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THE LAMINAR BOUNDARY LAYER
ON A FINITE ROTATING DISC

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Richard James Bodonyi, B.S., M.S.

The Ohio State University
1973

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INTRODUCTION

The problems associated with the interaction of rotating flows with both stationary and moving surfaces have been of wide interest for many years. Meteorologists have long been concerned with such problems in connection with the study of tornadoes, hurricanes, and planetary circulation models. Oceanographers are interested in those rotating flows which are relevant to the understanding of large scale oceanic circulation models, of the coastal currents such as the Gulf Stream, and many other phenomenon. In the design of rotating machinery such as turbines there are many surfaces which encounter rotating fluids, many of which can be analyzed as planar surfaces. Thus the problem of determining the flow characteristics and heat transfer in the boundary layer developed by vortex flows on rotating surfaces has been an important one for many years, and it still remains at the center of interest for many researchers.

There have been many papers published on different aspects of the problems associated with rotating flows.
A book by Dorfman\textsuperscript{1} summarizes much of the theoretical work done prior to 1963 concerning the fluid mechanics and heat transfer problems for both finite and infinite rotating discs. Discussions of bodies and fluids in rotation have also been given in a review article by F. K. Moore\textsuperscript{2}, while the general theory of rotating flows has been presented in a recent book by H. P. Greenspan\textsuperscript{3}.

Boundary layers in rotating flows provide excellent examples of fully three-dimensional boundary layers whose properties are still far from being completely understood. Due to the complexity of the governing equations for such flows, solutions are usually sought for in the form of similarity variables whereby the number of independent variables is reduced by one. Thus for the axisymmetric flows the governing equations can be reduced from a coupled set of nonlinear partial differential equations to a coupled fifth-order system of nonlinear ordinary differential equations if an appropriate similarity variable can be found. Not all problems, however, can be simplified by similarity transformations. Besides transforming the governing equations into a similarity form, the similarity variable must be such that the boundary conditions can also be reduced to a similarity form. If the original boundary conditions are not related in a specific way then it will, in general, not be possible to find similarity solutions
to the original problem. Thus similarity solutions usually represent some rather idealized situations, but due to the simplifications which arise when they can be found, they play an important role in the study of both two and three-dimensional boundary layers. Furthermore, in many instances similarity solutions can be regarded as either initiators or terminators of more general boundary layer flows. For these reasons it is important to determine whether the similarity equations, assuming that they can be derived, have physically acceptable solutions; and, if not, what can be said about the structure of the flow in the region where the equations fail to give a solution.

The problem most often treated by theoreticians is that of the boundary layer induced on a disc by a circulatory motion of the fluid above it. Both finite and infinite discs have been considered in various studies, and these studies have been both of a numerical and analytical nature. For the most part the analytical studies have been concerned with the behavior of the appropriate similarity equations whose properties are reviewed in an article by Rott and Lewellen.

While attempting to explain the nonexistence of the classical similarity solution for the boundary layer induced under a potential vortex, Burggraf, Stewartson, and Belcher resolved the terminal flow structure for the
boundary layer induced by a potential vortex on a finite, stationary disc. Their conclusions were that while no solutions to the classical similarity equations exist, there is a double-similarity structure for the flow near the center of the disc. Near the wall there is an inner region of axial extent $O(r)$ which is dominated by an inward radial flow produced by a strong pressure gradient. The tangential velocity in this region is much smaller than the radial velocity. Above this region is an outer region where the flow is mainly inviscid, with the speed of the fluid independent of the axial distance and the tangential velocity increasing from zero to that of the potential vortex as the axial distance increases from zero to infinity.

The above problem was generalized by Belcher, Burggraf, and Stewartson to include a generalized vortex, i.e., a circulatory motion in which the tangential velocity is proportional to $r^{-n}$, where $r$ is the distance from the axis. They found that for $0.1217 \leq n < 1.0$ a multiple-structure terminal boundary layer exists, while for values of $n < 0.1217$ the multiple structure reduced to the classical similarity-type boundary layer.

Similarity equations can also be formulated for the problem in which the outer flow is in a state of rigid rotation and an infinite disc is allowed to rotate at
varying rates different from the uniformly rotating fluid both in magnitude and direction. Evans\textsuperscript{8} has considered the numerical solution of these similarity equations for the values of the parameter $\alpha$, the ratio of the angular speed of the disc to that of the fluid, varying over all possible values. From his numerical results he concluded that solutions could not be obtained for values of $\alpha$ in the range $-6.211 < \alpha < -0.650$ unless suction was applied at the surface. His numerical results also indicated that large gradients were appearing in the flow and that the boundary layer moved away from the surface of the disc into the interior of the fluid. McLeod\textsuperscript{9,10}, while studying these same similarity equations, gave a general proof for the existence of solutions when $\alpha > 0$, and for the case $\alpha = -1$ he proved rigorously from the differential equations that no solution is possible whether there is blowing at the disc or not. Later Ockendon\textsuperscript{23}, using the method of matched asymptotic expansions, showed that for $-\infty < \alpha < -0.6961$ there are no finite solutions of the similarity equations unless suction is applied at the disc.

The question of whether physically acceptable similarity solutions exist for $-6.211 < \alpha < -0.650$ is still not fully resolved; and, even if the similarity solutions do break down, there still remains the problem of determining the nature of the terminal structure of the boundary layer.
in the vicinity of the center of the disc. In an attempt to answer these questions, the time-dependent laminar boundary-layer equations for a finite rotating disc submerged in a rigidly rotating fluid have been numerically integrated for the special case $\alpha = -1$ and the results of this investigation are discussed in Chapter II.

Based on the results of the finite disc calculations further study of both the steady and unsteady forms of the similarity equations for an infinite rotating disc seemed to be warranted. Such a study was carried out and the details are presented in Chapters III and IV.

In an attempt to assess the effects of a finite disc on the boundary layer similarity solutions, Stewartson was led to a different class of similarity equations applicable to the initial development of the boundary layer on a finite disc. Mack has solved these equations numerically for several cases. However, no systematic study of the Stewartson similarity equations has been made for those cases where the disc is rotating in its own plane. Since the numerical computations for a finite rotating disc use the Stewartson similarity solutions as edge boundary conditions, it was felt that such a study of the equations was in order to determine whether there was any anomalous behavior of the solutions for the different disc speeds. The nature of the solutions turned
out to be quite unexpected and the results are discussed in Chapter V.
CHAPTER I
EQUATIONS OF MOTION

In this chapter the equations governing the flow above a finite rotating disc submerged in an otherwise unbounded rigidly rotating fluid will be derived. From these equations the appropriate similarity equations for the center and edge of the disc can readily be obtained.

Consider a fixed set of cylindrical polar coordinates \( r, \theta, z \) with the origin 0 attached to the center of a finite rotating disc of radius \( a \) and the axis \( 0z \) normal to the plane of the disc. Let the fluid occupy all space except for the region occupied by the disc at \( z = 0 \). Furthermore, assume that the fluid at all points in space except in the neighborhood of the disc is in a state of rigid rotation, thus

\[
\hat{u}(\hat{r}, \hat{z}) = 0, \quad \hat{v}(\hat{r}, \hat{z}) = \hat{r}, \quad \hat{w}(\hat{r}, \hat{z}) = 0 \quad (1-1)
\]

where \( \hat{u}, \hat{v}, \hat{w} \) are the radial, tangential, and axial velocity components, respectively, and \( \hat{\Omega} \) is the angular speed of the rigidly rotating fluid. Allowing the disc to rotate
with the angular speed \( \omega \), and applying the no-slip condition at the surface of the disc results in

\[
\hat{u} = 0, \quad \hat{v} = \omega \hat{r}, \quad \hat{w} = 0 \quad \text{for } \hat{z} = 0 \quad \text{on } \hat{r} \leq a \quad (1-2)
\]

It will now be assumed that the discrepancy between (1-1) and (1-2) can be resolved by means of an axisymmetric, incompressible, laminar boundary layer which begins at the edge of the disc and grows inward; hence the governing differential equations are from Moore\(^2\)

\[
\begin{align*}
\hat{u}_t + \hat{u}\hat{u}_r + \hat{w}\hat{u}_z - \frac{1}{\hat{r}} \hat{v}^2 &= -\frac{1}{\rho} \hat{p}_r + \nu \hat{u}_{zz} \\
\hat{v}_t + \hat{u}\hat{v}_r + \hat{w}\hat{v}_z + \frac{1}{\hat{r}} \hat{u} \hat{v} &= \nu \hat{v}_{zz} \\
\frac{1}{\hat{r}} (\hat{r} \hat{u})_r + \hat{w}_z &= 0
\end{align*}
\] (l-3a, l-3b, l-3c)

where subscripts denote partial derivatives, \( \rho \) is the density, assumed constant, \( \hat{p} \) the pressure, and \( \nu \) the kinematic viscosity. Note that the time-dependent terms have been retained for later use. The boundary conditions appropriate to this set of equations are given by
While these initial conditions can be quite arbitrary, the motion is generally thought to start from a state of rest or rigid rotation.

Upon using the conventional hypotheses of boundary-layer theory it can be shown that the pressure is independent of $\hat{z}$ and is given by the inviscid theory. The radial pressure gradient, $\hat{p}_r$, therefore can be evaluated from (1-3a) by considering the limit $\hat{z} \to \infty$ and using (1-4b). The result is

$$\frac{1}{\rho} \hat{p}_r = \frac{1}{\hat{r}} \hat{v}^2 \bigg|_{\hat{z} \to \infty} = \Omega^2 \hat{r}$$  (l-5)
It is more convenient to consider the equations in a nondimensional form, and for this purpose the following nondimensionalization is used:

\[ r = \hat{r}/a \]
\[ u = (an)^{-1}\hat{u} \]
\[ z = (n^2k^2_v) \]
\[ v = (an)^{-1}\hat{v} \]
\[ t = \hat{\Omega} \]
\[ w = (vn)^{-1}\hat{w} \]
\[ p = (\rho a^2n^2)^{-1}\hat{p} \]  

(1-6)

With these definitions the boundary-layer equations in nondimensional form are

\[ u_t + uu_r + wu_z - \frac{1}{r} v^2 = -r + u_{zz} \]  

(1-7a)

\[ v_t + uv_r + wv_z + \frac{1}{r} uv = v_{zz} \]  

(1-7b)

\[ \frac{1}{r}(ru)_r + w_z = 0 \]  

(1-7c)

with boundary conditions

\[ z = 0; \quad r < 1; \quad t > 0: \quad u = 0, \quad v = \alpha r, \quad w = 0 \]  

(1-8a)

\[ z \to \infty; \quad \text{all } r; \quad \text{all } t: \quad u + 0, \quad v + r \]  

(1-8b)
\[ t = 0; \text{all } r, z > 0: \ u = u_r, \ v = v_z \quad (1-8c) \]

where

\[ \alpha = \omega / n \quad (1-8d) \]

It is advantageous to introduce the stream function at this point. Defining \( \psi(r,z,t) \) in the usual manner

\[ ru = \psi_z; \ rw = -\psi_r \quad (1-9a) \]

the equation of continuity, (1-7c), is identically satisfied while the radial and tangential momentum equations, (1-7a, b) become

\[ \psi_{zt} + \frac{1}{r} \psi_z \psi_{rr} - \frac{1}{r^2} \psi_z^2 - \frac{1}{r} \psi_r \psi_{zzz} - v^2 = -r + \psi_{zzz} \quad (1-10a) \]

\[ v_t + \frac{1}{r} \psi_z v_r - \frac{1}{r} \psi_r v_z + \frac{1}{r^2} v \psi_z = v_{zz} \quad (1-10b) \]

and the boundary conditions are transformed to

\[ z = 0; \ r \leq 1; \ t > 0: \ \psi = 0, \ \psi_z = 0, \ v = ar \quad (1-11a) \]

\[ z \rightarrow \infty; \ \text{all } r; \ \text{all } t: \ \psi_z \rightarrow 0, \ v \rightarrow r \quad (1-11b) \]
Equations (1-10) and (1-11) define the boundary-value problem for the development of a laminar, axisymmetric boundary layer on a finite rotating disc placed in a rigidly rotating fluid of infinite extent. These equations have been studied both numerically and analytically for several limiting cases, and the results of these studies are discussed in the chapters to follow.
CHAPTER II
BOUNDARY LAYER DEVELOPMENT ON A FINITE ROTATING DISC

The properties of the laminar boundary layer generated on a rotating disc of a finite radius in a rotating fluid are of interest for a variety of reasons. For example, there are many applications where a rotating fluid is enclosed within a body, e.g., in rotating machinery such as turbines. Many surfaces in such bodies can be analyzed as planar ones, and the question naturally arises as to the behavior of the boundary layer on these surfaces. Also, the question of the applicability and pertinence of the similarity solutions for both finite and infinite discs to more general flow fields is always present. Thus, even though similarity solutions can be found in many instances, it is not known a priori whether they actually represent an adequate approximation to the flow under consideration. One way to determine the validity and limitations of these solutions is to study the more general problem in some manner and compare the results with the known similarity solutions. Finally, the boundary layer generated on a finite rotating disc by a rotating fluid is of interest because it
is an example of a fully three-dimensional boundary layer whose properties are still far from being understood.

Rogers and Lance\textsuperscript{14}, in an attempt to ascertain the validity of the Bödewadt similarity solution for an infinite disc, investigated the boundary layer in a rotating fluid over a stationary finite disc. Using a series expansion derived by Stewartson\textsuperscript{11} which is valid near the outer edge of the disc, they deduced that Bödewadt's solution is a good approximation over the inner half of the disc and concluded that the boundary layer flow in the central region depends essentially on the local inviscid flow field.

Mack\textsuperscript{12} has investigated the steady laminar boundary layer on a stationary disc of finite radius in a rotating flow using the momentum integral method. The rotating outer flow was chosen such that the radial velocity was zero and the tangential velocity was a power of the radius $r^{-n}$. For the boundary-layer velocity profiles both polynomials and the Stewartson edge similarity solutions were used. In his numerical computations Mack calculated the radial mass flow in the boundary layer, the axial outflow velocity, boundary-layer thickness, and the direction of the surface streamline as functions of the radial position. For the potential vortex, $n = 1$, he found that the mass flow increased monotonically with decreasing radius. In all other cases he considered, however, the maximum radial
inflow existed at some radius greater than zero and was zero at the axis. Due to the approximate method used in his analysis there is no means of assessing the error involved unless the results are compared to an exact solution. Furthermore, the details of the flow field are not known with any accuracy.

More recently a complete study of the properties of the laminar boundary layer produced by a generalized vortex (one with the tangential velocity proportional to a power of the radius $r^{-n}$) on a fixed coaxial disc of radius $a$ has been carried out by Belcher, Burggraf, and Stewartson. Using a new method of integration to determine the flow field, numerical computations were made for $n = -1.0, 0.0, 0.5, +1.0$. From their computations they were able to deduce the terminal structure of the steady boundary layer even though they could not numerically integrate the equations all the way to the axis $r = 0$. It was also found from a steady asymptotic analysis that for values of $n < 0.1217$ the classical similarity solutions for the flow over an infinite disc described the appropriate terminal structure for the finite disc solution, whereas for $0.1217 < n < 1$ the classical similarity solutions failed to exist. They deduced that the terminal structure for these cases consisted of an infinite number of regions where viscosity can be
neglected, each separated by a thin viscous transition region.

The problems considered heretofore concerned the boundary layer on a stationary disc under a generalized vortex. It is also of interest to inquire as to the structure of the laminar boundary layer induced on a finite rotating disc by a rigidly rotating fluid far from the disc. This problem is of importance in its own right; and also, the results can be used to determine the applicability of the steady similarity solutions for both finite and infinite rotating discs, when they exist, to physical problems. The applicability of these solutions remains in doubt especially when the disc and outer flow are rotating in opposite directions. When the similarity solutions do not exist there still remains the problem of determining the terminal structure of the boundary layer.

The objective of this chapter is to consider, in detail, the numerical solution of the unsteady boundary-layer equations for a finite rotating disc placed in an otherwise unbounded rigidly rotating fluid. The numerical scheme used in the computations was developed by Belcher and will be reviewed below.

The appropriate boundary-layer equations in nondimensional form are given by equation (1-7). For the numerical computations the known similarity variables for the center
and edge of the disc are utilized by defining a new vertical coordinate in place of $z$. Thus let

$$y = \frac{z}{(1-r)^{l}} \tag{2-la}$$

The velocity components are also scaled to take advantage of the appropriate similarity forms, hence write

$$u(r,z,t) = r(l-r)^{\frac{1}{2}} U(r,y,t) \tag{2-lb}$$
$$v(r,z,t) = rV(r,y,t) \tag{2-1c}$$
$$w(r,z,t) = (1-r)^{-\frac{1}{2}} W(r,y,t) \tag{2-1d}$$

Using these transformations equations (1-7) become

$$(l-r)^{\frac{1}{2}} U_t = U_{yy} - (W+kryU)U_y$$
$$-r(1-r)UU_r - l + V^2 + (\frac{5}{2} . r - l)U^2 \tag{2-2a}$$

$$(l-r)^{\frac{1}{2}} V_t = V_{yy} - (W+kryU)V_y$$
$$-r(1-r)UV_r - 2(l-r)UV \tag{2-2b}$$

$$W_y + kryU_y + r(1-r)U_r + (2 - \frac{5}{2} r)U = 0 \tag{2-2c}$$
and from (1-8) the boundary conditions are

\[ y = 0; \quad r \leq l; \quad t > 0: \quad U = 0, \quad V = \alpha, \quad W = 0 \quad (2-3a) \]

\[ y \to \infty; \quad \text{all} \quad r; \quad \text{all} \quad t: \quad U \to 0, \quad V = 1 \quad (2-3b) \]

\[ t = 0; \quad \text{all} \quad r; \quad y > 0: \quad U = 0, \quad V = 1 \quad (2-3c) \]

It should be noted that for the numerical computations the initial conditions, (2-3c), have been specialized so that the inviscid solution holds everywhere except at the disc itself. This condition can be realized physically by allowing both the disc and the fluid above it to rotate with the same angular speed \( \Omega \) and at time \( t = 0 \) impulsively changing the disc's speed from \( \Omega \) to \( \omega \).

Since the major interest lies in the behavior of the steady boundary layer on a finite rotating disc one might be tempted to start with the steady boundary-layer equations and numerically integrate them using a forward marching procedure from the edge of the disc towards the axis. As the boundary-layer equations are parabolic this procedure would seem to be applicable. However, it has been shown by Belcher, Burggraf, and Stewartson\(^ {14} \) that if the square of the circulation in the outer flow increases with the radius then both the radial and tangential velocity profiles
oscillate about their inviscid values for any \( r < 1 \) with amplitudes that diminish exponentially as the outer edge of the boundary layer is approached. This result is valid whether or not the disc is rotating. Hence the steady boundary-layer equations will of necessity have regions of reverse flow, and these will cause the steady forward marching procedure to ultimately become unstable.

To avoid the difficulties encountered in the numerical solution of the steady boundary-layer equations the time-dependent form of the equations (2-2) were numerically integrated using the method developed by Belcher\(^1\), and the steady-state solutions were sought by letting time become large.

To numerically integrate the parabolic equations forward in time all exterior boundary conditions of the region of interest must be specified for \( t > 0 \). The appropriate conditions at the edge \( r = 1 \) can be deduced by noting that equation (2-2) indicates that the time derivatives are singular at the edge of the disc. This means that the steady-state solution is established instantaneously there. Physically this observation is consistent with the fact that the boundary-layer has zero thickness at \( r = 1 \), i.e., the boundary-layer starts at \( r = 1 \) and grows inward towards the axis. Consequently the Stewartson similarity profiles (to be
discussed in Chapter V) can be applied both as boundary and initial conditions at \( r = 1, \ t \geq 0 \). For \( r < 1 \) the no-slip condition (2-3a) is applied at \( y = 0 \) for all \( t \), while at the axis \( r = 0 \), the radial symmetry property, \( \left( \frac{\partial}{\partial r} \right)_{r=0} = 0 \), is applied for \( y > 0, \ t \geq 0 \). The true outer boundary condition of a rigidly rotating fluid (2-3b) is applied at \( y_e \), where \( y_e \) is a suitably large value of \( y \) taken to approximate the conditions at infinity. Finally, for the initial conditions at \( t = 0 \), the outer edge condition (2-3c) is applied all the way down to the disc.

Equations (2-2) were solved numerically using a partially implicit finite-difference scheme. The difference mesh is defined by uniform spacings in the \( r \) and \( y \) directions with the node points given by

\[
\begin{align*}
  r_i &= 1-(i-1)\Delta r \quad (i=1,2,...,M+1) \\
  y_j &= (j-1)\Delta y \\n  y_j &= (j-1)\Delta y 
\end{align*}
\]

where \( M = \frac{1}{\Delta r} \) and \( N = \frac{y_e}{\Delta y} \). Letting \( q_{ij} \) denote a velocity component at the node point \((r_i, y_j)\) for the current time and \( \bar{q}_{ij} \) the known velocity component at the previous time \( \bar{t} = t-\Delta t \), the difference equations are written for the
intermediate time \( t^* = \frac{1}{4}(t + \bar{t}) \) in the manner of Crank and Nicholson\(^\text{22} \) with \( q_{1j}^\# = \frac{1}{4}(\overline{q}_{1j} + q_{1j}) \) except for the continuity equation (2-2c), which is solved for \( W \) at the new time using the known values of \( U \). Thus the time derivatives are replaced by the centered difference

\[
\frac{3q}{\Delta t} \rightarrow \frac{1}{\Delta t} (q_{1j} - \overline{q}_{1j})
\]

\( y \) derivatives by the centered differences

\[
\frac{3q}{\Delta y} \rightarrow \frac{1}{2\Delta y} (q_{1,j+1}^\# - q_{1,j-1}^\#)
\]

\[
\frac{3^2q}{\Delta y^2} \rightarrow \frac{1}{(\Delta y)^2} \left(q_{1,j+1}^\# - 2q_{1j}^\# + q_{1,j-1}^\#\right)
\]

and the \( r \) derivatives by the uncentered differences

\[
\frac{3q}{\Delta r} \rightarrow \frac{1}{\Delta r} (q_{1,j+1} - \overline{q}_{1,j}) \quad \text{if } U_{1,j} > 0
\]

\[
\frac{3q}{\Delta r} \rightarrow \frac{1}{\Delta r} (q_{1,j} - \overline{q}_{1,j-1}) \quad \text{if } U_{1,j} < 0
\]

Note that the radial derivatives are evaluated using downwind differencing at the old time \( \bar{t} \), thus making the scheme
partially explicit.

