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The Ohio State University, Ph.D., 1973
Mathematics

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RESTRICTED REPRESENTATIONS OF CLASSICAL LIE ALGEBRAS
OF PRIME CHARACTERISTICS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Kwok Chi Wong, B.Sc., B.Sc.(Special), M.Sc.

The Ohio State University
1973

Reading Committee:
Robert B. Brown
Joseph C. Ferrar
Hans J. Zassenhaus
Duncan McConnell

Approved By

Hans J. Zassenhaus
Adviser
Department of Mathematics
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VITA

May 11, 1942 . . . . Born - Sumatra, Indonesia

1965 . . . . . . B.Sc., New Asia College, The Chinese University of Hong Kong, Hong Kong

1966 . . . . . . B.Sc. (Special), Hong Kong University, Hong Kong

1966-1967 . . . . Teaching Assistant, Department of Mathematics, The Ohio State University, Columbus, Ohio, U.S.A.

1967 . . . . . . M.Sc., The Ohio State University, Columbus, Ohio, U.S.A.

1967-1969 . . . . Teaching Assistant, Department of Mathematics, The Ohio State University, Columbus, Ohio, U.S.A.

1970-1971 . . . . Teaching Associate, Department of Mathematics, The Ohio State University, Columbus, Ohio, U.S.A.

1972 . . . . . . Dissertation Year Fellow, The Graduate School, The Ohio State University, Columbus, Ohio, U.S.A.

1973 . . . . . . Teaching Associate, Department of Mathematics, The Ohio State University, Columbus, Ohio, U.S.A.
FIELDS OF STUDY

Major Field: Mathematics

Studies in Algebra. Professors R. P. Bambah and A. Cronheim

Studies in Complex Analysis. Professor D. J. Eustice

Studies in Real Analysis. Professor B. M. Bishanski


Studies in Representation Theory of Finite Groups. Professor H. Zassenhaus

Studies in Lie Algebra. Professor J. C. Ferrar

Studies in Hopf Algebra. Professor H. P. Allen

Studies in Algebraic Group Theory. Professor H. Zassenhaus
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INTRODUCTION

Jacobson showed in [10, p.826] that complete reducibility holds for any representation \( \psi \) of a split 3-dimensional simple Lie algebra \( L \) with a basis \( e, f, h \) and \([e,f] = h, [e,h] = 2e, [f,h] = -2f\), over a field of characteristic \( p > 2 \), if \((\psi(e))^{p-1} = (\psi(f))^{p-1} = 0\). But this condition is not necessary, even for restricted representations. Similar to Jacobson's approach, Seligman gave in [13] a necessary and sufficient condition for complete reducibility of any restricted representation \( \psi \) of \( L \). Consequently a generating set for the radical of the \( u \)-algebra of \( L \) was obtained. \( L \) is known as classical Lie algebra of rank 1. A classical Lie algebra \( \mathcal{L} \) of rank \( l \geq 2 \) over an algebraically closed field of characteristic \( p > 7 \) has a canonical generating set \( \{e_1, \ldots, e_l, h_1, \ldots, h_l, e_{-1}, \ldots, e_{-1}\} \) such that \( e_1, h_1, e_{-1} \) form a basis of a split 3-dimensional simple Lie algebra \( \mathcal{L}(i) \) for each \( i = 1, \ldots, l \). It is not known as to what sort of conditions are both necessary and sufficient for a representation \( \psi \) of \( \mathcal{L} \) to be completely reducible. It was hoped that the complete reducibility of \( m \) as a
restricted module for \( L(i) \), \( i = 1,2,\ldots,1 \), would imply the complete reducibility of \( \mathfrak{m} \) as a restricted \( \mathfrak{L} \)-module, and that the complete reducibility of \( \mathfrak{m} \) as a restricted \( \mathfrak{L} \)-module would imply the complete reducibility of \( \mathfrak{m} \) as a restricted \( L(i) \)-module \( i = 1,\ldots,1 \). Hence Seligman's result could be used to give necessary and sufficient conditions for the complete reducibility of any restricted representation \( \mathcal{F} \) of \( L \). However, our Examples III.1 and III.2 show that neither direction of the implications is true. Since a finite dimensional restricted representation \( \mathcal{F} \) of \( L \) is completely reducible if and only if \( \mathcal{F} \) vanishes on a generating set of the radical \( \mathcal{R} \) of the \( U \)-algebra \( \mathcal{U} \) of \( L \) (Proposition I.18), the study of completely reducible restricted representations of \( L \) amounts to finding generating sets of the radical of \( U \). Reinvestigating Seligman's result and studying the Nielsen's minimal right ideals in \( U \) [11], we obtain an entirely different approach to the proof of Seligman's theorem, as well as a number of different sets of conditions (Theorem II.18) necessary and sufficient for the complete reducibility of any restricted representation of a split 3-dimensional simple Lie algebra of characteristic \( p > 2 \). Our approach involves only computations within \( U \) and is easily generalized to give necessary conditions (Theorem
III.4) for complete reducibility of any restricted representation \( \gamma \) of classical Lie algebras of rank \( l \geq 2 \). Examples III.5 and III.6 show that these conditions are not sufficient, whence a generating set of a proper two-sided nilpotent ideal \( \mathcal{N} \) in \( \mathcal{U} \) is found. We also show (Theorem III.9) that \( \mathcal{N} \) contains properly the sum of all the two-sided ideals generated by the nilpotent right ideals constructed by Nielsen [11]. Finally we give two possible directions for finding more elements in the radical of the \( \mathfrak{u} \)-algebra of a classical Lie algebra of type \( A_2 \) with the knowledge of the finite dimensional irreducible modules for a complex simple Lie algebra of type \( A_2 \).
CHAPTER I
PRELIMINARY

Throughout this paper unless otherwise stated module means right module, and all Lie algebras, vector spaces, modules and representations are finite dimensional.

Definition I.1. A restricted Lie algebra \( \mathfrak{L} \) over a field \( K \) of characteristic \( p \neq 0 \) is a Lie algebra over \( K \) together with a mapping \( x \rightarrow x^{[p]} \) of \( \mathfrak{L} \) into \( \mathfrak{L} \) satisfying

(i) \((\alpha x)^{[p]} = \alpha x^{[p]} \), \( \alpha \in K \), \( x \in \mathfrak{L} \)

(ii) \((\text{ad} x)^{[p]} = (\text{ad} x)^p \), \( x \in \mathfrak{L} \)

(iii) \((x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} iS_i(x,y) \), where

\( iS_i(x,y) \) is the coefficient of \( \lambda^{i-1} \) in \( x(\text{ad}(\lambda x+y))^{p-1} \).

If \( \mathcal{U} \) is an associative algebra over a field \( K \) of characteristic \( p \neq 0 \), then the Lie algebra \( \mathcal{U}_L([x,y] = xy-yx) \) of \( \mathcal{U} \) is a restricted Lie algebra with \( x^{[p]} = x^p \).
Theorem 1.2 (Jacobson [8 p.190]). Let \( \mathcal{L} \) be a Lie algebra over a field \( K \) of characteristic \( p \neq 0 \), with an ordered basis \( \{ u_i \} \) such that \( (\text{ad} u_i)^p \) is an inner derivation for all \( u_i \). Then there is a unique mapping \( x \mapsto x^p \) of \( \mathcal{L} \) into \( \mathcal{L} \), relative to which \( \mathcal{L} \) is a restricted Lie algebra.

Definition 1.3. Let \( \mathcal{L} \) be a restricted Lie algebra over a field \( K \) of characteristic \( p \neq 0 \). A restricted representation of \( \mathcal{L} \) is a Lie representation \( \varphi \) of \( \mathcal{L} \), such that \( \varphi(x^p) = (\varphi(x))^p \).

Definition 1.4. Let \( \mathcal{L} \) be a restricted Lie algebra over a field \( K \) of characteristic \( p \neq 0 \). A restricted universal associative algebra (u-algebra) \( \mathcal{U} \) for \( \mathcal{L} \) is an associative algebra over \( K \) together with a restricted Lie homomorphism \( \varphi : \mathcal{L} \rightarrow \mathcal{U} \), such that if \( \psi \) is any associative algebra and \( \psi : \mathcal{L} \rightarrow \nu \) a restricted Lie homomorphism, then there is a unique algebra homomorphism \( \eta : \mathcal{U} \rightarrow \nu \), such that the diagram

\[
\begin{align*}
\mathcal{L} & \xrightarrow{\psi} \nu \\
\mathcal{U} & \xrightarrow{\eta} \nu
\end{align*}
\]

is commutative.
Construction of a u-algebra \((U, \varphi)\) of a restricted Lie algebra \(L\): Let \((U', \varphi')\) be a universal associative algebra of \(L\), and let \(\mathcal{A}\) be the two-sided ideal in \(U'\) generated by the set \(\{\varphi'([x^p]) - (\varphi'(x))^p \mid x \in L\}\); let \(U = U'/\mathcal{A}\); and let \(\varphi\) be the composite of \(\varphi'\) and the canonical homomorphism of \(U'\) onto \(U\). Then \((U, \varphi)\) is a u-algebra for \(L\). Since all u-algebras of \(L\) are isomorphic and \(\varphi\) is injective, we call \((U, \varphi)\) the u-algebra of \(L\) and identify \(x\) with \(\mathcal{A} + x\).

**Theorem 1.5** (Jacobson [9 p.18]). If \(\{u_i\}_{1 \leq i \leq n}\) is an ordered basis for a restricted Lie algebra \(L\) over a field \(K\) of characteristic \(p \neq 0\), then \(\{u_1^{\lambda_1} \ldots u_n^{\lambda_n} \mid 0 \leq \lambda_i = p-1\}\) with \(u_1^0 \ldots u_n^0 = 1\), is a basis for the u-algebra \(U\) of \(L\) over \(K\).

**Definition 1.6** [12 p.28]. Let \(K\) be a field of characteristic not 2,3; a classical Lie algebra \(L\) over \(K\) is a Lie algebra satisfying

(i) the center of \(L\) is zero

(ii) \([L,L] = L\)

(iii) \(L\) has an abelian Cartan subalgebra \(\mathfrak{h}\) (called a classical Cartan subalgebra), relative to which:

(a) \(L = \sum L_\alpha\), where \([xh] = \alpha(h)x\) for all \(x \in L_\alpha\), \(h \in \mathfrak{h}\)
(b) if $\alpha \neq 0$ is a root, $[L_\alpha, L_{-\alpha}]$ is one-dimensional.

(c) if $\alpha$ and $\beta$ are roots and if $\beta \neq 0$, then not all $\alpha + k\beta$ are roots.

Proposition I.7. Every classical Lie algebra $L$ over a field $K$ of characteristic $p > 7$ has a unique mapping $x \to x^{[p]}$, relative to which $L$ is a restricted Lie algebra.

Proof. By Theorem 1 of [14 p.6] and text of [12 p.29] let $\{h_i, e_\alpha\}$ be a Chevalley basis for $L$. It follows that $(ade_\alpha)^p = 0$ and $(adh_i)^p = adh_i$. Hence $(ade_\alpha)^p$ and $(adh_i)^p$ are inner derivations, and by Theorem I.2 $L$ is restricted relative to the mapping $e_\alpha^{[p]} = 0$, $h_i^{[p]} = h_i$.

Let $L$ be a classical Lie algebra over a field $K$ of characteristic $p > 7$, let $\Gamma$ be the dimension of a classical Cartan subalgebra $\mathcal{H}$ of $L$, let $\Sigma$ be the set of all non-zero roots of $\mathcal{H}$ in $L$. By §II.5 of [12], there exist roots $\epsilon_1, \ldots, \epsilon_\Gamma$ such that $h_{\epsilon_1}, \ldots, h_{\epsilon_\Gamma}$ form a basis for $\mathcal{H}$. For each root $\alpha \in \Sigma$, let $\delta(\alpha)$ be the rational integer $0, \pm 1, \pm 2, \pm 3$ according as $\alpha(h_{\epsilon_1})$ is $0, \pm 1, \pm 2, \pm 3$, where $1$ is the residue class of $i$ modulo $p$, then the 1-tuple
For each $\alpha \in \Sigma$, because the field $K$ is of characteristic $p \neq 7$, and $a(h_1) \in \{0, \pm 1, \pm 2, \pm 3\}$. There is a bijection of $\Sigma$ onto the set $\{(\alpha(1), \ldots, \alpha(n)) \mid \alpha \in \Sigma\}$. We introduce a lexicographical order among elements of $\Sigma$ as follows:

- For $\alpha, \beta \in \Sigma$, we say that $\alpha \succ \beta$ if and only if the first non-zero integer $\alpha(1) - \beta(1)$ is positive, and a root $\alpha$ is said to be positive if the first non-zero integer $\alpha(1)$ is positive. Seligman has shown in [12 p.32] that this order relation has the following properties:

1. The principle of trichotomy holds
2. $\alpha \prec \beta$, $\beta \prec \gamma$ implies $\alpha \prec \gamma$
3. If $\alpha \succ 0$, $\beta \succ 0$ and if $\alpha + \beta$ is a root, then $\alpha + \beta > 0$
4. $\alpha \succ 0$ if and only if $-\alpha \succ 0$
5. If $\alpha, \beta$ and $\alpha - \beta$ are roots, then $\alpha \succ \beta$ if and only if $\alpha - \beta > 0$.

A positive root $\alpha$ is said to be simple if it cannot be written as the sum of two positive roots. Following Curtis [4], we define a simple system $\Pi$ to be a set of roots such that if $\alpha, \beta \in \Pi$, then $\alpha - \beta$ is not a root. A chain $C$ of roots (from $\Pi$) is an ordered set of roots $(\alpha_1, \ldots, \alpha_n)$ such that $\alpha_1, \alpha_1 + \alpha_2, \ldots, \sum_{i=1}^{n} \alpha_i$ are roots. The roots $\alpha_i, 1 \leq i \leq n$ are called the
The number of times a root $\alpha \in \Pi$ appears in a chain $C$ is called the multiplicity of $\alpha$ in $C$; and the total number of links in $C$ is called the length of $C$. A root $\alpha$ is said to be generated by a chain of length $n$ if there is at least one chain $C(\alpha_1, \ldots, \alpha_n)$ in $\Pi$ such that $\alpha = \sum_{i=1}^{n} \alpha_i$. Let $\Gamma(\Pi)$ denote the set of all roots generated by chains with links in a simple system $\Pi$ and let $\Pi^- = \{-\alpha \mid \alpha \in \Pi\}$. Seligman has proved in [12] that there exists a simple system $\Pi$ of $\lambda$ relative to $\Lambda$, such that

1. $\Gamma(\Pi) \cap \Gamma(\Pi^-) = \emptyset$ ,

2. $\Sigma = \Gamma(\Pi) \cup \Gamma(\Pi^-)$ .

