INFORMATION TO USERS

This material was produced from a microfilm copy of the original document. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the original submitted.

The following explanation of techniques is provided to help you understand markings or patterns which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting thru an image and duplicating adjacent pages to insure you complete continuity.

2. When an image on the film is obliterated with a large round black mark, it is an indication that the photographer suspected that the copy may have moved during exposure and thus cause a blurred image. You will find a good image of the page in the adjacent frame.

3. When a map, drawing or chart, etc., was part of the material being photographed the photographer followed a definite method in "sectioning" the material. It is customary to begin photoing at the upper left hand corner of a large sheet and to continue photoing from left to right in equal sections with a small overlap. If necessary, sectioning is continued again -- beginning below the first row and continuing on until complete.

4. The majority of users indicate that the textual content is of greatest value, however, a somewhat higher quality reproduction could be made from "photographs" if essential to the understanding of the dissertation. Silver prints of "photographs" may be ordered at additional charge by writing the Order Department, giving the catalog number, title, author and specific pages you wish reproduced.

5. PLEASE NOTE: Some pages may have indistinct print. Filmed as received.

Xerox University Microfilms
300 North Zeeb Road
Ann Arbor, Michigan 48106
REINHARD, Kenneth Lynn, 1940-
ADAPTIVE ANTENNA ARRAYS FOR CODED
COMMUNICATION SYSTEMS.

The Ohio State University, Ph.D., 1973
Engineering, electrical

University Microfilms, A XEROX Company, Ann Arbor, Michigan

THIS DISSERTATION HAS BEEN MICROFILMED EXACTLY AS RECEIVED.
ADAPTIVE ANTENNA ARRAYS FOR CODED COMMUNICATION SYSTEMS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By


*****

The Ohio State University
1973

Reading Committee:
Professor A.A. Ksieniski
Professor W.H. Peake
Assoc. Professor R.T. Compton, Jr.

Approved by

A.A. Ksieniski
Adviser
Department of Electrical Engineering
VITA

August 15, 1940... Born - Dayton, Ohio

June 1964........... B.E.E., The Ohio State University
                   Columbus, Ohio

1964-1965......... Research Assistant, Antenna Laboratory,
                   The Ohio State University, Columbus, Ohio

June 1967......... M.Sc., The Ohio State University,
                   Columbus, Ohio

1965-present..... Graduate Research Associate, ElectroScience
                   Laboratory (formerly Antenna Laboratory),
                   The Ohio State University, Columbus, Ohio

PUBLICATIONS

"Analysis of a Pseudo-Random Network Timing System for Time Division
University, 1967.

"A Sampled-Data Delay-Lock Loop for Synchronizing TDMA Space
Communications Systems," (co-author: R.J. Huff), 1968 IEEE

"A Delay-Lock Loop for Tracking Pulsed-Envelope Signals," (Co-author:
R.J. Huff), IEEE Transactions on Aerospace and Electronic
FIELDS OF STUDY

Major Field: Electrical Engineering

Studies in Communication Theory.
  Professors A.A. Ksieniak and C.E. Warren

Studies in Coding Theory.
  Professor R. Lackey

Studies in Antenna Theory.
  Professors C.H. Walter and R.C. Rudduck

Studies in Applied Mathematics.
  Professor S. Drobot
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>VITA</td>
<td>iii</td>
</tr>
<tr>
<td>TABLES</td>
<td>vii</td>
</tr>
<tr>
<td>ILLUSTRATIONS</td>
<td>viii</td>
</tr>
<tr>
<td>LIST OF SYMBOLS</td>
<td>xv</td>
</tr>
<tr>
<td>Chapter I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. MATHEMATICAL MODEL FORMULATION</td>
<td>4</td>
</tr>
<tr>
<td>A. Introduction</td>
<td>4</td>
</tr>
<tr>
<td>B. Signal Structure</td>
<td>4</td>
</tr>
<tr>
<td>C. Adaptive Array Structure</td>
<td>10</td>
</tr>
<tr>
<td>D. Performance Measures</td>
<td>16</td>
</tr>
<tr>
<td>III. THEORETICAL PERFORMANCE WITH AN IDEAL</td>
<td>19</td>
</tr>
<tr>
<td>REFERENCE SIGNAL</td>
<td></td>
</tr>
<tr>
<td>A. Array Transient Response</td>
<td>19</td>
</tr>
<tr>
<td>B. Array Steady-state Performance</td>
<td>26</td>
</tr>
<tr>
<td>C. The Effect of Control-loop Offsets</td>
<td>34</td>
</tr>
<tr>
<td>D. Steady-state Response to Pulse Interference</td>
<td>45</td>
</tr>
<tr>
<td>IV. THEORETICAL PERFORMANCE WITH A NON-IDEAL</td>
<td>58</td>
</tr>
<tr>
<td>REFERENCE SIGNAL</td>
<td></td>
</tr>
<tr>
<td>A. Introduction</td>
<td>58</td>
</tr>
<tr>
<td>B. Array Performance with a Non-ideal, Fixed</td>
<td>58</td>
</tr>
<tr>
<td>Reference Signal</td>
<td></td>
</tr>
<tr>
<td>C. Array Performance with an Ideal, Waveform-</td>
<td>65</td>
</tr>
<tr>
<td>processed Reference Signal</td>
<td></td>
</tr>
<tr>
<td>D. Effects of Waveform-processor Time Delay</td>
<td>74</td>
</tr>
<tr>
<td>E. Effects of Inadequate Waveform-processor Gain</td>
<td>79</td>
</tr>
<tr>
<td>Chapter</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>------</td>
</tr>
<tr>
<td>V. THEORETICAL PERFORMANCE WITH WIDEBAND NOISE INTERFERENCE</td>
<td>95</td>
</tr>
<tr>
<td>A. Introduction</td>
<td>95</td>
</tr>
<tr>
<td>B. Performance Analysis</td>
<td>95</td>
</tr>
<tr>
<td>C. Numerical Evaluation</td>
<td>99</td>
</tr>
<tr>
<td>VI. EXPERIMENTAL PERFORMANCE OF A FOUR-ELEMENT ADAPTIVE PROCESSOR</td>
<td>110</td>
</tr>
<tr>
<td>A. Introduction</td>
<td>110</td>
</tr>
<tr>
<td>B. Adaptive Processor Description</td>
<td>111</td>
</tr>
<tr>
<td>C. Reference Signal Processor Description</td>
<td>120</td>
</tr>
<tr>
<td>D. Bit Error Probability Performance Measure</td>
<td>120</td>
</tr>
<tr>
<td>E. Experimental Performance with Narrowband Signals</td>
<td>133</td>
</tr>
<tr>
<td>F. Experimental Performance with Pulse Interference</td>
<td>165</td>
</tr>
<tr>
<td>G. Experimental Performance with Wideband Noise Interference</td>
<td>175</td>
</tr>
<tr>
<td>VII. SUMMARY AND CONCLUSIONS</td>
<td>188</td>
</tr>
<tr>
<td>APPENDIX A</td>
<td>194</td>
</tr>
<tr>
<td>APPENDIX B</td>
<td>197</td>
</tr>
<tr>
<td>APPENDIX C</td>
<td>200</td>
</tr>
<tr>
<td>APPENDIX D</td>
<td>204</td>
</tr>
<tr>
<td>APPENDIX E</td>
<td>206</td>
</tr>
<tr>
<td>APPENDIX F</td>
<td>209</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>212</td>
</tr>
<tr>
<td>Table</td>
<td>Description</td>
</tr>
<tr>
<td>-------</td>
<td>------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>1</td>
<td>Worst-case Average Power Performance with Pulse Interference ($\alpha \lambda_1 \tau_j &lt;&lt; 1$)</td>
</tr>
<tr>
<td>2</td>
<td>The Degradation in dB of the Array Output Signal-to-Total Noise Ratio from its Ideal Value ($4S/\sigma^2$)</td>
</tr>
<tr>
<td>3</td>
<td>The Parameters of the Experimental System</td>
</tr>
</tbody>
</table>
### ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Array geometry and signal environment</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>Adaptive array structure</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>Weighting coefficient models with (a) real signals, (b) complex-envelope signals</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>Adaptive processor model</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>The improvement in signal-to-thermal noise ratio versus the separation parameter ( \psi ) for several input interference-to-thermal noise ratios</td>
<td>32</td>
</tr>
<tr>
<td>6</td>
<td>The normalized output interference-to-signal ratio versus the input interference-to-thermal noise ratio for several values of the separation parameter ( \psi )</td>
<td>33</td>
</tr>
<tr>
<td>7</td>
<td>Expanded adaptive processor model with control-loop d.c. offsets</td>
<td>35</td>
</tr>
<tr>
<td>8</td>
<td>An illustration of the array's steady-state response to pulse interference</td>
<td>48</td>
</tr>
<tr>
<td>9</td>
<td>The output interference-to-signal ratio versus time for several values of interference duty cycle; ( N_j = 10 )</td>
<td>51</td>
</tr>
<tr>
<td>10</td>
<td>The output signal-to-thermal noise ratio versus time for several values of interference duty cycle; ( N_j = 10 )</td>
<td>52</td>
</tr>
<tr>
<td>11</td>
<td>The output interference-to-signal ratio versus time for several values of interference duty cycle; ( N_j = 1 )</td>
<td>53</td>
</tr>
<tr>
<td>12</td>
<td>The ratio of average output interference power-to-average output signal power versus duty cycle for several values of normalized pulsewidth</td>
<td>55</td>
</tr>
<tr>
<td>13</td>
<td>An idealized model of the reference signal waveform processor</td>
<td>65</td>
</tr>
</tbody>
</table>
ILLUSTRATIONS (Continued)

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>A model of the reference waveform processor with timing offset in the bandspreading phase modulation</td>
</tr>
<tr>
<td>15</td>
<td>An illustration showing that the product of biphase pseudonoise modulations is real-valued</td>
</tr>
<tr>
<td>16</td>
<td>A model of the reference waveform processor which includes processing-time delay</td>
</tr>
<tr>
<td>17</td>
<td>A graphical illustration of the transcendental equation (168)</td>
</tr>
<tr>
<td>18</td>
<td>The model of a linear reference waveform processor which imperfectly rejects undesired input signals</td>
</tr>
<tr>
<td>19</td>
<td>Effect of inadequate waveform-processing gain on weight vector direction</td>
</tr>
<tr>
<td>20</td>
<td>The effect of correlated interference and thermal noise in the reference signal on the array output interference-to-signal ratio</td>
</tr>
<tr>
<td>21</td>
<td>The interference-to-signal ratio at the array output versus the correlated interference-to-signal ratio at the reference processor output</td>
</tr>
<tr>
<td>22</td>
<td>An illustration of the non-symmetry in the off-diagonal elements of the noise covariance matrix</td>
</tr>
<tr>
<td>23</td>
<td>The normalized output signal-to-total noise ratio versus the bandwidth parameter B; $J/\sigma^2 = 0$ dB</td>
</tr>
<tr>
<td>24</td>
<td>The normalized output signal-to-total noise ratio versus the bandwidth parameter B; $J/\sigma^2 = 10$ dB</td>
</tr>
</tbody>
</table>

ix
ILLUSTRATIONS (Continued)

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>The normalized ratio of output signal-to-total noise power versus the bandwidth parameter B; J/\sigma^2 = 20 dB</td>
<td>103</td>
</tr>
<tr>
<td>26</td>
<td>The normalized output signal-to-total noise ratio versus the input interference-to-thermal noise ratio; \psi = \pi/2</td>
<td>104</td>
</tr>
<tr>
<td>27</td>
<td>The normalized output signal-to-total noise ratio versus the input interference-to-thermal noise ratio; \psi = \pi/3</td>
<td>105</td>
</tr>
<tr>
<td>28</td>
<td>Synthesis of simulated array element output signals</td>
<td>112</td>
</tr>
<tr>
<td>29</td>
<td>Broadside pattern of a four-element, \lambda/2-spaced linear array</td>
<td>113</td>
</tr>
<tr>
<td>30</td>
<td>Block diagram of the four-element adaptive processor</td>
<td>115</td>
</tr>
<tr>
<td>31</td>
<td>The linearity of a signal weighting circuit versus the input signal amplitude</td>
<td>117</td>
</tr>
<tr>
<td>32</td>
<td>The phase shift-versus-frequency characteristics of a weight-control loop</td>
<td>118</td>
</tr>
<tr>
<td>33</td>
<td>The change in error multiplier output voltage versus input signal frequency</td>
<td>119</td>
</tr>
<tr>
<td>34</td>
<td>Block diagram of the reference waveform processor</td>
<td>121</td>
</tr>
<tr>
<td>35</td>
<td>The frequency response of the data bandwidth filter in the reference waveform processor</td>
<td>122</td>
</tr>
<tr>
<td>36</td>
<td>Waveforms illustrating biphase data modulation delay in reference waveform processor</td>
<td>123</td>
</tr>
<tr>
<td>37</td>
<td>The input-output amplitude characteristic of the reference waveform processor</td>
<td>124</td>
</tr>
<tr>
<td>Figure</td>
<td>Illustration Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>--------------------------</td>
<td>------</td>
</tr>
<tr>
<td>38</td>
<td>System configuration for evaluating the steady-state performance of the adaptive processor</td>
<td>125</td>
</tr>
<tr>
<td>39</td>
<td>Performance of the differential detector with and without PN coding applied to the input and local oscillators signals. Code timing offset $\epsilon = 0$.</td>
<td>127</td>
</tr>
<tr>
<td>40</td>
<td>Measurements showing the interference protection afforded by waveform processing at the differential detector for a band-spreading ratio of $10:1$.</td>
<td>128</td>
</tr>
<tr>
<td>41(a)</td>
<td>Performance of the limiter-detector subsystem versus the limiter input ratios of interference-to-signal and signal-to-thermal noise</td>
<td>129</td>
</tr>
<tr>
<td>41(b)</td>
<td>Performance of the limiter-detector subsystem versus the limiter input ratios of interference-to-signal and signal-to-thermal noise</td>
<td>130</td>
</tr>
<tr>
<td>41(c)</td>
<td>Performance of the limiter-detector subsystem versus the limiter input ratios of interference-to-signal and signal-to-thermal noise</td>
<td>131</td>
</tr>
<tr>
<td>42</td>
<td>Performance of the limiter-detector subsystem versus the limiter input signal-to-thermal noise ratio; no interference present</td>
<td>132</td>
</tr>
<tr>
<td>43</td>
<td>Transient response of weight-control voltages $w_5, w_6, w_7, w_8$ (top to bottom)</td>
<td>134</td>
</tr>
<tr>
<td>44</td>
<td>Transient response of weight-control voltages $w_5, w_6, w_7, w_8$ (top to bottom)</td>
<td>136</td>
</tr>
<tr>
<td>45</td>
<td>Transient response waveforms</td>
<td>137</td>
</tr>
<tr>
<td>46</td>
<td>Signal spectra at the array output</td>
<td>139</td>
</tr>
<tr>
<td>47</td>
<td>Signal spectra at the array output</td>
<td>140</td>
</tr>
<tr>
<td>48</td>
<td>Performance of the experimental system versus the input interference-to-thermal noise ratio for several values of the separation parameter $\psi$.</td>
<td>141</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>49</td>
<td>Performance of the experimental system versus the separation parameter $\psi$ for several input interference-to-thermal noise ratios</td>
<td>142</td>
</tr>
<tr>
<td>50</td>
<td>Measured array output power ratios versus the input interference-to-thermal noise ratio</td>
<td>144</td>
</tr>
<tr>
<td>51</td>
<td>Performance of the experimental system versus the separation parameter $\psi$ for several input signal-to-thermal noise ratios</td>
<td>146</td>
</tr>
<tr>
<td>52</td>
<td>Performance of the experimental system versus the frequency of the CW interfering signal</td>
<td>147</td>
</tr>
<tr>
<td>53</td>
<td>Measured improvement in signal-to-thermal noise ratio versus the input signal-to-thermal noise ratio (signal amplitude varied)</td>
<td>149</td>
</tr>
<tr>
<td>54</td>
<td>Effect of worst-case offset voltages in the control loops on system error rate</td>
<td>150</td>
</tr>
<tr>
<td>55</td>
<td>Transient response of weight-control voltages $w_5, w_6, w_7, w_8$ (top to bottom) with 40 mV offset in $w_6$ error multiplier</td>
<td>152</td>
</tr>
<tr>
<td>56</td>
<td>Waveforms illustrating the effect of reference processor delay</td>
<td>154</td>
</tr>
<tr>
<td>57</td>
<td>Waveforms illustrating the effect of reference processor delay. Same conditions as in Fig. 56(b) except horizontal sweep speed: uncalibrated, $\approx 21.4 \mu $sec/cm.</td>
<td>155</td>
</tr>
<tr>
<td>58</td>
<td>Waveforms illustrating the effect of reference processor delay</td>
<td>155</td>
</tr>
<tr>
<td>59</td>
<td>The effect of inhibiting array processing during data-delay periods. Other conditions as in Fig. 58 except horizontal sweep speed: 10 $\mu $sec/cm.</td>
<td>156</td>
</tr>
<tr>
<td>60</td>
<td>The effect of inhibiting processing during data-delay periods. Other conditions as in Fig. 56(b)</td>
<td>156</td>
</tr>
</tbody>
</table>
## ILLUSTRATIONS (Continued)

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>61</td>
<td>The effect of code timing offset on the bit error probability for various modes of operation</td>
<td>158</td>
</tr>
<tr>
<td>62</td>
<td>Waveforms illustrating the effect of code timing offset.</td>
<td>159</td>
</tr>
<tr>
<td>63</td>
<td>The effect of code timing offset on the array output signal amplitude.</td>
<td>160</td>
</tr>
<tr>
<td>64</td>
<td>The effect of offsetting the carrier frequency of the array input signal on the array output signal amplitude</td>
<td>161</td>
</tr>
<tr>
<td>65</td>
<td>The effect of signal carrier frequency offset on the weight-control voltages $w_5, w_6, w_7, w_8$.</td>
<td>163</td>
</tr>
<tr>
<td>66</td>
<td>The effect of offsetting the carrier frequency of the array input signal on the bit error probability</td>
<td>164</td>
</tr>
<tr>
<td>67</td>
<td>The method for estimating the average bit error probability of the experimental system with pulse interference present</td>
<td>167</td>
</tr>
<tr>
<td>68</td>
<td>Performance of the experimental system versus pulse interference parameters</td>
<td>168</td>
</tr>
<tr>
<td>69</td>
<td>Performance of the experimental system versus pulse interference parameters</td>
<td>169</td>
</tr>
<tr>
<td>70</td>
<td>Performance of the experimental system versus pulse interference parameters</td>
<td>170</td>
</tr>
<tr>
<td>71</td>
<td>The duty cycle which maximizes Eq. (117) versus the normalized pulselength for two values of the separation parameter $\Psi$</td>
<td>171</td>
</tr>
<tr>
<td>72</td>
<td>Performance of the experimental system versus pulse interference parameters</td>
<td>173</td>
</tr>
<tr>
<td>73</td>
<td>Performance of the experimental system versus pulse interference parameters</td>
<td>174</td>
</tr>
<tr>
<td>74</td>
<td>Spectra of the array input noise interference.</td>
<td>176</td>
</tr>
</tbody>
</table>
ILLUSTRATIONS (Continued)

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>Performance of the experimental system versus the input interference-to-thermal noise ratio for several values of the separation parameter $\psi$.</td>
</tr>
<tr>
<td>76</td>
<td>Performance of the experimental system versus the separation parameter $\psi$ for three values of interfering signal bandwidth.</td>
</tr>
<tr>
<td>77</td>
<td>Performance of the experimental system versus the input interference-to-thermal noise ratio for three values of the bandwidth parameter $B$.</td>
</tr>
<tr>
<td>78</td>
<td>Spectra of desired signal plus noise interference. (a) Output from one element, (b) array output in adapt mode.</td>
</tr>
<tr>
<td>79</td>
<td>Spectra at the array output. (a) Initial condition mode, equal weights, (b) adapt mode.</td>
</tr>
<tr>
<td>80</td>
<td>Spectra at the array output. $J/\sigma^2 = 10$ dB. Other conditions as in Fig. 79.</td>
</tr>
<tr>
<td>81</td>
<td>Spectra at the array output. $J/\sigma^2 = 20$ dB. Other conditions as in Fig. 79.</td>
</tr>
<tr>
<td>82</td>
<td>Performance of the experimental system versus the separation parameter $\psi$ (desired signal's angle-of-arrival varied).</td>
</tr>
<tr>
<td>83</td>
<td>Performance of the experimental system versus the bandwidth parameter $B$ (angles-of-arrival of desired and interfering signals varied).</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS

\[ m = \text{the number of array elements} \]

\[ \xi_i(t) = \text{ith signal present at the phase center of the array} \]

\[ \tilde{\xi}_i(t) = \text{complex envelope of the signal } \xi_i(t) \text{ at the center frequency, } f_c, \text{ of the array} \]

\[ \xi_k(t) = \text{thermally generated noise process in the receiving} \]

\[ \text{electronics associated with the } k^{th} \text{ array element} \]

\[ \tau_{ik} = \text{incremental time delay experienced by i th signal in} \]

\[ \text{propagating from the array phase center to the } k^{th} \text{ array element} \]

\[ \nu_{ij} = \text{column vector of incremental phase delays, } \exp(-j\omega_c \tau_{ik}), \]

\[ k=1,2,\ldots,m \text{ associated with the ith input signal} \]

\[ \hat{v}_i = \text{unit vector in the direction of the vector } v_i \]

\[ <v_i, v_j> = \text{inner product of the vectors } v_i \text{ and } v_j \]

\[ x_k(t) = \text{composite received signal plus noise process at the output} \]

\[ \text{of the } k^{th} \text{ array element} \]

\[ \tilde{x}(t) = \text{column vector of complex envelopes } \tilde{x}_k(t), k=1,2,\ldots,m \]

\[ K_x = \text{covariance matrix associated with the vector, } \tilde{x}(t), \]

\[ \text{of complex envelopes} \]

\[ N_1 = \text{mean-square value of the ith signal } \xi_i(t); \]

\[ (N_1 \equiv S, N_2 \equiv J) \]

\[ \sigma^2 = \text{mean-square value of the thermal noise processes } \xi_k(t), \quad k=1,2,\ldots,m \]

\[ w_k(t) = \text{complex-valued weighting coefficient associated} \]

\[ \text{with the } k^{th} \text{ channel of the adaptive processor} \]

\[ w(t) = \text{column vector of weighting coefficients } w_k(t), \quad k=1,2,\ldots,m \]

\[ y(t) = \text{composite signal plus noise process at the output} \]

\[ \text{of the array} \]

\[ \text{xv} \]
SYMBOLS (Continued)

\[ P_i(t) = \text{mean-square value (ensemble expectation at time } t) \]
\[ \text{of the } i\text{th signal component in the array output } \gamma(t); \]
\[ (P_1 = P_S, P_2 = P_J) \]

\[ P_{nt}(t) = \text{mean-square value of the thermal noise components} \]
\[ \text{in the array output } \gamma(t) \]

\[ R = \text{mean-square value of the reference signal } r(t) \]

\[ r_x(t) = \text{input-reference cross-correlation vector} \]

\[ \alpha = \text{gain constant associated with the weight-control} \]
\[ \text{processing loops} \]

\[ \lambda_i = i\text{th distinct eigenvalue of the covariance matrix } K_X \]

\[ \rho = \text{element spacing for equispaced linear array} \]

\[ \theta_s, \theta_j = \text{spatial angles of desired signal and interference,} \]
\[ \text{respectively, as measured from the axis of a linear,} \]
\[ \text{equispaced array} \]

\[ \psi = \text{difference in element-to-element phase shift of} \]
\[ \text{desired and interfering signals (equispaced linear array)} \]

\[ \Delta = \text{reciprocal of the desired signal's pseudonoise code rate} \]

\[ \epsilon = \text{timing offset between received and locally generated} \]
\[ \text{pseudonoise codes} \]

\[ T_b = \text{the desired signal's data bit period} \]

\[ \delta_r = \text{processing-time delay in the reference waveform-processor} \]

\[ \delta_f = \text{processing-time delay in the weighting coefficient} \]
\[ \text{feedback-control loops} \]

\[ \Delta \omega_s = \text{offset in the angular frequency of the desired signal} \]
\[ \text{carrier from the design frequency } \omega_C \]

\[ \Delta \omega_r = \text{angular frequency of periodic variations in the} \]
\[ \text{weight vector } w(t) \]

\[ \Omega_v = \text{column vector of d.c. offset voltages in the weight-} \]
\[ \text{control loops} \]
SYMBOLES (Continued)

\(\tau_j, T_j\) = pulsewidth and repetition period, respectively, of a pulsed-envelope interfering signal

\(b_j\) = bandwidth of a noise interfering signal

\(b_n\) = bandwidth of the element thermal noise processes \(\xi_k(t), \ k=1,2,\ldots,m\)
Adaptive antenna arrays have been the subject of considerable research in the past few years. The motivation for this research stems from the fact that adaptive arrays can automatically adjust to the received signal and noise environment. Through spatial and temporal processing, they optimize the reception of the desired signal while suppressing undesired signals at the array output. Among the fundamental advantages of adaptive array processors compared to conventional phased arrays are 1) the capability of rejecting high-level, directional interference, 2) the capability of maximizing the desired signal-to-thermal noise ratio, 3) the ability to compensate for phase disturbances (wavefront distortion) in the transmission medium, and 4) the capability of providing optimum aperture illumination when element patterns, mutual coupling, and aperture blockage effects are taken into account. Applebaum[1,2] and Shor[3] were among the first to consider adaptive antenna processing as an optimal control problem. They chose as a performance criterion the maximization of output signal-to-noise ratio. Widrow, et al[4], suggested the basic feedback algorithm for minimizing the mean-square error between the array output signal and a reference signal which closely approximates the desired component of the array output. LaCoss[5] proposed an algorithm based on minimizing the output noise variance. Baird and Zahm[6], and Baird, et al[7] have studied the relationships between the different performance criteria for adaptive arrays. Their results suggest that adaptive processors developed under any of the above criteria will satisfy the other criteria as well under narrow-band processing conditions.

Research activities in the field of adaptive arrays have developed generally along two parallel but distinct lines to satisfy two major applications: radar and communications. The radar problem is characterized by a priori knowledge of the angle-of-arrival of the received signal utilizing the fact that the direction of the transmitted beam is known. This a priori knowledge is translated into a set of conditions on the weighting coefficients of the receiving array (i.e., a beam-steering vector) which permits reception from the prescribed look-angle while rejecting undesired signals or clutter arriving from other angles. Brennan and Reed have studied the use of adaptive arrays for clutter rejection[8,9], the effects of envelope limiting in the control loops[10], and, with Pugh, the effects of control loop noise[11]. Other references pertaining to radar applications are[12,13,14,15]. The use of adaptive arrays for communications
presupposes a priori knowledge of desired signal characteristics without assuming prior knowledge of the angle-of-arrival. In general, omnidirectional reception and tracking of desired signals with the maximum gain obtainable from the array is desired. Early experimental measurements of adaptive array performance in a communications environment are described by Riegler and Compton[16]. Compton[17] also discusses the results of extensive measurements with a four-element adaptive array in the 200-400 MHz band, where the array elements are mounted on an irregular surface. The problem of establishing initial code timing for the array when a coded communication signal and high-level interference are present has also been under investigation. In particular, one approach that appears promising is a power equilization technique described by Compton and Lee[18]. Zahm[19], and Baird, et al[7], also discuss techniques closely related to power equalization for obtaining interference rejection during the prelockup phase, before code timing has been established at the receiver.

In the present study, the performance of an adaptive receiving array is investigated, both analytically and experimentally, under realistic conditions encountered in practical communication links. The desired communication signal is assumed to be spread in bandwidth by a deterministic (pseudonoise) code prior to transmission. Knowledge of the code is utilized at the array receiver to provide signal discrimination. Types of interference that would exist in a realistic environment -- broadband noise, variable-frequency CW, and pulse interference -- are considered. Modest values of signal-to-element thermal noise ratio are employed in the experimental evaluation. The purpose of the study was to investigate the potential advantages and performance limitations of adaptive array processors under these conditions.

In Chapter II, the complex envelope representation of input signals, the mathematical model of the array processor, and appropriate performance measures are formulated. In order to provide a basis for comparison, the ideal performance of the array is calculated in Chapter III. The degradation from ideal performance resulting from d.c. circuit imbalance in the feedback-control system is then determined. The chapter concludes with an analysis of the array response to periodic pulse interference. In Chapter IV, array performance with practical, i.e., non-ideal, reference signals is examined. First, the array response is analyzed for the case of a fixed reference signal containing the desired signal modulation but having offset errors in carrier frequency and time-base synchronization. Then, performance with a reference signal generated by waveform-processing of the array output signal is examined. The degradation which results from imperfect rejection of undesired signals in the waveform processor and from processor time delay are determined. In Chapter V, array performance with wideband noise interference is calculated. The experimental performance of a four-element, adaptive processor is
described in Chapter VI. Measurements of the average bit error probability (BEP) which occur upon detecting a bandpass-limited version of the array output signal in a differential (DPSK) detector are presented. These measurements are compared with calculated values derived from the theoretical performance of the array and from calibration measurements of the limiter-detector BEP performance. The results are summarized and conclusions drawn in Chapter VII.
A. Introduction

In this chapter the basic mathematical representations that are required for analysis in later chapters are established. A description of the signal structure at the array is given for an arbitrary geometrical distribution of antenna elements. The temporal characteristics of the signals and element thermal noise processes are expressed in terms of complex envelopes. Then, the basic model of the adaptive array processor is introduced, and measures to be used in evaluating array performance are discussed.

B. Signal Structure

A three-dimensional array of m omni-directional antennas, distributed in an arbitrary but known configuration, is assumed as illustrated in Fig. 1. The location of the kth antenna element is

Fig. 1.—Array geometry and signal environment.
defined by the vector \( d \) specifying both distance and direction in a cartesian coordinate frame from the geometric (phase) center of the array. A signal environment consisting of \( p \) signal sources (\( p < m \)) located at distant points in the far-field of the array is also postulated. The direction of propagation of the signal from the \( i \)th source is denoted by the unit vector \( \hat{a}_i \) in Fig. 1. The sources are assumed to be spatially separated and distinct, i.e., \( \hat{a}_i \neq \hat{a}_j \) for \( i \neq j \). The signal emitted from the first source (\( i = 1 \)) is defined to be the desired signal while the \( p-1 \) signals from the other sources represent directional interference. These signals are assumed to propagate toward the antenna elements as uniform plane waves through a non-dispersive transmission medium whose only effect on the signals is time delay. The output of each array element will be modeled by time-delayed versions of the signals arriving at the array phase center plus an element thermal noise process. The effects of non-ideal antenna elements or anomalies in the transmission medium will not be considered here; however, the signal representations to be developed in this section may be modified to include these effects.

The signals present at the array phase center may be represented in general as amplitude- and phase-modulated carrier signals of the form

\[
\xi_i(t) = \alpha_i(t) \cos(\omega_c t + \phi_i(t) + \theta_i) \quad i = 1, 2, \ldots, p
\]

where \( \xi_i(t) \) is the signal from the \( i \)th source. This representation is valid for all real signals which are bandlimited between d.c. and twice the array center frequency, \( f_c \). The analytic signal associated with \( \xi_i(t) \) is defined by

\[
z_i(t) = \xi_i(t) + j\hat{\xi}_i(t)
\]

where \( \hat{\xi}_i(t) \) is the Hilbert transform of \( \xi_i(t) \). The complex envelope of \( \xi_i(t) \) at the frequency \( f_c \), denoted as \( \hat{\xi}_i(t) \), is defined in terms of the analytic signal, \( z_i(t) \), as

\[
\hat{\xi}_i(t) = \frac{1}{\sqrt{2}} z_i(t) e^{-j\omega_c t}
\]

The real part of Eq. (1b), i.e.,

\[
\xi_i(t) = \text{Re} \, z_i(t) = \text{Re}[\hat{\xi}_i(t) \cdot \sqrt{2} \exp(j\omega_c t)]
\]

gives the representation for the real signals, Eq. (1a) in terms of their complex envelopes, Eq. (1c). Upon comparing Eq. (1a) and (1d), the desired expression for the complex envelope signals follows as
The element thermal noise processes may also be represented as modulated carriers at the frequency $f_c$,

$$\zeta_k(t) = \beta_k(t) \cos[\omega_c t + \psi_k(t)],$$

$$\bar{\zeta}_k(t) = \frac{\beta_k(t)}{\sqrt{2}} \exp[j\psi_k(t)], \quad k=1,2,\ldots,m$$

where

- $\zeta_k(t)$ = thermal noise process in $k$th array element
- $\beta_k(t)$ = amplitude modulation of $\zeta_k(t)$
- $\psi_k(t)$ = phase modulation of $\zeta_k(t)$
- $\bar{\zeta}_k(t)$ = complex envelope of $\zeta_k(t)$.

In terms of the signals, Eqs. (1)-(4), the output of the $k$th array element, denoted by $x_k(t)$ and having a complex envelope $\bar{x}_k(t)$, may be written as

$$x_k(t) = \sum_{i=1}^{p} \xi_i(t-\tau_{ik}) + \bar{\zeta}_k(t)$$

$$\bar{x}_k(t) = \sum_{i=1}^{p} \bar{\xi}_i(t-\tau_{ik}) \exp[-j\omega_c \tau_{ik}] + \bar{\zeta}_k(t)$$

where $\tau_{ik}$ is the differential delay of the $i$th signal to the $k$th array element relative to the array phase center. The delay $\tau_{ik}$ may be expressed in terms of the projection of the direction vector $\hat{a}_i$ onto the element position vector $d_k$ by the inner product

$$\tau_{ik} = \frac{\langle \hat{a}_i, d_k \rangle}{v}, \quad i=1,2,\ldots,p$$

$$k=1,2,\ldots,m$$

---

1 In Eq. (6), $\xi_i(t-\tau_{ik})$ denotes the complex envelope, $\xi_i(t)$, time-delayed by $\tau_{ik}$. 
where $v$ is the velocity of propagation in the transmission medium.

For purposes of analysis in later chapters, the input signals will be modeled as sample functions of stationary, uncorrelated, zero-mean random processes. The complex envelope processes are assumed to be partially characterized by the following ensemble expectations:

\begin{align}
(8) \quad & E[\xi_i(t)] = 0 \\
(9) \quad & E[\xi_i(t) \overline{\xi}_j(t-\tau)] = \begin{cases} R_{\xi_i}(\tau) & , \quad i=j \\
0 & , \quad i \neq j \end{cases} \\
(10) \quad & E[\xi_i(t) \overline{\xi}_i(t)^\dagger] = E[|\xi_i(t)|^2] = R_{\xi_i}(0) = N_i
\end{align}

It can be shown\[21\] that the real part of the product of Eq. (9) and $\exp[j\omega_c \tau]$ is the cross-correlation function of the real signals, i.e.,

$$E[\xi_i(t) \xi_j(t-\tau)] = \text{Re} \{e^{j\omega_c \tau} E[\xi_i(t) \overline{\xi}_j(t-\tau)^\dagger]\}.$$

The element thermal noise processes are assumed to be independent, bandlimited Gaussian processes which are also independent of the signals. These zero-mean processes have identical variances, $\sigma^2$, and a constant spectral density over a bandwidth of $b_H$ Hertz about the carrier frequency $f_c$:

\begin{align}
(11) \quad & E[\xi_k(t)] = 0 \quad k=1,2,\ldots,m \\
& \quad \frac{\sin \pi b_H \tau}{\alpha 2 \pi b_H n^2} , \quad j=k \\
(12) \quad & E[\xi_j(t) \overline{\xi}_k(t-\tau)] = \begin{cases} \frac{\sin \pi b_H \tau}{\alpha 2 \pi b_H n^2} & , \quad j=k \\
0 & , \quad j \neq k \end{cases}
\end{align}

The symbol $\dagger$ denotes the adjoint (complex-conjugate transpose) operation. The adjoint of a complex-valued scalar function, as in Eq. (9) and Eq. (10), is simply the complex conjugate of the scalar function.

\[2\] The symbol $\dagger$ denotes the adjoint (complex-conjugate transpose) operation. The adjoint of a complex-valued scalar function, as in Eq. (9) and Eq. (10), is simply the complex conjugate of the scalar function.
To ensure wide-sense stationarity, the independent random variables \( \theta_i \) and \( \psi_k \) in Eqs. (1a) and (3) have uniform probability densities over the interval \([0, 2\pi]\).