The finite difference analog of equations (2-2) for any radial station \( r_1 \) can be expressed in matrix notation as

\[
\mathbf{M} \mathbf{U} = \mathbf{R} \quad \text{and} \quad \mathbf{M} \mathbf{V} = \mathbf{S}
\]  

(2-4)

where \( \mathbf{M} \) is a square tridiagonal matrix whose elements are the coefficients in the difference equations for \( U \) and \( V \), while \( \mathbf{U} \) and \( \mathbf{V} \) are column vectors representing the \( U_{ij} \) and \( V_{ij} \) velocity components at the radial station of interest. \( \mathbf{R} \) and \( \mathbf{S} \) are also column vectors whose elements contain both known and unknown values of the flow properties. If the matrix \( \mathbf{M} \) is defined by

\[
\begin{bmatrix}
  b_1 & c_1 & 0 & \cdots & 0 \\
  a_2 & b_2 & c_2 & 0 & \cdots & 0 \\
  & & & & & \\
  & & & & & \\
  & & & & & \\
  0 & 0 & 0 & \cdots & a_j & b_j & c_j & 0 & \cdots & 0 \\
  & & & & & & & & & \\
  & & & & & & & & & \\
  & & & & & & & & & \\
  0 & \cdots & \cdots & \cdots & 0 & a_{N+1} & b_{N+1}
\end{bmatrix}
\]

(2-5)

then the elements of \( \mathbf{M} \) are given by
\( a_j = -1 + \frac{\Delta y}{2}[W_{1j}^* + kr_1 y_j U_{1j}^*] \)
\[ b_j = 2[1 + (1-r_1)^2 \frac{(\Delta y)^2}{\Delta t}] \quad j=2,\ldots,N \]
\( c_j = -(2 + a_j) \)  \hspace{1cm} (2-6)

while the column vectors \( \vec{R} \) and \( \vec{S} \) have the elements

\[
R_j = -2[1-(1-r_1)^2 \frac{(\Delta y)^2}{\Delta t}]\bar{U}_{1j} + \bar{U}_{1,j-1} + \bar{U}_{1,j+1}.
\]

\[
- [\bar{U}_{1,j+1} - \bar{U}_{1,j-1}] \cdot [W_{1j}^* + kr_1 y_j U_{1j}^*] \frac{\Delta y}{2} \quad (2-7)
\]

\[
+2(\Delta y)^2 \left\{ -1 + V_{1j}^* - u_{1j}^* [r_1(1-r_1)U_{r1j} + (1-\frac{3}{2}r_1)U_{1j}^*] \right\}
\]

\[
S_j = -2[1-(1-r_1)^2 \frac{(\Delta y)^2}{\Delta t}]\bar{V}_{1j} + \bar{V}_{1,j-1} + \bar{V}_{1,j+1}
\]

\[
- [\bar{V}_{1,j+1} - \bar{V}_{1,j-1}] \cdot [W_{1j}^* + kr_1 y_j U_{1j}^*] \frac{\Delta y}{2} \quad (2-8)
\]

\[
-2(\Delta y)^2 [r_1 V_{r1j} + 2V_{1j}^*(1-r_1)U_{1j}^*]
\]

The first and last rows of \( \vec{R} \) and the elements of \( \vec{R} \)
and $\mathcal{S}$ are defined such that the appropriate boundary conditions (2-3) are satisfied, thus

\[ b_1 = 1, \quad c_1 = 0, \quad R_1 = 0, \quad S_1 = a \]

\[ a_{N+1} = 0, \quad b_{N+1} = 1, \quad R_{N+1} = 0, \quad S_{N+1} = 1 \]

The $N+1$ unknowns $U_{ij}$, $V_{ij}$ at each radial station $r_i$ are solved for iteratively using Gaussian elimination with the latest values for $U_{ij}$ and $V_{ij}$ used in the matrices $\mathcal{M}$, $\mathcal{R}$, and $\mathcal{S}$. After each iteration the continuity equation is integrated to obtain the new value of the axial velocity $W_{ij}$ from

\[ W_{i,j+1} = W_{ij} - k r_i (y_1 + \Delta y) [U_{i,j+1} - U_{ij}] \]

\[ -\Delta y \left\{ r_i (1-r_i) [U_{r_1,j+1} + U_{r_1 j}] \right\} \]

\[ + \frac{1}{4} (4 - 5 r_i) (U_{i,j+1} + U_{ij}) \]

In order to save on both computer time and storage, the radial derivatives appearing in the equations are evaluated using the old time step values. This allows the iteration at each radial station to be completed without sweeping the entire flow field before iteration at each
radial station. Once convergence was obtained at a given radial station the next radial station was advanced until the entire flow field had been computed. Then time was advanced and the entire process was repeated until, hopefully, a steady solution was found or until the iterations failed to converge.

The finite-difference equations (2-4) contain one parameter, namely, $\alpha$, the ratio of the disc's angular speed to that of the angular speed of the rigidly rotating outer flow. Owing to a singularity on the axis in the generalized vortex, Belcher used a transformed radial coordinate $x = -\ln r$ and solved the corresponding finite-difference equations in $(x,y,t)$ variables for the special case $\alpha = 0$. With his choice of variables however, he was not able to consider the entire region $0 \leq r \leq 1$; rather he considered the solution in the annulus $0.145 \leq r \leq 1$. For $t = 35.0$ he found that his solution had essentially reached its steady-state behavior, and at $r = 0.145$ it agreed favorably with the Bödewadt similarity solution for an infinite stationary disc submersed in a rigidly rotating fluid. In fact the similarity solution seems to be applicable over the inner 40% of the disc.

Steady similarity solutions for an infinite disc in a rigidly rotating fluid can also be found for values of $\alpha \neq 0$, and presumably these solutions describe the behavior
of the flow near the center of a finite rotating disc. It is known however, that for α = -1.0 no solution exists for the steady similarity equations appropriate near the center of the disc (McLeod^9), and the question naturally arises as to the structure of the steady flow field near the center of the rotating disc in this case.

In an attempt to answer this question, the differential equations (2-2) and (2-3) with α = -1.0 were solved numerically using the finite-difference analogs (2-4). To achieve a balance between accuracy and reasonable computer time and storage the various step sizes were taken to be \( \Delta r = 0.05, \Delta y = 0.30 \) and \( \Delta t = 0.02, 0.04 \).

Acceptable numerical solutions for the entire disc were found for all \( t < 2.28 \); while for \( t = 2.28 \) the numerical computations did not converge in 500 iterations at \( r = 0 \), and at this point the numerical procedure was terminated. The numerical results indicate that the flow field is approaching a steady solution for the outer 40% of the disc in a relatively short time period. However, near the center of the disc the velocity components show no signs of approaching a steady solution.

Figures 1 through 4 show the transformed radial and tangential velocity profiles at \( r = 0.80 \) and \( r = 0.60 \) for several values of \( t \). As can be seen in Figures 1 and 2 for \( r = 0.80 \) both the radial and tangential velocities have
essentially reached their steady values for $t = 2.0$. It should be noted that the computed profiles overshoot their outer values and exhibit damped oscillations for large $y$ as required by the steady asymptotic analysis of Belcher, Burggraf, and Stewartson. These oscillations, however, have very small amplitudes and are not discernible on the scale used in the figures.

Figures 3 and 4 depict the transformed velocity profiles at $r = 0.60$ and the trend is similar to that for $r = 0.80$. Figure 3 shows that the tangential component has reached a steady solution for all $y$ when $t > 2.0$. Figure 4, on the contrary, indicates that the steady solution has essentially been reached by $t = 2.0$ only in the vicinity of the disc; when $y > 1.8$ the radial component still exhibits a time dependence for $t > 2.0$.

The numerical computations indicate that a steady solution is not obtained over the inner half of the disc. Evidence of this can be seen by considering the time-dependent radial and tangential wall shears as functions of radius which are given in Figures 5 and 6 respectively. For $r > 0.60$ and $t > 2.0$ both wall shears are essentially independent of time implying that, near the disc at least, the solutions have reached their steady-state. This trend is confirmed by the velocity profiles given in Figures 1 through 4 and discussed above. For $r < 0.60$, however, no
such trend is evident. As shown in Figures 5 and 6 both the radial and tangential wall shears vary significantly with radius and time. Note that for \( r < .40 \) and \( t > .20 \), \( V_y(r,o,t) \) has become negative, and concurrently \(-U_y(r,o,t)\) has decreased towards zero.

Representative radial and tangential velocity profiles for the inner half of the disc are given at \( r = 0.20 \) and \( r = 0.0 \) in Figures 7 through 10 for several values of \( t \). The tangential velocity component clearly shows no signs of approaching a steady solution for either value of \( r \); in fact, quite the contrary is implied by the figures. Likewise, the radial velocity component at \( r = 0.0 \) is not converging to a steady solution, and I feel that the behavior of the radial component at \( r = 0.20 \) for \( t = 2.0 \) and \( 2.2 \) is more of a coincidence rather than an indication of the approach to a steady solution.

The behavior of the flow field near the outer edge of the boundary layer can be inferred from Figure 11 which gives axial outflow, \( W(r,\omega,t) \), as a function of time for selected radii. Near the edge of the disc, \( w(r,\omega,t) \) approaches its steady value quickly; at \( r = 0.90 \) \( W(r,\omega,t) \) is essentially independent of \( t \) for \( t > 1.0 \), while for \( r = 0.80 \) the steady-state is reached for \( t > 1.6 \). Also, the time dependence on \( W(r,\omega,t) \) is small for \( t > 2.0 \) at \( r = 0.60 \). These results support the earlier conclusions that a
Figure 1. Time Development of the Radial Velocity at \( r = 0.8 \)
Figure 2. Time Development of the Tangential Velocity at $r = 0.8$
Figure 3. Time Development of the Tangential Velocity at $r = 0.6$
Figure 4. Time Development of the Radial Velocity at $r = 0.6$
Figure 5. Radial Development of the Radial Wall Shear
Figure 6. Radial Development of the Tangential Wall Shear
steady-state solution is obtained for the outer 40% of the disc. The results further indicate that in the steady-state fluid is being drawn into the boundary layer for \( r > 0.60 \) while it is being ejected from the boundary layer into the rigidly rotating fluid above the disc when \( r < 0.60 \). Finally, the variation of \( W(r,\omega,t) \) for values of \( r < 0.60 \) gives further evidence that a steady solution is not being approached over the inner 60% of the disc.

The behavior of the numerical solutions in the vicinity of the axis \( r = 0.0 \) is especially interesting. Figures 9 and 10 clearly show that the boundary layer thickens considerably with time, doubling in thickness as \( t \) increases from 2.0 to 2.2. Also, the magnitudes of \( U(r,y,t) \) and \( V(r,y,t) \) are increasing substantially. Although not shown in the figures, it is found that \( |U|_{\text{max}} = 5.13 \) and \( |V|_{\text{max}} = 3.45 \) at \( t = 2.2 \), doubling from their respective values at \( t = 2.0 \).

The axial outflow from the boundary layer at \( r = 0 \), shown in Figure 11, gives further evidence of the growth occurring in the boundary layer at the center of the disc. For example, \( W(0,\omega,t) = 36 \) at \( t = 2.0 \) and has increased to 146 for \( t = 2.2 \), indicating that the time-dependent boundary layer erupts in a relatively short time. In fact, the numerical computations with the edge condition \( y = \omega \) approximated by \( y = 60.0 \) failed to converge at \( r = 0 \) in 500
Figure 7. Time Development of the Radial Velocity at $r = 0.2$
Figure 8. Time Development of the Tangential Velocity at $r = 0.2$
Figure 9. Time Development of Radial Velocity at $r = 0$

$U = (1-r)^{-\frac{1}{2}} r^{-1} u$
Figure 10. Time Development of Tangential Velocity at $r = 0$
Figure 11. Time development of Axial Velocity at $y \to \infty$. 
iterations for $t = 2.28$.

The results of the numerical study indicate that no acceptable steady solution is possible over the entire disc, although a steady solution is seemingly reached in the vicinity of the edge of the disc. Near the center of the disc fluid is being expelled from the boundary-layer in ever increasing amounts as shown in Figure 11, suggesting a possible eruption of the boundary-layer at the axis in a finite time. Thus it seems that a laminar boundary layer will not permit a stable solution to exist for the flow field over a finite rotating disc submerged in a rigidly rotating fluid when the disc and fluid are rotating with the same speed in opposite directions.

The numerical computations discussed up to now have only been for the special case $\alpha = -1$. It is also of interest to consider other values of $\alpha$, in particular those lying in the range $-1 < \alpha < 0$. From Evans' numerical study it is known that solutions of the appropriate steady similarity equations can be found for $\alpha \geq -0.65$, and the question naturally arises as to the behavior of the boundary layer on a finite rotating disc when $\alpha$ lies in this range. While a complete study of the boundary layer equations was not undertaken for values of $\alpha$ differing from $-1$, a preliminary study for $\alpha = -0.50$ indicated that the behavior was similar to that for the case $\alpha = -1$. 
Based on these findings it was decided to investigate both the steady and unsteady forms of the similarity equations near the center of a rotating disc more fully, and the results of these studies are explained in detail in Chapters III and IV to follow.
The results of the last chapter have shown that the unsteady boundary-layer equations for a finite rotating disc are not well behaved in the vicinity of the axis, \( r = 0 \), when the disc rotates in the opposite sense to that of the rigidly rotating outer flow. Although only the case \( \alpha = -1 \) was considered in detail, a preliminary investigation showed that a similar behavior occurred at the axis when the disc and the fluid at infinity were rotating in the opposite sense with different angular speeds.

If, in the boundary layer equations (2-2) \( r \) is set equal to zero, the unsteady similarity equations for an infinite rotating disc submersed in a rigidly rotating fluid are obtained; and based on the results of Chapter II, a detailed study of these similarity equations in terms of the parameter \( \alpha \) seems warranted to determine the range of values of \( \alpha \) for which meaningful solutions exist for both the steady and unsteady forms of the similarity equations.

The existence of self-similar solutions for the steady incompressible boundary layer developed under a fluid in circulatory motion bounded by an infinite rotating disc
has been known for some time. Von Kármán, in 1921, found an exact solution of the steady, incompressible Navier-Stokes equations for an infinite disc rotating in its plane with a uniform angular velocity in a fluid initially at rest by reducing the Navier-Stokes equations to a set of ordinary differential equations via a similarity transformation. Due to the rotation of the disc, the fluid near the disc spirals outward under the action of the centrifugal force, thereby causing an axial inflow from the stationary fluid. An important feature of Von Kármán's solution is that it does not exhibit any oscillatory behavior in the boundary layer. Actually, it is the only solution of the steady similarity equations which does not oscillate. Any amount of rotation far from the disc is sufficient to produce an oscillatory motion in the boundary layer.

Bödewadt in 1940 investigated the opposite extreme of a stationary infinite disc placed in a fluid undergoing rigid rotation. The same similarity equations hold as for Kármán's problem, except now the boundary conditions are different. The resulting solution is quite different, however. The rigidly rotating fluid far from the disc imposes an inward radial pressure gradient on the fluid near the disc which, due to viscous effects, produces a radial inflow near the disc in contrast to the radial outflow in Von Kármán's solution. Furthermore, Bödewadt's numerical
results showed that significant overshoots occur in the radial and tangential velocities followed by damped oscillations about their limiting values. As a result the boundary layer at a fixed radius consists of alternate layers of fluid, some moving inward along the disc, others moving outward. The applicability of this solution for a finite disc was questioned for sometime and is discussed in detail by Moore. The problem is that the flow begins at infinity and moves inward through the boundary layer and is ejected from the boundary layer in a continuous manner. Also, the inwardly moving fluid must lose all information about its original state and fall under the control of the local inviscid flow outside the boundary layer. Evidence that the Bödewadt similarity solution is the proper terminal solution near the center of the disc has been given by Belcher, Burggraf, and Stewartson.

Batchelor considered in a qualitative manner a class of solutions of the Navier-Stokes equations representing steady axisymmetric flow of which Kármán's and Bödewadt's solutions are special cases. Allowing the disc to rotate with angular speed \( \omega \) and the outer flow to be rigidly rotating with angular speed \( \Omega \), he was able to reduce the Navier-Stokes equations to a one-parameter, coupled set of ordinary differential equations using Kármán's similarity variables. He discussed, qualitatively, the
nature of the solutions for values of the parameter, $\alpha = \omega/\Omega$, ranging from minus infinity to plus infinity, erroneously concluding that there is a solution for each value $\alpha$. He further deduced that a boundary layer exists near the disc and that when the disc rotates faster than the fluid far from it, but in the same sense, $\alpha > 1$, there is a net axial flow inward towards the disc, and when $0 < \alpha < 1$ there is a net outflow of fluid away from the disc.

When $\alpha < 0$ he anticipated that the radial velocity would be outwards near the disc independent of the relative sizes of $|\Omega|$ and $|\omega|$, and inwards sufficiently far from the disc, i.e., the disc would act locally like a centrifugal fan. Apparently the critical plane on which the axial velocity vanished divided the flow field into two self-contained regions; the dividing plane coinciding with the disc for $\alpha = 0$, and moving outwards toward infinity as $\alpha$ decreases from zero to minus infinity. It must be noted, however, that later McLeod proved conclusively that the flow will be radially outward near the disc only if $|\omega| > |\Omega|$. Furthermore, the radial velocity far from the disc will not always be radially inward as envisioned by Batchelor since the asymptotic structure indicates an oscillatory behavior.

Rogers and Lance investigated numerically the
one-parameter boundary-value problem discussed by Batchelor. From their results they concluded that when the fluid and disc are rotating in the same direction, $\alpha > 0$, physically acceptable similarity solutions exist for all cases. However, when $\alpha < 0$ they were not able to find physically acceptable solutions unless suction was applied at the wall to keep the boundary layer attached to the disc.

Evans also considered the problem formulated by Batchelor. The main objective of his study was to consider the effect of suction on the boundary layer. However, as part of his numerical study he confirmed the results of Rogers and Lance for $\alpha > 0$, and he found that there do exist physically acceptable solutions when the fluid and the disc are rotating in opposite directions, contrary to the results of Rogers and Lance. In terms of the parameter $\alpha$, Evans was able to find numerical solutions without suction for the following range of values:

$$-\infty < \alpha < -6.211 \text{ and } -0.65 < \alpha < 0$$

Solutions for the range $-6.211 < \alpha < -0.65$ could not be found numerically due to the large gradients in the radial velocity at the wall and the formation of a shear layer in the interior of the flow.

The questions of the existence and uniqueness of the
solution of the boundary-value problem formulated by Batchelor have been investigated in detail by McLeod. When the fluid at infinity and the disc are rotating in the same direction \((a > 0)\), the existence of solutions to the boundary-value problem with or without suction at the surface has been conclusively proved by McLeod; and uniqueness of the solution has been shown for the special case \(a = 1\). Existence theorems for the cases when the fluid at infinity and the disc are rotating in opposite directions with different angular speeds have not been given yet. However, the nonexistence of a solution to the boundary-value problem has been proved by McLeod for the special case when the disc and the outer flow rotate in opposite directions with the same speed \((a = -1)\) and there is no suction at the wall, although blowing at the wall is permitted in the proof. This nonexistence of a solution for \(a = -1\) is especially interesting if one recalls that Evans could not obtain numerical solutions for \(-6.211 < a < -0.65\). Hence, at least for \(a = -1\), there is a breakdown of the similarity equations governing the flow near the center of a rotating disc; and the question naturally arises whether there is a breakdown of the equations for other values of \(a\).

Ockendon studied the similarity equations for small values of a suction parameter and found, using asymptotic methods, the first-order solution for all values of \(a\) in the
range $|\alpha^{-1}| < 1.436$. Her results have confirmed the non-existence of a finite solution for $\alpha = -1$ when there is no suction applied at the surface.

Since the question of the limiting value of $\alpha$ for which steady similarity solutions exist without suction has not yet been completely resolved and the structure of the flow field in the vicinity of the breakdown is not known, a further study of the steady similarity equations seems to be in order. Such a study was undertaken in this thesis, the results of which will be presented in this chapter.

In order to assess the acceptability of the steady similarity solution corresponding to a particular value of $\alpha$, the unsteady similarity equations for the same value of $\alpha$ were integrated forward in time from a specified initial state. In those cases where the steady solution could not be reached by the time-dependent process, the unsteady similarity equations were reformulated as a linearized eigenvalue problem to ascertain whether the steady solutions were stable or unstable to small perturbations. The details of the unsteady calculations will be deferred to Chapter IV.

**Governing Equations for $r = 0$ Similarity**

The unsteady similarity equations for an infinite
rotating disc placed in an otherwise unbounded rigidly rotating fluid are easily obtained from the incompressible axisymmetric Navier-Stokes equations in cylindrical coordinates which are given by Lagerstrom:

\[ u_t + uu_r + wu_z - \frac{1}{r}v^2 = -p_r + \frac{1}{r}u_r + u_{zz} - \frac{1}{r^2}u \quad (3-1a) \]

\[ v_t + uv_r + wv_z + \frac{1}{r}uv = v_{rr} + \frac{1}{r}v_r + v_{zz} - \frac{1}{r^2}v \quad (3-1b) \]

\[ w_t + uw_r + ww_z = -p_z + w_{rr} + \frac{1}{r}w_r + w_{zz} \quad (3-1c) \]

\[ \frac{1}{r}(ru)_r + w_z = 0 \quad (3-1d) \]

where subscripts denote partial derivatives. Equations (3-1) were nondimensionalized using equations (1-6). For the infinite disc, however, there is no geometric length scale occurring in the formulation of the problem, and for this reason the viscous length scale, \((\nu/\eta)^{1/2}\), should be used in place of \(a\) in equations (1-6) for the proper nondimensionalization of variables.

Following von Kármán, similarity solutions are sought in the form

\[ u(r,z,t) = r f_z(z,t) \quad (3-2a) \]
\[ v(r,z,t) = rg(z,t) \quad (3-2b) \]
\[ w(r,z,t) = -2f(z,t) \quad (3-2c) \]
\[ p(r,z,t) = 2r^2 + h(z,t) \quad (3-2d) \]

Substituting (3-2) into (3-1) the governing equations reduce to

\[ f_{zt} = f_{zzz} + 2ff_{zz} - f_z^2 + g^2 - 1 \quad (3-3a) \]
\[ g_t = g_{zz} + 2fg_z - 2gf_z \quad (3-3b) \]
\[ h_z = 2(f_t + ff_z - f_{zz}) \quad (3-3c) \]

with boundary and initial conditions given by

\[ z = 0; \ t > 0: \ f = f_z = 0; \ g = \alpha \quad (3-4a) \]
\[ z \to \infty; \ \text{all} \ t: \ f_z \to 0, \ g \to 1, \ h \to 0 \quad (3-4b) \]
\[ t = 0; \ \text{all} \ z: \ f_z = f^*(z); \ g = g^*(z). \quad (3-4c) \]

The pressure term, \( h(z,t) \), is uncoupled from the equations for \( f(z,t) \) and \( g(z,t) \); thus \( f \) and \( g \) are found
first from (3-3a, b), and then \( h(z,t) \) is found from (3-3c) by integration. It should also be noted that apart from the equation giving \( h(z,t) \), the unsteady similarity equations (3-3a, b) derived above from the Navier-Stokes equations are identical to the similarity equations obtained by substituting equations (3-2) into the boundary-layer equations (1-7). In this instance the terms neglected in boundary-layer theory, viz.

\[
\frac{1}{r} u_r + u_{rr} - \frac{u}{r^2}; \quad \frac{1}{r} v_r + v_{rr} - \frac{v}{r^2}
\]

are identically zero when the similarity transformations are made. Thus solutions of the similarity equations (3-3) are solutions of both the boundary-layer equations and the Navier-Stokes equations.

In order to deduce the properties of the solution to the similarity equations as a function of the parameter \( \alpha \), both the steady and unsteady form of equations (3-3) have been investigated; the results of the former study will be discussed in the remainder of this chapter.

**Steady-State Similarity**

The appropriate steady-state similarity equations are obtained from equation (3-3) by setting the partial
derivatives with respect to \( t \) equal to zero, thus

\[
\frac{d^3f_s}{dz^3} + 2f_s \frac{df_s}{dz} - \left( \frac{df_s}{dz} \right)^2 + g_s^2 - 1 = 0 \quad (3-5a)
\]

\[
\frac{d^2g_s}{dz^2} + 2f_s \frac{dg_s}{dz} - 2f_s' g_s = 0 \quad (3-5b)
\]

with boundary conditions

\[
z = 0: f_s = f_s' = 0; g_s = \alpha \quad (3-6a)
\]

\[
z \rightarrow \infty: f_s' \rightarrow 0; g_s \rightarrow 1 \quad (3-6b)
\]

where the subscript \( s \) denotes a steady-state function and \((\quad)\)' = \( \frac{d}{dz} \).