A simple system $\Pi$ satisfying the above conditions (1) and (2) is called a maximal simple system of roots. If $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ is a maximal simple system of roots, and if $\alpha$ is a positive root, then $\alpha = \sum_{i=1}^{l} n_i \alpha_i$ where $n_i$ are uniquely determined non-negative integers, and we define $\sum_{i=1}^{l} n_i$ to be the length of $\alpha$ and denote it by $\ell(\alpha)$. If $\alpha = 0$, we define $\ell(\alpha) = -\ell(-\alpha)$. Note that positive roots have positive lengths and negative roots have negative lengths. We now introduce a second ordering in $\Sigma$ by defining roots of length $n+1$ greater than roots of length $n$ and for
roots of the same length, we keep the same order relationship as before. Nielsen has shown in [11 p.6] that the second ordering in $\Sigma$ also satisfies (01)-(05). In the latter ordering we see that if $\alpha_1, \alpha_2$ and $\alpha_1 + \alpha_2$ are roots and if $\alpha_1 > 0$, $\alpha_2 > 0$, then $\alpha_1 + \alpha_2 > \alpha_i$, $i=1,2$. And if $\alpha_1 < 0$, $\alpha_2 < 0$ and $\alpha_1 + \alpha_2$ are roots then $\alpha_1 + \alpha_2 < \alpha_i$, $i=1,2$. From now on, unless otherwise stated, the order relation we use will be the second one.

Let $\mathcal{L} = \mathfrak{h} \oplus \sum_{\alpha \in \Sigma} \mathcal{L}_\alpha$ be a classical Lie algebra of rank 1 over an algebraically closed field $K$ of characteristic $p > 7$, where $\mathfrak{h}$ is a classical Cartan subalgebra in $\mathcal{L}$. Seligman has proved in [12 p.33] that there exists a maximal simple system of roots of $\mathcal{L}$ in $\mathfrak{h}$ which has exactly 1 elements. Let $\Pi = \{\alpha_1, \ldots, \alpha_1\}$ be such a maximal simple system and let $\Sigma = \{+\alpha_1, \ldots, +\alpha_1, +\alpha_1 + \alpha_1, \ldots, +\alpha_m\}$ be the set of all non-zero roots. Assume the roots are ordered such that $\alpha_1 < \alpha_2 < \ldots < \alpha_m$, hence $-\alpha_m < \ldots < -\alpha_2 < -\alpha_1$. $\mathcal{L}$ has an ordered basis $\{e_{\alpha_1}, \ldots, e_{\alpha_m}; h_{\alpha_1}, \ldots, h_{\alpha_1}; e_{-\alpha_1}, \ldots, e_{-\alpha_m}\}$ such that for each $i=1,2,\ldots, l$, $\{e_{\alpha_1}, h_{\alpha_1}, e_{-\alpha_1}\}$ is a basis of a classical 3-dimensional Lie algebra $\mathcal{L}(i)$ of type $A_1$ over $K$, such that.
\[ [e_{\alpha_i}, e_{-\alpha_i}] = h_\alpha, \quad [e_{\alpha_i}, h_{\alpha_i}] = 2e_{\alpha_i}, \quad [e_{-\alpha_i}, h_{\alpha_i}] = -2e_{-\alpha_i}, \quad h = \sum_{i=1}^{\infty} kh_{\alpha_i}, \quad \ell = Ke_\alpha. \]

Let \( e_{+\alpha_i} = e_{+i}, \quad i=1, \ldots, m; \quad h_{\alpha_j} = h_j, \quad j=1, \ldots, l \). By Theorem 1.5 the \( \mathfrak{u} \)-algebra \( \mathfrak{u} \) of \( \mathbb{C} \) has a basis consisting of standard monomials \( e_{p_1}^{p_2} \ldots e_m^{p_m} h_{q_1}^{q_2} \ldots h_{r_1}^{r_2} \ldots e_{-m}^{r_m} \), \( 0 \leq p_i, q_i, r_i \leq p-1 \) with \( 1 = e_{1}^{0} \ldots e_{m}^{0} h_{1}^{0} \ldots h_{l}^{0} \). This is often denoted by \( P(p) = P(p_1, \ldots, p_m), \quad p = (p_1, \ldots, p_m), \quad h_{q_1}^{q_2} \ldots h_{q_l}^{q_2} \) by \( Q(q) = Q(q_1, \ldots, q_1), \) and \( e_{r_1}^{r_2} \ldots e_{r_m}^{r_m} \) by \( R(r) = R(r_1, \ldots, r_m) \).

**Definition 1.8.** Let \( u = P(p)Q(q)R(r) \) be a standard monomial in \( \mathfrak{u} \), the extent of \( u \), denoted by \( \varepsilon(u) \), is defined as the rational integer
\[
\sum_{i=1}^{m} (p_i - r_i) \ell(\alpha_i), \quad \text{where} \quad \ell(\alpha_i) \quad \text{is the uniquely determined length of the positive root} \quad \alpha_i.
\]

For any standard monomial \( u \in \mathfrak{u} \), we have
\[
\varepsilon(P^{-1}) \leq \varepsilon(u) \leq \varepsilon(P^{P-1}), \quad \text{where} \quad P^{P-1} = e_{p_1}^{p_2} \ldots e_m^{p_m}, \quad P^{-1} = e_{-p_1}^{p_2} \ldots e_m^{-p_m}. \quad \text{An element} \quad u \quad \text{in} \quad \mathfrak{u} \quad \text{can be written by Theorem 1.5 as linear combination of standard monomials in} \quad \mathfrak{u} \quad \text{with coefficients in} \quad K; \quad \text{the standard monomials which appear in} \quad u \quad \text{with non-zero coefficients are called standard monomials of} \quad u. \]
Definition 1.9. If $0 \neq u \in \mathfrak{u}$ is a linear combination of standard monomials of the same extent, then $u$ is called an extent vector, and the common extent is defined to be the extent of $u$ and is denoted by $\mathcal{E}(u)$.

Proposition 1.10 (Nielsen [11 p.11]). If $u$ and $u'$ are two standard monomials in $\mathfrak{u}$, then each standard monomial of $uu'$ has extent equal to $\mathcal{E}(u) + \mathcal{E}(u')$.

Let \{\(e_1, \ldots, e_m; h_1, \ldots, h_1, e_{-1}, \ldots, e_{-m}\)\} be an ordered Chevalley basis of a classical Lie algebra $\mathfrak{g}$ over an algebraically closed field $K$ of characteristic $p > 7$, where \{\(h_1, \ldots, h_1\)\} is a basis of a classical Cartan subalgebra $\mathfrak{h}$. If $c_i$ is an element of the prime field $\mathbb{Z}_p$ of $K$, following Nielsen [11 p.16] we define
\[
H(h_i, c_i) = \sum_{j=1}^{p-1} \left(\frac{h_i}{c_i}\right)^j, \text{ if } c_i \neq 0 \text{ and } H(h_1, 0) = 1 - h_1^{p-1}.
\]
Then $H(h_1, c_1)h_1 = \sum_{j=1}^{p-1} \left(\frac{h_i}{c_i}\right)^j h_1 = p_{1} c_1 \sum_{j=1}^{p-1} \left(\frac{h_i}{c_i}\right)^j h_1 = c_1 h(h_1, c_1)$ because $h_1^{p} = h_1$, $c_i^{p} = c_i$. Also $H(h_1, 0)h_1 = 0$. Hence $H(h_1, c_1)$ is a weight vector for $h_1$ with weight $c_i$. Define, for each $c = (c_1, \ldots, c_1) \in \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p$, $H(c) = \prod_{i=1}^{1} H(h_i, c_1)$; then we have:

...
Proposition I.11. \( H(c)h_i = c_i H(c) \) for \( i = 1, \ldots, l \).

Proof. \[ H(c)h_i = H(h_1, c_i) \ldots H(h_i, c_i)h_i \ldots H(h_l, c_i) \]
\[ = c_i H(h_1, c_i) \ldots H(h_l, c_i) \]
\[ = c_i H(c) \).

Theorem I.12 (Nielsen [11 p.17]). Let \( \mathcal{L} \) be a classical Lie algebra of rank 1, with an ordered Chevalley basis \( \{ e_1, \ldots, e_m, h_1, \ldots, h_l, e_1, \ldots, e_m \} \) over an algebraically closed field \( K \) of characteristic \( p > 7 \), and let \( \mathcal{U} \) be the \( \mathcal{U} \)-algebra of \( \mathcal{L} \). Let \( E^{p-1} = e_1^{p-1} \ldots e_m^{p-1} \), \( p^{p-1} = e_1^{-1} \ldots e_m^{-1} \) and let \( H(c) \) be defined as in Proposition I.11 for each \( c = (c_1, \ldots, c_l) \in \mathbb{Z}_p^x \ldots \times \mathbb{Z}_p^x = (\mathbb{Z}_p)^l \). Then \( E^{p-1}H(c)p^{p-1} \) generates a minimal right ideal \( J(c) \) in \( \mathcal{U} \), and \( J(c) \) is \( \mathcal{U} \)-isomorphic to \( J(d) \) if and only if \( c = d \).

Theorem I.13 (Nielsen [11 p.32]). Let the hypotheses and notations be as in Theorem I.12. Then \( J(c) \) is nilpotent whenever some \( c_i \neq -1 \).

Lemma I.14 (Nielsen [11 p.17]). Let the hypotheses and notations be as in Theorem I.12. Then the ideal \( J(c) \) is spanned by elements of the form \( E^{p-1}H(c)R(r) \), where \( R(r) = R(r_1, \ldots, r_m) = e_{-1}^{r_1} \ldots e_{-m}^{r_m}, \ 0 \leq r_i \leq p-1 \).
Lemma I.15 (Nielsen [11 p.32]). Let the hypotheses and notations be as in Theorem I.12. Then
\[ E_{p-1}^{H(c)F_{p-1}E_{p-1}} = 0 \text{ if } c_i \neq -1 \text{ for some } i \in \{1,2,\ldots,l\}. \]

Theorem I.16 (Curtis [5 p.315]). Let \( \mathfrak{L} \) be a classical Lie algebra of rank 1 over an algebraically closed field \( K \) of characteristic \( p > 7 \), let \( \mathfrak{h} \) with a basis \( \{h_1,\ldots,h_l\} \) be a classical Cartan subalgebra of \( \mathfrak{L} \), and let \( \mathcal{U} \) be the \( \mathcal{U} \)-algebra of \( \mathcal{L} \). Then there is a 1-1 correspondence between the set of integral linear functions on \( \mathfrak{h} \) (a linear function \( \lambda \) on \( \mathfrak{h} \) is called integral if \( \lambda(h_i) \) is in the prime field of \( K \) for \( i=1,2,\ldots,l \) ), and the set of all equivalence classes of isomorphic irreducible \( \mathcal{U} \)-modules. So there are exactly \( p^l \) non-isomorphic irreducible \( \mathcal{U} \)-modules.

Combining Theorems I.12 and I.16, we see that the set \( \{J(c) \mid c \in (\mathbb{Z}_p)^l\} \) contains representatives of all isomorphism classes of irreducible \( \mathcal{U} \)-modules. Since the concept of restricted \( \mathcal{L} \)-modules and that of \( \mathcal{U} \)-modules are equivalent, a complete set of non-equivalent irreducible restricted representations of \( \mathcal{L} \) are afforded by these \( p^l \) minimal right ideals \( J(c) \).
Definition 1.17. Let \( U \) be a finite dimensional associative algebra with 1. Then the sum of all nilpotent right ideals of \( U \) is called the radical of \( U \) and is denoted by \( R \).

By Theorem 18 in [7 p.66] \( R \) is the intersection of all maximal right ideals of \( U \), hence \( R \) is the intersection of the kernels of all the finite dimensional irreducible representations of \( U \).

Let \( \mathcal{L} \) be a finite dimensional restricted Lie algebra over an algebraically closed field \( K \) of characteristic \( p > 2 \), and let \( U \) be the \( \mathcal{L} \)-algebra of \( \mathcal{L} \). Let \( \psi \) be a restricted representation of \( \mathcal{L} \) with representation space \( m \), then \( m \) is a \( U \)-module. Assume that \( \psi \) is completely reducible, \( m = m_1 + \ldots + m_n \), where \( m_i \) are irreducible \( U \)-modules. Since \( R \) is a two-sided nilpotent ideal in \( U \), \( m_i R \) is a proper submodule of \( m_i \) for \( U \), \( m_i R = 0 \), \( i = 1, \ldots, n \).

Hence \( m R = 0 \). This shows that every completely reducible representation \( \psi \) of \( \mathcal{L} \) vanishes on \( R \). Conversely, if \( \psi \) is a restricted representation of \( \mathcal{L} \), hence a representation of \( U \), vanishing on the radical \( R \) of \( U \), then \( \psi \) is a representation of \( U/R \). But \( U/R \) is a semi-simple finite dimensional associative algebra with 1, and by 25.8 of [6] \( \psi \) is a completely reducible representation of \( U/R \), since \( \psi(R) = 0 \).
\( \psi \) is a completely reducible representation of \( \mathfrak{u} \), and hence a completely reducible restricted representation of \( \mathfrak{L} \). Therefore we have established:

**Proposition I.18.** A restricted representation \( \psi \) of a finite dimensional restricted Lie algebra \( \mathfrak{L} \) over an algebraically closed field \( K \) of characteristic \( p > 2 \) is completely reducible if and only if \( \psi \) vanishes on the radical \( \mathfrak{r} \) of the \( \mathfrak{u} \)-algebra \( \mathfrak{u} \) of \( \mathfrak{L} \).

If we can find a generating set of \( \mathfrak{r} \), then we will have a criterion for complete reducibility of restricted representations of \( \mathfrak{L} \). To check if an element \( \mathfrak{u} \in \mathfrak{u} \) is in \( \mathfrak{r} \), we have the following:

**Proposition I.19.** Let \( \mathfrak{u} \in \mathfrak{u} \), then \( \mathfrak{u} \in \mathfrak{r} \) if and only if \( J(c)\mathfrak{u} = 0 \) for all \( c \in (\mathbb{Z}_p)^1 \), where \( J(c) \) and \( \mathfrak{u} \) are defined as in Theorem I.12, and \( \mathfrak{r} = \text{radical of } \mathfrak{u} \).