For convenience in later analysis, a complex vector notation will be employed. The complex envelopes, Eq. (6), of the outputs from the array elements will be defined as elements of a column vector

\[
X(t) = \begin{bmatrix}
\hat{x}_1(t) \\
\hat{x}_2(t) \\
\vdots \\
\hat{x}_n(t)
\end{bmatrix} = \sum_{i=1}^{P} \hat{c}_i(t) + \hat{n}_t(t)
\]

where, from Eq. (6),

\[
\hat{c}_i(t) = \begin{bmatrix}
\xi_i(t-\tau_{i1}) \exp(-j\omega_c \tau_{i1}) \\
\vdots \\
\xi_i(t-\tau_{im}) \exp(-j\omega_c \tau_{im})
\end{bmatrix}
\]

and

\[
\hat{n}_t(t) = \begin{bmatrix}
\xi_1(t) \\
\xi_2(t) \\
\vdots \\
\xi_n(t)
\end{bmatrix}
\]

The vector \( \hat{c}_i(t) \) is the set of complex envelopes associated with the \( i \)th input signal; the vector \( \hat{n}_t(t) \) is the set of complex envelopes of the element thermal noises. The vector \( X(t) \) in Eq. (13) may also be expressed as the sum of a desired component vector, \( \hat{y}(t) \), and an undesired component vector, \( \hat{n}(t) \), according to

\[
X(t) = \hat{y}(t) + \hat{n}(t)
\]
where \( \hat{s}(t) = \hat{c}_i(t) \)

\[
\hat{n}(t) = \sum_{i=2}^{P} \hat{c}_i(t) + \hat{\nu}_t(t).
\]

The relationships among the different elements of the vector \( \hat{c}_i(t) \) in Eq. (14a) is a consequence of the assumptions regarding the transmission medium and the antenna elements. For non-isotropic but identical array elements, the only modification required in Eq. (13) is to multiply each vector \( c_i \) by a complex-valued scalar \( f_i(\hat{a}_i) \) which is functionally dependent upon the arrival direction \( \hat{a}_i \) of the \( i \)th signal. Other effects such as aperture blockage, mutual coupling, or wavefront distortion alter the relative magnitudes and phases of the elements in the vector \( c_i \). In these cases, a more involved description of the vector \( c_i \) is required in terms of the appropriate parameters of the problem.

With the exception of the analysis in Chapter V, a signal environment comprised of narrowband emitters compared to array bandwidth is assumed. The narrowband assumption is that the differential delays, \( \tau_{ik} \), between elements are small compared to the reciprocal of the highest frequency components in the complex envelopes, (Eq. (2)), permitting the approximation

\[
\hat{\xi}_i(t-\tau_{ik}) \approx \hat{\xi}_i(t) ; \quad i=1,2,\ldots,p
\]

\( k=1,2,\ldots,m. \)

A set of direction-delay vectors may be defined in the narrowband case as

\[
V_i = \begin{bmatrix}
\exp(-j\omega_c^\tau_{i1}) \\
\vdots \\
\exp(-j\omega_c^\tau_{im})
\end{bmatrix}, \quad i=1,2,\ldots,p
\]

where the \( i \)th vector contains the relative phase delays experienced by the \( i \)th input signal in propagating to the array elements. Substitution of Eq. (16) and Eq. (17) into Eq. (14a) yields the representation for a narrowband input vector,

\[
\hat{x}(t) = \sum_{i=1}^{P} \hat{\xi}_i(t) V_i + \hat{\nu}_t(t).
\]
The covariance matrix associated with this narrowband input vector may be determined using Eqs. (8)-(12) as follows:

\[
E[\tilde{x}(t) \tilde{x}(t-t)^\dagger] \bigg|_{t=0} = K_X (0) \\
= E \left[ \sum_{i=1}^{p} \tilde{\xi}_i(t) \mathbf{v}_i + \tilde{\eta}_c(t) \right] \left[ \sum_{j=1}^{p} \tilde{\xi}_j(t)^\dagger \mathbf{v}_j^\dagger + \tilde{\eta}_c(t)^\dagger \right] \\
= E \left[ \sum_{i=1}^{p} \tilde{\xi}_i(t) \tilde{\xi}_i(t)^\dagger \mathbf{v}_i \mathbf{v}_i^\dagger \right] + E[\tilde{\eta}_c(t) \tilde{\eta}_c(t)^\dagger] \\
= \sum_{i=1}^{p} N_i \mathbf{v}_i \mathbf{v}_i^\dagger + \sigma^2 I. \tag{19}
\]

In this expression, \( N_i \) is the mean-square value (power) of the \( i \)th narrowband input signal.

C. Adaptive Array Structure

The adaptive array processor to be investigated is represented functionally by the block diagram shown in Fig. 2. Each individual antenna element is shown connected to a device having two outputs which differ in phase by 90° at every frequency of the input signal. The physical device modeled is a broadband quadrature hybrid; mathematically, the quadrature output signals are a Hilbert transform pair. The in-phase and quadrature signals are multiplied individually by real-valued weighting coefficients and then summed to form the weighted output signal from the element. This process is illustrated in Fig. 3(a) where the weighted output from the \( k \)th array element is given by

\[
\gamma_k(t) = w_{1k}(t)x_k(t) + w_{2k}(t)\tilde{x}_k(t). \tag{20}
\]

In complex form, the weighting coefficient model is shown in Fig. 3(b) where the complex envelope of the weighted element signal is

\[
\tilde{\gamma}_k(t) = w_{1k}(t)\tilde{x}_k(t) + w_{2k}(t)[-j \tilde{x}_k(t)] = [w_{1k}(t)-j w_{2k}(t)] \tilde{x}_k(t) = w_k(t)\tilde{x}_k(t) \tag{21}
\]

\[^3\text{If the signals are correlated, terms in Eq. (19) of the form } E[\tilde{\xi}_i(t) \tilde{\xi}_j(t)^\dagger] \mathbf{v}_i \mathbf{v}_j^\dagger \text{ with } i \neq j \text{ would be retained.}\]
Fig. 2.--Adaptive array structure.
and the complex weighting coefficient is defined as

\[ w_k(t) = w_{1k}(t) + jw_{2k}(t). \]

\[_{4}^{4}\text{In order to be mathematically rigorous, the step from Eq. (20) to Eq. (21) requires that the lowpass spectrums of the weighting coefficients do not overlap the highpass spectrum of the input signal[30].} \]
Using Eqs. (21) and (22), the complex envelope of the array output signal may be expressed as the sum

\[ \tilde{\gamma}(t) = \sum_{k=1}^{m} \tilde{\gamma}_k(t) = \sum_{k=1}^{m} w_k(t)^\dagger \tilde{x}_k(t) \]

\[ = w(t)^\dagger \tilde{\chi}(t) = \langle w(t), \tilde{\chi}(t) \rangle \]

where the weight vector in the inner product is defined as a column vector of complex weighting coefficients:

\[ w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_m(t) \end{bmatrix} \]

The error signal shown in Fig. 2 is formed by subtracting the array output signal from a locally-generated reference signal \( r(t) \). The complex envelope of the error signal is represented by

\[ \tilde{\epsilon}(t) = \tilde{r}(t) - \tilde{\gamma}(t) = \tilde{r}(t) - w(t)^\dagger \tilde{\chi}(t) \]

\[ = \tilde{r}(t) - \sum_{k=1}^{m} (w_{1k} - jw_{2k}) \tilde{x}_k(t). \]

The error signal and the signals present at the inputs to the weighting coefficients are applied to the adaptive weight-control processor which determines the values of the weighting coefficients in Eq. (24).

The adaptive weight-control processor is designed to minimize the instantaneous squared-magnitude of the error signal. The design is based on a steepest-descent optimization procedure\[4,7\] where the time derivative of the weight vector is set proportional to the negative of the gradient of the squared-error magnitude, i.e.,

\[ \frac{dw(t)}{dt} = - \lambda \nabla_w |\tilde{\epsilon}(t, w)|^2 \]

where \( \lambda \) is a real constant. The weighting coefficients are forced to move along the path of steepest descent on the magnitude-squared error surface, \( |\tilde{\epsilon}(w_1, w_2, \ldots, w_m, t)|^2 \), defined as a functional of the \( m \) complex weights (2\( m \) real weights). The vector equation (26) has, as its \( k^{th} \) component, the equation
Using Eq. (25), the right-hand side of Eq. (27) may be evaluated as follows:

\[
\frac{\partial}{\partial w_{1k}} \dot{\hat{e}}(t) \dot{\hat{e}}(t)^\dagger = \dot{\hat{e}}(t) \frac{\partial}{\partial w_{1k}} \dot{\hat{e}}(t)^\dagger + \dot{\hat{e}}(t)^\dagger \frac{\partial}{\partial w_{1k}} \dot{\hat{e}}(t) \\
= \dot{\hat{e}}(t) [-\hat{x}_k(t)^\dagger] + \dot{\hat{e}}(t)^\dagger [-\hat{x}_k(t)] \\
= -2 \text{ Re} [\dot{\hat{e}}(t)^\dagger \hat{x}_k(t)];
\]

\[
\frac{\partial}{\partial w_{2k}} \dot{\hat{e}}(t) \dot{\hat{e}}(t)^\dagger = \dot{\hat{e}}(t) [-j \dot{\hat{x}}_k(t)^\dagger] + \dot{\hat{e}}(t)^\dagger [+j \dot{\hat{x}}_k(t)] \\
= -2 \text{ Re} [\dot{\hat{e}}(t)^\dagger \{-j \dot{\hat{x}}_k(t)^\dagger\}].
\]

These results define the processing operations required in a physical processor. A multiplication of the error signal with each of the quadrature input signals (in separate multipliers) is indicated. The presence of the complex conjugate operation on the error signal’s complex envelope indicates that the difference in the phase modulations of input and error signals is retained. The weighting coefficient, \(w_{1k}(t)\), for the in-phase channel of the \(k\)th array element satisfies the equation

\[
\frac{d}{dt} w_{1k}(t) = 2\lambda \text{ Re}[\dot{\hat{e}}(t)^\dagger \hat{x}_k(t)] \\
= 2\lambda |\dot{\hat{e}}(t)| \cdot |\dot{\hat{x}}_k(t)| \cdot \cos[\text{Arg} \ \dot{\hat{x}}_k(t) - \text{Arg} \ \dot{\hat{e}}(t)].
\]

The right-hand side of this equation can be implemented with a four-quadrant (phase-sensitive) multiplier followed by a lowpass zonal filter which passes only the baseband (difference frequency) signals at the multiplier output. The processing operations required to generate the derivative of the quadrature channel weighting coefficient, \(w_{2k}(t)\), are identical. For analysis purposes the right-hand side of Eq. (27) may be written in the form...
With the aid of Eq. (25), this equation may be expressed in vector notation as

\[
\frac{d}{dt} w_k(t) = 2 \lambda \{ \text{Re} [\hat{e}(t) \hat{x}_k(t)] + j \text{Im} [\hat{c}(t) \hat{x}_k(t)] \} \\
= \alpha \hat{x}_k(t) \hat{c}(t)^\dagger, \quad k=1,2,\cdots,m
\]

\[
\alpha = 2 \lambda.
\]

The processing model defined by Eq. (29) is shown in Fig. 4. The

![Diagram of Adaptive Processor Model](image)

Fig. 4.—Adaptive processor model.
weight vector is a random vector whose mean value is governed by the ensemble average of Eq. (29) over all random variables present,

\[ \frac{d}{dt} \mathbb{E}[w(t)] = \alpha \left( \mathbb{E}[\dot{x}(t) \dot{r}(t)^\dagger] - \mathbb{E}[\ddot{x}(t) \dot{x}(t)^\dagger] \right) w(t). \]

The mean weight vector may be separated out of the right-hand side of this equation using a well-known identity for products of random variables:

\[ \mathbb{E}[\dot{x}(t) \dot{x}(t)^\dagger w(t)] = \mathbb{E}[\dot{x}(t) \dot{x}(t)^\dagger] \cdot \mathbb{E}[w(t)] + \mathbb{E}\left[ \left( \dot{x}(t) \dot{x}(t)^\dagger - \mathbb{E}[\dot{x}(t) \dot{x}(t)^\dagger] \right) \left( w(t) - \mathbb{E}[w(t)] \right) \right]. \]

In order to obtain an analytical solution to Eq. (30), it will be necessary to assume that the second term in the right-hand side of Eq. (31) is negligible compared to the first term. This approximation appears to be justified when the fluctuations in $\dot{x}(t) \dot{x}(t)^\dagger$ about its mean value, the covariance matrix Eq. (19), are small, or when the spectral widths of the input signals are much larger than the bandwidth of the control loops. In the first case the resulting variances of the weights about their mean values are presumably small; in the second case, the narrowband fluctuations in the weights are not expected to correlate significantly with the wideband fluctuations in $\dot{x}(t) \dot{x}(t)^\dagger$. General conditions on the statistics of $\dot{x}(t)$ and control loop gain $\alpha$ under which these heuristic arguments are valid have not been determined. Difficulties are encountered in attempting to formulate matrix differential equations satisfied by the weight autocovariance matrix $K_w(\tau)$ and the cross-covariance matrix $K_{wx}(\tau)$ as a result of the multiplicative nature of the (known) input and (unknown) weight vector processes in Eq. (30). The approach adopted in this research is to solve the approximate mean-value equations

\[ \frac{1}{\alpha} \frac{d}{dt} \mathbb{E}[w(t)] = \mathbb{E}[\dot{x}(t) \dot{r}(t)^\dagger] - \mathbb{E}[\ddot{x}(t) \dot{x}(t)^\dagger] \cdot \mathbb{E}[w(t)] \]

for various choices of input and reference signals and to compute the array performance from these approximate solutions. The performance obtained in this manner is then compared with the measured performance of an experimental adaptive processor to establish the validity of the approximate solutions.

D. Performance Measures

The measure of performance that will be used to describe array behavior is the output signal-to-noise power ratio. This measure is chosen over others relating to the smallness of the error signal
since, in later chapters where non-ideal reference signals containing undesired components are considered, minimization of the error may not correspond to maximization of output signal-to-noise ratio. The total power at the array output is defined as the expectation of the magnitude-squared array output signal, Eq. (24), i.e.,

\[ P_T(t) = \mathbb{E}[|\gamma(t)|^2] = \mathbb{E}[w(t)^\dagger \check{x}(t)\check{x}(t)^\dagger w(t)]. \]

This is an ensemble-average definition of power which is applicable at any time instant during adaptation. As in the preceding section, the assumption of small weight variances about their mean values will be made permitting the expectation of the product in Eq. (33) to be approximated by the product of expectations,

\[ P_T(t) = \mathbb{E}[w(t)^\dagger] \cdot \mathbb{E}[\check{x}(t)\check{x}(t)^\dagger] \cdot \mathbb{E}[w(t)] \]

\[ = \langle \mathbb{E}[w(t)], K_x \mathbb{E}[w(t)] \rangle. \]

For uncorrelated input processes, the total output power is comprised exclusively of components attributed to each input source, i.e.,

\[ P_T(t) = \sum_{i=1}^{P} P_i(t) + P_{nt}(t) \]

where, from Eqs. (13)-(15),

\[ P_i(t) = \mathbb{E}[|\gamma_i(t)|^2] = \mathbb{E}[w(t)^\dagger \check{c}_i(t)\check{c}_i(t)^\dagger w(t)] \]

\[ = \langle \mathbb{E}[w(t)], \mathbb{E}[\check{c}_i(t)\check{c}_i(t)^\dagger] \mathbb{E}[w(t)] \rangle \]

\[ i=1,2,\ldots,P \]

and

\[ P_{nt}(t) = \mathbb{E}[|\gamma_{nt}(t)|^2] = \mathbb{E}[w(t)^\dagger \check{n}_t(t)\check{n}_t(t)^\dagger w(t)] \]

\[ = \langle \mathbb{E}[w(t)], \mathbb{E}[\check{n}_t(t)\check{n}_t(t)^\dagger] \mathbb{E}[w(t)] \rangle \]

\[ = \langle \mathbb{E}[w(t)], \sigma^2 I \mathbb{E}[w(t)] \rangle \]

\[ = \sigma^2 |\mathbb{E}[w(t)]|^2. \]
When the input signals are narrowband, the covariance matrix of the vector \( \mathbf{z}_i(t) \) in Eq. (36) reduces to the form \( N \mathbf{V}_i \mathbf{V}_i^\dagger \), as in Eq. (19), permitting Eq. (36) to be written as

\[
\begin{align*}
P_i(t) &= \langle E[\mathbf{w}(t)], N \mathbf{V}_i \mathbf{V}_i^\dagger \rangle E[\mathbf{w}(t)] > \\
&= N \langle \mathbf{v}_1, \mathbf{w}(t) \rangle^2.
\end{align*}
\]

A general expression for the desired signal-to-total noise power ratio at the array output follows from Eqs. (36) and (37) as

\[
\begin{align*}
P_s &= \frac{P_1}{P_n} = \frac{\langle E[\mathbf{w}(t)], K_S E[\mathbf{w}(t)] \rangle}{\langle E[\mathbf{w}(t)], K_n E[\mathbf{w}(t)] \rangle} \\
&= \frac{\sum_{i=2}^{P} P_i}{P_1 + P_{nt}}
\end{align*}
\]

where

\[
K_S = E[\mathbf{z}_1(t) \mathbf{z}_1(t)^\dagger] \quad \text{and} \quad K_n = \sum_{i=2}^{P} E[\mathbf{z}_i(t) \mathbf{z}_i(t)^\dagger] + \sigma^2 I
\]

are covariance matrices of the desired and undesired input vectors defined in Eq. (15). These covariance matrices can be computed for general input signals using Eq. (14a) and the second-order statistics defined in Eq. (9). For the case of narrowband input signals, the output powers required in Eq. (39) are given by the expressions in Eq. (37) and Eq. (38). In either case, the mean weight vector must be determined before the output signal-to-noise ratio can be calculated. The solution of Eq. (32) for the mean weight vector is examined in the next chapter under the assumption of a narrowband signal environment.
CHAPTER III
THEORETICAL PERFORMANCE WITH AN IDEAL REFERENCE SIGNAL

In this chapter array performance is investigated assuming narrowband input signals and an ideal reference signal equal to the desired input signal. The solution of the vector differential Eq. (32) for the mean weight vector response is expressed in terms of eigenvalues and eigenvectors (or projection operators) of the covariance matrix $K_x$ in Eq. (19). Expressions for the transients in the array output powers are then developed for an environment consisting of a desired signal and a single interfering signal. The output power ratios under steady-state conditions are calculated in Section B where specific numerical results are presented for the case of a four-element linear array. The differential Eq. (32) is modified in Section C to include d.c. offsets in the feedback-control loops; the performance degradation resulting from these offsets is determined. In Section D, array performance under a potentially worst-case type of interference - periodic pulse interference - is examined.

A. Array Transient Response

The differential Eq. (32) may be written in the form

$$\frac{d w(t)}{dt} + \alpha K_x w(t) = \alpha r_x(t); \quad w(t_0) = w_0 \tag{41}$$

where the symbol notations for vectors, complex envelopes, and expected values have been deleted for convenience and where

$$r_x(t) = E[x(t)^n r(t)^n] \tag{42}$$

and

$$K_x = E[x(t)^x(t)^n]$$

are the input-reference cross-correlation vector and input vector covariance matrix, respectively. For the initial part of the following analysis, only the assumption of a stationary covariance matrix is required. The unforced response ($r_x=0$) of Eq. (41) is well-known from linear, time-invariant system theory[22] to be given by the exponential matrix

$$w_u(t) = \exp[-\alpha K_x (t-t_0)]w_0 \tag{43}$$
The forced response is found by multiplying Eq. (41) by an integrating factor, \( \exp[+\alpha K_x(t-t_0)] \), and integrating from time \( t_0 \) to \( t \):

\[
\exp[+\alpha K_x(t-t_0)]w_f(t) = \int_{t_0}^{t} \alpha \exp[+\alpha K_x(t-t)] r_x(\tau) d\tau,
\]

(44) \( w_f(t) = \int_{t_0}^{t} \alpha \exp[-\alpha K_x(t-\tau)] r_x(\tau) d\tau. \)

The total solution for \( w(t) \) is the sum of Eq. (43) and Eq. (44), i.e.,

(45) \( w(t) = \exp[-\alpha K_x(t-t_0)]w_0 + \int_{t_0}^{t} \alpha \exp[-\alpha K_x(t-\tau)] r_x(\tau) d\tau. \)

This result will now be expanded into a form suitable for numerical calculation. The positive-definite, Hermitian, covariance matrix will be represented by the orthonormal series expansion

(46) \( K_x = \sum_{i=1}^{\ell} \lambda_i \left( \sum_{j=1}^{n_i} \hat{e}_{ij} \hat{e}_{ij}^\dagger \right) \equiv \sum_{i=1}^{\ell} \lambda_i E_i \)

where \( \{\hat{e}_{ij}, j=1,2,\ldots,n_i\} \) is the set of orthonormal eigenvectors of \( K_x \) associated with the distinct eigenvalue \( \lambda_i \) of multiplicity \( n_i \).

The orthonormality of the eigenvectors is expressed by the relation

(47) \( \hat{e}_{ij}^\dagger \hat{e}_{kl} = \langle \hat{e}_{ij}^\dagger \hat{e}_{kl} \rangle = \begin{cases} 1, & i=j=k=l \\ 0, & \text{otherwise} \end{cases} \)

The sum over \( j \) in Eq. (46) of the orthonormal matrices associated with \( \lambda_i \) is defined as the \( i \)th "projection operator" \( E_i \) of the matrix \( K_x \). In linear space terminology, the \( m \)-dimensional complex Euclidean space \( H \) which contains the input vector \( x(t) \) and the weight vector \( w(t) \) is represented by a direct sum of orthogonal eigenspaces

(48) \( H = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_{\ell} \)

\[ \text{This expansion follows from the spectral decomposition theorem for normal operators (matrices)[23]. To ensure positive-definiteness of } K_x \text{ the thermal noise variances, } \sigma^2, \text{ will be assumed non-zero.} \]
where the \(i^{th}\) eigenspace \(Q_i\) is of dimension \(n_i\) and has the eigenvector basis \(\{e_{ij}, j=1,2,\ldots, n_i\}\). The projection operator \(E_i\) acts on any vector \(V\) in \(H\) and produces that component of \(V\) in \(Q_i\):

\[
E_i V = \sum_{j=1}^{n_i} \hat{e}_{ij} \hat{e}_{ij}^\dagger V = \sum_{j=1}^{n_i} \langle \hat{e}_{ij}, V \rangle \hat{e}_{ij}.
\]

The orthonormality properties of the projection operators,

\[
E_i E_j = \begin{cases} E_i & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}
\]

follow from the orthonormality of the eigenvectors in Eq. (47). The matrix inverse to \(K_X\) may also be represented in terms of projection operators by

\[
K_X^{-1} = \sum_{i=1}^{d} \frac{1}{\lambda_i} E_i
\]

as verified by multiplying Eq. (46) and Eq. (52) then applying Eq. (50) and Eq. (51). The exponential matrix

\[
\exp[-\alpha K_X t] = I - \alpha K_X t + (\alpha K_X)^2 \frac{t^2}{2!} - \cdots + (-1)^n (\alpha K_X)^n \frac{t^n}{n!} + \cdots
\]

has a projection operator representation found by substituting Eq. (51) for \(I\) in the first term of the series, and Eq. (46) for \(K_X\) in the remaining terms, then applying Eqs. (50) and (51). The result is

\[
\exp[-\alpha K_X t] = \sum_{i=1}^{d} \left[1 - \alpha \lambda_i t + \frac{(\alpha \lambda_i t)^2}{2!} - \cdots\right] E_i
\]

\[
= \sum_{i=1}^{d} \exp[-\alpha \lambda_i t] E_i.
\]
This expression may be used for the exponential matrix in Eq. (45) after substituting \((t-x)\) for the variable \(t\). Under the assumption that the cross-correlation vector \(r_x(t)\) defined in Eq. (42) is stationary, i.e., independent of \(t\), the weight vector response in Eq. (45) may be integrated yielding

\[
(55) \quad w(t) = \sum_{i=1}^{k} \left\{ \exp[-\alpha_{i}(t-t_0)](E_i w_0 - \frac{1}{\lambda_i} E_i r_x) + \frac{1}{\lambda_i} E_i r_x \right\},
\]

\(t \geq t_0\).

This general result is applicable whenever the covariance matrix \(K_x\) and the cross-correlation vector \(r_x\) are stationary. Further evaluation of Eq. (55) requires a determination of the eigenvalues and eigenvectors of \(K_x\) and specific assumptions regarding the ideal nature of the reference signal.

For narrowband, uncorrelated signals, the eigenvalues and eigenvectors of \(K_x\) satisfy the eigenvector equation

\[
(56) \quad (K_x - \lambda I)e = \left[ \sum_{i=1}^{p} N_i V_i V_i^\dagger + (\sigma^2 - \lambda)I \right] e = 0.
\]

A method for solving this equation which is applicable when \(p \leq m\) and when the direction-delay vectors \(V_i\) are linearly independent is presented in Appendix A. For the case of two signals \((p=2)\) the eigenvalues and (unnormalized) eigenvectors are shown to be

\[
\lambda_{1,2} = \sigma^2 + \frac{m^2}{2}(S+J) \pm \left[ \frac{m^2}{4} (S-J)^2 + 4SJ <V_1, V_2>^2 \right]^{1/2}
\]

\[
(57) \quad \lambda_k = \sigma^2, \quad k > 2
\]

\[
\begin{align*}
\epsilon_{1,2} &= \left\{ \left[ m(S-J) \pm \left[ m^2(2-J)^2 + 4SJ <V_1, V_2>^2 \right]^{1/2} \right] \right\} v_1 + v_2 \\
\epsilon_k &= \perp \epsilon_{1,2}, \quad k > 2
\end{align*}
\]

where \(S = N_1, \quad J = N_2, \quad \text{and} \quad <V_1, V_2> \neq 0\).

The inner product of the vectors \(V_1\) and \(V_2\), which were defined earlier in Eq. (17), may be expressed in terms of the differential time delays, Eq. (7), as
<V_1, V_2> = \sum_{k=1}^{m} e^{j\omega_c(\tau_1k-\tau_2k)}.

This inner product is real-valued when the coordinate origin in Fig. 1 is at the array phase center. Its magnitude is related to the angular separation of desired signal and interference as will be noted later in Eq. (82). For certain preferred angular separations, the vectors \(V_1\) and \(V_2\) are orthogonal and the inner product is zero. In this case Eq. (57) reduces to the simpler form

\[
\begin{align*}
\lambda_1 &= \sigma^2 + mj \\
\lambda_2 &= \sigma^2 + ms \\
\lambda_k &= \sigma^2 \\
e_1 &= V_2 \\
e_2 &= V_1 \\
e_k &\perp e_1, e_2 \quad ; k > 2
\end{align*}
\]

\[(58) \quad \langle V_1, V_2 \rangle = 0.
\]

The projections of any vector \(V\) in the space \(H\) onto the eigenspaces corresponding to Eq. (57) or Eq. (58) are given by

\[
E_i V = \check{e}_i, V \check{e}_i = \frac{\langle e_i, V \rangle}{|e_i|^2} e_i \quad , \quad i=1,2
\]

\[
E_3 V = (I - E_1 - E_2) V = V - \frac{\langle e_1, V \rangle}{|e_1|^2} e_1 - \frac{\langle e_2, V \rangle}{|e_2|^2} e_2 .
\]

In the latter result, the projection operator \(E_3\) associated with the thermal noise eigenspace (of dimension \(m-2\)) is determined using Eq. (51) with \(\xi = p + 1 = 3\). When the reference signal is an amplitude-scaled replica of the desired signal at the array phase center, i.e., when

\[\text{In this case, the interference direction-of-arrival coincides with the direction of a null in the array pattern when the array is cophased to the desired input signal.}\]
\[ \vec{r}(t) = \left[ \frac{\vec{r}}{S} \right] \vec{r}_1(t) \]

\[ E[\vec{r}(t) \vec{r}(t)^\dagger] = E[|\vec{r}(t)|^2] = R \]

\[ r_x = E[\vec{x}(t) \vec{r}(t)^\dagger] = \sqrt{RS} V_1 , \]

the weight vector transient response, Eq. (55), may be evaluated as

\[ w(t) = \left[ \frac{<e_1,w_0>}{|e_1|^2} - \frac{\sqrt{RS} <e_1,V_1>}{\lambda_1 |e_1|^2} \right] e^{-\alpha \lambda_1 (t-t_0)} e_1 \]

\[ + \left[ \frac{<e_2,w_0>}{|e_2|^2} - \frac{\sqrt{RS} <e_2,V_1>}{\lambda_2 |e_2|^2} \right] e^{-\alpha \lambda_2 (t-t_0)} e_2 \]

\[ - \alpha \sigma^2 (t-t_0) \]

\[ + [E_3w_0] e \]

\[ + \frac{\sqrt{RS} <e_1,V_1>}{\lambda_1 |e_1|^2} e_1 + \frac{\sqrt{RS} <e_2,V_1>}{\lambda_2 |e_2|^2} e_2 \]

The fact that the noise and signal eigenspaces are orthogonal,

\[ E_3 e_1 = E_3 e_2 = E_3 V_1 = E_3 V_2 = 0, \]

has been used in Eq. (55) to eliminate terms involving \( E_3 r_x \). The transients in the expected powers of the array output signal and noise components are obtained by using Eqs. (61), (37), and (38):

\[ P_j(t) = J|w(t),V_2|^2 = J|ae^{-\alpha \lambda_1 (t-t_0)} + be^{-\alpha \lambda_2 (t-t_0)} + c|^2 \]
where \( a = \left[ \frac{e_1^* w_0 - \sqrt{RS} e_1^* V_1}{|e_1|^2} \right] e_1^* V_2 \)

\( b = \left[ \frac{e_2^* w_0 - \sqrt{RS} e_2^* V_1}{|e_2|^2} \right] e_2^* V_2 \)

\( c = \frac{\sqrt{RS}}{\lambda_1} \frac{e_1^* V_1 \cdot e_1^* V_2}{|e_1|^2} + \frac{\sqrt{RS}}{\lambda_2} \frac{e_2^* V_1 \cdot e_2^* V_2}{|e_2|^2} \)

(64) \[ P_s(t) = S |w(t), V_1|^2 = S \left| de^{-\alpha \lambda_1 (t-t_0)} + fe^{-\alpha \lambda_2 (t-t_0)} + g \right|^2 \]

where \( d = \left[ \frac{e_1^* w_0 - \sqrt{RS} e_1^* V_1}{|e_1|^2} \right] e_1^* V_1 \)

\( f = \left[ \frac{e_2^* w_0 - \sqrt{RS} e_2^* V_1}{|e_2|^2} \right] e_2^* V_1 \)

\( g = \frac{\sqrt{RS}}{\lambda_1} \frac{e_1^* V_1^2}{|e_1|^2} + \frac{\sqrt{RS}}{\lambda_2} \frac{e_2^* V_1^2}{|e_2|^2} \)

(65) \[ P_n(t) = \sigma^2 |w(t)|^2 = \]

\[ \sigma^2 \left( \left| \frac{e_1^* w_0 - \sqrt{RS} e_1^* V_1}{|e_1|^2} \right| e^{-\alpha \lambda_1 (t-t_0)} + \frac{\sqrt{RS}}{\lambda_1} \frac{e_1^* V_1}{|e_1|^2} \right)^2 |e_1|^2 \]

\[ + \sigma^2 \left( \left| \frac{e_2^* w_0 - \sqrt{RS} e_2^* V_1}{|e_2|^2} \right| e^{-\alpha \lambda_2 (t-t_0)} + \frac{\sqrt{RS}}{\lambda_2} \frac{e_2^* V_1}{|e_2|^2} \right)^2 |e_2|^2 \]

\[ + \sigma^2 |E_3 w_0 e^{-\alpha \sigma^2 (t-t_0)}|^2 \]
Note that the weight vector in Eq. (61) and output thermal noise power in Eq. (65) contain the exponential term $\exp[-\omega^2(t-t_0)]$ associated with the array response to thermal noise. The output signal powers in Eqs. (63) and (64) do not contain this term as a result of the orthogonality property, Eq. (62). The implication of this result is that the output signals approach steady-state values much sooner than the output thermal noise when the input signals are large, $S,J \gg \sigma^2$.

Any performance index involving the output thermal noise power, however, will converge at the slower rate.

B. Array Steady-state Performance

When the covariance matrix $K_X$ and cross-correlation vector $r_X$ are stationary, the mean weight vector asymptotically approaches the steady-state value

$$w(t)|_{t=\infty} = K_X^{-1}r_X = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} E_i r_X$$

as determined by setting the derivative of the mean weight vector equal to zero in Eq. (41) or taking the limit as $t \to \infty$ in Eq. (55). The projection operator approach of the preceding section may be used to evaluate Eq. (66), or the inverse matrix may be expanded directly using an identity attributed to Woodbury[24]:

$$(A+URV^\dagger)^{-1} = A^{-1} - A^{-1}U(R^{-1}+V^\dagger A^{-1}U)^{-1}V^\dagger A^{-1}$$

where

$A$ = nonsingular $m \times m$ matrix
$U,V$ = $m \times r$ matrices ($r \leq m$)
$R$ = nonsingular $r \times r$ matrix.