As mentioned previously, numerical solutions of these equations for various values of the parameter \( \alpha \) have been obtained by both Rogers and Lance\textsuperscript{21} and Evans\textsuperscript{8}. For values of \( \alpha > 0 \), no difficulties were encountered in obtaining solutions; however, for \( \alpha < 0 \) Rogers and Lance were unable to find acceptable solutions and Evans was able to find solutions only for \( \alpha \geq -0.65 \). For \( \alpha < -0.65 \) Evans found that, due to the large gradients involved, his integration step size became so small that computation times became excessive. His numerical method was essentially of the initial-value type except that he guessed values of the
unknown functions and gradients at both ends of the range of integration. Numerical integrations of the differential equations were carried out using the Kutta-Merson method in two directions; starting at the boundaries \( z = 0 \) and \( z = 12 \), and proceeding inwards to the mid-point of the interval. At the midpoint corrections to the initial guesses were made using the Newton-Raphson iterative method; the process being repeated until the differences in the numerical solutions and their derivatives were less than \( 1 \times 10^{-6} \) at the mid-point of the interval.

In order to study the behavior of the similarity equations in the vicinity of \( a = -0.65 \) more closely, it was decided to carry out further numerical integrations of the similarity equations using a different numerical method than that used by Evans. Hopefully the problems of small step sizes and excessive computer time could be avoided and further solutions of the equations could be found. The numerical computations are simplified if the similarity equations (3-5) are rewritten in the following form:

\[
\begin{align*}
\frac{d^2 u_s}{ds^2} + 2f_s u_s' - u_s^2 + g_s^2 - 1 &= 0 \quad (3-7a) \\
\frac{d^2 g_s}{ds^2} + 2f_s g_s' - 2u_s g_s &= 0 \quad (3-7b)
\end{align*}
\]
with boundary conditions

\[ z = 0: \quad u_s = 0; \quad g_s = a \quad (3-8a) \]

\[ z \to \infty: \quad u_s \to 0; \quad g_s \to 1 \quad (3-8b) \]

Numerical approximations for the solution of equations (3-7) subject to (3-8) are found using a computer program developed by Dr. Odus R. Burggraf. The finite difference analog of the governing equations are obtained using centered differences, and the resulting nonlinear difference equations are solved recursively by assuming initial profiles for \( u_s \) and \( g_s \). Using the notation

\[
\mathbf{Y}_j = \begin{bmatrix} u_{sj} \\ g_{sj} \end{bmatrix} \quad j = 1, 2, \ldots, N \quad (3-9a)
\]

and defining successive iterates by

\[
\mathbf{Y}_j^{(\nu+1)} = \mathbf{Y}_j^{(\nu)} + \delta \mathbf{Y}_j^{(\nu)} \quad \nu = 0, 1, \ldots \quad (3-9b)
\]
the difference equations yield the following vector-matrix equation for the correction terms \( \delta Y_j^{(v)} \), where quadratic terms in \( \delta Y_j^{(v)} \) have been neglected

\[
M_{ij}^{(v)} \delta Y_j^{(v)} = R_i^{(v)} \quad i, j = 1, 2, \ldots, N \quad (3-9c)
\]

The matrix \( M_{ij}^{(v)} \) is a block tridiagonal matrix defined by

\[
M_{ij}^{(v)} = \begin{bmatrix}
B_1^{(v)} & C_1^{(v)} & 0 & 0 & \cdots & 0 \\
A_2^{(v)} & B_2^{(v)} & C_2^{(v)} & 0 & \cdots & 0 \\
0 & \cdots & A_j^{(v)} & B_j^{(v)} & C_j^{(v)} & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & A_N^{(v)} & B_N^{(v)} & 0 \\
\end{bmatrix} \quad (3-9d)
\]

and \( A_j^{(v)} \), \( B_j^{(v)} \), \( C_j^{(v)} \), are 2 x 2 matrices with elements given by

\[
A_j^{(v)} = \begin{bmatrix}
-1 + (\Delta z) f_{s_j}^{(v)} & 0 \\
0 & -1 + (\Delta z) f_{s_j}^{(v)} \\
\end{bmatrix} \quad (3-9e)
\]
\[
B_j(v) = \begin{bmatrix}
2(1-2(\Delta z)^2)u_{sj}(v) & -2(\Delta z)^2g_{sj}(v) \\
2(\Delta z)^2g_{sj}(v) & 2(1+2(\Delta z)^2)u_{sj}(v)
\end{bmatrix}
\]

\[j = 2, \ldots, N-1 \] (3-9f)

\[
C_j(v) = \begin{bmatrix}
-(1+\Delta z)f_{sj}^{(v)} & 0 \\
0 & -(1+\Delta z)f_{sj}^{(v)}
\end{bmatrix}
\]

\[i = 2, \ldots, N-1 \] (3-9g)

\[R_1^{(v)} \] is a \((2 \times 1)\) matrix defined by

\[
R_1^{(v)} = \begin{bmatrix}
-(\Delta z)^2+(1-(\Delta z)f_{s1}^{(v)})u_{s1-1}(v) & -2(1-(\Delta z)^2)u_{s1}(v) \\
+(1+(\Delta z)f_{s1}^{(v)})u_{s1+1}(v) & +(\Delta z)^2(g_{s1}^{(v)})^2 \\
(1-(\Delta z)f_{s1}^{(v)})g_{s1-1}(v) & -2(1+2(\Delta z)^2)u_{s1}(v)g_{s1}(v) \\
+(1+(\Delta z)f_{s1}^{(v)})g_{s1}(v)
\end{bmatrix}
\]

\[i = 2, \ldots, N-1 \] (3-9h)

From the specification of the boundary conditions (3-8) we find

\[
B_1^{(v)} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}; \quad C_1^{(v)} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

(3-9i)
Given an initial profile $\Psi_j^{(0)}$ the system of linear equations (3-9c) is solved for $\delta \Psi_j^{(v)}$ and a new correction for $\Psi_j^{(v)}$ is found from equation (3-9b). The iteration procedure is continued until $|\delta \Psi_j^{(v)}|$ is less than some prescribed amount, taken to be $1 \times 10^{-5}$ or $1 \times 10^{-6}$ in this study; at this point the solutions are said to converge and the results, $\Psi_j^{(v)}$, are taken as the solution to the original boundary-value problem defined by equations (3-7) and (3-8).

Numerical solutions were found for values of $\alpha$ ranging from +1.0 to -0.689043. For reference the numerical values of $u_s'(0)$, $g_s'(0)$, and $f_s(\alpha)$ have been listed in Table 1 as functions of the parameter $\alpha$. The results for $\alpha \geq 0$ agreed well with the earlier results of Rogers and Lance\textsuperscript{21} and Evans\textsuperscript{8}, and will not be discussed further.

The behavior of the solutions for $\alpha < 0$ is of primary concern. With a step size $\Delta z = 0.05$ and the outer
edge taken to be $z = 20$, solutions found using the method described above for $-.65 < \alpha < 0$ agreed favorably with those of Evans. However, whereas Evans was not able to obtain solutions for smaller values of $\alpha$, the method used in this study permitted new solutions for $-.689043 < \alpha < -.65$ to be found; and the resulting solutions clearly indicated that a critical value of $\alpha$ exists for which finite solutions of equations (3-7) can be found. In an analysis to be discussed below it will be shown that this critical value of $\alpha$ is $\alpha_{cr} = -0.6961$.

Figures 12 and 13 give the transformed radial and tangential velocity profiles for several values of $\alpha$ while Figure 14 shows the behavior of the transformed stream function for the same values of $\alpha$. It is clearly seen from these profiles that the amplitudes of both velocity components and the stream function are becoming unbounded as $\alpha \to \alpha_{cr}$, suggesting that no finite solution exists for $\alpha = \alpha_{cr}$.

Further evidence for the breakdown of the similarity equations as $\alpha$ approaches its critical value is given in Figure 15 in which the radial and tangential wall shears are given as functions of $\alpha$. The singular behavior in the wall shears is readily apparent only for values of $\alpha$ quite close to $\alpha_{cr}$; and even then only the radial wall shear strongly shows the presence of the singular behavior.
### TABLE 1

Transformed Radial and Tangential Wall Shear and Stream Function at $z \to \infty$, from the Steady $r = 0$ Similarity Equations

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$u'_s(0)$</th>
<th>$g'_s(0)$</th>
<th>$f'_s(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
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<td>0.90000</td>
<td>-0.10101</td>
<td>0.09600</td>
<td>-0.05045</td>
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<td>0.80000</td>
<td>-0.19956</td>
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<td>-0.10419</td>
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<tr>
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<td>0.27676</td>
<td>-0.16152</td>
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<td>0.36119</td>
<td>-0.22272</td>
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<tr>
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<td>0.44138</td>
<td>-0.28806</td>
</tr>
<tr>
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<td>-0.35770</td>
</tr>
<tr>
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<td>-0.43161</td>
</tr>
<tr>
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<td>0.65443</td>
<td>-0.50932</td>
</tr>
<tr>
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<td>0.71558</td>
<td>-0.58968</td>
</tr>
<tr>
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<td>0.77156</td>
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<td>-0.68904</td>
<td>-13.8474</td>
<td>1.80000</td>
<td>1.29744</td>
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Figure 12. Steady $r = 0$ Similarity, Radial Velocity

Asymptotic Theory
Eq'n (3-32c)
Figure 13. Steady $r = 0$ Similarity, Tangential Velocity

Asymptotic Theory
Eq'n (3-32c)
Figure 14. Steady r = 0 Similarity, Stream Function
Figure 15. Steady $r = 0$ Similarity, Radial and Tangential Wall Shears
As can be seen from Figure 15, the changes in \( g'(0) \) with \( \alpha \) are much less than the changes in \( u_s'(0) \), indicating that to first order the radial velocity will dominate the tangential velocity in the vicinity of the wall.

**Limiting Solution for \( r = 0 \) Similarity**

The singular behavior of the numerical solutions as \( \alpha \) approaches its critical value suggests that asymptotic methods for small values of the parameter \( \alpha - \alpha_{cr} \) may be useful in deducing both the critical value of \( \alpha \), \( \alpha_{cr} \), and the limiting structure of the flow field. The asymptotic analysis to be discussed below is similar to that used by Ockendon\(^2\) to study the flow above an infinite rotating disc with suction.

Near the surface of the disc there is a thin layer of fluid where the viscous and inertia terms are in balance. To examine this region in detail the normal coordinate, \( z \), is stretched by the following scaling

\[
\xi = e^{-k}z \quad \text{(3-10a)}
\]

while the appropriate scalings for the dependent variables are

\[
f_a(z) = e^{-k} \left[ f_0(\xi) + e f_1(\xi) + \ldots \right] \quad \text{(3-10b)}
\]
where

$$g_s(z) = g_0(\xi) + \varepsilon g_1(\xi) + \ldots$$

(3-10c)

$$\varepsilon = \alpha - \alpha_{cr}$$

(3-10d)

and $\alpha_{cr}$ is to be found.

Upon substituting equations (3-10) into the steady similarity equations (3-5) and equating like powers of $\varepsilon$, the following set of differential equations is obtained

\[

t'' + 2t't'' - (t')^2 = 0 \tag{3-11a}
\]

\[

g'' + 2f_0g_0' - 2f_0'f_0 g_0 = 0 \tag{3-11b}
\]

\[
\begin{align*}
  f_1'' + 2f_0f_1'' + 2f_0'f_1 - 2f_0f_1' &= 1 - g_0^2 \tag{3-12a} \\
  g_1'' + 2f_0g_1' - 2f_0'g_1 + 2f_1g_0' - 2f_1'g_0 &= 0 \tag{3-12b}
\end{align*}
\]

The boundary conditions at $\xi = 0$ are found by comparing equations (3-6) and (3-10), thus
The boundary conditions as \( \xi \to \infty \) are replaced by the requirement that the solutions do not become exponentially large as this would preclude any chance of matching the solutions to an outer layer.

The solution of equation (3-lla) which satisfies the appropriate boundary conditions is easily found to be

\[
f_0(\xi) = -\frac{1}{2} A\xi^2 \tag{3-14}
\]

While \( A \) is an arbitrary constant at this stage, in the appendix to Ockendon's paper it is shown that the boundary conditions on \( f_s(z) \) and \( g_s(z) \) as \( z \to \infty \) cannot be satisfied unless \( A > 0 \). Thus in what follows it will be assumed that \( A \) is positive. Using the result for \( f_0(\xi) \) in equation (3-llb), the governing equation for \( g_0 \) is

\[
ge_0'' - A\xi^2 g_0' + 2A\xi g_0 = 0 \tag{3-15a}
\]

subject to

\[
ge_0(0) = a_{cr} \tag{3-15b}
\]
The general solution of (3-15a) can be written down in terms of confluent hypergeometric functions. To see this, make the transformation

\[ t = \frac{1}{3} A \xi^3 \]  
(3-16a)

\[ g_0(t) = \bar{g}_0(t) \]  
(3-16b)

then equation (3-15) becomes

\[ t\bar{g}_0'' + \left( \frac{2}{3} - t \right) \bar{g}_0' + \frac{2}{3} \bar{g}_0 = 0 \]

This is Kummer's form of the confluent hypergeometric equation and the general solution is given by

\[ \bar{g}_0(t) = \bar{a}_0 \, _1F_1\left( -\frac{2}{3}; \frac{2}{3}; t \right) + \bar{b}_0 \, t^{1/3} \, _1F_1\left( -\frac{1}{3}; \frac{4}{3}; t \right) \]

where \( \bar{a}_0 \) and \( \bar{b}_0 \) are arbitrary constants and

\[ _1F_1(a; b; t) = \sum_{n=0}^{\infty} \left( \frac{a}{b} \right)_n \frac{t^n}{n!} \]

Thus in terms of the original variables the general solution of (3-15a) is
The boundary condition (3-15b) requires that $a_o = a_{cr}$, and a linear relation between $a_o$ and $b_o$ is found from the requirement that $g_o(\xi)$ not be exponentially large for $\xi \to \infty$. Using the asymptotic expansion of $\text{I}_1$ for large $\xi$ given by Abramowitz and Stegun\textsuperscript{31}, equation (3-19) becomes

$$g_o(\xi) = a_o \text{I}_1\left(-\frac{2}{3};\frac{2}{3};\frac{1}{3} A\xi^3\right) + b_o \text{I}_1\left(-\frac{4}{3};\frac{4}{3};\frac{1}{3} A\xi^3\right)$$

$$g_o(\xi) \sim \xi^{-4/3} e^{1/3 A\xi^3} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{4}{3}\right)^n \left(\frac{1}{3} A\xi^3\right)^{-n}$$

$$\left\{a_o \left(\frac{1}{3} A\right)^{-4/3} \frac{\text{I}_1\left(\frac{2}{3}\right)}{\text{r}\left(-\frac{2}{3}\right)} + b_o \left(\frac{1}{3} A\right)^{-5/3} \frac{\text{I}_1\left(\frac{4}{3}\right)}{\text{r}\left(-\frac{1}{3}\right)}\right\}$$

$$+ \xi^2 \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{1}{3}\right)^n (-\frac{2}{3})^n (-\frac{1}{3} A\xi^3)^{-n} \left\{a_o \left(\frac{1}{3} A\right)^{2/3} e^{-2/3\pi i} \frac{\text{I}_1\left(\frac{2}{3}\right)}{\text{r}\left(\frac{2}{3}\right)} + b_o \left(\frac{1}{3} A\right)^{1/3} e^{-\pi i/3} \frac{\text{I}_1\left(\frac{4}{3}\right)}{\text{r}\left(\frac{4}{3}\right)}\right\}$$

(3-17)
Thus in order to avoid exponential growth in (3-17) it is required that

\[ b_o = -2a_o 3^{2/3} \frac{A^{4/3}}{r(\frac{\eta_0}{3})} \left( \frac{r(\frac{\eta_0}{3})}{r(\frac{1}{3})} \right)^2 \]  

(3-18)

and \( g_o(\xi) \) is therefore given by

\[ g_o(\xi) = \alpha_{cr} \left\{ 1 F_1(-\frac{2}{3},\frac{2}{3};\frac{1}{3} A\xi^3) - 2 \cdot 3^{2/3} A^{1/3} \left( \frac{r(\frac{\eta_0}{3})}{r(\frac{1}{3})} \right)^2 \right\} \]

(3-19)

To leading order the solution for large \( \xi \) is

\[ g_o(\xi) \sim \beta_0 \xi^2 + O(\xi) \]  

(3-20a)

where

\[ \beta_0 = -\alpha_{cr} 3^{1/3} \frac{r(\frac{2}{3})}{r(\frac{1}{3})} A^{2/3} \]  

(3-20b)
With the solutions for $f_0(\xi)$ and $g_0(\xi)$ known we now proceed to the equation for $f_1(\xi)$ given by

$$f_1''' - A\xi^2 f_1' + 2A\xi f_1' - 2Af_1 = 1 - g_0^2$$  \hspace{1cm} (3-21)

with boundary conditions

$$f_1(0) = 0 = f_1'(0)$$

As can be verified by direct substitution the complementary solution of (3-21) is

$$f_{1c}(\xi) = c_{11}\xi + c_{12}\xi^2 + c_{13}\xi \int \xi^{-3} e^{1/3A\xi^3} d\xi d\xi$$  \hspace{1cm} (3-22)

To find a particular solution try the substitution

$$f_{1p} = -\frac{1}{2} \xi r(\xi)$$

then the differential equation (3-21) becomes

$$\xi r'' + (3 - A\xi^3)r'' = 2(g_0^2 - 1)$$

and a solution is given by
Finally the general solution for $f_1(\xi)$ is

$$f_1(\xi) = c_{11}\xi + c_{12}\xi^2 + c_{13}\xi^3 \int_{x}^{\xi} e^{1/3A\xi^3} \, d\xi$$

$$-\xi \int_{x}^{\xi} \frac{y}{n-3} e^{1/3A\xi^3} \, dn \int_{x}^{\xi} x^2 (g_{0}^2-1)e^{-1/3A\xi^3} \, dx \quad (3-23)$$

To determine the constants the boundary conditions at $\xi = 0$ must be applied. Note that for $\xi \to 0$

$$\int_{x}^{\xi} e^{1/3A\xi^3} \, d\xi = \frac{1}{2} \xi^2 + O(\xi^4)$$

$$f_1 - \frac{1}{12}(a_{cr}^2-1)\xi^3 + O(\xi^4)$$

Applying the appropriate boundary conditions for $f_1(\xi)$ it is readily seen that
The solution for \( f'(\xi) \) is, therefore,

\[
f_1(\xi) = c_{12}\xi^2 - \xi \int_0^\xi \int_0^n \int_0^n e^{1/3A\xi^3} \, dn \, \int_0^n x^2 (g_0^2 - 1) e^{-1/3A\xi^3} \, dx
\]

(3-24)

In particular it is found from equation (3-24) that

\[
f_1''(\xi) = e^{1/3A\xi^3} \left\{ -A \int_0^\xi x^2 g_0^2 e^{-1/3A\xi^3} \, dx + 1 \right\} - g_0^2(\xi)
\]

(3-25)

Equation (3-25) indicates that \( f_1(\xi) \) will become exponentially large as \( \xi + \infty \) unless

\[
\int_0^\infty x^2 g_0^2 e^{-1/3A\xi^3} \, dx = \frac{1}{A}
\]

or, using the change of variables (3-16)

\[
\int_0^\infty g_0^2(t) e^{-t} \, dt = 1
\]

(3-26)
where \( g_o(t) \) satisfies

\[
t g_o'' + \left( \frac{2}{3} - t \right) g_o' + \frac{2}{3} g_o = 0
\]

\( g_o(0) = \alpha_{cr} \)

Defining \( g_o(t) = \alpha_{cr} g_o^*(t) \) equation (3-26) reduces to

\[
\int_0^\infty g_o^* e^{-t} dt = \alpha_{cr}^{-2}
\] (3-27)

The integral appearing in (3-27) has been evaluated by Ockendon\(^{23}\), and its value is 2.064.

Hence the critical value of \( \alpha \) is given by

\[
\alpha_{cr} = -0.6961
\]

This number defines a limiting value of \( \alpha \) for which finite solutions of the similarity equations can be found. It is not surprising, therefore, that numerical solutions could not be found for values of \( \alpha < -0.68904 \).

Finally, the asymptotic form of \( f_1(\xi) \) for large \( \xi \) is found to be of the form
Summarizing the results for the inner region when $\xi$ is large, we find

$$f_1(\xi) \sim \frac{1}{6A} \beta_0 \eta^4 + O(\xi)$$

The expansions in the inner region fail when $\xi = O(\varepsilon^{-1/2})$ and the variables must be rescaled in order to study the intermediate region. The appropriate scalings are given by

$$z = \varepsilon^{-1/4} \eta$$

$$f_\beta(z) = \varepsilon^{-5/4} (\overline{F}_0(\eta) + \varepsilon^{3/2} \overline{F}_1(\eta) + \ldots)$$

$$g_\beta(z) = \varepsilon^{-1} (\overline{G}_0(\eta) + \varepsilon^{3/2} \overline{G}_1(\eta) + \ldots)$$
Substituting equations (3-29) into equations (3-5) and equating the coefficients of like powers of $\varepsilon$ to zero, it is found that, to leading order, the equations are inviscid, i.e.,

\[ 2\overline{T}_o T'_o - (T'_o)^2 + \overline{E}_o^2 = 0 \] (3-30a)

\[ \overline{T}_o \overline{E}_o' - T'_o \overline{E}_o = 0 \] (3-30b)

The boundary conditions on $\overline{T}_o$ and $\overline{E}_o$ as $n \to 0$ are found by matching the functions with the solutions in the inner region as the inner variable $\xi \to \infty$. Thus

\[ \overline{T}_o(n) = - \frac{1}{2} A n^2 ; \quad T'_o(n) = - A n ; \quad \overline{E}_o(n) = B_o n^2 \quad \text{as} \quad n \to 0 \] (3-31)

The solutions to (3-30) satisfying (3-31) are

\[ \overline{T}_o(n) = - \lambda^2 A [1 - \cos(\frac{n}{\lambda})] \] (3-32a)

\[ \overline{E}_o(n) = - \lambda A [1 - \cos(\frac{n}{\lambda})] \] (3-32b)
\[ F'_0(\eta) = -\lambda A \sin \left( \frac{\eta}{\lambda} \right) \]  \hspace{1cm} (3-32c)

where

\[ \lambda = -\frac{A}{2\beta_0} = 0.6857a_{cr}^{-1} A^{1/3} \]  \hspace{1cm} (3-32d)

The expansions in the intermediate region breakdown when \( \frac{\eta}{\lambda} \rightarrow 2\pi \) because then the functions \( F'_0(\eta) \) and \( g_0(\eta) \) both are zero in the limit. In order to continue the solution further the normal coordinate must be stretched in the neighborhood of \( \eta = 2\lambda \pi \) where there will be another viscous layer. As before the appropriate scalings for variables are

\[ z = e^{-k_2} 2\lambda \pi + \epsilon k_1 \zeta \]  \hspace{1cm} (3-33a)

\[ f_b(z) = e^{-k_2} (\hat{f}_0(\zeta) + \epsilon \hat{f}_1(\zeta) + ...) \]  \hspace{1cm} (3-33b)

\[ g_b(z) = \hat{g}_0(\zeta) + \epsilon \hat{g}_1(\zeta) + ...) \]  \hspace{1cm} (3-33c)

The differential equations for this region are identical to those of the inner region; the equations for the
leading terms being given by

\[ \hat{f}_0''' + 2\hat{f}_0\hat{f}_0'' - (\hat{f}_0')^2 = 0 \]
\[ \hat{g}_0'' + 2\hat{f}_0\hat{g}_0' - 2\hat{f}_0'\hat{g}_0 = 0 \]

The boundary conditions for \( \zeta \to -\infty \) are found by matching with the intermediate region for \( \eta \to 2\lambda \pi \). Thus as \( \zeta \to -\infty \)

\[ \hat{f}_0 \to -\frac{1}{2} A\zeta^2 \]
\[ \hat{g}_0 \to \frac{1}{2} \frac{A}{\lambda} \zeta^2 \]
\[ \hat{f}_0' \to -A\zeta \]

As discussed by Ockendon\(^2\), the boundary condition on \( \hat{f}_0(\zeta) \) as \( \zeta \to +\infty \) can take one of three possible forms:

(a) \( \hat{f}_0 \) can approach a single solution with the asymptotic form

\[ \hat{f}_0(\zeta) \sim \hat{c}_0 + \hat{d}_0 e^{-\hat{c}_0 \zeta} + \ldots \quad \text{as } \zeta \to +\infty \]

where \( \hat{c}_0, \hat{d}_0 \), are constants to be found.
(b) \( \hat{f}_o \) has the single solution
\[
\hat{f}_o(\xi) = -\frac{1}{2} \Lambda \xi^2
\]

(c) \( \hat{f}_o \) has an infinite number of solutions such that
\[
\hat{f}_o(\xi) = \beta \xi^2
\]
where \( \beta \) is an arbitrary positive constant.