**Proof.** Let \( \psi \) be an irreducible representation of \( \mathfrak{u} \) with representation space \( \mathfrak{m} \). Then by Theorems I.12 and I.16 there is some \( c \in (\mathbb{Z}_p)^1 \), such that \( g : J(c) \to \mathfrak{m} \) is a \( \mathfrak{u} \)-module isomorphism, \( g(J(c)) = \mathfrak{m} \). \( \mathfrak{mu} = g(J(c))\mathfrak{u} = g(J(c)\mathfrak{u}) = g(0) \), hence \( \mathfrak{u} \) belongs to the kernel of \( \psi \). Since \( \psi \) is any finite dimensional
irreducible representation of \( U \), \( u \) is in the intersection of the kernels of all the finite dimensional irreducible representations of \( U \). Hence \( u \in \mathcal{R} \).

Conversely, for any \( c \in (Z_p)^1 \) and \( u \in \mathcal{R} \), \( J(c) \) is an irreducible \( u \)-module; hence \( J(c) \) is a completely reducible \( u \)-module. By Proposition 1.18 \( J(c)u = 0 \).

\[ 1.20. \text{ The weight diagram of } L(r,s). \]  

Let \( \mathfrak{g}_c \) be a complex classical Lie algebra of type \( A_2 \) with a Chevalley basis \( \{e'_1, e'_2, e'_3, h'_1, h'_2, e'_1, e'_2, e'_3\} \) and a Cartan subalgebra \( \mathfrak{h}_c = Ch'_1 + Ch'_2 \). Let \([a_1, a_2]\) be a maximal simple system of roots (for definition, see paragraph following Proposition 1.7) and let \( \{\pm a_1, \pm a_2, \pm a_3 = \pm (a_1 + a_2)\} \) be the complete system of roots of \( \mathfrak{h}_c \) in \( \mathfrak{g}_c \). Let \( \Lambda \) be a linear function on \( \mathfrak{h}_c \) such that \( \Lambda(h'_1) = r \); and \( \Lambda(h'_2) = s \) are non-negative integers. Then there exists a unique finite dimensional irreducible \( \mathfrak{g}_c \)-module \( L(r,s) \) having \( \Lambda \) as its highest weight, and all other weights of \( L(r,s) \) in \( \mathfrak{h}_c \) are of the form \( \Lambda - \sum_{i=1}^{2} k_i a_i \), \( k_i \) a non-negative integer. Let \( G \) be the infinite abelian group consisting of all integral linear functions on \( \mathfrak{h}_c \) (a linear function \( \lambda \) on \( \mathfrak{h}_c \) is called integral if \( \lambda(h'_1) \) and \( \lambda(h'_2) \) are integers). \( G \) contains all the weights of \( \mathfrak{h}_c \) in \( L(r,s) \); though not every element of \( G \) is
a weight of \( \mathfrak{h}_C \) in \( L(r,s) \). Let \( \alpha = \mathbb{C}[G] \) be the group algebra of \( G \) over the complex number field \( \mathbb{C} \). \( \alpha \) has a basis \( \{ e(M) \mid M \in G \} \), \( e(M)e(M') = e(M+M') \), \( e(0) = 1 \). The mapping \( M \rightarrow e(M) \) embeds \( G \) into \( \alpha \). Define the character \( \chi_M \) of \( L(r,s) \) to be the formal exponential \( \chi_M = \sum_{M \in G} n_M e(M) \), where \( n_M \) is the dimension of the weight space of the weight \( M \), and \( n_M = 0 \) if \( M \) is not a weight of \( \mathfrak{h}_C \) in \( L(r,s) \). \( \sum_{M \in G} n_M e(M) \) is a finite sum, since \( n_M = 0 \) for all but a finite number of \( M \) in \( G \). Let \( \lambda_1, \lambda_2 \) be the fundamental weights of \( \mathfrak{h}_C \), i.e. that \( \lambda_i(h_j') = \delta_{ij} \), \( i=1,2 \), \( j=1,2 \), and let \( (\mathfrak{h}_C)_0^* \) be the rational span of \( \{ \lambda_1, \lambda_2 \} \). By Chapter IV of [8], \( (\mathfrak{h}_C)_0^* \) is equal to the rational span of \( \{ a_1, a_2 \} \), and \( G \subseteq (\mathfrak{h}_C)_0^* \). Following Antoine's idea ([1 p.155] and [2 p.303]) we identify \( e(M) \) with the point \( M \) in \( (\mathfrak{h}_C)_0^* \). Since \( M \) is an integral linear combination of \( \lambda_1 \) and \( \lambda_2 \), and \( \{ \lambda_1, \lambda_2 \} \) is used as a basis for \( (\mathfrak{h}_C)_0^* \), the point \( M \) has integer coordinates \( (M(h_1'), M(h_2')) \). To each weight \( M \) of \( \mathfrak{h}_C \) in \( L(r,s) \) there corresponds a point on the integral lattice in \( (\mathfrak{h}_C)_0^* \) determined by \( \{ \lambda_1, \lambda_2 \} \). The element \( s_M e(M) \) in \( \alpha \) is identified with the point \( M \) together with the coefficient \( s_M \) attached to it. To add two elements \( a = \sum s_M e(M) \) and \( b = \sum t_M e(M') \) we simply
superimpose the corresponding configurations, where the coefficients of lattice points appearing in both terms are added. To multiply $a = \sum s_M e(M)$ by $t_M e(M')$ we shift the configuration corresponding to $a$ in $(\mathfrak{h}_G)_0^*$ by the vector $M'$ and attach the coefficient $s_M t_M$ to the image of $M$. Let $F(r,s)$ denote the configuration in $(\mathfrak{h}_G)_0^*$ consisting of all lattice points of the form $\lambda = \sum_{i=1}^{2} k_i \alpha_i$, which are inside or on the hexagon determined by $\lambda = r\lambda_1 + s\lambda_2$ (i.e., the hexagon whose vertexes are determined by $\lambda, \sigma_1(\lambda)$, $i=1, \ldots, 5$, where $\sigma_1$ are elements of the Weyl group of the root system of $\mathfrak{h}_G$ in $\mathfrak{g}_C$), each point with coefficient $+1$. By Antoine's character formula for $A_2$ ([1 p.157] and [2 p.40]), we have

\[ (*) \chi_\lambda = F(r,s) + F(r-1,s-1) + F(r-2,s-2) + \ldots + F(0,0) \text{ if } r=s \]

But $\chi_\lambda = \sum n_M e(M)$, where $n_M$ is the dimension of the $M$-weight space in $L(r,s)$. Hence (*) gives a diagrammatic presentation of the weight space decomposition of $L(r,s)$. The weight diagram consists of concentric hexagonal layers of lattice points, namely the boundaries of the figures $F(r-i,s-i)$, where the multiplicities are constant on each layer. The multi-
plicity of the weights on the \( i^{\text{th}} \) layer is \( i \), for \( i \leq \min\{r,s\} \). But the coefficients of the points of \( F(0,s-r) \), when \( r < s \), are all equal to \( r+1 \). The coefficients of the points of \( F(r-s,0) \), when \( r > s \), are all equal to \( s+1 \), and the coefficient of the point \( F(0,0) \), when \( r = s \), is \( r+1 \). The weight diagrams of \( L(3,2) \), \( L(2,4) \), and \( L(6,6) \) are shown in Figure 1, Figure 2, and Figure 3 respectively.
Figure 1. The weight diagram of $L(3,2)$. 

$\Lambda = 3\lambda_1 + 2\lambda_2$
Figure 2. The weight diagram of $L(2,4)$. 
Figure 3. The weight diagram of $L(6,6)$.

Note: (1) The linear polynomial in $h_1$ shown in the diagram attached to a point is an annihilating polynomial of the weight space belonging to the weight represented by the point; e.g., $h_1-6$ annihilates the highest weight space, and $h_2-12$ annihilates the weight space of weight $\Lambda-6\alpha_1$.
(2) Not all the weights for each point of the diagram are written down, but one can tell easily from the diagram which point corresponds to which weight.
CHAPTER II

COMPLETE REDUCIBILITY OF RESTRICTED REPRESENTATIONS
OF CLASSICAL LIE ALGEBRAS OF TYPE $A_1$

Throughout this chapter, we let $\mathcal{L}$ be a classical Lie algebra of type $A_1$ over a field $K$ of characteristic $p > 7$, with a basis $\{e, h, f\}$ satisfying $[e, f] = h$, $[e, h] = 2e$, $[f, h] = -2f$, $e^{[p]} = 0$, $h^{[p]} = h$, and $f^{[p]} = 0$; we let $U$ be the $u$-algebra of $\mathcal{L}$, and we let $R$ be the radical of $U$.

The main result in this chapter is Theorem II.18, which gives a number of different generating sets of $R$, all in terms of $e, h, f$. Hence we obtain a number of different criteria for complete reducibility of restricted representations of $\mathcal{L}$. Our proof of the theorem involves only computations within $U$ and some of the ideas can be generalized to obtain similar results concerning classical Lie algebras of rank $1 \geq 2$. As a corollary of the theorem, we obtain Seligman's Theorem [13] which was proved through a completely different approach. Before proving Theorem II.18, we shall
prove a few lemmas. Unless otherwise stated, all lemmas in this chapter are assumed to take place in $U$.

**Lemma II.1.** If $A(h)$ is a polynomial in $h$ over $K$ and $n$ is a positive integer, then $A(h)e^n = e^nA(h-2n)$ and $e^nA(h) = A(h+2n)e^n$.

*Proof.* Since $he^h = -[e,h] = -2e$, $he^{-2e} = e(h-2)$. Assume $he^m = e^m(h-2m)$, then $he^{m+1} = (he^m)e = (e^m(h-2m))e = e^m(he-2me) = e^m(eh-2e-2me) = e^{m+1}(h-2(m+1))$. Hence $he^n = e^n(h-2n)$ for all $n > 0$. Assume $h^me^n = e^n(h-2n)^m$, then $h^{m+1}e^n = h^m(he^n) = h^m(e^n(h-2n)) = (h^me^n)(h-2n) = e^n(h-2n)^m(h-2n) = e^n(h-2n)^{m+1}$. Hence $h^me^n = e^n(h-2n)^m$ for all positive integers $m$ and $n$.

Now let $A(h) = a_0 + a_1h + \ldots + a_mh^m$, $a_i \in K$, then $A(h)e^n = a_0e^n + a_1he^n + \ldots + a_mh^me^n = a_0e^n + a_1e^n(h-2n) + \ldots + a_m e^n(h-2n)^m = e^n(a_0 + a_1(h-2n) + \ldots + a_m(h-2n)^m) = e^nA(h-2n)$.

Replacing $A(h)$ by $A(h+2n)$, $A(h+2n)e^n = e^nA(h+2n-2n) = e^nA(h)$, we have $e^nA(h) = A(h+2n)e^n$.

**Lemma II.2.** If $A(h)$ is a polynomial in $h$ over $K$, and $n$ is a non-negative integer, then $f^nA(h) = A(h-2n)f^n$ and $A(h)f^n = f^nA(h+2n)$.

*Proof.* Similar to the proof of Lemma II.1, with $fh-hf = [f,h] = -2f$, $fh = (h-2)f$ and $hf = f(h+2)$.
Lemma II.3. If \( n \) is a non-negative integer, then \( e^n f - n e^{n-1}(h-(n-1)) \).

**Proof.** Since the mapping \( x \rightarrow [x,f] \) is a derivation in \( U \),
\[
e^n f - n e^n = [e^n,f]
\]
\[
= e^{n-1}[e,f] + e^{n-2}[e,f]e + \ldots + e[e,f]e^{n-2} + [e,f]e^{n-1}
\]
\[
= e^{n-1}h + e^{n-2}he + \ldots + he^{n-2} + he^{n-1}
\]
\[
= e^{n-1}h + e^{n-2}e(h-2) + \ldots + eye^{n-2}(h-2(n-2)) + e^{n-1}(h-2(n-1))
\]
by Lemma II.1
\[
= e^{n-1}(nh-2 \cdot \frac{1}{2} \cdot n(n-1))
\]
\[
= ne^{n-1}(h-(n-1)) .
\]
Hence \( e^n f - ne^{n-1}(h-(n-1)) \).

Lemma II.4. If \( m \) is a non-negative integer, then \( f^m e = ef^m - m(h-(m-1))f^{m-1} \).

**Proof.** Since the mapping \( y \rightarrow [e,y] \) is a derivation in \( U \),
\[
e^m f - f^m e = [e,f^m]
\]
\[
= [e,f]f^{m-1} + f[e,f]f^{m-2} + \ldots + f^{m-2}[e,f]f + f^{m-1}[e,f]
\]
\[
= hf^{m-1} + fhf^{m-2} + \ldots + f^{m-2}hf + f^{m-1}h
\]
\[
= hf^{m-1} + (h-2)ff^{m-2} + \ldots + (h-2(m-2))f^{m-2}f + (h-2(m-1))f^{m-1}
\]
by Lemma II.2
\[
= (mh-2 \cdot \frac{1}{2} \cdot m(m-1))f^{m-1}
\]
\[
= m(h-(m-1))f^{m-1} .
\]
Hence \( f^m e = ef^m - m(h-(m-1))f^{m-1} \).
Lemma II.5. If \( n \) is an integer such that \( 0 \leq n \leq p-2 \), then 
\[
e^n(h-n) = \frac{1}{n+1}(e^{n+1}f-fe^{n+1}).
\]

Proof. Since \( 0 \leq n \leq p-2 \), \( 1 \leq n+1 \leq p-1 \), 
\( 0 \neq n+1 \in K \). By Lemma II.3, we have 
\[
e^{n+1}f-(n+1)e^n(h-n).\]
Hence 
\[
e^n(h-n) = \frac{1}{n+1}(e^{n+1}f-fe^{n+1}).
\]

Lemma II.6. If \( m \) is an integer such that 
\( 0 \leq m \leq p-2 \), then 
\[
(h-m)f^m = \frac{1}{m+1}(e^{m+1}f^{m+1}e)
\]

Proof. Since \( 0 \leq m \leq p-2 \), \( 1 \leq m+1 \leq p-1 \), 
\( 0 \neq m+1 \in K \). By Lemma II.4, we have 
\[
e^{m+1}f-(m+1)(h-m)f^m.\]
Hence 
\[
(h-m)f^m = \frac{1}{m+1}(e^{m+1}f^{m+1}e)
\]

Lemma II.7. If \( m \geq n \) are positive integers, 
then 
\[
\sum_{j=0}^{n-1} e^{-j}A_j(h)f^{m-j}(-1)^{m(m-1)...(m-n+1)} \left( \begin{array}{c} m-1 \\ j \end{array} \right) \left( \begin{array}{c} m-1 \\ n-j \end{array} \right) f^{m-n}.
\]

where the \( A_j(h) \)'s are polynomials in \( h \) over \( K \).