The latter approach was suggested by Baird, et al[7]. Application of the Woodbury identity for the case of a narrowband desired signal yields

$$K_X^{-1} = (K_s + K_n)^{-1} = (K_n + V_1 S V_1^\dagger)^{-1}$$

$$= K_n^{-1} - K_n^{-1} V_1 \left( \frac{1}{S} + V_1^\dagger K_n^{-1} V_1 \right)^{-1} V_1^\dagger K_n^{-1}$$

$$= K_n^{-1} - \frac{S K_n^{-1} V_1 V_1^\dagger K_n^{-1}}{(1 + S V_1^\dagger K_n^{-1} V_1)}.$$
When the inverse matrix in Eq. (68) multiplies the specific vector \( V_1 \), as in the case of the ideal cross-correlation vector, Eq. (60), the final weight vector is given by

\[
(69) \quad w(t)\bigg|_{t=\infty} = \sqrt{RS} \left( \frac{1}{1 + SV_1^\dagger K_n^{-1}V_1} \right)^{-1} K_n^{-1}V_1
\]

The Woodbury identity may be applied repeatedly to expand the inverse of the noise covariance matrix \( K_n \) when the interfering signals are also narrowband.\(^3\) For the case of a single uncorrelated interfering signal, the inverse is obtained after one application of Eq. (67);

\[
(70) \quad K_n^{-1} = (\sigma^2 I + J V_2 V_2^\dagger)^{-1} = \sigma^{-2} I - \frac{J \sigma^{-4} V_2 V_2^\dagger}{1 + m J \sigma^{-2}}.
\]

The mean weight vector in this case has the asymptotic value

\[
(71) \quad w(t)\bigg|_{t=\infty} = g \left[ V_1 - \frac{m J \sigma^{-2}}{1 + m J \sigma^{-2}} \left\langle V_1, V_2 \right\rangle \frac{V_2}{m} \right]
\]

where the scalar (gain) multiplier \( g \) is given by

\[
g = \frac{\sqrt{RS}}{\sigma^2} \left( \frac{1}{1 + SV_1^\dagger K_n^{-1}V_1} \right)
\]

with

\(^3\)Correlated narrowband signals may also be considered since the vectors \( U \) and \( V \) in Eq. (68) need not be identical. (See Eq. (19)).
The relative magnitudes of the weight vector components along the vectors \( V_1 \) and \( V_2 \) determine array output performance. Consider, for example, the output signal-to-thermal noise power ratio which, from Eq. (37) and Eq. (38), is given by

\[
P_{\text{s}(t)} = \frac{\left| w(t), V_1 \right|^2}{\sigma^2 |w(t)|^2} = \frac{mS}{\sigma^2} \left| \hat{w}(t), \hat{V}_1 \right|^2.
\]

This ratio is maximum when the unit weight vector, \( \hat{w}(t) \), is equal to the unit vector \( V_1 \), i.e., when the \( V_2 \)-component of the weight vector is zero. In this case the array is co-phased to the desired input signal. This condition occurs in Eq. (71) when the power in the interfering signal is very small or when the angular separation of sources is such that \( V_1 \) and \( V_2 \) are orthogonal:

\[
\left(73\right) \quad w(t) \bigg|_{t \to \infty} = g V_1 = \frac{\sqrt{R S \sigma^2}}{1 + mS \sigma^2} V_1,
\]

\[
\left. \frac{P_{\text{s}(t)}}{P_{\text{nt}(t)}} \right|_{t \to \infty} = \frac{mS}{\sigma^2} ; \quad J \ll \sigma^2 \quad \text{or} \quad \left<V_1, V_2\right> = 0.
\]

For non-orthogonal angular separations, the weight vector in Eq. (71) has a non-zero component along \( V_2 \) whose magnitude is dependent on input interference power. This \( V_2 \)-component changes the direction of the weight vector from the direction of \( V_1 \) and, as a result, the output ratio, Eq. (72), decreases from its maximum value. In the general case where the vectors \( V_1 \) and \( V_2 \) are neither identical or orthogonal, the loss in the output ratio, Eq. (72), is maximum when the \( V_2 \)-component of the weight vector, Eq. (71), is maximum. This condition occurs when the power in the interfering signal is very large compared to element noise power (\( J \gg \sigma^2 \)). For this case, the asymptotic, mean weight vector in Eq. (71) approaches the value

\[
\left(74\right) \quad w(t) \bigg|_{t \to \infty} = g \left[ V_1 - \frac{<V_1, V_2>}{m} V_2 \right] , \quad J \gg \sigma^2
\]

\[
\begin{align*}
V_1 & \neq V_2 \\
<V_1, V_2> & \neq 0.
\end{align*}
\]
This weight vector is orthogonal to the interference direction-delay vector \( V_2 \); hence, the interference component in the array output is nulled in steady-state. The latter result is clearly demonstrated from the expression for the output interference power in steady-state,

\[
P_{J}(\infty) = J|\langle w(\infty), V_2 \rangle|^2 = (\text{input power})(\text{power pattern gain})
\]

\[
= J \sigma^2 g(K_n^{-1}V_1)^{+} V_2 |^2 = J \sigma^2 g^2 |V_1^{-1} V_2 |^2
\]

\[
= J g^2 \left[ 1 - \frac{mJ \sigma^{-2}}{1 + mJ \sigma^{-2}} \right]^2 <V_1, V_2>^2
\]

\[
= J \left[ \frac{1}{1 + mJ \sigma^{-2}} \right]^2 g^2 <V_1, V_2>^2 \approx \frac{1}{J} \text{ as } J \to \infty.
\]

The larger the interfering signal is at the array input, the more it is suppressed at the array output: the so-called "reciprocal suppression" effect noted elsewhere\[7\]. Expressions similar to Eq. (75) for the powers in the desired signal and thermal noise components at the array output are readily calculated as follows:

\[
P_{S}(\infty) = S |\langle w(\infty), V_1 \rangle|^2 = S \sigma^2 g^2 |V_1^{-1} V_1 |^2
\]

\[
= S g^2 m^2 \left[ 1 - \left( \frac{mJ \sigma^{-2}}{1 + mJ \sigma^{-2}} \right) \frac{<V_1, V_2>^2}{m^2} \right]^2
\]

\[
= S g^2 m^2 \left[ \frac{1}{1 + mJ \sigma^{-2}} \right]^2 \left[ 1 + mJ \sigma^{-2} \right]^2 \left( 1 - \frac{<V_1, V_2>^2}{m^2} \right)^2
\]
The total noise power at the array output in steady-state is given by the sum of Eqs. (75) and (77), or more simply from Eq. (40) by

\[
(P_n(t)) = (w(\omega), K_n w(\omega)) = g^2 \sigma^2 \left[ \frac{1}{m^2} \left( \frac{V_1^2}{1+m\sigma^{-2}} \right) - \frac{1}{m^2} \right]
\]

The relative magnitudes of the array output components are determined by taking appropriate ratios of Eqs. (75) to (78), i.e.,

\[
\frac{P_S(\omega)}{P_n(\omega)} = \frac{mS}{\sigma^2} \left[ 1 - \frac{1}{m^2} \left( \frac{V_1^2}{1+m\sigma^{-2}} \right) \right]
\]

\[
\frac{P_S(t)}{P_n(t)} = \frac{mS}{\sigma^2} \left[ \frac{1}{m^2} \left( 1 - \frac{V_1^2}{1+m\sigma^{-2}} \right) \right]^2
\]

and
The latter two power ratios are plotted in Figs. 5 and 6, respectively, for the case of a four-element linear array. For equispaced linear arrays, the inner product of $V_1$ and $V_2$ corresponds to the spatial pattern factor of the uniformly-illuminated array:

\begin{equation}
\langle V_1, V_2 \rangle = \frac{\sin[m \frac{\rho}{\lambda_c} (\cos \theta_S - \cos \theta_J)]}{\sin[\pi \frac{\rho}{\lambda_c} (\cos \theta_S - \cos \theta_J)]} \frac{\sin[\frac{m \psi}{2}]}{\sin[\frac{\psi}{2}]} ; \quad \psi = 2\pi \frac{\rho}{\lambda_c} |\cos \theta_S - \cos \theta_J|,
\end{equation}

where $\frac{\rho}{\lambda_c} = \text{element separation in center-frequency wavelengths}$

$\theta_S, \theta_J = \text{arrival angles of the desired and interfering signals (as measured from the array axis)}$

$\psi = \text{difference in element-to-element phase shift of desired signal and interference.}$

As the angular separation of sources approaches zero, maximum improvement in the desired signal-to-thermal noise ratio is obtained as indicated in Fig. 5; however, the interference-to-desired signal ratio at the output of the array approaches the input ratio $J/S$, i.e., no improvement. This condition of zero angular separation is indicated by the dotted line in Fig. 6. The other curves in this figure "bend over" at input interference power levels which are sufficiently large to initiate a response from the array processor. That is, for smaller interference powers, there is no response by the array (except for co-phasing on the desired signal); for larger interference powers the array responds strongly to null the output interference.

The performance illustrated in Figs. 5 and 6 represents a compromise between desired signal-to-thermal noise ratio improvement and output interference rejection. This compromise is such that the squared-error signal is minimized in steady-state.
Fig. 5.--The improvement in signal-to-thermal noise ratio versus the separation parameter $\psi$ for several input interference-to-thermal noise ratios.
Fig. 6.—The normalized output interference-to-signal ratio versus the input interference-to-thermal noise ratio for several values of the separation parameter \( \psi \).
C. The Effect of Control-loop Offsets

The array performance discussed in the preceding section is based on a model which assumes ideal components in the feedback control loops. In particular, the input $x$ error multiplier in each loop is assumed to have zero output when the signal at either input to the multiplier, or at both inputs, is zero. In practice, however, neither of these conditions are satisfied as a result of imperfect balancing in the active devices and circuits used to implement the multipliers. Small d.c. offset voltages are present at the error multiplier outputs which prevent the ideal performance from being realized. Moreover, the loop integrators are not ideal in practice and may be represented as having small d.c. offsets at their inputs, i.e., at the outputs of the error multipliers. In the analysis to follow, the effect of these offset voltages on the steady-state weight vector and on the relative powers in the array output components will be examined. From the results of the analysis, practical methods for minimizing the effect of offsets will be evident. The set of offset voltages at the outputs of the error multipliers will be denoted by the complex vector $\mathbf{R}_v$ as shown in Fig. 7. In this model the loop gain constant $\alpha$ has been separated into factors identifiable with the parameters of the physical processor. In particular,

\begin{equation}
\alpha = G_r G_{i} G_{m} G_{d} G_{s} G_{w}, \ [\text{rms-volts}^2 \text{sec}]^{-1}
\end{equation}

where

- $G_r = \text{voltage gain of rf amplifiers preceding processor inputs}$
- $G_i = (RC)^{-1} = \text{integrator gain constant, (sec)}^{-1}$
- $G_m = \text{error multiplier gain constant, (dc-volts)}$
- $G_d = \text{net voltage gain in difference (error) path}$
- $G_s = \text{net voltage gain of summing circuit}$
- $G_w = \text{weight multiplier gain constant, (dc-volts)}^{-1}$.

The vector differential equation describing this model is given by

\begin{equation}
\frac{d\mathbf{w}_v(t)}{dt} = G_i \left\{ G_m G_r \mathbf{x}(t) G_d \mathbf{r}_a(t)^+ - G_s G_r \mathbf{x}(t)^+ G_{w_v}(t) \right\} + \mathbf{\Omega}_v.
\end{equation}
Fig. 7.—Expanded adaptive processor model with control-loop d.c. offsets.

where

\( \mathbf{w}_v(t) \) = complex vector of integrator output voltages, dc-volts

\( \Omega_v(t) \) = complex vector of error multiplier offset voltages, dc-volts

\( \tilde{r}_a(t) \) = complex envelope of actual reference signal, rms-volts

\[ E[\tilde{r}_a(t)\tilde{r}_a(t)^+] = R_a, \text{ (rms-volts)}^2. \]
When Eq. (84) is multiplied by $G_{rfGcGw}$ and the gain factors comprising $\alpha$ are collected on the right-hand side of the equation, the result is

$$
\frac{dw_v(t)}{dt} = \alpha \left\{ \hat{x}(t) \left[ \hat{r}_a(t) - \hat{x}(t)^\dagger G_{rfGcGw} w_v(t) \right] + \frac{\Omega_v}{G_{rfGcGd}} \right\}.
$$

With the definitions

$$
G_{rfGcGw} w_v(t) = w(t)
$$

$$
\frac{\Omega_v}{G_{rfGcGd}} \equiv \Omega_v,
$$

$$
\hat{r}_a(t) \equiv \hat{r}(t), \text{ and } R_a \equiv R,
$$

the expected value of Eq. (85) reduces to

$$
\frac{dw(t)}{dt} + \alpha K x w(t) = \alpha (w(t) + \Omega).
$$

It is apparent from this result that the analysis model in Fig. 4 is equivalent to the "physical" model in Fig. 7 if a normalized offset vector $\Omega$ is added to the error multiplier outputs in Fig. 4.

The offset vector $\Omega$ in Eq. (87) changes or perturbs the cross-correlation vector $r_x$ from its ideal value, Eq. (60). As a result, the direction-delay information ($V_1$) about the desired signal is affected. The effect of this perturbation might be expected to be small when the perturbation itself is small, i.e.,, when

$$
|\Omega| << |r_x|
$$

$$
|\Omega_v| \ll \sqrt{RS} |V_1|
$$

which is

$$
\frac{|\Omega_v|}{G_{rfGcGd}} \ll \sqrt{S} \frac{G_m G_d}{G_{rf} \sqrt{R}}
$$

or

$$
(88) \quad \frac{|\Omega_v|}{\sqrt{m} G_{rf} \sqrt{S} G_m G_d \sqrt{R}} \ll 1.
$$

This will be the case, however, only if the steady-state weight vector,
is also near its zero-offset value (the first term), i.e.,

\[
|K_x^{-1}\Omega| \ll |K_x^{-1}r_x| = \sqrt{RS} |K_x^{-1}V_1|
\]

The question of interest, therefore, is whether or not Eq. (88) implies Eq. (90). An approach in answering this question is to evaluate Eq. (89) for the analytically tractable case of a narrowband signal environment. The weight vector in Eq. (89) is first expanded using the results, Eq. (60), and Eq. (68), for a narrowband desired signal,

\[
w(\infty) = K_x^{-1}(r_x + \Omega) = K_x^{-1}r_x + K_x^{-1}\Omega
\]

where \(Q = V_1^+K_n^{-1}\Omega\) is a bi-linear form. Then, under the assumption that a single, narrowband interfering signal is present, the inverse matrix, Eq. (70), is applied to further expand Eq. (91):

\[
w(\infty) = \frac{\sqrt{RS}-SQ}{1+S V_1^+K_n^{-1}V_1} K_n^{-1}V_1 + \left[ K_n^{-1}\Omega - \frac{S K_n^{-1}V_1V_1^+K_n^{-1}\Omega}{1+S V_1^+K_n^{-1}V_1} \right]
\]

where \(Q = V_1^+K_n^{-1}\Omega\) is a bi-linear form. Then, under the assumption that a single, narrowband interfering signal is present, the inverse matrix, Eq. (70), is applied to further expand Eq. (91):

\[
w(\infty) = \frac{(\sqrt{RS}-SQ)\sigma^{-2}}{1+S V_1^+K_n^{-1}V_1} V_1
\]

\[
- \frac{J\sigma^{-2}}{1+m\sigma^{-2}} \left[ \frac{(\sqrt{RS}-SQ)\sigma^{-2} V_1 V_2}{1+S V_1^+K_n^{-1}V_1} + <V_1,\sigma^{-2}\Omega> V_2 
\]

The four offset-induced terms in Eq. (92) are those containing the vector \(\Omega\) or the scalar \(Q\). The two terms containing \(Q\) will have negligible effect when the condition
is satisfied. The magnitude of $Q$ is bounded by

$$|Q| = \sqrt{m} \sigma^{-2} |\Omega| \cdot \left| \hat{V}_1 \hat{\sigma} \right| - \frac{m \sigma^{-2}}{1 + m \sigma^{-2}} \left| \hat{V}_1 \hat{V}_2 <\hat{V}_2, \hat{\sigma}\right|$$

\[ \leq 2 \sqrt{m} \sigma^{-2} |\Omega| \]

since the inner products of unit vectors in this expression have magnitudes bounded by unity. Therefore, the above condition on the magnitude of $Q$ may be replaced by the condition

$$2 \sqrt{m} \sigma^{-2} |\Omega| \leq \frac{R}{S} \cdot$$

If the remaining two offset terms in Eq. (92) can also be shown to be negligible under Eq. (94), then Eq. (94) is a sufficient condition for the inequality, Eq. (90), to hold. It must be shown that the inequalities

$$|\hat{V}_1 \hat{\sigma} \cdot v| \leq \frac{\sqrt{R S \sigma^{-2}}}{1 + S V_1^T K_n^{-1} V_1} \left( V_1 - \frac{m \sigma^{-2}}{1 + m \sigma^{-2}} \left| \hat{V}_1 \hat{V}_2 <\hat{V}_2, \hat{\sigma}\right| \right)$$

are satisfied under Eq. (94). The lower term on the left-hand side need not be considered since it is less than or equal to the upper term on the same side. By manipulating the right-hand side, this inequality may be written as

$$|\hat{V}_1 \hat{\sigma} \cdot v| \leq \frac{1}{\sqrt{m}} \sqrt{R S \sigma^{-2}} \left( \frac{m S \sigma^{-2}}{1 + S V_1^T K_n^{-1} V_1} \right) \left| \hat{V}_1 - \frac{m \sigma^{-2}}{1 + m \sigma^{-2}} \left| \hat{V}_1 \hat{V}_2 <\hat{V}_2, \hat{\sigma}\right| \right|$$

or as
This expression is the same as Eq. (94) except for the two additional factors on the right-hand side. The magnitude of these factors will be on the order of unity except for the extreme conditions of very small signal-to-thermal noise ratio or very small angular separation of a large interfering signal. Consider, for example, the case of a four-element, $\lambda/2$-spaced linear array where the angular separation is less than a half-beamwidth:

$$m = 4 \quad \theta_S = 90^\circ \quad \theta_J = 80^\circ$$

$$\psi = 32.8^\circ \quad <\hat{v}_1, \hat{v}_2> = 0.806.$$

For input ratios $S/\sigma^2 = -12$ dB and $J/\sigma^2 = 30$ dB, the above factors equal 0.85 and 0.62 respectively. Thus, Eq. (94) is a sufficient condition for small weight vector deviation except under very adverse input conditions where the inequality in Eq. (95) must be satisfied as well. Using Eq. (86), the condition in Eq. (94) may be re-written in the form

$$2 \sqrt{m} \sigma^{-2} |\Omega| < \frac{R}{S} \left( \frac{2m\sigma^{-2}}{1+m\sigma^{-2}} \left( 1 - \frac{mJ\sigma^{-2}}{1+mJ\sigma^{-2}} \right) <\hat{v}_1, \hat{v}_2> \right) \frac{\hat{v}_1 - \frac{mJ\sigma^{-2}}{1+mJ\sigma^{-2}} <\hat{v}_1, \hat{v}_2> \hat{v}_2}{}$$

which is the same as Eq. (88) except for the additional factor proportional to the desired signal-to-thermal noise ratio. The loop gain factors in Eq. (96) must be increased as the input signal-to-noise ratio is increased in order to maintain small deviation in the weight vector.

The output power in the desired signal component may be determined from Eq. (76) and Eq. (91) as
The offsets have a negligible effect on output signal power when

\[ |\Omega| \leq 2\sqrt{\eta_0} |\omega| < \frac{R}{S} (S V_1^{+} K_n^{-1} V_1) \]

Since the quadratic form on the right-hand side of Eq. (98) is of the same order of magnitude as the maximum signal-to-thermal noise ratio at the array output, this condition is essentially the same as

\[ 2\sqrt{m} \sigma^{-2} |\omega| < \frac{R}{S} (mS\sigma^{-2}) \]

which is

\[ \frac{2 |\omega|}{\sqrt{m} G_{rf} \sqrt{S} G_d} \frac{\sqrt{R}}{\sqrt{S}} \ll 1. \]

Note that this condition does not contain the signal-to-thermal noise ratio factor as in Eq. (96).

The effect of offsets on the total noise power at the array output may be determined using Eqs. (78) and (91):

\[ P_s(\omega) = S |\Omega| \cdot w(\omega), V_1^{+} |^2 \]

\[ = S \left( \frac{\sqrt{RS}}{1+S V_1^{+} K_n^{-1} V_1} + \Omega^{+} K_n^{-1} V_1 - \frac{(\Omega^{+} K_n^{-1} V_1) S V_1^{+} K_n^{-1} V_1}{1+S V_1^{+} K_n^{-1} V_1} \right)^2 \]

\[ = \frac{S}{(1+S V_1^{+} K_n^{-1} V_1)^2} \left( \frac{\sqrt{RS}}{1+S V_1^{+} K_n^{-1} V_1} + Q^{+} \right)^2 \]

\[ = \frac{RS}{(1+S V_1^{+} K_n^{-1} V_1)^2} S (V_1^{+} K_n^{-1} V_1)^2 \left( 1 + \frac{Q^{+}}{\sqrt{RS}} \right)^2 \]
\[ P_n(\infty) = w(\infty)^\dagger K_n w(\infty) \]
\[ = \left[ \frac{\sqrt{RS}}{\beta} V_1^\dagger K_n^{-1} + \Omega^\dagger K_n^{-1} - \frac{Q^\dagger SV_1^\dagger K_n^{-1}}{\beta} \right] \]
\[ \cdot \left[ \frac{\sqrt{RS}}{\beta} V_1 + \Omega - \frac{Q SV_1}{\beta} \right] \]

where \( \beta \equiv 1 + SV_1^\dagger K_n^{-1} V_1 \).

This expression may be evaluated in a straightforward manner and put into the form

\[ P_n(\infty) = \frac{RS V_1^\dagger K_n^{-1} V_1}{\beta^2} \left| 1 - \frac{\sqrt{S}}{\sqrt{R}} Q \right|^2 \]
\[ + \frac{2R}{\beta} \text{Re} \left[ \frac{\sqrt{S}}{\sqrt{R}} Q^\dagger (1 - \frac{\sqrt{S}}{\sqrt{R}} Q) \right] + \Omega^\dagger K_n^{-1} \Omega. \]  

It is evident that the condition, Eq. (94), or equivalently, Eq. (96), must again be imposed to maintain small offset effects in the first two terms. The third term, a quadratic form, must also be small compared to the scalar multiplier of the first term, i.e.,

\[ \Omega^\dagger K_n^{-1} \Omega \ll R \left[ \frac{S V_1^\dagger K_n^{-1} V_1}{(1+S V_1^\dagger K_n^{-1} V_1)^2} \right]. \]  

Using Eq. (70), the left-hand side of Eq. (101) may be evaluated as

\[ \Omega^\dagger K_n^{-1} \Omega = \left| \frac{\Omega}{\sigma} \right|^2 - \frac{mJ_\sigma - 4}{1 + mJ_\sigma - 2} \left| \langle V_2, \hat{\Omega} \rangle \right|^2 \]
\[ = \sigma^2 \left| \sigma^{-2} \Omega \right|^2 \left| 1 - \frac{mJ_\sigma - 2}{1 + mJ_\sigma - 2} \right| \left| \langle \hat{V}_2, \hat{\Omega} \rangle \right|^2 \]
\[ \leq 2\sigma^{-2} \left| \Omega \right|^2. \]
The right-hand side of Eq. (101) is of the same order of magnitude as
\[
R \frac{m \sigma^{-2}}{(1+m \sigma^{-2})^2}
\]
as noted previously in Eq. (95). Therefore, it follows that Eq. (101)
is nearly equivalent to the condition
\[
2 \left(1 + \frac{m \sigma^2}{2}\right) |\Omega|^2 \frac{\sigma}{m R S} << 1
\]
or
\[
\left\{ \frac{\sqrt{2} \left(1 + \frac{m \sigma^2}{2}\right) |\Omega_v|}{\sqrt{m} G_{rf} \sqrt{S} G_{m} G_{d} \sqrt{R}} \right\}^2 << 1
\]
which is essentially the square of Eq. (94) or Eq. (96). Note that
for fixed circuit gains, it is increasingly more difficult to satisfy
Eq. (102) as the input signal-to-thermal noise ratio becomes large.
As the input signal power S becomes large, the output noise power in
Eq. (100) does not approach zero, as in the zero-offset case, but
instead approaches the lower limit
\[
P_n(\omega) \xrightarrow{S \to \infty} |\Omega|^2 \Omega^\dagger \Omega - \left| \frac{1_{\Omega^+\Omega^{-1}}}{V \Omega^+\Omega^{-1} V_1} \right|^2.
\]
This behavior is explained by the fact that an error signal must be
present in steady-state to cancel the offset voltages. There is no
desired signal error component in the limit of large S, however,
since from Eq. (97)
\[
\lim_{S \to \infty} P_s(\omega) = R,
\]
so the error must consist entirely of output noise in this limit.
As a final step in the analysis, the effect of offsets on the array output signal-to-total noise ratio will be evaluated. In deriving this result, only the assumption of a narrowband desired signal and an ideal reference signal will be necessary; the results in Eqs. (97) and (100) for a general noise covariance matrix $K_n$ will be used. The ratio of Eqs. (97) and (100) is

$$
p_s(\infty) = \frac{p_n(\infty)}{1 + \frac{Q^+}{\sqrt{RS}} v_1^T K_n^{-1} v_1} \left[ 1 + \frac{Q^+}{\sqrt{RS}} v_1^T K_n^{-1} v_1 \right] \left[ 1 - \frac{\sqrt{S}}{\sqrt{R}} Q \right] - \frac{2(1+S v_1^T K_n^{-1} v_1)}{S v_1^T K_n^{-1} v_1} \left[ \text{Re} \frac{\sqrt{S}}{\sqrt{R}} Q^+ - \frac{S |Q|^2}{R} \right] + \frac{(1+S v_1^T K_n^{-1} v_1)^2}{RS v_1^T K_n^{-1} v_1} K_n^{-1} \Omega
$$

where common leading coefficients have been factored out of each expression. The first two of the three denominator terms will be denoted by "D" and manipulated as follows:

$$D = 1 - 2 \text{Re} \frac{\sqrt{S}}{\sqrt{R}} Q + \frac{S |Q|^2}{R v_1^T K_n^{-1} v_1} + 2 \text{Re} \frac{\sqrt{S}}{\sqrt{R}} Q^+ + 2 \text{Re} \frac{Q^+}{\sqrt{RS} v_1^T K_n^{-1} v_1}$$

$$= 1 + 2 \text{Re} \frac{Q^+}{\sqrt{RS} v_1^T K_n^{-1} v_1} + \frac{|Q|^2}{RS(v_1^T K_n^{-1} v_1)^2} - \frac{|Q|^2}{RS(v_1^T K_n^{-1} v_1)^2} [1 + 2S v_1^T K_n^{-1} v_1 + S^2(v_1^T K_n^{-1} v_1)^2]$$

$$= \left[ 1 + \frac{Q^+}{\sqrt{RS} v_1^T K_n^{-1} v_1} \right]^2 + (1+S v_1^T K_n^{-1} v_1)^2 \left[ - \frac{|Q|^2}{RS(v_1^T K_n^{-1} v_1)^2} \right].$$
Substituting this result in Eq. (103) and then dividing numerator and denominator by the leading magnitude-squared term yields the desired result

(104) \[
\frac{P_{S}(\omega)}{P_{n}(\omega)} = \frac{S V_{l}^{+} K_{n}^{-1} V_{1}}{1 + \varepsilon}
\]

where \( \varepsilon \) is the offset-degradation factor given by

\[
\varepsilon = (1 + S V_{l}^{+} K_{n}^{-1} V_{1})^2 \left[ \frac{\Omega^{+} K_{n}^{-1} \Omega}{R S V_{l}^{+} K_{n}^{-1} V_{1}} - \frac{|Q|^2}{R S (V_{l}^{+} K_{n}^{-1} V_{1})^2} \right]
\]

A 3 dB performance degradation occurs when \( \varepsilon = 1 \). By substituting for \( Q \) from Eq. (91) and factoring out the magnitudes of the vectors in the quadratic and bi-linear forms, a more desirable form for \( \varepsilon \) is obtained as follows:

(105) \[
\varepsilon = (1 + S V_{l}^{+} K_{n}^{-1} V_{1})^2 \left[ \frac{\hat{\Omega}^{+} K_{n}^{-1} \hat{\Omega} - |\hat{V}_{1}^{+} K_{n}^{-1} \hat{\Omega}|^2}{m R S V_{l}^{+} K_{n}^{-1} V_{1}} \right]
\]

The first two factors of \( \varepsilon \) in Eq. (105) strongly influence the amount of degradation for arbitrary directions of the unit offset vector \( \hat{\Omega} \). The remaining factor in Eq. (105) - a ratio of scalars - represents the effect on performance of various directions in \( \hat{\Omega} \), i.e., the effect of various "patterns" in the set of offset voltages at the error multiplier outputs. This factor is zero when \( \hat{\Omega} \) has the direction of \( V_{1} \); there is no degradation in this case since the offset vector has the same direction as the ideal cross-correlation vector \( r_{x} \) in Eq. (87). The direction of \( \hat{\Omega} \) for worst-case performance is difficult to determine in the general case, Eq. (105); however, when no interfering signals are present, the expression for \( \varepsilon \) reduces to
Worst-case degradation occurs when \( \hat{\Omega} \) is orthogonal to \( \hat{V}_1 \). This result is as expected since it was shown in Eq. (72) that performance degrades as the angle between the unit weight vector \( w \) and \( V \) increases; this angle is maximized when the weight vector, given by Eq. (92) with \( J=0 \), has an offset component \( \omega \) orthogonal to \( V_1 \). By proper assignment of values to the gain factors in Eq. (106), even the worst-case degradation can be held to acceptable limits over the operating range of input signal-to-noise ratio. These factors can be seen in Fig. 7 to represent the feedback path gains from the point where the reference signal is generated to the point where the offset vector is introduced.\(^4\) An upper limit on antenna amplifier gains, \( G_{\text{rf}} \), is imposed in practice by the dynamic range limitations of the weighting coefficients. As to be shown later in the experimental results, performance degrades severely when the (active) weight multiplier circuits are overdriven by excessive input interfering signal. The non-linearity of a weight circuit to the composite signal at its input results in the suppression of desired signal and in the generation of undesired intermodulation products which are not removed at the array output. The rf amplifier gains must therefore be sufficiently small under worst-case conditions of maximum interference power and maximum weight circuit gain so that saturation effects in the weighting circuits are avoided.

D. Steady-state Response to Pulse Interference

The response of the adaptive array to a periodic pulse interfering signal will be examined in this section. The pulse interference is modeled as a narrowband random process having a variance given by the periodic function

\[
J(t) = \sum_{n=-\infty}^{\infty} \beta \frac{P_a(t-\tau_j/2-nT_j)}{2}
\]

where

\[
P_a(t) = \begin{cases} 
1 & ; |t| < a \\
0 & ; |t| \geq a 
\end{cases}
\]

\(4G_{\text{rf}} \sqrt{S} G_m\) is the conversion gain of the error multiplier to the desired signal component of the rf error signal.
is a unit pulse function. The variance is constant at the value $J$ during each pulse of $\tau_j$ seconds duration, and is zero for the remainder of each repetition period of $T_j$ seconds. The narrowband approximation includes the assumptions of a large pulse width $\tau_j$ compared to the differential delays between array elements, $\tau_{jk}$, and narrowband phase modulation within each pulse. With this approximation, the covariance matrix of the composite input vector $x(t)$ may be expressed as

$$K_x(t) = E[x(t)x(t)^\dagger] = S V_1 V_1^\dagger + J(t) V_2 V_2^\dagger + \sigma^2 I.$$  

This matrix is non-stationary as a result of the power variations $J(t)$ of the pulse interference process. The matrix elements are constants, however, over each time interval where the pulse is present, and a different set of constants over each interval where the pulse is absent. This property of local (piecewise) stationarity of the covariance matrix, Eq. (108), permits mean weight vector solutions of Eq. (32) to be formulated in each time interval where the statistics of the input vector are invariant. The additional property of weight vector (integrator output) continuity allows initial values at the transition instants between adjacent time intervals to be determined.

Consider the time interval from $t=0$ to $t=\tau_j$ where the interference pulse is present. Under the assumption of an ideal cross-correlation vector as in Eq. (60), the weight vector response over this interval is the same as in Eq. (61). That is, for

$$0 \leq t \leq \tau_j,$$

$$w(t) = \begin{bmatrix} \frac{<e_1,w(0)> \ e^{-\alpha_1 t}}{|e_1|^2} & \frac{-b_1(1-e^{-\alpha_1 t})}{e_1} \\ \frac{<e_2,w(0)> \ e^{-\alpha_2 t}}{|e_2|^2} & \frac{-b_2(1-e^{-\alpha_2 t})}{e_2} \end{bmatrix} + E_3 w(0) e^{-\alpha \sigma^2 t}$$

where
The eigenvalues and eigenvectors in these expressions are given in Eq. (57); the non-noise eigenvectors, $e_1$ and $e_2$, are linear combinations of the vectors $V_1$ and $V_2$. The initial weight vector, $w(0)$, is arbitrary at present. The weight vector response from $t=\tau_j$ to $t=T_j$ is similar to Eq. (109):

\[ w(t) = \begin{cases} 
\frac{<V_1,w(\tau_j)>}{m} e^{-\alpha(\sigma^2+mS)(t-\tau_j)} + \frac{\sqrt{RS}}{\sigma^2+mS} \left[ 1 - e^{-\alpha(\sigma^2+mS)(t-\tau_j)} \right] e^{2(t-\tau_j)} \left\{ w(\tau_j) - \frac{<V_1,w(\tau_j)>}{m} V_1 \right\} & \text{for } \tau_j < t < T_j 
\end{cases} \]

During this interval between interference pulses, the weight vector approaches the eigenvector $V_1$, which corresponds to an array co-phased to desired signal. The second term in Eq. (111) represents the transient decay of components of $w(\tau_j)$ which are not along $V_1$, i.e., in the thermal noise eigenspace orthogonal to $V_1$. Note that $w(\tau_j)$ may be obtained from Eq. (109) after the initial condition vector $w(0)$ has been specified. Similarly, the value of $w(T_j)$ obtained by evaluating Eq. (111) at $t=T_j$ is the initial value of the weight response over the next pulse period which ends at $t=2T_j$. The weight vector response for this interval has the same basic form as in Eq. (109) and Eq. (111); only the initial conditions and time-base delays in Eqs. (109) and (111) must be altered.

\[ (110) \quad w(0) = \frac{<e_1,w(0)>}{|e_1|^2} e_1 + \frac{<e_2,w(0)>}{|e_2|^2} e_2 + E_3 w(0) \]

and

\[ b_1 = \frac{\sqrt{RS}}{\lambda_1} \frac{<e_1,V_1>}{|e_1|^2}, \quad b_2 = \frac{\sqrt{RS}}{\lambda_2} \frac{<e_2,V_1>}{|e_2|^2}. \]

5The weight vector must be continuous at $t=\tau_j$ to satisfy integrator output continuity conditions.
Thus, given the initial vector $w(0)$, the weight vector $w(t)$ at each time instant thereafter can be (computer) calculated using a sequential, period-by-period computation process. After sufficient adaptation time, the weight vector approaches periodic, steady-state behavior characterized by the condition

$$\tag{112} w(t) = w(t + T_j) \quad t \to \infty.$$  

The weight vector is a repetitive transient in steady-state as illustrated in Fig. 8(b) where it has been assumed that processing

![Image](image_url)

**Fig. 8.**--An illustration of the array's steady-state response to pulse interference.

is initiated prior to $t = 0$ so that steady-state conditions are established for positive time. The only additional constraint that
must be imposed in Eqs. (109) and (111) in this case is the condition Eq. (112), evaluated at \( t = 0 \): \( w(0) = w(T_j) \). It is shown in Appendix B that in order to satisfy this constraint, the components of \( w(0) \) along \( e_1 \) and \( e_2 \) must be solutions of a pair of simultaneous equations whose coefficients are functions of all the array input parameters. Furthermore, the remaining components of \( w(0) \) not along \( e_1 \) or \( e_2 \) must vanish in steady-state:

\[
E \cdot w(0) = 0.
\]

This latter result follows since only weight vector components in the \( V_1V_2 \)-subspace are asymptotically retained in Eqs. (109) and (111).

A computer program was written to solve the equations in Appendix B for \( w(0) \) and to compute the output power ratios

\[
\frac{P_s(t)}{P_{n_t}(t)} = \frac{S}{\sigma^2} \frac{|<w(t),V_1>|^2}{|w(t)|^2} \quad ; \quad 0 \leq t \leq T_j
\]

(113)

\[
\frac{P_j(t)}{P_s(t)} = \begin{cases} 
\frac{J}{S} \frac{|<w(t),V_2>|^2}{|<w(t),V_1>|^2} & ; \quad 0 \leq t \leq \tau_j \\
0 & ; \quad \tau_j \leq t \leq T_j
\end{cases}
\]

(114)

over one period of the interference using the results in Eqs. (109) and (111). In order to facilitate comparison with experimental results to be discussed in Chapter VI, the time axis in the program was divided into data bit periods, \( T_b \), defined by the relationship

\[
\alpha(\sigma^2 + mS) N_b T_b \equiv 1
\]

(115)

where \( N_b \) was a known integer.\(^6\) In this expression, the array time constant associated with the response to desired signal only is assumed to equal a known number \( N_b \) of data bit periods \( T_b \). The interference pulse width was also expressed in the program as an integer number, \( N_j \), of data bit periods,

---

\(^6\)For the data bit rate, power levels, and relatively small loop gain used in the experimental implementation, the integer \( N_b \) was approximately one-hundred.
(116) \[ N_j = \frac{\tau_j}{T_b} \]

Figures 9 and 10 illustrate typical results for a four-element, linear array where the angular separation of sources corresponds to a value of \( \psi \) of sixty electrical degrees in Eq. (82). The interference pulse width is fixed at ten data bit periods, and each curve in these figures corresponds to the time response for a different duty cycle (repetition period). The array responds when desired signal only is present with a time constant of 100 data bit periods; the response when pulse interference is present corresponds to a minimum time constant of 1.25 data bit periods as noted from Eq. (57) and Eq. (115):

\[
\frac{\lambda_1/\sigma^2}{(\sigma^2 + \omega^2)/\sigma^2} = \frac{401.752}{5} = 80.35,
\]

\[
\frac{1}{a \lambda_1} = \frac{N_b T_b}{80.35} = \left(\frac{100}{80.35}\right) T_b.
\]

The interference pulse spans approximately eight minimum time constant periods in this case,

\[ a \lambda_1 \tau_j = 8.035; \]

thus, the interference rejection near the end of each pulse approaches that of continuous interference. When the pulse repetition period is decreased, the initial value of each output pulse decreases as shown in Fig. 9. This behavior results from pulse-to-pulse "memory" in the feedback loop integrators; i.e., the array does not "relax" completely to a co-phased condition on desired signal in the time intervals between pulses. When the repetition period is very large, complete relaxation occurs; the initial output ratio in this case equals the input ratio (20 dB) minus the interference rejection provided by the co-phased array pattern (7.3 dB). Figure 10 shows the loss in output signal-to-thermal noise ratio for the same conditions as in Fig. 9. The output ratio versus time is plotted over one repetition period, \( T_j \), of the interference; the repetition period is varied to obtain the family of curves. A maximum loss of 0.875 dB occurs when the duty cycle approaches unity (see Fig. 5).