The solution for the thin viscous region corresponding to case (c) above will not occur because then it is impossible to satisfy the boundary conditions on \( f'(z) \) and \( g(z) \) as \( z \to \infty \). However, both cases (a) and (b) above are acceptable, and hence the behavior of the limiting solution as \( \alpha \to \alpha_{cr} \) is not unique. In fact, using case (b) it is possible to find solutions containing any number of thick inviscid regions each bounded by a thin transition layer before the boundary conditions on \( f'(z) \) and \( g(z) \) are satisfied.

The numerical solutions given in Figures 12, 13, and 14 indicate that \( f'(z) \), \( g(z) \), and \( f(z) \) have only one inviscid region followed by a viscous region in which the solutions asymptotically approach their outer values. This behavior suggests that (a) above is the appropriate form for \( \hat{f}_o(\xi) \) as \( \xi \to \infty \). Thus for the outer region the boundary-
value problem to be solved is given by

\[ \hat{f}'''' + 2\hat{f}'\hat{f}'' - (\hat{f}')^2 = 0 \]  
(3-34a)

\[ \hat{g}'''' + 2\hat{f}'\hat{g}'' - 2\hat{f}'\hat{g} = 0 \]  
(3-34b)

with boundary conditions

\[ \hat{f}_0 \to -\frac{1}{2} \lambda \zeta^2 \quad \text{as} \quad \zeta \to -\infty \]  
(3-35a)

\[ \hat{g}_0 + \frac{1}{2} \frac{\lambda}{\lambda} \zeta^2 \quad \text{as} \quad \zeta \to -\infty \]  
(3-35b)

\[ \hat{f}_0' \to 0; \quad \hat{g}_0 \to 1 \quad \text{as} \quad \zeta \to +\infty \]  
(3-35c)

The properties of the solution to equation (3-34a) subject to boundary conditions (3-35a, c) have been investigated by Ockendon who found that the solution exists and is unique apart from an arbitrary shift in \( \zeta \).

The parameter \( A \) appearing in the differential equation for \( \hat{f}_0 \) can be removed from the problem by the following change of variables:

\[ \hat{f}_0(\zeta) = -\frac{1}{2} A^{1/3} y(x) \]

\[ x = A^{1/3} \zeta \]
then
\[ y'''' - yy'' + \frac{1}{2}(y')^2 = 0 \quad (3-36a) \]

\[ y + x^2 \quad \text{as} \quad x \to -\infty; \quad y' \to 0 \quad \text{as} \quad x \to +\infty \quad (3-36b) \]

Kuiken has studied equation (3-36) both analytically and numerically. The asymptotic solutions for \( y(x) \) as \( x \to \pm \infty \) are given by

\[ y_{x \to \infty} \sim x^2 + yx \int_{-\infty}^{x} dt \int_{-\infty}^{t} \frac{1}{s^3} e^{1/3s^3} ds + ... \]

\[ y_{x \to +\infty} y(\infty) + \delta e^{y(\infty)} x + ... \]

and from his numerical solutions \( y(\infty) = -1.209625 \). Thus the asymptotic forms of \( \hat{f}_o(t) \) are

\[ \hat{f}_o(t)_{t \to \infty} \sim -\frac{1}{2} A t^2 + b_o A^{2/3} t \int_{-\infty}^{t} \frac{1}{s^3} e^{1/3s^3} ds + ... \quad (3-37) \]

\[ \hat{f}_o(t)_{t \to +\infty} 0.604812 A^{1/3} + c_o A^{1/3} e^{-1.209625A^{1/3}t} \quad (3-38) \]
where \( \hat{b}_o \) and \( \hat{c}_o \) are constants to be determined.

A comparison of the first-order asymptotic theory for the intermediate region with the numerical solutions can be made once the value of \( A \) has been determined. It is not possible to find an expression for \( A \) using only the first-order theory, although such an expression might be found if the theory were extended to include the second-order terms. However, for the purposes of this study it was felt that adequate values of \( A \) could be found by using the results of the numerical computations. From equations (3-10b) and (3-14) it is found that

\[
\dot{u}_s'(0;\alpha) = -(\alpha - \alpha_{cr})^{-3/4} A + O[(\alpha - \alpha_{cr})^{1/4}]
\]

or, solving for \( A(\alpha) \)

\[
A(\alpha) = -(\alpha - \alpha_{cr})^{3/4} \dot{u}_s'(0;\alpha) + O[(\alpha - \alpha_{cr})]
\]

(3-39)

Utilizing the values of \( \dot{u}_s'(0;\alpha) \) given in Table 1 the variation of \( A \) with \( \alpha \) is easily found and the results are given in Table 2.

As a check on the accuracy of \( A(\alpha) \) the tangential wall shear, \( \tau_s'(0;\alpha) \), was computed using the asymptotic theory, thus
TABLE 2

Variation of A with $\alpha$ from the Asymptotic Solution of Steady $r = 0$ Similarity Equations as $\alpha \to \alpha_{cr}$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$A(\alpha)$</th>
<th>$g'<em>s(0;\alpha)</em>{\text{theory}}$</th>
<th>$g'<em>s(0;\alpha)</em>{\text{numerical}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-.68904</td>
<td>.33727</td>
<td>1.77675</td>
<td>1.80000</td>
</tr>
<tr>
<td>-.68646</td>
<td>.35785</td>
<td>1.67643</td>
<td>1.70000</td>
</tr>
<tr>
<td>-.68296</td>
<td>.37455</td>
<td>1.57528</td>
<td>1.60000</td>
</tr>
<tr>
<td>-.67802</td>
<td>.38910</td>
<td>1.47307</td>
<td>1.50000</td>
</tr>
<tr>
<td>-.67075</td>
<td>.40276</td>
<td>1.36938</td>
<td>1.40000</td>
</tr>
<tr>
<td>-.66460</td>
<td>.41138</td>
<td>1.30619</td>
<td>1.34000</td>
</tr>
<tr>
<td>-.66000</td>
<td>.41700</td>
<td>1.26815</td>
<td>1.30432</td>
</tr>
<tr>
<td>-.65000</td>
<td>.42779</td>
<td>1.20316</td>
<td>1.24428</td>
</tr>
<tr>
<td>-.64000</td>
<td>.43752</td>
<td>1.15415</td>
<td>1.19996</td>
</tr>
<tr>
<td>-.62000</td>
<td>.45537</td>
<td>1.08379</td>
<td>1.13812</td>
</tr>
<tr>
<td>-.60000</td>
<td>.46343</td>
<td>1.02839</td>
<td>1.10176</td>
</tr>
<tr>
<td>-.55000</td>
<td>.51043</td>
<td>.95643</td>
<td>1.03315</td>
</tr>
<tr>
<td>-.20000</td>
<td>.68278</td>
<td>.77631</td>
<td>0.86869</td>
</tr>
</tbody>
</table>
\[ g'_s(0;\alpha) = -1.06292 \, \alpha_{cr}(\alpha-\alpha_{cr})^{-1/4} A^{1/3} + 0[(\alpha-\alpha_{cr})^{3/4}] \]

(3-40)

A comparison between the values of \( g'_s(0;\alpha) \) obtained from equation (3-40) and the numerical computations is also given in Table 2. The difference between the two values is about 1.3\% for \( \alpha = -0.68904 \) and only 10.6\% for \( \alpha = -0.20 \), indicating that, near the wall at least, the theory yields quite good results for \( \alpha \) close to \( \alpha_{cr} \) and acceptable values even for \( \alpha-\alpha_{cr} \).50.

Further comparisons of the asymptotic solutions for \( u_s(z;\alpha) \) and \( g_s(z;\alpha) \) in the intermediate region with the numerical solutions are illustrated in Figures 12 and 13. The agreement is quite good over the entire range of integration with the exception of the regions near the outer boundary wherein the solutions must be adjusted to accommodate the appropriate boundary conditions.

In summary a first-order asymptotic solution of the steady \( r = 0 \) similarity equations has been found in terms of the small parameter \( (\alpha-\alpha_{cr}) \); where \( \alpha_{cr} \) is the limiting value of \( \alpha \) for which finite solutions are possible and whose value was determined as part of the analysis. The solution consists of a thin viscous region of non-dimensional thickness \( 0((\alpha-\alpha_{cr})^{+1/4}) \) in which the wall boundary conditions
are satisfied. This inner region joins onto a thick inviscid region of thickness \(0((\alpha-\alpha_{cr})^{-1/4})\) which is in turn bounded by another thin viscous region in which the boundary conditions at infinity are satisfied. The analytical solution given by this analysis fits the numerical solution given in Figures 12 through 14 quite closely; and, furthermore, the value of \(\alpha_{cr}\) given by the analysis agrees well with the limiting value of \(\alpha\) for which numerical solutions could be found. It should be recalled, however, that this solution is not unique since there can be any number of thick inviscid regions, each separated by a thin viscous region, before the outer boundary condition on \(f_s\) and \(g_s\) are attained.

The family of solutions of the steady similarity equations for an infinite rotating disc placed in a rigidly rotating fluid can be characterized by the appropriate value of the parameter \(\alpha\). Acceptable solutions can be found for all values of \(\alpha > 0\); \(\alpha = 0\) corresponding to the Bödewadt solution and \(\alpha \to \infty\) the Kármán solution. For \(\alpha < 0\), however, the situation is quite different. Numerical solutions without suction at the wall have been found only for values of \(\alpha\) in the range \(-.68904 < \alpha < 0\) and \(-\infty < \alpha \leq -6.211\).

The results of this chapter have shown that no finite solutions of the steady similarity equations are possible when \(\alpha = -.6961\), and also McLeod proved that no solution.
exists when $\alpha = -1$. Thus as $\alpha$ decreases from zero, finite solutions are possible only for $\alpha > -0.6961$. In the range $-1.0 \leq \alpha < -0.6961$ there does not seem to be any solution of the boundary-value problem defined by equations (3-5) and (3-6). For $\alpha$ lying in the range $-\infty < \alpha < -6.211$ solutions with and without suction have been found numerically by Evans. However, for $-6.211 < \alpha < -1.0$ Evans was able to find finite solutions only if an appropriate amount of suction was applied at the disc. Ockendon's asymptotic analysis confirmed the results of Evans for small values of the suction parameter, and in addition she found that solutions for non-zero suction were possible when $-1.0 < \alpha < -0.6961$, although numerical solutions for this range of $\alpha$ have not yet been given.

The reasons for the breakdown of the numerical solutions without suction in the neighborhood of $\alpha = -6.211$ are still not understood. Evans' method failed to yield results due to the excessive amounts of computer time needed, and the numerical method described in this chapter fails to converge for no apparent reason when $\alpha = -6.211$. Therefore, it appears from Ockendon's analysis and the present work that solutions of the steady similarity equations without suction do not exist when $\alpha$ lies in the range $-6.211 < \alpha < -0.6961$. This nonexistence, however, has been rigorously proved by McLeod only for the single case $\alpha = -1$. 
CHAPTER IV
UNSTEADY BOUNDARY LAYER SIMILARITY FOR AN INFINITE ROTATING DISC

The numerical computations for the development of the boundary layer on a finite rotating disc which were discussed in Chapter II indicate that, near the center of the disc at least, the unsteady boundary-layer equations do not approach a steady-state limit. In particular, for $\alpha = -1$ the computations failed to converge at the center of the disc, $r = 0$, when $t = 2.28$. Furthermore, a preliminary study for $\alpha = -0.50$ indicated that a similar behavior could be expected at the center of the disc. It has also been noted that if $r$ is set equal to zero in equations (2-2) governing the boundary-layer development on a finite rotating disc, then the resulting equations are identical to the unsteady similarity equations for an infinite rotating disc derived in Chapter III. Thus, in order to study the behavior of the boundary layer near the center of the disc more fully, it was decided to examine, in detail, the unsteady similarity equations for an infinite rotating disc.

The unsteady similarity equations for infinite rotating discs have been studied by several authors.
Thiriot\textsuperscript{25} in 1940 considered the time-dependent motion of an initially quiescent fluid over an infinite disc which was impulsively started from rest with a constant angular velocity. He formulated the problem such that the velocity components are expanded in powers of time with the coefficients being functions of the similarity variable, \( z/\sqrt{t} \); and he was able to obtain a closed-form solution for the leading terms of his expansions, which are valid for small times. Benton\textsuperscript{26} has extended this scheme by solving for the higher order terms numerically, and he was thus able to examine the important features of the early transient flow. He concluded that the steady state (Kármán's similarity solution) is almost completely established within the first few revolutions of the plate, i.e., for \( \omega t \approx 2 \). After this initial phase steady boundary-layer theory applies, although small inertial oscillations do persist for a much longer time. Homsy and Hudson\textsuperscript{27} also considered this same problem except that they used the method of finite differences to numerically compute the entire transient motion. Their results showed that the tangential velocity exhibited a single overshoot of its steady value while the growth of the secondary flow was monotonic. Furthermore, they deduced that the overshoot was a direct consequence of the lag in the development of the secondary flow.
This writer is not aware of any other studies wherein the time-dependent similarity equations for a single infinite disc have been studied for different values of the parameter \( \alpha \). There have been, however, studies on the time-dependent flow between two infinite rotating discs a finite distance apart. Greenspan and Howard\textsuperscript{28} considered the linearized time-dependent problem for the flow between two infinite rotating discs by assuming that the discs, initially rotating in unison, have their common angular velocity impulsively altered a like amount. They found that the state of rigid rotation was restored in a nondimensional time of the order \( \text{Re}^{\frac{1}{2}} \), where \( \text{Re} \) is the Reynolds number based on the distance between the discs. During this time the Ekman boundary layers develop near the surfaces; superimposed on this strictly decaying mode are additional decaying modes of very small amplitudes which oscillate with twice the frequency of rotation.

Pearson\textsuperscript{29} numerically integrated the time-dependent equations for the flow between two infinite rotating discs when each of the discs is given a specified angular velocity. In all cases considered the fluid was always initially at rest and the angular velocity of one or both of the discs was impulsively changed and the computations were carried out until the steady-state solution was obtained. In all but one of the cases considered the solutions obtained were
in overall agreement with the steady-state results of Lance and Rogers\textsuperscript{30}. Furthermore, it was found that the Reynolds number has a strong influence on the character of the solutions. Of particular interest was the result that at higher Reynolds numbers, Pearson could find no stable symmetrical solution when the two discs were rotating in opposite directions; however, an unsymmetrical solution in which the main body of fluid rotates with one of the discs is possible.

It was shown in Chapter III that no finite solutions of the steady similarity equations for an infinite rotating disc without suction exist when \(-6.211 < \alpha < -0.6961\); and, therefore, it is perhaps not surprising that the unsteady computations for \(\alpha = -1\) failed to reach a steady-state behavior since no steady solution exists. However, a steady solution for \(\alpha = -.50\) is possible, and yet the unsteady computations for the finite disc seemed to indicate that no steady-state solution exists near the center of the disc. The question which naturally arises is, what are the values of \(\alpha\) for which the unsteady similarity equations fail to approach an appropriate steady solution; and, in those cases, why don't the unsteady computations reach a steady behavior?

To numerically integrate the unsteady similarity equations (3-3) they were rewritten in terms of the velocity components \(u(z,t) = f_z(z,t), g(z,t), \) and \(w(z,t) = -\frac{1}{2}f(z,t);\)
thus

\[ u_t = u_{zz} - wu_z - u^2 + g^2 - 1 \]  \hspace{1cm} (4-1a)

\[ g_t = g_{zz} - wg_z - 2ug \]  \hspace{1cm} (4-1b)

\[ 0 = w_z + 2u \]  \hspace{1cm} (4-1c)

with boundary conditions

\[ z = 0; \ t > 0: \ u = w = 0; \ g = a \]  \hspace{1cm} (4-1d)

\[ z \to \infty; \ \text{all} \ t: \ u \to 0; \ g \to 1 \]  \hspace{1cm} (4-1e)

The initial conditions can, in general, be specified quite arbitrarily. However, for the purposes of this study the velocity profiles were initialized by assuming that the entire flow field was in a state of rigid rotation unless otherwise specified.

The governing equations (4-1) are written in their finite-difference form in a manner similar to that used in Chapter II in connection with the finite-disc equations. In fact, by setting \( r = 0 \) in the finite-difference equations given in Chapter II, the finite-difference analogs of
equations (4-1) are obtained directly. Moreover, the numerical scheme used to solve the resulting equations is identical to that used in Chapter II and therefore will not be discussed further here. Computations were carried out for the following values of the parameter $\alpha$: 0, -0.10, -0.25, and -0.50; and the time-dependent computations were said to have reached the steady-state solution when both of the time derivatives, $u_t(z,t)$ and $g_t(z,t)$, were less than $0.001$ in absolute value for all values of $z$.

The time-dependent computations for the case $\alpha = 0$ correspond to the unsteady form of the Bödewadt similarity equations\textsuperscript{19} for the flow over an infinite stationary disc placed in an otherwise unbounded rigidly rotating fluid. Belcher\textsuperscript{15} considered, indirectly, the approach to the steady Bödewadt solution via the time-dependent solution of the boundary-layer equations for a generalized vortex on a finite disc. He could not extend the computations all the way to the center of the disc, $r = 0$, due to his formulation of the problem. However, computations were made at $r = 0.145$ and it was found that the maximum time derivative was reduced to 0.048 by the time $t = 35.0$; and furthermore, for $t = 35.0$ the time-dependent solutions agreed favorably with the steady Bödewadt solution. Based on these results he concluded that the Bödewadt solution is the appropriate steady-state limit near the center of the disc.
The problem formulated in this thesis allows the unsteady similarity equations to be considered directly. Thus as a check on both Belcher's conclusion and the numerical scheme used herein the unsteady Bödewadt similarity equations, corresponding to the case $\alpha = 0$, were integrated forward in time using a time step $\Delta t = 0.05$ from an initial state of rigid rotation. At each time step the equations were solved for $z$ in the range 0 to 60 using a step size $\Delta z = 0.30$. The results of the numerical computations are summarized in Figures 16, 17, and 18.

Figure 16, which shows the tangential wall shear $e_z(0,t)$ plotted as a function of the radial wall shear $u_z(0,t)$ using time as a parameter, clearly indicates that the time-dependent wall shears are spiraling into the values given by the steady Bödewadt solution. Figures 17 and 18, giving the time development of the radial and tangential velocity profiles respectively, give further evidence that the Bödewadt solution is the large-time limit of the time-dependent solutions.

The time required for the computations to reach the steady-state condition was much longer than expected. For $t = 50.0$ the maximum time derivative was still 0.299, and by the time $t$ had reached 100.0 the maximum value had only decreased to 0.0662. As indicated in Figures 17 and 18 the solutions for $t = 200.0$ were in excellent agreement with
Figure 16. Unsteady $r = 0$ Similarity, Phase Plane, $g_z$ and $u_z$ for $\alpha = 0$
Figure 17. Unsteady $r = 0$ Similarity, Radial Velocity, $\alpha = 0$
Figure 18. Unsteady $r = 0$ Similarity, Tangential Velocity, $\alpha = 0$
the steady solution for all values of \( z \); and by this time the maximum time derivative had been reduced to only 0.00515. The calculations were assumed to have reached a steady-state when the maximum time derivative was less than 0.001 and this condition was finally met for \( t \geq 270.0 \). As will be discussed later, the large time associated with the approach to the steady solution appears to be related to an instability in the governing equations for values of \( \alpha < \alpha_c < 0 \), where \( \alpha_c \) denotes the critical value of \( \alpha \) for which stable solutions are possible.

In an attempt to deduce the value of \( \alpha_c \) numerical solutions were carried out for \( \alpha = -0.50, -0.25, \) and \(-0.10\) assuming that initially the entire flow field was in a state of rigid rotation. In none of the cases, however, did the unsteady computations approach a steady-state limit. For \( \alpha = -0.10 \) the calculations were carried out for \( 0.0 \leq t \leq 126.0 \) by which time it was apparent that the unsteady solutions had evolved into a limit cycle. Figure 19, giving \( g_z(0,t) \) as a function of \( u_z(0,t) \) for \( \alpha = -0.10 \) with time as a parameter, clearly indicates the presence of the limit cycle for \( t \geq 53.5 \). Furthermore, this behavior is not confined to the region near the disc. As shown in Figures 20 and 21 the radial and tangential velocity profiles also evolve into a limit cycle for all values of \( z \).

It is possible that the critical value for stability
Figure 19. Unsteady $r = 0$ Similarity, Phase Plane, $g_z$ and $u_z$, $\alpha = -0.10$
Figure 20. Unsteady $r = 0$ Similarity, Radial Velocity, $\alpha = -0.1$
Figure 21. Unsteady $r = 0$ Similarity, Tangential Velocity, $\alpha = -0.1$
is given by $\alpha_c = -0.10$. However, if $\alpha = -0.10$ does correspond to the marginally stable case then the unsteady solutions for all $\alpha < -0.10$ should diverge. To test this hypothesis the unsteady similarity equations with the initial conditions again taken to be those of rigid rotation were integrated for the two cases $\alpha = -0.50$ and $\alpha = -0.25$. In both cases the computations failed to converge for finite values of $t$; when $\alpha = -0.25$ failure occurred at $t = 8.25$, while for $\alpha = -0.50$ the solutions failed when $t = 2.75$. The behavior of the radial and tangential wall shears with time, shown in Figures 22 and 24 for $\alpha = -0.25$ and $-0.50$ respectively, clearly indicates the divergence of the unsteady solutions from the steady-state limit. Based on these results it can be concluded that, for $\alpha = -0.25$ and $-0.50$ at least, stable solutions of the steady similarity equations are not possible; and therefore, these solutions will not occur in practice. It is also reasonable to suppose that a similar behavior will occur for all values of $\alpha$ between $-0.25$ and $-0.68904$, implying that no stable solutions of the steady similarity equations are possible when $\alpha \leq -0.25$. For $-0.25 < \alpha < -0.10$ further numerical studies must be conducted to determine the precise value of $\alpha$ for which unstable solutions first appear.

To study the effect of the initial conditions used in the preceding calculations for the unsteady similarity
Figure 22. Unsteady $r = 0$ Similarity, Phase plane, $g_z$ and $u_z$, $\alpha = -0.25$
Figure 23. Unsteady $r = 0$ Similarity, Phase Plane, $\xi_z$ and $u_z$, $\alpha = -0.25$. 