Proof. We shall prove the lemma by mathematical induction on \( n \). By Lemma II.4, the lemma holds when 
\( n = 1 \). Assume \( m \geq n > 1 \) and assume for all \( n' \leq n-1 \), 
\[
\sum_{j=0}^{n'-1} e^{-j}A_j(h)f^{m-j}(-1)^{m(m-1)...(m-n'+1)} \left( \begin{array}{c} m-1 \\ j \end{array} \right) \left( \begin{array}{c} m-1 \\ n'-j \end{array} \right) f^{m-n'}.
\]
Since $f_{m}^{n} e^{n} = f_{m}^{n+1} e$

$$= \left( \sum_{j=0}^{n-2} e^{n-1-j} A_{j} (h) f_{m}^{j} \right)$$

$$+ (-1)^{n-1} m (m-1) \ldots (m-n+2) \left( \prod_{j=m-n+1}^{m-1} (h-j) \right) f_{m-n+1}$$

$$= \sum_{j=0}^{n-2} e^{n-1-j} A_{j} (h) f_{m}^{j}$$

$$+ (-1)^{n-1} m (m-1) \ldots (m-n+2) \left( \prod_{j=m-n+1}^{m-1} (h-j) \right) f_{m-n+1}$$

by Lemma II.4

$$= \sum_{j=0}^{n-2} e^{n-1-j} A_{j} (h) \left[ e f_{m-j}^{j-(m-j)} (h-(m-j-1)) f_{m-j-1} \right]$$

$$+ (-1)^{n-1} m (m-1) \ldots$$

$$(m-n+2) \left( \prod_{j=m-n+1}^{m-1} (h-j) \right) \left[ e f_{m-n+1-j}^{m-n-1-j} (h-(m-n)) f_{m-n} \right]$$

by Lemma II.1

$$= \sum_{j=0}^{n-2} e^{n-j} A_{j} (h-2) f_{m}^{j}$$

$$- \sum_{j=0}^{n-2} e^{n-1-j} (m-j) A_{j} (h) (h-(m-j-1)) f_{m-1-j}$$

$$+ (-1)^{n-1} m (m-1) \ldots (m-n+2) e \left( \prod_{j=m-n+1}^{m-1} (h-j-2) \right) f_{m-n+1}$$

$$+ (-1)^{n} m (m-1) \ldots (m-n+1) \left( \prod_{j=m-n}^{m-1} (h-j) \right) f_{m-n}$$
\[ 3 = 0 \]

\[ (+1)^{n} \binom{m}{m-n+1} \left( \prod_{j=m-n}^{m-1} (h-j) \right) f^{m-n}, \]

where the \( B_{j}(h) \)'s are polynomials in \( h \), the lemma holds for all \( m, n \) with \( m \geq n \). Our proof is finished.

**Lemma II.8.** If \( m < n \) are positive integers,

then \( f^{m} e^{n} = \sum_{j=0}^{m} e^{n-j} A_{j}(h) f^{m-j} \), where the \( A_{j}(h) \)'s are polynomials in \( h \) over \( K \).

**Proof.** We shall prove the lemma by mathematical induction on \( m \). By Lemma II.3, the lemma holds when \( m = 1 \). Assume \( n > m > 1 \) and assume for all \( m' \leq m-1 \),

\[ f^{m'} e^{n} = \sum_{j=0}^{m'} e^{n-j} A_{j}(h) f^{m'-j}, \]

where \( A_{j}(h) \)'s are polynomials in \( h \) over \( K \). Since

\[ f^{m} e^{n} = ff^{m-1} e^{n} \]

\[ = f \left( \sum_{j=0}^{m-1} e^{n-j} A_{j}(h) f^{m-1-j} \right) \]

\[ = \sum_{j=0}^{m-1} f e^{n-j} A_{j}(h) f^{m-1-j} \]

by Lemma II.3
$$\sum_{j=0}^{m-1} \left( e^{n-j} f - (n-j) e^{n-j-1} (h-(n-j-1)) \right) A_j(h) x^{m-1-j}$$

by Lemma II.2

$$= \sum_{j=0}^{m-1} e^{n-j} A_j(h-2) f^{m-j} - \sum_{j=0}^{m-1} e^{n-j-1} (n-j) (h-(n-j-1)) A_j(h) x^{m-1-j}$$

$$= \sum_{j=0}^{m} e^{n-j} B_j(h) x^{m-j}$$

where the $B_j(h)$'s are polynomials in $h$ over $K$, the lemma holds for all $m,n$ with $m \leq n$. Our proof is finished.

In this chapter, the classical Lie algebra $\mathfrak{g}$ is of rank 1, hence the complete set of non-isomorphic minimal right ideals in $\mathfrak{g}$ constructed by Nielsen in Theorem I.12 consists of $p$ members $J(0), J(1), \ldots, J(p-1)$. $J(i) = e^{p-1} H(i) f^{p-1} \mathfrak{g}$, where $H(0) = 1 - h^{p-1}$, $H(i) = \sum_{j=1}^{p-1} (h/k)^j$ if $k \neq 0$, and $H(i)h = iH(i)$. Hence $\{ J(0), J(1), \ldots, J(p-1) \}$ is a complete set of non-isomorphic irreducible right $\mathfrak{g}$-modules.

If $i \in \{ 0, 1, \ldots, p-2 \}$, $(e^{p-1} H(i) f^{p-1}) x = 0$, so $e^{p-1} H(i) f^{p-1}$ is a minimal vector (an element $x$ in a $\mathfrak{g}$-module is called a minimal vector if $xf = 0$, and $x$ is called a maximal vector if $xe = 0$) of the
irreducible \( u \)-module \( f(i) \):

\[
(e^{p-1}H(i)f^{p-1})h
= e^{p-1}H(i)(h-2(p-1))f^{p-1}
= (i-2(p-1))e^{p-1}H(i)f^{p-1}
= -(p-2-i)e^{p-1}H(i)f^{p-1};
\]

\[
(e^{p-1}H(i)f^{p-1})e^{p-2-i+1}
= e^{p-1}H(i) \left( \sum_{j=0}^{p-2-i} e^{p-1-i-j}A_j(h)f^{p-1-j} \right)
\]

\[
+(-1)^{p-1-i}(p-1)(p-2)...
\]

\[
(p-1-(p-1-i)+1) \left( \prod_{j=p-1-(p-1-i)}^{p-2} (h-j) \right) f^i
\]

by Lemma II.7

\[
= (-1)^i(p-1)(p-2)...(i+1)e^{p-1}H(i) \left( \prod_{j=1}^{p-2} (h-j) \right) f^i
\]

by Lemma II.1 and \( e^p = 0 \)

\[
= 0, \text{ since } H(i)(h-i) = 0.
\]

Since \( e^p = f^p = 0 \), by Lemmas 2 and 3 of [10 pp. 827-828], the elements \( e^{p-1}H(i)f^{p-1} = m_{p-2-i} \), \( m_{p-2-i}e^{p-2-i} \) form a basis for \( f(i) \). Since \( m_{p-2-i}e^{p-2-i} \) is a maximal vector, \( m_{p-2-i}e^{p-2-i}h = \)
(p-2-i)^m_{p-2-i}te^{p-2-i}; and \quad e^p = f^p = 0 ; by Lemmas 2 and 3 of [10 pp.827-828], the elements \ m_{p-2-i}e^{p-2-i},
\ m_{p-2-i}e^{p-2-i}f, \ldots, m_{p-2-i}e^{p-2-i}f^{p-2-i} \ form a basis
for \ J(i) . Since \ m_{p-2-i} and \ m_{p-2-i}e^{p-2-i}f^{p-2-i} \ are
in the same weight space which is one-dimensional,
\ m_{p-2-i}e^{p-2-i}f^{p-2-i} \ = \ \delta_{p-2-i}m_{p-2-i} \ for some \ 0 \neq \ \delta_{p-2-i} \in K .

In the preceding cases, the maximal weight of \ J(i)
is \ p-2-i \ instead of \ i ; while in the following case
the maximal weight of \ J(p-1) \ is \ p-1 ; therefore we
discuss \ J(p-1) \ separately.

In \ J(p-1) ; \ (e^{p-1}f^{p-1})f = 0 ; so
\ e^{p-1}f^{p-1} \ is a minimal vector;

\ (e^{p-1}f^{p-1})h
= e^{p-1}(h-2(p-1))f^{p-1} \quad \text{by Lemma II.2}
= -(p-1)e^{p-1}f^{p-1} ; and

\ (e^{p-1}f^{p-1})e^{p-1}
= e^{p-1}(p-1) \left( \sum_{j=0}^{p-2} e^{p-1-j}A_j(h)f^{p-1-j}+(-1)^{p-1}(p-1) \prod_{j=0}^{p-2} (h-j) \right)
= (p-1)! \left( \prod_{j=0}^{p-2} (p-1-j) \right) e^{p-1}f^{p-1} \neq 0 .
Hence \( \{e^{p-1}h(p-1)f^{p-1} = m_{p-1}, m_{p-1}e, \ldots, m_{p-1}e^{p-1}\} \) and 
\( \{m_{p-1}e^{p-1}, m_{p-1}e^{p-1}f, \ldots, m_{p-1}e^{p-1}f^{p-1}\} \), by Lemmas 2 and 3 of [10 pp.827-828], are two bases for \( J(p-1) \). Let 
\( L(p-1) = J(p-1) \) and \( L(p-2-i) = J(i) \) if \( i = 0, 1, \ldots, p-2 \).

We therefore have the following:

**Lemma II.9.** \( L(i) \) is an irreducible \( U \)-module

for each \( i \in \{0, 1, \ldots, p-1\} \). \( m_i \) is a minimal vector in \( L(i) \), \( m_i h = -im_i \), \( \{m_i, m_i e, m_i e^2, \ldots, m_i e^i\} \) is a basis for \( L(i) \). \( m_i e^i \) is a maximal vector in \( L(i) \), 
\( m_i e^i h = im_i e^i \) and \( \{m_i e^i, m_i e^i f, m_i e^i f^2, \ldots, m_i e^i f^i\} \) is a basis for \( L(i) \), and \( m_i e^i f^i = \delta_i m_i \) for some \( 0 \neq \delta_i \in K \), Note that the extent \( \varepsilon(m_i) = 0 \).

**Lemma II.10.** Let \( N_1 \) be the two-sided ideal in \( U \) generated by the two elements \( e^{p-1}(h+1) \) and 
\( (h+1)f^{p-1} \). Then \( N_1 \subseteq K \).

**Proof.** By Lemma II.9, for each \( i \in \{0, 1, \ldots, p-2\} \), \( L(i) \) has a basis \( \{m_i e^n \mid n = 0, 1, \ldots, i\} \). \( (m_i e^n)e^{p-1}(h+1) = m_i e^{p-1+n}(h+1) = 0 \), because \( p-1+n = i \), and \( m_i e^i \) is a maximal vector in \( L(i) \), \( m_i e^{p-1+n} = 0 \). \( L(p-1) \) has a basis \( \{m_{p-1} e^n \mid n = 0, 1, \ldots, p-1\} \). \( m_{p-1} e^{p-1}(h+1) = (p-1+i)m_{p-1} e^{p-1} = p m_{p-1} e^{p-1} = 0 \); and \( (m_{p-1} e^n)e^{p-1}(h+1) = 0 \) if \( n = 1, 2, \ldots, p-1 \); because \( e^{p-1+n} = 0 \). Hence
L(i)e^{p-1}(h+1) = 0 \text{ for } 0 \leq i \leq p-1. \text{ By Proposition I.19, we have } e^{p-1}(h+1) \in \mathcal{R}.

By Lemma II.9 for each \( i \in \{0, 1, \ldots, p-2\} \), L(i) has a basis \( \{m_i e^j f^n | n=0, 1, \ldots, i\} \) and \( m_i e^j f^j = 0 \) for \( j > i \). \( (m_i e^j f^n)(h+1)e^{p-1} = m_i e^j f^n e^{p-1}(h+2(p-1)+1) \) by Lemma II.2] = 0 because \( p-1+n > i \). L(p-1) has a basis \( \{m_{p-1} e^{p-1} f^n | n=0, 1, \ldots, p-1\} \). Further, \( (m_{p-1} e^{p-1})(h+1)e^{p-1} = (p-1+1)m_{p-1} e^{p-1} f^{p-1} = pm_{p-1} e^{p-1} f^{p-1} = 0 \), and if \( n=1, \ldots, p-1 \), \( (m_{p-1} e^{p-1} f^n)(h+1)e^{p-1} = m_{p-1} e^{p-1} f^n e^{p-1}(h+2(p-1)+1) = 0 \), because \( p-1+n > p-1 \). Hence \( (h+1)e^{p-1} \) annihilates L(i) for all \( i=0, 1, \ldots, p-1 \). By Proposition I.19, \( (h+1)e^{p-1} \in \mathcal{R} \). \( \mathcal{R} \) is a two-sided ideal in \( \mathcal{U} \); hence \( \mathcal{N}_1 \subseteq \mathcal{R} \).

Recall that an extent vector is an element \( u \in \mathcal{U} \), such that all its standard monomials are of the same extent and the common extent is the extent of \( u \), denoted by \( \mathcal{E}(u) \). We shall prove a few lemmas concerning extent vectors.

**Lemma II.11.** If \( x = u_1 + u_2 + \ldots + u_n \in \mathcal{R} \), where \( u_j \) is an extent vector of extent \( \mathcal{E}(u_j) \), and \( \mathcal{E}(u_i) \neq \mathcal{E}(u_j) \) for \( i \neq j \), then \( u_j \in \mathcal{R} \) for \( j=1, 2, \ldots, n \).

