 4 11 10 2 6 8 3 12 9 | 41 | 0
| 264 | 1 7 4 5 10 11 2 8 3 6 9 12 | 41 | 1
| 265 | 1 12 7 4 5 10 2 11 8 3 6 9 | 41 | 0
| 266 | 1 12 7 4 10 5 2 11 8 3 9 6 | 41 | 1
| 267 | 1 6 7 4 12 10 2 5 8 3 11 9 | 41 | 0
| 268 | 1 7 4 6 10 12 2 8 3 5 9 11 | 41 | 1
| 269 | 1 11 7 4 6 10 2 12 8 3 5 9 | 41 | 0
| 270 | 1 11 7 4 10 6 2 12 8 3 9 5 | 41 | 1
| 271 | 1 5 7 11 4 10 2 6 8 12 3 9 | 41 | 1
| 272 | 1 7 5 4 11 10 2 8 6 3 12 9 | 41 | 1
| 273 | 1 6 7 12 4 10 2 5 8 11 3 9 | 41 | 0
| 274 | 1 7 6 4 12 10 2 8 5 3 11 9 | 41 | 1
| 275 | 1 9 5 7 11 4 2 10 6 8 12 3 | 41 | 0
| 276 | 1 9 7 5 4 11 2 13 8 3 9 6 | 41 | 1
| 277 | 1 9 6 7 12 4 2 10 5 8 11 3 | 41 | 0
| 278 | 1 9 7 6 4 12 2 10 8 5 3 11 | 41 | 0
| 279 | 1 9 7 5 11 4 2 10 8 6 12 3 | 41 | 1
| 280 | 1 9 7 6 12 4 2 10 8 5 11 3 | 41 | 1
| 281 | 1 7 3 9 5 11 2 8 4 10 6 12 | 44 | 0
| 282 | 1 7 3 9 6 12 2 6 4 10 5 11 | 44 | 1 |
| 283 | 1 7 5 11 3 9 2 8 6 12 4 10 | 44 | 0 |
| 284 | 1 7 6 12 3 9 2 8 5 11 4 10 | 44 | 0 |
| 285 | 1 7 4 10 5 11 2 8 3 9 6 12 | 44 | 0 |
| 286 | 1 7 4 10 6 12 2 8 3 9 5 11 | 44 | 0 |
| 287 | 1 7 5 11 4 10 2 8 6 12 3 9 | 44 | 0 |
| 288 | 1 7 6 12 4 10 2 8 5 11 3 9 | 44 | 0 |
4.4 Stability

4.4.1 Introduction

When a legged vehicle is moving, it is essential that it not fall during its motion. From a practical point of view, it is likely that for some time the only type of legged vehicle capable of operating under automatic control will be those in which all gait phases are statically stable [23]. There has been some research on static stability for quadruped and hexapod gaits [16,17,23]. This work basically consists of the following two steps:

(1) Find the support patterns of a gait formula at different instants of the motion.

(2) Find the relative position between the center of gravity of the body and the support pattern to see if it is stable.

The work in this section concerns six-legged regular symmetric gait and constitutes a further elaboration of the work of Bessonov and Umnov [17]. A computer program has been written to do the complicated analysis work. This work can be easily extended to more than six-legged regular symmetric gaits.

To begin with, a description of the mathematical model of the vehicle and a brief summary of the previous work is given. The notion of stability margin is introduced and discussed. Finally, a display simulation program is described.

4.4.2 Ideal Six-Legged Locomotion Vehicle

The stability properties associated with a gait formula cannot
be studied out of the context of a particular vehicle geometry. In this study, the ideal legged locomotion machine defined by McGhee and Frank [16] is used as the mathematical model for six-legged regular symmetric gait stability analysis. The following are some of the definitions and results worked out by the above mentioned authors.

The first is the definition of an ideal legged locomotion machine.

Definition 37: An ideal legged locomotion machine is a rigid body to which are attached a specified number K of massless legs. The length of each leg is arbitrarily controllable. Each leg contacts the supporting surface at a point and can exert an arbitrary force directed into this surface. Arbitrary moments can be applied to the body by any leg subject only to the constraint that no moment be applied to the supporting surface at any leg contact point.

The support pattern for any phase of a gait is defined as follows:

Definition 38: The support pattern associated with any phase of a given gait of an ideal legged locomotion machine is the minimum area convex point set in the support plane such that all of the leg contact points are contained.

Figure 4.2 shows the sequence of support patterns for an arbitrarily selected six-legged regular symmetric gait. The horizontal lines within each pattern show the motion of the vehicle
Figure 4.2. Support Pattern for a Hexapod. \( k = (0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 1.5, 0.94, 0.28, -0.23, -1.56, -1, 0.5, -0.5, 0.5, -0.5, 0.5, -0.5, 0.5, 0.8, 0.3, 0.5, 0.55). \)
center of gravity during that phase. Support patterns are related
to gait stability for ideal legged locomotion machines by the following
definition and theorem.

**Definition 39:** An ideal legged locomotion machine is
statically stable at time $t$ if and only if all legs in contact with
the support plane at the given time remain in contact with that plane
when all legs of the machine are fixed at their locations and length
at time $t$ and the translational and rotational velocities of the
resulting rigid body are simultaneously reduced to zero.

**Theorem 4-4:** An ideal legged locomotion machine supported
by a stationary horizontal plane surface is statically stable at
time $t$ if and only if the vertical projection of the center of
gravity of the machine onto the supporting surface lies within its
support pattern at the given time.

The following definition provides a measurement of the rela-
tive stability between different gait formulas.

**Definition 40:** The magnitude of the static stability margin
at time $t$ for an arbitrary support pattern is equal to the shortest
distance from the vertical projection of the center of gravity to any
point on the boundary of the support pattern. If the pattern is
statically stable, the stability margin is positive. Otherwise, it
is negative. The longitudinal stability margin is the static stabi-
liy margin measured in the direction of travel; i.e., the distance
from the center of gravity to the rear or front boundary of the
pattern when measured in the direction of travel.

A gait formula can completely describe the sequential characteristics of a gait, but omits all of its spatial properties. The following definitions extend the idea of a gait formula to include certain kinematic aspects of locomotion.

Definition 41: The stride length of a gait is the distance $\lambda$ by which the body of a locomotion machine is translated during any complete leg cycle.

Definition 42: The dimensionless foot position $(x_i, y_i)$ for leg $i$ of a legged locomotion machine is a pair of coordinate values that specifies the position of the contact point of any supporting leg. The origin of the $xy$ coordinate system is the center of gravity of the machine. The $x$ coordinate axis is aligned with the direction of motion with positive $x$ directed forward. The $y$ coordinate axis is normal to the $x$-axis and is oriented so that it is positive on the left side of the machine. The scale of the coordinate system is chosen so that $\lambda = 1$.

Definition 43: The dimensionless initial foot position $(y_i, \delta_i)$ is the value for the pair, $x_i, y_i$, that exists at the time leg $i$ first contacts the supporting surface during any locomotion cycle.

Definition 44: A kinematic gait formula $K$ for a $k$-legged locomotion machine is the $(4k-1)$-tuple.

$$K = (\beta_1, \beta_2, \ldots, \beta_k, \gamma_1, \gamma_2, \ldots, \gamma_k, \delta_1, \delta_2, \ldots, \delta_k, \phi_1, \phi_2, \ldots, \phi_k).$$ 

(4-15)
An example of a kinematic gait formula for \( k=6 \) is provided by Figure 4.2.

4.4.3 Support Patterns

There are two steps involved in finding the support pattern at any instant \( t \) during a legged locomotion cycle. The first step is to determine which feet are on the ground at that instant of time. The next step is to then determine the position of these feet. The following are the detailed procedures of the above two steps.

Whether leg \( i \) is on the ground or in the air at time \( t \) depends on the relations among leg placing time (\( \phi_i \)), leg lifting time (\( \phi_{i+k} \)) and \( t \), where \( k \) is the number of legs of the legged locomotion machine. In Figure 4.3, there are two diagrams to show these relationships.