**x:** Steady-State Solution

Initial Condition $\alpha = -0.2$
Figure 24. Unsteady $r = 0$ Similarity, Phase Plane, $g_z$ and $u_z$, $a = -0.5$
equations, the calculations were repeated using as initial conditions the known steady similarity solutions at neighboring values of \( \alpha \). In this manner the stability of the governing equations to small perturbations from the steady-state could be considered. For \( \alpha = -0.10 \) the initial condition was taken as the steady solution for \( \alpha = -0.05 \); and for \( \alpha = -0.25 \) the initial condition corresponded to the steady solution at \( \alpha = -0.20 \). In addition, for the case \( \alpha = -0.25 \) the steady solution for \( \alpha = -0.25 \) correct to two decimal places was used as an initial condition. With this choice the perturbations from the steady solution are initially very small. Finally, for \( \alpha = -0.50 \) the steady solutions for \( \alpha = -0.45 \) and \( \alpha = -0.48 \) were used as initial conditions. For the cases \( \alpha = -0.25 \) and \(-0.50 \) the time-dependent computations ultimately failed to converge giving further evidence that the steady similarity solutions for \( \alpha < -0.10 \) are unstable, and therefore not physically realizable. For completeness the variations of the wall shears with time obtained using the above initial conditions are shown in Figures 23 and 24 for \( \alpha = -0.25 \) and \(-0.50 \), respectively. For \( \alpha = -0.10 \), however, the computations did not diverge. Instead, the unsteady solutions ultimately evolved into the same limit cycle solution found earlier when the initial condition was taken to be that of rigid rotation, as can be seen by comparing Figures 19 and 25.
Figure 25. Unsteady $r = 0$ Similarity, Phase Plane, $g_z$ and $u_z$, $\alpha = -0.1$
The numerical solution of the unsteady non-linear similarity equations indicates that when \( \alpha > 0 \) the corresponding steady solution is stable in the sense that the time-dependent solution approaches the appropriate steady-state solution in the limit \( t \to \infty \). Conversely, for values of \( \alpha < -0.10 \) the unsteady solutions diverge as \( t \to \infty \) for both small and large perturbations from the steady-state solution indicating that the steady solutions are unstable. It is probable that a band of values of \( \alpha \) about \( \alpha = -0.10 \) will yield limit-cycle solutions of the unsteady similarity equations, and this band should define those values of \( \alpha \) for which marginally stable solutions are possible.

**Linearized Stability Analysis**

The stability of flows over infinite rotating discs has been considered by several authors. Gregory, Stuart, and Walker\(^{36}\) analyzed the inviscid equations of motion for a rotating disc using von Kármán's similarity solution as the basic flow. The instabilities were found to occur as vortex rolls in the form of spiral bands with the axes of symmetry lying approximately parallel to the mean flow outside the boundary layer. Postulating that the instability was centered near the inflection point of the velocity component normal to the bands, they were able to predict a relation between the orientation angle of the bands and
their velocity which agreed favorably with the available experimental data.

Baricilon\textsuperscript{37}, Faller and Kaylor\textsuperscript{38}, and Lilly\textsuperscript{39} have all considered the stability of the Ekman layer. In each case the extent of the fluid domain is taken as infinite, bounded by a plane surface. From the experimental data available it is known that the observed instabilities depend mainly on the local flow conditions. Furthermore, the boundary layer instabilities are assumed to consist of two-dimensional rolls and therefore the horizontal coordinates are aligned so that one of them measures distance along the bands and the other is in the normal direction. In each case the disturbances are assumed to be two-dimensional waves that are independent of the coordinate measured along the bands. Baricilon has obtained analytic solutions of the resulting perturbation equations. He does not, however, obtain values for the critical Reynolds number since his method of solution is not accurate for the small critical Reynolds numbers observed.

Lilly approximates the linear perturbation equations by a set of finite-difference equations. The complex eigenvalue wave velocities are found numerically as functions of three parameters: the wave number, the band orientation angle relative to the geostrophic flow, and the Reynolds number. A critical Reynolds number of \(\approx 55\) is found, where
the Reynolds number is based on the boundary layer-length scale and the relative velocity of the fluid far from the disc. In addition, the inclusion of the viscous terms omitted in the work of Gregory et al., leads to a new mechanism of instability which is dependent on the coriolis force and viscosity.

The numerical solutions of the unsteady similarity equations for an infinite rotating disc suggest that another mode of instability is possible, the nature of which is now discussed.

A steady viscous laminar motion is said to be stable to infinitesimal disturbances if all such disturbances decay to zero leaving the basic steady flow unchanged. Conversely, a flow is said to be unstable if any of the disturbances do not decay to zero, or if the flow evolves into another steady-state. To derive the appropriate linearized perturbation equations write

\[ \begin{align*}
  u(r,z,t;\alpha) &= r f_s'(z;\alpha) + \bar{u}(r,z,t;\alpha) \quad (4-2a) \\
  v(r,z,t;\alpha) &= r g_s(z;\alpha) + \bar{v}(r,z,t;\alpha) \quad (4-2b) \\
  w(r,z,t;\alpha) &= -2f_s(z;\alpha) + \bar{w}(r,z,t;\alpha) \quad (4-2c)
\end{align*} \]
\[ p(r,z,t;\alpha) = kr^2 + h_s(z;\alpha) + \tilde{p}(r,z,t;\alpha) \quad (4-2d) \]

Substituting equations (4-2) into the Navier-Stokes equations (3-1) and neglecting squares and products of all perturbation quantities yields the following system of linear equations for the perturbation functions:

\[ \ddot{u}_t + rf_s'\ddot{u}_r - 2f_s\ddot{u}_z + f'_s\ddot{u} + rf_s'w - 2g_s\ddot{v} = \]
\[ - \ddot{p}_r + \ddot{u}_{rr} + \frac{1}{r} \ddot{u}_r + \ddot{u}_{zz} - \frac{1}{r^2} \ddot{u} \quad (4-3a) \]

\[ \ddot{v}_t + rf_s'\ddot{u}_r - 2f_s\ddot{v}_z + 2g_s\ddot{u} + g'_sr\ddot{w} + f'_s\ddot{v} = \]
\[ = \ddot{v}_{rr} + \frac{1}{r} \ddot{v}_r + \ddot{v}_{zz} - \ddot{v}/r^2 \quad (4-3b) \]

\[ \ddot{w}_t + rf_s'\ddot{w}_r - 2f_s\ddot{w}_z - 2f'_sw = \]
\[ = - \ddot{p}_z + \ddot{w}_{rr} + \frac{1}{r} \ddot{w}_r + \ddot{w}_{zz} \quad (4-3c) \]

\[ \frac{1}{r}(r\ddot{u})_r + \ddot{w}_z = 0 \quad (4-3d) \]
subject to homogeneous boundary conditions at \( z = 0 \) and \( z \rightarrow \infty \).

In order to examine the nature of the instabilities which were demonstrated in the numerical integration of the unsteady similarity equations for negative values of \( \alpha \), it is sufficient to consider the following forms for the perturbation functions:

\[
\begin{align*}
\tilde{u}(r,z,t;\alpha) &= r\tilde{f}_z(z,t;\alpha) \quad (4-4a) \\
\tilde{v}(r,z,t;\alpha) &= r\tilde{g}(z,t;\alpha) \quad (4-4b) \\
\tilde{w}(r,z,t;\alpha) &= -2\tilde{f}(z,t;\alpha) \quad (4-4c) \\
\tilde{p}(r,z,t;\alpha) &= \tilde{h}(z,t;\alpha) \quad (4-4d)
\end{align*}
\]

Putting equations (4-4) into (4-3) yields

\[
\begin{align*}
\tilde{f}_{zt} &= \tilde{f}_{zzz} + 2f_s\tilde{f}_{zz} + 2f_s'\tilde{f}' - 2f_s'\tilde{f}_z + 2g_s\tilde{g} \quad (4-5a) \\
\tilde{g}_t &= \tilde{g}_{zz} + 2[f_s\tilde{g}_z + g_s'\tilde{f}' - f_s'\tilde{g} - g_s\tilde{f}_z] \quad (4-5b) \\
\tilde{h}_z &= 2[\tilde{f}_t - (f_s\tilde{f}_z + f_s'\tilde{f}) - \tilde{f}_{zz}] \quad (4-5c)
\end{align*}
\]
Equation (4-5c) is uncoupled from (4-5a; b), thus once \( \tilde{f} \) and \( \tilde{g} \) are found, \( \tilde{h} \) is immediately found from (4-5c) by a single integration, if needed.

Note that the perturbation mode functions are of the same similarity type as the steady-state solutions. In particular, \( \tilde{f}_z \) and \( \tilde{g} \) grow linearly in \( r \) while the perturbation in the axial direction is independent of \( r \). Also, there is no radial pressure perturbation entering into these calculations. This type of disturbance corresponds in some manner to low radial wave numbers, or equivalently to disturbances of very long radial wavelengths.

The boundary conditions for equations (4-5) are given by

\[
z = 0: \quad \tilde{f} = \tilde{f}_z = \tilde{g} = 0 \quad \text{(4-5e)}
\]

\[
z \to \infty: \quad \tilde{f}_z \to 0; \quad \tilde{g} \to 0 \quad \text{(4-5f)}
\]

The precise form of the initial conditions at \( t = 0 \) are of no concern here as the interest in the present study lies in the behavior of \( \tilde{f} \) and \( \tilde{g} \) for large times.

Solutions of the above linear system of equations are sought in the following form:
\[ \tilde{f}(z,t) = R1 \left[ t^{-c} e^{\lambda t} \sum_{k=0}^{\infty} \tilde{f}_k(z)t^{-k} \right] \quad (4-6a) \]

\[ \tilde{g}(z,t) = R1 \left[ t^{-c} e^{\lambda t} \sum_{k=0}^{\infty} \tilde{g}_k(z)t^{-k} \right] \quad (4-6b) \]

where \( c \) is a real constant and \( \lambda \) is a complex eigenvalue, both of which must be found. Also, \( \tilde{f}_k(z) \) and \( \tilde{g}_k(z) \) are complex-valued functions of the real variable \( z \). Substituting (4-6) into (4-5) yields, to lowest order, the following system of equations:

\[ \tilde{f}_o''' + 2 [ f_s \tilde{f}_o'' + f_s' \tilde{f}_o' - f_s' \tilde{f}_o + g_s \tilde{g}] = \lambda \tilde{f}_o \quad (4-7a) \]

\[ \tilde{g}_o''' + 2 [ f_s \tilde{g}_o'' + \tilde{f}_o \tilde{g}_s' - g_s \tilde{f}_o' - f_s' \tilde{g}_o ] = \lambda \tilde{g}_o \quad (4-7b) \]

with the boundary conditions given by

\[ \tilde{f}_o(0) = \tilde{f}_o'(0) = \tilde{g}_o(0) = \tilde{f}_o'(\infty) = \tilde{g}_o(\infty) = 0 \quad (4-7c) \]
The problem defined above is in the form of a conventional eigenvalue problem, and the complex eigenvalues, \( \lambda \), are obtained numerically by solving the matrix form of the finite-difference analog of (4-7) using a computer program for the complex eigenvalues of a real matrix made available to me by Mr. Peter Williams of University College, London.

The results for the lowest two eigenvalues are given in Table 3 for various values of \( \alpha \). In each case the boundary condition for \( z \to \infty \) was approximated at the point \( z = 30.3 \), and Richardson's \( h^2 \)-extrapolation\(^{33}\) was used to determine the appropriate value of \( \lambda_r \) and \( \lambda_i \).

Of special interest is the value of \( \lambda r_1 \) since it governs the growth or decay of the perturbation functions \( \tilde{f}(z,t) \) and \( \tilde{g}(z,t) \) depending on whether \( \lambda_r \) is greater than or less than zero. As seen from the table \( \lambda r_1 \) is negative for all \( \alpha > 0 \) indicating that the perturbations decay to zero as \( t \to \infty \). Therefore, the steady-state solutions of the \( r = 0 \) similarity equations for \( \alpha \geq 0 \) are stable to small perturbations. These results are also in agreement with the large-time behavior of the unsteady similarity solutions when \( \alpha = 0 \), both solutions indicating that the steady-state solutions are stable. When \( \alpha \leq -0.10 \), \( \lambda r_1 \) is positive implying the steady-state solutions are ultimately unstable to infinitesimal disturbances. These results are also in
qualitative agreement with the numerical solutions of the unsteady similarity equations for \( \alpha < 0 \). However, whereas the numerical eigenvalue analysis suggests that the marginally stable solution \( (\lambda r_1 = 0) \) occurs for \( \alpha = -0.033 \), the earlier computations suggest that \( \alpha = -0.10 \) defines the marginally stable solution.

The important conclusion to be drawn from these results is that there exists some finite negative value of \( \alpha = \alpha_c \) lying between -0.10 and 0 which divides the steady-state similarity solutions into two classes. Solutions for \( \alpha > \alpha_c \) are stable and therefore possible in nature, while those for \( \alpha < \alpha_c \) are presumably unstable and not likely to occur in nature.

From equation (4-6) it is found that

\[
\tilde{f}_z(z,t) = [\tilde{f}_r \cos \lambda_1 t - \tilde{f}_i \sin \lambda_1 t] e^{\lambda r t} t^{-c} \quad (4-8a)
\]

\[
\tilde{g}(z,t) = [\tilde{g}_r \cos \lambda_1 t - \tilde{g}_i \sin \lambda_1 t] e^{\lambda r t} t^{-c} \quad (4-8b)
\]

where \( \tilde{f}_r, \tilde{f}_i, \tilde{g}_r, \) and \( \tilde{g}_i \) are the real and imaginary parts of the complex-valued functions \( \tilde{f}_r(z) \) and \( \tilde{g}_r(z) \), respectively, defined in equations (4-7). With the eigenvalues \( \lambda_r, \lambda_1 \) known, approximations for the perturbation
### TABLE 3
Eigenvalues, \( r = 0 \) Stability Analysis

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \lambda_{r_1} )</th>
<th>( \pm \lambda_{l_1} )</th>
<th>( \lambda_{r_2} )</th>
<th>( \pm \lambda_{l_2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>-0.010821</td>
<td>2.000000</td>
<td>-0.043063</td>
<td>2.000000</td>
</tr>
<tr>
<td>0.00</td>
<td>-0.010830</td>
<td>2.000013</td>
<td>-0.043541</td>
<td>2.000248</td>
</tr>
<tr>
<td>0.95</td>
<td>-0.011373</td>
<td>2.000068</td>
<td>-0.045092</td>
<td>2.000551</td>
</tr>
<tr>
<td>0.90</td>
<td>-0.013274</td>
<td>2.000153</td>
<td>-0.052450</td>
<td>2.001394</td>
</tr>
<tr>
<td>0.80</td>
<td>-0.021623</td>
<td>2.000403</td>
<td>-0.085131</td>
<td>2.004288</td>
</tr>
<tr>
<td>0.70</td>
<td>-0.036923</td>
<td>2.000843</td>
<td>-0.148424</td>
<td>1.998185</td>
</tr>
<tr>
<td>0.60</td>
<td>-0.060028</td>
<td>2.001849</td>
<td>-0.205858</td>
<td>1.997731</td>
</tr>
<tr>
<td>0.50</td>
<td>-0.088023</td>
<td>2.003907</td>
<td>-0.398458</td>
<td>1.993969</td>
</tr>
<tr>
<td>0.40</td>
<td>-0.107229</td>
<td>1.990147</td>
<td>-0.573065</td>
<td>1.992243</td>
</tr>
<tr>
<td>0.30</td>
<td>-0.109428</td>
<td>1.947608</td>
<td>-0.65960</td>
<td>2.002710</td>
</tr>
<tr>
<td>0.20</td>
<td>-0.019509</td>
<td>1.706753</td>
<td>-0.469251</td>
<td>1.997731</td>
</tr>
<tr>
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<td>+0.007277</td>
<td>1.656185</td>
<td>-0.398458</td>
<td>1.908035</td>
</tr>
<tr>
<td>-0.10</td>
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<td>1.590742</td>
<td>-0.573065</td>
<td>1.993969</td>
</tr>
<tr>
<td>-0.55</td>
<td>+0.712433</td>
<td>0.411789</td>
<td>-0.176904</td>
<td>1.992243</td>
</tr>
</tbody>
</table>

* values not \( h^2 \)-extrapolated
eigenfunctions, (4-8) are found numerically using a shooting technique to solve the differential equations (4-8) for \( \tilde{r}_{r_0}, \tilde{r}_1, \tilde{g}_{r_0}, \) and \( \tilde{g}_1. \) The resulting mode shapes \( \tilde{r}_{r_0} \) and \( \tilde{g}_{r_0} \) are shown in Figures 26 and 27 for the case \( \alpha = 0.50. \)

Of special interest is the near-linear growth in \( z \) in both \( \tilde{r}_z \) and \( \tilde{g}. \) The sharp fall back to zero for \( z = 15 \) is due to the requirement that \( \tilde{r}_z \) and \( \tilde{g} \) vanish at the outer boundary which was taken to be at the point \( z = 15. \) in this case.

The boundary conditions should in fact be applied at \( z + \infty, \) and the numerical results for a thicker mesh show that the eigenfunctions will continue to grow linearly with \( z; \) the behavior of the functions in the vicinity of the outer edge being due entirely to the finite mesh used in the numerical computations. Hence these mode shapes are inappropriate for the problem under consideration since it is necessary for the perturbation functions to asymptotically approach zero as the outer edge of the mesh is approached. In fact the nearly-linear growth of \( \tilde{r}_z \) and \( \tilde{g} \) with increasing \( z \) suggests that an outer layer is necessary in which the mode shapes are adjusted so that the boundary conditions as \( z \to \infty \) can be satisfied.

A similar behavior of the eigenfunctions occurred for different values of \( \alpha \) and, therefore, it was decided to investigate the linearized perturbation equations (4-5)
Figure 26. Unsteady $r = 0$ Similarity, Radial Velocity Eigenfunction for $\alpha = 0.5$
Figure 27. Unsteady $r = 0$ Similarity, Tangential Velocity Eigenfunction for $a = 0.5$
and (4-6) analytically for several limiting cases. First consider the perturbations of the steady-state solution defined by \( \alpha = 1 \), i.e., solid-body rotation. For this case \( f_s(z) = 0 \) and \( g_s(z) = 1 \). Thus from equations (4-7) it follows that

\[
\tilde{f}_o''' + 2\tilde{g}_o = \lambda \tilde{f}_o
\]

\[
\tilde{g}_o'' - 2\tilde{f}_o' = \lambda \tilde{g}_o
\]

These equations can be combined into a single second-order equation by multiplying the second equation by 1 and adding it to the first. Hence

\[
(\tilde{f}_o' + i\tilde{g}_o)''' - (\lambda + 2i)(\tilde{f}_o' + i\tilde{g}_o) = 0
\]

and the general solution is

\[
(\tilde{f}_o' + i\tilde{g}_o) = \tilde{A}_0 e^{-\sqrt{\lambda+2i} z} + \tilde{B}_0 e^{\sqrt{\lambda+2i} z}
\]

(4-9)

However, this solution is inappropriate for \( \lambda = -2i \) as is shown to be true in equation (4-13). In this case the correct solution is
There are no nontrivial solutions of (4-9) or (4-10) which satisfy all the boundary conditions (4-7c). But if the outer boundary condition is applied at a finite value of \( z \), say \( z = \delta \), then it is possible to solve (4-9) and to determine the values of \( \lambda \). The results are

\[
\tilde{f}_o' + i\tilde{g}_o = -2iB_0\sin\left(\frac{n\pi}{\delta}z\right) \quad n = 1, 2, \ldots \quad (4-11a)
\]

\[
\lambda_{r_n} = -\frac{n^2\pi^2}{\delta^2} \quad ; \quad \lambda_{\tilde{r}_1} = -2 \quad (4-11b)
\]

The values of \( \lambda_{r_1} \) and \( \lambda_{r_2} \) obtained from this exact solution are in excellent agreement with those found from the numerical solutions for \( \alpha = 1.0 \). With \( \delta = 30.3 \) it follows from (4-11b) that \( \lambda_{r_1} = -0.010750 \) and \( \lambda_{r_2} = -0.043000 \) while from Table 3 \( \lambda_{r_1} = -0.010821 \) and \( \lambda_{r_2} = -0.043063 \).

It is also of importance to note that \( \lambda_{r_n} \) can be made arbitrarily small by taking \( \delta \) sufficiently large but
finite. Thus it seems that for $\alpha \approx 1$ at least, the eigenvalues calculated numerically depend strongly on the thickness of the mesh used in the computations. One must question, therefore, whether the eigenvalues given in Table 3 adequately approximate those of the physical problem under consideration. The boundary condition given in (4-7c) corresponds to $\delta \to \infty$ in which case $\lambda_n \to 0$ and the solution given by (4-9) or (4-10) breaks down since the boundary conditions as $z \to \infty$ cannot be satisfied. It follows, therefore, that for $\alpha = 1$ at least, there are no solutions of the form assumed in equations (4-6). The time dependence in the perturbation functions cannot be separated from the spatial dependence. Dr. Odus R. Burggraf has suggested the following form of solution.

Returning to equations (4-5) and considering the case $\alpha = +1$ again, it is found that the perturbation functions must satisfy

\[(\hat{r}_z + i\hat{g})_t = (\hat{r}_z + i\hat{g})_{zz} - 2i(\hat{r}_z + i\hat{g})\]

This equation can be reduced to the heat-conduction equation by the transformation

\[(\hat{r}_z + i\hat{g}) = e^{-2it} \hat{\varphi}(z,t)\]
where $\phi$ satisfies the following equation

$$\dot{\phi}_t = \phi_{zz}$$

with boundary conditions

$$\phi(0,t) = \phi(\pi,t) = 0$$

The complete solution of this boundary-value problem can be found in integral form by using the appropriate Green's function. The final solution is

$$\tilde{\phi}_z + i\tilde{g} = \frac{1}{\sqrt{\pi \sqrt{t}}} e^{-\frac{z^2}{4t}} \int_0^\infty \left[ \tilde{\phi}_z(\xi,0) + i\tilde{g}(\xi,0) \right] e^{-\frac{\xi^2}{4t}} \sinh\left(\frac{\xi \sqrt{t}}{2t}\right) d\xi$$

(4-12)

where $\tilde{\phi}_z(\xi,0)$ and $\tilde{g}(\xi,0)$ correspond to the arbitrary initial conditions.

The behavior of (4-12) for large times can be found by expanding the appropriate functions in (4-12) for $t \to \infty$ and $z = O(1)$. The large-time behavior is thus given by

$$(\tilde{\phi}_z + i\tilde{g}) \sim \frac{1}{2\pi \sqrt{t}^{3/2}} e^{-2it} \int_0^\infty \left[ \tilde{\phi}_z(\xi,0) + i\tilde{g}(\xi,0) \right] e^{-\frac{\xi^2}{2t}} d\xi + ...$$

(4-13)
Note in particular that the perturbation functions grow linearly with \( z \), based on the assumption that \( z^2/t \to 0 \). Furthermore, the disturbances approach zero as \( t \to \infty \), although the decay is only algebraic. This result fits in with the observation that \( \lambda_r \to 0 \) as \( \delta \to \infty \) discussed above. Also, from (4-13) it is seen that \( \lambda_1 = -2 \), and in addition there is the linear growth in \( z \) present which is necessary for matching this solution with the wall layer given by (4-10), and indicated in Figures 26 and 27 for \( \alpha = 0.50 \).

Hence, from the above results we can conclude that solutions of the perturbation equations of the form given by (4-6) are appropriate in a wall layer, however, they must match to an outer layer with a solution of the type given by (4-13) in order to satisfy all the boundary conditions.