**Proof.** By Lemma II.9, for each \( i \in \{0, 1, \ldots, p-1\} \), \( \{m_i e^j f^n | n=0, 1, \ldots, i\} \) is a basis for \( L(i) \), and
since $x \in \mathcal{R}$, $0 = m_i e^\gamma x = \sum_{j=1}^{n} m_i e^\gamma u_j$. If $\varepsilon(m_i e^\gamma u_j) > p-1$ is maximal extent, then $m_i e^\gamma u_j = 0$. If $\varepsilon(m_i e^\gamma u_j) \leq p-1$, since $\varepsilon(m_i) = 0$, by Proposition 1.10 $\varepsilon(m_i e^\gamma u_j) = \nu + \varepsilon(u_j) \neq \nu + \varepsilon(u_k) = \varepsilon(m_i e^\gamma u_k)$ for $j \neq k$.

Each $m_i e^\gamma u_j$, $j=1,2,\ldots,n$, is an extent vector, and elements of $\mathcal{U}$ which are of distinct extents are linearly independent. Hence we have $m_i e^\gamma u_j = 0$, $\nu=0,1,2,\ldots,i$. Hence $u_j$ annihilates $L(i)$ $i=0,1,\ldots,p-1$, and $u_j \in \mathcal{R}$, $j=1,2,\ldots,n$.

**Lemma II.12.** Let $x \in \mathcal{R}$. If $x$ is an extent vector of non-positive extent, then $x \in \langle\langle(1+h)f^p\rangle\rangle$ the two-sided ideal in $\mathcal{U}$ generated by $(1+h)f^p$.

**Proof.** Let $\varepsilon(x) = -d$, $d \in \{0,1,2,\ldots,p-1\}$. For each $j \in \{d,d+1,\ldots,p-1\}$, let $S_j = \{e^{j-d}A(h)f^j | A(h) \text{ is a polynomial in } h \text{ over } K\}$, then $S_j$ is a $p-1$ vector space over $K$, and $\sum_{j=d}^{p-1} S_j$ is the set of all extent vectors in $\mathcal{U}$ of extent equal to $-d$. Our proof is carried out by mathematical induction in the following manner: First we show that $x \in \mathcal{R} \cap S_{p-1}$ implies $x \in \langle\langle(1+h)f^{p-1}\rangle\rangle$. Our next step is to assume that $x \in \mathcal{R} \cap \sum_{j=k+1}^{p-1} S_j$ for $k \geq d$ implies
\( x \in \langle (1+h)^{-1} \rangle \) and then to infer that \( x \in \mathcal{R} \cap \sum_{j=k}^{p-1} S_j \) implies \( x \in \langle (1+h)^{-1} \rangle \).

When \( x \in \mathcal{R} \cap S_{p-1} \), \( x = e^{p-1-d}A(h)f^{p-1} \). Let \( m_{p-1} \) be the minimal vector in \( L(p-1) \) as defined in Lemma II.9. Since \( \mathcal{R} \) is a two-sided ideal in \( \mathcal{U} \), \( e^d x \in \mathcal{R} \), and \( 0 = m_{p-1} e^d x = m_{p-1} e^{p-1}A(h)f^{p-1} = A(p-1)m_{p-1} e^{p-1}f^{p-1} = \delta_{p-1} A(-1)m_{p-1} \). Hence \( A(-1) = 0 \) and \( A(h) = (h+1)A_1(h) \), where \( A_1(h) \) is some polynomial in \( h \) over \( K \). Hence \( x = e^{p-1-d}A_1(h)(h+1)f^{p-1} \) \( \in \langle (1+h)^{-1} \rangle \).

When \( x \in \mathcal{R} \cap \sum_{j=k}^{p-1} S_j \), \( k \in \{d, d+1, \ldots, p-1\} \), \( x = \sum_{j=k}^{p-1} e^{j-d}A_j(h)f^j \). Let \( m_i \) be the minimal vector in \( L(i) \) for \( i \in \{k, k+1, \ldots, p-1\} \). Since \( \mathcal{R} \) is a two-sided ideal in \( \mathcal{U} \), \( e^{i-(k-d)} x \in \mathcal{R} \), and

\[
0 = m_i e^{i-(k-d)} x
= m_i \sum_{j=k}^{p-1} e^{j+i-k}A_j(h)f^j
= m_i e^{A_k(h)} f^k + m_i e^{i+1} A_{k+1}(h) f^{k+1} + \ldots + m_i e^{i+p-1-k} A_{p-1}(h) f^{p-1}
= m_i e^{A_k(h)} f^k
\]

for \( m_i e^{i+1} = m_i e^{i+2} = \ldots = m_i e^{i+p-1-k} = 0 \) by Lemma II.9

\[ = A_k(i) m_i e^{i} f^k.\]
By Lemma II.9, $m_1 e^{i f_k} \neq 0$. Hence $A_k(i) = 0$ for
$i \in \{k, k+1; \ldots, p-1\}$ and $A_k(h) = B(h) \prod_{j=k}^{p-1} (h-j)$,
where $B(h)$ is a polynomial in $h$ over $K$. Hence

\[ x = e^{k-d} B(h) \left( \prod_{j=k}^{p-1} (h-j) \right) f^k + \sum_{j=k+1}^{p-1} e^{j-d} A_j(h) f^j. \]

Now we claim that for each $\nu \in \{1, 2, \ldots, p-k-1\}$

\[ x \equiv (-1)^\nu \cdot \frac{1}{k+1} \cdot \frac{1}{k+2} \ldots \frac{1}{k+\nu} e^{k-d} B(h) \left( \prod_{j=k+\nu}^{p-1} (h-j) \right) f^{k+\nu} e^\nu \mod \sum_{j=k+1}^{p-1} S_j. \]

Since $x$

\[ = e^{k-d} B(h) \left( \prod_{j=k}^{p-1} (h-j) \right) f^k + \sum_{j=k+1}^{p-1} e^{j-d} A_j(h) f^j \]
\[ = e^{k-d} B(h) \left( \prod_{j=k+1}^{p-1} (h-j) \right) (h-k) f^k \mod \sum_{j=k+1}^{p-1} S_j \]
\[ = e^{k-d} B(h) \left( \prod_{j=k+1}^{p-1} (h-j) \right) \cdot \frac{1}{k+1} (e^{k+1} - e^{k+1}) \]
\[ \text{by Lemma II.6} \]
\[ = \frac{1}{k+1} e^{k+1-d} B(h-2) \left( \prod_{j=k+1}^{p-1} (h-j-2) \right) f^{k+1} \]
\[ - \frac{1}{k+1} e^{k-d} B(h) \left( \prod_{j=k+1}^{p-1} (h-j) \right) f^{k+1} e. \]
\[ \equiv \frac{-1}{k+1}e^{k-d_B(h)} \left( \prod_{j=k+1}^{p-1} (h-j) \right) f^{k+1} e \pmod{\sum_{j=k+1}^{p-1} S_j}; \]

(\dagger) is true for \( v = 1 \).

Assume for \( 1 < v < p-k-1 \)

\[ x = (-1)^v \cdot \frac{1}{k+1} \cdot \frac{1}{k+2} \cdots \frac{1}{k+v} e^{k-d_B(h)} \left( \prod_{j=k+1}^{p-1} (h-j) \right) f^{k+v} e^v \pmod{\sum_{j=k+1}^{p-1} S_j}; \]

then \( x \)

\[ = (-1)^v \cdot \frac{1}{k+1} \cdot \frac{1}{k+2} \cdots \frac{1}{k+v} e^{k-d_B(h)} \left( \prod_{j=k+1}^{p-1} (h-j) \right) (h-(k+v)) f^{k+v} e^v \pmod{\sum_{j=k+1}^{p-1} S_j} \]

\[ = (-1)^v \cdot \frac{1}{k+1} \cdot \frac{1}{k+2} \cdots \frac{1}{k+v} e^{k-d_B(h)} \left( \prod_{j=k+1}^{p-1} (h-j) \right) \frac{1}{k+v+1} (e^{k+v+1} f^{k+v+1} e) e^v \]

by Lemma II.6
\[
(-1)^\nu \cdot \frac{1}{k+1} \cdot \frac{1}{k+2} \ldots \cdot \frac{p-1}{k^+\nu+1} e^{k+1-dB(h-2)} \left( \prod_{j=k+\nu+1}^{p-1} (h-j-2) \right) f^{k+\nu+1} e^\nu
\]

\[+(-1)^{\nu+1} \cdot \frac{1}{k+1} \cdot \frac{1}{k+2} \ldots \cdot \frac{p-1}{k^+\nu+1} e^{k-dB(h)} \left( \prod_{j=k+\nu+1}^{p-1} (h-j) \right) f^{k+\nu+1} e^{\nu+1}
\]

\[= (-1)^{\nu+1} \cdot \frac{1}{k+1} \cdot \frac{1}{k+2} \ldots \cdot \frac{1}{k^+\nu+1} e^{k-dB(h)} \left( \prod_{j=k+\nu+1}^{p-1} (h-j) \right) f^{k+\nu+1} e^{\nu+1}
\]

\[\mod \left( \sum_{j=k+1}^{p-1} S_j \right) \]

because \( e^{k+1-dC(h)} f^{k+\nu+1} e^\nu \)

\[= e^{k+1-dC(h)} \sum_{j=0}^{\nu} e^{\nu-j} A_j(h) f^{k+1+\nu-j} \]

by Lemma II.7

\[= \sum_{j=0}^{\nu} e^{k+1-d+\nu-j} C(h-2(\nu-j)) A_j(h) f^{k+1+\nu-j}
\]

\[\equiv 0 \mod \left( \sum_{j=k+1}^{p-1} S_j \right) \]

where \( C(h) \) is any polynomial in \( h \) over \( K \). Hence (\( \dagger \)) holds for all \( \nu \in \{1,2,\ldots, p-k-1\} \). Setting \( \nu = p-1-k \) in (\( \dagger \)).
we have \( x \)

\[= (-1)^{p-1-k} \cdot \frac{1}{k+1} \cdot \frac{1}{k+2} \cdots \frac{1}{p-1} \cdot e^{k-d} B(h)(h-(p-1)) e^{p-1} e^{p-1-k} \]

\[\pmod {\sum_{j=k+1}^{p-1} S_j} \]

Let \( y = (-1)^{p-1-k} \cdot \frac{1}{k+1} \cdot \frac{1}{k+2} \cdots \frac{1}{p-1} \cdot e^{k-d} B(h)(h+1) e^{p-1} e^{p-1-k} \), \( y \in R \) by Lemma II.10. \( x-y = s \) for some \( s \in \sum_{j=k+1}^{p-1} S_j \).

But \( x-y \in R \), so \( s \in R \cap \sum_{j=k+1}^{p-1} S_j \). By induction hypothesis, \( s \in \langle (h+1)^{p-1} \rangle \); hence \( x = s+y \in \langle (h+1)^{p-1} \rangle \) for \( y \in \langle (h+1)^{p-1} \rangle \). The proof is completed.

**Lemma II.13.** Let \( x \in R \); if \( x \) is an extent vector of non-negative extent, then \( x \in \langle e^{p-1}(h+1) \rangle \) is the two-sided ideal in \( U \) generated by \( e^{p-1}(h+1) \).

**Proof.** Let \( E(x) = d \in \{0,1,\ldots,p-1\} \). For each \( j \in \{d,d+1,\ldots,p-1\} \), let \( T_j = \{e^j A(h)f^{j-d} | A(h) \text{ is a polynomial in } h \text{ over } K\} \), then \( T_j \) is a vector space over \( K \); and \( \sum_{j=d}^{p-1} T_j \) is the set of all extent vectors in \( U \) of extent equal to \( d \). Our proof is carried out
by mathematical induction in the following manner: First we show that $x \in R \cap T_{p-1}$ implies $x \in \langle\langle e^{p-1}(h+1) \rangle\rangle$. Our next step is to assume that $x \in R \cap \sum_{j=k+1}^{p-1} T_j$ implies $x \in \langle\langle e^{p-1}(h+1) \rangle\rangle$ and then to infer that $x \in R \cap \sum_{j=k}^{p-1} T_j$ implies $x \in \langle\langle e^{p-1}(h+1) \rangle\rangle$.

When $x \in R \cap T_{p-1}$, $x = e^{p-1}A(h)f^{p-1-d}$. Let $m_{p-1}$ be the minimal vector in $L(p-1)$ as in Lemma II.9. Since $x \in R$, $0 = m_{p-1}x = m_{p-1}e^{p-1}A(h)f^{p-1-d} = A(p-1)m_{p-1}e^{p-1}f^{p-1-d}$, $m_{p-1}e^{p-1}f^{p-1-d} \neq 0$, we have $A(p-1) = 0$ and $A(h) = (h+1)A_1(h)$ where $A_1(h)$ is some polynomial in $h$ over $K$. Hence $x = e^{p-1}(h+1)A_1(h)f^{p-1-d} \in \langle\langle e^{p-1}(h+1) \rangle\rangle$.

When $x \in R \cap \sum_{j=k}^{p-1} T_j$, $k \in \{d, d+1, \ldots, p-1\}$, $x = \sum_{j=k}^{p-1} e^j A_j(h)f^j-d$. Let $m_i$ be the minimal vector in $L(i)$ for $i \in \{k, k+1, \ldots, p-1\}$ as defined in Lemma II.9. Since $R$ is a two-sided ideal in $U$, $e^{i-k}x \in R$ and

$0 = m_i e^{i-k}x$

$= \sum_{j=k}^{p-1} m_i e^{i-k+j} A_j(h)f^{j-d}$
\[ m_i e^{i A_k(h) r^{k-d} + m_i e^{i+1 A_{k+1}(h) r^{k+1-d} + \ldots} + m_i e^{i-k+p-1 A_{p-1}(h) r^{p-1-d} = m_i e^{i A_k(h) r^{k-d}} \quad \text{for } m_i e^{i+1} = m_i e^{i+2} = \ldots = m_i e^{i-k+p-1} = 0 \]

\[ A_k(i) m_i e^{i r^{k-d}}, \]

and \( m_i e^{i r^{k-d}} \neq 0 \), we have \( A_k(i) = 0 \) for \( i = k, k+1, \ldots, p-1 \).

Hence \( A_k(h) = \left( \prod_{j=k}^{p-1} (h-j) \right) C(h), \) where \( C(h) \) is some polynomial in \( h \) over \( K \).