![Diagram of support and transfer phases of a leg](image)

Figure 4.3. Support and Transfer Phases of a Leg

Figure 4.3a shows the case when \( \phi_{i+k} > \phi_i \). Figure 4.3b shows the case when \( \phi_{i+k} < \phi_i \). The region shown by the heavy line is the time interval while leg \( i \) is on the ground. The general rule can be stated as follows:

Leg \( i \) is on the ground at \( t \), if

\[
\begin{align*}
(1) \quad & \phi_{i+k} > \phi_i \quad \text{and} \quad \phi_{i+k} < t < \phi_i, \quad \text{or} \\
& \tag{4-16}
\end{align*}
\]
Figure 4.4 shows the support pattern of an ideal six-legged locomotion machine in stationary standing position. The vertical projection of the center of gravity of the machine is taken as the origin of the coordinate system of the foot support plane. The direction of motion is limited to the positive x direction. It is assumed that these legs are equally spaced in both the x and y directions. The distance between neighboring legs in x and y direction is normalized to be 1. The mechanical stroke of each leg is also assumed to be 1. The stride length, \( \lambda \), in this case is 1/\( \beta \) where \( \beta \) is the duty factor of all the legs.

\[ (i) \phi_{1+k} < \phi_1 \text{ and } 1 \leq t \leq \phi_1, \text{ or } 0 \leq t < \phi_{1+k} \quad (4-17) \]

When the machine starts to move, each leg moves forward one half of the stroke at its first movement. The initial foot position of all six legs can thus be determined as follows:

\[ (v_1, \delta_1) = (1.5, 0.5) \]
Since legged locomotion is a cyclic process, it is only necessary to consider the motion in one period of time. The period is normalized to be 1. The foot positions of all six legs at time \( t \) can be determined by the following equations:

\[
\begin{align*}
  x^i &= v^i + \lambda \cdot (t - \phi_i) \mod 1. \\
  y^i &= 0.5 \cdot (-1)^{i-1} \quad \text{for } i=1,2,\ldots,6.
\end{align*}
\]

The above equations are only good for those legs which are on the ground at the specified time \( t \). With the above assumptions, the velocity of the center of gravity, \( v_{cg} \), is 1/8.

4.4.4 Stability Margin Analysis

To accomplish a general analysis of stability margin, the following notation is helpful. Let \( R \) and \( L \) indicate the right or left side of the machine relative to the center of gravity along the direction of motion. Let \( F \) and \( H \) denote the front or rear (hind) legs in contact with the ground along the direction of motion. Then \( x_{RF} \) designates the foot position of the right front leg in contact with the ground. Note that this leg is not always leg number 2, but could also be leg 4 or even leg 6. Likewise, \( x_{RH} \) stands for
the foot position of the right hind leg, $x_{RF}$ means the foot position of the left front leg, and $x_{LF}$ means the foot position of the left hind leg, all relative to the center of gravity.

The line connected between $x_{RF}$ and $x_{LF}$ will be called the front boundary line of the support pattern. Similarly, the line between $x_{RH}$ and $x_{LH}$ is called the rear boundary line of the support pattern. Let $x_{CG}$ mean the position of the center of gravity. Then the distance between the center of gravity and the front boundary line, $D_F$, can be expressed as

$$D_F = \frac{1}{2} (x_{RF} + x_{LF}) - x_{CG} \tag{4-20}$$

The distance between the center of gravity and the rear boundary line, $D_R$, is

$$D_R = x_{CG} - \frac{1}{2} (x_{RH} + x_{LH}) \tag{4-21}$$

The longitudinal stability margin at any time $t$ is the smaller of $D_F$ and $D_R$. The stability margin of a given stable gait formula is the minimum $D_F$ or $D_R$ during the whole cycle of motion.

The work to find the stability margin of a gait formula is tedious if it is based on finding $D_F$ and $D_R$ during the whole locomotion cycle. Because of the forward motion of the center of gravity of a locomotion machine, the minimum value of the longitudinal stability margin associated with any support pattern of any gait occurs either at the moment the pattern is established or at the end of the interval.
in which the pattern exists. For six-legged locomotion gaits, there
are at most 12 different support patterns. Therefore, there are 12
critical times at which the minimum longitudinal stability margin
may occur. These 12 critical times are the lifting and placing
moments of these six legs during a locomotion cycle.

The above procedures have been used in a computer program to
measure the stability margin of multi-legged regular symmetric gaits.
This program is presented in Appendix IV. A modified version of the
program which can provide a continuous display of support patterns
and motion of the center of gravity on a CRT display, is also in­
cluded in this appendix. Either of the above programs permits a com­
parison between the static stability margins among different gaits.

Figure 4.5 presents the static stability margin for quadruped
regular symmetric gaits. Since no quadruped gaits are stable for
$\beta < .75$ or for $\phi_3 < .5$, only the upper right quadrant of the entire
quadruped regular symmetric gait space is shown in this figure. No
results of this kind have been presented before. However, in [16],
the analytic solution for the stability margin of the optimal quad­
ruped crawl gait, which is a kind of regular symmetric gait, has been
given. The results are that the following equation is the optimal
stability margin for a given $\beta$:

$$S_1 = \frac{3}{4} - \beta \quad (4-22)$$

This solution is based on an analysis using a normalization to
unity stride length, $\lambda$, of a quadruped locomotion machine. Since in
Figure 4.5. Stability Margin of Regular Symmetric Quadruped Gaits
the present work, the leg stroke, \( d \), is normalized to be unity, the following relation as given in [23] is required to correlate the quantities in these two systems for comparison:

\[
d = \beta \lambda . \quad (4-23)
\]

From the above relation, the value given by Eq. (4-22) must be transformed into the following form for the present system:

\[
S_2 = \frac{\beta - \frac{3}{4} \lambda}{\beta} = 1 - \frac{3}{4\beta} . \quad (4-24)
\]

Taking \( \beta = 0.9 \) in Eq. (4-24), then

\[
S_2 = 1 - \frac{3}{4 \times 0.9} = 1 - \frac{10}{12} = 0.167 .
\]

As it must, this value agrees with the values given in Figure 4.6.

The same program has also been used to calculate static stability margin for six-legged regular symmetric gaits. The results are shown in Appendix IV.

A very careful and extensive study of stability margin for six-legged locomotion system was also presented in a recently published paper by Bessonov and Umnov [17]. They made an extensive computer study to examine the stability margin of all possible regular hexapod gaits. From the results of their study, they observed the following two facts:

(1) The stability margin increases with the increase of duty factor.

(2) Optimal gaits are symmetric gaits.
From there on they have presented a special study on regular symmetric hexapod gaits. It is further discovered that for a given duty factor, the optimal hexapod gait has the following phase relations:

\[ \phi_3 = \beta, \quad \phi_5 = 2\beta - 1, \quad \beta \geq 0.5 \] (4-25)

where \( \phi_3 \) is the time delay of the left middle leg and \( \phi_5 \) is the delay of the left rear leg. They note that with these phase variables the footfalls on each side of the machine are shifted in a wave mode from rear to front. A detailed discussion of such "wave gaits" is given in the next section of this dissertation.

The results in this dissertation were worked out independently before the publication of Bessonov and Umnov [17]. The above noted optimality of wave gaits can also be noted in Appendix IV. The figures in Appendix IV have been compared quantitatively with similar figures in [17] involving a more complicated coordinate system. The results checked exactly, thereby lending confidence to the correctness of both investigations.

4.5 Wave Gaits

From the above study of the stability of regular symmetric gait, and also from the previously mentioned work of Bessonov and Umnov [17], Eq. (4-25) is found to be necessary and sufficient for a hexapod gait to have maximum static stability margin for a given duty factor. From a physical point of view, this equation implies that for optimally stable gaits, a wave of placing events runs
from the rear to the front along either side of an animal or vehicle with a constant time interval between the action of adjacent legs on the same side. This phenomenon coincides with the mechanism of "wave gait" proposed by Hughes [24] and Wilson [12] for insect motion. The following hypotheses from [24] can be used as a definition for wave gait.

(1) A wave of protractions (forward movements of the legs relative to the body) runs from posterior to anterior (and no leg protracts until the one behind is placed in a supporting position).

(2) Contralateral legs of the same segment alternate in phase.

(3) Protraction time is constant.

(4) Frequency varies (retraction time decreases as frequencies increase).

(5) The intervals between steps of the hind leg and middle leg and between the middle leg and fore leg are constant, while the interval between the fore leg and hind leg steps varies inversely with frequency.