Guided by the exact solutions obtained when \( \alpha = 1 \) the asymptotic structure of the perturbation equations for \( z \) large and \( \alpha \neq 1 \) has also been examined. Before this can be done, however, the asymptotic form of the solution of the steady similarity equations (3-5) is required. It is easily shown that for \( z \) large

\[
g_s(z; \alpha) = 1 - e^{p_z} [B \cos yz - A \sin yz] + \ldots \quad (4-14a)
\]
where \(-\beta(a)\) is the limiting value of the stream function as \(z \to \infty\) and is a function of \(a\); \(A\) and \(B\) are integration constants which are determined by comparing this solution with the solution of the full equations; and

\[
\rho = \beta - \frac{1}{\sqrt{2}} \left[ \sqrt{\beta^4 + 4} + \beta^2 \right]^4
\]  

(4-14c)

\[
\gamma = \frac{1}{\sqrt{2}} \left[ \sqrt{\beta^4 + 4} - \beta^2 \right]^4
\]  

(4-14d)

Note that \(\rho < 0\) for all values of \(\beta\) irrespective of its sign.

In order to describe the behavior of the outer layer it is convenient to introduce new variables, the form of which is suggested by the integrand in (4-12). Thus write

\[
\sigma = \frac{z}{4\beta t}; \quad \tau = t
\]  

(4-14a)

\[
\tilde{f}(z,t) = 4\beta \tau F(\sigma, \tau)
\]  

(4-14b)
\[ g(z,t) = G(\sigma, \tau) \quad (4-14c) \]
\[ f_s(z) = F_s(\sigma, \tau) \quad (4-14d) \]
\[ g_s(z) = G_s(\sigma, \tau) \quad (4-14e) \]

Substituting (4-14) into (4-5) and simplifying yields

\[ 16\beta^2 \tau^2 \ddot{F}_{\sigma\tau} = F_{\sigma\sigma\sigma} + 8\beta\tau(2\beta\sigma + F_s)F_{\sigma\sigma} + 8\beta\tau F_{s,s\sigma}F \]
\[ - 8\beta\tau F_{s,s\sigma}F_s + 32\beta^2 \tau^2 G_{s\sigma} \]
\[ (4-15a) \]

\[ 16\beta^2 \tau^2 G_{\tau} = G_{\sigma\sigma} + 8\beta\tau(2\beta\sigma + F_s)G_{\sigma} + 32\beta^2 \tau^2 G_{s\sigma}F \]
\[ - 8\beta\tau F_{s,s\sigma}G - 32\beta^2 \tau^2 G_{s\sigma}F_s \]
\[ (4-15b) \]

Solutions of equations (4-15) are sought for \( \tau \) large and \( \sigma = 0(1) \), neglecting exponentially small terms for the first approximation. Hence we must solve

\[ 16\beta^2 \tau^2 \ddot{F}_{\sigma\tau} = F_{\sigma\sigma\sigma} + 8\beta^2 \tau[2\sigma - 1]F_{\sigma\sigma} + 32\beta^2 \tau^2 G \]

\[ 16\beta^2 \tau^2 G_{\tau} = G_{\sigma\sigma} + 8\beta^2 \tau(2\sigma - 1)G_{\sigma} - 32\beta^2 \tau^2 F_s \]
16\beta^2 t^2 \phi = \phi_{\sigma \sigma} + 8\beta^2 \tau (2\sigma - 1) \phi_{\sigma} - 32\beta^2 t^2 \psi \quad (4-16a)

where

\[ \phi(\sigma, \tau) = F_{\sigma} + 10 \quad (4-16b) \]

The boundary conditions are such that \( \phi(\sigma, \tau) \to 0 \) as \( \sigma \to \infty \) and \( \phi \) must match with the solution in the wall layer as \( \sigma \to 0 \).

To solve (4-16) first consider the following transformation

\[ \phi(\sigma, \tau) = e^{-\beta^2 \tau - 2i\tau - 4\beta^2 \tau (\sigma^2 - \sigma) - 2i\ln \tau} \psi(\sigma, \tau) \quad (4-17a) \]

then \( \psi \) satisfies the equation

\[ \psi_{\sigma \sigma} = 16\beta^2 t^2 \psi_{\tau} \quad (4-17b) \]

and a solution is given by

\[ \psi(\sigma, \tau) = \int_{0}^{\infty} \psi_{\xi}(\xi) e^{-\frac{\xi^2}{4\tau}} \sinh(2\beta \sigma \xi) d\xi \]
Thus it follows that

\[ F_0 + iG = \tau^{-\frac{1}{2}} e^{-(\beta^2 + 21)\tau} \int_0^\infty \phi_1(\xi) e^{-\frac{\xi^2}{4\tau}} \sinh(2\beta\xi) d\xi \]

(4-18)

To match (4-18) with the appropriate solution in the wall layer in the sense of Van Dyke, the solution is rewritten in terms of the inner variables \( z \) and \( t \) and then expanding for large \( t \):

\[ t^{-\frac{1}{2}} e^{-(\beta^2 + 21) t + \beta z - \frac{z^2}{4t}} \int_0^\infty \phi_1(\xi) e^{-\frac{\xi^2}{4t}} \sinh(\frac{z\xi}{2t}) d\xi \]

keeping the leading term only the result is

\[ \int \frac{z\xi}{2t} + \frac{z^3\xi^3}{48t^3} + \ldots \] d\( \xi \)

(4-19)
Anticipating that a match is possible only if \( \lambda = -(\beta^2 + 2i) \), which is confirmed by equation (4-24), the governing equations in the wall layer are, to leading order,

\[
\dddot{f}_0 - 2\beta \ddot{f}_0 + (\beta^2 + 2i)\dot{f}_0 + 2\tilde{g}_0 = 0 \quad (4-20a)
\]

\[
\dddot{\tilde{g}}_0 - 2\beta \ddot{\tilde{g}}_0 + (\beta^2 + 2i)\dot{\tilde{g}}_0 - 2\tilde{f}_0 = 0 \quad (4-20b)
\]

and the general solution is

\[
\ddot{f}_0 = (A_1 + A_2 z)e^{\beta z} + A_3 e^{(\beta + \sqrt{2}(1+i))z} + A_4 e^{(\beta - \sqrt{2}(1-i))z} \quad (4-21a)
\]

\[
\ddot{\tilde{g}}_0 = -(A_1 + A_2 z)e^{\beta z} + iA_3 e^{(\beta + \sqrt{2}(1-i))z} + iA_4 e^{(\beta - \sqrt{2}(1-i))z} \quad (4-21b)
\]

Combining equations (4-8) and (4-21) yields

\[
\ddot{f}_z + i\ddot{g} = t^{-\alpha} e^{-\beta^2 t + \beta z} \left\{ (A_1 + A_2 z)e^{-2it} + A_3 e^{\sqrt{2}(1+i)z + 2it} + A_4 e^{-\sqrt{2}(1+i)z + 2it} \right\} \quad (4-22)
\]
In order to complete the solution it is necessary to match the solutions given by (4-19) and (4-22) in their region of overlap. Following Van Dyke, equation (4-22) is written in terms of $\sigma$ and $\tau$ and then expanded for large $\tau$ keeping the first term only, viz.:

\[
\tilde{f}_z + i\tilde{g} = \tau^{-c} e^{-\beta^2 \tau + 4\beta^2 \sigma \tau} \left\{ (A_1 + 4\beta \sigma \tau A_2) e^{-21\tau} + A_3 e^{\sqrt{2}(1+i)4\beta \sigma \tau + 21\tau} + A_4 e^{-\sqrt{2}(1+i)4\beta \sigma \tau + 21\tau} \right\}
\]

Rewriting in terms of $z$ and $t$ variables:

\[
\tilde{f}_z + i\tilde{g} \sim \tau^{-c} e^{-\beta^2 \tau + 4\beta^2 \sigma \tau} \left\{ 4\beta \sigma \tau A_2 e^{-21\tau} + A_3 e^{\sqrt{2}(1+i)4\beta \sigma \tau + 21\tau} + \ldots \right\}
\]

Comparing (4-19) and (4-23) it is readily seen that a match is possible only if

\[
\lambda = -(\beta^2 + 21)
\]

(4-24a)

\[
c = 3/2
\]

(4-24b)
These results show that a consistent first-order asymptotic theory can be formulated to describe the behavior of the eigenfunctions $f_\xi$ and $g$. The theory also yields a single complex eigenvalue $\lambda$. Hence the theory predicts that the steady similarity solutions for an infinite rotating disc are stable for all values of $\alpha$, whether positive or negative. Therefore, there is a contradiction between the theory and the numerical solutions.

A comparison between the real part of the eigenvalues computed numerically and that given by the asymptotic theory is shown in Figure 28. Note in particular the qualitative agreement between the value for $\lambda_r$ obtained from the asymptotic theory, and $\lambda_{r2}$ from the numerical computations for values of $\alpha \leq 0.40$. The discrepancy between these results for $\alpha = 1$ can readily be explained by recalling equation (4-11b) which gives the values $\lambda_{rn}$ based on a finite domain $0 \leq z \leq \delta$. The value of $\lambda_{rn}$ for $\alpha = 1$ can be made as small as desired providing $\delta$ is taken sufficiently large. It appears, therefore, that the deviation between

\[
A_2 = \frac{1}{2} \int_0^\infty \phi_1(\xi) \xi d\xi \quad (4-24c)
\]

\[
A_3 = 0 \quad (4-24d)
\]
Figure 28. Eigenvalues from Linearized Stability Analysis
the two results for $\alpha$ near unity is a result of specifying the boundary condition at infinity at a finite point. Presumably the numerical computations will yield good results for finite domains, however the results must be questioned when using them to approximate the semi-infinite domain $0 \leq z < \infty$.

The numerical computations for $\lambda_{r1}$ as a function of $\alpha$ indicate that for $\alpha < -0.03$ there are unstable modes present in the flow field. On the contrary, the asymptotic theory shows that $\lambda_r$ is negative for all values of $\alpha$ indicating that the similarity solutions are stable to small perturbations for all choices of $\alpha$. One must question, therefore, whether the positive values of $\lambda_{r1}$ are physically meaningful or whether they are extraneous roots introduced by using a finite mesh in the numerical computations.

Unfortunately, at this time the discrepancy is still not fully resolved, especially in light of the fact that the time-dependent solution of the nonlinear $r = 0$ similarity equations clearly diverge for $\alpha < -0.10$ and converge for $\alpha > 0$. It may be that the numerical scheme used to solve the equations is inappropriate in view of the relationship between $z$ and $t$ in the outer layer found from the asymptotic theory since this relationship was not taken into account in the computations. It is equally possible that a second-order asymptotic theory would bring in these unstable modes.
Alternatively, it is possible that there is another mode of instability which is yet to be found.

In this regard Professor K. Stewartson has suggested in discussions with this author concerning this problem that there is possibly a bifurcation of solutions at some critical value of $\alpha > 0$. Along one branch there are eigenfunctions of the form found by the present asymptotic solution, and along the other branch there may be eigenfunctions of the form found by separating variables which permit an unstable mode to exist. It must be emphasized, however, that these are just suggestions as to how the problem might be resolved. Further research is necessary in order to deduce with certainty the appropriate behavior of the similarity solutions for negative values of $\alpha$. 
CHAPTER V

BOUNDARY LAYER SIMILARITY NEAR THE EDGE OF A ROTATING DISC

The boundary-layer equations near the edge of a finite rotating disc placed in a rotating fluid can be reduced to a set of similarity equations which govern the initial growth of the boundary layer on the disc. However, these similarity equations have not been examined in any systematic way to determine whether their solutions are realistic or not. In particular, it is important to know if there are conditions under which the similarity solutions fail to exist, especially in view of the nonexistence of solutions of the $r = 0$ similarity equations for a range of values of $\alpha$. For these reasons, the edge similarity equations have been studied both numerically and analytically in this chapter.

Shultz-Grunow in 1935 first considered the problem of the boundary-layer development on a finite, stationary disc in a rotating fluid, finding an approximate solution using momentum-integral methods. Assuming that the boundary layer has zero thickness at the edge of the disc, he found that initial growth of the boundary layer is pro-

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portional to \((1 - r)^\frac{1}{2}\), where \(r\) denotes the nondimensional distance from the axis of the disc. The boundary layer was found to spread over the entire disc, but the convergence of the series he used was not satisfactory near the disc's center.

Some years later Stewartson\(^{11}\) considered the problem in more detail, deducing from the laminar boundary-layer equations an infinite sequence of ordinary differential equations, the first of which gives rise to a similarity solution with non-oscillatory velocity profiles valid in the region near the edge of the disc.

The similarity equations derived by Stewartson have not been studied extensively in their own right, although Mack\(^{12}\), Rogers and Lance\(^{14}\), Belcher\(^{15}\), and Burggraf, Stewartson, and Belcher\(^{5}\) as well as the present author have made use of them in numerical studies of the laminar boundary layer on a finite disc in a rotating fluid, the similarity velocity profiles providing the appropriate boundary conditions on the boundary layer at the edge of the disc.

As will be shown below, Stewartson's similarity equations constitute a coupled fifth order system of ordinary differential equations with two-point boundary conditions; and, as formulated, there is a one parameter family of solutions to the system of equations. It is the objective of this chapter to examine in detail the nature of the solu-
tions of the boundary-value problem as the parameter ranges over all allowable values.

The similarity equations which govern the boundary-layer development near the edge of the disc can be derived from the boundary-layer equations (1-10) and (1-11) given in Chapter I. Considering the steady form of these equations and following Stewartson, the stream function and tangential velocity are expanded in the following forms:

\[ \psi(r,z) = \xi^{3/4} \sum_{n=0}^{\infty} \xi^n F_n(\eta) \quad (5-1a) \]

\[ v(r,z) = \sum_{n=0}^{\infty} \xi^n V_n(\eta) \quad (5-1b) \]

where

\[ \xi = 1-r \quad (5-1c) \]

and

\[ \eta = \frac{z}{(1-r)^{1/4}} \quad (5-1d) \]

It then follows from equations (1-9) and (5-1a) that the radial and axial velocity components are given by
\[ u(r,z) = \frac{\xi^b}{1-\xi} \sum_{n=0}^{\infty} \xi^n F_n'(n) \quad (5-2a) \]

\[ w(r,z) = \frac{1}{1-\xi} \sum_{n=0}^{\infty} \left[ \xi^{-\frac{3}{4}} \left( \frac{3}{4} F_n - \frac{1}{4} \eta F_n' \right) + (n+1) \xi^{3/4} F_n+1 \right] \xi^n \quad (5-2b) \]

where primes denote differentiation with respect to the similarity variable \( \eta \).

By substituting equations (5-1) into (1-10) and (1-11) and equating the coefficients of like powers of \( \xi \), an infinite number of ordinary differential equations for the unknown functions \( F_n(n) \) and \( V_n(n) \) is obtained. The leading equations are

\[ F'''' - \frac{3}{4} F' F' + \frac{1}{2} (F'_n)^2 + V^2 - 1 = 0 \quad (5-3a) \]

\[ V'' - \frac{3}{4} F V' = 0 \quad (5-3b) \]

with boundary conditions

\[ \eta = 0: \quad F_0 = 0, \quad F'_0 = 0, \quad V_0 = \alpha \quad (5-4a) \]

\[ \eta \to \infty: \quad F'_0 + 0, \quad V_0 + 1 \quad (5-4b) \]
Higher order equations will not be considered in this work and, therefore, will not be written down here.

Equations (5-3) are Stewartson's similarity equations for the edge of a finite disc in a fluid undergoing a circulatory motion, and their solution gives the structure of the laminar boundary layer near the edge to within an error of $O(1 - r)$. The remainder of this chapter will consider the solution of the boundary-value problem defined by (5-3) and (5-4).

**General Properties of the Edge Similarity Solutions**

It is possible to deduce from the differential equations two interesting properties of the solutions to Stewartson's similarity equations. First it will be shown that $V_o(\eta)$ is a monotonic function of $\eta$. This proof was first given by Stewartson\textsuperscript{32}. From (5-3b) after one integration

$$V'_o(\eta) = V'_o(0) \, e^{3/4} \int_0^\eta F_o(\xi) \, d\xi$$

If $V'_o(0) = 0$, then $V_o(\eta)$ is a constant, and a contradiction occurs if $\alpha \neq 1$ because the boundary conditions for $V_o$ at $\eta = 0$ and $\eta \to \infty$ cannot both be satisfied. Hence for $\alpha \neq 1$, $V'_o(0)$ is necessarily non-zero. Furthermore
for all finite \( F_o \), and thus it follows from (5-5) that \( V_o'(\eta) \) has only one sign and cannot vanish at a finite value of \( F_o \). Therefore \( V_o(\eta) \) is a monotonic function of \( \eta \).

Secondly it can be shown that no solution of the boundary-value problem exists for \( \alpha > 1 \). First consider an integration of (5-3a) from \( \eta \) to infinity

\[
F_o''(\infty) - F_o''(\eta) - \frac{3}{4} \left\{ F_o'(\infty)F_o'(\infty) - F_o(\eta)F_o''(\eta) \right\} = \int_{\eta}^{\infty} \left( 1 - V_o^2 - \frac{5}{4}(F_o')^2 \right) d\xi
\]

(5-6)

Using the boundary condition \( F_o'(\infty) = 0 \) and assuming that \( F_o''(\infty) = 0 \) also, (5-6) becomes

\[
-(F_o')' + \frac{3}{8}F_o'' = \int_{\eta}^{\infty} \left\{ 1 - V_o^2 - \frac{5}{4}(F_o')^2 \right\} d\xi
\]

(5-7)

Integrating (5-7) from zero to infinity and applying the appropriate boundary conditions yields

\[
\frac{3}{8}F_o^2(\infty) = \int_0^{\infty} \int_{\eta}^{\infty} \left\{ 1 - V_o^2 - \frac{5}{4}(F_o')^2 \right\} d\eta d\xi
\]

(5-8)
Suppose that $V_0(0) = \alpha > 1.0$. But $V_0(\infty) = 1$, and it was shown above that $V_0(\eta)$ is a monotonic function of $\eta$, therefore $V_0(\eta) > 1$ for all finite $\eta$. Let

$$H_0^2 = V_0^2 - 1$$

then from (5-8)

$$\frac{3}{8} F_0^2 (\infty) = - \int_0^\infty \int_0^\xi H_0^2 + \frac{5}{4} (F_0')^2 \, d\eta$$

(5-9)

The left-hand side of (5-9) is a positive quantity and the right-hand side is a negative quantity; hence, there is a contradiction. Therefore the supposition that $\alpha > 1$ is false, and it follows that $\alpha < 1$ for solutions to exist to the edge similarity equations. Note that for $\alpha = 1$ (5-9) reduces to an identity because the exact solution is $F_0(\eta) = 0$ and $V_0(\eta) = 1.0$ for all $\eta$.

**Numerical Solution of the Edge Similarity Equations**

The edge similarity equations form a coupled set of non-linear ordinary differential equations with two-point boundary conditions of the boundary-layer type. Numerical solutions for all permissible values of the parameter $\alpha$ were sought using the shooting technique developed by
Nachtsheim and Swigert\textsuperscript{16}, the details of which are described in Appendix I.

Using this method, numerical solutions of the edge similarity equations were obtained for values of the parameter $\alpha$ ranging between 1.0 and -2.06626; and for completeness the values of the radial and tangential wall shears and the stream function at infinity are given in Table 4 as functions of the parameter $\alpha$. The step size, $\Delta n$, used in the computations was 0.05, 0.01, or 0.005; while the boundary-layer edge, $\eta_0$, was found to be 12.0, 14.0, or 30.0, depending on the value of $\alpha$. Interestingly it was discovered that solutions could not be obtained for values of $\alpha < -2.06626$, but by suitably modifying the numerical method a new set of solutions was found for values of $\alpha$ increasing from -2.06626 towards the limiting value of -1.0. The behavior of these solutions as $\alpha$ approaches its limiting value will be discussed below.

Due to the circular streamlines of the flow far from the disc, the radial pressure gradient is balanced by the centrifugal force corresponding to the streamline curvature of the outer flow; and according to boundary-layer theory this radial pressure gradient is impressed on the boundary-layer flow. However, the flow in the boundary layer is decelerated by viscous effects so that the balance between the local centrifugal-pressure gradient and that imposed
### TABLE 4

Transformed Radial and Tangential Wall Shears, and Stream Function at Infinity as Functions of $\alpha$ for the Edge Similarity Equations

<table>
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<th>$\alpha$</th>
<th>$U_0'(0;\alpha)$</th>
<th>$V_0'(0;\alpha)$</th>
<th>$F_0(\infty;\alpha)$</th>
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<td>0.000000</td>
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<td>0.074561</td>
<td>0.000750</td>
<td>-1.86262</td>
</tr>
<tr>
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<td>0.049750</td>
<td>0.000350</td>
<td>-1.86426</td>
</tr>
<tr>
<td>-1.001417</td>
<td>0.045808</td>
<td>0.000300</td>
<td>-1.86440</td>
</tr>
<tr>
<td>-1.000722</td>
<td>0.036831</td>
<td>0.000200</td>
<td>-1.86645</td>
</tr>
<tr>
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<td>0.025295</td>
<td>0.000100</td>
<td>-1.86829</td>
</tr>
<tr>
<td>-0.999926</td>
<td>0.022399</td>
<td>0.000080</td>
<td>-1.868547</td>
</tr>
<tr>
<td>-0.999810</td>
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<td>-1.86556</td>
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<tr>
<td>-0.999710</td>
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<td>0.000040</td>
<td>-1.86570</td>
</tr>
</tbody>
</table>
externally to the boundary layer is achieved through substantial changes in local streamline curvature. Thus the flow in the boundary layer differs from the outer flow in both magnitude and direction. This change in direction results in a transverse velocity component normal to the streamlines of the outer flow; the direction and magnitude of which strongly depends on the value of $\alpha$. Consequently, there will be either a radial inflow or outflow near the disc depending on the value of $\alpha$.

For $\alpha$ in the range $0 \leq \alpha < 1.0$, the tangential velocity at the disc is less than at any other point in the fluid, so that the pressure gradient imposed on the boundary layer is more than sufficient to keep the streamlines circular near the disc; consequently, a radial inflow develops near the disc with a corresponding axial inflow as required by mass conservation. Figures 29 and 30 show the radial and tangential velocity profiles, respectively, for $\alpha = 0.80, 0.40, \text{ and } 0.0$. As $\alpha$ decreases from 1, the radial wall shear, $U'_0(0)$, shown in Figure 31 becomes larger in absolute value indicating that the radial inflow in the boundary layer is getting stronger as $\alpha$ decreases, as is evidenced in the radial velocity profiles in Figure 29. Figures 30 and 32 show a similar behavior for the tangential velocity component in the boundary layer.

For $\alpha < 0$ the disc and the fluid at infinity are
rotating in opposite directions. As can be seen from Figure 31, the radial wall shear increases in absolute value to a maximum of 1.11883 at \( \alpha = -.30 \) and then decreases to zero as \( \alpha \) approaches -1.775, indicating that the radial pressure gradient appropriate to the outer flow is sufficient to provide a radial inflow only for values of \( \alpha \) in the range -1.775 to 1. The tangential wall shear, however, continues to increase as long as \( U_0'(0) \) is negative, reaching a maximum value of 1.13776 for \( \alpha = -1.775 \).

Velocity profiles for \( \alpha = -0.30, -1.0, -1.60, \) and -1.775 are given in Figures 33 and 34. The effects of the decreasing radial wall shear and increasing tangential wall shear are clearly shown in the radial and tangential velocity profiles. Furthermore, as evidenced by the two possible values of \( U_0'(0) \) and \( V_0'(0) \) for \( \alpha < -1.0 \), the solutions of the edge similarity equations are not unique for \( \alpha \leq -1 \).