Now we claim that for each \( \nu \in \{1, 2, \ldots, p-k-1\} \)

\[ x \equiv (-1)^\nu \cdot \prod_{k+1}^{k+\nu} f^{\nu} e^{k+\nu} \left( \prod_{j=k+\nu}^{p-1} (h-j) \right) C(h) f^{k-d} \left( \text{mod } \sum_{j=k+1}^{p-1} T_j \right). \]

Since \( x \)

\[ = \sum_{j=k}^{p-1} e^J A_j(h) f^j d \]

\[ = e^k \left( \prod_{j=k}^{p-1} (h-j) \right) C(h) f^{k-d} \left( \text{mod } \sum_{j=k+1}^{p-1} T_j \right) \]

\[ = e^k (h-k) \left( \prod_{j=k+1}^{p-1} (h-j) \right) C(h) f^{k-d} \]

\[ = \frac{1}{k+1} \left( e^{k+1} f - f e^{k+1} \right) \left( \prod_{j=k+1}^{p-1} (h-j) \right) C(h) f^{k-d} \]

by Lemma II.5.
\[
\frac{1}{k+1} \cdot e^{k+1} \left( \prod_{j=k+1}^{p-1} (h-j-2) \right) C(h-2) f^{k+1-d} - \frac{1}{k+1} \cdot e^{k+1} \left( \prod_{j=k+1}^{p-1} (h-j) \right) C(h) f^{k-d}
\]

by Lemma II.2

\[\equiv -\frac{1}{k+1} \cdot e^{k+1} \left( \prod_{j=k+1}^{p-1} (h-j) \right) C(h) f^{k-d} \quad \left( \mod \sum_{j=k+1}^{p-1} T_j \right), \]

(\dagger\dagger) holds for \( \nu = 1 \). Assume for \( 1 \leq \nu \leq p-k-1 \)

\[
x = (-1)^{\nu} \cdot \frac{1}{k+1} \ldots \frac{1}{k+\nu} \cdot e^{k+\nu} \left( \prod_{j=k+\nu+1}^{p-1} (h-j) \right) C(h) f^{k-d} \quad \left( \mod \sum_{j=k+1}^{p-1} T_j \right),
\]

then \( x \)

\[
= (-1)^{\nu} \cdot \frac{1}{k+1} \ldots \frac{1}{k+\nu} \cdot e^{k+\nu} (h-(k+\nu)) \left( \prod_{j=k+\nu+1}^{p-1} (h-j) \right) C(h) f^{k-d} \quad \left( \mod \sum_{j=k+1}^{p-1} T_j \right)
\]

\[
= (-1)^{\nu} \cdot \frac{1}{k+1} \ldots
\]

\[
\frac{1}{k+\nu} \cdot \frac{1}{k+\nu+1} \cdot e^{k+\nu+1} f_{-2} e^{k+\nu+1} \left( \prod_{j=k+\nu+1}^{p-1} (h-j) \right) C(h) f^{k-d} \quad \text{by Lemma II.5}
\]
\[
(-1)^{\nu} \cdot \frac{1}{k+1} \cdots \frac{1}{k+\nu+1} \cdot \nu \text{e}^{k+\nu+1} \left( \sum_{j=0}^{p-1} C(h)^{j} \right) f^{k-d} \\
+ \nu \text{e}^{k+\nu+1} \left( \sum_{j=0}^{\nu} C^{j}(h) f^{k-d} \right) \\

= (-1)^{\nu+1} \cdot \frac{1}{k+1} \cdots \frac{1}{k+\nu+1} \cdot \nu \text{e}^{k+\nu+1} \left( \sum_{j=0}^{p-1} C^{j}(h) f^{k-d} \right)

= (-1)^{\nu+1} \cdot \frac{1}{k+1} \cdots \frac{1}{k+\nu+1} \cdot \nu \text{e}^{k+\nu+1} \left( \sum_{j=0}^{p-1} C^{j}(h) f^{k-d} \right) \\
\mod \left( \sum_{j=k+1}^{p-1} T_{j} \right) \\

\text{because } \nu \text{e}^{k+\nu+1} f B(h) f^{k-d}

= \left( \sum_{j=0}^{\nu} e^{k+\nu+1-j} A_{j}(h) f^{\nu-j} \right) B(h-2) f^{k+1-d}

\text{by Lemmas II.8 and II.2}

= \sum_{j=0}^{\nu} e^{k+\nu+1-j} A_{j}(h) B(h-2-(\nu-j)) f^{k+1+\nu-j} \\
= 0 \\
\mod \left( \sum_{j=k+1}^{p-1} T_{j} \right) \\

\text{where } B(h) \text{ is any polynomial in } h \text{ over } K. \text{ Hence} \n
(\dagger \dagger) \text{ holds for all } \nu \in \{1, 2, \ldots, p-k-1\}. \text{ Setting} \n
\nu = p-k-1 \text{ in } (\dagger \dagger), \text{ we have} \n
x = (-1)^{p-1-k} \cdot \frac{1}{p+1} \cdots \frac{1}{p+1} \cdot \nu^{p-1-k} e^{p-1} (h-(p-1)) C(h) f^{k-d} \\
\mod \left( \sum_{j=k+1}^{p-1} T_{j} \right).
Let \( z = (-1)^{p-1-k} \frac{1}{k+1} \cdots \frac{1}{p-1} \cdot p^{-1-k} e^{p-1} (h+1) c(h) r^{k-d} \), 
\( z \in R \) by Lemma II.10. \( x-z = t \) for some \( t \in \sum_{j=k+1}^{p-1} T_j \).
But \( x-z \in R \), so \( t \in R \cap \sum_{j=k+1}^{p-1} T_j \); by induction hypothesis \( t \in \langle \langle e^{p-1}(h+1) \rangle \rangle \), and \( z \in \langle \langle e^{p-1}(h+1) \rangle \rangle \).
Hence \( x = t+z \in \langle \langle e^{p-1}(h+1) \rangle \rangle \). This completes the proof.

**Lemma II.14.** \( R \subseteq N_1 = \langle \langle e^{p-1}(h+1), (h+1) r^{p-1} \rangle \rangle \).

**Proof.** Since every element in \( U \) is a finite sum of extent vectors: Let \( 0 \neq x \in R \), then \( x = u_1 + \cdots + u_n \), where \( u_j \) are extent vectors of extent \( \epsilon(u_j) \) and \( \epsilon(u_j) \neq \epsilon(u_k) \) if \( j \neq k \). By Lemma II.11 \( u_j \in R \). Since \( \epsilon(u_j) \in \{0, \pm 1, \pm 2, \ldots, \pm (p-1)\} \), by Lemmas II.12 and II.13 \( u_j \in N_1 \). This is true for \( j=1, 2, \ldots, n \). Hence \( x \in N_1 \).

**Lemma II.15.** If \( S \) is a proper subset of \( Z_p = \{0, 1, \ldots, p-1\} \), then
\[
-1 = \left( \prod_{j \in S} (h+j) \right) g(h) + \sum_{i \in S} \prod_{j \in Z_p \setminus \{i\}} (h+j),
\]
where \( g(h) \) is some polynomial in \( h \) over \( K \).

**Proof.** Since \( K \) is of characteristic \( p 
eq 0 \), let \( x \) be an indeterminate,
then

\[ x^p - x = \prod_{j \in \mathbb{Z}_p} (x+j) . \]

Computing the derivatives of both sides, we have

\[-1 = \sum_{i \in \mathbb{Z}_p} \prod_{j \in \mathbb{Z}_p-i} (x+j) . \]

Replacing \( x \) by \( h \), we have

\[-1 = \sum_{i \in \mathbb{Z}_p} \prod_{j \in \mathbb{Z}_p-i} (h+j) \]

\[ = \sum_{i \in \mathbb{Z}_p-S} \prod_{j \in \mathbb{Z}_p-i} (h+j) + \sum_{i \in S} \prod_{j \in \mathbb{Z}_p-i} (h+j) \]

\[ = \sum_{i \in \mathbb{Z}_p-S} \left( \prod_{j \in \mathbb{Z}_p-S} (h+j) \right) \prod_{j \in \mathbb{Z}_p-(S\cup\{i\})} (h+j) + \sum_{i \in S} \prod_{j \in \mathbb{Z}_p-i} (h+j) \]

\[ = \left( \prod_{j \in \mathbb{Z}_p-S} (h+j) \right) \prod_{j \in \mathbb{Z}_p-(S\cup\{i\})} (h+j) + \sum_{i \in S} \prod_{j \in \mathbb{Z}_p-i} (h+j) , \]

where \( g(h) = \sum_{i \in \mathbb{Z}_p-S} \prod_{j \in \mathbb{Z}_p-(S\cup\{i\})} (h+j) \).

**Lemma II.16.** For each \( \nu \in \{1, 2, \ldots, \frac{p}{2}(p-1)\} \), let \( \mathcal{N}_\nu \) be the two-sided ideal in \( \mathcal{U} \) generated by the two elements \( e^{p-\nu} \prod_{j=1}^{2\nu-1} (h+j) \) and \( \left( \prod_{j=1}^{2\nu-1} (h+j) \right) x^{p-\nu} ; \) then \( \mathcal{R} \subseteq \mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \cdots \subseteq \mathcal{N}_{\frac{p}{2}(p-1)} \).
Proof. By Lemma II.14, \( \mathcal{R} \subseteq \mathcal{N}_1 \).

Since \( e^{p-1}(h+1) \)

\[
= -e^{p-1}(h+1) \left[ (h+2)(h+3)g(h) + \sum_{i=2}^{3} \prod_{j \in \mathbb{Z}_p^* \setminus \{i\}} (h+j) \right]
\]

by Lemma II.15

\[
= -e^{p-2}(h+1)(h+2)(h+3)g(h) - e^{p-2}e(h+1) \sum_{i=2}^{3} \prod_{j \in \mathbb{Z}_p^* \setminus \{i\}} (h+j)
\]

by Lemma II.1

\[
= -e^{p-2}(h+2)(h+3)g(h) + e(h+1) \sum_{i=2}^{3} \prod_{j \in \mathbb{Z}_p^* \setminus \{i+2, 1, 2\}} (h+j) \quad (\text{mod } \mathcal{N}_2)
\]

\[
= 0 \quad (\text{mod } \mathcal{N}_2),
\]

\[
(h+1)e^{p-1}
\]

\[
= - \left[ g(h)(h+2)(h+3) + \sum_{i=2}^{3} \prod_{j \in \mathbb{Z}_p^* \setminus \{i\}} (h+j) \right] (h+1)e^{p-1}
\]

by Lemma II.15

\[
= -g(h)(h+1)(h+2)(h+3)e^{p-2}f^{p-2} \left( \sum_{i=2}^{3} \prod_{j \in \mathbb{Z}_p^* \setminus \{i\}} (h+j) \right) (h+1)e^{p-2}
\]

by Lemma II.2

\[
= f \left( \sum_{i=2}^{3} \prod_{j \in \mathbb{Z}_p^* \setminus \{i+2\}} (h+j) \right) (h+3)e^{p-2} \quad (\text{mod } \mathcal{N}_2)
\]
\[
= f \left( \sum_{i=2}^{3} \prod_{j \in \mathbb{Z}_p - \{1+2,1,2\}} (h+j) \right) (h+1)(h+2)(h+3) r^{p-2} \pmod{N_2},
\]

we have \( N_1 \subseteq N_2 \).

Assuming \( N_1 \subseteq N_2 \subseteq \ldots \subseteq N_\nu \) for \( \nu \in \{1,2,\ldots, \frac{1}{2}(p-3)\} \), we claim that \( N_\nu \subseteq N_{\nu+1} \). Since

\[
\begin{align*}
N_\nu &= \langle \langle e^{p-\nu} \prod_{j=1}^{2^{\nu-1}} (h+j), \left( \prod_{j=1}^{2^{\nu-1}} (h+j) \right) r^{p-\nu} \rangle \rangle, \\
&= e^{p-\nu} \prod_{j=1}^{2^{\nu-1}} (h+j) \\
&= -e^{p-\nu} \left( \prod_{j=1}^{2^{\nu-1}} (h+j) \right) \left( (h+2^\nu)(h+2^\nu+1) g(h) \right)
\end{align*}
\]

by Lemma II.15

\[
= -e \cdot e^{p-(\nu+1)} \left( \prod_{j=1}^{2^{\nu+1}} (h+j) \right) g(h) \\
= e^{p-(\nu+1)} \left( \prod_{j=1}^{2^{\nu+1}} (h+j) \right) \left( \sum_{i=2^{\nu}}^{2^{\nu+1}} \prod_{j \in \mathbb{Z}_p - \{1+2,1,2\}} (h+j) \right) \cdot
\]

\( \equiv 0 \pmod{N_{\nu+1}} \).

and
\[
\left( \prod_{j=1}^{2^\nu-1} (h+j) \right) f^{p-\nu} \\
= -\left[ g(h)(h+2\nu)(h+2^\nu+1) \\
+ \sum_{i=2^\nu}^{2^\nu+1} \prod_{j \in \mathbb{Z}_p \setminus \{1\}} (h+j) \right] \left( \prod_{j=1}^{2^\nu-1} (h+j) \right) f^{p-\nu}
\]
by Lemma II.15

\[
= -g(h) \left( \prod_{j=1}^{2^\nu+1} (h+j) \right) f^{p-\nu-1} \\
- f \left( \sum_{i=2^\nu}^{2^\nu+1} \prod_{j \in \mathbb{Z}_p \setminus \{i+2\}} (h+j) \right) \left( \prod_{j=3}^{2^\nu+1} (h+j) \right) f^{p-\nu-1}
\]
by Lemma II.2

\[
= f \left( \sum_{i=2^\nu}^{2^\nu+1} \prod_{j \in \mathbb{Z}_p \setminus \{i+2,1,2\}} (h+j) \right) \left( \prod_{j=1}^{2^\nu+1} (h+j) \right) f^{p-\nu-1}
\]
(mod \( \mathcal{N}_{\nu+1} \))

\[
\equiv 0
\]
(mod \( \mathcal{N}_{\nu+1} \)),
we have \( \mathcal{N}_\nu \subseteq \mathcal{N}_{\nu+1} \). Hence \( R \subseteq \mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \cdots \subseteq \mathcal{N}_{\frac{p}{2}}(p-1) \).