Since in wave gait the placing events on either side of the body are ordered from rear to front, one after the other, with a constant interval, the following formula holds for the phase relations of 2K-legged regular symmetric wave gaits. This relation is also shown in [23].

\[ \phi_{2k+1} = R(n\phi_0), \quad n=1, 2, 3, \ldots, K-1, \quad 0 < \phi_0 < 1 \]  

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where $R(x)$ represents the fractional part (residue) of a real number $x$ and the subscripts $n$ denote successive legs on the left side numbered from front to back [16]. This relation implies that regardless of the number of legs, the independent variables for a $2K$-legged wave gait are two---a duty factor, $\beta$, and a constant phase interval, $\phi_o$. This small number of independent control variables makes it a good selection for a multi-legged locomotion system. This is observed by Sindall who has shown that wave mode motion is typified in nature by millipedes and centipedes [18]. Since static stability plays an important role in legged locomotion, the next few paragraphs describe a general study of the static stability of wave gaits.

The same computer programs used to calculate stability margin for general regular symmetric gaits have also been used to calculate the stability margin for wave gaits. The maximum stability margin found for a given duty factor is shown in Figure 4.6 for $2K = 4, 6, 8, 10, 12$. All of these curves are for the phase interval

$$\phi_o = \beta \quad (4-27)$$

The meaning of the result is that the optimal phase increment or phase between successive legs is independent of the number of legs for wave gait locomotion. This is a new result in the theory of legged locomotion. Moreover, it has been shown in this dissertation and elsewhere, [16,17] that Eq. (4-27) is a unique optimum for four-legged and six-legged gait. It is not known at present whether
Figure 4.6. Optimal Wave Gait Stability Margin for Fixed Leg Spacing as a Function of Duty Factor and Number of Legs.

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or not uniqueness holds for more than six legs.

It is interesting to note on Figure 4.6 that successive curves are separated in the vertical direction by a distance of exactly 0.5. The reason for this has been explained in [23]. Specifically, with the assumed model for leg placement, the addition of another pair of legs to a 2K-legged vehicle increases the vehicle length by d/2 in front and d/2 in the back so that the value for the stability margin is correspondingly increased so long as the original 2K-legged gait remains stable.

Another interesting point observable from Figure 4.6 is that the stability margin reaches zero at $\beta = 3/2K$. This agrees with the previous necessary condition that for k-legged locomotion systems, the minimum duty factor for a stable gait is $3/k$. Further, it proves that this bound on $\beta$ is also sufficient as well as necessary.

4.6 Animal Gait

Most of the preceding work in this dissertation is concerned with the use of mathematical models for gait analysis and synthesis. This section is devoted to a comparison of these theoretical results with the gaits used by natural multi-legged animals. In an earlier work, McGhee and Jain [9] tabulated the observed connected quadruped regular gaits. They collected 19 different quadruped gaits, and found that all but one of them were compatible and can be grouped into 14 different canonical row and column forms. It is also very
interesting to note that 12 of these 14 are also regularly realizable while 7 are in addition symmetric.

In general, the legs of arthropods move at a faster frequency than quadruped animals, but the step is much shorter. This makes it very hard to identify the gait event sequences from their recorded motion data. But arthropods, except in a few special cases, always use wave gaits in their motion \([12,13,18,24]\). Under this condition, the lifting order of the legs is sufficient to identify different footfall patterns. This is also the most general method used by biologists in their research to record arthropod gaits. In the following discussion, the available results for hexapod, arachnid and millipede gaits are separately discussed.

Cockroaches and locusts are the animals most often used in studying hexapod gaits \([10,13,24,25]\). The five rules for wave gait motion as given in the last section are the general rules used by these insects in their motion. The most commonly used gait pattern is the tripod gait in which the triangle formed by the front and hind legs of one side and the middle leg of the other side moves alternately. This gait has the following gait parameters: \( \beta = 0.5, \phi_0 = 0.5 \). The following Table 4.4 is a list of some of the observed lifting orders of insects. These patterns are all reported in \([24]\).

Wilson \([14]\) has also made a study of arachnid gaits. He notes that wave gait is the general rule for their motion. The normal gaits are the ones in which odd legs of one side work together
Table 4.4: Insect Foot Lifting Order

<table>
<thead>
<tr>
<th>Insect Name</th>
<th>Foot Lifting Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chrysomela</td>
<td>( (L_3 R_2 L_1) (R_1 L_2 R_3) )</td>
</tr>
<tr>
<td>Dytiscus</td>
<td>( (12-9-8)(7-10-11) )</td>
</tr>
<tr>
<td>Hydrophilus</td>
<td></td>
</tr>
<tr>
<td>Balps</td>
<td></td>
</tr>
<tr>
<td>Cockroach</td>
<td>( R_3 R_2 R_1 L_3 L_2 L_1 )</td>
</tr>
<tr>
<td>Periplaneta</td>
<td>( (11-9-7-12-10-8) )</td>
</tr>
<tr>
<td>Periplaneta</td>
<td>( L_3 L_2 R_3 L_1 R_2 R_1 )</td>
</tr>
<tr>
<td></td>
<td>( (12-10-11-8-9-7) )</td>
</tr>
<tr>
<td>Mantis</td>
<td>( (L_3 R_1) (L_2 R_3) (L_1 R_2) )</td>
</tr>
<tr>
<td>Mantis</td>
<td>( L_3 L_2 R_3 R_2 )</td>
</tr>
<tr>
<td>Mantis</td>
<td>( L_2 R_2 L_3 R_3 )</td>
</tr>
</tbody>
</table>

Table 4.5: Arachnid Leg Lifting Order

<table>
<thead>
<tr>
<th>Gait No.</th>
<th>Lifting Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4 1 2 3</td>
</tr>
<tr>
<td>2</td>
<td>4 1 3 2</td>
</tr>
<tr>
<td>3</td>
<td>4 2 3 1</td>
</tr>
<tr>
<td>4</td>
<td>4 3 2 1</td>
</tr>
</tbody>
</table>
with even ones on the other side. In Table 4.5 are some of the observed lifting orders of legs in one side of crayfish and tarantula. The legs are numbered from rear to front [13].

The only information about centipedes and millipedes is that wave gait is the dominant mode of motion [18].

4.7 Summary

Symmetric gaits have been observed often in animal locomotion. The relative phase of the two legs of a symmetric pair is exactly half of the locomotion period. This makes the placing events (or lifting events) of a symmetric pair placed at the two ends of a diameter if the gait event sequence is shown on a circle. This geometrical property has been used to derive a general formula for enumerating canonical row form symmetric gaits. The result for quadrupeds is 48 and for hexapods is 3840.

The special set of symmetric gaits which are also regularly realizable are discussed in this chapter. Regular symmetric gait enumeration is accomplished by searching all of the different regular symmetric gait event sequences generated by quantized values of duty factor and relative phases over their allowable ranges. The quantization interval is determined by looking over the gait separation planes. It is known that there are 16 row canonical form quadruped regular symmetric gaits. These can be further grouped into 3 interval invariant canonical forms. For hexapods, there are 288 canonical row
form symmetric gaits. They belong to 6 interval invariant equivalence classes.

Static stability margin has been examined for hexapod regular symmetric gaits. This analysis shows that, for a given duty factor, wave gaits possess the maximum static stability margin. Static stability margins of wave gaits for legged locomotion machines possessing up to 12 legs have been calculated and plotted for comparison. In general, static stability margin increases with increasing duty factor and with increasing number of legs. With constant leg spacing, for every added pair of legs the stable wave gait stability margin will increase by 0.5. The analysis of wave gaits also shows that no statically stable gaits exist for \( \beta \) less than \( 3/k \), where \( k \) is the number of legs.

In the last section of this chapter, some observed hexapod, arachnid, and millipede animal gaits are presented for comparison with the mathematically derived results. It seems to be a general rule that under normal conditions, wave gaits are the most common mode for arthropod motion. Part of the reason for this situation could be the optimal static stability of wave gaits and the simple control algorithm inherent to wave gaits, since duty factor and a constant phase difference are the only two independent variables needed to define and control such a gait.
5.1 Summary of the Work of This Dissertation

Gait studies have been pursued by many interested scientists for quite a long time. Most of their work and results have been on a particular class of gait. The work of this dissertation is basically a summary of the results and methods used in previous gait studies together with a further development and extension to a more general theory. This theory includes methods which ease the work of analysis and selection of a gait for multi-legged locomotion machines. A detailed analysis of six-legged locomotion gaits has been presented to illustrate the work.