The steady-state response of the array for a smaller interference pulse width of one data bit period is shown in Fig. 11. All other parameters are unchanged from Fig. 9. Although the output ratio at the beginning of the pulse is smaller in Fig. 11 than in Fig. 9 for a
Fig. 9.—The output interference-to-signal ratio versus time for several values of interference duty cycle; 
\( N_j = 10 \).

\[
m = 4 \\
\frac{P}{\lambda_c} = 0.5 \\
\theta_s = 90^\circ \\
\theta_d = 70.5^\circ \\
\frac{S}{\sigma^2} = 0 \text{ dB} \\
\frac{J}{\sigma^2} = 20 \text{ dB} \\
N_j = \frac{T_i}{T_b} = 10 \\
N_b = 100
\]
Fig. 10.—The output signal-to-thermal noise ratio versus time for several values of interference duty cycle; $N_j = 10$. 

$$0.01 = \frac{\tau_i}{T_i}$$
fixed value of duty cycle, this improvement is more than offset by the greater occurrence rate of the narrower pulses. Consider, for example, the case in which the duty cycle is 0.01 and the performance over a time interval spanning one-thousand data bits is to be examined. In Fig. 9, where the pulse is present for ten data bits and absent for nine hundred and ninety data bits, the first five data bits are adversely affected (i.e., $P_J/P_S$ greater than -20 dB). For the same one-thousand data bit interval in Fig. 11, ten data bits are affected - one every one-hundred data-bit periods. This example shows that performance degrades as the interference pulsewidth becomes small compared to the array response time required to null the output interference.
Worst-case performance under small pulsewidth conditions is also indicated from computations of the time-averaged performance measure

\[
\frac{\text{av. } P_j}{\text{av. } P_s} = \frac{\frac{1}{T_j} \int_0^{T_j} p_j(t) \, dt}{\frac{1}{T_j} \int_0^{T_j} p_s(t) \, dt} = \frac{\int_0^{T_j} p_j(t) \, dt}{\int_0^{T_j} p_s(t) \, dt}.
\]

The expressions for expected output powers in Eq. (114) were integrated (analytically) over one pulse repetition period in steady-state and the ratio of the resulting averages computed as in Eq. (117). This ratio is a function of five parameters,\( \psi, \tau_j, T_j, S/\sigma^2, J/\sigma^2 \), i.e., angular separation, pulsewidth, repetition period, and the input power levels. Figure 12 shows the results of a computer evaluation of this ratio for a particular set of input conditions. The ratio is plotted as a function of the duty cycle with the pulsewidth, normalized in minimum time constant periods, as the parameter. It can be noted that for any fixed pulsewidth, a duty cycle exists such that the average power ratio, Eq. (117), is maximum. Also, as the pulsewidth is decreased and the duty cycle simultaneously adjusted for worst-case (maximum) response, an upper bound to the average performance degradation is approached in the limit of very short pulses. The absolute maximum value of the average power ratio in Fig. 12 is -7.3 dB; this value occurs at a worst-case duty cycle of 0.01. These numbers are listed in the third line of Table 1. Other entries in this table correspond to computed worst-case performance in Eq. (117) for different input power levels and angular separations. These results indicate that the average power ratio, Eq. (117), does not exceed an upper bound dependent only on angular separation and input desired signal-to-thermal noise ratio - at least for angular separations not approaching zero. \( \psi > 27^\circ \). Performance for very small angular separations was not investigated.\(^7\)

\(^7\)In the limit as \( \psi \) approaches zero, the worst-case duty cycle should approach unity and the maximum value of Eq. (117) should approach the input ratio \( J/S \).
Fig. 12.—The ratio of average output interference power-to-average output signal power versus duty cycle for several values of normalized pulsewidth.
Table 1
Worst-case Average Power Performance with Pulse Interference
$(\alpha \lambda \tau_j \ll 1)$

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>$s/\sigma^2$</th>
<th>$J/\sigma^2$</th>
<th>$\tau_j/T_j$ (worst-case)</th>
<th>$\frac{\text{av. } P_j}{\text{av. } P_s}$ (max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.7°</td>
<td>0 dB</td>
<td>30 dB</td>
<td>-25.2 dB</td>
<td>-2.0 dB</td>
</tr>
<tr>
<td>27.2</td>
<td>0 dB</td>
<td>10</td>
<td>-10</td>
<td>-7.3</td>
</tr>
<tr>
<td>27.2</td>
<td>0 dB</td>
<td>20</td>
<td>-20</td>
<td>-7.3</td>
</tr>
<tr>
<td>27.2</td>
<td>0 dB</td>
<td>30</td>
<td>-30</td>
<td>-7.3</td>
</tr>
<tr>
<td>27.2</td>
<td>10 dB</td>
<td>20</td>
<td>-20</td>
<td>-17.3</td>
</tr>
<tr>
<td>27.2</td>
<td>10 dB</td>
<td>40</td>
<td>-40</td>
<td>-17.3</td>
</tr>
<tr>
<td>31.2</td>
<td>10 dB</td>
<td>30</td>
<td>-31.5</td>
<td>-18.7</td>
</tr>
<tr>
<td>60</td>
<td>10 dB</td>
<td>30</td>
<td>-34.8</td>
<td>-28.4</td>
</tr>
<tr>
<td>60</td>
<td>0 dB</td>
<td>20</td>
<td>-24.8</td>
<td>-18.4</td>
</tr>
</tbody>
</table>

Note that the sum in decibels of the entries in the third and fourth columns of the table represents the ratio of interference average power-to-thermal noise power at the array inputs. The range of this ratio from -5 dB to +5 dB in the table is the same range as occurs in Fig. 6 when continuous interference is maximum in the array output. Thus, the input-output average power relationships are similar in the two cases. If the input average interference power is increased significantly above this range either by an increase in duty cycle or peak pulse power, the average interference power in the array output will decrease as a result of either better "memory" (less relaxation) in the feedback loops, better adaption (more rejection) within each pulse, or a combination of both.

The interference-to-signal ratio at the array output during the pulse may be computed for the condition in which the average output power ratio, Eq. (117), is worst-case. The interference pulsewidth is very small compared to array response time in this case, resulting in a weight vector which is essentially constant in steady-state. It follows from Eq. (117) that under small pulsewidth, steady-state conditions,

$$\frac{\text{av. } P_j}{\text{av. } P_s} = \left(\frac{\tau_j}{T_j}\right) \frac{P_j}{P_s}, \quad \tau_j \ll \frac{1}{\alpha \lambda}$$

where $P_j$ is the constant output interference power during the pulse. The output interference-to-signal ratio during the pulse, therefore, is equal to the average power ratio times the reciprocal of the duty
cycle. This ratio, in decibels, is given by the difference in the last two columns of Table 1. For example, in the last line of the table, the output interference-to-signal ratio during the pulse is +6.4 dB. This value is consistent with the results in Fig. 11, where the input power levels and angular separation are identical and the small pulsewidth condition is approximately satisfied. At a duty cycle of -24.8 dB (0.0033 numeric), the output ratio during the pulse varies from approximately +9 dB to +3 dB. For smaller pulsewidths, this ratio approaches 6.4 dB at the above worst-case duty cycle.

The performance characteristics discussed above in conjunction with the average power ratio, Eq. (117), are also expected to be evident when another measure of average performance, the average bit error probability, is assumed as the figure of merit. This measure is appropriate when applied to the desired signal's (detected) data bit stream, i.e., after array processing and subsequent data demodulation operations. The calculated power ratios at the array output, Eqs. (113) and (114), will be used in Chapter VI, Sec. F to determine the expected error probability of each data bit in a pulse repetition period. An average of the resulting bit error probabilities will then be compared with a measured value for the average bit error probability under the same conditions of pulse parameters, input power levels, and angular separation.
CHAPTER IV
THEORETICAL PERFORMANCE WITH A NON-IDEAL REFERENCE SIGNAL

A. Introduction

The performance analyses of the previous chapter assumed an ideal reference signal equal to the desired signal at the phase center of the array. In practice, however, the reference signal will be either a fixed, prespecified signal generated on the basis of a priori information about the desired signal, or a derived-signal obtained by waveform-processing of the received array signal. The fixed reference signal will not be ideal if the a priori information is incomplete or inaccurate, while the derived-reference will be imperfect because of limitations in signal-processing. In the following section, the effect on performance of offset errors in the phase, frequency, or time delay (synchronization) of the fixed reference signal will be determined. The more general case of a reference signal which is partially correlated with the interference (e.g., multipath interference) is also discussed. Performance analyses are presented in Sections C, D, and E for the case in which the reference signal is generated by waveform-processing of the array output signal. The effects of time delay in the waveform processor and imperfect removal of undesired components from the processed-reference signal are determined using appropriate analytic models.

B. Array Performance with a Non-ideal, Fixed Reference Signal

Consider first the effect of adding a phase offset of $\phi$ degrees to the complex envelope of the ideal reference signal, Eq. (60),:

$$r(t) = j f \xi(t) e^{-j\phi}$$

The cross-correlation vector is trivially modified by an additional complex scalar, $\exp(+j\phi)$; therefore, the solution to

$$\frac{1}{\alpha} \frac{dE[w(t)]}{dt} = r_X - K_X E[w(t)]$$

is

$$r_X = \sqrt{RS} e^{+j\phi} v_I.$$
is still given by Eq. (61) when reference signal terms (containing \( \sqrt{R/S} \)) include this factor. Each component of the mean weight vector is phase-shifted by \( \phi \) degrees, in steady-state, to align the phase of the output signal component with that of the fixed reference signal:

\[
E[w(t)]_{t \to \infty} = \sqrt{RS} e^{+j\phi} \mathbf{K}^{-1}_X \mathbf{v}_1,
\]

\[
\gamma_S(t)_{t \to \infty} = E[w(t)]^\dagger \xi_1(t) \mathbf{v}_1 = \sqrt{RS} \mathbf{v}_1^\dagger \mathbf{K}^{-1}_X \mathbf{v}_1 \xi_1(t) e^{-j\phi}, \quad t \to \infty.
\]

Since any ratio of output powers, Eqs. (75), (76), or (77) does not contain the complex scalar multipliers in the expression for the weight vector, the array performance is invariant with respect to reference phase offset.

An offset in the center frequency of the fixed reference signal, however, does result in performance degradation. In this case, the representation is

\[
\gamma(t) = \frac{1}{\sqrt{RS}} \xi_1(t) e^{-j\Delta \omega_s t}
\]

\[
\gamma_X(t) = \sqrt{RS} e^{-j\Delta \omega_s t} \mathbf{v}_1
\]

where it is evident that the cross-correlation vector is non-stationary. Using the results from Eqs. (45) and (54), the mean weight vector response may be written as

\[
E[w(t)] = \sum_{i=1}^{\mathbf{w}} e^{-\alpha \lambda_i (t-t_0)} E_i w_0
\]

\[
+ \sum_{i=1}^{\mathbf{w}} \frac{\sqrt{RS}}{\lambda_i} \int_{t_0}^{t} \alpha \lambda_i e^{-\alpha \lambda_i (t-\tau)+j\Delta \omega_s \tau} E_i v_1 d\tau
\]

and integrated to yield

\[
E[w(t)] = \sum_{i=1}^{\mathbf{w}} e^{-\alpha \lambda_i (t-t_0)} E_i w_0
\]

\[
+ \sum_{i=1}^{\mathbf{w}} \frac{\sqrt{RS}}{\lambda_i} \frac{j\Delta \omega_s t}{e^{1+ \frac{\Delta \omega_s}{\alpha \lambda_i}}} \left[ 1 - e^{-(\alpha \lambda_i + \Delta \omega_s) (t-t_0)} \right] E_i v_1.
\]
In steady state,

\[ E[w(t)]_{t \to \infty} = \sqrt{RS} e^{j\omega_s t} \sum_{i=1}^{\infty} \frac{E_i v_1}{\lambda_i (1 + j \frac{\Delta \omega_s}{\alpha \lambda_i})} \cdot \]

The periodic, phase-rotation factor in this result, \( \exp[j\omega_s t] \), effectively shifts the frequency spectrum of the received signal by an amount necessary to compensate for the reference frequency offset. For very small frequency offsets, the ideal weight vector is obtained in steady-state since the denominator factors in the summation approach \( \lambda_i \):

\[ \Delta \omega_s \ll \alpha \lambda_i. \]

For large frequency offset, however, the weight vector loses its component along the vector \( v_2 \),

\[ E[w(t)]_{t \to \infty} = \alpha \sqrt{RS} e^{j\omega_s t} \sum_{i=1}^{\infty} \frac{E_i v_1}{\lambda_i} = \sqrt{RS} e^{j\omega_s t} k x^{-1} v_1, \]

\[ \Delta \omega_s \gg (\alpha \lambda_i)_{\text{max}} = \alpha \lambda_1. \]

The array is co-phased on desired signal in this case with the array output signal component in phase quadrature with the reference signal. The interfering signal is rejected only to the extent provided by the co-phased array pattern differential in the signal and interference directions. It is apparent that the frequency offset of the reference signal must be small compared to reciprocal time constants of the feedback loops in order to avoid this loss in interference rejection.

The effect of an error in synchronizing the time base of the reference signal with that of the received signal may be determined using the representation
\[
(126) \quad \mathbf{r}(t) = \sqrt{\frac{R}{S}} \xi_1(t-\delta_r)
\]

\[
\hat{\mathbf{r}}(t) = \sqrt{\frac{R}{S}} \xi_1(t-\delta_r) e^{-j\omega_c \delta r},
\]

where \(\delta_r\) is the time-base offset. Then, the cross-correlation vector is given by

\[
(127) \quad \mathbf{r}_x = \sqrt{\frac{R}{S}} e^{j\omega_c \delta r} E[\xi_1(t) \xi_1^*(t-\delta_r) + \xi_2(t) \xi_2^* + \mathbf{n}_r(t) \cdot \xi_1(t-\delta_r)^+] \]

\[
= \sqrt{\frac{R}{S}} e^{j\omega_c \delta r} E[\xi_1(t) \xi_1^*(t-\delta_r)] \mathbf{v}_1
\]

\[
= \sqrt{\frac{R}{S}} e^{j\omega_c \delta r} \rho(\delta_r) \mathbf{v}_1
\]

where

\[
\rho(\tau) = \frac{E[\xi_1(t) \xi_1^*(t-\tau)]}{E[|\xi_1(t)|^2]}
\]

is the desired signal's normalized autocorrelation function. The reference delay reduces the magnitude of the cross-correlation vector and introduces a phase offset factor, but the direction of this vector remains unchanged. As a result, the mean weight vector response is again given by Eq. (61) with \(\sqrt{R/S}\) factors replaced by the factors multiplying \(\mathbf{v}_1\) in Eq. (127). Only the complex gain of the steady-state weight vector is affected by reference timing error:

\[
(128) \quad E[\mathbf{w}(t)]_{t \to \infty} = \sqrt{\frac{R}{S}} \rho(\delta_r) e^{j\omega_c \delta r} \mathbf{v}_1
\]

Since Eq. (119) is an approximation to the complete differential equation, Eq. (30), this conclusion of performance invariance with timing error is expected to be valid only when the weight-control loops have narrow bandwidth compared to the bandwidth of the input signal:

\[
(129) \quad [\lambda_{\text{max}}]^{-1} \gg \Delta
\]

or

\[
\alpha \lambda_{\text{1}} \Delta \ll 1
\]
where $\Delta$ is the width of the desired signal's autocorrelation function,

$$\rho(\tau) \neq 0; \quad |\tau| < \Delta$$

$$= 0; \quad |\tau| > \Delta .$$

Consider, for example, a constant envelope desired signal which is bi-phase modulated by a binary pseudorandom code. If the code is modeled as a random binary process of bit duration $\Delta$, the autocorrelation is a triangular function

$$\rho(\tau) = \begin{cases} 1 - \frac{|\tau|}{\Delta} & ; \quad |\tau| < \Delta \\ 0 & ; \quad |\tau| > \Delta \end{cases}$$

The condition in Eq. (129) indicates that many of the $\delta_r$-second, error periods which occur due to misalignment of local and received codes will be averaged in one response time constant of the control loops. The result in Eq. (128) may be used to calculate the steady-state output signal,

$$\gamma_s(t)|_{t+\infty} = \sqrt{RS} v_1^\dagger k^{-1} v_1 \left[ \rho(\delta_r) e^{-j\omega c \delta_r} \xi_1(t) \right].$$

It is interesting to compare the amplitude of this signal with the amplitude of the correlated component of the reference signal, given by the first term in

$$\gamma(t) = \sqrt{RS} \left[ \rho(\delta_r) e^{-j\omega c \delta_r} \xi_1(t) \right]$$

$$+ \sqrt{RS} e^{-j\omega c \delta_r} \left[ \xi_1(t-\delta_r) - \rho(\delta_r) \xi_1(t) \right].$$

The ratio of amplitudes is the same as when there is no delay error, $\delta_r = 0$. The uncorrelated component of the delayed reference signal multiplies the input signals in the error multipliers and produces zero-mean, noise-like fluctuations at the loop integrator inputs. These fluctuations are assumed to have negligible effect on the mean value of the weight vector when adequate smoothing is provided by the integrators.

When the fixed reference signal is partially correlated with the interfering signal - presumably as a consequence of correlation between the input desired signal and interference - an appropriate representation is
(133) \[ \hat{r}(t) = g_1 \frac{\hat{\xi}_1(t)}{\sqrt{S}} + g_2 \frac{\hat{\xi}_2(t)}{\sqrt{J}} \]

where \( g_1 \) and \( g_2 \) are complex correlation constants. The power in this reference signal is

\[
E[\hat{r}(t)\hat{r}(t)^+] = |g_1|^2 + 2 \text{Re}(g_1g_2^c_{12}) + |g_2|^2
\]

where

\[
c_{12} \equiv \frac{E[\hat{\xi}_1(t)\hat{\xi}_2(t)^{+}]}{\sqrt{SJ}}
\]

is the correlation coefficient of the input processes. The cross-correlation vector may be determined for this case as follows:

\[
(134) \quad r_x = E\left[\{\hat{\xi}_1(t)v_1 + \hat{\xi}_2(t)v_2 + \hat{n}_x(t)\} \begin{bmatrix} g_1^+ & g_2^+ \\ \sqrt{S} & \sqrt{J} \end{bmatrix} \begin{bmatrix} \hat{\xi}_1(t)^+ \\ \hat{\xi}_2(t)^+ \end{bmatrix} \right]
\]

\[
= \sqrt{S} (g_1^+g_2^+c_{12})v_1 + \sqrt{J} (g_2^+g_1^+c_{12}^+)v_2.
\]

An offset vector component along \( v_2 \) is present as expected. Using Eq. (19), with cross-terms retained, an expression for the mean weight vector in steady-state may be formulated as

\[
E[w(t)]|_{t=\infty} = K^{-1} \quad r_x
\]

\[
= [S_{v_1}v_1^+ + J_{v_2}v_2^+ + \sqrt{SJ} \quad c_{12}v_1v_2^+ + \sqrt{SJ} \quad c_{12}^+v_2v_1^+ 
\]

\[
+ \sigma^2 I]^{-1} [\sqrt{S}(g_1^+g_2^+c_{12})v_1 + \sqrt{J} (g_2^+g_1^+c_{12}^+)v_2].
\]

In principle, this result can be simplified analytically using repeated applications of Woodbury’s identity, Eq. (67), to evaluate the inverse covariance matrix. The algebraic expressions become lengthy, however, and require numerical evaluation to be interpreted. Some information regarding performance may be obtained, however, by assuming uncorrelated input signals (\( c_{12}=0 \)) and retaining the interference component in the reference signal, Eq. (133): \( g_2 \neq 0 \). In this case, the results of the offset voltage analysis in Chapter III are applicable since the weight vector reduces to
\( E[w(t)] \mid_{t \to \infty} = K_x^{-1}(\sqrt{S} \, v_1 + \sqrt{J} \, g_2^\dagger \, v_2) \)

\[ e^{-j \, \text{Arg} \, g_1} K_x^{-1} \left( \left| g_2 \right| \sqrt{S} \, v_1 + j[\text{Arg} \, g_1 - \text{Arg} \, g_2] \right) \]

which is of the form

\[ E[w(t)] \mid_{t \to \infty} = e^{-j \, \text{Arg} \, g_1} K_x^{-1} \left( \sqrt{R S} \, v_1 + \hat{\Omega} \right) \]

where

\[ \sqrt{R} = |g_1| \]
\[ \hat{\Omega} = \hat{v}_2 \]
\[ \frac{|\Omega|^2}{m R S} = \frac{|g_2|^2 m J}{m R S} = \frac{|g_2|^2}{|g_1|^2} \frac{J}{S} . \]

The array output signal-to-total noise ratio is given by the expressions in Eqs. (104) and (105). Near-ideal performance is obtained when

\[ \frac{|\Omega|^2}{m R S} \ll 1 \]

which implies

\[ \frac{|g_2|^2}{|g_1|^2} \ll \frac{s}{J} ; \]

i.e., the ratio of correlated interference power-to-signal power in the reference signal must be much less than the reciprocal of the input interference-to-signal power ratio.
C. Array Performance with an Ideal, Waveform-processed Reference Signal.

In this and the following two sections, array performance will be investigated for the case in which the reference signal is derived from the array output signal. The analyses to be presented are primarily intended for those applications involving phase-modulated communication signals which are constant in amplitude, or nearly so. The steady amplitude approximations

\[ E[|\xi_1(t)|] \approx \sqrt{S} \]

\[ \text{Var}[|\xi_1(t)| - E[|\xi_1(t)|]] \approx 0 \]

are considered realistic for continuous-envelope desired signals transmitted from ground or airborne terminals to a synchronous-satellite receiving array. The desired signal re-transmitted by the satellite to a ground-based receiving array might also be expected to be reasonably constant in amplitude provided sufficient protection of the up-link desired signal is ensured. An ideal waveform-processed reference signal for the array will be defined as the signal which yields a mean weight vector proportional to the ideal vector \( \frac{K}{\lambda} \) asymptotically as \( t \to \infty \). Under a set of assumptions no more restrictive than those employed previously, it will be shown that a processed reference signal which is constant in amplitude and which has the same phase variations as the desired signal component in the array output is ideal in this sense. This signal,

\[ (137) \quad \tilde{r}(t) = \sqrt{R} \frac{w(t)^+ \tilde{s}(t)}{|w(t)^+ \tilde{s}(t)|} = \sqrt{R} \frac{w(t)^+ \xi_1(t) v_1}{|w(t)^+ \xi_1(t) v_1|} , \]

is obtained from the ideal waveform processor and bandpass limiter shown in Fig. 13. The reason for fixing the amplitude of the

\[ \tilde{\gamma}_s = w(t)^+ \tilde{s}(t) \]

\[ \tilde{\gamma}(t) = w(t)^+ \tilde{X}(t) \]

\[ = w(t)^+ [\tilde{s}(t) + \tilde{n}(t)] \]

\[ \tilde{r}(t) = \sqrt{R} \frac{w(t)^+ \tilde{s}(t)}{|w(t)^+ \tilde{s}(t)|} \]

Fig. 13.—An idealized model of the reference signal waveform processor.
reference signal will become apparent later in Section E where linear processing models are considered.

The cross-correlation vector corresponding to Eq. (137) is given by

\[
\mathbf{r}_X(t) = E \left[ \left( \hat{s}(t) + \hat{n}(t) \right) \sqrt{R} \frac{\hat{s}(t)^+ w(t)}{|\hat{s}(t)^+ w(t)|} \right]
\]

\[
= \sqrt{R} \left[ E \left( |\hat{\xi}_1(t)| \frac{v_1^+ w(t)}{|v_1^+ w(t)|} \right) v_1
\right] + \sqrt{R} \left[ E \left( \frac{\hat{\xi}_1(t)^+ v_1^+ w(t)}{|\hat{\xi}_1(t)^+ v_1^+ w(t)|} \right) \right].
\]

The expectation of the product in the second term may be expanded using the same approach as in Eq. (31):

\[
E \left[ \frac{\hat{\xi}_1(t)^+ j \text{Arg} v_1^+ w(t)}{|\hat{\xi}_1(t)|} e^{j \text{Arg} \frac{\hat{\xi}_1(t)^+ v_1^+ w(t)}{|\hat{\xi}_1(t)^+ v_1^+ w(t)|}} \right] =
\]

\[
E \left[ \frac{\hat{n}(t) \frac{\hat{\xi}_1(t)^+}{|\hat{\xi}_1(t)|}}{E \left[ j \text{Arg} v_1^+ w(t) \right]} \right] - E \left[ \frac{\hat{n}(t) \frac{\hat{\xi}_1(t)^+}{|\hat{\xi}_1(t)|}}{E \left[ j \text{Arg} v_1^+ w(t) \right]} \right].
\]

The first term on the right-hand side of Eq. (139) is zero by hypothesis of uncorrelated input processes,
The second term in Eq. (139), which involves the correlation between zero-mean fluctuation processes, is also expected to be negligible compared to the dominant (first) term of Eq. (138) when the variances of the weighting coefficients about their expected values are small. This, of course, is the basic premise assumed in Chapter II. The next step in the derivation is to approximate the remaining component in Eq. (138) by

\begin{equation}
\mathbf{r}_x(t) = \sqrt{R} \sqrt{S} \mathbf{e} \left[ j \text{Arg} \: v_1^\dagger \: w(t) \right] .
\end{equation}

This approximation that Eq. (141) is large compared to

\begin{equation}
\sqrt{R} \mathbf{e} \left[ j \text{Arg} \: v_1^\dagger \: w(t) \right] E\left\{ e^{-j \text{Arg} \: v_1^\dagger \: w(t)} \right\}
\end{equation}

is also satisfied under the conditions of small weight variances and steady signal amplitude, Eq. (136). The expectation over the phase variable in Eq. (141) may be expressed in terms of the joint probability density \( p(w) \) of the complex weighting coefficients as

\begin{equation}
E \left[ e^{j \text{Arg} \: v_1^\dagger \: w(t)} \right] = \int_{\text{all } w} \mathbf{e}^{j \text{tan}^{-1} \left( \frac{\text{Im} \: v_1^\dagger \: w(t)}{\text{Re} \: v_1^\dagger \: w(t)} \right)} p(w) dw .
\end{equation}

When the variances of the weights about their mean values are small, the joint density approaches a product of delta functions,

\begin{equation}
P_w(w) \rightarrow \prod_{i=1}^{m} \delta(w_i - E[w_i])
\end{equation}

\begin{equation}
\sigma_{w_i}^2 \rightarrow 0, \: i = 1, \: 2, \: \ldots, \: m
\end{equation}

permitting Eq. (142) to be approximated by

\begin{equation}
E \left[ e^{j \text{Arg} \: v_1^\dagger \: w(t)} \right] \approx e^{j \text{Arg} \: v_1^\dagger \: E[w(t)]} = \frac{v_1^\dagger E[w(t)]}{|v_1^\dagger E[w(t)]|} .
\end{equation}
Therefore, the nominal result for the cross-correlation vector is

\[ r_x(t) = \sqrt{R} \sqrt{S} \frac{v_1^* E[w(t)]}{|v_1^* E[w(t)]|} v_1. \]  

(144)

The differential equation governing the time response of the mean weight vector is, from Eqs. (119) and (144),

\[ \frac{1}{\alpha} \frac{d}{dt} E[w(t)] = \sqrt{R} \sqrt{S} \frac{v_1^* E[w(t)]}{|v_1^* E[w(t)]|} v_1 - K_x E[w(t)]. \]  

(145)

It is apparent, by direct substitution, that the steady-state solution is

\[ E[w(t)] = 0, \quad E[w(t)] = \sqrt{R} \sqrt{S} e^{j\phi} K_x^{-1} v_1. \]  

(146)

where \( \phi \) is an arbitrary constant. Note that no constraints are imposed on the phase of the array output signal by the ideal processor in Fig. 13. The reference phase is always aligned regardless of the value of a common phase shift added to all the complex weights. Except for this arbitrary phase condition, the solution in Eq. (146) is identical to that obtained with a fixed, ideal reference signal. The performance of the array, therefore, is ideal in steady-state.

The transient response of the mean weight vector may be investigated using the trial solution

\[ E[w(t)] = h_1(t) v_1 + h_2(t) v_2 + h_n(t) v_n \]

(147)

where

\[ h_1(t) = g_1(t) e^{j\phi_1(t)} \]

\[ h_2(t) = g_2(t) e^{j\phi_2(t)} \]

\[ h_n(t) = g_n(t) e^{j\phi_n(t)} \]
are complex functions which are unknown except initially at time $t_0$ and where $v_0$ is any vector in the noise eigenspace orthogonal to $v_1$ and $v_2$, i.e.,

$$<v_n, v_1> = <v_n, v_2> = 0.$$  

Substitution of this trial solution into Eq. (145) yields the conditions which must be satisfied by the unknown coefficients:

(148) \[ \frac{1}{\alpha} \dot{h}_1(t) = -[(mS+\sigma^2)h_1(t) + <v_1, v_2> S h_2(t)] + \sqrt{R} \sqrt{S} e^{j \text{Arg}[m h_1(t) + <v_1, v_2> h_2(t)]} \]

(149) \[ \frac{1}{\alpha} \dot{h}_2(t) = -[(mJ+\sigma^2)h_2(t) + <v_1, v_2> J h_1(t)] \]

and

(150) \[ \frac{1}{\alpha} \dot{h}_n(t) = -\sigma^2 h_n(t) \Rightarrow h_n(t) = h_n(t_0) e^{-\alpha \sigma^2 (t-t_0)} \]

An analytic solution for $h_1(t)$ and $h_2(t)$ does not appear possible, in general, as a result of the non-linearity in Eq. (148) due to amplitude limiting in the reference processor. However, analytic solutions can be obtained for a restricted (incomplete) set of initial weight vectors satisfying

(151) \[ E[w(t_0)] = e^{j \phi_0} [g_1(t_0) v_1 + g_2(t_0) v_2] + h_n(t_0) v_n \]

where $\phi_0$, $g_1(t_0)$, and $g_2(t_0)$ are arbitrary real constants and $h_n(t_0)$ is an arbitrary complex constant. For this set of initial weight vectors, the initial values of the desired signal and interference complex envelopes at the array output are given by

$$\tilde{\gamma}_s(t_0) = \tilde{\xi}_1(t_0) E[w(t_0)]^\dagger v_1 = e^{j \phi_0} p \tilde{\xi}_1(t_0)$$

$$\tilde{\gamma}_i(t_0) = \tilde{\xi}_2(t_0) E[w(t_0)]^\dagger v_2 = e^{j \phi_0} q \tilde{\xi}_2(t_0)$$

where $p$ and $q$ are real constants,
\[ p = m g_1(t_0) + \langle v_2, v_1 \rangle g_2(t_0) \]

\[ q = \langle v_1, v_2 \rangle g_1(t_0) + m g_2(t_0). \]

It may be noted that the initial phase arguments of the array output complex envelopes differ from the initial phase arguments at the array phase center by a common phase shift \( \phi_0 \). This condition, which results from the equi-phase arguments of the scalars multiplying \( v_1 \) and \( v_2 \) in Eq. (151), is exactly the condition required in steady-state. As verified from Eq. (146) and the expansion of \( K_x^{-1} v_1 \) in Eq. (71), the scalar multipliers of \( v_1 \) and \( v_2 \) in the expression for the steady-state weight vector are also equal in phase. The significance of these observations is that for initial weight vectors as in Eq. (151), phase transients in the array output signal and interference components are not required and do not occur. The unknown coefficients in Eq. (147) reduce in this case to the form

\[ h_1(t) = g_1(t) e^{j\phi_0} \]
\[ h_2(t) = g_2(t) e^{j\phi_0}, \]

and the cross-correlation vector, Eq. (144), is time-independent:

\[ r_x = \sqrt{R} \sqrt{S} e^{j \text{Arg}[e^{j\phi_0}(m g_1(t) + \langle v_1, v_2 \rangle g_2(t))] v_1} \]
\[ = \sqrt{R} \sqrt{S} e^{j\phi_0} v_1. \]

Consequently, the exponential transient response in Eq. (61) applies for this set of initial weight vectors. For other initial weight vectors where \( h_1(t_0) \) and \( h_2(t_0) \) in Eq. (147) are unequal in phase, the differential equations (148) and (149) must be solved. Since the final value of the weight vector in Eq. (146) and the output power ratios are invariant with respect to the choice of initial weight vector, the transient performance is not expected to differ substantially in these cases from that computed with the ideal, fixed reference signal for the same array input conditions and initial weight vector.

The next step in the development is to modify the ideal waveform processor in Fig. 13 such that only the information-bearing (data) components of the desired signal are retained at the waveform processor's output. In typical waveform-processing techniques,
the data-carrying signal, $\xi_d(t)$, is spread in bandwidth, prior to transmission, by a deterministic, wideband phase modulation $\phi_c(t)$:

$$\tilde{\xi}_1(t) = e^{j\phi_c(t)} \xi_d(t).$$

(152)

The spectrum of the desired signal is then compressed in the waveform processor by multiplying the received signal with a locally-generated signal containing the same phase modulation. As shown in Fig. 14, the locally-generated signal can be used again, after filtering and bandpass limiting the received data-carrying signal, to reinsert the phase modulation on the reference signal. In this model, the waveform processor will also be assumed to reject undesired signals perfectly - even when there are small timing offsets, $\varepsilon$, in the locally-generated phase modulation. The data-carrying signal at the waveform processor's output has a complex envelope modeled as

$$\tilde{y}(t) = w(t)^\dagger e^{j\phi_c(t)} \xi_d(t) v_1$$

(153)

where $a(t, 0) = 1$.

---

1The effects of imperfect rejection of undesired signals in the waveform processor are considered in Section E of this chapter.
The function \( a(t, \varepsilon) \) in this equation represents the effect of the timing offset. The characteristics of \( a(t, \varepsilon) \) depend, in practice, on the type of phase modulation employed, the ratio of spread bandwidth to information bandwidth, and the filter characteristics in the waveform processor. For the case of special interest here—bi-phase pseudonoise code modulation and bi-phase data modulation—it is sufficient to assume that \( a(t, \varepsilon) \) is real-valued. The justification for this assumption is that

\[
\begin{align*}
  e^{j\phi_c(t)} &- e^{j\phi_c(t-\varepsilon)} = e^{j\pi m_1(t)} &- j\pi m_1(t-\varepsilon) \\
  &= e^{j\pi[m_1(t) - m_1(t-\varepsilon)]} \\
  &= e^{j\pi m_1(t) - j\pi m_1(t-\varepsilon)}
\end{align*}
\]

is a real-valued \((+1, -1)\) process, as illustrated in Fig. 15, when

\[m_1(t)\] is a binary \((0,1)\) pseudonoise code. This process is averaged in the waveform processor's filter and results in a real-valued output process, \( a(t, \varepsilon) \), whose mean value is less than unity when \( \varepsilon \) is non-zero. The amplitude fluctuations about the mean are suppressed in the limiter yielding the reference signal

\[
\gamma(t) = \sqrt{R} e^{+j\phi_c(t-\varepsilon)} w(t) \xi_d(t) v_1
\]

\[
|w(t) \xi_d(t) v_1|
\]

Fig. 15.—An illustration showing that the product of biphase pseudonoise modulations is real-valued.
The expected cross-correlation vector may be calculated for this case as

\[
(155) \quad r_X(t) = E[\hat{x}(t) \hat{x}(t)^+] = E[\xi_1(t) \xi_1^* + E[\hat{n}(t)\hat{n}(t)^*]]
\]

\[
= \sqrt{R} \mathbb{E} \begin{bmatrix}
j \phi_c(t) - j \phi_c(t - \epsilon) \\
\xi_d(t) \\
\hat{n}(t) - j \phi_c(t - \epsilon) \xi_d(t)^* \\
\end{bmatrix} \begin{bmatrix}
v_1^+ w(t) \\
|v_1^+ w(t)| \\
v_1^+ w(t) \\
|v_1^+ w(t)| \\
\end{bmatrix}
\]

It is apparent that this result is similar in form to that in Eq. (138) for the previous model (Fig. 13) when the code timing error is zero: \( \epsilon = 0 \). Upon modeling the bi-phase code and data modulations as independent random binary processes, and employing the approximations in Eqs. (139) - (143), the nominal cross-correlation vector is obtained as

\[
(156) \quad r_X(t) = \sqrt{R} \mathbb{E} \left[ \begin{bmatrix}
j \phi_c(t) - j \phi_c(t - \epsilon) \\
x(t) + j x(t - \epsilon) \\
|\xi(t)| \\
|\xi(t)| \\
\end{bmatrix} \begin{bmatrix}
v_1^+ E[w(t)] \\
|v_1^+ E[w(t)]| \\
v_1^+ E[w(t)] \\
|v_1^+ E[w(t)]| \\
\end{bmatrix}
\right] v_1
\]

This approximate result is the same as Eq. (144) to within a real scalar constant indicating that the weight vector will be ideal in steady-state with only a gain adjustment occurring due to timing error. The essential requirement for this conclusion of performance invariance with small code timing offset is that the scalar factors in Eq. (156) which are independent of the mean weight vector must be real-valued. It will be shown in the following section that the ideal weight vector is not obtained, in general, when these factors are complex. Thus, for other types of phase modulation, it can be argued that
must have an average value

\[
E\left[ e^{j\phi_c(t)} - e^{j\phi_c(t-\epsilon)} \right] = R_c(\epsilon)
\]

which is a real-valued function of \( \epsilon \). This condition is expected to be satisfied for discrete (digital) phase modulations whose allowed phase states are symmetrically distributed over \((0, 2\pi)\) radians and occur equally often in time, e.g., quadrature pseudonoise modulation. For other bandspreading techniques such as frequency-hopping, a more careful analysis is necessary. The above condition essentially requires the phase error (fluctuation) processes which result from timing offset to have zero mean values. There must be adequate smoothing of these fluctuation processes at the error multiplier outputs and in the waveform processor in order for the expected value results derived here to apply.