For \( \alpha < -1.775 \) the external pressure gradient impressed on the boundary layer is not sufficient to overcome the local centrifugal force applied to the fluid by the rotating disc; and a radial outflow results as is evidenced by the positive radial wall shear for \( \alpha < -1.775 \) in Figure 31. Following the lower branch of this curve as \( \alpha \) decreases towards -2.06626, it is seen that \( U_0'(0) \) continues to increase rapidly, indicating a significant radial outflow near the disc. It should be noted, however, that even though there
Figure 29. Edge Similarity, Radial Velocity

$0 \leq \alpha \leq 0.8$
Figure 30. Edge Similarity, Tangential Velocity $0 \leq \alpha \leq 0.8$
Figure 31. Edge Similarity, Radial Wall Shear
Figure 32. Edge Similarity, Tangential Wall Shear
Figure 33. Edge Similarity, Radial Velocity $-1.775 \leq \alpha \leq -0.3$
Figure 34. Edge Similarity, Tangential Velocity
-1.775 \leq \alpha \leq -0.3
is a significant outward radial velocity in terms of the similarity variable \( U_0(\eta) \), in terms of the physical variables the radial velocity goes to zero as the edge of the disc is approached. This is readily seen by the relationship between \( u(r,z) \) and \( U_0(\eta) \):

\[
u(r,z) = (1-r)^\frac{1}{2} U_0(\eta) \quad r \xrightarrow{+1} 0
\]

Figure 32 shows that \( V'_o(0) \) is decreasing as \( \alpha \) decreases along the upper branch from -1.775 towards its minimum value of -2.06626, although the change is not as pronounced as for the radial wall shear. Typical tangential velocity profiles for this range of \( \alpha \) are given in Figure 35.

Numerical solutions of the edge similarity equations using the method described in Appendix I became increasingly difficult to obtain as \( \alpha \) approached its minimum value of -2.06626. The slopes of the \( U'_o(0) \) and \( V'_o(0) \) versus \( \alpha \) curves were becoming very large, indicating that there was a limiting value of \( \alpha \) for which solutions could be obtained. For no apparent physical reason the solutions seemed to be terminating with the value of \( \alpha = -2.06626 \).

However, by reformulating the numerical scheme so that the initial guess was on \( V_o(0) = \alpha \) rather than \( V'_o(0) \),
as was done previously, and treating \( V'_o(0) \) as a parameter, a new set of solutions of the edge similarity equations was found with the value of \( \alpha \) increasing from its minimum value of \(-2.06626\) towards the limiting value of \(-1\). Thus for \( \alpha \) lying between \(-1.0\) and \(-2.06626\) there are two solutions of the boundary-value problem posed by equations (5-3) and (5-4). Representative profiles of the tangential velocity, radial velocity and the stream function for these new solutions are given in Figures 35, 36, and 37, respectively, for several values of \( \alpha \) while the variations of \( U'_o(0) \) and \( V'_o(0) \) with \( \alpha \) are shown on the upper branch of Figure 31 and the lower branch of Figure 32.

It is possible to deduce from the numerical solutions several interesting features concerning the structure of the flow as \( \alpha \) approaches its limiting value of \(-1.0\). The tangential velocity profiles given in Figure 35 clearly indicate the development of a region near the disc wherein the tangential velocity of the fluid is approximately the same as that of the disc. Furthermore, this region is becoming larger as \( \alpha \) tends to \(-1\). Also, the shear layer in which the tangential velocity is adjusted to its outer value is being continuously displaced outwards. If the mid-point of this shear layer is defined by the value \( \eta = \hat{\eta}(\alpha) \) such that \( V'_o(\hat{\eta};\alpha) = 0 \), then the outward movement is clearly indicated in Table 5 by the variation of \( \hat{\eta}(\alpha) \) as \( \alpha \) decreases towards \(-1\).
Figure 31 indicates that $U_0'(0)$ is approaching zero as $\alpha \to -1$ in such a manner that there is a significant radial outflow near the disc for values of $\alpha$ close to -1, the magnitude of which can be seen in Figure 36. Note that the maximum value of the radial outflow is decreasing while the location of the second zero of $U_0(\eta)$, defined by the value of $\eta = \eta^*(\alpha)$ such that $U_0(\eta^*; \alpha) = 0$, is moving outwards as $\alpha \to -1$ as is shown in Table 5.

The outer half of the radial velocity profiles closely resemble one another in form for values of $\alpha$ close to -1, all giving the appearance of an inwardly directed radial jet. Furthermore, this region of radial inflow coincides with the region in which the tangential velocity is adjusted to its outer value, giving further evidence of the development of a free shear layer far from the disc as $\alpha \to -1$. Finally Figure 36 indicates that the minimum value of $U_0(\eta; \alpha)$ is approaching a limit as $\alpha \to -1$, and the location of this minimum is continuously moving outward as $\alpha \to -1$, presumably going to infinity on the scale of the boundary layer.

The transformed stream function, $F_0(\eta; \alpha)$, defined by

$$F_0(\eta; \alpha) = \int_0^{\eta} U_0(\xi; \alpha) d\xi$$  \hspace{1cm} (5-10)
Figure 35. Edge Similarity, Tangential Velocity $-0.99971 \leq \alpha \leq -2.06626$
Figure 36. Edge Similarity, Radial Velocity
\(-0.99971 \leq \alpha \leq -2.06625\)
Figure 37. Edge Similarity, Stream function $-0.99971 \leq \alpha \leq -2.06626$


TABLE 5

Zeroes of $F_0(n;\alpha)$, $V_0(n;\alpha)$ and $F_0''(n;\alpha)$ as functions of $\alpha$ from the edge similarity solutions

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\bar{n}(\alpha)$</th>
<th>$\hat{n}(\alpha)$</th>
<th>$n^*(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.496545</td>
<td>3.292711</td>
<td>3.516115</td>
<td>3.730783</td>
</tr>
<tr>
<td>-1.496545</td>
<td>3.817175</td>
<td>3.953321</td>
<td>4.157511</td>
</tr>
<tr>
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<td>5.497609</td>
<td>5.501022</td>
<td>5.650772</td>
</tr>
<tr>
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<td>6.645835</td>
<td>6.624217</td>
<td>6.738724</td>
</tr>
<tr>
<td>-1.028618</td>
<td>8.565529</td>
<td>8.537308</td>
<td>8.613860</td>
</tr>
<tr>
<td>-1.004492</td>
<td>11.741556</td>
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</tr>
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<td>-1.000050</td>
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<tr>
<td>-0.999710</td>
<td>18.345760</td>
<td>18.330827</td>
<td>18.362163</td>
</tr>
</tbody>
</table>
Figure 38. Edge Similarity, Radial and Tangential Wall Shears
is shown plotted in Figure 37 for selected values of $\alpha$. The trend as $\alpha \to -1$ is quite like that discussed for the radial and tangential velocity profiles. The presence of a shear layer within the fluid which is moving outward as $\alpha \to -1$ is again clearly indicated by the outer portions of the stream function profiles, and also by the values of $\bar{\eta}(\alpha)$, defined such that $F_0(\bar{\eta};\alpha) = 0$, given in Table 5.

**Limiting Solution of Edge Similarity Equations for $\alpha = -1$.**

The structure of the solutions in the limit $\alpha = -1$ is especially intriguing. The numerical results indicate the formation of a free shear layer in the fluid which moves outwards as $\alpha \to -1$. Presumably in the limit $\alpha = -1$ all the fluid in the boundary layer is rotating rigidly with the same speed as the disc; and infinitely far from it, on the boundary-layer scale, there is a free shear layer in which the flow is adjusted to the rigidly rotating outer flow.

While attempting to construct an asymptotic solution valid as $\alpha \to -1$ using the method of matched asymptotic expansions it was discovered that a solution was possible only if $\alpha$ approached its limiting value from the right. The numerical solutions discussed above, however, seemed to indicate that the limiting value was being approached from the left. This trend was evident for values of $\alpha$ very close to $-1$ and for very small values of the wall shears. Thus if
α were to become greater than -1 and then turn around and approach -1 from the right it would have to do so for very small values of \( V'_0(0) \). Although numerical solutions became increasingly more difficult to obtain for values of \( V'_0(0) < .00075 \), solutions with \( α > -1 \) were found for \( V'_0(0) < .00010 \). The radial and tangential wall shears in the neighborhood of \( α = -1 \) are given in Figure 38 wherein it is clearly seen that solutions are possible for \( α \) slightly greater than -1. With the existence of such solutions demonstrated numerically the limiting form of the solution as \( α \to -1 \) will now be discussed.

One conclusion to be drawn from the discussion of the numerical results is that the outer portions of the \( V_0(η;α) \), \( U_0(η;α) \) and \( F_0(η;α) \) curves for different values of \( α \) close to -1 can each be collapsed onto a single limiting curve by a shift along the \( η \) axis. If a new coordinate, \( x' \), is defined by \( x = η - \hat{n}(α) \) where \( \hat{n}(α) \) is defined such that \( V'_0(\hat{n};α) = 0 \), and the \( V_0(η;α) \), \( U_0(η;α) \) and \( F_0(η;α) \) profiles are replotted as functions of \( x \) for different values of \( α \), then the collapse of the outer portions of the profiles onto limiting curves is clearly shown in Figures 39, 40, and 41. Also, with \( \overline{η}(α) \) and \( η^*(α) \) defined such that \( F_0(\overline{η};α) = 0 = U'_0(η^*;α) \), Table 5 shows that \( \overline{η}(α), \hat{n}(α), \text{and} η^*(α) \) all have approximately the same value for a given \( α \). Furthermore this value is increasing as \( α \to -1 \) indicating the outward
Figure 39. Edge Similarity, Tangential Velocity in the Outer Shear Layer

\[ V_0 \]

\[ x \]

Equation (5-12)
Figure 40. Edge Similarity, Stream function in the Outer Shear Layer

Eq'n (5-12)

α = -1.091734
-1.028618
-1.004492
-0.999710

F₀
Figure 41. Edge Similarity, Radial Velocity in the Outer Shear Layer

Equation (5-12)

- $x$ values: -1.091734, -1.028618, -1.00449, -1.0050, -0.999710
displacement of the shear layer. Presumably the values of \( \overline{\eta}, \hat{\eta}, \) and \( \eta^* \) will be moved off to infinity in the limit \( \alpha = -1 \).

The above results suggest that the limiting structure of the outer shear layer is of the form

\[
F_0(\eta) = f_0(x) \tag{5-11a}
\]

\[
V_0(\eta) = v_0(x) \tag{5-11b}
\]

\[x = \eta - \eta_0 \tag{5-11c}\]

where \( f_0(x) \) and \( v_0(x) \) satisfy the differential equations

\[
f_0''' - \frac{3}{4} f_0 f_0'' + \frac{1}{2} (f_0')^2 + v_0^2 - 1 = 0 \tag{5-12a}
\]

\[v_0''' - \frac{3}{4} f_0 v_0' = 0 \tag{5-12b}\]

with boundary conditions

\[x = 0: \ f_0 = 0; \quad f_0' = 0, \quad v_0 = 0 \tag{5-12c}\]

\[x \rightarrow \infty: \ f_0' \rightarrow 0; \quad v_0 \rightarrow 1 \tag{5-12d}\]
and \( \eta_0 \to \infty \) as \( \alpha \to -1 \) as suggested by the numerical computations. Using the symmetry properties of the differential equations, solutions for \( x < 0 \) are found from the relations

\[
f_0(-x) = -f_0(x)
\]

\[
v_0(-x) = -v_0(x)
\]

The numerical solution of (5-12) was also obtained using the numerical scheme described in Appendix I and the results for \( v_0(x), f_0(x), \) and \( u_0(x) = f'_0(x) \) are given as the circled points in Figures 39, 40, and 41, respectively. The agreement between this limiting solution and the shifted profiles is excellent when \( x > 0 \), indicating that this is the correct structure for the outer region as \( \alpha \to -1 \).

For \( x < 0 \) only the tangential velocity profiles are in close agreement for all values of \( x \). The agreement for \( x < -0.20 \) is not good for the radial velocity and the stream function, indicating that if a limiting solution for \( \alpha = -1 \) is obtainable there must be an inner layer in which the flow is adjusted from its wall condition to that of the outer shear layer. The following analysis for the inner layer was suggested by K. Stewartson.

To find the governing equations in the inner region
for the terminal solution as \( \alpha \rightarrow -1 \) consider the transformations

\[
n = (2\epsilon)^{-k} \xi \quad (5-13a)
\]

\[
F_0(\eta) = (2\epsilon)^k f_1(\xi) \quad (5-13b)
\]

\[
V_0(\eta) = -1 + \epsilon + v_1(\xi) \quad (5-13c)
\]

where \( \epsilon = 1 + \alpha \) and is assumed to be small and positive.

Substituting equations (5-13) into (5-3) results in the following differential equations for the inner region:

\[
f_1''' - \frac{3}{4} f_1 f_1'' + \frac{1}{2}(f_1')^2 = 1 + \frac{v_1}{\epsilon} \quad (5-14a)
\]

\[
v_1'' - \frac{3}{4} f_1 v_1' = 0 \quad (5-14b)
\]

with boundary conditions at the wall

\[
\xi = 0: \quad f_1 = 0, \quad f_1' = 0, \quad v_1 = 0 \quad (5-14c)
\]

The boundary conditions as \( \xi \rightarrow \infty \) are replaced by the requirement that the solutions not become exponentially large as \( \xi \rightarrow \infty \). It should be noted that the term \((\epsilon + v_1)^2\)
has been neglected in comparison to \((v_1 + v)\) in equation \(5-14a\).

Equation \((5-14b)\) can be integrated immediately to yield

\[
v_1(\xi) = v_1'(0) \int_0^\xi e^{3/4} f_1(y) dy
\]

The numerical results shown in Figures 35 and 39 suggest that to a first approximation \(V_0\) is essentially constant throughout the inner region with its value given by the wall boundary condition. Thus using the method of successive approximations to find solutions of \((5-14)\) take as the first approximation \(v_1(\xi) = 0\). From \((5-14a)\) it follows that

\[
f_1'''' - \frac{3}{4} f_1 f_1'' + \frac{1}{2}(f_1')^2 = 1
\]

with the solution given by

\[
f_1(\xi) = \frac{1}{6} \xi^3
\]

To find the next approximation put \((5-16)\) into \((5-15)\) to obtain
\[ v_1(\xi) = v_1'(0) \int_0^\xi e^{1/32} y^4 \, dy \]

and thus

\[ f_1''' - \frac{3}{4} f_1 f_1'' + \frac{1}{2} (f_1')^2 = 1 + \frac{v_1'(0)}{\xi} \int_0^{\xi/32} y^4 \, dy \]

(5-17)

To solve equation (5-17) write

\[ f_1(\xi) = \frac{1}{6} \xi^3 + \frac{1}{2} \delta \xi^2 + \frac{1}{2} \delta^2 (\xi - \bar{F}_1) + \bar{G}_1 \]

(5-18)

where \( \delta(\xi) \) is a constant to be found and \( \bar{F}_1(\xi) \) and \( \bar{G}_1(\xi) \) satisfy the following linear differential equations:

\[ \bar{F}_1''' - \frac{1}{8} \xi^3 \bar{F}_1'' + \frac{1}{2} \xi^2 \bar{F}_1' - \frac{3}{4} \xi \bar{F}_1 = 0 \]

(5-19a)

\[ \bar{F}_1(0) = 0; \quad \bar{F}_1'(0) = 1; \quad \bar{F}_1''(0) = 0 \]

(5-19b)

\[ \bar{G}_1''' - \frac{1}{8} \xi^3 \bar{G}_1'' + \frac{1}{2} \xi^2 \bar{G}_1' - \frac{3}{4} \xi \bar{G}_1 = \frac{1}{\xi} v_1'(0) \int_0^{\xi/32} y^4 \, dy \]

(5-20a)
\( \bar{g}_1(0) = 0, \; \bar{g}_1'(0) = 0, \; \bar{g}_1''(0) = 0 \) \hfill (5-20b)

A series solution of equation (5-19) is easily found using standard methods and the result is

\[
\bar{F}_1(\xi) = \frac{(k)\Gamma}{8\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(n-3/2)!}{(n+k)! (n-k)! n!} \frac{1}{32^n} \xi^{4n+1}
\]

This solution can be rewritten in terms of the confluent hypergeometric function by noting that

\[
\frac{1}{4n-1} \left( \frac{\xi}{32} \right)^{4n-1} = \int_0^\xi \left( \frac{x}{32} \right)^{4n-2} d\left( \frac{x}{32} \right)
\]

Thus the final result for \( \bar{F}_1(\xi) \) is

\[
\bar{F}_1(\xi) = -\xi^2 \int_0^\xi x^{-2} \, \, _1F_1\left( -\frac{1}{2}; \frac{5}{4}; \frac{\xi^4}{32} \right) dx \hfill (5-21)
\]

In order to match \( f_1(\xi) \) as \( \xi \to \infty \) with the appropriate solution in the outer region the behavior of \( \bar{F}_1(\xi) \) for \( \xi \) large is required. From Abramowitz and Stegun, the
asymptotic form of the confluent hypergeometric function is found to be

\[ {}_1F_1\left(-\frac{1}{2}, \frac{5}{4}, \frac{x^4}{32}\right)_{x \to \infty} \sim \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(-\frac{1}{2}\right)} \frac{x^4}{32^4} e^{x^2/32} \left\{1 + O(x^{-8})\right\} \]

Using this result in equation (5-21) it can be shown that for \( \xi \to \infty \)

\[ F_1(\xi)_{\xi \to \infty} \sim \frac{1}{\sqrt{\pi}} \left(\frac{1}{4}\right)! 2^{4/3} \xi^{-10} e^{1/32} \xi^4 \]  

(5-22)

To complete the solution for \( f_1(\xi) \) the solution for \( \bar{g}_1(\xi) \) must be found from (5-20), and in particular its asymptotic form for large \( \xi \) is required. For large \( \xi \) the differential equation (5-20) can be written in the form

\[
\frac{d^2}{d\xi^2} \bar{g}_1 + \frac{1}{\xi} \frac{d}{d\xi} \bar{g}_1 - \frac{3}{4} \xi \bar{g}_1 = \frac{8}{e^v_1(0)} \xi^{-3} e^{1/32} \xi^4 \left\{1 + \frac{24}{\xi^4} + O(\xi^{-8})\right\}
\]

(5-23)

Solutions for \( \bar{g}_1(\xi) \) are sought in the form
and the resulting solution is

\[
\bar{v}_1(\xi) = e^{1/32} \xi^4 \sum_{n=0}^{\infty} a_n \xi^{-n}
\]

where \(a\) is a free constant at this stage but independent of \(\varepsilon\).

Thus the asymptotic form of the solution for the second approximation in the inner region is given by

\[
v_1(\xi) \sim 8v_1(0) \xi^{-3} e^{1/32} \xi^4 \left(1 + 24\xi^{-4} + o(\xi^{-8})\right)
\]

(5-25)

\[
f_1(\xi) \sim \frac{1}{6} \xi^3 + \frac{1}{2} \delta \xi^2 + \frac{1}{2} \delta^2 [\xi - \frac{1}{\sqrt{\pi}} (\frac{3}{4})! 2^{43/4} \xi^{-10} e^{1/32} \xi^4]
\]

+ \[\frac{v_1(0)}{\varepsilon} e^{1/32} \xi^4 \left[256\xi^{-8} + a\xi^{-10} + o(\xi^{-11})\right]
\]

(5-26)

To continue the solution further the asymptotic form
of the solution in the outer shear layer must be determined. The differential equations for the terminal structure in the outer shear layer when \( \alpha = -1 \) have already been given in (5-12). Perturbations from this limiting solution for small values of \( \epsilon \) are now sought by writing

\[
F_o(\eta) = f_o(x) + \epsilon^{1/3} F_0(x) \tag{5-27a}
\]

\[
V_o(\eta) = v_o(x) + \epsilon^{1/3} V_0(x) \tag{5-27b}
\]

Substituting (5-26) into (5-3) and collecting like powers of \( \epsilon \) leads to the following set of differential equations for the perturbation functions \( F_0(x) \) and \( V_0(x) \):

\[
F_0''' = \frac{3}{4}[f_o F_0'' + f_o' F_0'] + f_o' F_0 + 2v_o V_o = 0 \tag{5-28a}
\]

\[
V_0'' = \frac{3}{4}[v_o' F_0 + f_o V_0'] = 0 \tag{5-28b}
\]

with boundary conditions as \( x \to \pm \infty \) given by

\[
x \to \pm \infty: \quad F_0' + 0; \quad V_0 + 0 \tag{5-28c}
\]
The constant $\bar{K}$ appearing in the boundary condition (5-28d) is found by matching the solution for $\bar{f}_o(x)$ in the outer shear layer with the appropriate solution in the inner region; while the constant $p$ has to be found from a numerical integration of the perturbation equations. It should also be noted that the functions $\bar{f}_o(x)$ and $\bar{v}_o(x)$ given by the solution of the boundary-value problem (5-28) are arbitrary to within the additive terms $\bar{a}f'_o(x)$ and $\bar{a}v'_o(x)$, respectively, since $f'_o(x)$ and $v'_o(x)$ are also solutions of the perturbation-differential equations.

In order to match the solutions between the outer shear layer and the inner region the asymptotic forms of $f'_o(x)$ and $v'_o(x)$ as $x \to -\infty$ are required. Let $f'_o(-\infty) = C_o > 0$, then it can easily be shown from the differential equations (5-12) that

$$f'_o(x) \sim C_o + \frac{32D_o}{9C_o^2} x e^{3/4} C_o^x + \ldots \quad (5-29a)$$

$$f'_o(x) \sim C_o + \left\{ E_o + \frac{32D_o}{9C_o^2} x \right\} e^{3/4} C_o^x + \ldots \quad (5-29b)$$
The values of the constants \( C_0, D_0, \) and \( E_0 \) are found by comparing (5-29a,b) with the numerical solution of (5-12). The results are

\[
C_0 = 1.87042; \quad D_0 = 2.71828; \quad E_0 = -1.08132 \quad (5-29c)
\]

The solution given by (5-29) above for \( x \to -\infty \) must match with the solution given by (5-24) in the limit \( \epsilon \to 0 \) since \( x \) large and negative corresponds to \( \xi \) tending to infinity. It should be noted, however, that this statement is based on the assumption that \( (2\epsilon)^{k_0} \eta_0(\epsilon) \to \infty \) as \( \epsilon \to 0 \), an assumption which can be shown to be consistent with the expression for \( \eta_0 \) in terms of \( \epsilon \) obtained in the subsequent analysis. First consider \( V_0(\eta) \). From (5-13c) and (5-25) it is found that

\[
V_0(\eta) \sim 1 + \epsilon + \frac{4V_0'(0)}{\epsilon \eta^3} \epsilon^{\frac{\epsilon}{16}} \eta^4 \left\{ 1 + \frac{12}{\epsilon \eta^4} + O\left(\frac{1}{\epsilon^2 \eta^3}\right) \right\}
\]

(5-30)

In terms of the outer variable \( x \) defined by (5-11c) the solution for \( V_0 \) can be written as

\[
V_0 = -1 + \epsilon + \frac{4V_0'(0)}{\epsilon \eta^3} \left\{ \epsilon^{\frac{\epsilon}{16}} \eta^4 + \frac{\epsilon}{16} \eta^3 x \left\{ 1 + O(\eta_0^{-4}) \right\} \right\}
\]

(5-31)
In order to match the terms involving exponentials in (5-29a) and (5-31) it is necessary that

\[ n_0(\epsilon) = \left( \frac{3C_0}{\epsilon} \right)^{1/3} \quad (5-32a) \]

\[ C = C_0 + \nu \epsilon^{1/3} + ... \quad (5-32b) \]

where \( \nu \) is an unknown constant at this point but independent of \( \epsilon \). Using (5-32) in (5-31) yields

\[ V_0 = -1 + \epsilon + \frac{4V_0'(0)}{3C_0} \frac{(3C_0)^{4/3}}{e^{16\epsilon^{1/3}}} + \frac{\nu(3C_0)^{1/3}}{4} e^{3/4} C_0^x + ... \quad (5-33) \]

Finally, comparing the exponential terms in (5-33) and (5-29a) it is readily seen that a match is possible if

\[ D_0 = \frac{4V_0'(0)}{3C_0} \frac{(3C_0)^{4/3}}{e^{16\epsilon^{1/3}}} + \frac{\nu(3C_0)^{1/3}}{4} \]

Hence, the variation of the tangential wall shear with \( \epsilon \) is known apart from the value of \( \nu \), and is given by
\[ V_0'(0; \varepsilon) = \frac{3}{4} \varepsilon \frac{C_0 D_0 e}{16 \varepsilon^{1/3}} - \frac{v(3C_0)^{1/3}}{4} \]  \hspace{1cm} (5-34) 

Similarly, for \( n \) large it is found from (5-13b) and (5-26) that

\[ F_0(n) = \frac{1}{3} \varepsilon n^3 + \frac{1}{2} \delta (2\varepsilon)^{3/4} n^2 + \frac{1}{2} \varepsilon^2 (2\varepsilon)^{1/2} n \]

\[ + \frac{e^{15}}{16} n^4 \left\{ \frac{256 V_0'(0)}{4 \varepsilon^3 n^3} - \frac{\frac{1}{2} \delta^2 \sqrt{\pi} \frac{1}{1!} \cdot 2^{43/4} (2\varepsilon)^{1/4} \cdot \frac{av_0'(0)}{\varepsilon}}{(2\varepsilon)^{5/2} n^{10}} \right\} \]  \hspace{1cm} (5-35)

When \( n = n_0 = (\frac{3C_0}{\varepsilon})^{1/3} \) the behavior of \( F_0(n) \) will be exponentially large to leading order as \( \varepsilon \to 0 \) unless the exponential term appearing in (5-42) is cancelled out by the appropriate choice of \( \delta = \delta(\varepsilon) \). Thus write

\[ \frac{1}{2} \left\{ \varepsilon^{-1/2} \left( \frac{1}{4} \right)! \cdot 2^{43/4} \right\} \delta^2 = (2\varepsilon)^{-1/4} \left\{ 256 \cdot 2^{1/2} (3C_0)^{2/3} \right\} \]

\[ \cdot \varepsilon^{-7/6} V_0'(0)(1 + \varepsilon^{1/3}) + \frac{av_0'(0)}{\varepsilon} \]
where $b$ is a constant, independent of $\epsilon$, which must be found by matching with the outer solution.