The first task involved in gait study is to reduce the possible number of gaits potentially useful for a given application. At the beginning of Chapter III, four very useful permutation groups, i.e., \( G_1 \), \( G_2 \), \( G_3 \), and \( G_4 \), were discussed. The first three were previously used to reduce the number of equivalence classes of quadruped gaits. The fourth one, \( G_4 \), is newly developed in this dissertation. It can group together a greater number of similar gaits in one equivalence class than any of the previous notions of gait equivalence.

A new canonical representation of gaits, interval invariant canonical form, has also been introduced to identify the equivalence classes.
under $G_4$. This work provides a more compact set of gait classes for analysis. Counting procedures based on group theory are also introduced to determine the number of equivalence classes under each permutation group. This approach provides a more formal and general method for enumerating gaits.

Compatibility is a necessary condition for a gait to be regularly realizable. No prior work has been done on the counting of compatible gaits for more than four legs. The method given in Chapter III makes compatible gait counting practical. The method used is based on iterative counting. As the number of legs increases, the number of compatible gaits expands even faster. The computer storage used to keep this data will increase at a fast rate also, and one may run out of computer memory capacity. In this case, some secondary storage device may be needed to solve this difficulty. However, it appears for computing compatible gait up to 8 legs, this method is still a very good and simple one. This computation has been carried out and is included in this dissertation.

It is reasonable to say that each leg should function equally in one locomotion cycle. This makes regular gaits interesting for use in locomotion. A complete investigation of quadruped regular gaits has been accomplished previously [9]. The method used for regular quadruped gait counting is by using a linear programming method to test compatible gaits. If this sufficient condition for regularly realizable gait is satisfied, the tested compatible gait
is then also regularly realizable. This work has been extended in this dissertation to hexapod regular gait counting. The complexity of the result is reduced by using the interval invariant equivalence relations introduced in this dissertation.

Symmetric gait is another kind of gait which is often used by animals. In Chapter IV, symmetric gait and regular symmetric gait are generally discussed. A general formula for symmetric counting is derived in Section 4.2. Counting of regular symmetric gaits is accomplished next in Section 4.3. No general formula has been found to count regular symmetric gaits. The simplest method so far is using a computer program to search all possible cases. Use of the concept of separation planes helps to reduce searching in such a calculation. This work has been carried out to identify all of the regular symmetric hexapod gaits. This is a new result in the theory of legged locomotion.

Static stability is another very important factor for a gait to be practically usable. In this dissertation, for the first time static stability has been generally evaluated for quadruped and hexapod regular symmetric gaits. One of the interesting outcomes is that for a given duty factor, wave gaits have the largest stability margin. This result has also been independently verified by others using different methods [16,17]. It is also found that wave gaits are used by arthropods in their normal walking [12,13,18,24]. The static stability margin of up to 12 legged optimal wave gaits
has been determined in this dissertation. A general result is that for a given duty factor, the stability margin is increased by 0.5 for every pair of legs added.

5.2 Extensions

Although much work has been done on animal gaits, it is nevertheless true that mathematical analysis of gait is still in an early stage of development. Many problems are waiting to be stated and solved. The following is just a short list of them.

5.2.1 Singular and Wave Gait Counting

With one exception [15], all of the previous work on gait counting has been limited to connected gaits. This means that no two legs are allowed to lift or fall at the same time. But many of the observed animal gaits have more than two legs changing state at the same time as, for example, in the tripod gait for insects. Singular gait, then, is deserving of further study. The counting theory discussed in Chapter III might be used in singular gait enumeration.

Wave gaits have been shown to have optimal static stability properties. Their enumeration has never been attempted. This is another interesting problem in gait study.

5.2.2 Effect of Stride and Leg Geometry on Stability Margin

In the present study, a very simplified mathematical model has been used for stability analysis. In this model, all legs have equal length and all legs have equal stride length. Adjacent legs
have also been assumed to be separated by equal distances. This is not in general true for animals. Further work on gait stability could consider stride length and leg geometry as variable parameters.

5.2.3 Turning and Uneven Terrain Considerations

All of the gaits studied in this dissertation have been assumed to be utilized on an even and straight road. This kind of ideal condition cannot, in general, be found for natural conditions. A determination of stability margin under turning conditions and uneven road conditions should be an interesting problem. The related problem of controlling the gait under such conditions also needs to be investigated.

5.2.4 Dynamics

Only static stability has been discussed in this dissertation. At low speed, static stability is a good measure of optimality for gaits. However, at higher speeds, the stability of the vehicle is not necessarily provided by static stability margin. Some other criteria should be studied to measure stability conditions and also to determine the optimal gait.

The above is certainly not an exhaustive list. Hopefully, some of those who read this dissertation will become sufficiently interested to do still further work on the mathematical theory of locomotion. It is the author's hope that such work will eventually lead on the one hand to useful legged vehicles with entirely unique locomotion characteristics, and on the other hand to a better understanding of animal locomotion.
### APPENDIX I: Table of Equivalence Classes of Hexapod Connected Gaits Under Various Permutation Groups

<table>
<thead>
<tr>
<th>Permutation Group</th>
<th>$P_G$</th>
<th>$P_H$</th>
<th>No. of Equivalence Classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>$x^{12}$</td>
<td>$\frac{1}{12}\left{x_1^{12} + x_2^6 + 2x_3^4\right}$</td>
<td>39,916,800</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{1}{12}\left{x_1^6 + 2x_3^2 + 2x_6^2 + 4x_1^{12}\right}$</td>
<td></td>
</tr>
<tr>
<td>$G_2$</td>
<td>$\frac{1}{120}\left{x_1^{12} + 10x_1^6x_2^2 + 20x_1^4x_2^3\right}$</td>
<td>$\frac{1}{12}\left{x_1^{12} + x_2^6 + 2x_3^4\right}$</td>
<td>332,640</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{120}\left{+ 16x_1^4x_2^4 + 29x_1^2x_2^6 + 2x_1^2x_2^8\right}$</td>
<td>$\frac{1}{12}\left{+ 2x_4^3 + 2x_6^2 + 4x_1^{12}\right}$</td>
<td></td>
</tr>
<tr>
<td>$G_3$</td>
<td>$\frac{1}{240}\left{x_1^{12} + 26x_2^6 + 10x_1^8x_2^2\right}$</td>
<td>$\frac{1}{12}\left{x_1^{12} + x_2^6 + 2x_3^4\right}$</td>
<td>166,736</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{240}\left{+ 20x_1^6x_2^4 + 15x_1^4x_2^6 + 2x_1^2x_2^8\right}$</td>
<td>$\frac{1}{12}\left{+ 2x_4^3 + 2x_6^2 + 4x_1^{12}\right}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{240}\left{+ 30x_1^2x_2^6 + 24x_1^2x_3^4 + 2x_1x_2x_3^2\right}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{240}\left{+ 40x_1^2x_2^6 + 30x_1^2x_4^2 + 2x_1x_2x_4^2\right}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{240}\left{+ 24x_1x_2x_3^2 + 20x_1x_2x_4^2\right}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_4$</td>
<td>$\frac{1}{1440}\left{x_1^{12} + 91x_2^6 + 40x_3^4\right}$</td>
<td>$\frac{1}{12}\left{x_1^{12} + x_2^6 + 2x_3^4\right}$</td>
<td>27,974</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{1440}\left{+ 280x_1^6 + 8x_2^8 + 2\right}$</td>
<td>$\frac{1}{12}\left{+ 2x_4^3 + 2x_6^2 + 4x_1^{12}\right}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{1440}\left{+ 40x_1^2x_2^6 + 90x_1^2x_3^4 + 2x_1x_2x_3^2\right}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{1440}\left{+ 44x_1^2x_2^6 + 144x_1^2x_4^2 + 2x_1x_2x_4^2\right}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{1440}\left{+ 160x_1^2x_2^6 + 270x_1^2x_4^2 + 2x_1x_2x_4^2\right}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{1440}\left{+ 145x_1x_2x_3^2 + 120x_1x_2x_4^2 + 2x_1x_2x_4^2\right}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX II: Computer Program for Iterative Counting of Compatible Gaits

The following program is to find iteratively the number of (k+1)-legged compatible gaits from the data of k-legged compatible gaits. It starts counting from the provided data of biped compatible gaits. The following is a partial listing of symbols used in this program.