D. Effects of Waveform-processor Time Delay

The effect on performance of time delay in the waveform processor may be determined using the model shown in Fig. 16. Perfect synchronization of the bandspreading modulation in the waveform processor is assumed, and the undesired signals are assumed to be perfectly rejected. The data-carrying signal present at the input to the \( \delta_r \)-second delay has the complex envelope

\[
\gamma(t) = w(t) + x(t)
\]

\[
\gamma_s(t) e^{j\phi_c(t)}
\]

\[
\gamma_d(t)
\]

\[
w(t - \delta_r)^\dagger \xi_d(t - \delta_r) v_1
\]

Fig. 16.—A model of the reference waveform processor which includes processing-time delay.
The delayed signal, 
\begin{equation}
\gamma_d(t) = \text{Re} \left[ w(t-\delta_r)^+ \xi_d(t-\delta_r) e^{-j\omega_c \delta_r} \frac{v_1}{\sqrt{2}} e^{+j\omega_c t} \right],
\end{equation}
is assumed to be compensated only at the center frequency of the waveform processor: \( \omega = \omega_c \). Therefore, the reference signal which results has the complex envelope 
\begin{equation}
\tilde{r}(t) = e^{+j\phi_c(t)} \frac{w(t-\delta_r)^+ \xi_d(t-\delta_r) v_1}{|w(t-\delta_r)^+ \xi_d(t-\delta_r) v_1|}.
\end{equation}

Under the same assumptions of uncorrelated input signals, narrow loop bandwidths, and small amplitude fluctuations in the desired signal as used in the preceding analyses, the cross-correlation vector, 
\begin{equation}
r_x = \sqrt{R} \text{E} \left[ \left( e^{+j\phi_c(t)} \xi_d(t) v_1 + \tilde{n}(t) \right) \tilde{r}(t)^+ \right],
\end{equation}
can be approximated by 
\begin{equation}
r_x(t) \approx \sqrt{R} \text{E} \left[ \frac{\xi_d(t) \xi_d(t-\delta_r)^+}{|\xi_d(t-\delta_r)|} \right] \cdot \frac{v_1^+ \text{E}[w(t-\delta_r)]}{|v_1^+ \text{E}[w(t-\delta_r)]|} \cdot v_1.
\end{equation}

The autocorrelation function of the data-carrying signal in this nominal result reflects the loss in correlation between the input and the reference signals as a result of the processing delay. The remaining term shows that variations in the phase of the desired signal at the array output - which occur as a consequence
of mean weight vector adaption — are transferred with transport
lag to the reference signal, and hence to the cross-correlation
vector.

If the desired signal is assumed to contain bi-phase random
data modulation and to be offset in carrier frequency from the
center frequency, \( \omega_c \), of the array, then its complex envelope
may be represented as

\[
\xi_i(t) = e^{+j\phi_c(t)} \xi_d(t) = e^{+j\phi_c(t)} \sqrt{S} e^{j(m_2(t)+\Delta \omega_s t+\theta)}
\]

where \( m_2(t) \) = random (data) binary process of bit duration \( T_b \),
\( \Delta \omega_s \) = carrier frequency offset.

In Eq. (162) then,

\[
E[\xi_d(t)\xi_d(t-\delta_r)] = S e^{+j\Delta \omega_s \delta_r} E[jn(m_2(t)-m_2(t-\delta_r))]
\]

\[
= S \left(1 - \frac{\delta_r}{T_b}\right) e^{+j\Delta \omega_s \delta_r}, \delta_r < T_b.
\]

The delayed bi-phase data modulation in the reference signal
produces only an amplitude reduction in the cross-correlation
vector. However, the combination of carrier frequency offset
and processor delay introduces a complex-valued factor in Eq. (162).
The resulting weight vector equation in this case,

\[
\frac{1}{\alpha} \frac{d}{dt} E[w(t)] = \sqrt{S} \left(1 - \frac{\delta_r}{T_b}\right) e^{+j\Delta \omega_s \delta_r} \frac{v_1^+E[w(t-\delta_r)]}{|v_1^+E[w(t-\delta_r)]|} v_1
\]

\[- K_x E[w(t)],
\]

has the ideal, asymptotic solution

\[
\frac{d}{dt} E[w(t)] = 0, E[w(t)] = \sqrt{S} \left(1 - \frac{\delta_r}{T_b}\right) e^{j\phi} K_x^{-1} v_1,
\]

\( t \to \infty \),

where \( \phi \) is an arbitrary constant, only when the carrier frequency
offset is zero. This can be seen by substituting Eq. (166) into
Eq. (165):
For non-zero frequency offsets, the solution to Eq. (165) can be expressed in the form

\[ \sqrt{R/S}\left(1 - \frac{\delta r}{r_b}\right)e^{j\phi}\left(e^{j\Delta\omega_s\delta r} - 1\right)v_1 \neq 0 \]

when \( \Delta\omega_s \neq 0 \).

The conditions which must be satisfied by the three unknown parameters in this expression are derived in Appendix C. When interference is not present (J=0), the parameters are given by

\[ \Delta\omega_r = \Delta\omega_r(\Delta\omega_s) = \text{angular rotation rate of weight vector}, \]

\[ d = d(\Delta\omega_s) = \text{complex-valued parameter representing deviation from ideal weight vector}, \]

\[ \phi = \text{arbitrary constant}. \]

The rotation rate of the weight vector which satisfies the transcendental equation, Eq. (168), is less than the input frequency offset as illustrated in Fig. 17. This inequality is also preserved when interference is present although the transcendental equation is more complicated in this case. For a given interference power, the magnitude of the degradation parameter \( d \) is monotonically related to the weight vector rotation rate as shown in the appendix. The weight vector loses its component along \( v_2 \) as the frequency offset is increased since
Fig. 17.—A graphical illustration of the transcendental equation (168).

\[
\frac{\Delta \omega_r}{\alpha (\sigma^2 + m)} \quad \tan \left[ (\Delta \omega_s - \Delta \omega_r) \delta_r \right]
\]

(169) \[ K^{-1}_x v_1 + dv_2 = \]

\[ \frac{K^{-1}_n v_1}{1 + S v_1 + K^{-1}_n v_1} + \frac{J \sigma^{-4} \langle v_1, v_2 \rangle \ Q(\Delta \omega_r)}{(1 + S v_1 + K^{-1}_n v_1)(1 + mJ \sigma^{-2})} \ v_2 \]

\[ = \left( \frac{\sigma^{-2}}{1 + S v_1 + K^{-1}_n v_1} \right) \left[ v_1 - \frac{mJ \sigma^{-2} \langle v_1, v_2 \rangle}{m} \ v_2 \right] \left( 1 - Q(\Delta \omega_r) \right) \]

where \( Q(\Delta \omega_r) \to 1, \Delta \omega_r \gg \alpha(\sigma^2 + mJ) \)

\( \to 0, \Delta \omega_r \to 0. \)

Consequently, the rejection of the interference decreases since the array phasing tends toward the co-phased condition (on the
desired signal) as the frequency offset becomes large. The carrier phase delay introduced in the reference processor,

\[ \phi_c = (\omega - \omega_c) \delta_r = (\omega_c + \Delta \omega_s - \omega_c) \delta_r - \Delta \omega_r \delta_r \]

approaches ninety degrees as noted in Fig. 17 where the tangent function curve shifts toward the right as \( \Delta \omega_s \) is increased.

E. Effects of Inadequate Waveform-processing Gain

In this section, the reference waveform processor will be assumed to only partially reject undesired signals present at its input. The effect on performance of interference and thermal noise components in the derived-reference signal will be determined using both linear and non-linear processor models. The linear model is shown in Fig. 18. The array output signal,

\[ y(t) = x(t) w(t) + v + f(t) w(t) + v_2 + w(t) + \tau(t) \]

is transformed to a reference signal

\[ \tilde{r}(t) = \tilde{\xi}_1(t) w(t) + v_1 + \tilde{\xi}_2(t) w(t) + v_2 + \tilde{n}_t(t) \]

where primes denote the effect of waveform processing on each component of the reference signal. This reference signal is multiplied by the input signals in the error multipliers and produces a cross-correlation vector whose expected value is given by

\[ \gamma(t) = \xi_1(t) w(t) + v_1 + \xi_2(t) w(t) + v_2 + \tilde{n}_t(t) \]
(171) \[ r_X(t) = E[(\hat{\xi}_1(t)v_1 + \hat{\xi}_2(t)v_2 + \hat{n}_t(t)) r(t)^+] \]

\[ = E[\hat{\xi}_1(t)^{\dagger}v_1]_1 E[w(t)] \]

\[ + E[\hat{\xi}_2(t)^{\dagger}v_2]_2 E[w(t)] \]

\[ + E[\hat{n}_t(t)^{\dagger}w(t)] \]

\[ + \text{terms containing } E[\hat{\xi}_i(t)^{\dagger}v_j]_i E[w(t)] \text{ for } i \neq j \]

\[ E[\hat{\xi}_i(t)^{\dagger}w(t)] v_i \text{ for } i = 1, 2 \]

\[ E[\hat{n}_t(t)^{\dagger}w(t)] v_i \text{ for } i = 1, 2 \]

The latter terms will be approximated by zero under the assumption that the uncorrelated array input processes remain uncorrelated after the operations of weighting and waveform-processing. The weighting coefficient responses are assumed here to be sufficiently smoothed, as in preceding analyses, so as to be ineffective in altering correlation properties of the input signals. Thus, Eq. (171) may be further approximated as

(172) \[ r_X(t) = E[\hat{\xi}_1(t)^{\dagger}v_1]_1 E[w(t)] v_1 \]

\[ + E[\hat{\xi}_2(t)^{\dagger}v_2] E[w(t)] v_2 \]

\[ + E[\hat{n}_t(t)^{\dagger}w(t)] E[w(t)] \]

\[ = k_1 S v_1 v_1^{\dagger} E[w(t)] + k_2 J v_2 v_2^{\dagger} E[w(t)] \]

\[ + k_3 \sigma^2 I E[w(t)] \]

\[ = [k_1 S v_1 v_1^{\dagger} + k_2 J v_2 v_2^{\dagger} + k_3 \sigma^2 I] E[w(t)] \]

where correlation coefficients have been defined as

(173) \[ k_1 = E[\hat{\xi}_1(t)^{\dagger}v_1]/S \]

\[ k_2 = E[\hat{\xi}_2(t)^{\dagger}v_2]/J \]

\[ k_3 I = E[\hat{n}_t(t)^{\dagger}w(t)]/\sigma^2. \]
These coefficients provide an ensemble-average measure of the similarity between corresponding input and reference waveforms at the error multipliers. Their relative values depend on the quality of the waveform processing while their absolute values are influenced by overall (circuit) gain in the reference processor. The processing gain to each component in the array output is assumed to be time-independent (i.e., stationary statistics in Eq. (173)) and also independent of array output conditions (which enter in Eqs. (170) and (172) as inner products involving the weight vector). Note that the matrix which multiplies the mean weight vector in Eq. (172) has the same form as the input covariance matrix. Consequently, the mean weight vector response is governed by a homogeneous differential equation:

\[
\frac{1}{\alpha} \frac{dE[w(t)]}{dt} = r_x - K_x E[w(t)]
\]

\[
= - \left[ (1-k_1)S v_1 v_1^\dagger + (1-k_2)J v_2 v_2^\dagger \right] E[w(t)]
\]

\[
= -K_x E[w(t)].
\]

The steady-state solution to Eq. (174) will be zero unless at least one of the eigenvalues of the primed matrix is zero. Thus, it is necessary to determine which values of processing gain constants \(k_1, k_2, \) and \(k_3\) generate a non-trivial result.

\[
\frac{dE[w(t)]}{dt} \bigg|_{t=\infty} = 0, \quad E[w(t)] \bigg|_{t=\infty} = g[v_1 + bv_2 + cv_n]
\]

where

\[
(a,b,c) \neq (0,0,0)
\]

\[
g \neq 0
\]

\[
v_n \perp v_1, v_2
\]

Substitution of Eq. (175) into Eq. (174) yields a system of homogeneous equations,

\[
[mS(k_1-1) + \sigma^2(k_3-1)]a + [S<v_1, v_2>(k_1-1)]b = 0
\]

\[
[J<v_1, v_2> (k_2-1)] a + [mJ(k_2-1) + \sigma^2(k_3-1)]b = 0
\]

\[
\sigma^2(k_3-1)c = 0
\]
whose solution is non-trivial when the system determinant $D$ is zero, i.e.,

$$D = \sigma^2(k_3 - 1) f(k_1, k_2, k_3) = 0$$

where

$$f(k_1, k_2, k_3) = (mS)(mJ) \left[ 1 - \frac{<v_1, v_2>^2}{m^2} \right] (k_1 - 1)(k_2 - 1)$$

$$+ \left[ mS(k_1 - 1) + mJ (k_2 - 1) \right] \sigma^2(k_3 - 1)$$

$$+ \left[ \sigma^2(k_3 - 1) \right]^2.$$  

When $k_3$ is not equal to unity, the coefficient $c$ in Eq. (176) is zero, and the gain constants must lie on the quadric surface $f = 0$ for a non-zero solution to be obtained.² That is, the gain constants must be related by

$$k_1 = 1 + \frac{\sigma^2}{mS} \left( 1 - k_3 \right) \frac{mJ}{\sigma^2} \left( \frac{1 - k_2}{1 - k_3} \right) + 1$$

$$\left( k_3 \neq 1 \right)$$

which is equivalent to the condition $f = 0$. Even when the waveform processing is perfect, i.e., $k_2 = k_3 = 0$, the gain $k_1$ to desired signal must exactly satisfy Eq. (177) to prevent the weights from decaying to zero in steady-state. In previous reference processor models which incorporated the bandpass limiter and rejected undesired signals perfectly, the processing gain to desired signal was adjusted (by the array) by varying the amplitude

²When the error signal contains no thermal noise ($k_3 = 1$), the coefficient $c$ along the noise vector $v_n$ is arbitrary. The condition $f=0$ must still be imposed for non-zero array output signals, however, since $v_n$ is orthogonal to $v_1$ and $v_2$. 
of the array output signal component (through weighting coefficient changes) relative to the fixed amplitude of the limited reference signal. In the present case when Eq. (177) is satisfied, the ratio of coefficients $b$ to $a$ in Eq. (175) is found to be given by

$$\frac{b}{a} = \frac{-\frac{m^2}{2} \frac{(1-k_2)}{(1-k_3)} \frac{<v_1,v_2>}{m}}{1 + \frac{m^2}{2} \frac{(1-k_2)}{(1-k_3)}}$$

It follows that

$$E[w(t)|_{t\to\infty} = g_0 \left[ \begin{array}{c} m^2 \frac{(1-k_2)}{(1-k_3)} \frac{<v_1,v_2>}{m} \\ 1 + \frac{m^2}{2} \frac{(1-k_2)}{(1-k_3)} \end{array} \right] v_2$$

when $f(k_1,k_2,k_3) = 0$, $k_3 \neq 1$.

The value of the scalar multiplier $g_0$ is related to initial conditions. An ideal weight vector is obtained when $k_2$ and $k_3$ have identical (but non-unity) values provided $k_1$ is on the quadric surface, Eq. (177). Other points $(k_1,k_2,k_3)$ on this surface correspond to non-ideal array performance. For example, when

$$\begin{cases} k_3 < k_2 < 1 \\
 k_3 > k_2 > 1 \end{cases} \quad \Rightarrow 0 < \frac{(1-k_2)}{(1-k_3)} < 1$$

and

$$k_1 = \text{Eq. (177)},$$

the weight vector has a smaller component along $v_2$ as illustrated in Fig. 19(b). The weight vector is more aligned with both $v_1$ and $v_2$ in this case implying a larger signal-to-thermal noise ratio at the array output, but a smaller signal-to-interference ratio.
Fig. 19.—Effect of inadequate waveform-processing gain on weight vector direction.

\[ k_2 < k_3 < 1 \]

\[ k_2 > k_3 > 1 \]

and

\[ \frac{1-k_2}{1-k_3} < \infty \]

\[ \frac{1-k_2}{1-k_3} < 1 \]

\[ k_1 = \text{Eq. (177)}, \]
the weight vector is more nearly orthogonal to $v_2$ as shown in Fig. 19(c). Interference rejection improves relative to the ideal (MMSE) performance, but the output signal-to-thermal noise ratio degrades. In both cases, the array adapts as if the interference power were

$$J' = J \frac{(1-k_2)}{(1-k_3)}$$

rather than the actual value, $J$, as seen by comparing Eq. (178) with the ideal solution, Eq. (71). Using this information, the expressions for the output power ratios are obtained by appropriately modifying Eqs. (80) and (81), i.e.,

$$\frac{P_s}{P_n} = \frac{mS}{\sigma^2} \frac{1 + m\sigma^{-2} \frac{(1-k_2)}{(1-k_3)} \left(1 - \frac{\langle v_1, v_2 \rangle^2}{m^2}\right)}{1 + m\sigma^{-2} \frac{(1-k_2)}{(1-k_3)} \left(2 + m\sigma^{-2} \frac{(1-k_2)}{(1-k_3)} \right) \left(1 - \frac{\langle v_1, v_2 \rangle^2}{m^2}\right)}$$

and

$$\frac{P_J}{P_S} = \frac{J}{S} \frac{\langle v_1, v_2 \rangle}{m}$$

where $k_1$ satisfies Eq. (177). The signal-to-total noise power ratio can be found from these expressions, if desired, using the identity

$$\frac{P_n}{P_S} = \frac{P_J}{P_S} + \frac{P_{nt}}{P_S} = \frac{P_J}{P_S} + \frac{P_{nt}}{P_S}.$$ 

The corresponding power ratios in the reference signal may be obtained after first separating each signal in Eq. (170) into a correlated component plus an uncorrelated component:

$$\hat{\xi}_1(t) = k_1 \xi_1(t) + \hat{\xi}_n(t); \quad \mathbb{E}[\hat{\xi}_1(t)\hat{\xi}_1(t)^+] = 0$$

$$\hat{\xi}_2(t) = k_2 \xi_2(t) + \hat{\xi}_n(t); \quad \mathbb{E}[\hat{\xi}_2(t)\hat{\xi}_2(t)^+] = 0$$

$$\hat{n}_t(t) = k_3 \hat{n}_t(t) + \hat{n}_n(t); \quad \mathbb{E}[\hat{n}_t(t)\hat{n}_t(t)^+] = 0.$$
The uncorrelated components contribute only to the power in the composite reference signal but do not affect performance since their products with the input signals (in the error multipliers) have zero expected values. Therefore, it follows that the power in the desired signal component of the reference signal is

\[
(P_S)_{\text{Ref}} = E[|\xi_1(t)w(t) + y_1|^2] = E[|\xi_1(t)|^2]|E[w(t)]v_1|^2
\]

\[
= k_1^2 P_S + P''_1
\]

where

- \( P_S \) = power in array output desired signal
- \( k_1^2 P_S \) = power in correlated component of reference
- \( P''_1 \) = power in uncorrelated component of reference.

Similarly,

\[
(P_J)_{\text{Ref}} = k_2^2 P_J + P''_2
\]

\[
(P_{nt})_{\text{Ref}} = k_3^2 P_{nt} + P''_{nt}.
\]

Hence, the ratio of powers in correlated components only is given by

\[
\left(\frac{P_J}{P_S}\right)_{\text{Ref.}} = \left(\frac{k_2}{k_1}\right)^2 \frac{P_J}{P_S}
\]

\[
\left(\frac{P_S}{P_{nt}}\right)_{\text{Ref.}} = \left(\frac{k_1}{k_3}\right)^2 \frac{P_S}{P_{nt}}
\]

where the ratio of squared gain constants in each expression represents the improvement provided by the waveform processor. Figure 20 shows a plot of the output ratio, Eq. (180), which incorporates these improvement ratios. Each point on the lower curve in this figure was obtained by setting \( k_3 \) equal to zero, choosing a particular value for \( k_2 \) in Eq. (180), and then calculating \( k_1 \) from
Eq. (177) to obtain the abscissa value of the point. The other curves were obtained in a similar manner. For the given array input conditions, it is apparent that the reference processing must improve the ratio of signal-to-correlated interference by at least 10 dB to maintain reasonable performance at the array output. When this condition is satisfied, the signal-to-correlated interference ratio in the reference is greater than 40 dB as noted in the figure.
Consider now the effect of bandpass limiting the imperfectly-processed signal in Eq. (170). The reference signal in this case may be represented as

$$r(t) = \sqrt{R} \frac{\xi_1'(t)w(t)^+v_1 + \xi_2'(t)w(t)^+v_2 + w(t)^+n^t(t)}{|\xi_1'(t)w(t)^+v_1 + \xi_2'(t)w(t)^+v_2 + w(t)^+n^t(t)|}.$$

The cross-correlation vector,

$$r_x(t) = E[\xi(t) r(t)^+]$$

is not readily calculated because of the magnitude-of-sum non-linearity in the denominator of Eq. (187). A justifiable approximation allowing separation of the denominator terms in the expectation is not apparent. In view of this difficulty, an approximate reference signal will be considered having the form

$$r(t) = \sqrt{R} \left[ h_1 \frac{\xi_1'(t)w(t)^+v_1}{|\xi_1'(t)w(t)^+v_1|} + h_2 \frac{\xi_2'(t)w(t)^+v_2}{|\xi_2'(t)w(t)^+v_2|} \right]$$

where $h_1$ and $h_2$ are assumed to be real-valued, constant parameters. The term contributed by thermal noise has been omitted for simplicity. This reference signal is the sum of two constant-amplitude, component signals which contain the phase-modulation of the desired signal and interfering signal at the array output. Assuming uncorrelated array input signals, the power in the reference signal,

$$E[r(t)r(t)^+] = R[h_1^2 + h_2^2],$$

is constant under the power-sharing constraint

$$h_1^2 + h_2^2 = 1.$$
is the interference-to-desired signal power ratio in the reference, as seen from Eq. (189). Assuming this ratio to be fixed in the reference signal, the interference-to-desired signal ratio which results at the array output in steady-state will be determined in the analysis to follow. The difference in the ratios obtained, i.e.,

\[ \frac{h_2}{h_1} = \left( \frac{p_j}{p_s} \right)_{\text{Ref}} \]

will then be interpreted as the amount of improvement required in the reference processor to maintain the equilibrium (steady-state) condition.

The cross-correlation vector corresponding to the reference signal, Eq. (188), may be calculated as follows:

\[ r_x(t) = \sqrt{R} \mathbb{E} \left[ \frac{v_1^+w(t)}{|v_1^+w(t)|} v_1 + \frac{v_2^+w(t)}{|v_2^+w(t)|} v_2 \right] \]

The expected value of the envelope processes will be approximated by the square-root of their mean-square values:

\[ \mathbb{E}[|\xi_1(t)|] \approx \sqrt{S}, \quad \mathbb{E}[|\xi_2(t)|] \approx \sqrt{J} \]

where the error in the interference approximation is, e.g.,
A weight vector having the form
\[ E[w(t)]_{t \to \infty} = e^{j\phi} [bv_1 + cv_2] \]
where \( \phi, b, \) and \( c \) are real constants is a solution to the differential equation (119) where Eq. (193) is used for \( r_x \) when the following condition is satisfied in steady-state:

\[
\frac{dE[w(t)]}{dt} \bigg|_{t \to \infty} = 0 = h_1 R^{1/2} e^{j \arg [b m + c v_1, v_2]} v_1 \\
+ h_2 R^{1/2} e^{j \arg [b v_2, v_1] + c m} v_2 \\
- (S v_1 v_1^\dagger + J v_2 v_2^\dagger + \sigma^2 I)(b v_1 + c v_2).
\]

Since the inner product of \( v_1 \) and \( v_2 \) is real as discussed previously, this condition reduces to a pair of non-homogeneous algebraic equations

\[
v_1: \quad b (m S + \sigma^2) + c S <v_1, v_2> = h_1 R^{1/2} \\
v_2: \quad b J <v_1, v_2> + c (m J + \sigma^2) = h_2 R^{1/2}.
\]

The solution to this system is

\[
b = \frac{h_1 R^{1/2} (m S + \sigma^2) - h_2 R^{1/2} S <v_1, v_2>}{D} \\
c = \frac{h_2 R^{1/2} (m S + \sigma^2) - h_1 R^{1/2} J <v_1, v_2>}{D}
\]

where the determinant \( D \) is

\[
D = (m S)(m J) \left[ 1 - \frac{<v_1, v_2>^2}{m^2} \right] + m(S+J)\sigma^2 + \sigma^4 > 0.
\]
After some algebraic manipulations, the weight vector may be written as

\[
E[w(t)] \bigg|_{t \to \infty} = \frac{e^{j\phi_h} h_1 \sqrt{R} \sigma^{-2} \left[ 1 - \frac{h_2}{h_1} \frac{m \sqrt{S} \sigma^{-2}}{1 + m \sigma^{-2}} \frac{<v_1, v_2>}{m} \right]}{1 + m \sigma^{-2} \left[ 1 - \frac{m \sigma^{-2}}{1 + m \sigma^{-2}} \frac{<v_1, v_2>^2}{m^2} \right]}
\]

\[
\begin{pmatrix}
    v_1 - \frac{m \sigma^{-2}}{1 + m \sigma^{-2}} \frac{<v_1, v_2>}{m} \left( \frac{h_2}{h_1} \right) v_2
\end{pmatrix}
\]

where the degradation factor \( Z \) due to interference in the reference signal is given by

\[
Z \left( \frac{h_2}{h_1} \right) = \left[ 1 - \frac{h_2}{h_1} \frac{(1 + m \sigma^{-2})}{m \sqrt{S} \sigma^{-2}} \frac{<v_1, v_2>}{m} \right] \left[ 1 - \frac{h_2}{h_1} \frac{m \sqrt{S} \sigma^{-2}}{1 + m \sigma^{-2}} \frac{<v_1, v_2>}{m} \right].
\]

Under ideal conditions,

\[ h_2 = 0, h_1 = 1 \Rightarrow Z(0) = 1, \]

and the ideal weight vector, Eq. (146), is obtained. As \( h_2 \) is increased from zero, the factor \( Z \) decreases from unity since the numerator falls off faster than the denominator in Eq. (198). The weight vector has a smaller \( v_2 \)-component and reduced overall gain as a result. The output power ratio is affected as follows:
This result is plotted in Fig. 21 as a function of the interference-to-signal ratio in the reference. The array input conditions are the same as in Fig. 20. The dotted curve in Fig. 21 represents the improvement, \( G_{rJ} \), required in the reference processor to maintain a given ratio at the array output. Upon comparing this result with the lower curve in Fig. 20, it is noted that performance is identical for equal values of reference processing gain; that is

\[
G_{rJ}
\]

implies

\[
\left| \frac{P_J}{P_S} \right|_{\text{Fig. 21}} = \left| \frac{P_J}{P_S} \right|_{\text{Fig. 20}}
\]

It follows that the steady-state weight vector has \( v_1 \) and \( v_2 \) components that are in the same ratio in both cases. The weight vector in Eq. (197), however, has a unique overall magnitude which explicitly depends on the values of the reference amplitude constants, \( h_1 \) and \( h_2 \). Recall that with linear processing there was no specific
Fig. 21.—The interference-to-signal ratio at the array output versus the correlated interference-to-signal ratio at the reference processor output.
constraint on the magnitude of the weight vector or on array output and reference signal absolute amplitudes. The constraints that were imposed yielded a steady-state error signal having desired signal, interference, and thermal noise components whose relative amplitudes and phases were such that a net voltage of zero resulted at each of the integrator inputs. If the undesired signals experienced phase shift in the linear processor, i.e., $k_2$ and $k_3$ complex-valued in Eq. (173), then a compensating phase shift of the desired signal (i.e., complex-valued $k_1$ satisfying Eq. (177)) was required to maintain non-zero equilibrium conditions. In the present analysis for the case of non-linear processing, the parameters $h_1$ and $h_2$ have been restricted to real values. Note that the algebraic equations, Eq. (195), do not have a real-valued solution $(b, c)$ when $h_1$ and/or $h_2$ are complex. If complex values are assumed for the unknowns, then non-linear (phase) factors must be retained on the right-hand side of Eq. (195), as in Eq. (194). Although a solution has not been obtained for this case, it is conjectured that no solution exists in the form assumed in Eq. (194). A time-periodic solution might be required judging from the results in Section D.
CHAPTER V
THEORETICAL PERFORMANCE WITH WIDEBAND NOISE INTERFERENCE

A. Introduction

In this chapter, array performance is analyzed for the case in which the interfering signal is a wideband noise process. The desired objective in the analysis is to calculate the degradation in signal-to-total noise ratio at the array output as the bandwidth of the interference is increased. This degradation results from array bandwidth limitations, i.e. complete cancellation of the differentially-delayed interfering signals in successive elements of the array is not possible with weighting and combining operations. In addition, small processing delays in the weighting and combining circuits, in the array output amplifier, and in the error combiner and dividers may have an appreciable effect on the correlation of wideband interference in the error multiplier. These processing delays will be modeled in the analysis by a lumped time delay of $\delta_f$-seconds in the feedback path of the error signal. In order to focus attention exclusively on the effects of interfering signal bandwidth, the desired signal will be assumed narrowband with respect to array bandwidth. Also, a replica of the narrow-band desired signal will be assumed as a fixed reference signal for the array. Following the derivation of a general expression for the output signal-to-total noise ratio, specific numerical results will be presented for the case of a four-element, linear array having negligible processing loop delay ($\delta_f = 0$).

B. Performance Analysis

The array model in Fig. 4 may be modified to include loop delay by inserting the transfer function

$$H_d(\omega) = e^{-j(\omega - \omega_c)\delta_f}$$

in the error signal's path. The delay is phase-compensated so as to introduce zero phase-shift at the array center frequency. The phase-compensated, delayed error signal has a complex envelope given by

$$[\tilde{\varepsilon}(t-\delta_f)e^{-j\omega_c\delta_f}] e^{+j\omega_c\delta_f} = \tilde{\varepsilon}(t-\delta_f)$$

$$= \tilde{r}(t-\delta_f) - w(t-\delta_f)^\dagger \tilde{\gamma}(t-\delta_f)$$
where
\[ \tilde{r}(t - \delta_f) = \sqrt{\frac{R}{S}} \tilde{\xi}_1(t - \delta_f) \]

and
\[ \tilde{x}_k(t - \delta_f) = \tilde{\xi}_1(t - \delta_f - \tau_{1k}) e^{-j\omega_c \tau_{1k}} \]
\[ + \tilde{\xi}_2(t - \delta_f - \tau_{2k}) e^{-j\omega_c \tau_{2k}} \]
\[ + \tilde{\xi}_k(t - \delta_f) \quad k = 1, 2, \ldots, m. \]

The appropriate differential equation for this delayed-error model is
\[ \frac{1}{\alpha} \frac{dE[w(t)]}{dt} = E[\tilde{x}(t) \tilde{r}(t - \delta_f)^\dagger] \]
\[ = E[\tilde{x}(t) \tilde{r}(t - \delta_f)^\dagger] - E[\tilde{x}(t) \tilde{x}(t - \delta_f)^\dagger] \cdot E[w(t - \delta_f)]. \]

The asymptotic, mean weight vector response follows from
\[ \frac{d}{dt} E[w(t)] \bigg|_{t \to \infty} = 0 \]
as
\[ E[w(t)] \bigg|_{t \to \infty} = E[w(t - \delta_f)] \bigg|_{t \to \infty} = \]
\[ E[\tilde{x}(t) \tilde{x}(t - \delta_f)^\dagger]^{-1} E[\tilde{x}(t) \tilde{r}(t - \delta_f)^\dagger] = K_x(\delta_f)^{-1} r_x(\delta_f). \]

Under the assumptions of uncorrelated input signals and a narrowband desired signal, the cross-correlation vector may be approximated as
This, of course, is the ideal cross-correlation vector. The matrix of correlation functions evaluated at lag $\delta_f$ in Eq. (203) has an $ij$th element which can be expressed as follows:

\begin{equation}
[K_x(\delta_f)]_{ij} = E[\xi_1(t-\tau_{1i})\xi_1(t-\tau_{1j}-\delta_f)] e^{-j\omega_c(\tau_{1i}-\tau_{1j})} + E[\xi_2(t-\tau_{2i})\xi_2(t-\tau_{2j}-\delta_f)] e^{-j\omega_c(\tau_{2i}-\tau_{2j})} + E[\xi_1(t)\xi_1(t-\delta_f)]
\end{equation}

\begin{equation}
= S[v_1v_1^+]_{ij} + R_{\xi_2}^\nu (\tau_{2i}-\tau_{2j}-\delta_f) [v_2v_2^+]_{ij} + R_{\xi_1}^\nu (\delta_f) \cdot \delta_{ij}.
\end{equation}

The narrowband approximation has been applied to the desired signal term of this expression. The interference and thermal noise processes are assumed to have constant power spectral densities over bandwidths of $b_j$ and $b_n$ Hertz, respectively, about the center frequency $\omega_c$; therefore,

\begin{equation}
R_{\xi_2}^\nu (\tau) = \frac{\sin \pi b_j \tau}{\pi b_j \tau},
\end{equation}

\begin{equation}
R_{\xi_1}^\nu (\tau) = \sigma^2 \frac{\sin \pi b_n \tau}{\pi b_n \tau}, \ i = 1, 2, \cdots, m
\end{equation}

where the condition $b_j \leq b_n$ is imposed in practice since the rf amplifiers in the array elements limit the bandwidth of the interfering signals applied to the adaptive processor. For purposes of analysis, the bandwidths will be assumed equal. Substitution of Eq. (206) into Eq. (205) yields a matrix with elements
The diagonal elements have reduced interference and thermal noise components as a result of loop delay. The interference components in off-diagonal pairs of elements are affected unequally by loop delay as illustrated in Fig. 22. The interference component is increased in one off-diagonal element $(ij)$ and decreased in the conjugate element $(ji)$. Thus, the symmetric (Hermittian) character of the matrix is not retained with non-zero loop delay. The weight vector may still be expanded, however, as in Eqs. (68) and (69):

\[(207) \quad [K_x(\delta_f)]_{ij} = S[v_1v_1^+]_{ij} + \frac{\sin \pi b_j (\tau_{2i} - \tau_{2j} - \delta_f)}{\pi b_j (\tau_{2i} - \tau_{2j} - \delta_f)} [v_zv_2^+]_{ij} + \sigma^2 \frac{\sin \pi b_x \delta_f}{\pi b_x \delta_f} \delta_{ij} .\]

Fig. 22.--An illustration of the non-symmetry in the off-diagonal elements of the noise covariance matrix.
The expressions for the signal power and the total noise power at the array output follow from Eqs. (208), (34), and (39) as

\[
\begin{align*}
P_n &= E[w(t)]^\dagger S v_1 v_1^\dagger E[w(t)] \\
&= [g(\delta_f)\sigma^2]S [K_n(\delta_f)^{-1}v_1]^\dagger v_1 v_1^\dagger [K_n(\delta_f)^{-1}v_1] \\
&= [g(\delta_f)\sigma^2]S [v_1^\dagger K_n(\delta_f)^{-1}v_1]^\dagger [v_1^\dagger K_n(\delta_f)^{-1}v_1] \\
&= [g(\delta_f)\sigma^2]S [v_1^\dagger v_1^\dagger K_n(\delta_f)^{-1}v_1]^\dagger [v_1^\dagger K_n(\delta_f)^{-1}v_1]
\end{align*}
\]

and

\[
\begin{align*}
P_s &= E[w(t)]^\dagger K_n(0) E[w(t)] \\
&= [g(\delta_f)\sigma^2]S [K_n(\delta_f)^{-1}v_1]^\dagger K_n(0)[K_n(\delta_f)^{-1}v_1].
\end{align*}
\]

The ratio of Eqs. (209) and (210) is the desired result,

\[
\frac{P_s}{P_n} = \left(\frac{mS}{\sigma^2}\right) \frac{|v_1^\dagger \sigma^2 K_n(\delta_f)^{-1}v_1|^2}{m[\sigma^2 K_n(\delta_f)^{-1}v_1]^\dagger [\sigma^2 K_n(\delta_f)^{-1}v_1]}.
\]

C. Numerical Evaluation

The output signal-to-total noise ratio has been evaluated numerically for the special case of a four-element, equispaced linear array having zero loop delay. Under these conditions, Eq. (211) simplifies to
\[
\frac{P_S/P_n}{mS\sigma^2} = \frac{1}{m} \left. v_1^+ \sigma^2 K_n(0)^{-1} v_1 \right|_{\delta_f=0}
\]

where \( m = 4 \),

\[
K_n(0)_{ik} = \frac{j \sin \pi(j-k)B}{\sigma^2} \left( \frac{1}{(i-k)B} e^{-j[(i-k)Z_2]} + \delta_{ik} \right)
\]

\[ B = b_j \frac{\rho}{c} \cos \theta_j, \]

\[ Z_2 = 2\pi \frac{\rho}{\lambda_c} \cos \theta_j, \]

and

\[ v_1^+ = \left[ e^{-j\frac{3}{2}Z_1} e^{-j\frac{1}{2}Z_1} e^{+j\frac{1}{2}Z_1} e^{+j\frac{3}{2}Z_1} \right] \]

\[ Z_1 = 2\pi \frac{\rho}{\lambda_c} \cos \theta_s. \]

The parameter \( B \) defined in the noise covariance matrix, Eq. (213), is the product of interference bandwidth times the differential element delay at the angle-of-arrival of the interfering signal. For this special case of zero loop delay, the matrix, Eq. (213), is Hermitian, and the quadratic form, Eq. (212), is dependent only on the absolute difference in the parameters \( Z_1 \) and \( Z_2 \):

\[
\frac{P_S/P_n}{mS\sigma^2} = D(|Z_1-Z_2|, J/\sigma^2, B),
\]

where \( \psi = |Z_1 - Z_2| = 2\pi \frac{\rho}{\lambda_c} |\cos \theta_s - \cos \theta_j| \).