**Lemma II.17.** \( \mathcal{N}_{\frac{p}{2}}(p-1) \subseteq R \).

**Proof.** It is sufficient to show that

\[
(2 \cdot \frac{1}{2}(p-1))^{-1} e^{p-\frac{1}{2}(p-1)} \prod_{j=1}^{p-2} (h+j) = e^{\frac{1}{2}(p+1)} \prod_{j=1}^{p-2} (h+j) \quad \text{and}
\]

\[
\left( \prod_{j=1}^{p-2} (h+j) \right) f^\frac{1}{2}(p+1) \quad \text{annihilate } L(i) \text{ for } i=0,1,2,
\]

\[
\ldots,p-1. \quad L(i) \text{ is defined as in Lemma II.9.} \quad \text{Note that}
\]
\[ p^{-2} \prod_{j=1}^{j-2} (h+j) = \prod_{j=1}^{j-2} (h-j) \text{ for } K \text{ is of characteristic } p \neq 0. \]

By Lemma II.9, for each \( i \in \{0, 1, \ldots, \frac{p}{2}(p-1)\} \), \( L(i) \) has a basis \( \{m_i e^n | n=0, 1, \ldots, i\} \), and \( m_i e^j = 0 \) for all \( j > i \). \( (m_i e^n) e^{\frac{3}{2}(p+1)} \prod_{j=1}^{j-2} (h+j) = 0 \) for \( n=0, 1, \ldots, i \), because \( n+\frac{1}{2}(p+1) > i \). For \( i \in \{\frac{3}{2}(p+1), \frac{5}{2}(p+3), \ldots, p-1\} \),

\[
(m_i e^n) e^{\frac{3}{2}(p+1)} \prod_{j=1}^{j-2} (h+j) = m_i e^{n+\frac{3}{2}(p+1)} \prod_{j=2}^{j-2} (h-j).
\]

If \( n+\frac{3}{2}(p+1) > i \), then \( m_i e^{n+\frac{3}{2}(p+1)} = 0 \). If \( n+\frac{3}{2}(p+1) \leq i \), then \( p+1 = 2n+p+1 \leq 2i \leq 2(p-1) \), which then implies \( 2 = (p+1)-(p-1) \leq 2n+(p+1)-i \leq i \leq p-1 \). Hence \( 2 \leq 2n+p+1-i \leq p-1 \). Since \( j \) ranges from 2 to \( p-1 \),

\[
\prod_{j=2}^{j-2} (2n+p+1-i-j) = 0. \text{ Therefore } e^{\frac{3}{2}(p+1)} \prod_{j=1}^{j-2} (h+j)
\]

annihilates all \( L(i) \) for \( i=0, 1, \ldots, p-1 \). By Proposition I.19, \( e^{\frac{3}{2}(p+1)} \prod_{j=1}^{j-2} (h+j) \in K \). Again by Lemma II.9,

for each \( i \in \{0, 1, \ldots, \frac{p}{2}(p-1)\} \), \( L(i) \) has a basis \( \{m_i e^{i/n} | n=0, 1, \ldots, i\} \), and \( m_i e^{i/n} = 0 \) for all \( j > i \).
\[
(m_1 e^{i \pi n}) \left( \sum_{j=1}^{p-2} \frac{\pi^j (h+j)}{\pi^j (h+j+p+1)} \right)\frac{i^n}{j=1} = m_1 e^{i \pi n + \frac{1}{2}(p+1)} \left( \sum_{j=1}^{p-2} \frac{\pi^j (h+j+p+1)}{\pi^j (h+j+p+1)} \right)
\]

\[= 0 \text{ for } n = 0, 1, \ldots, i, \text{ because } n + \frac{1}{2}(p+1) > i. \]

For \( i \in \{\frac{1}{2}(p+1), \frac{3}{2}(p+3), \ldots, p-1\} \), \( (m_1 e^{i \pi n}) \left( \sum_{j=1}^{p-2} \frac{\pi^j (h+j)}{\pi^j (h+j+p+1)} \right)\frac{i^n}{j=1} = m_1 e^{i \pi n + \frac{1}{2}(p+1)} \)

\[= m_1 e^{i \pi n + \frac{1}{2}(p+1)} \left( \sum_{j=2}^{p-1} \frac{\pi^j (h-j-2n)}{\pi^j (h+j-2n)} \right)\frac{i^n}{j=2} \]

by Lemma II.2 and \( \pi^j (h+j) = \pi^j (h-j) \)

\[= \left( \pi^j (i-j-2n) \right) m_1 e^{i \pi n + \frac{1}{2}(p+1)} . \]

If \( n + \frac{1}{2}(p+1) > i \), then \( m_1 e^{i \pi n + \frac{1}{2}(p+1)} = 0 \). If \( n + \frac{1}{2}(p+1) \leq i \), then by the same argument given above we have \( \pi^j (i-2n-j) = 0 \). Therefore \( \left( \sum_{j=1}^{p-2} \frac{\pi^j (h+j)}{\pi^j (h+j)} \right)\frac{i^n}{j=1} \)

annihilates \( L(i) \); \( i = 0, 1, \ldots, p-1 \). By Proposition I.19, \( \left( \sum_{j=1}^{p-2} \frac{\pi^j (h+j)}{\pi^j (h+j)} \right)\frac{i^n}{j=1} \) is a two-sided ideal in \( U \); hence \( N_{\frac{1}{2}(p+1)} \subseteq \mathcal{K} \).

**Theorem II.18.** Let \( \mathcal{L} \) be a classical Lie algebra of type \( A_1 \) over a field \( K \) of characteristic \( p > 7 \).

A finite dimensional restricted representation \( \gamma \) of \( \mathcal{L} \) is completely reducible if and only if \( \gamma \) vanishes on one of the \( \frac{1}{2}(p-1) \) sets
\[
\left\{ e^{p^{-\nu}} \prod_{j=1}^{2^\nu-1} (h+j) ; \left( \prod_{j=1}^{2^\nu-1} (h+j) \right)^{p^{\nu-\gamma}} \right\}, \quad \nu = 1, 2, \ldots, \frac{1}{3}(p-1).
\]

Proof. The two-sided ideal in \( U \) generated by
\[
\left\{ e^{p^{-\nu}} \prod_{j=1}^{2^\nu-1} (h+j) ; \left( \prod_{j=1}^{2^\nu-1} (h+j) \right)^{p^{\nu-\gamma}} \right\}
\]
is \( N_\nu \) in Lemma II.16, where we proved that \( \mathcal{R} \leq N_1 \leq \cdots \leq N_{\frac{1}{3}(p-1)} \), and Lemma II.17 shows \( N_{\frac{1}{3}(p-1)} \leq \mathcal{R} \). Therefore \( \mathcal{R} = N_1 = \cdots = N_{\frac{1}{3}(p-1)} \). By Proposition I.18 and the fact that \( \varphi \) vanishes on \( N_\nu \) if and only if \( \varphi \) vanishes on a generating set of \( N_\nu \), the theorem holds.

Corollary II.19 (Seligman [13]). Let \( \varphi \) be a restricted representation of \( \mathcal{L} \). Then \( \varphi \) is completely reducible if and only if \( \varphi(e)^{p^{-1}} \varphi(h) = -\varphi(e)^{p^{-1}} \) and \( \varphi(h)\varphi(f)^{p^{-1}} = -\varphi(f)^{p^{-1}} \).

Proof. In Theorem II.18, taking \( \nu = 1 \),
\( \mathcal{R} = N_1 = \langle \langle e^{p^{-1}}(h+1) ; (h+1)^{p^{-1}} \rangle \rangle \), the corollary follows from Proposition I.18.

In order to have consistency concerning the characteristic of the ground field \( K \) throughout our entire work, we have proved our results in this chapter under the hypothesis that the characteristic of \( K \) is greater than 7. In fact all results in this chapter also hold
true for $K$ of characteristic $p > 2$; the reason being the following: that the characteristic $p$ of $K$ being greater than 7 is needed in introducing a lexicographical order among the roots (see page 8) when $\mathcal{L}$ is of rank $l \geq 2$. Nielsen [11] constructed the minimal right ideals $J(c)$ in the general case when $l \geq 2$. The construction needed the lexicographical order of the roots; hence it required $p > 7$. We employed the $J(c)$ in this chapter (see page 30), therefore we also assume $p > 7$. But if $\mathcal{L}$ is of type $A_1$, a complete system of roots has only two elements; one being a positive root, the other being negative. We no longer need $p > 7$ to introduce an order among the roots; and Nielsen's construction of the $J(c)$ also works when $p > 2$. Hence all results in this chapter hold true for $K$ of characteristic $p > 2$. 
CHAPTER III

NECESSARY CONDITIONS FOR COMPLETE REDUCIBILITY
OF RESTRICTED REPRESENTATIONS OF CLASSICAL
LIE ALGEBRAS OF RANK 1 \geq 2

In this chapter we consider the restricted rep­
resentations of a classical Lie algebra \( \mathfrak{L} \) of rank
1 \geq 2. When the ground field is of characteristic 0,
by Theorem 8 page 79 of [8] we know that every finite
dimensional representation of a classical Lie algebra
of rank 1 \geq 1 is completely reducible. However,
when the ground field is of characteristic \( p \neq 0 \),
two examples are given here to show that a completely
reducible restricted \( \mathfrak{L} \)-module need not be a completely
reducible restricted \( \mathfrak{L}(i) \)-module, where \( \mathfrak{L}(i) \) is, as
described on page 10, the split 3-dimensional simple
subalgebra of \( \mathfrak{L} \), \( i=1,2,\ldots,1 \). And a restricted
\( \mathfrak{L} \)-module which is completely reducible as restricted
\( \mathfrak{L}(i) \)-module for \( i=1,2,\ldots,1 \) need not be completely
reducible. It is not known if there exists an \( \mathfrak{L} \)-module
which is not completely reducible but is completely

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reducible for every split 3-dimensional simple subalgebra. We shall obtain some necessary conditions for complete reducibility of restricted $\mathcal{L}$-modules, and we also give two examples to show that these conditions are not sufficient. Hence we obtain a two-sided ideal $\mathcal{N}$ in the $u$-algebra $\mathcal{U}$ of $\mathcal{L}$ with the following properties: $\mathcal{N}$ contains properly the sum of all the two-sided ideals generated by the nilpotent right ideals $j(c)$ constructed by Nielsen in Theorem 1.12. $\mathcal{N}$ is properly contained in the radical $\mathcal{K}$ of $\mathcal{U}$ if $l = 2$, and $\mathcal{N} = \mathcal{K}$ if $l = 1$.

Example III.1. To show that a completely reducible restricted $\mathcal{L}(i)$-module need not be a completely reducible restricted $\mathcal{L}(i)$-module for $i=1,2$. Let $\mathcal{L}$ be the set of all $3 \times 3$ matrices of trace 0, with entries in an algebraically closed field $K$ of characteristic $p > 7$, then $\mathcal{L}$ is a classical Lie algebra of type $A_2$. Relative to the matrix units $E_{ij}$, $\mathcal{L}$ has a Chevalley basis $e_1 = E_{21}, e_2 = E_{32}, e_3 = E_{31}, h_1 = E_{11} - E_{22}, h_2 = E_{22} - E_{33}, e_{-1} = -E_{12}, e_{-2} = -E_{23}, e_{-3} = -E_{13}$. Let $\Sigma$ be the complete system of roots of $\mathcal{L} = kh_1 + kh_2$ in $\mathcal{L}$, $\Sigma = \{a_1, a_2, a_3\}$, $\alpha_1(\text{diag}(a_{11}, a_{22}, a_{33})) = a_{11} - a_{22}$, $\alpha_2(\text{diag}(a_{11}, a_{22}, a_{33})) = a_{22} - a_{33}$, $\alpha_3(\text{diag}(a_{11}, a_{22}, a_{33}))$
= a_{11} - a_{33}, where \( \text{diag}(a_{11}, a_{22}, a_{33}) = a_{11}E_{11} + a_{22}E_{22} + a_{33}E_{33} \). \( h_3 = h_1 + h_2 \) and \( h_2 \) form a basis of the Cartan subalgebra \( \mathfrak{h} \). Relative to the ordered basis \( (h_3, h_2) \) of \( \mathfrak{h} \), the roots \(-\alpha_3, -\alpha_2, -\alpha_1, \alpha_1, \alpha_2, \alpha_3\) uniquely determine the ordered pairs of rational integers \((-2, -1), (-1, -2), (-1, 1), (1, -1), (1, 2), (2, 1)\). The six roots are then ordered lexicographically (cf. the paragraph following Proposition 1.7) as \(-\alpha_3 < -\alpha_2 < -\alpha_1 < \alpha_1 < \alpha_2 < \alpha_3\). \( e_1 \) and \( e_{-1} \) are root vectors respectively for \( \alpha_1 \) and \(-\alpha_1 \) relative to the Cartan subalgebra \( \mathfrak{h} \). The multiplication table relative to the basis \( \{e_1, e_2, e_3, h_1, h_2, e_{-1}, e_{-2}, e_{-3}\} \) is as follows:

\[
\begin{align*}
[e_1, e_2] &= -e_3, & [e_1, h_1] &= 2e_1, & [e_1, h_2] &= -e_1, \\
[e_1, e_{-1}] &= h_1, & [e_1, e_{-3}] &= e_{-2}, & [e_2, h_1] &= -e_2, \\
[e_2, h_2] &= 2e_2, & [e_2, e_{-2}] &= h_2, & [e_2, e_{-3}] &= -e_1, \\
[e_3, h_1] &= e_3, & [e_3, h_2] &= e_3, & [e_3, e_{-1}] &= -e_2, \\
[e_3, e_{-2}] &= e_1, & [e_3, e_{-3}] &= h_1 + h_2, & [h_1, e_{-1}] &= 2e_{-1}, \\
[h_1, e_{-2}] &= -e_{-2}, & [h_1, e_{-3}] &= e_{-3}, & [h_2, e_{-1}] &= -e_{-1}, \\
[h_2, e_{-2}] &= 2e_{-2}, & [h_2, e_{-3}] &= e_{-3}, & [e_{-1}, e_{-2}] &= -e_{-3},
\end{align*}
\]

all other products are zero.