ILNL1, JLN1: Event intervals between the last lifting event and the first lifting event of current k-legged compatible gaits and generated (k+1)-legged compatible gaits respectively.

IPN1, JPN1: Event intervals between the last placing event and the first placing event of current k-legged compatible gaits and generated (k+1)-legged compatible gaits respectively.

ILNL1, JLN1: Event intervals between the last lifting event and the first placing event of k- and (k+1)-legged compatible gaits respectively.

ILNPN, JLNPN: Event intervals between the last lifting event and the last placing event of k- and (k+1)-legged compatible gaits respectively.

DX1: Index for present k-legged compatible gaits.

DXO: Index for generated (k+1)-legged compatible gaits.

A complete listing of this program is provided on the following pages.
DIMENSION ILNL1(1650), ILNP1(1650), ILNPN(1650), IPNPI(1650)
DIMENSION JLNLI(1650), JLNPI(1650), JLNPNC1650), JPNPl(1650)

ILNL1(1) = 3
ILNL1(2) = 2
ILNL1(3) = 2
ILNL1(4) = 1
IPNPI(1) = 3
IPNPI(2) = 2
IPNPI(3) = 2
IPNPI(4) = 1
ILNP1(1) = 1
ILNP1(2) = 3
ILNP1(3) = 1
ILNP1(4) = 3
ILNPN(1) = 2
ILNPN(2) = 1
ILNPN(3) = 3
ILNPN(4) = 2
IDX1 = 4
DO 800 N = 2, 5
WRITE(6, 803)

803 FORMAT (1X, 71H * GATE INDEX * * NO. OF GATES
ICAN * SUBTOTAL OF GATES *)
N = N + 1
WRITE(6, 802) N

802 FORMAT (1X, 9H FOR N=, 12, 21H, * LN--LI * PN--PI *,
139H BE GENERATED. * CAN BE GENERATED. *
118H LN--PI * LN--PN *)
IDX0 = 0
KLS = 0
DO 100 I = 1, IDX1
M = ILNL1(I) - ILNPN(I)
IF (M) 20, 20, 30
20 M = - M
GO TO 30
30 IPNPI = M

THE FOLLOWING PROCEDURE IS TO GENERATE A NEW COMPATIBLE
GAIT BY PLACING A COMPATIBLE EVENT OF A NEW LEG TO A
PREVIOUS 2K-LEGGED COMPATIBLE GAIT EVENT SEQUENCE.
IT ALSO COUNT HOW MANY COMPATIBLE GAIT EVENT SEQUENCE
CAN BE GENERATED BY THE NEW COMPATIBLE GAIT, IF
ONE MORE LEG IS ADDED.

IJL = ILNL1(I)
DO 200 JL = 1, IJL
IJP = IPNPI(I)
DO 300 JP = 1, IJP
IDX0 = IDX0 + 1
JLNPN(IDX0) = ILNPN(I) + 1 + JP - JL
IF (ILNP1(I) .GT. ILNL1(I)) GO TO 111

119
IF(ILNPNI(I) .GT. ILNLI(I)) GO TO 112
IF(JL .GT. ILNL1(I)) GO TO 113
JPNPI(IDX0) = IPNP1(I)+2-JP
JLNPI(IDX0) = ILNP1(I)+1-JL
GO TO 500
113 JPNPI(IDX0) = IPNP1(I)+1-JP
JLNPI(IDX0) = 2*N+2+ILNP1(I)-JL
GO TO 500
500 IF (JP .GT. IPNL1) GO TO 114
JLNLI(IDX0) = ILNL1(I)+2-JL
GO TO 400
114 JLNLI(IDX0) = ILNL1(I)+1-JL
GO TO 400
112 JLNLI(IDX0) = ILNL1(I)+1-JL
IF(JL .GT. ILNPNI(I)) GO TO 115
JPNPI(IDX0) = IPNP1(I)+2-JP
JLNPI(IDX0) = ILNP1(I)+1-JL
GO TO 400
115 JPNPI(IDX0) = IPNP1(I)+1-JP
JLNPI(IDX0) = 2*N+2+ILNP1(I)-JL
GO TO 400
111 JPNPI(IDX0) = IPNP1(I)+1-JP
JLNPI(IDX0) = ILNP1(I)+2-JL
IF (ILNPNI(I) .GT. ILNL1(I)) GO TO 116
GO TO 500
116 JLNLI(IDX0) = ILNL1(I)+1-JL
GO TO 400
400 KLS1 = JLNLI(IDX0)*JPNPI(IDX0)
KLS = KLS+KLS1
WRITE(6,700) IDX0, JLNLI(IDX0), JPNPI(IDX0), KLS1, KLS, JLNPI(IDX0),
I, JLNPN(IDX0)
700 FORMAT(6X, I4, 6X, I4, 5X, I4, 10X, I6, 10X, I6, 10X, I6, 3X, I6)
300 CONTINUE
200 CONTINUE
100 CONTINUE
DO 600 IJ=1, IDX0
IPNP1(IJ) = JPNPI(IJ)
ILNL1(IJ) = JLNLI(IJ)
ILNP1(IJ) = JLNPI(IJ)
ILNPNI(IJ) = JLNPN(IJ)
600 CONTINUE
WRITE(6,801)
801 FORMAT(5X, 5HS$$)
IDX1 = IDX0
800 CONTINUE
STOP
END

120
APPENDIX III: Hexapod Regular Symmetric Gaits

III.1 Introduction

This appendix consists of two parts. The first part is a listing of a Fortran program used to find the hexapod regular symmetric gaits. The second part consists of 10 pages of diagrams. Each shows the distribution of regular symmetric gaits on a given BETA (duty factor) plane.

III.2 Listing of Program

This program is written to find all regular symmetric hexapod gaits. For an accurate calculation, integer arithmetic is used. The duty factor (BETA) and relative phases of events (PHA(i)'s) are assumed to vary between 0 and 100 instead of 0 and 1 as described in the previous text of this dissertation. The values of BETA, PHA(3), and PHA(5) are sampled at an interval of 5. Based on this sampling, the various distinct gait event sequences are calculated and tabulated.

The following is a partial listing of the symbols used in the program:

- BETA Duty factor.
- PHA(I) Relative phase of event I. I < 6 is used to represent placing events. I > 6 is used for lifting events.
- IG Current sampled gait event sequence.
- IGl Table for all of the different regular symmetric event sequences.

The detailed program listing and resulting diagrams are presented on the following pages.
INTEGER PHA,BETA
DIMENSION PHA(12), IG(12), IGI(300,12)
IDX1=0
PHA(1)=0
PHA(2)=50

C TAKING VALUES OF BETA BETWEEN 5 AND 95 AT AN INTERVAL OF 5.
C
DO 101 II=1,19
BETA=5*II
PHA(7)=BETA
PHA(8)=PHA(2)+BETA
IF (PHA(8).LT.100) GO TO 201
PHA(8)=PHA(8)-100

C TAKING VALUES OF PHA(3) BETWEEN 5 AN 95 AT AN INTERVAL OF 5.
C
201 DO 102 I2=1,19
PHA(3)=5*I2
PHA(4)=PHA(3)+50
IF (PHA(4).LT.100) GO TO 202
PHA(4)=PHA(4)-100

202 PHA(9)=PHA(3)+BETA
IF (PHA(9).LT.100) GO TO 203
PHA(9)=PHA(9)-100

203 PHA(10)=PHA(4)+BETA
IF (PHA(10).LT.100) GO TO 204
PHA(10)=PHA(10)-100

C TAKING VALUES OF PHA(5) BETWEEN 5 AND 95 AT AN INTERVAL OF 5.
C
204 DO 103 I3=1,19
PHA(5)=5*I3
PHA(6)=PHA(5)+50
IF (PHA(6).LT.100) GO TO 205
PHA(6)=PHA(6)-100

205 PHA(11)=PHA(5)+BETA
IF (PHA(11).LT.100) GO TO 206
PHA(11)=PHA(11)-100

206 PHA(12)=PHA(6)+BETA
IF (PHA(12).LT.100) GO TO 207
PHA(12)=PHA(12)-100

207 DO 104 I4=1,12
M=I4
IGI(M)=M

104 CONTINUE
COMPARING ALL PHA(I)'S, AND ARRANGING THEM IN CORRECT GAIT EVENT SEQUENCE ORDER

IDX=1
DO 105 I5=1,11
I6=I5+1
L1=IG(I5)
L2=IG(I6)
IF (PHA(L1).EQ.PHA(L2)) GO TO 103
IF (PHA(L1).GT.PHA(L2)) GO TO 209
IDX=IDX+1
GO TO 105

IWA=IG(I5)
IG(I5)=IG(I6)
IG(I6)=IWA

CONTINUE

COMPARING PRESENTLY GENERATED GAIT EVENT SEQUENCE WITH PREVIOUSLY GENERATED GAIT EVENT SEQUENCES TO DETERMINE WHETHER A NEW REG. SYMM. GAIT IS GENERATED.