The parameter \( \psi \) defines the angular separation of sources as noted previously in Eq. (82). Figures 23-25 illustrate the calculated performance versus \( \psi \) and \( B \) for fixed values of interference power. The interference power is varied in Figs. 26 and 27. In the limit as the parameter \( B \) approaches zero corresponding to narrowband interference (\( b_j = 0 \)), or wideband interference broadside to the array (\( \theta_j = 90^\circ \)), the output performance approaches that of the narrowband case given by Eqs. (79) and (82). In this limit, the output ratio is bounded by the values
Fig. 23.—The normalized output signal-to-total noise ratio versus the bandwidth parameter $B$; $J/\sigma^2 = 0$ dB.
Fig. 24.—The normalized output signal-to-total noise ratio versus the bandwidth parameter $B$; $J/\sigma^2 = 10$ dB.
Fig. 25.--The normalized ratio of output signal-to-total noise power versus the bandwidth parameter $B$; $m = 4, \frac{J}{\sigma^2} = 20 \text{ dB}$.
Fig. 26.—The normalized output signal-to-total noise ratio versus the input interference-to-thermal noise ratio;  
\[\psi = \pi/2.\]
Fig. 27.--The normalized output signal-to-total noise ratio versus the input interference-to-thermal noise ratio; 
\[ \psi = \pi/3. \]
The parameter $B$ increases toward unity as the bandwidth of the (off-broadside) interfering signal is increased. The normalized covariance matrix, Eq. (213), approaches a diagonal matrix in this case, and the mean weight vector, Eq. (208), approaches the co-phased array solution

\[
P_{\text{sn}} = \begin{cases} 
\frac{mS}{\sigma^2} \cdot 1; & \psi = 90^\circ, \ B = 0 \\
\frac{mS}{\sigma^2} \cdot \frac{1}{1 + mJ\sigma^{-2}}; & \psi = 0^\circ, \ B = 0.
\end{cases}
\]

In this limit, the interfering signals in different elements are completely uncorrelated; the weight-control loops are uncoupled with respect to the interference and respond as if an additional independent element noise of power $J$ were present in each element. The uncorrelated interference processes combine (incoherently) at the array output resulting in an output ratio

\[
P_{\text{sn}} = \frac{mS}{\sigma^2 + J} = \frac{mS}{\sigma^2} \cdot \frac{1}{1 + J\sigma^{-2}}; \quad B = 1.
\]

The curves in Figs. 23-25 all converge to this value. Note that the lowest curve ($\psi=0^\circ$) in each figure represents the case of zero angular separation of sources. For this case, the weight vector is along $v_1$ when the parameter $B$ is either zero or unity. Array output performance actually improves as the bandwidth of the interfering signal is increased due to the transition from coherent addition of narrowband (correlated) interference ($B=0$) to incoherent addition of wideband (uncorrelated) interference ($B=1$). This transition is apparent upon comparing Eq. (215) with Eq. (216). It is also evident in Figs. 26 and 27 that for fixed interference bandwidth, the performance degradation increases as the power in the interfering signal is increased. For a 20 dB input interference-to-thermal noise ratio as in Fig. 25, the performance for small values of $B$ is given in Table 2.
TABLE 2

The Degradation in dB of the Array Output Signal-to-Total Noise Ratio from its Ideal Value (4S/σ^2)

<table>
<thead>
<tr>
<th>Ψ</th>
<th>90°</th>
<th>60°</th>
<th>45°</th>
<th>30°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.9</td>
<td>2.4</td>
<td>5.2</td>
</tr>
<tr>
<td>0.00555</td>
<td>0.2</td>
<td>1.3</td>
<td>2.9</td>
<td>5.6</td>
</tr>
<tr>
<td>0.025</td>
<td>0.7</td>
<td>3.2</td>
<td>5.0</td>
<td>7.5</td>
</tr>
<tr>
<td>0.05</td>
<td>1.2</td>
<td>5.0</td>
<td>7.3</td>
<td>11.2</td>
</tr>
</tbody>
</table>

m = 4, \quad Jσ^{-2} = 20 \text{ dB}

The top line of the table gives the ideal, narrowband interference performance. The additional degradation due to non-zero interference bandwidth is less than or equal to 0.5 dB in the second line of the table, and less than 3 dB in the third line. These values of B, i.e.,

\[
B = \frac{b_J}{f_c} \frac{\rho}{\lambda_c} \cos \theta_J,
\]

correspond to amplifier noise bandwidths in the array elements of approximately one-percent and five-percent of f_c, respectively, for half-wavelength element spacing and worst-angle (endfire) interference. (Recall that b_J/f_c is equal to the normalized rf noise bandwidth b_R/f_c). Additional degradation will occur with larger interfering signal powers and with desired signals which are not narrowband as assumed here. Thus, a practical upper limit for the parameter B appears to be somewhere in the range 0.005 to 0.025. Of course, the amount of signal-to-noise ratio loss that can be tolerated at the array output without significantly disrupting communications will influence the choice of the parameter B. For an array located at the input to a hard-limiting, repeater satellite, the array output (limiter input) signal-to-noise ratio must exceed approximately five decibels to avoid significant signal power suppression in the limiter. The desired signal-to-thermal noise
ratio in each array element and array size determine the maximum array output ratio \((mS/\sigma^2)\) and hence the margin available. A reduction in amplifier bandwidth \((b_n)\) in the array elements improves the element signal-to-thermal noise ratio \((S/\sigma^2)\) and decreases the (maximum) bandwidth of the interfering signals applied to the adaptive processor, i.e., the parameter \(B\) decreases. Thus, interference rejection and output margin can both be increased at the expense of system bandwidth. The bandsplreasing ratio in the reference waveform processor is reduced as the system bandwidth is lowered, however, so that a compromise appears necessary under low signal-to-thermal noise ratio conditions.

With the above results as an analytical base, a restriction on the maximum permissible value of loop delay will now be imposed. At frequencies corresponding to the bandedge of the array element amplifiers, the phase shift introduced by the loop delay must not exceed ninety degrees:

\[
(217) \quad \left| (\omega - \omega_c) \delta_f \right| = \left| \pm \pi b_n \delta_f \right| < \frac{\pi}{2}.
\]

If the phase shift were greater than ninety degrees at these frequencies, an interfering signal having only these two spectral components would not be rejected. The feedback around each weight-control loop would be positive rather than negative in this case since the interference products at the error multiplier outputs would have the wrong polarity. Consequently, an upper limit on loop delay for stability is

\[
(218) \quad \delta_f \leq \frac{1}{2b_n}
\]

When loop delay is equal to this maximum value, the diagonal elements of the correlation matrix, Eq. (207), have interference and thermal noise components which are reduced in magnitude by the factor

\[
\frac{\sin \pi b_j \delta_f}{\pi b_j \delta_f} = \frac{\sin \pi b_n \delta_f}{\pi b_n \delta_f} = \frac{1}{\pi/2} = \frac{2}{\pi}.
\]

The off-diagonal elements in Eq. (207) have interference components reduced by the factors
where \( b_j = b_n \) has been assumed. With the recommended maximum value of 0.025 for the parameter \( B \), the phase arguments, \( \pi(i-k)B \), in Eq. (219) due to array differential delays are multiples of 4.5°. The maximum loop delay in Eq. (218) adds ninety degree phase arguments in Eq. (219) thus shifting the magnitude of the off-diagonal interference components well down the correlation characteristic of Fig. 22. It is apparent that loop delay must be much less than the maximum value in Eq. (218) in order to negligibly affect the matrix elements. Although a detailed evaluation of Eq. (211) have not been completed at present, there is some experimental evidence which indicates that additional degradation in wideband noise rejection due to the effects of loop delay are minimal. In section G of Chapter VI, it will be shown that the measured performance in many experiments is nearly the same as calculated performance for zero loop delay - even when the product \( b_j\delta_f \) is near the maximum value of one-half in Eq. (218).
A. Introduction

The performance obtained with an experimental implementation of a four-element adaptive processor will be discussed in this chapter. The results to be presented were obtained by bench-testing the adaptive processor using signals which simulate antenna element output signals. This approach simplified the experimental investigations by eliminating from consideration those performance characteristics associated exclusively with the antenna subsystem, e.g., antenna element patterns and mutual coupling effects. Moreover, the parameters of the input signals were readily changed and accurately controlled with this approach. The method of generating the simulated element-output signals is discussed in the following section. The configuration and parameters of the adaptive processor are also described. Section C contains a description of the signal processing electronics implemented to generate the reference signal. In order to model the case of an adaptive antenna system operating in a hard-limiting repeater satellite, the signal at the adaptive processor's output was applied to a bandpass limiter. The limiter output signal, in turn, was attenuated and applied to a differential (DPSK) detector to simulate reception and detection of the down-link signal from the satellite. The configuration of this post-array subsystem is described in Section D. Calibration measurements relating the average bit error probability (BEP) performance of this subsystem to the power ratios at the array output (Pj/Ps and Ps/Pnt) are also presented in this section. These calibration measurements are employed in later sections to transform the theoretical performance of the array into an estimated BEP performance for purposes of comparison with the measured BEP performance. The experimental results, beginning in Section E, are presented in approximately the same order as the analyses of the previous chapters. For example, transient response times and steady-state performance with CW interference are discussed first. The effects of multiplier offset voltages, data delay errors and code timing offsets in the reference processor, and signal frequency offsets are then described. The measured BEP performance with pulse interference is presented in Section F, and the effect of wideband noise interference on array performance is discussed in Section G.
B. Adaptive Processor Description

1. Input signal synthesis

The simulated element output signals were synthesized as shown in Fig. 28. The output of a signal generator (the desired signal 30 MHz CW source) was bi-phase modulated by the modulo-two sum of a pseudonoise code and a pseudonoise data sequence. The clocking rates of the code and data sequences were 500 KHz and 50 KHz, respectively, corresponding to a bandspreading ratio of ten. Seven-stage shift registers were employed in both cases to generate sequences of maximal-length: 127. The interfering signal source was also a CW signal generator for the narrowband interference experiments. Narrowband pulse interference was generated by amplitude modulating the source output signal as shown. The signal at the output of each modulator was split into four equi-amplitude, equi-phase signals in wideband (10-100 MHz) hybrid dividers. For the narrowband interference tests, the connections in Fig. 28 were "A" to "1" and "B" to "2". Equal lengths of coaxial cable were used at the outputs of the upper divider to simulate the condition of broadside incidence of the desired signal. The narrowband interfering signals (CW or pulsed) were progressively phased in manual phase-shifters to simulate off-broadside incidence of interference on a linear array of half-wavelength spacing. The desired and interfering signals associated with a given element were summed in wideband hybrids with the output of a (simulated element) noise source having a 3 dB-bandwidth of 1.5 MHz, centered at 30 MHz. The range of the manual phase-shifters permitted element-to-element phase shifts, \( \psi \), from zero to seventy electrical degrees. Consequently, it was possible to vary the simulated arrival angle of the interference within the main lobe of the array broadside pattern as shown in Fig. 29. For an angular separation of desired signal and interference of 13.1 spatial degrees - one-half the half-power beamwidth in Fig. 29 - the value of \( \psi \) is 41 electrical degrees/element.

A noise source having a 3 dB-bandwidth of 9 MHz, centered about 30 MHz, was the interfering signal source for the wideband interference experiments. The ten percent bandwidth (3 MHz) of the manual phase-shifters was insufficient to permit use of these devices with the wideband interfering signals.\(^1\) Consequently, the connections in

\(^1\)The amplitude roll-off of the phase-shifters over a thirty percent frequency range was minimal; however, the phase dispersion introduced by these devices effectively decorrelated the interfering signals almost completely as indicated by a result in Section G.
Fig. 28.—Synthesis of simulated array element output signals.
Fig. 29.--Broadside pattern of a four-element, $\lambda/2$-spaced linear array.
Fig. 28 were reversed, i.e., "A" to "2" and "B" to "1", permitting the phase-shifters to progressively phase the more narrowband desired signal (whose \( \sin x/x \)-type spectrum had first nulls separated by twice the code rate or 1 MHz). The interfering signals were progressively delayed using selected, unequal lengths of coaxial cable to simulate off-broadside arrival directions. By changing these cable lengths, the bandwidth parameter, \( B \), of the previous chapter was effectively varied. This parameter was also varied by reducing the 9 MHz noise spectrum down to 3 MHz by passively filtering the output of the noise source. As in the narrowband experiments, the phase shifters were used to vary the separation parameter, \( \psi \).

2. Adaptive processor configuration

A simplified block diagram of the experimental adaptive processor is shown in Fig. 30. Wideband hybrid dividers/combiners were employed extensively in lieu of active circuits to minimize packaging and impedance mismatch problems. The quadrature hybrids at the processor inputs were thirty percent bandwidth devices having outputs within one degree of quadrature from 25 MHz to 35 MHz. All other hybrids in Fig. 30 were 10-100 MHz devices. A schematic of the circuitry employed to weight each input signal and to generate the corresponding weight-control voltage is given in Appendix D. Four-quadrant, variable-transconductance multipliers were employed to implement both the signal weighting and error-by-signal multiplication functions. The operational integrator in each channel was capable of being operated in one of three modes - initial condition set, integrate, or hold - depending on the states of two digital, mode-control voltages applied to associated control circuitry. All of the (eight) integrators in the processor were in a common operating mode at any given instant of time. Short lengths of coaxial cable were used to interconnect the various components in Fig. 30. The length of the cable connecting the error monitor hybrid to the eight-way error divider was selected to provide a total phase shift of 180° around the feedback loops at 30 MHz.

Measurements were conducted to determine the parameters of the experimental processor. The measured values are shown in Table 3. The gain constants associated with the multipliers were measured under conditions where the signal amplitudes at the multiplier inputs did not exceed the linear operating range of these circuits. Fig. 31 illustrates the linearity versus input amplitude of a typical weight multiplier having a fixed weight-control voltage of one volt. The output is compressed approximately one decibel when the signal amplitude at the quadrature hybrid input is 50 mV rms. When the weight-control voltage is increased, equivalent compression occurs at a smaller input level since the maximum linear output of the weight multiplier circuit is approximately constant. For most experiments, the amplitude of the desired input signal was selected to be much less than the maximum value, i.e.,
Fig. 30.—Block diagram of the four-element adaptive processor.
Table 3
The Parameters of the Experimental System

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Notation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight-multiplier gain constant</td>
<td>$G_W$</td>
<td>0.25 (dc-volts)$^{-1}$</td>
</tr>
<tr>
<td>Net voltage gain of summing network</td>
<td>$G_S$</td>
<td>6.0</td>
</tr>
<tr>
<td>Net voltage gain of error generation and distribution network</td>
<td>$G_d$</td>
<td>0.155</td>
</tr>
<tr>
<td>Error-multiplier gain constant</td>
<td>$G_m$</td>
<td>$5550 \frac{dc-volts}{(rms-volts)^2}$</td>
</tr>
<tr>
<td>Integrator gain constant</td>
<td>$G_i$</td>
<td>$10^4$ (sec)$^{-1}$</td>
</tr>
<tr>
<td>Loop gain constant</td>
<td>$\alpha$</td>
<td>$1.29 \times 10^7 \frac{(rms-volts)^2 sec}{sec}$</td>
</tr>
<tr>
<td>Loop delay</td>
<td>$\delta_f$</td>
<td>56 nsec</td>
</tr>
<tr>
<td>Error multiplier output bandwidth</td>
<td></td>
<td>4 MHz (-3 dB)</td>
</tr>
<tr>
<td>Limited reference signal amplitude</td>
<td>$\sqrt{R}$</td>
<td>35 mV rms</td>
</tr>
</tbody>
</table>
Fig. 31. The linearity of a signal weighting circuit versus the input signal amplitude.

\[
\left( \frac{A_s}{V_{\text{nominal}}} \right) = (\sqrt{5})_{\text{nominal}} = 3.5 \, \text{mV rms}.
\]

The effect of input interference exceeding the dynamic range capabilities of the weight multipliers will be noted later. The time delay around the feedback loops was computed from measurements shown in Figs. 32 and 33. The phase shift-versus-frequency characteristic of the network beginning at a weight multiplier input (quadrature hybrid output) and ending at various points around the loop is illustrated in Fig. 32. The upper curve in this figure represents the total phase shift around a feedback loop except for the phase shift within the error multiplier itself. This curve is nearly linear with a slope of 72°/4 MHz at 30 MHz indicating a total loop delay of approximately fifty nanoseconds. Figure 33 shows the change in error multiplier output voltage versus the input signal frequency. The reference signal and the input signals to the seven other weights.
Fig. 32.--The phase shift-versus-frequency characteristics of a weight-control loop.
Fig. 33.--The change in error multiplier output voltage versus input signal frequency.
were set to zero for this measurement. At frequencies corresponding to ±90° phase shift, the output voltage is zero. These frequencies are separated by 8.9 MHz indicating that loop delay is approximately equal to the reciprocal of 17.8 MHz, or 56.2 nsec.

C. Reference Signal Processor Description

A block diagram of the waveform-processing electronics employed to generate the reference signal is shown in Fig. 34. Linear integrated circuits (the TI SN76514L and RCA CA3049) were used to instrument all active stages in the path of the input signal. The bandpass filter had a 3 dB bandwidth of approximately 200 KHz (four times the data rate) and a phase shift-versus-frequency slope of 90°/50 KHz at midband as evidenced by the measured characteristics in Fig. 35. The overall slope of the waveform processor's phase characteristic was 90°/45 KHz near the 30 MHz center frequency. Bi-phase data transitions in the desired signal at the array output were delayed slightly more than six microseconds in the waveform processor as shown by the upper and lower oscilloscope traces of Fig. 36. The output-versus-input amplitude characteristic of the waveform processor is illustrated in Fig. 37. This curve was obtained with desired signal only at the waveform processor's input and with the reference PN code timing offset adjusted to zero. Complete limiting occurred for an input amplitude equaling the limited output amplitude: 50 mV peak. The voltage gain was unity in this case and increased to a maximum value of 1.5 for smaller input amplitudes. For larger input amplitudes, the gain was less than unity. The filter at the waveform processor's output was tuned to 30 MHz and rejected the upper sideband components (around 40.7 MHz) of the output signal from the second mixer. A selected length of coaxial cable at the waveform processor's output provided the 180° phase relationship (at 30 MHz) required of the two signals at the error hybrid inputs.

D. Bit Error Probability Performance Measure

1. Post-array subsystem configuration

In addition to oscilloscope and spectrum analyzer measurements at the adaptive processor output, the subsystem configured as shown in Fig. 38 was used to evaluate array performance under steady-state conditions. The array output signal was successively bandpass limited, attenuated, summed with wideband noise, and applied to the input of a differential (DPSK) detector. A detailed description of the differential detector and a documentation of its basic performance is given in Reference [25]. The measurements of detector performance in the configuration of Fig. 38 will be presented below. Note that the pseudonoise (PN) codes applied to the detector's local oscillator signal and to the array's reference waveform processor had a common
Fig. 34.--Block diagram of the reference waveform processor.
Fig. 35.—The frequency response of the data bandwidth filter in the reference waveform processor.
Fig. 36.—Waveforms illustrating biphase data modulation delay in reference waveform processor.

Top: Reference monitor, 50 mV/cm
Middle: Array output, 50 mV/cm
Bottom: Error Monitor, 20 mV/cm
Horizontal: 2 μsec/cm.
Fig. 37.--The input-output amplitude characteristic of the reference waveform processor.
Fig. 38.--System configuration for evaluating the steady-state performance of the adaptive processor.
time base. The time base of the array input signal was variable to permit investigation of the effects of code timing offsets. Also note that the PN data sequence applied to the input signal's bi-phase modulator was not differentially encoded. In lieu of this operation, the detected sequence was differentially-decoded prior to its application to the error counter.

2. Post-array subsystem performance

The measurements to be discussed in this section were obtained with the adaptive processor operating in the initial condition mode (i.e., with constant weight-control voltages). In this mode, the array acts simply as a fixed transmission path for the input sources of desired signal, interference, and thermal noise. The measurements in Fig. 39 show that applying PN coding to the differential detector's input and local oscillator signals causes only a slight increase in bit error probability (BEP) when the code timing offset, $e$, is adjusted to zero. The data points shown were obtained by incrementing the post-limiter attenuator in 0.5 dB steps with only desired signal present at the limiter input (array output). With the attenuator fixed at a value corresponding to a detector input energy-to-noise density ratio, $\text{E}_b/\text{N}_0$, of approximately 10.2, and a measured BEP of $4.5 \times 10^{-5}$ in Fig. 39, CW interference was added to the array output signal. The measurements in Fig. 40 indicate the interference immunity afforded by waveform processing at the differential detector. The BEP was measured as a function of interference-to-signal ratio ($P_I/P_S$) at the limiter input -- both with and without PN coding applied to the signal. The results show that a bandwidth spreading ratio of ten provides only about 3 dB of interference protection: the horizontal separation of the two curves. This amount of waveform processing gain is clearly inadequate since the BEP increases three orders of magnitude as the limiter input ratio increases to unity (0 dB).

The next set of measurements shown in Figs. 41 and 42 were obtained with the post-limiter attenuator set at a smaller value corresponding to a differential detector input ratio ($\text{E}_b/\text{N}_0$) of thirteen in Fig. 39 and a measured BEP of $4 \times 10^{-6}$. The latter BEP can be noted at the extreme left of the lowermost curve in Fig. 41(a). Noise from the (element) noise sources was added to the array output signal and the BEP was measured as a function of both the signal-to-thermal noise ratio ($P_S/P_{nt}$) and the interference-to-signal ratio ($P_I/P_S$) at the limiter input. The measured error probabilities

\[ \text{BEP} = \frac{2}{A} \]  

A 50 KHz data clock signal having this same fixed time base was applied to the differential detector for bit timing purposes.
Fig. 39. -- Performance of the differential detector with and without PN coding applied to the input and local oscillators signals. Code timing offset ε = 0.
Fig. 40.—Measurements showing the interference protection afforded by waveform processing at the differential detector for a bandspreading ratio of 10:1.
Fig. 41(a).--Performance of the limiter-detector subsystem versus the limiter input ratios of interference-to-signal and signal-to-thermal noise.
Fig. 41(b).--Performance of the limiter-detector subsystem versus the limiter input ratios of interference-to-signal and signal-to-thermal noise.
Fig. 41(c).--Performance of the limiter-detector subsystem versus the limiter input ratios of interference-to-signal and signal-to-thermal noise.
Fig. 42.--Performance of the limiter-detector subsystem versus the limiter input signal-to-thermal noise ratio; no interference present.
were very regular versus these two ratios enabling smooth curves to be drawn as shown in Figs. 41(a), (b), and (c) through the ensemble of data points. PN coding was applied to the desired signal and to the detector local oscillator signal during these measurements and the code timing offset was adjusted to zero. The frequency of the interfering signal could be varied several kilohertz about 30 MHz without any noticeable effect on the measured error probabilities. Figure 42 illustrates detector performance when the interfering signal was not present at the input to the limiter: the left end of the curves in Fig. 41. The increase in BEP as the signal-to-thermal noise ratio \( \frac{P_s}{P_{nt}} \) was reduced reflects both the suppression of desired signal at the limiter output (the power-sharing effect) and the increase in limiter output noise. At the differential detector input, the ratio of the spectral density of the added noise components to the spectral density of the limiter noise components was greater than 10 dB for limiter input ratios exceeding 3 dB. This ratio of spectral densities decreased only to 4.4 dB (2.75 numeric) when the limiter input was thermal noise only \( \left( \frac{P_s}{P_{nt}} = -\infty \, \text{dB} \right) \). Thus, the energy-to-noise density ratio at the detector input in the absence of interference was given approximately by

\[
\frac{E_b}{N_0} \approx 13 \, \frac{\alpha^2}{1 + \frac{(1-\alpha^2)}{2.75}}
\]

where \( \alpha^2 \) is the limiter power suppression of desired signal,

\[
0 \leq \alpha(p) \leq 1, \quad p = \frac{P_s}{P_{nt}}.
\]

When Davenport's result[26] for the limiter suppression factor, \( \alpha(p) \), is used in Eq. (220) and the BEP is calculated from the measured BEP-versus-\( E_b/N_0 \) characteristic in Fig. 39, the resulting BEP-versus-\( p \) curve agrees favorably with the measured result as shown in Fig. 42.

E. Experimental Performance with Narrowband Signals

1. Transient performance

The transient response of the adaptive processor was evaluated by alternately switching between initial condition and integrate modes of operation. The duration of each mode was longer than the response time of the processor. Figure 43(a) illustrates oscilloscope traces of the fifth through eighth weighting coefficients when only element noise is present at the inputs having an amplitude of 3 mV rms. At the beginning of each trace, the array is switched to
Fig. 43.--Transient response of weight-control voltages $w_5, w_6, w_7, w_8$ (top to bottom).

Desired signal amplitude:
(a) zero, (b) 5 mV peak, (c) 16 mV peak

Vertical:
5 volts/cm

Horizontal:
(a) 10 msec/cm, (b) 2 msec/cm, (c) 0.5 msec/cm
the initial condition mode and the weight-control voltages assume a value of -4 volts. The response time constant when the array is switched to the adapt mode is approximately 13 msec compared to the calculated value of 8.6 msec:

\[ [a\sigma^2]^{-1} = [1.29 \times 10^7 (3 \times 10^{-3})^2]^{-1} \text{ sec} \]
\[ = 8.6 \text{ msec}. \]

In Fig. 43(b), a desired signal with amplitude equal to 3.5 mV rms was added to the array inputs resulting in an input signal-to-thermal noise ratio \((S/\sigma^2)\) of 1.5 dB. The response time constant in this case is approximately 2 msec compared to the calculated value of 1.3 msec:

\[ [a(\sigma^2 + mS)]^{-1} = [a\sigma^2]^{-1} \left[1 + \frac{mS}{\sigma^2}\right]^{-1} \]
\[ = 8.6 \text{ msec} \left[1 + 4(1.41)\right]^{-1} \]
\[ = 1.3 \text{ msec}. \]

The calculated response, Eq. (61), for this case is a single exponential with the above time constant since the initial weight vector is in the broadside direction: the same direction as required in steady-state. Only a change in the overall gain of the weight vector is required during the transient.\(^3\) The signal amplitude was increased 10 dB in Fig. 43(c) resulting in a 270 usec response time constant compared to the calculated value of 150 usec. An additional 10 dB amplitude increase (35 mV rms input amplitude) yielded a weight response having a 50 usec time constant compared to the calculated value of 15 usec. The increased deviation between calculated and measured response times as the input signal amplitude is increased is attributed primarily to circuit saturation effects and secondarily to component bandwidth (rise-time) limitations, e.g., integrator slew-rate limiting.

When CW interference was added to the array inputs, the weighting coefficients responded as shown in Fig. 44. An interference amplitude of 35 mV rms in Fig. 44(b) resulted in a very fast initial response to null the array output interference. The duration of this initial transient was approximately 150 usec as observed on an

---
\(^3\)The constants \(g_2(t_0)\) and \(h_\eta(t_p)\) in Eq. (151) are zero in this case and the constant \(\phi_0\) equals forty-five degrees.
Fig. 44.—Transient response of weight-control voltages \( w_5, w_6, w_7, w_8 \) (top to bottom).

Vertical: 5 volts/cm;
Horizontal: 2 msec/cm;
Desired signal: \( A_s = 5 \text{ mV peak}, S/\sigma^2 = 1.5 \text{ dB} \)
CW Interference: frequency offset 10 KHz,
\( \psi = 60^\circ \) phase shift/element
(a) \( J/S = 10 \text{ dB} \)  (b) \( J/S = 20 \text{ dB} \).
expanded time scale. Thus, the minimum response time constant of approximately 50 μsec is in agreement with the above result for desired signal only. Following the initial response, the weights change at a slower rate to reduce output signal amplitude error and thermal noise errors. This behavior is evident in the oscilloscope traces of the array output and error signals in Fig. 45.

![Fig. 45.—Transient response waveforms.](image)

Top: Error monitor, 0.5 volt/cm  
Middle: Array output, 0.5 volt/cm  
Bottom: Weight-control voltage $w_B$, 5 volts/cm  
Horizontal: 2 msec/cm  
Other conditions as in Fig. 44(b).

These results indicate that the response time constants of the experimental processor are in approximate agreement with those predicted theoretically for an ideal system.
2. Steady-state performance

The spectrum analyzer measurements in Figs. 46 and 47 illustrate CW interference rejection at the adaptive processor output. The processor was in the initial condition mode in Fig. 46(a) with non-zero, weight-control voltages present in only one element; thus only this element's signal was present at the array output. The data modulation for the desired signal and the element thermal noises were not applied in order to display the 500 KHz pseudonoise code spectrum of the desired signal. The interfering signal present at the 30 MHz center frequency was rejected in the adapt mode as shown in Fig. 46(b). In Fig. 47, the spectrum width setting of the analyzer was decreased to 10 KHz/cm to illustrate the fine structure in the desired signal's spectrum. Thermal noise was also added to the element inputs. The frequency components observable in Fig. 47(a) are separated by the code repetition rate of 4 KHz (corresponding to the 254 usec period of the 127-bit code). In addition, a large interference component is present at 30.010 MHz. The ideal performance in steady-state may be calculated from Eq. (81) for this case as

$$\frac{P_J}{P_S} = -26.2 \, \text{dB}; \quad \psi = 30^\circ, \quad J/S = 20 \, \text{dB}, \quad J/\sigma^2 = 21.5 \, \text{dB},$$

indicating a 46.2 dB reduction in the interference-to-signal ratio. The observed reduction in Fig. 47 is somewhat in excess of 42 dB; a precise value cannot be determined due to the presence of thermal noise in the array output and inadequate spectrum analyzer resolution.

The BEP measurements in Figs. 48 and 49 show that steady-state performance with CW interference and reference waveform-processing is nearly the same as the calculated performance with an ideal, fixed reference signal. The solid curves in these figures were obtained by transforming the ideal, narrowband performance results in Figs. 5 and 6 using the calibration curves in Fig. 41. The expected performance curves in Fig. 48 approach limiting values of BEP as the input interference power becomes large. The interference at the array output is strongly surpressed in this case and the expected BEP is determined by the signal-to-thermal noise ratio (SNR) at the array output. This ratio decreases as the angular separation $\psi$ is reduced accounting for the increase in the BEP "plateaus". The upper curve in Fig. 49 also illustrates this behavior versus the separation parameter $\psi$. The interference component in the array output is negligibly small except at very small angular separations and moderate input interference power. For example, when

$$S/\sigma^2 = J/\sigma^2 = 0 \, \text{dB}, \psi = 20^\circ,$$
Fig. 46. -- Signal spectra at the array output.

Horizontal: 300 KHz/cm;
(a) initial condition mode,
(b) adapt mode;
J/S = 20 dB, $\psi = 30^\circ$. 
Fig. 47.---Signal spectra at the array output.

Horizontal: 10 KHz/cm;
(a) initial condition mode,
(b) adapt mode.
Fig. 48.—Performance of the experimental system versus the input interference-to-thermal noise ratio for several values of the separation parameter $\psi$. 
Fig. 49.--Performance of the experimental system versus the separation parameter $\psi$ for several input interference-to-thermal noise ratios.
the calculated output ratios, Eqs. (80) and (81), are

\[
\frac{P_J}{P_S} = -4.62 \text{ dB} \quad \text{and} \quad \frac{P_S}{P_{nt}} = 3.48 \text{ dB.}
\]

These ratios yield an expected BEP of \(5 \times 10^{-3}\) in Fig. 41(a) as compared with the measured BEP of \(7.3 \times 10^{-3}\) in Fig. 49. The contribution of the array output interference to this error probability may be noted as the change in BEP which occurs when the input interference is removed after "freezing" the weight-control voltages at their adapted, steady-state values. This corresponds to decreasing the interference-to-signal ratio from \(-4.62 \text{ dB}\) to \(-\infty \text{ dB}\) in Fig. 41(a) with the expected BEP decreasing to \(1.6 \times 10^{-4}\). When these operations were performed experimentally, the measured BEP decreased to \(8.3 \times 10^{-4}\) - a value approximately five times larger than expected. Apparently, the array output SNR loss at this very small angular separation was greater than the calculated loss of 2.52 dB, possibly due to the presence of error multiplier offset voltages causing the weights to deviate from calculated values.

The measured error probabilities in Fig. 48 depart sharply from calculated values for input interference amplitudes exceeding 35 mV rms (\(J/\sigma^2 = 20 \text{ dB}\)). This behavior results from signal suppression and intermodulation product generation in the weighting multipliers as the dynamic range capabilities of these circuits are exceeded. The magnitude of the largest intermodulation component in the array output is indicated in Fig. 50; a CW desired signal was used here to facilitate spectrum analyzer measurements.\(^4\) The power contained in this intermodulation component is comparable to that in the desired signal when the input interference is very large (\(J/\sigma^2 = 30 \text{ dB}\)). An analysis contained in Appendix E shows that the dominant intermodulation product is generated from the square of the interfering signal component times the desired signal component. In addition, it is shown that this intermodulation component cannot be nulled at the array output by weighting coefficient adjustments. The obvious method of avoiding weighting circuit non-linearities is to reduce pre-processor rf gains under high-level interference conditions. Two problems are encountered with this approach: 1) the steady-state performance degradation due to offsets is increased and 2) the transient response time for desired signal adaption is increased. An increase in array output amplifier gain

\(^4\) The PN code was omitted in the reference waveform-processor for this case.
Fig. 50.—Measured array output power ratios versus the input interference-to-thermal noise ratio.
compensates for the loss in response time as well as ensuring that the amplitude of the desired signal at the output is sufficiently large in steady-state to match the fixed amplitude of the reference signal. Although saturation of the output amplifier is more likely to occur during the initial part of the transient when the amplifier gain is increased, the time duration of these non-linearities will be small because of the fast processor response to null the interference. The more serious problem appears to be that of SNR degradation in steady-state due to the effects of offset voltages. As was shown in Chapter III, this degradation increases as the rf element gains are reduced and is independent of the gain in the sum channel. Thus, the task of increasing the dynamic range of the processor without sacrificing output SNR centers around the problem of eliminating unwanted offsets in the error multipliers and d.c. weight-control circuitry.

The error rate versus interference angle performance may be improved by increasing the array input SNR as shown in Fig. 51. The amplitudes of both the desired and interfering signals were increased (together) to maintain a constant input J/S ratio of +10 dB. The solid curves were obtained in the same manner as discussed previously for Figs. 48 and 49. The results show that smaller angular separations of desired signal and interference may be tolerated when the array input SNR is increased. The limiter's output signal power versus input SNR characteristic is the key to the observed behavior (see Fig. 42). When the array output SNR is sufficiently large in the absence of interference, the loss in array output SNR when low-angle interference is present does not significantly reduce the signal power at the limiter's output. The deviation in the experimental data from the calculated curves at large angular separations \( \psi > 40^\circ \) is attributed to the effects of offset voltages, i.e., the calculated SNR improvement indicated in Fig. 5 was not realized. Spectrum analyzer observations under these conditions indicated that the output interference component was sufficiently small so as not to be a source of error.