Therefore, to leading order

\[
\delta^2 = \frac{512}{(1/4)!} \frac{\pi^{1/2}}{2^{21/2}} \frac{\left(\frac{3}{5}\right)^{2/3}}{\epsilon - 17/12} v_0'(0)
\]

and using the results for $v_0'(0)$ and $C(\epsilon)$ given by equations (5-32b) and (5-34)

\[
\delta^2 = \frac{384}{(1/4)!} \frac{\pi^{1/2}}{2^{21/2}} \frac{3^{2/3} C_0^{5/3} D_0}{\epsilon - 17/12} e^{16\epsilon^{1/3}} \frac{(3C_0)^{4/3}}{4}
\]

(5-36)

Finally, rewriting the solution for $F_\eta(\eta)$ given by (5-35) in terms of the outer variable $x$ and simplifying, it is found that

\[
F_\eta - C_\eta + D_\eta e^{3/4} C_\eta x \left\{ \frac{32 x}{9 C_\eta^2} - \left[ \frac{256 v}{(3C_\eta)^{8/3}} + \frac{16 b}{(3C_\eta)^{5/3}} \right] \right\}
\]

\[
+ \epsilon^{1/3} \left\{ (3C_\eta)^{2/3} [x + \frac{v}{(3C_\eta)^{2/3}}] \right\} + \ldots
\]
Similarly the solution for $F_0$ obtained from the outer shear layer as $x \to \infty$ is given by

$$F_0 = C_0 + D_0 e^{3/4 C_0 x} \frac{32}{9 C_0^2} x + \frac{E_0}{D_0} + \varepsilon^{1/3} \overline{F}(x+p) + \ldots$$

Hence a match between the inner region and the outer shear layer is possible if

$$\overline{F} = (3C_0)^{2/3} \quad (5-37a)$$

$$v = (3C_0)^{2/3} p \quad (5-37b)$$

$$b = -\frac{(3C_0)^{5/3}}{16} \left\{ \frac{E_0}{D_0} + \frac{256 \nu}{(3C_0)^{8/3}} \right\} \quad (5-37c)$$

To complete the match the value of $p$ must be determined, and as previously mentioned it is found from the numerical integration of the perturbation equations (5-28). In order to find a numerical solution, however, another boundary condition must be added to those given by (5-28c,d). For the purposes of this study the condition $\overline{F}_0(0) = 0$ was used. Actually, since the solution to (5-28) is arbitrary to within the additive terms $\overline{f}_0'(x)$ and $\overline{v}_0'(x)$ any value
for \( \bar{f}_o(0) \) could have been used without affecting the asymptotic structure of the solution as \( x \to \pm \infty \). As a check the solution for \( \bar{f}_o(0) = -1.0 \) was also found numerically and the results for \( x \to \pm \infty \) were identical to those obtained when the boundary condition was \( \bar{f}_o(0) = 0 \). With the numerical solution for \( \bar{f}_o(x) \) and \( \bar{v}_o(x) \) known the value of \( p \) is easily found to be \( p = 2.90525 \); whence it follows from (5-37) that

\[
\nu = 9.17396, \quad b = 25.71819, \quad \bar{K} = 3.15772
\]

An expression for the tangential wall shear, \( V_o'(0) \), as a function of \( \epsilon \) has already been given by equation (5-34); and a similar expression for the radial wall shear, \( U_o'(0) \), can be found from equations (5-18) and (5-36), the result being

\[
U_o'(0) = \frac{\pi^{1/4}}{2\sqrt{(1/4)^{1/4}}} \left(3C_o\right)^{5/6} D_o^{1/2} \epsilon^{1/24} e^{-\frac{(3C_o)^{4/3}}{32\epsilon^{1/3}}} - \frac{\nu(3C_o)^{1/3}}{8}
\]

(5-38)

Using the numerical values for \( C_o, D_o, \) and \( \nu \) defined above the variations of \( U_o'(0) \) and \( V_o'(0) \) with \( \epsilon \) were determined and are shown in Figure 38 along with the values
of the wall shears obtained from the numerical solutions. As can be seen from the figure the values of \( U'_o(0) \) and \( V'_o(0) \) calculated from the above analysis do not agree with those found from the numerical solutions, although it appears possible that they are in asymptotic agreement. Thus it cannot be said with complete certainty that the wall shears calculated from the numerical solutions will approach zero as \( \varepsilon \to 0 \) in the manner indicated by the asymptotic analysis. The numerical results do, however, suggest that the \( U'_o(0) \) and \( V'_o(0) \) versus \( \varepsilon \) curves are beginning to turn towards the origin; and presumably, if numerical solutions could be obtained for smaller values of \( V'_o(0) \), the results for the wall shears would agree more closely with those from the asymptotic analysis.
CONCLUSIONS

The existence of a steady laminar boundary layer which begins at the edge of a finite rotating disc placed in an otherwise unbounded rigidly rotating fluid is found to be limited by the value of the parameter $\alpha$; the explicit value depending on the structure of the similarity equations applicable at the center and edge of the disc.

The numerical solutions of the edge similarity equations show that a boundary layer which begins at the edge of the disc and grows inward towards the center is possible only if $-2.06626 < \alpha < 1$. Furthermore, for $\alpha < -1$ the solutions of these equations are not unique, but instead there exist two possible branches in this range. Along one branch the solutions approach a singular structure as $\alpha \to -1$. In the limit $\alpha = -1$ this solution has an inner region in which the flow is rotating rigidly with the disc. While at infinity, on the scale of the boundary layer, there is a free shear layer in which the flow is adjusted to the outer boundary conditions. Along the second branch the flow field is well-behaved for all values of $\alpha$; and presumably, this is the branch which will occur in nature, although no proof has been given for this. In addition the radial flow near
the surface of the disc is radially inward for $\alpha = -1$ along the second branch while along the other branch it is outward.

Numerical solutions of the steady $r = 0$ similarity equations applicable near the center of the disc could not be found for values of $\alpha < -0.689043$, and the numerical solutions obtained indicated that a critical value of $\alpha$ exists for which finite solutions can be found. Also, a first-order asymptotic theory was considered assuming that such a critical value of $\alpha$ exists. A consistent expansion was obtained, involving a thin viscous region near the wall followed by a thick inviscid intermediate region, and yet another thin viscous outer region. As a result of this analysis the critical value of $\alpha$ was found to be $\alpha_{cr} = -0.6961$. From these results and those of Evans$^8$ and McLeod$^9$ it is concluded that the steady similarity equations for $r = 0$ do not possess solutions for values of $\alpha$ lying in the range $-6.211 < \alpha < -0.6961$. Therefore, for this range of $\alpha$ no steady laminar boundary layer is possible near the center of a finite rotating disc.

The pertinence of self-similar solutions to physically realizable flows is always in doubt due to the generally restrictive assumptions in the similarity studies. One way to examine the relevance of the similarity solutions is to
consider a more general problem in some manner and then compare the results with those obtained from the similarity analysis.

To examine the nonexistence of the steady similarity solutions when $\alpha = -1$ the time-dependent boundary-layer equations were integrated numerically for the disc of radius $a$ using an implicit finite-difference scheme. The computations fail to approach a steady-state behavior near the axis of the disc; in fact, at $r = 0$ the numerical computations failed to converge for $t > 2.28$. By this time the axial outflow from the boundary layer has become very large indicating an eruption of the boundary layer near the center of the disc. Heuristically the nonexistence of the steady similarity solution when $\alpha = -1$ can be explained by the fact that the boundary-layer solutions in the vicinity of the axis do not approach a steady-state.

In addition, numerical computations of the unsteady similarity equations for $\alpha = -0.5$, $-0.25$, and $-0.10$ show that the appropriate steady-state solutions are not reached, even though they are known to exist. For $\alpha = -0.5$ and $-0.25$ the computations diverge while for $\alpha = -0.10$ a numerical limit cycle is reached. These results indicate that the $r = 0$ steady similarity solutions are physically unacceptable for values of $\alpha < -0.10$ since they cannot be
reached via a time-dependent process and are therefore not likely to occur in nature. Thus the existence of a steady laminar boundary layer on a finite rotating disc when $\alpha \leq -0.10$ is in doubt.

A linearized stability analysis for these same similarity equations has led to contradictory results. Numerical computations for the eigenvalues show that unstable modes are possible for $\alpha < -0.03$ while an asymptotic theory shows that all disturbances ultimately decay to zero for all choices of $\alpha$, whether positive or negative. A completely satisfactory explanation of the contradiction between these two results is still not available. The instability may be entirely numerical in origin due to the finite-mesh thickness used in the computations to approximate the semi-infinite domain. In fact, there is some evidence to support this claim. However, there may be some unstable mode which is lost in this first-order asymptotic theory since the unsteady numerical computations are clearly unstable for large times when $\alpha < -0.10$.

There are several ways in which the problems discussed in this study can be extended. First of all, for the steady $r = 0$ similarity equations the nature of the breakdown in the neighborhood of $\alpha = -6.211$ is still not understood. Further numerical and analytical studies can be carried out to examine the failure of the convergence of
the numerical computations for no apparent reason when \( \alpha > -6.211 \).

The stability analysis discussed in Chapter IV is still not complete. In this study only one class of disturbances was examined. A second-order asymptotic theory could be carried out as this might bring in an unstable mode which is missing in the first-order theory. K. Stewartson has suggested that there is possibly a bifurcation of solutions occurring in this problem which would help explain the appearance of the unstable mode in the eigenvalue analysis. Also, the stability analysis shows that a linear relationship between \( z \) and \( t \) is important, and possibly it would be of value to reconsider the unsteady calculations explicitly taking into account this relationship in the formulation of the problem.

For the edge similarity solutions the agreement between the numerical results and the first-order asymptotic theory in the limit \( \alpha \to -1 \) is not as good as one would like. It might be possible to carry out further numerical solutions using a different numerical scheme than that used in this work. Alternatively, a second-order asymptotic theory could be attempted to determine whether the numerical and theoretical values of the wall shears could be brought into closer agreement.
Finally, the approach to the steady-state solution of the edge similarity equations can be considered using the time-dependent equations to determine whether the steady solutions along one or both of the branches is unstable to small perturbations. This analysis could rule out one of the two solution branches as physically unrealizable.
APPENDIX I

The shooting technique developed by Nachtsheim and Swigert is especially suited to problems of the boundary-layer type. As applied to the problem considered in Chapter V, the method involves guessing initially the values of $F'_o(0)$ and $V'_o(0)$ (or $V_o(0)$), numerically integrating the differential equations to some finite value of $n$ which is taken to be the outer edge of the boundary layer, and then suitably varying the initial guesses until the edge conditions have been satisfied to within some prescribed small amount. Also, in boundary-layer type problems where the outer conditions are applied, in principle, at infinity, the "edge" of the boundary layer ($n_e$) used in the numerical computations must be chosen sufficiently large.

The computations are very sensitive to both the choice of the values used for $F'_o(0)$ and $V'_o(0)$ and the value of $n_e$ used. In particular, if $n_e$ is too large and there is a small error in the unknown wall values, then any growing terms in the differential equations will quickly dominate the computations and cause the solutions to diverge. Nachtsheim and Swigert's method eliminates this problem (where to stop the integration), and it is capable of satisfying
the boundary conditions at the edge of the boundary layer correctly; that is, the boundary values are approached asymptotically.

Let \( x = F'(0) \) and \( y = V'(0) \) be the unknown initial conditions at the wall which are to be determined so that the appropriate boundary conditions at the outer edge of the boundary layer are satisfied; i.e., values of \( x \) and \( y \) are sought that will simultaneously satisfy the nonlinear equations

\[
F'[\eta_e, x, y] = 0 \tag{I-1a}
\]

\[
V'[\eta_e, x, y] = 1 \tag{I-1b}
\]

Corrections to \( x \) and \( y \) are found by considering the Taylor series expansions of (I-1):

\[
0 = F_o' + F_o' \Delta x + F_o'y \Delta y + \frac{1}{2} F_o''(\Delta x)^2 + F_o' y \Delta x \Delta y + \frac{1}{2} F_o'' y (\Delta y)^2 + \ldots \tag{I-2a}
\]

\[
1 = V_o + V_o' \Delta x + V_o' y \Delta y + \frac{1}{2} V_o''(\Delta x)^2 + V_o y \Delta x \Delta y + \frac{1}{2} V_o'' y (\Delta y)^2 + \ldots \tag{I-2b}
\]
where all quantities in (5-11) are evaluated at \( n = n_e \) and 
\[
\left( \frac{\partial}{\partial x} \right)_x \equiv \frac{\partial}{\partial x}( ), etc.
\]

The original boundary-value problem has asymptotic boundary conditions in that the outer conditions are satisfied in the limit \( n \to \infty \). Numerically, however, the boundary conditions are applied at a finite value of \( n \); and in order to approximate the asymptotic conditions at a finite value of \( n \) the conditions \( F_o''(n \text{ edge}) = 0 = V_o'(n \text{ edge}) \) are applied along with the original boundary conditions. Hence equations (5-11) are supplemented by

\[
0 = F''_o + F''_{ox} \Delta x + F''_{oy} \Delta y + \frac{1}{2} F''_{oxy} (\Delta x)^2 + F''_{oxo} \Delta x \Delta y + \frac{1}{2} F''_{ooy} (\Delta y)^2 + \ldots
\]

(I-2c)

\[
0 = V''_o + V''_{ox} \Delta x + V''_{oy} \Delta y + \frac{1}{2} V''_{oxo} (\Delta x)^2 + V''_{oxy} \Delta x \Delta y + \frac{1}{2} V''_{ooy} (\Delta y)^2 + \ldots
\]

(I-2d)

In general there will be no solution to the problem defined by (I-2) since there are four equations and only two unknowns, namely \( \Delta x \) and \( \Delta y \). However, a satisfactory solution that is consistent with the idea that the boundary conditions cannot be satisfied exactly at a finite value
of \eta is to seek the least-squares solution of \((I-2)\). To do this consider the differences, \(\delta_1, \ldots, \delta_4\), defined by the difference between the calculated values of \(F'_o, F''_o, V'_o,\) and \(V''_o\) at \(\eta = \eta_e\) and their true values:

\[
\delta_1 = F'_o + F'_{ox} \Delta x + F'_{oy} \Delta y + \frac{1}{2} F''_{oxx} (\Delta x)^2 + F''_{oxy} \Delta x \Delta y + \frac{1}{2} F''_{oyy} (\Delta y)^2 + \ldots
\]

\[
\delta_2 = F''_o + F''_{ox} \Delta x + F''_{oy} \Delta y + \frac{1}{2} F''_{oxx} (\Delta x)^2 + F''_{oxy} \Delta x \Delta y + \frac{1}{2} F''_{oyy} (\Delta y)^2 + \ldots
\]

\[
\delta_3 = (V'_o - 1) + V'_{ox} \Delta x + V'_{oy} \Delta y + \frac{1}{2} V''_{oxx} (\Delta x)^2 + V''_{oxy} \Delta x \Delta y + \frac{1}{2} V''_{oyy} (\Delta y)^2 + \ldots
\]

\[
\delta_4 = V''_o + V''_{ox} \Delta x + V''_{oy} \Delta y + \frac{1}{2} V''_{oxx} (\Delta x)^2 + V''_{oxy} \Delta x \Delta y + \frac{1}{2} V''_{oyy} (\Delta y)^2 + \ldots
\]

The values \(\Delta x\) and \(\Delta y\) are sought such that the sum

\[
S = \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2
\]

is minimized. Thus, set \(\frac{S}{\partial \Delta x} = 0 = \frac{S}{\partial \Delta y}\) and solve for \(\Delta x\) and \(\Delta y\) by neglecting terms which are quadratic in \(\Delta x\) and \(\Delta y\).

The results are given by the solution of the following linear equations:
\[ A_1 \Delta x + A_2 \Delta y = A_3 \]  
(I-3a)

\[ A_2 \Delta x + A_4 \Delta y = A_5 \]  
(I-3b)

where

\[ A_1 = (F'_{ox})^2 + (F''_{ox})^2 + (V_{ox})^2 + (V'_{ox})^2 \]
\[ + [F'_{ox}F'_{ox} + F''_{ox}F''_{ox} + V_{ox}V'_{ox} + V'_{ox}V''_{ox} - V_{ox}] \]  
(I-3c)

\[ A_2 = F'_{ox}F'_{oy} + F''_{ox}F''_{oy} + V_{ox}V_{oy} + V'_{ox}V'_{oy} \]
\[ + [F'_{ox}F'_{oxy} + F''_{ox}F''_{oxy} + V_{ox}V_{oxy} + V'_{ox}V'_{oxy} - V_{oxy}] \]  
(I-3d)

\[ A_3 = - \left\{ F'_{ox}F'_{ox} + F''_{ox}F''_{ox} + V_{ox}V_{ox} + V'_{ox}V'_{ox} - \dot{V}_{ox} \right\} \]  
(I-3e)

\[ A_4 = (F'_{oy})^2 + (F''_{oy})^2 + (V_{oy})^2 + (V'_{oy})^2 \]
\[ + [F'_{oy}F'_{oy} + F''_{oy}F''_{oy} + V_{oy}V'_{oy} + V'_{oy}V''_{oy} - V_{oy}] \]  
(I-3f)
where all the terms appearing in (I-3) are to be evaluated at \( \eta = \eta_c \).

The partial derivatives \( F'_{\eta x}, F'_{\eta xx}, \text{etc.} \), appearing in the above equations, are obtained by solving the perturbation-differential equations, which, in turn, are formed by taking the appropriate \( \eta \) and \( \eta \) derivatives of the original differential equations (5-3). A considerable savings in computer time can be achieved, however, by noting that second-order derivatives in \( x \) and \( y \) will generally be much smaller in magnitude than the corresponding first-order derivatives for values of \( x \) and \( y \) reasonably close to their correct values. Therefore, if the bracketed terms in \( A_5 \), \( A_2 \), and \( A_4 \) are neglected in comparison with other terms in the expressions, then only the first-order perturbation-differential equations need be formulated and solved along with the original differential equations. This was the approach taken in the present numerical study, and it should be noted that Nachtsheim and Swigert have implicitly made the same assumption in their report because they neglected second-order effects in \( \Delta x \) and \( \Delta y \) from the very beginning.

The values of \( \Delta x \) and \( \Delta y \) used in the numerical computations are thus given by the solution to the following:

\[
A_5 = -\left[ F'_o F'_{\eta y} + F''_o F''_{\eta y} + V_o V_{\eta y} + V'_o V'_{\eta y} - V_{\eta y} \right] (I-3g)
\]
matrix equation:

\[
\begin{bmatrix}
(F_{ox})^2 + (F_{ox})^2 + (V_{ox} V_{oy})^2 & F_{ox} F_{oy} + F_{ox} F_{oy} + V_{ox} V_{oy} + V_{ox} V_{oy} \\
F_{ox} F_{oy} + F_{ox} F_{oy} + V_{ox} V_{oy} + V_{ox} V_{oy} & (F_{oy})^2 + (F_{oy})^2 + (V_{oy})^2
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta y
\end{bmatrix}
\]

\[
= - \begin{bmatrix}
F_{ox} F_{oy} + F_{ox} F_{oy} + V_{ox} V_{oy} + V_{ox} V_{oy} - V_{ox} \\
F_{ox} F_{oy} + F_{ox} F_{oy} + V_{ox} V_{oy} + V_{ox} V_{oy} - V_{oy}
\end{bmatrix}
\]

where all terms are to be evaluated at \( n = n_e \).

The first-order perturbation-differential equations are

\[
F_{ox}''' - \frac{3}{4} [F_{ox} F_{ox} + F_{o} F_{ox}] + F_{ox} F_{ox}' + 2V_{ox} V_{ox} = 0 \quad (I-4a)
\]

\[
F_{ox}''' - \frac{3}{4} [F_{o} F_{ox} + F_{o} F_{ox}] + F_{o} F_{ox}' + 2V_{o} V_{ox} = 0 \quad (I-4b)
\]

\[
V_{ox}'' - \frac{3}{4} [F_{ox} V_{ox} + F_{ox} V_{ox}'] = 0 \quad (I-4c)
\]

\[
V_{oy}'' - \frac{3}{4} [F_{o} V_{o} + F_{o} V_{o}'] = 0 \quad (I-4d)
\]
with boundary conditions at \( n = 0 \) given by

\[
\begin{align*}
F_{ox} &= F'_{ox} = V_{ox} = V'_x = 0; \quad F''_{ox} = 1 \\
F_{oy} &= F'_{oy} = F''_{oy} = V_{oy} = 0; \quad V_{oy} = 1
\end{align*}
\]  (I-4e)

The differential equations (5-3) and the perturbation equations (I-4) were integrated simultaneously using either fourth-order Runge-Kutta or an Adams-Moulton scheme; and the process was repeated until \( \Delta x/k \) and \( \Delta y/y \) were less than \( 1 \times 10^{-8} \) in absolute value.

Let the error, \( E \), be defined as the sum of the squares of the deviations of the computed quantities from their asymptotic values, thus

\[
E = (F_{o}')^2 + (F_{o}'')^2 + (1-V_{o})^2 + (V_{o}')^2
\]

The magnitude of \( E \) evaluated at \( n = n_e \) gives an indication of how unsatisfactory the asymptotic boundary conditions are in a mean square sense. If \( E \) is greater than some pre-assigned amount, taken to be \( 1 \times 10^{-8} \) in this study, the value of \( n_e \) is increased and the whole process is repeated, continuing until the condition on \( E \) is satisfied. At this
point the solutions are said to have converged, and the results are taken as the solution of the original boundary-value problem.
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