IF (IDX.NE.12) GO TO 212
IF (IDX1.EQ.0) GO TO 110
DO 106 I8=1,IDX1
DO 107 19=1,12
IF (IG(19).NE.IG(I8,19)) GO TO 106

CONTINUE
GO TO 103
CONTINUE

IDX1=IDX1+1
DO 109 JJ=1,12
IG1(IDX1,JJ)=IG(JJ)
CONTINUE
WRITE(6,400) (IG(I7),I7=1,12)
FORMAT(5X,12I3)
CONTINUE
CONTINUE
CONTINUE
STOP
END
Figure IIIa: Distribution of Regular Symmetrical Hexapod Gaits in Unit 3-Cube for $\beta = .05$. 

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Figure IIIb: Distribution of Regular Symmetrical Hexapod Gaits in Unit 3-Cube for \( \beta = .15 \).
Figure IIIc: Distribution of Regular Symmetrical Hexapod Gaits in Unit 3-Cube for $\beta = .25$. 

126
Figure IIId: Distribution of Regular Symmetrical Hexapod Gaits in Unit 3-Cube for $\beta = .35$. 

127
Figure IIIe: Distribution of Regular Symmetrical Hexapod Gaits in Unit 3-Cube for $\beta = .45.$

128
Figure IIIf: Distribution of Regular Symmetrical Hexapod Gaits in Unit 3-Cube for $\beta = .55$. 

129
Figure IIIg: Distribution of Regular Symmetrical Hexapod Gaits in Unit 3-Cube for $\beta = .65$. 

130
Figure IIIh: Distribution of Regular Symmetrical Hexapod Gaits in Unit 3-Cube for $\beta = .75$. 

131
Figure IIIi: Distribution of Regular Symmetrical Hexapod Gaits in Unit 3-Cube for $\beta = .85$. 

132
Figure IIIj: Distribution of Regular Symmetrical Hexapod Gaits in Unit 3-Cube for $\beta = .95$. 

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APPENDIX IV: Stability Margin for Hexapod Gaits

IV.1 Introduction

The programs and results of calculations related to six-legged regular symmetric gait static stability analysis are collected in this appendix. It consists of the following three parts:

Part 1: This part contains a listing of HSM, a Fortran program, which is used to calculate static stability margin of six-legged regular symmetric gaits for a given value of BETA. The values of $\phi_3$ and $\phi_5$ are taken at 20 equally spaced sampling points between .05 and .95.

Part 2: A Fortran program called HGTDIS is listed. This program is used to present the continuous succession of support patterns for a given gait on a CRT display. The motion of the center of gravity and the value of static stability margin are also shown on the display.

Part 3: The results of Part 1 are shown on a given BETA plane, for five different values of BETA. Contours of constant stability margin are indicated by dotted lines on this figure. The distribution of different regular symmetric gaits on these planes is also shown.

IV.2 Program for Regular Symmetric Gait Stability Margin Calculation

A partial listing of symbols used in this program is given below to help understand the following program listing:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>BETA</td>
<td>Duty factor.</td>
</tr>
<tr>
<td>PHI(2)</td>
<td>Relative phase of event I. $I &lt; 6$ is used to represent placing events. $I &gt; 6$ is used for lifting events.</td>
</tr>
</tbody>
</table>
XFP(2)  X-coordinate of footfall position of leg I.

IG     Gait event sequence.

XMIN1  Minimal x-directional distance between center of gravity and a support line at the beginning of the present phase.

XMIN2  Minimal x-directional distance between center of gravity and a support line at the end of previous phase.

MOD    \((t - \phi_i) \mod 1\).

SM     Static stability margin

A listing of this program follows.
INTEGER PHI, PHTIME, DETI
INTEGER DIF1, DIF2, DIF3, DIF4, DIF5, DIF6, DIF7, BETA1
DIMENSION XFP(6), XMIN(I0), PHI(12), I0(15)
DIMENSION XMIN1(12), XMIN2(12), IFR(4,2), NP(12), IG(12)
PHI(1) = 0
PHI(2) = 50
FORMAT(I7)

C
C SETTING A VALUE FOR BETA
C
WRITE(4,75)
75 FORMAT(5X,5HBETA=)
READ(4,71) BETA
BETA = BETA1

C
C TAKING VALUES OF PHI(3) BETWEEN 0 AND 95 AT AN INTERVAL OF 5
C
DO 700 1700:1,5
PHI(3) = 1700 + 5*

C
C TAKING VALUES OF PHI(5) BETWEEN 0 AND 95 AT AN INTERVAL OF 5
C
DO 701 J700 = 1,5
PHI(5) = J700 + 5 *
C
C CALCULATING PHI(4), PHI(6)
C
DO 96 I96 = 1,2
N1 = 2 * I96 + 2
N2 = 2 * I96 + 1
PHI(N1) = PHI(N2) + 50
IF (PHI(N1).LT.100) GO TO 96
PHI(N1) = PHI(N1) - 100
CONTINUE

96
C
C CALCULATING LIFTING EVENTS PHASE
C
DO 1 11 = 1,6
J1 = 11 + 6
PHI(J1) = PHI(11) + BETA
IF (PHI(J1).LT.100) GO TO 1
PHI(J1) = PHI(J1) - 100
CONTINUE

C
C ORDERING EVENTS AND TRANSFORMING IT INTO ROW AND COLUMN CANONICAL FORM GAIT EVENT SEQUENCE
C
DO 104 I104 = 1,12
MI04 = I104
IG(MI04) = MI04
CONTINUE

104
IDX=1
DO 105 1105=1,11
II06=II05+1
LI=IG(II05)
L2=IG(II06)
IF (PHI(L1).GT.PHI(L2)) GO TO 209
IDX=IDX+1
GO TO 105
209 IWA=IG(II05)
IG(II05)=IG(II06)
IG(II06)=IWA
105 CONTINUE
IF (IDX.NE.12) GO TO 212
C CALCULATING STATIC STABILITY CONDITION AT THE
C INSTANT OF THE 12 PLACING AND LIFTING EVENTS
C I400=1
SM=100.
444 IF (I400.GT.12) GO TO 500
L400=IG(I400)
PHTIME=PHI(L400)
XTIME=PHTIME
C CHECKING OUT THOSE ODD NUMBERED LEGS WHICH ARE
C ON THE GROUND AT THE CURRENT TIME
C J7=1
J6=1
J8=0
J9=0
42 IF (J6.GT.5) GO TO 5000
DIR2=PHTIME-PHI(J6)
IF (DIR2) 46,49,43
43 DIR3=PHI(J6)-PHI(J6+6)
IF (DIR3) 44,44,49
44 DIR4=PHTIME-PHI(J6+6)
IF (DIR4) 49,48,48
49 I0(J7)=J6
J7=J7+1
J8=J8+1
J6=J6+2
GO TO 42
46 DIR5=PHI(J6+6)-PHI(J6)
IF (DIR5) 47,47,48
47 DIR6=PHTIME-PHI(J6+6)
IF (DIR6.GE.0) GO TO 48
GO TO 49
48 J6=J6+2
GO TO 42
5000 IRX=J7-1
50 J6=J6-1
C CHECKING OUT THOSE EVEN NUMBERED LEGS WHICH ARE ON
C THE GROUND AT THE CURRENT TIME
IF (I6.EQ.0) GO TO 60
IF (PHIIME-PHI(I6)) 55,58,52
IF (PHI(I6)-PHI(I6+6)) 53,53,58
IF (PHIIME-PHI(I6+6)) 58,57,57
10(J7)=I6
J7=J7+1
J9=J9+1
GO TO 57
IF (PHI(I6+6)-PHI(16)) 56,56,57
DIF7 = PHIIME-PHI(I6+6)
IF (DIF7.GE.0) GO TO 57
GO TO 58
I6=I6-2
GO TO 51
60 XCG=XTIME/(BETA*100.)
J7=J7-1