The frequency of the interfering signal was varied plus or minus 1 MHz about 30 MHz with no appreciable change in the amount of interference rejection at the array output. This insensitivity to interference frequency was noted at various interference amplitudes and angular separations. A few measurements of BEP-versus-interference frequency also confirmed this behavior; Fig. 52 illustrates typical BEP performance with a large interfering signal. 5 A dramatic change

5The baseline BEP shown by the dashed line in Fig. 52 is larger compared to previous results because 1) a smaller input SNR was employed and 2) the output SNR decreased 1.3 dB in the adapt mode presumably because of offset voltage effects.
Fig. 51. -- Performance of the experimental system versus the separation parameter $\psi$ for several input signal-to-thermal noise ratios.
in performance occurred at interference frequency offsets greater than approximately 4.5 MHz where positive feedback occurred in the weight-control loops (as indicated by the results in Fig. 33). The integrator output voltages increased to their positive saturation limits, the array output interference component became large, and the BEP approached one-half. This behavior occurred at an input J/S ratio of +20 dB, an input SNR of 0 dB, and a separation \( \psi \) equal to sixty degrees. Unfortunately, no measurements or observations were made at smaller input interference levels; these measurements would be useful in defining the attenuation required in element band-pass amplifiers at frequencies where control-loop phase shift is excessive.

3. Effect of offset voltages on performance

In the early phases of the experimental research, it was noted that with element noise only at the processor inputs, the integrator outputs would saturate (at plus or minus 12-13 volts) when the input noise levels were reduced below 1 mV rms. Input noise levels of 1-2 mV rms were required to bring the integrator outputs down to quasi-stable, mid-range values and, as verified in Fig. 43(a), noise levels of 3 mV rms or greater resulted in steady weight-control voltages of
one volt or less. This behavior was typical when the error multiplier outputs had been previously nulled to within 15 mV of zero (i.e., at least 60 dB down from the maximum output of 15 volts) in the absence of signal or noise at the processor inputs. These results are consistent with the measured gain constants in Table 3. The amplitude of the array output noise component produced by 3 mV rms input noise and a weight multiplier control-voltage of one volt in steady-state is

\[ e_0 = \sigma G_w w v G_s = (3 \text{ mV rms})(.25)(1)(6) \]
\[ = 4.5 \text{ mV rms}. \]

The reference noise component resulting from this array output noise may be neglected since it is small and essentially uncorrelated due to the signal processing. Hence, a feedback voltage is generated at the error multiplier output having the value

\[ e_m = \sigma G_d G e = 11.6 \text{ mV dc} \]

which is the correct order-of-magnitude to cancel a presumed net offset voltage of the same value.

With a desired signal also present at the array inputs, the output SNR improvement was measured versus input SNR (signal amplitude varied) as shown in Fig. 53. The SNR improvement was 5.2 dB at low input SNR and decreased to 1.9 dB at an input SNR of 10.6 dB. The measurement procedure consisted of freezing the weights at their adapted values, then alternately removing the input signal and noise processes and measuring the output levels in a wideband, true rms-reading voltmeter. These measurements confirm the analytical result that offset degradation is more pronounced at higher input SNR's. When the critical importance of dc offset balance in the control loops became apparent upon completion of the offset analysis, a more stringent balancing procedure was initiated. The error multiplier outputs were nulled to within 3 mV with both desired signal and noise present at the array inputs; the error signal was set to zero by opening the feedback path at the error output and properly terminating the input to the eight-way error splitter. In this open-loop mode, the integrator balances were then adjusted for minimum integrator output voltage deflection in the integrate mode. With this procedure, nearly the same error probabilities were obtained in both adapt and initial condition modes as evidenced by the BEP measurements at the extreme left in Fig. 54. The results in this figure illustrate performance when worst-case offset voltages were deliberately added at the error multiplier outputs after completion of the above balancing procedure. For example, the data point labeled "6, 8" was obtained when opposite polarity 20 mV offsets were present at the
Fig. 53.—Measured improvement in signal-to-thermal noise ratio versus the input signal-to-thermal noise ratio (signal amplitude varied).
Fig. 54.--Effect of worst-case offset voltages in the control loops on system error rate.

inputs to sixth and eighth integrators. This corresponds to a (worst-case) offset vector which is orthogonal to the desired signal's direction-delay vector $v_1$:

$$v_1 = \begin{pmatrix} 1 + j 0 \\ 1 + j 0 \\ 1 + j 0 \\ 1 + j 0 \end{pmatrix}, \quad \Omega_v = \begin{pmatrix} 0 + j 0 \\ 0 + j 0 \\ 0 + j 20 \text{ mV} \\ 0 - j 20 \text{ mV} \end{pmatrix}.$$

For a given magnitude of the offset vector, there are many other ways to obtain this orthogonal relationship; the other data points shown illustrate some of the possibilities. The calculated degradation was obtained using the BEP-versus-output SNR transformation of Fig. 42 and the theoretical results in Eqs. (104) and (106):
These calculations predict a 3 dB reduction in output SNR when two channels have opposite polarity offsets of 30 mV. The experimental measurements in Fig. 54 indicate the degradation was larger than the calculated worst-case degradation. An accumulation of errors in measuring the signal amplitudes and gain constants is the most likely reason for the discrepancy. An average of the experimental data follow a curve corresponding to an error gain reduced by a factor of approximately 1.7 in the above equation. Other tests with offset vectors which were not worst-case yielded smaller BEP values for the same offset vector magnitude. Figure 55 illustrates the transient response of the weight-control voltages when one channel was offset by 40 mV. The complex weighting coefficients undergo significant phase changes in reaching steady-state values accounting for the difference in transient response compared to Fig. 43(b).

With the error signal set to zero, the input signal amplitude could be varied from zero to approximately 10 mV rms without changing the error multiplier output balances by more than 2 mV dc. For an input level of 35 mV rms, offsets in the range 1-10 mV were typical. The offsets increased sharply with further increase in input amplitude due to imperfect cancellation of the leakage component appearing at the other (error) input of the multipliers. Offsets as large as 100 mV were generated when the input amplitude was 100 mV rms. As noted previously in Fig. 50, severe distortion in the weight multipliers also occurred at this input level. Only slightly better balance conditions were noted with the processor inputs zero and the error signal non-zero. The error signal at the error monitor output could be varied from zero to 35 mV rms producing changes of less than 5 mV in the error multiplier outputs. Normally, the error signal was forced to a small value in steady-state - except during data-delay

\[
\min \left( \frac{P_{s}}{P_{nt}} \right)_{J=0} = \frac{mS}{\sigma^2} \left( 1 + \frac{mS}{\sigma^2} \right)^2 \left( \frac{\Omega_v}{\sqrt{m} \sqrt{S} G_m G_d \sqrt{R}} \right)^2
\]

\[
= \frac{4}{1 + \left( \frac{\Omega_v}{42 \, \text{mV dc}} \right)^2} \quad ; \quad <v_1, \Omega_v> = 0.
\]

\[\text{6The CA 3049 used to perform the multiplication had to be replaced in two of the eight circuits before obtaining this degree of input signal by attenuated (leakage) input signal balance.}\]
Fig. 55.—Transient response of weight-control voltages \( w_5, w_6, w_7, w_8 \) (top to bottom) with 40 mV offset in \( w_6 \) error multiplier.

Desired signal amplitude: 5 mV peak
Vertical: 5 volts/cm
Horizontal: 10 msec/cm

error periods as noted in Fig. 36 - so that error multiplier balance with respect to the error input was less critical than that associated with the element input.

In addition to the above offsets attributed to imperfect isolation of error multiplier inputs, a steady drift in output balance occurred due to offset sources normally present in active dc circuitry. These drifts were present even after a lengthy warm-up period (2-3 hours) and were noted to be particularly sensitive to ambient temperature change. Changes on the order of 10-30 mV were typical in a one-hour period; frequent rebalancing was necessary to maintain offsets at less than 5 mV.

It is apparent from these results that the low end of the experimental processor's dynamic input range was limited to approximately 3 mV rms (-7.5 dBm) by offset voltage effects. The analytical and experimental results suggest that input signal and noise levels 20 dB smaller might be accommodated by increasing the error circuit gain by the same amount. An IF implementation of the error multiplier has also been suggested by Huff[27] as a possible means of reducing offset voltage magnitudes.
4. Effects of data delay and code timing errors

Under conditions of small offset voltages in the error multipliers, no interference, and no data modulation on the desired signal, (i.e., a data bit stream of all zeros) the error signal was quite small as evidenced by the oscilloscope traces in Fig. 56(a). The PN code was applied to the desired signal, and the input SNR was set at 10 dB to obtain a good indication of the signal amplitude at the processor's output (middle trace). When PN data modulation was added to the input signal, the output signal amplitude decreased nearly 3 dB as illustrated by the middle traces in Figs. 36, 56(b), and 57 at different sweep speed settings. Note that the analytical result, Eq. (166), predicts a 3 dB gain reduction for a reference time delay of 6 μsec:

$$1 - \frac{\delta r}{r_b} = 1 - \frac{6 \text{ μsec}}{20 \text{ μsec}} = 0.7.$$ 

The middle trace in Fig. 58 shows the variations in a weight-control voltage about its average value in steady-state. The 40 mV peak-to-peak variations resulting from the response to data-delay errors are about twice as large as the variations normally present due to thermal noise (i.e., the spread in the trace). The effect of data delay errors on BEP performance was minimal as a result of the long averaging time in the control-loops. That is, the data-delay, burst error durations of 6 μsec were small compared to the response time constant for desired signal which was between 2 msec and 270 μsec depending on input signal amplitude. Typically, when PN data modulation was added, the BEP increased only slightly, corresponding to a loss in detector input energy-to-noise density ratio of less than 0.2 dB in Fig. 39. The same behavior was noted when an interfering signal was present and the PN data modulation was alternately applied and removed. These results confirm the analysis result that bi-phase data modulation delay in the reference processor does not affect SNR performance under narrow loop-bandwidth conditions.

It is quite possible that the above performance will not be observed in a higher gain processor where the signal response time constant is comparable with the data bit period. A technique which is applicable in this case is to inhibit processing (i.e., hold the weights constant) during periods when data-delay errors exist. This technique was implemented in the experimental processor by switching the integrator mode-control voltages to the hold mode during the initial portion of each data bit period. The weight-control voltage variations due to data-delay errors were eliminated as shown by the middle trace in Fig. 59; the array output signal amplitude also increased to its original value as shown in Fig. 60. The presence of switching waveforms in the integrator control circuitry, however,
Fig. 56.—Waveforms illustrating the effect of reference processor delay.

- Top: Reference monitor, 50 mV/cm
- Middle: Array output, 50 mV/cm
- Bottom: Error monitor, 20 mV/cm

Horizontal: 10 nsec/cm

(a) PN code only  (b) PN code plus PN data
Fig. 57.—Waveforms illustrating the effect of reference processor delay. Same conditions as in Fig. 56(b) except horizontal sweep speed: uncalibrated, $\pm 21.4$ µsec/cm.

Fig. 58.—Waveforms illustrating the effect of reference processor delay.

Top: Error monitor, 50 mV/cm
Middle: Weight-control voltage $w_1$, 20 mV/cm (ac-coupled)
Bottom: Reference monitor, 50 mV/cm
Horizontal: Uncal., $\pm 23.0$ µsec/cm
Fig. 59.—The effect of inhibiting array processing during data-delay periods. Other conditions as in Fig. 58 except horizontal sweep speed: 10 μsec/cm.

Fig. 60.—The effect of inhibiting processing during data-delay periods. Other conditions as in Fig. 56(b).
resulted in offset voltages being generated having a d.c. component referred to the integrator inputs.\(^7\) The effect of these offsets was noted upon removing the PN data modulation in Fig. 60, i.e., returning to the conditions of Fig. 56(a) except for the addition of the switching waveforms. The amplitude of the array output signal did not remain unchanged as expected, but increased to a value greater than the reference signal amplitude. The error signal that developed resulted in opposite sense "bucking" voltages being generated at the error multiplier outputs which effectively cancelled the integrator offsets. Based on observations of the error signal when equal magnitude, same polarity offsets were deliberately introduced in the error multiplier outputs, the integrator offsets were estimated to be approximately 10-20 mV in each channel and of the same polarity. As a result of the presence of these offsets, and the fact that signal response times could not be decreased to approach the data bit period due to input dynamic range limitations, the usefulness of this technique in higher gain systems could not be evaluated.

The effect of PN code timing offset on the composite (array and post-array) system in Fig. 38 was determined as shown in Fig. 61. The lower curve in this figure was obtained with the array in the initial condition mode (equal weights) and with desired signal only at the array inputs. For these measurements without element noise, the post-limiter attenuator was adjusted to give a BEP at zero code timing offset which was (nearly) equal to the BEP occurring at an array output SNR of 6 dB in Fig. 41(a). Noise was added to the array input (0 dB input SNR), the post-limiter attenuator setting was returned to the value used in Fig. 41, and the BEP measurements were repeated for selected code timing offsets (see Fig. 61). Both of these sets of measurements indicate the degradation in performance of the differential detector caused by code timing offset. The array was then placed in the adapt mode and the measurements again repeated. These latter results show the joint effect of (equal) code timing offsets at the adaptive processor and the differential detector on system performance. The lack of a significant change in BEP indicates that array output SNR is essentially unaffected by small code timing offsets at the processor. The oscilloscope traces in Fig. 62 and the spectrum analyzer measurements in Fig. 63 show that the amplitude of the array output signal decreased as the magnitude of the timing offset was increased.\(^8\) The output noise was also observed to

\(^7\)These offsets are attributed to charge-storage effects in the FET switching circuitry.

\(^8\) The effect of code timing errors appears to be present in every code bit period in Fig. 62(a) since the oscilloscope was synchronized to the data-delay error transitions; the code timing errors are actually distributed pseudorandomly in time.
Fig. 61.--The effect of code timing offset on the bit error probability for various modes of operation.
Fig. 62.--Waveforms illustrating the effect of code timing offset.

(a) $c/\Delta = 0.1$ (compare with Fig. 36)
(b) $c/\Delta = 0.25$ (compare with Fig. 57)
Fig. 63.--The effect of code timing offset on the array output signal amplitude.
decrease with code timing offset. Spectrum analyzer measurements with CW interference present indicated that the output signal-to-interference ratio did not change with code timing offset. All of these results confirm the analytical result, Eq. (156), that only array gain is affected by small code timing offsets.

5. Effect of desired signal frequency offset

Experiments were conducted with the carrier frequency of the desired signal offset from center frequency and with no interfering signal present. Figure 64 illustrates the reduction in output signal amplitude as a function of frequency offset for three values of input signal amplitude. When the input signal amplitude was small (5 mV peak), the input signal frequency offsets resulting in a 3 dB reduction in output signal amplitude were -22 KHz and +26 KHz. For an input signal amplitude of 50 mV peak, frequency offsets of -30 KHz and +35 KHz produced a 3 dB output reduction. These results are in approximate agreement with those calculated from Eq. (168) using measured
values for loop gain, input noise amplitude, and reference processor phase delay. The calculated results are as follows:

\[
\frac{2\pi \Delta f_r}{[1 + 4 \frac{s}{\sigma^2}]116 \text{ Hz}} = \tan \left[ 2\pi \frac{\Delta f_s - \Delta f_r}{180 \text{ KHz}} \right]
\]

\[a = \cos \left[ 2\pi \frac{\Delta f_s - \Delta f_r}{180 \text{ KHz}} \right]\]

<table>
<thead>
<tr>
<th>(s/\sigma^2)</th>
<th>(\Delta f_s)</th>
<th>(\Delta f_r)</th>
<th>(a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.414 (1.5 dB)</td>
<td>22.5 KHz</td>
<td>130 Hz</td>
<td>-3 dB</td>
</tr>
<tr>
<td>141.4 (21.5 dB)</td>
<td>32.95 KHz</td>
<td>10.45 KHz</td>
<td>-3 dB</td>
</tr>
</tbody>
</table>

The observed rotation rate of the weight vector \((\Delta f_r)\) was smaller than calculated, e.g., approximately 8 KHz compared to the calculated value of 10.45 KHz for the 50 mV input signal case and the input frequency offsets stated above. Figure 65 illustrates the cyclic nature of the weight-control voltages when the input signal frequency was offset. The oscillation frequencies of 25 Hz in (a) and 57 Hz in (b) compare with calculated values from the above equation of 45 Hz and 102 Hz, respectively. This indication of smaller loop processing bandwidths than calculated was also noted in the transient response results in Section E1.

Several performance characteristics were noted which are not explained by the theoretical results in Chapter IV. The weight-control voltage variations in Fig. 65 are non-sinusoidal and do not average zero, i.e., d.c. pedestals are present. In addition, amplitude fluctuations were observed in the array output signal having the same frequency as the weight variations. The magnitude of these amplitude variations increased as the signal frequency offset was increased; approximately 1-2 dB variations were observed at large input frequency offsets. The most significant behavior noted was a reduction in array output SNR with input signal frequency offset. The BEP data given in Fig. 66 verify this behavior. The frequency offset in the differential detector was maintained at 50 Hz here by

\[9\text{The theoretical degradation parameter, } d, \text{ in Eq. (168) is zero.}\]
Fig. 65.--The effect of signal carrier frequency offset on the weight-control voltages $w_5$, $w_6$, $w_7$, $w_8$.

Vertical: 5 volts/cm
Horizontal: 10 msec/cm
Input signal amplitude: $A_s = 5$ mV peak
Input signal frequency offset from 30 MHz:
(a) 10 KHz, (b) 20 KHz
Fig. 66.--The effect of offsetting the carrier frequency of the array input signal on the bit error probability.
varying the detector's local oscillator frequency and the array input signal frequency together. At offsets of -19 KHz and +29 KHz (a 48 KHz separation), the BEP increased from 2.3x10^-5 to 1.3x10^-4; this amount of change in Fig. 42 indicates that the output SNR loss was 2.3 dB. The asymmetry in the frequency offset data is attributed to non-linearities in the waveform processor's phase shift versus frequency characteristic (Fig. 35) and to inaccuracy in setting the phase shift of the waveform processor to exactly 180° at the center frequency. All of the above performance characteristics appear to be due to the presence of offset voltages in the control-loops which were not assumed in the analysis. The offset voltages apparently result in the addition of a constant vector component to the time-harmonic (rotating) component of the weight vector. This constant component results in the generation of an error signal component (having the input signal frequency) which then multiplies the input signals in the error multipliers to produce constant, offset-cancelling voltages. As the rotating component of the weight vector becomes small with increasing input frequency offset, the offset-related component remains invariant and the effects of offsets are accentuated. The analytical results in Appendix F for a worst-case offset vector provide evidence that this interpretation is correct. A solution to the weighting coefficient differential equations has not been found for the case of a general offset vector and input frequency offset.

F. Experimental Performance with Pulse Interference

Measurements of the average BEP performance of the system in Fig. 38 were conducted with a pulse interfering signal present at the processor inputs. The pulsewidth (τₐ), repetition period (Tₐ), and input power during the pulse (J/σ²) were varied over a 20 dB range in these measurements. The minimum pulsewidth employed was one data bit period (20 µsec) in duration, corresponding to an interfering signal spectrum with first nulls separated in frequency by 100 KHz. The carrier frequency and angular separation of the interfering signal were fixed at 30 MHz + 100 Hz and ψ = 60°, respectively. Other fixed parameters included the carrier frequency of the desired signal (30 MHz) and detector local oscillator signal (30 MHz + 50 Hz), the code timing offset (ε = 0), and the power contained in the desired signal (s/σ² = 0 dB or +10 dB). Before presenting the experimental results, the method used to transform the theoretical response of the array into calculated, average BEP performance will first be described.

In the absence of analytical results or experimental calibration data for the BEP performance of a differential detector which incorporates waveform-processing and has array-processed pulse interference at its input, the following approximation was employed. The time-varying power ratios at the array output, calculated in Eqs. (113) and (114) using Eqs. (109) and (111), were entered into Fig. 41 to obtain a BEP-versus-time function, i.e.,
Figure 67 illustrates the result of this calculation using the output power ratios plotted in Figs. 9 and 10. Three BEP curves are shown where each curve corresponds to a selected value of the duty-cycle in Figs. 9 and 10. The BEP which occurs at the midpoint of a given data bit period (denoted by a dot on a calculated curve in Fig. 67) was taken to be the BEP of that particular data bit. That is, the BEP of a data bit occurring during the pulse was approximated by the average BEP which would occur if constant power (continuous) interference were at the array output having the same amplitude as that calculated at the midpoint of the data bit period. An average BEP was computed for each curve in Fig. 67 as the arithmetic average of the resulting BEPs determined for each data bit in a pulse repetition period. Note that the average BEP is smaller when the duty-cycle is 0.01 as compared to a duty cycle of 0.025, even though the BEP during the pulse is larger in the former case. The longer time period where the BEP is low (pulse absent) in the 0.01 duty cycle case more than compensates for the increased BEP during the pulse. At a duty cycle of 0.02 (i.e., between the above values), the calculated average BEP has a maximum value of 7.5x10^{-4}.

The calculated and measured values of average BEP are given in Figs. 68-70. Each figure corresponds to a different fixed value of pulsewidth: 20 μsec, 200 μsec and 2 msec. The average BEP values calculated in Fig. 67 are three points on the upper solid curve in Fig. 69. A dashed curve has been drawn through the experimental data points for purposes of comparison. The agreement is very good near the peaks of the curves and for smaller values of duty cycle. The agreement at large duty cycles (i.e., small BEP) would have been better if the post-limiter attenuation had been adjusted to exactly the same value used in Fig. 41. A slightly smaller attenuator setting was used in the pulse measurements corresponding to a BEP of approximately 1.2x10^{-5} in the absence of interference (as indicated by the arrow at the lower left edge of Figs. 68-70). A BEP of 2.3x10^{-5} under these conditions would have been more appropriate for purposes of comparing calculated and experimental results. The measured curves

\[ \frac{p_J(t)}{p_s(t)} \Rightarrow \text{Fig. 41} \rightarrow p_E(t); \quad 0 \leq t \leq T_j. \]

\[ \frac{p_s}{p_{nt}} \]

\[ \text{The measurements in Figs. 68-70 were completed prior to the theoretical analysis of pulse interference and calibration measurements in Fig. 41.} \]
Fig. 67.—The method for estimating the average bit error probability of the experimental system with pulse interference present.
Fig. 68.—Performance of the experimental system versus pulse interference parameters.
Fig. 69.--Performance of the experimental system versus pulse interference parameters.
Fig. 70.--Performance of the experimental system versus pulse interference parameters.
for $J/\sigma^2 = 20$ dB are maximum at a value of duty cycle which is in close agreement with the duty cycle at which the average power ratio, Eq. (117), is maximum. The curves in Fig. 71 illustrate how the computed, worst-case duty cycle changes with pulsewidth. The upper curve corresponds to the same conditions as in Fig. 12, while the lower curve corresponds to the value of angular separation simulated experimentally. The computed, worst-case duty cycles in Fig. 71 at the pulsewidths used experimentally are 0.0037, 0.0019, and 0.14; the corresponding values at the peaks of the upper measured curves in Figs. 68-70 are 0.005, 0.02, and 0.14, respectively. The agreement is not as good for smaller input interference power during the pulse. For example, the middle curve in Fig. 68 (where $J/\sigma^2 = +10$ dB) has a peak at a duty cycle of 0.007 whereas the average power ratio, Eq. (117), is maximum at a calculated duty cycle of 0.033 (-14.8 dB).

Fig. 71.--The duty cycle which maximizes Eq. (117) versus the normalized pulsewidth for two values of the separation parameter $\psi$. 
Nevertheless, the calculated BEP curve in this case agrees closely with the measured results. It can be concluded, therefore, that pulse parameters which produce a maximum in the average power ratio, Eq. (117), do not necessarily produce a maximum in average BEP, and conversely.

It can also be noted in Figs. 68-70 that only a small increase occurs in the maxima of the BEP curves as the pulsewidth is decreased. The performance for pulsewidths smaller than one data bit period was observed to be only slightly worse than that in Fig. 68. The average BEP did not exceed $2 \times 10^{-3}$ under any conditions of pulsewidth or duty cycle when the input interference ratio, $J/\sigma^2$, was +20 dB. For this input ratio, the worst-case duty cycle was observed to be in the range 0.002 to 0.005 for pulsewidths between 2 usec and 20 usec; i.e., the duty cycle was near the calculated value of 0.0033 in Fig. 71. These results indicate that both the worst-case duty cycle and average BEP approach limiting values as the pulsewidth is decreased. This behavior was noted in the analytical results of Fig. 12.

A pessimistic estimate of the average BEP can be calculated for the case of narrow pulses (less than a data bit period in duration) separated by many data bit periods. This estimate assumes that 1) the interference pulse does not overlap two data bits, 2) the probability of error of the data bit occurring during the pulse and the data bit immediately following is one-half, and 3) the probability of error of the remaining data bits in the pulse repetition period is equal to $P_0$, the BEP obtained in the absence of interference. The estimate is given by

$$A V P_E = \frac{1}{K} \left[ \sum_{n=1}^{K} \begin{array}{c} P_r \text{[error in } n^{th} \text{ data bit]} \\ 0.5 + 0.5 + (K-2) P_0 \end{array} \right]$$

$$= \frac{0.5 + 0.5 + (K-2) P_0}{K} = \frac{1}{K} + \frac{(K-2)P_0}{K}$$

where $K = \frac{T}{T_b}$

is the number of data bit periods in a pulse repetition period ($K \gg 1$). The line labeled "bound" in Fig. 68 illustrates this estimate of performance. It is anticipated that measured performance would more closely approach this bound at very small duty cycles if the pulse power were increased 10 dB ($J/\sigma^2 = 30$ dB) and the pulsewidth decreased by 10 dB ($\tau_j = 2$ usec).$^{11}$ The average power ratio, Eq. (117), under

$^{11}$An insufficient on-off ratio in the pulse-amplitude modulator prevented measurements from being obtained for this interference power ratio.
these conditions would be maximum at a calculated duty cycle of 0.00033 (-34.8 dB), suggesting a maximum BEP somewhat to the left of the maximum in the upper curve of Fig. 68. The effects of signal suppression and limiting in the weighting circuits at this input interference level would also influence average BEP performance. For example, at unity duty cycle (CW interference), the average BEP is greater than 1x10^-2 as noted by extending the measurements in Fig. 48 to the case $\psi = 60^\circ$ and $J/\sigma^2 = +30$ dB. For values of duty cycle decreasing from unity, the average BEP would most likely decrease from this large value and approach the bound in Fig. 68 at small duty cycles. The performance at intermediate duty cycles is not easily predicted. It is certainly expected, however, that in-band intermodulation products generated in the weighting coefficients will have a dominant effect.\(^{12}\)

The measurements in Figs. 72 and 73 indicate performance when

![Fig. 72. -- Performance of the experimental system versus pulse interference parameters.](image)

\(^{12}\) A substantial reduction in intermodulation distortion could have been achieved using available devices if sufficient attention had been focused on making the circuits linear.
the desired signal amplitude is increased 10 dB. The post-limiter attenuation was increased to give an average BEP of $1 \times 10^{-5}$ in the absence of interference. As a result of this latter change (to shorten measurement times at low BEP), the calibration measurements in Fig. 41 are not applicable for obtaining calculated performance curves. The results in Figs. 72 and 73 do show, however, that average BEP decreases as the input SNR is increased. In view of the good agreement between calculated and measured performance in Figs. 68-70, it is presumed that predicted performance at other input SNR and angular separations might also be fairly accurate. This hypothesis assumes, of course, that input power levels producing non-linearities are avoided and that other system parameters are at fixed and/or ideal values. These parameters include the desired signal carrier frequency offset, the code timing offset (and jitter) in the differential detector, the bandspreading (code-to-data rate) ratio in the detector, and the post-limiter (downlink) attenuation. It is conceivable that a sequence of calibration measurements, each as in Fig. 41, could be obtained for selected values of these parameters in an exhaustive investigation of pulse interference performance. An alternate approach, of course, is to investigate post-array subsystem performance analytically.

Fig. 73.--Performance of the experimental system versus pulse interference parameters.
G. **Experimental Performance with Wideband Noise Interference**

Experiments were conducted with wideband noise interference having a 3 dB bandwidth of either 9 MHz or 3 MHz as shown by the spectrum analyzer displays in Fig. 74. The measurements in Figs. 75 and 76 illustrate BEP performance when this interfering signal was incident at broadside to the array (θ_j = 90°). The interference power was varied in Fig. 75, and the desired signal's phase shift per element was varied in both figures to obtain different values for the separation parameter ψ. For purposes of comparison, Fig. 76 also shows the measured performance with broadside, CW interference and the calculated performance for narrowband interference (the upper solid curve). The latter curve was obtained from the calculated performance in Fig. 48 at J/σ^2 = 10 dB. The measured values of BEP with CW interference are somewhat smaller than calculated and are smaller than corresponding measured values in Fig. 48. This is not an expected result since, ideally, the performance with narrowband interference is the same for all combinations of spatial angles θ_S and θ_j which yield a common value for ψ in Eq. (82). In spite of this difference between measured and calculated performance with CW interference, it is apparent that broadside noise interference is less effective in producing errors. The BEP with noise interference is smaller than with equal power, CW interference for all input levels not producing limiting in the weighting circuits. This improvement in performance is expected since the bandwidth of the noise interference in the differential detector is greater than the spectrum spreading experienced by CW interference there, i.e., the detector waveform-processing gain is larger. For input ratios, J/σ^2, exceeding approximately 16 dB in Fig. 75, the effects of weighting circuit saturation are noticeable. The non-linearities which occur during peak excursions in the input noise voltage are apparently the reason why saturation effects are appreciable at a smaller value of average input power than in Fig. 48.

The results in Fig. 77 indicate performance when the interfering signal experiences delays between elements corresponding to an angle-of-arrival of 70.5° spatial. At this angle, the phase shift of the center frequency-component of the interfering signal is sixty degrees per element. The desired signal is broadside to the array so that the parameter ψ also equals sixty degrees. The four calculated curves in this figure were obtained by transforming the theoretical results in Fig. 27 into BEP curves using the calibration measurements in Fig. 42. That is, for given values of the parameters B and J/σ^2, the signal-to-total noise ratio at the array output, P_s/P_n, as determined from Fig. 27, was entered as the abcissa value, P_s/P_n, in Fig. 42 to obtain the BEP. This approach essentially treats the interference noise at the array output as part of the summed, element-noise process. The resulting BEP curves are only
Fig. 74.--Spectra of the array input noise interference.

Vertical: 10 dB/cm
Horizontal: 3 MHz/cm

(a) $f_j = 9$ MHz, (b) $f_j = 3$ MHz
Fig. 75.--Performance of the experimental system versus the input interference-to-thermal noise ratio for several values of the separation parameter $\psi$. 

\[
\begin{align*}
\theta_d &= 90^\circ \ (B = 0) \\
\frac{S}{\sigma^2} &= 0 \text{ dB} \\
\beta_j &= \begin{cases} 
3 \text{ MHz} & \Delta \Delta \\
9 \text{ MHz} & \circ \circ \circ
\end{cases}
\end{align*}
\]
Fig. 76.--Performance of the experimental system versus the separation parameter $\psi$ for three values of interfering signal bandwidth.

- $\theta_j = 90^\circ (B = 0)$
- $\frac{S}{\sigma_j} = 0 \text{ dB}$
- $\frac{J}{\sigma_j} = 10 \text{ dB}$

- $b_j = 0$ (CW INTERFERENCE)
- $3 \text{ MHz } \Delta \Delta \Delta$
- $9 \text{ MHz } \circ \circ \circ$

$P_E$ vs. $\psi$ (Desired Signal Phase Shift/Element)
Fig. 77.—Performance of the experimental system versus the input interference-to-thermal noise ratio for three values of the bandwidth parameter $B$. 
approximate since the conditions under which the measurements in Fig. 42 were obtained are not duplicated here. In particular, the total noise process at the limiter input is not confined to a bandwidth of 1.5 MHz and does not have constant spectral density. The spectrum analyzer display in Fig. 78(b) shows that the array output interference spectrum is small near the center frequency; only desired signal components are present in this region. This result was obtained with the element noises removed from the array inputs. The element noise power density is large around center frequency and effectively "fills in" this notch in the interfering signal's spectrum. This can be observed in the displays of Figs. 79-81 which illustrate array output spectra before and after adaption for three different ratios of input interference power-to-thermal noise power. The BEP obtained in the adapt mode in each case is given by an experimental data point in Fig. 77, along the curve labelled "B = 1/20". The parameter B has this value when the interference bandwidth is 9 MHz as can be easily demonstrated:

\[ B = b_j \frac{P}{c} \cos \theta_j = \left( \frac{j}{f_c} \right) \left( \frac{P}{I_0} \right) \cos \theta_j \]

\[ = \left( \frac{9}{30} \right) \left( \frac{1}{2} \right) \left( \frac{1}{3} \right) = \frac{1}{20}. \]

The measured points in Fig. 77 along the B = 1 curve were also obtained with 9 MHz interference but with one important change: the coaxial cables used to delay the input interfering signals were replaced with manual phase shifters adjusted for the same phase delay at center frequency: sixty degrees per element. The phase shift-versus-frequency characteristics of these 3 MHz-bandwidth devices were sufficiently different in successive elements (i.e., with different dial settings) so as to produce uncorrelated interfering signals at the processor inputs. This conclusion follows from the closeness of the measured BEPs to the calculated curve for completely uncorrelated noises (B = 1). In effect, the phase shifters produced the same conditions as would be expected when noise interference of 60 MHz bandwidth (d.c. to 60 MHz) is incident from the endfire direction on a linear array with half-wavelength spacing at 30 MHz.

The effect of changing the angular separation of desired and interfering sources was determined as shown in Fig. 82. The angle-of-arrival of the interference was fixed at 70.5° spatial, as in Fig. 77, and the input power ratio, \( J/\sigma^2 \), was set at 10 dB. The desired signal's angle-of-arrival was varied such that the parameter \( \psi \) in Fig. 82 equaled sixty degrees when the desired signal was at broadside and decreased as the desired signal approached coincidence with the interference (\( \psi = 0° \)). The calculated BEP performance in this figure was obtained using Figs. 24 and 42 and the procedure described above. The very good agreement here between calculated and measured
Fig. 78.—Spectra of desired signal plus noise interference.
(a) Output from one element, (b) array output in adapt mode.

No element noise, $\sqrt{S} = \sqrt{J} = 11$ mV rms,
$\theta_s = 90^\circ$, $\theta_j = 70.5^\circ$, $\psi = 60^\circ$.
Vertical: 10 dB/cm
Horizontal: Uncal. = 1.5 MHz/cm
Fig. 79.--Spectra at the array output.
(a) Initial condition mode, equal weights, (b) adapt mode.
\[ \sqrt{S} = 3.5 \text{ mV rms}, \frac{S}{\sigma^2} = 0 \text{ dB}, \frac{J}{\sigma^2} = 0 \text{ dB} \]
\[ \theta_s = 90^\circ, \theta_j = 70.5^\circ, \psi = 60^\circ \]
Horizontal: 3 MHz/cm
Vertical: 10 dB/cm
Fig. 80.—Spectra at the array output.

$J/\sigma^2 = 10$ dB. Other conditions as in Fig. 79.
Fig. 81.—Spectra at the array output. $J/\sigma^2 = 20$ dB. Other conditions as in Fig. 79.
Fig. 82.--Performance of the experimental system versus the separation parameter $\psi$ (desired signal's angle-of-arrival varied).
results is attributed, in part, to the fact that only the relative phases of the narrowband desired signals were varied in the experiment. The input powers were fixed at values within dynamic range limits, and the (performance-sensitive) bandwidth parameter $B$ associated with the larger input interfering signal was constant.

The final result in Fig. 83 illustrates performance when the

$$\frac{S}{\sigma^2} = 0 \text{ dB}$$
$$\frac{J}{\sigma^2} = 10 \text{ dB}$$
$$b_J = 9 \text{ MHz}$$

VARIABLE $\theta_s$ AND $\theta_j$; $\psi = 60^\circ$

Fig. 83.—Performance of the experimental system versus the bandwidth parameter $B$ (angles-of-arrival of desired and interfering signals varied).
angles-of-arrival of both the desired and interfering signals were varied together in such a manner as to maintain a constant value for the separation parameter $\phi$ of sixty degrees. For example, interference and desired signal phase shifts per element (at 30 MHz) of 180° and 120°, respectively, were selected to simulate the case of endfire interference ($\theta_j = 0°$) and desired signal at $\theta_d = 48.2°$. For this case of endfire interference, the bandwidth parameter $B$ in Fig. 83 has the value 0.15:

$$B = \frac{b_j}{f_c} \cdot \frac{P_j}{\lambda_c} \cos \theta_j = \frac{9}{30} \cdot \frac{1}{2} \cdot 1 = \frac{3}{20}.$$

The increasing difference between calculated and measured performance as the interfering signal approaches endfire may be due to the effects of non-zero loop delay which are not included in the calculated performance curves. Note that the measured loop delay of 56.2 nsec is significant compared to the reciprocal of the interfering signal's bandwidth:

$$b_j \delta_f = (9 \text{ MHz})(56.2 \text{ nsec}) = 0.506.$$

An evaluation of the BEP performance using the general analytical result in Eq. (211) for this value of loop delay would be expected to result in a calculated performance curve more closely following the measured values in Fig. 83.