C TESTING FOOT SUPPORT PATTERNS:
C 940: NO FOOT ON THE GROUND
C 941: ONE SIDE WITHOUT FOOT ON THE GROUND
C 943: ONE FOOT ON EACH SIDE
C 945: MORE THAN 3 FEET ON THE GROUND,
C EACH SIDE HAVING AT LEAST ONE FOOT ON
C THE GROUND.
C
DET1=J8-1
IF (DET1) 991,992,993
991 IF (J9.EQ.0) GO TO 940
GO TO 941
992 IF (J9.EQ.0) GO TO 941
IF (J9.EQ.1) GO TO 943
GO TO 945
993 IF (J9.EQ.0) GO TO 941
GO TO 945
940 NP(J700)=0
XMIN(J100)=500
GO TO 701
941 NP(J700)=1
XMIN(J700)=500
GO TO 701
943 NP(J700)=3
GO TO 720
945 NP(J700)=5
GO TO 720

C FINDING FRONT AND REAR SUPPORT FOOT AT EACH SIDE OF
C THE LEGGED MACHINE AT THE CURRENT PHASE AND CALCULATING
C THEIR LOCATION.
C
720 IFR(1,1)=I0(1)
IFR(2,1)=I0(J7)
IFR(3,1)=I0(IRX)
IFR(4,1)=I0(IRX+1)
CALCULATING THE MINIMAL DISTANCE BETWEEN THE CENTER OF GRAVITY AND A SUPPORT LINE AT THE BEGINNING OF CURRENT PHASE.

DO 17 M1 = I, A
N1 = IFR(M1, 1)
MOD = PHTIME - PHI(N1)
IF (MOD .GE. 0) GO TO 10
MOD = MOD + 100

10 XMOD = MOD
IF (N1 .GT. 2) GO TO 11
XFP(N1) = 1.5 + (XTIME - XMOD) / (100 .* BETA)
GO TO 17

11 IF (N1 .GT. 4) GO TO 12
XFP(N1) = 0.5 + (XTIME - XMOD) / (100 .* BETA)
GO TO 17

12 XFP(N1) = -0.5 + (XTIME - XMOD) / (100 .* BETA)
CONTINUE

IFR1 = IFR(1, 1)
IFR2 = IFR(2, 1)
IFR3 = IFR(3, 1)
IFR4 = IFR(4, 1)
XD1 = ((XFP(IFR1) + XFP(IFR2)) / 2) - XCG
XD2 = XCG - ((XFP(IFR3) + XFP(IFR4)) / 2.
DIF = XD1 - XD2
M400 = IG(1400)
IF (DIF .LE. 0.) GO TO 14
XMIN1(M400) = XD2
GO TO 16

14 XMIN1(M400) = XD1

16 IF (I400 .EQ. 1) GO TO 430
DO 401 I401 = 1, 4
N2 = IFR(I401, 2)
MOD = PHTIME - PHI(N2)
IF (MOD .GE. 0) GO TO 270
MOD = MOD + 100

270 XMOD = MOD
IF (N2 .GT. 2) GO TO 110
XFP(N2) = 1.5 + (XTIME - XMOD) / (100 .* BETA)
GO TO 401

110 IF (N2 .GT. 4) GO TO 120
XFP(N2) = 0.5 + (XTIME - XMOD) / (100 .* BETA)
GO TO 401

120 XFP(N2) = -0.5 + (XTIME - XMOD) / (100 .* BETA)
CONTINUE

CALCULATING THE MINIMAL DISTANCE BETWEEN THE CENTER OF GRAVITY AND A SUPPORT LINE AT THE END OF PREVIOUS PHASE.

IFR1 = IFR(1, 2)
IFR2 = IFR(2, 2)
IFR3 = IFR(3, 2)
IFR4 = IFR(4, 2)
XD1 = ((XFP(IFR1) + XFP(IFR2)) / 2) - XCG
XD2 = XCG - ((XFP(IFR3) + XFP(IFR4)) / 2.
DIF = XD1 - XD2
L401 = I400 - 1
M401 = IG(L401)
IF (DIF LE 0.) GO TO 140
XMIN = (M401) = XD2
GO TO 160
140
XMIN (M401) = XD1
160
DO 403 I403 = 1, 4
IFR(I403, 2) = IFR(I403, 1)
CONTINUE
403
IFR1 = IFR(1, 1)
IFR2 = IFR(2, 1)
IFR3 = IFR(3, 1)
IFR4 = IFR(4, 1)
XD1 = ((XFP(IFR1) + XFP(IFR2)) / 2. - XCG
XD2 = XCG - ((XFP(IFR3) + XFP(IFR4)) / 2.
DIF = XD1 - XD2
M12 = IG(12)
IF (DIF LE 0.) GO TO 440
XMIN(12) = XD2
GO TO 460
440
XMIN(12) = XD1
GO TO 460
C
C CALCULATING THE MINIMAL DISTANCE BETWEEN THE CENTER OF
C GRAVITY AND A SUPPORT LINE AT THE END OF PHI(12).
C
PHI1 = PHI(1)
XTIME = PHI1
XCG = XTIME / (BETA * 100.)
DO 405 I405 = 1, 4
N3 = IFR(I405, 1)
MOD = PHI1 - PHI(N3)
IF (MOD LE 0.) GO TO 190
MOD = MOD + 100
190
XMOD = MOD
IF (N5 GT 2) GO TO 410
XFP(N5) = 1.5 + (XTIME - XMOD) / (100 * BETA)
GO TO 405
410
IF (N5 GT 4) GO TO 420
XFP(N5) = 0.5 + (XTIME - XMOD) / (100 * BETA)
GO TO 405
420
XFP(N5) = -0.5 + (XTIME - XMOD) / (100 * BETA)
CONTINUE
405
IFR1 = IFR(1, 1)
IFR2 = IFR(2, 1)
IFR3 = IFR(3, 1)
IFR4 = IFR(4, 1)
XD1 = ((XFP(IFR1) + XFP(IFR2)) / 2. - XCG
XD2 = XCG - ((XFP(IFR3) + XFP(IFR4)) / 2.
DIF = XD1 - XD2
M12 = IG(12)
IF (DIF LE 0.) GO TO 440
XMIN(12) = XD2
GO TO 460
440
XMIN(12) = XD1
GO TO 460
C
C SAVE THIS DATA FOR PHI(2) TO USE.
C
500
DO 504 I404 = 1, 4
IFR(I404) = IFR(I404, 1)
IFR(I404, 2) = IFR(I404, 1)
CONTINUE
SELECTING THE MINIMAL XMINI'S AND MINJ'S AS THE STATIC
STABILITY MARGIN.

DO 300 I300=1,12
DIF=XMIN(J700)-XMIN(I300)
IF (DIF.GE.0.) GO TO 301
DIF=XMIN(J700)-XMIN(I300)
IF (DIF.LE.0.) GO TO 300
XMIN(J700)=XMIN(I300)
GO TO 300

301 DIF=XMIN(J700)-XMIN(I300)
IF (DIF.LE.0.) GO TO 300
XMIN(J700)=XMIN(I300)
300 CONTINUE

WRITE(6,909) (XMIN(L300),NP(L300),L300=1,5)
909 FORMAT(3X,10(F5.2,I2))
700 CONTINUE
500 STOP
END
IV.3 Display Programs

The symbol listing in the previous section can be used as a symbol reference for the following program.
INTEGER PHI, RTIME, PTIME, PHTIME
INTEGER DIF1, DIF2, DIF3, DIF4, DIF5, DIF6, DIF7, BETA1
LOGICAL K
DIMENSION XI(6), XF(6), PHI(12), AT(6), YFP(6), IO(15)
DIMENSION A(300), P1(3), P2(3)
PHI(1) = 0
PHI(2) = 50

READING IN PHI(3), PHI(5), AND BETA.

TYPE 73
FORMAT(5X, 'PHI3 = ')
READ(5, 71) PHI(3)

TYPE 74
FORMAT(5X, 'PHI5 = ')
READ(5, 71) PHI(5)

TYPE 75
FORMAT(5X, 'BETA = ')
READ(5, 71) BETA1
TBETA = BETA1 * 0.01
BETA = TBETA * 0.01

CALCULATING PHI(4), PHI(6).

DO 96 I96 = 1, 2
NI = 2 * I96 + 2
N2 = 2 * I96 + 1
PHI(N1) = PHI(N2) + 50
IF (PHI(N1) .LT. 100) GO TO 96
PHI(N1) = PHI(N1) - 100
96 CONTINUE

CALCULATING LIFTING ENENT PHASES.

DO 1 I1 = 1, 6
J1 = I1 + 6
PHI(J1) = PHI(I1) + BETA1
IF (PHI(J1) .LT. 100) GO TO 1
PHI(J1) = PHI(J1) - 100
1 CONTINUE

Y-COORDINATES OF LEGS. 0.5 FOR ODD NUMBERED LEGS, AND -0.5 FOR EVEN NUMBERED LEGS.

YFP(1) = 0.5
YFP(3) = 0.5
YFP(5) = 0.5
\( YFP(2) = -0.5 \)
\( YFP(4) = -0.5 \)
\( YFP(6) = -0.5 \)

IRTIME = 0

[type 70]

70 FORMAT(5X, 'IPT=')
READ(5,7) IPT

[type 72]

72 FORMAT(5X, 'N=')
READ(5,7) N
XIPT = IPT
XBETA = BETA*XIPT

C

INITIALIZATION OF DISPLAY ROUTINE.

C

CALL DSPINI(A,300)
CALL DCLPIC(P1,3)
CALL DCLPIC(P2,3)
CALL SETWIN(-5,10,-4,2)
CALL SETPORT(-5,1023,0,1023)

L = 1
K = .FALSE.
RTIME = -1
XCG = -1./XBETA
SM = 100.
GO TO 28

C

GENERATING PICTURES FOR DISPLAY.

C

21 GO TO (22,23,L
22 CALL GENINI(P1)
GO TO 24
23 CALL GENINI(P2)
24 CALL PARM(1,7)

K1 = 1
K2 = IO(K1)
K3 = IO(K1+1)

K4 = K2/2
K5 = K2 - K4*2
IF (K5 .EQ. 0) GO TO 140
XX = XFP(K2) - 0.4
CALL PARM(4,7)
CALL PGFX(XX,0.9,0)
CALL TEXTIP
TYPE 94,K2
94 FORMAT(13)
GO TO 141
140 CALL PARM(4,7)
XX = XFP(K2) - 0.4
CALL PGFX(XX,-0.6,0)
CALL TEXTP
TYPE 94,K2
CALL PARMS(1,7)
  CALL PGEM(XFP(K2),YFP(K2),7)
  IF (K1.EQ.J7) GO TO 31
  CALL PARMS(1,1)
  CALL VGEN((XFP(K3)-XFP(K2)),(YFP(K3)-YFP(K2)),1)
  K1=K1+1
  GO TO 30
K3=10(1)
CALL PARMS(1,1)
  CALL VGEN((XFP(K3)-XFP(K2)),(YFP(K3)-YFP(K2)),1)
  CALL PARMS(1,7)
  CALL PGEM(XCG,0.,7)
  CALL PARMS(4,7)
  CALL PGEM(-4.5,-1.5,0)
  CALL TEXTP
TYPE 90,PHI(3),PHI(5),BETA1
  FORMAT(5X,'PHI3=',F9.4,2X,'PHI5=',F9.4,2X,'BETA=',F9.4)
  CALL PARMS(4,7)
  CALL PGEM(-4.5,-2.,0)
  CALL TEXTP
TYPE 92,XMIN,SM
  FORMAT(5X,'XMIN=',F9.4,2X,'SM=',F9.4)
  GO TO (25,26),L
25 CALL GENEND(P1)
  IF (K) CALL DESTROY(P2)
  DO 100 I=1,N
    K100=I100
  CONTINUE
  K=.TRUE.
  GO TO 27
26 CALL GENEND(P2)
  CALL DESTROY(P1)
C
TIME DELAY LOOP USED TO STABLE DISPLAY PICTURE.
C
DO 200 I=1,N
K200=I200
  CONTINUE
200  L=3-L
C
INCREMENT TIME, AND CALCULATING THE CORRESPONDING PHASE
TIME IN A LOCOMOTIVE CYCLE.
C
RTIME=RTIME+1
RTIME1=RTIME
XTIME=((RTIME1*100.)/XIPT
  PRTIME=RTIME/IPT
  PHTIME=((RTIME-PRTIME*IPT)*100)/IPT
J7=1
J6=1
C
TESTING THE SUPPORTING FOOT OF ODD NUMBERED LEGS.

IF (I6.GT.5) GO TO 5000

TESTING THE SUPPORT FEET OF EVEN NUMBERED LEGS.

DIF2 = PHTIME - PHI(I6)
I. (DIF2) 46, 49, 43
DIF3 = PHI(I6) - PHI(I6+6)
IF (DIF3) 44, 44, 49
DIF4 = PHTIME - PHI(I6+6)
IF (DIF4) 49, 48, 48
I0(J7) = I6
J7 = J7 + 1
I6 = I6 + 2
GO TO 42

DIF5 = PHI(I6+6) - PHI(I6)
IF (DIF5) 47, 47, 48
DIF6 = PHTIME - PHI(I6+6)
IF (DIF6) GO TO 48
GO TO 49

I6 = I6 + 2
GO TO 42

5000
IRX = J7 - 1
51
I6 = I6 - 1
IF (I6.EQ.0) GO TO 60
IF (PHTIME - PHI(I6)) 55, 58, 52
IF (PHI(I6) - PHI(I6+6)) 53, 53, 58
55
IF (PHTIME - PHI(I6+6)) 58, 57, 57
58
I0(J7) = I6
J7 = J7 + 1
GO TO 57

55
IF (PHI(I6+6) - PHI(I6)) 56, 56, 57
56
DIF7 = PHTIME - PHI(I6+6)
IF (DIF7.GE.0) GO TO 57
GO TO 58

57
I6 = I6 - 2
GO TO 51

CALCULATING X-COORDINATE OF CENTER OF GRAVITY.

XCG = XCG + (1./XIPT)*(1./BETA)
J7 = J7 - 1
DO 15 MI = 1, J7
N1 = I0(M1)
MOD = PHTIME - PHI(N1)
IF (MOD.GE.0) GO TO 10
MOD = MOD + 100
XMOD = MOD
88
FORMAT(5X, 'XMOD=', F8.4)
CALCULATING X-COORDINATES OF SUPPORTING FEET.

IF (NL.GT.2) GO TO 11
XFP(N1)=1.5+(XTIME-XMOD)/(100.*BETA)
GO TO 15
11 IF (NL.GT.4) GO TO 12
XFP(N1)=0.5+(XTIME-XMOD)/(100.*BETA)
GO TO 15
12 XFP(N1)=-0.5+(XTIME-XMOD)/(100.*BETA)
89 FORMAT(5X, F8.4, 3X, 2I)
15 CONTINUE

IFI: LEFT SIDE FRONT LEG.
IR2: RIGHT SIDE FRONT LEG.
IR1: LEFT SIDE REAR LEG.
IR2: RIGHT SIDE REAR LEG.

IFI=IO(1)
IR1=IO(IRX)
IR2=IO(J7)
IR2=IO(IRX+1)

CALCULATING STABILITY MARGIN.
XD1=((XFP(IFI)+XFP(IR2))/2.-XCG
XD2=XCG-((XFP(IR2)+XFP(IR1))/2.
DIF=XD1-XD2
IF (DIF.LE.0.) GO TO 14
XMIN=XD2
GO TO 16
14 XMIN=XD1
16 IF ((SM-XMIN).LE.0.) GO TO 18
SM=XMIN
GO TO 19
18 SM=SM
19 GO TO 21
500 STOP
END
Figure IVa: Stability Margin of Hexapod Gait as a Function of $\phi_3$
and $\phi_5$ for $\beta = .55$. 

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Figure IVb: Stability Margin of Hexapod Gait as a Function of $\phi_3$ and $\phi_5$ for $\beta = .65$. 

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Figure IVc: Stability Margin of Hexapod Gait as a Function of $\phi_3$
and $\phi_5$ for $\beta = .75$. 
Figure IVd: Stability Margin of Hexapod Gait as a Function of $\phi_3$ and $\phi_5$ for $\beta = .85$. 

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**Figure IVe:** Stability Margin of Hexapod Gait as a Function of $\phi_3$ and $\phi_5$ for $\beta = .95$. 

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REFERENCES