In view of the relatively good agreement between calculated and measured results in this section, it is concluded that practical array performance with noise interference can be adequately predicted from the theoretical result, Eq. (211), provided the power in the interfering signal is not excessive. Even in a hypothetical processor with unlimited input dynamic range, the effects of weighting coefficient jitter as the input noise power is increased are expected to degrade performance beyond that predicted by the narrow loop-bandwidth analysis.
CHAPTER VII
SUMMARY AND CONCLUSIONS

The performance of a coded communication system consisting of an adaptive antenna array, bandpass limiter, and differential (DPSK) detector has been documented in this report. Feedback-control circuits in the adaptive array processor are employed to adjust the array's weighting coefficients in real time so that a desired communication signal is received with gain and interfering signals are suppressed at the array output. The communication signal is generated by biphase modulating a carrier signal with the modulo-two sum of a differentially-encoded data stream and a pseudonoise (PN) code. An identical, locally generated PN code is used in waveform (correlation) processing of the array output signal to generate a reference signal for the array. The local PN code is also used to remove the PN modulation from the desired signal in the differential detector prior to data detection.

An analysis of array performance was carried out in several steps. Initially, an idealized model of the array processor was analyzed under the assumptions that 1) the received signals are constant in power and are narrowband relative to array bandwidth, 2) the reference signal is ideally generated, and 3) the effects of jitter in the array weighting coefficients about their mean values may be neglected to a first-order approximation. The transient and steady-state response of the array was calculated as a function of the array geometry, the angular separation of desired and interfering signal sources, and the powers of desired signal, interference, and thermal noise at the array processor inputs. Additional analyses were performed to determine the array response to pulsed-envelope interference as a function of pulsewidth and repetition period parameters, and to wideband noise interference as a function of the spectral width of the noise process. The next step in the analysis consisted of modifying the idealized model to include imperfections present in practical adaptive array processors. Analytical results were obtained which indicate the effect on performance of

- voltage offsets in the baseband portion of the feedback control loops,
- an offset in the carrier frequency of the desired signal from the design frequency,
the presence of interference and thermal noise components in the processed reference signal which are correlated with the input interference and thermal noise processes, and

- non-zero delays in the processing circuits.

These results are useful for specifying parameter ranges or tolerances in practical array implementations. Additional work is desirable in this area, however, to generalize the results to the case where several or all of the above imperfections are simultaneously present.

In order to evaluate the usefulness of the approximate analytical results, a number of experimental measurements were made showing the bit error probability (BEP) performance of the differential detector as a function of system and signal parameters. These measurements were compared with estimates of the average BEP obtained from the calculated power ratios at the array output and measured performance characteristics of the limiter-detector subsystem (Figs. 41 and 42). Several sets of estimated performance curves were computed which relate the average error rate to array input power levels, angular separation of sources, pulse interference parameters, and noise interference bandwidth. In general, the measured error rates were in close agreement with predicted values over most of the range of parameter variations employed in the experiments. Good agreement was obtained with a cw interfering signal having a frequency within the band occupied by the desired signal and a power level not exceeding the dynamic range limits of the signal weighting circuits. It was shown that increasing the offset of the interfering signal's frequency to the point where distributed time delays in the control loops cause a ninety-degree differential phase change results in failure of the control loops. In a practical system, bandpass filtering can be employed in the circuitry preceding the processing loops to reject high-level, out-of-band interference. Thus, rejection by the array processor at these frequencies would not be required. A large increase in the average error rate was observed with very large interference as a result of desired signal suppression and intermodulation component generation in the weighting circuits. This failure mechanism can be circumvented by incorporating automatic gain control in the array element circuitry preceding the adaptive processor. The response of the array to pulse interference -- a repetitive transient under steady-state conditions -- was also verified experimentally by average error rate measurements. It was shown that the average BEP can be accurately predicted using the analytical results obtained over at least a one-hundred to one range of interference pulsewidths, duty cycles, and peak pulse powers. The experimental measurements with noise interference having a bandwidth of one-tenth and three-tenths of the array center frequency agreed moderately well with theoretical calculations based on the idealized array processor model. These
measurements verify the analytical result that array output signal-to-
total noise ratio decreases with an increase in either the power or
bandwidth of the interfering noise process.

Additional experiments were performed to substantiate analytical
results regarding the effect of imperfections in the array processor.
The degradation in system error rate with worst-case offset voltages
in the feedback-control loops was quantitatively verified. It
was also demonstrated that offsets in local PN code timing which are
small relative to a code bit duration result in a reduction of signal,
interference, and thermal noise component amplitudes at the array
output without affecting relative magnitudes, i.e., the signal-to-
total noise ratio is unchanged. Similar behavior was observed as a
result of the time delay introduced in the reference waveform pro-
cessor. This delay causes the desired signal's biphase data modulation
to be delayed in the reference signal and results in the generation of
an undesired error signal after each transition in the binary data
stream. Under conditions of zero offset in the desired signal
carrier frequency and large time constants compared to a data bit
period in the feedback-control loops, the average BEP was not sub-
stantially different from the BEP obtained with no data modulation
applied to the desired signal, i.e., with a data stream consisting of
binary zeros. It was not possible experimentally to reduce the
minimum loop time constant to a value less than a data bit period
(by increasing input powers) without producing severe non-linearities
in the weighting circuits which dominated performance. It is expected,
however, that the performance of an array processor designed to
respond to desired signal in a few data bit periods, i.e., a high-
gain processor, will be adversely affected by biphase data-delay
errors when high-level (but non-saturating) interference is present at
the inputs to the weighting circuits. One approach to preventing an
undesired response due to waveform processor delay is to inhibit
array processing during the initial portion of each data bit inter-
val (where biphase data-delay errors occur). For this approach to
be effective, offset voltages introduced by integrate-hold switching
circuitry in the feedback loops must be minimized. In addition, the
uncertainties in the carrier frequencies of the desired signal and
the reference processor local oscillator signal must be small in
comparison to the data bandwidth of the waveform processor. The
combination of reference processor delay and carrier frequency offset
was shown to result in performance degradation which increased with
the frequency offset. Basically, this degradation occurs because
the phase shift introduced by the waveform processor is a multiple
of $2\pi$ radians only at the array center frequency, i.e., the waveform
processor estimates the phase of the desired signal's carrier perfectly
at only one carrier frequency. The magnitude of the degradation was
shown analytically to depend on 1) the amount of phase error intro-
duced by the waveform processor (given nominally by the product of
carrier frequency offset and time delay), 2) the input power levels
and control-loop gain, and 3) the values of offset voltages in the control loops. Experimental results obtained with the interfering signal absent (Fig. 66) indicate that performance degrades gracefully with carrier frequency offset provided the offset is removed in the differential detector through adjustment of the detector's local oscillator frequency.

In view of the general agreement (under favorable offset conditions) between measured performance with a processed reference signal and calculated performance with an ideal reference signal, it is concluded that reference signal generation based on correlation processing of the array output signal is a promising technique. For practical application of the technique, an additional subsystem must be provided to acquire and track the time base of the received signal's PN code, i.e., to synchronize the local reference code. A prototype subsystem which performs these functions has recently been developed and incorporated into the experimental system[28]. Initial results obtained with this subsystem demonstrate that the reference code can be synchronized in the presence of interference at the array processor inputs.

The experimental results in Chapter VI also indicate that the ratio of code rate to data filter bandwidth in the waveform processor -- a modest factor of 2.5 experimentally -- can be significantly less than the ratio of input interference-to-desired signal without degrading interference rejection at the array output. This behavior stems from the fact that the interfering signal is spread in bandwidth twice in the waveform processor: once in the multiplier preceding the data filter, and a second time in the multiplier which inserts the code modulation on the reference signal. The spread interference component in the reference signal undergoes a second correlation-filtering operation when it is multiplied by the (non-processed) input interference in the error multipliers and the product subsequently averaged in the loop integrators. Thus, the bandspreading ratio in the waveform processor need not be large when the corresponding ratio in the feedback loops, i.e., the ratio of code rate-to-loop processing bandwidth, is large. That is, the correlation between the interference component in the reference signal and the input interference component will be small when the correlation time interval, interpreted as the minimum loop time constant $(a\lambda_1)^{-1}$, is significantly larger than the code bit duration $\Delta$, i.e.,

$$(a\lambda_1)^{-1} > \Delta.$$  

This condition was satisfied in the experimental processor where the minimum response time constant always exceeded approximately twenty-five code bit periods. There is supporting evidence from computer simulations of a higher-gain adaptive array processor which also
indicates that proper operation is not obtained when loop processing bandwidths exceed the bandwidth of the desired signal[29]. The array prefers to minimize the rms error under such conditions by switching the output interfering signal with wideband modulation introduced through weighting coefficient variations rather than forming the desired pattern. This behavior, which occurs with high-level interference, is indicative of conditions where the approximation of small weight variances in the analysis is invalid. The condition of large processing bandwidths also violates stochastic approximation arguments which form the basis for the feedback-control algorithm.\(^1\)

In order to prevent this undesired response from occurring, it appears necessary to reduce the loop gain, \( \alpha \), under large interference conditions so that small processing bandwidths are maintained. This implies a constraint on the time constant \((\alpha \lambda_2)^{-1}\) associated with the response to a low-level desired signal,

\[
(\alpha \lambda_2)^{-1} = (\alpha \lambda_1)^{-1} \frac{\lambda_1}{\lambda_2} > \Delta \frac{\lambda_1}{\lambda_2} \leq \frac{J}{S} \Delta
\]

i.e.,

\[
\frac{1}{(\alpha \lambda_2)^{-1}} > \frac{J/S}{\Delta}
\]

That is, the nominal response time to desired signal, normalized in data bit periods, must exceed the ratio of input interference-to-signal divided by the bandspreading ratio for proper operation of the array. It is concluded, therefore, that the array processing technique considered provides the capability for rejecting large interfering signals of adequate angular separation at a cost of increased acquisition time to desired signal.

A final conclusion to be drawn from the closeness of predicted and measured BEP performance concerns the effect of array processing on the phase coherence of the desired signal. The instantaneous phase fluctuations in the desired signal at the array output which result from time-variations in the weighting coefficients did not significantly degrade signal coherence — at least for the range of experimental parameters investigated and for time intervals of two data bit periods required for differential detection. Some caution should be exercised, however, in attempting to extrapolate the results obtained from approximate power ratio calculations to the performance

\(^1\)These arguments are discussed in Reference [7].
of systems having larger processing bandwidths or employing coherent (PSK) detection. The fact that desired signal phase is not constrained to have a preferred value at the array output indicates that variations in the mean output signal phase may occur in practical, i.e., non-ideal, array processors. Moreover, when non-zero offsets are present in the desired signal's carrier frequency, the array processor reduces the frequency offset by an amount which is dependent on interfering signal parameters and hence is not completely controllable (see Eq. (167) and Eq. (239)). These characteristics must be considered in designing the phase-tracking loop bandwidth of a coherent detector.

In summary, analytical and experimental results have been obtained which provide a basis for the design of adaptive antenna array processors operating with coded communication signals.
APPENDIX A

The eigenvector equation, Eq. (56), is solved in this appendix for the case of linearly independent, direction-delay vectors \( v_j, \ i=1,2,\ldots,p \) where \( p \leq m \). A candidate eigenvector is assumed as the linear combination

\[
(221) \quad e = \sum_{i=1}^{p} \beta_i \ v_i \ ; \quad \beta_p = 1
\]

where only \( p-1 \) of the coefficients need to be independently variable. Substitution of Eq. (221) into Eq. (56) yields

\[
(222) \quad \sum_{i=1}^{p} \left[ \left( \sum_{j=1}^{p} N_i \beta_j <v_i, v_j> \right) + (\sigma^2 + \lambda) \beta_i \right] v_i = 0, \quad \beta_p = 1.
\]

The linear independence condition

\[
\sum_{i} \alpha_i v_i = 0 \rightarrow \alpha_i = 0, \ i=1,2,\ldots,p
\]

may be applied to set each coefficient in the sum, Eq. (222), equal to zero:

\[
(223) \quad \sum_{j=1}^{p} N_i \beta_j <v_i, v_j> + (\sigma^2 - \lambda) \beta_i = 0 \quad \text{for} \ i=1,2,\ldots,p
\]

This is a system of non-linear, homogeneous equations in the unknowns \( \beta_1,\ldots,\beta_{p-1} \) and \( \lambda \). Using \( p-1 \) of the equations, the \( \beta \)'s may be found in terms of the parameter \( \lambda \); substitution of the \( \beta_i(\lambda)'s \) into the \( p \)th equation then yields a \( p \)th degree polynomial in \( \lambda \) whose \( p \) distinct roots \( \lambda_1, \lambda_2,\ldots,\lambda_p \) are the non-noise eigenvalues corresponding to the unnormalized eigenvectors.
The remaining \( m-p \) eigenvectors must be orthogonal to those in Eq. (224), and are therefore not in the subspace \( \{v_1\} \). These noise eigenvectors correspond to the eigenvalue \( \sigma^2 \) as seen from Eq. (56):

\[
\langle v_i, e_k \rangle = 0 \quad k = p+1, \ldots, m
\]

\[
(K_x - \lambda I)e_k = 0 \quad \Rightarrow (\sigma^2 - \lambda)e_k = 0
\]

\[
\Rightarrow \lambda_k = \sigma^2
\]

Consider, for example, the case \( p=2 \). The equations in Eq. (222) in this case are

\[
N_1 \beta_1 \langle v_1, v_1 \rangle + N_1 \langle v_1, v_2 \rangle + (\sigma^2 - \lambda) \beta_1 = 0
\]

\[
N_2 \beta_1 \langle v_2, v_1 \rangle + N_2 \langle v_2, v_2 \rangle + (\sigma^2 - \lambda) = 0.
\]

The second equation yields

\[
\beta_1(\lambda) = \frac{\lambda - \sigma^2 - mN_2}{N_2 \langle v_2, v_1 \rangle}
\]

which, when substituted into the first equation, gives

\[
\lambda^2 - \lambda [m(N_1+N_2)+2\sigma^2] + m^2N_1N_2 + m(N_1+N_2)\sigma^2 + \sigma^4
\]

\[
- N_1N_2 |\langle v_1, v_2 \rangle|^2 = 0.
\]

The roots of this equation are

\[
\lambda_{1,2} = \sigma^2 + \frac{m(N_1+N_2)}{2} \pm \frac{1}{2} \left[ m^2(N_1-N_2)^2 + 4N_1N_2 |\langle v_1, v_2 \rangle|^2 \right]^{1/2}
\]
and the corresponding eigenvectors are

\[ e_{1,2} = \frac{\lambda_{1,2} - \sigma^2 - mN_2}{N_2 \langle v_2, v_1 \rangle} \left( v_1 + v_2 \right) \]

\[
= \left\{ \frac{m(N_1 - N_2) \pm \left[ m^2(N_1 - N_2)^2 + 4N_1N_2 \langle v_1, v_2 \rangle^2 \right]^{1/2}}{2N_2 \langle v_2, v_1 \rangle} \right\} \left( v_1 \right) ; + v_2.
\]
APPENDIX B

In this appendix, equations will be derived for the initial value of the periodic, steady-state weight vector at the beginning of each pulse interference repetition period. The steady-state condition,

\[ w(0) = w(T_j), \]

may be written as

\[ \frac{\langle e_1, w(0) \rangle}{|e_1|^2} e_1 + \frac{\langle e_2, w(0) \rangle}{|e_2|^2} e_2 + E_3 w(0) = \]

\[ + \frac{\sqrt{RS}}{\sigma^2 + mS} \left( 1 - e^{-\alpha(\sigma^2 + mS)(T_j - \tau_j)} \right) v_1 \]

\[ + \frac{\sqrt{RS}}{\sigma^2 + mS} \left( \frac{\langle v_1, w(\tau_j) \rangle}{m} v_1 - e^{-\alpha^2(T_j - \tau_j)} \right) \]

using Eqs. (110) and (111). In order to solve for the unknown coefficients on the left-hand side of this expression, the vectors on the right-hand side must be expressed in terms of the eigenvectors \( e_1 \) and \( e_2 \). From Eq. (57) it is apparent that

\[ e_1 = \beta_{11} v_1 + v_2 \]
\[ e_2 = \beta_{12} v_1 + v_2 \]

where

\[ \beta_{11,12} \equiv \frac{\lambda_{1,2} - \sigma^2 - mj}{\langle v_1, v_2 \rangle} \]
so that the desired expression for \( v_1 \) is

\[
(228) \quad v_1 = \frac{(e_1 - e_2)}{\beta_{11} - \beta_{12}} = \frac{(e_1 - e_2)}{\Delta \beta}
\]

\( \Delta \beta \equiv \beta_{11} - \beta_{12} \).

Also, from Eq. (109), the weight vector \( w(\tau_j) \) is given by

\[
(229) \quad w(\tau_j) = \left[ \begin{array}{c}
\langle e_1, w(0) \rangle e^{-\alpha \lambda_1 \tau_j} + b_1 (1 - e^{-\alpha \lambda_1 \tau_j}) e_1 \\
\langle e_2, w(0) \rangle e^{-\alpha \lambda_2 \tau_j} + b_2 (1 - e^{-\alpha \lambda_2 \tau_j}) e_2 \\
+ E_3 w(0) e^{-\alpha \sigma^2 \tau_j}
\end{array} \right]
\]

When Eqs. (228) and (229) are substituted in Eq. (226) and the coefficients of \( e_1, e_2, \) and \( E_3 w(0) \) on both sides are equated, the results are that

\[
(230) \quad E_3 w(0) = 0
\]

and that the initial components along \( e_1 \) and \( e_2 \) satisfy a pair of simultaneous equations,

\[
(231) \quad g_{11} \frac{\langle e_1, w(0) \rangle}{|e_1|^2} + g_{12} \frac{\langle e_2, w(0) \rangle}{|e_2|^2} = h_1
\]

\[
\quad g_{21} \frac{\langle e_1, w(0) \rangle}{|e_1|^2} + g_{22} \frac{\langle e_2, w(0) \rangle}{|e_2|^2} = h_2.
\]

The coefficients in these equations are given in the following expressions where the auxiliary variables
\[ A = e^{-\alpha \sigma^2 (T_j - \tau_j)} \]
\[ B = 1 - e^{-\alpha m S (T_j - \tau_j)} \]
\[ C = 1 - e^{-\alpha (\sigma^2 + m S) (T_j - \tau_j)} \]

have been defined:

\[ g_{11} + 1 = e^{-\alpha \lambda_1 \tau_j} \left[ A \left( 1 - \frac{\langle e_1, v_1 \rangle}{\Delta \beta m} \right) B \right] \]
\[ g_{22} + 1 = e^{-\alpha \lambda_2 \tau_j} \left[ A \left( 1 + \frac{\langle e_2, v_1 \rangle}{\Delta \beta m} \right) B \right] \]
\[ g_{12} = e^{-\alpha \lambda_2 \tau_j} \left[ A B \frac{\langle e_2, v_1 \rangle}{\Delta \beta m} \right] \]
\[ g_{21} = e^{-\alpha \lambda_1 \tau_j} \left[ A B \frac{\langle e_1, v_1 \rangle}{\Delta \beta m} \right] \]

\[ h_1 = -\frac{\sqrt{RS}}{\sigma^2 + m S} \cdot \frac{C}{\Delta \beta} - A(1-e^{-\alpha \lambda_1 \tau_j}) \left( 1 - \frac{\langle e_1, v_1 \rangle}{\Delta \beta m} \right) B \frac{\sqrt{RS}}{\lambda_1} \frac{\langle e_1, v_1 \rangle}{|e_1|^2} \]
\[ + A(1-e^{-\alpha \lambda_2 \tau_j}) B \frac{\sqrt{RS}}{\lambda_2} \frac{\langle e_2, v_1 \rangle^2}{|e_2|^2} \cdot \frac{1}{\Delta \beta m} \]

\[ h_2 = +\frac{\sqrt{RS}}{\sigma^2 + m S} \cdot \frac{C}{\Delta \beta} - A(1-e^{-\alpha \lambda_2 \tau_j}) \left( 1 + \frac{\langle e_2, v_1 \rangle}{\Delta \beta m} \right) B \frac{\sqrt{RS}}{\lambda_2} \frac{\langle e_2, v_1 \rangle}{|e_2|^2} \]
\[ - A(1-e^{-\alpha \lambda_1 \tau_j}) B \frac{\sqrt{RS}}{\lambda_1} \frac{\langle e_1, v_1 \rangle^2}{|e_1|^2} \cdot \frac{1}{\Delta \beta m} \]
APPENDIX C

It will be shown in this appendix that the mean weight vector given in Eq. (167) is a solution to the differential equation, Eq. (165), when certain constraints are satisfied by the parameters a, d, and \( \Delta \omega_r \). Substitution of Eq. (167) into Eq. (165) yields

\[
\sqrt{R} \sqrt{S} \left( 1 - \frac{\delta r}{T_b} \right) a \frac{\Delta \omega_r}{\alpha} e^{j(\Delta \omega_r t + \phi)} [K^{-1}v_1 + d v_2]
\]

for the left-hand side, and

\[
v_1 \sqrt{R} \sqrt{S} \left( 1 - \frac{\delta r}{T_b} \right) e^{j[\Delta \omega_s \delta r + \Delta \omega_r t + \phi - \Delta \omega_r \delta r + \text{Arg}(v_1^+ K^{-1}v_1 + d<v_1,v_2>)]}
\]

\[
\sqrt{R} \sqrt{S} \left( 1 - \frac{\delta r}{T_b} \right) a e^{j(\Delta \omega_r t + \phi)} [v_1 + d K_x v_2]
\]

for the right-hand side. When factors common to both sides are deleted, Eq. (165) becomes

\[
(232) \quad j \frac{a \Delta \omega_r}{\alpha} [K^{-1}v_1 + d v_2] = v_1 e^{j[\text{Arg}(v_1^+ K^{-1}v_1 + d<v_1,v_2>) + (\Delta \omega_s - \Delta \omega_r) \delta r]}
\]

\[
\quad - a[v_1 + d(S_v v_1^+ J v_2 v_2^+ + \sigma^2 I) v_2]
\]

The left-hand side may be expanded using

\[
(233) \quad K^{-1}v_1 = \frac{K_n^{-1}v_1}{(1 + S_v v_1^+ K_n^{-1}v_1)} = \frac{\sigma^{-2} v_1 - \frac{J \sigma}{(1 + m \sigma)^2} <v_1,v_2>v_2}{(1 + S_v v_1^+ K_n^{-1}v_1)}
\]

from Eq. (71). The components of Eq. (232) along \( v_2 \) are therefore given by
This equation is satisfied when the parameters \( d \) and \( \Delta \omega_r \) are related such that

\[
(235) \quad d = \frac{j \sigma^{-4} \langle v_1, v_2 \rangle}{(1 + S v_1^T K_n^{-1} v_1)(1 + m J_0^{-2})} \cdot Q(\Delta \omega_r)
\]

where

\[
Q(\Delta \omega_r) = \frac{j \frac{\Delta \omega_r}{\alpha}}{(m J + \sigma^2) + j \frac{\Delta \omega_r}{\alpha}}
\]

\[
= \frac{\Delta \omega_r [\Delta \omega_r + j \alpha (\sigma^2 + m J)]}{(\Delta \omega_r)^2 + [\alpha (\sigma^2 + m J)]^2}
\]

Similarly, the components of Eq. (232) along the vector \( v_1 \) are given by

\[
(236) \quad v_1 \left\{ j \Delta \omega_r \left[ \frac{\sigma^{-2}}{1 + S v_1^T K_n^{-1} v_1} \right] + 1 + d S \langle v_1, v_2 \rangle \right\} a = j [\text{Arg}(v_1^T K_n^{-1} v_1 + d \langle v_1, v_2 \rangle) + (\Delta \omega_s - \Delta \omega_r) \delta_r] v_1 \text{e}
\]

The real part of this equation is satisfied when

\[
(237) \quad a = \frac{\text{Re}(d) \cos[\text{Arg}(v_1^T K_n^{-1} v_1 + d \langle v_1, v_2 \rangle) + (\Delta \omega_s - \Delta \omega_r) \delta_r]}{1 + S \langle v_1, v_2 \rangle}.
\]

This expression may be used in place of the parameter \( a \) in writing the equation for the imaginary part of Eq. (236):
Substitution of Eq. (235) into Eq. (238) yields

\[
\frac{\Delta \omega_r}{\alpha^2 (1 + S_{v_1}^+ K_n^- v_1)} = \tan \left[ \arctan \left( \frac{1 + \alpha \sigma^2 (1 + S_{v_1}^+ K_n^- v_1) S\langle v_1, v_2 \rangle}{1 + S\langle v_1, v_2 \rangle} \frac{\text{Im}(d)}{\Delta \omega_r} \right) - \frac{1}{2} \right] + \left( \Delta \omega_s - \Delta \omega_r \right) \xi_r .
\]

Substitution of Eq. (235) into Eq. (238) yields

\[
\frac{\Delta \omega_r}{\alpha^2 (1 + S_{v_1}^+ K_n^- v_1)} P(\Delta \omega_r) = \tan \left[ \arctan \left( \frac{S_{v_1}^+ K_n^- v_1 + \frac{S\sigma^4 \langle v_1, v_2 \rangle}{(1 + m\sigma^2)^2}}{1 + \frac{S\sigma^4 \langle v_1, v_2 \rangle}{(1 + m\sigma^2)^2}} \frac{\text{Im}(Q(\Delta \omega_r))}{\Delta \omega_r} \right) + \left( \Delta \omega_s - \Delta \omega_r \right) \xi_r \right] .
\]

where

\[
P(\Delta \omega_r) = \begin{bmatrix}
1 + \frac{\alpha S\sigma^2 \langle v_1, v_2 \rangle^2}{(1 + m\sigma^2)^2} & \text{Im}(Q(\Delta \omega_r)) \\
S\sigma^4 \langle v_1, v_2 \rangle^2 & \text{Re}(Q(\Delta \omega_r))
\end{bmatrix}.
\]

The latter function can be algebraically simplified to

\[
P(\Delta \omega_r) = \frac{(\Delta \omega_r)^2 + C(C+A)}{(\Delta \omega_r)^2(1+B) + C^2}
\]

where

\[
A = \frac{\alpha S\sigma^2 \langle v_1, v_2 \rangle^2}{(1 + m\sigma^2)^2}
\]

\[
B = \frac{S\sigma^4 \langle v_1, v_2 \rangle^2}{(1 + S_{v_1}^+ K_n^- v_1)(1 + m\sigma^2)^2}
\]

\[
C = \alpha (\sigma^2 + mJ).
\]
It can be seen that the effect of this function is to change the left-hand side of Eq. (239) from a linear function of $\Delta \omega_r$. For a given value of input frequency offset, $\Delta \omega_s$, the value of $\Delta \omega_r$ can be obtained only by solving the transcendental equation, Eq. (239), numerically. The parameter $d$ then follows from Eq. (235), and the parameter $a$ from Eq. (237). When there is no interfering signal present, the parameters are given by simple expressions:

$$d = 0$$

For $J = 0$:

$$a = \cos[(\Delta \omega_s - \Delta \omega_r)\delta_r]$$

$$\frac{\Delta \omega_r}{\alpha(\sigma^2 + mS)} = \tan[(\Delta \omega_s - \Delta \omega_r)\delta_r] .$$
APPENDIX D
A SCHEMATIC OF THE EXPERIMENTAL SIGNAL-WEIGHTING AND WEIGHT-CONTROL (FEEDBACK) CIRCUITS
* DIODES ARE FROM A CA3039

RESISTANCE IN OHMS AND CAPACITANCE IN μF UNLESS NOTED OTHERWISE

OPERATIONAL AMPLIFIERS ARE BURR-BROWN
This appendix contains an analysis of the intermodulation distortion components produced in the weighting coefficients when excessively large interference is present at the inputs to these circuits. The objective of the analysis is to identify the dominant intermodulation product in the array output and to show that this product cannot be eliminated by weighting coefficient adjustment.

Constant-envelope signals are assumed in the analysis. Upon denoting the total input to a weighting circuit by the symbol "x", and the weight-control voltage input by "w", the output "y" may be represented in general by a series

\[ y = k_{00}w + k_0x + k_1wx + k_2w^2 + k_3x^2 + k_4x^3 + k_5w^3 + k_6wx^2 + \cdots. \]

Considering the first eight terms, only the third and sixth have in-band frequency components; the rest are rejected in the array output bandpass amplifier centered at \( \omega_c \). The output terms which are retained then are given by

\[ y = k_1wx + k_4x^3 + \cdots = k_1w(s+n) + k_4(s+n)^3 + \cdots = k_1w(s+n) + k_4[s^3 + 3sn^2 + 3s^2n + n^3] + \cdots \]

where the input \( x \) has been split into a desired signal-plus-thermal noise component "s" and an interference component "n". The most significant feature of this result is that the \( x^3 \) term is independent of \( w \), i.e., it cannot be controlled by weight-voltage adjustment. Not all of the components of \( x^3 \) are necessarily undesirable, however. For example, the \( s^3 \)-factor has an in-band component - the d.c. part of \( s^2 \) times \( s \) - which has the same phase information as in "s". The out-of-band component of \( s^3 \) centered at \( 3\omega_c \) is rejected in the output amplifier. If the in-band components of \( x^3 \) are denoted by the subscript "i", then Eq. (241) may be written as
\[ y = \left[ k_1 w_s + k_4 s_i^3 \right] + \]
\[ \left[ k_1 w_n + k_4 n_i^3 \right] + \]
\[ 3k_4 \left[ (s^2)_i + (s^2 n)_i \right] + \ldots . \]

The final step is to expand the cross-product (sxn) terms using the bandpass signal representations

\[ s = A_s(t) \cos(\omega_c t + \phi_s(t)) \]
\[ n = A_n \cos(\omega_c t + \phi_n(t)). \]

The thermal noise components of \( s \) are responsible for the time-variations in the envelope function \( A_s(t) \). It follows then that

\[ (s^2n)_i + (s^2n)_i = s' + n' + \frac{1}{2} A_s A_n \left\{ \cos(\omega_c t + 2\phi_n - \phi_s) + \frac{A_s}{A_n} \cos(\omega_c t + 2\phi_n - \phi_s) \right\} \]

where
\[ s' = \frac{1}{2} A_s A_n^2 \cos(\omega_c t + \phi_s) \]
\[ n' = \frac{1}{2} A_s A_n^2 \cos(\omega_c t + \phi_n). \]

Under large interference conditions, the second cosine term in Eq. (244) is negligibly small compared to the first cosine term. Thus, the dominant, in-band intermodulation product at the output of a weight multiplier is generated from the squared interference-by-signal plus noise product. In this output \( y \),

\[ y = \left[ k_1 w_s + k_4 s_i^3 + 3k_4 s'n' \right] + \]
\[ \left[ k_1 w_n + k_4 n_i^3 + 3k_4 n'n' \right] + \]
\[ \frac{3}{2} k_4 A_s A_n^2 \cos(\omega_c t + 2\phi_n - \phi_s) + \ldots , \]

the signal and interference components, represented in brackets, have amplitudes which are adjustable with weight-control voltage, \( w \),
whereas the amplitude of the dominant intermodulation product is fixed. The sum of the intermodulation components from each of the weighting circuits in the array, i.e., the output intermodulation component, will be non-zero in general.
APPENDIX F

The solution of the weight vector equation, Eq. (165), modified by the addition of an offset vector \( \Omega \), will be discussed in this appendix. The interfering signal will be assumed absent for simplicity so that the equation of interest is

\[
\frac{1}{\alpha} \frac{d}{dt} \mathbb{E}[w(t)] = \Omega + r_X(t) - K_X \mathbb{E}[w(t)]
\]

where

\[
r_X(t) = \sqrt{R} \sqrt{S} (1 - \frac{\delta r}{T_b}) e^{+j \Delta \omega s \delta r} + j \text{Arg} v_1 \mathbb{E}[w(t-\delta r)] v_1^T
\]

and

\[
K_X = S \frac{v_1}{v_1^*} v_1 + \sigma^2 I.
\]

In general, the offset vector has components both along and orthogonal to the vector \( v_1 \),

\[
\Omega = \frac{\langle v_1, \Omega \rangle}{m} v_1 + \frac{\langle v_0, \Omega \rangle}{|v_0|^2} v_0;
\]

\[
\langle v_0, v_1 \rangle = 0.
\]

A trial solution may be assumed of the form

\[
\mathbb{E}[w(t)] \big|_{t \to \infty} = \sqrt{R} \sqrt{S} (1 - \frac{\delta r}{T_b}) c(t) v_1 + \gamma_2 \Omega
\]

where \( c(t) \) is an unknown complex function and \( \gamma_2 \) is an unknown parameter. Substitution of Eq. (249) into Eq. (246) yields the following equation:
In order that this equation be satisfied along the vector component $v_0$, the condition

\begin{equation}
(251) \quad \gamma_2 = \frac{1}{\sigma^2}
\end{equation}

must be imposed. In addition, the function $c(t)$ must be chosen so that

\begin{equation}
(252) \quad \sqrt{\frac{S}{S'}} (1 - \frac{\delta_r}{T_b}) \left[ c(t) \frac{\dot{c}(t)}{\alpha} + (mS+\sigma^2)c(t) \right] + \frac{S}{\sigma^2} \langle v_1, \Omega \rangle
\end{equation}

is satisfied. When the offset vector is orthogonal to $v_1$, a solution for $c(t)$ may be obtained in the form

\begin{equation}
(253) \quad c(t) = e^{j(\Delta \omega \cdot t + \phi)} v_1 \quad ; \quad \langle v_1, \Omega \rangle = 0.
\end{equation}

Substitution of this expression into Eq. (252) yields the condition

\begin{align*}
\frac{\sqrt{R/S}}{a} (1 - \frac{\delta_r}{T_b}) \dot{c}(t) v_1 &= \sqrt{R/S} (1 - \frac{\delta_r}{T_b}) e^{j[\Delta \omega \cdot \delta_r + \theta(t)]} v_1 \\
- \sqrt{R/S} (1 - \frac{\delta_r}{T_b}) (mS+\sigma^2) c(t) v_1 \\
- \gamma_2 S \langle v_1, \Omega \rangle v_1 \\
+ (1 - \sigma^2 \gamma_2) \Omega, \quad t \to \infty
\end{align*}
\begin{equation}
\gamma_1 \left[ mS + \sigma^2 + j \frac{\Delta_{\omega} r}{\alpha} \right] = e^{\left[ \Delta_{\omega} r - \Delta_{\omega} r_s \right] \delta_r}
= \cos(\Delta_{\omega} r - \Delta_{\omega} r_s) \delta_r
+ j \sin(\Delta_{\omega} r - \Delta_{\omega} r_s) \delta_r.
\end{equation}

It follows then that \( \Delta_{\omega} r \) and \( \gamma_1 \) must satisfy

\begin{equation}
\frac{\Delta_{\omega} r}{\alpha (\sigma^2 + mS)} = \tan (\Delta_{\omega} r_s - \Delta_{\omega} r) \delta_r
\end{equation}

and

\begin{equation}
\gamma_1 = \frac{\cos (\Delta_{\omega} r_s - \Delta_{\omega} r) \delta_r}{\sigma^2 + mS} = \frac{a(\Delta_{\omega} r_s)}{\sigma^2 + mS}
\end{equation}

where the amplitude reduction parameter, \( a(\Delta_{\omega} r_s) \), is as defined previously in Eq. (168). The weight vector for this case is

\begin{equation}
E[w(t)]_{t \to \infty} = \sqrt{R} \delta_r (1 - \frac{\delta_r}{\frac{\sigma^2}{a}}) e^{j(\Delta_{\omega} r \cdot t + \phi_0)} \frac{a}{\sigma^2 + mS} \gamma_1 + \frac{\Omega}{\sigma^2}
= \sqrt{R} \delta_r (1 - \frac{\delta_r}{\frac{\sigma^2}{a}}) e^{j(\Delta_{\omega} r \cdot t + \phi_0)} \frac{a}{\sigma^2 + mS} \gamma_1 + \frac{\Omega}{\sigma^2},
\end{equation}

\[ \langle \gamma_1, \Omega \rangle = 0. \]

As the frequency offset of the desired signal approaches a large value, the weight vector reduces to the constant offset vector component, and the array output SNR approaches zero:

\begin{equation}
P_s = \frac{S \langle w, \gamma_1 \rangle^2}{\sigma^2 |w|^2} \quad \text{as} \quad \Delta_{\omega} r_s \to \infty.
\end{equation}

A solution has not been obtained for the case of more general offset vector with a component also along \( \gamma_1 \). It is apparent from these results, however, that SNR degradation will occur with input signal frequency offset whenever the offset vector has a component orthogonal to \( \gamma_1 \).
REFERENCES