For \( i=1, 2 \), let \( \mathcal{L}(i) \) be the split 3-dimensional simple subalgebra of \( \mathcal{L} \), and let \( \{e_i, h_i, e_{-i}\} \) be a basis of \( \mathcal{L}(i) \). Let \( \mathcal{U} \) be the \( \mathbb{u} \)-algebra of \( \mathcal{L} \). By Theorem I.12, \( (p-1, p-1) \in \mathbb{Z}_p \times \mathbb{Z}_p \); \( f((p-1, p-1)) \) is a
minimal right ideal in \( \mathcal{U} \) generated by
\[ e_1^{p-1}e_2^{p-1}e_3^{p-1}H((p-1,p-1))e_1^{p-1}e_2^{p-1}e_3^{p-1}. \]
By the proof of
Theorem 2 of Nielsen [11 p. 22], \( J((p-1,p-1)) \) is also
generated by \( e_1^{p-1}e_2^{p-1}e_3^{p-1}H((p-1,p-1)) \).
Now
\( J((p-1,p-1)) \) is an irreducible \( \mathcal{U} \)-module, hence
\( J((p-1,p-1)) \) is a completely reducible restricted
\( \mathcal{L} \)-module. However \( J((p-1,p-1)) \) is not a completely
reducible restricted \( \mathcal{L}(1) \)-module, \( i = 1, 2 \).
Since
\[ J((p-1,p-1)) = e_1^{p-1}e_2^{p-1}e_3^{p-1}H((p-1,p-1))\mathcal{U}, \]
\( 0 \neq e_1^{p-1}e_2^{p-1}e_3^{p-1}H((p-1,p-1))e_3 \) (for \( H((p-1,p-1)) \) is a
non-zero polynomial in \( h_1, \ldots, h_1 \).
\[ e_1^{p-1}e_2^{p-1}e_3^{p-1}H((p-1,p-1))e_3 \] is a finite sum of non-zero linearly independent standard monomials in \( \mathcal{U} \),
hence is not zero) is in \( J((p-1,p-1)) \) and \( e_3^{p-1}h_1 \]
\[ = h_1 e_3 + [e_3, h_1] = (h_1 - 1)e_3, \]
we have
\[ (e_1^{p-1}e_2^{p-1}e_3^{p-1}H((p-1,p-1))e_3)(1 + h_1)e_1^{p-1} \]
\[ = e_1^{p-1}e_2^{p-1}e_3^{p-1}H((p-1,p-1))(1 + h_1 - 1)e_3 e_1^{p-1} \]
\[ = (p-1)e_1^{p-1}e_2^{p-1}e_3^{p-1}H((p-1,p-1)) e_1^{p-1} e_3 \]
\( \neq 0. \)
\( (1 + h_1)e_1^{p-1} \) does not annihilate \( J((p-1,p-1)) \), and
\( J((p-1,p-1)) \) is a restricted \( \mathcal{L}(1) \)-module. By Corollary
II.19, \( J((p-1,p-1)) \) is not a completely reducible re-
stricted \( \mathcal{L}(1) \)-module. Similarly, since
\[ (e_1^{p-1}e_2^{p-1}e_3^{p-1}H((p-1,p-1))e_3)(1 + h_2)e_2^{p-1} \]
\[(p-1)e_1^{p-1}e_2^{p-1}e_3^{p-1}h((p-1,p-1))e_2^{p-1}e_{-3} \neq 0,\]

\[(1+h_2)e_{-2}^{p-1}\] does not annihilate \(J((p-1,p-1))\), \(J((p-1,p-1))\) is not a completely reducible restricted \(\mathcal{L}(2)\)-module.

Example III.1 also serves as an example supporting the following remark of Seligman [12 p.99]: Curtis has given sufficient conditions for a restricted representation \(\varphi\) of a classical Lie algebra to be completely reducible; conditions which require at least that \(\varphi(e_\alpha)^{p-1} = 0\) for all root vectors \(e_\alpha\) relative to a Cartan subalgebra; ... these conditions are not necessary. If we let \(\varphi\) be the right regular representation of \(U\) restricted to \(J((p-1,p-1))\) as in Example III.1, then \(\varphi\) is a completely reducible restricted representation of \(\mathcal{L}\). We showed that \(J((p-1,p-1))(h_1+1)e_{-1}^{p-1} \neq 0\), hence \(\varphi(e_{-1})^{p-1} \neq 0\). Hence complete reducibility does not necessitate \(\varphi(e_\alpha)^{p-1} = 0\) for all root vectors.

In [10 p.830] Jacobson gave the following conjecture: For any representation \(\varphi\) of a classical Lie algebra \(\mathcal{L}\) of characteristic \(p \neq 0\) to be completely reducible, it is sufficient that \((\varphi(e_\alpha))^{p-1} = 0\) for
all roots $\alpha \neq 0$ of $\mathcal{L}$. Braden's counter-example in [3 p.485] shows that these conditions are not sufficient. Our Example III.1 here shows that these conditions are not necessary either.

Example III.2. To show that a restricted $\mathcal{L}$-module that is completely reducible as a restricted $\mathcal{L}(1)$-module for $i=1,2,3$ need not be completely reducible. Let $\mathcal{L}$ be defined as in Example III.1. Let $\mathcal{L}_c$ be the set of all $3 \times 3$ matrices of trace 0, with entries in the complex number field $\mathbb{C}$, we obtain, in exactly the same way as in Example III.1, a Chevalley basis $\{e_1',e_2',e_3',h_1',h_2',e_{-1}',e_{-2}',e_{-3}'\}$, a Cartan subalgebra $\mathfrak{h}_c = \mathfrak{h}_1 + \mathfrak{h}_2$, the complete root system $\Sigma' = \{\pm \alpha_1', \pm \alpha_2', \pm \alpha_3'\}$ of $\mathcal{L}_c$ in $\mathfrak{h}_c$, the ordering of roots $-\alpha_3' < -\alpha_2' = -\alpha_1' < \alpha_1' < \alpha_2' = \alpha_3'$, and the same multiplication table relative to the Chevalley basis $\{e_{+1}',h_1'\}$. By Theorem of [4 p.104], $\mathcal{L} = K \otimes \mathcal{L}_Z$, where $\mathcal{L}_Z$ is the $z$-span of the Chevalley basis $\{e_{+1}',h_1'\}$, $\mathfrak{h} = K \otimes \mathfrak{h}_Z$, $\mathfrak{h}_Z = \mathfrak{h}_c \cap \mathcal{L}_Z$. Now let the characteristic of $K$ be 13, let $\Lambda$ be a linear functional on $\mathfrak{h}_c$ with $\Lambda(h_1) = \Lambda(h_2) = 6$. Let $L(6;6)$ be a finite dimension irreducible $\mathcal{L}_c$-module with highest weight $\Lambda$, and a maximal vector $x$. Let $\tilde{L}(6,6) = K \otimes x\mathbb{Z}_u$, where $u_2$ is the $z$-algebra generated by $\{e_{+1}'/m! \mid i=1,2,3 ; m=0,1,2,\ldots\}$. Then
$L(6;6)$ is a restricted $L$-module, by Theorem 2 of [2 p. 16]. $L(6,6)$ is indecomposable; and by Theorem 2 of [2 p. 54], $L(6;6)$ is reducible. Hence $L(6;6)$ is not a completely reducible restricted $L$-module. By examining the weight diagram of $L(6,6)$ (see Figure 3), we see that $e_i^1 (h_i^1-12)$ and $(h_i^1-12)e_i^i$, $i=1,2,3$, annihilate $L(6;6)$ where $h_i^3 = h_i^1 + h_2^1$. Since $e_i^1$ sends every vector of any weight space into the next weight space along the $\alpha_i$ direction (see Figure 3), and if a weight vector $y$ is sent outside the weight diagram by $e_i^1$, then it means that $ye_i^1 = 0$. Hence all weight spaces except the one belonging to the weight $\Lambda - 6\alpha_1 - 6\alpha_3$ are annihilated by $e_i^1$. If $y$ is a weight vector belonging to the weight $\Lambda - 6\alpha_1 - 6\alpha_3$, then $ye_i^1$ belongs to the weight space of the weight $\Lambda - 6\alpha_2$ which is annihilated by $h_i^1 - 12$. Hence $ye_i^1 (h_i^1 - 12) = 0$. Hence $e_i^1 (h_i^1 - 12)$ annihilates all the weight spaces of $L(6;6)$.

Since $e_{-1}^i$ sends every vector of one weight space into the next weight space along the $-\alpha_1$ direction (see Figure 3); and if a weight vector $y$ is sent outside the weight diagram by $e_{-1}^i$, then $ye_{-1}^i = 0$. Hence all weight spaces except the one belonging to $\Lambda - 6\alpha_2$ are annihilated by $e_{-1}^i$. All weight spaces are invariant under $h_i^1 - 12$; and the one belonging to
\( \Lambda \cdot 6 \alpha_2 \) is annihilated by \( h_1^{-12} \). Hence \( (h_1^{-12})e_{-1}^{12} \) annihilates all the weight spaces of \( L(6,6) \).

Similarly we can show that \( e_i^{12} (h_1^{-12}) \) and \( (h_1^{-12})e_{-1}^{12} \); \( i=2,3 \), annihilate \( L(6,6) \). Hence
\[
L(6,6)e_i^{12} (h_1^{-12}) = 0, \quad L(6,6)(h_1^{-12})e_{-1}^{12} = 0, \quad \text{for } i=1,2,3.
\]
Reducing modulo 13, we have
\[
L(6,6)e_i^{12} (h_1+1) = L(6,6)(h_1+1)e_{-1}^{12} = 0 \quad \text{for } i=1,2,3.
\]

By Corollary II.19, \( L(6,6) \) is completely reducible restricted \( \mathcal{L}(i) \)-module, \( i=1,2,3 \).

Through the above two examples we see that no information about the complete reducibility of a restricted \( \mathcal{L} \)-module \( \mathcal{M} \) can be derived from the complete reducibility of \( \mathcal{M} \) viewed as an \( \mathcal{L}(i) \)-module, though one might still hope for information by looking at all subalgebras of type \( A_1 \). However, generalizing the idea in Lemma II.10, we obtain some necessary conditions for complete reducibility of restricted representations of \( \mathcal{L} \).

**Theorem III.3.** Let \( \mathcal{L} \) be a classical Lie algebra of rank 1; with an ordered Chevalley basis \( \{e_1, \ldots, e_m; h_1, \ldots, h_l; e_{-1}, \ldots, e_{-m}\} \) over an algebraically closed
field $K$ of characteristic $p > 7$; let $U$ be the $u$-algebra of $\zeta$; and let $\mathcal{N}$ be the two-sided ideal in $U$ generated by the 21 elements $E_{i+1}(h^i+1)$, $(h^i+1)E_{i+1}$, $i=1,2,\ldots,1$, where $E_{i+1} = e_{i+1}^{1} \ldots e_{m}^{1}$, 
$E_{i-1} = e_{i-1}^{1} \ldots e_{m}^{1}$. Then $\mathcal{N}$ is contained in the radical $\mathfrak{r}$ of $U$.

**Proof.** By Proposition I.19, it suffices to show that these 21 elements annihilate $J(c)$ for all $c \in (Z_p)^1$ upon right multiplication. Let $c = (-1,-1,\ldots,-1) \in (Z_p)^1$, by Lemma I.14, $J(c)$ is spanned by elements of the form $E_{i+1}\zeta(c)R(r)$; 
$R(r) = R((r_1^1,\ldots,r_m^1)) = e_{i-1}^{r_1} \ldots e_{m}^{r_m};$ $0 \leq r_i \leq p-1$. If for some $i^*$, $0 \leq r_i^* < p-1$, then the extent of $R(r)$ is 
$\varepsilon(R(r)) = -\sum_{i=1}^{m} r_i \zeta(a_i^*) = -(p-1)\sum_{i=1}^{m} \zeta(a_i) = -\varepsilon(E_{i+1})$;
and $\varepsilon(E_{i+1}) + \varepsilon(R(r)) > 0$; by Proposition I.10, then 
$\varepsilon(E_{i+1}\zeta(c)R(r)) + \varepsilon(E_{i+1}(h^i+1)) = \varepsilon(E_{i+1}) + \varepsilon(R(r)) + \varepsilon(E_{i+1})$ 
$= \varepsilon(E_{i+1})$; but $E_{i+1}$ is of maximal extent, hence 
$E_{i+1}\zeta(c)R(r)E_{i+1}(h^i+1) = 0$. If $r_i^* = p-1$ for all $i=1,2,\ldots,m$, then $R(r) = F_{i+1}$; by Theorem I.5 and Proposition I.10, $E_{i+1}\zeta(c)E_{i+1}E_{i+1}(h^i+1) = 
E_{i+1}\zeta(c)\left(\sum_{T;T';T''} a(t,t',t'')p(t)q(t')r(t'')\right)(h^i+1)$;
where $a(t,t',t'') \in K$ and $\varepsilon(p(t)) + \varepsilon(r(t'')) = 0$. If $t \neq (0,0,\ldots,0)$, then $\varepsilon(p(t)) > 0$, by Proposition I.10.
\( \varepsilon(E^{p-1}H(c)P(t)) = \varepsilon(E^{p-1}) + \varepsilon(P(t)) = \varepsilon(E^{p-1}) \), but \( E^{p-1} \) is of maximal extent; so \( E^{p-1}H(c)P(t) = 0 \), and if \( t = (0,0,...,0) \); then \( \varepsilon(R(t^*)) = -\varepsilon(P(t)) = 0 \), hence \( t^* = (0,0,...,0) \). Hence \( E^{p-1}H(c)F^{p-1}E^{p-1}(h_{i+1}) = a(0,t^*,0)E^{p-1}H(c)Q(t^*)(h_{i+1}) \)

\[ = a(0,t^*,0)E^{p-1}H(c)(h_{i+1})Q(t^*) = 0 \), because \( H(c)(h_{i+1}) = (-1+1)H(c) = 0 \). Next let \( c = (c_1,c_2,...,c_1) \in (Z_p)^1 \) such that \( c_i \neq -1 \), for some \( i \). By Lemma 1.14, \( f(c) \) is spanned by elements of the form \( E^{p-1}H(c)R(r) \).

If \( R(r) \neq F^{p-1} \), by Proposition 1.10 \( \varepsilon(E^{p-1}H(c)R(r)E^{p-1}) = \varepsilon(E^{p-1}) + \varepsilon(R(r)) + \varepsilon(E^{p-1}) = \varepsilon(E^{p-1}) \), but \( E^{p-1} \) is of maximal extent, so \( E^{p-1}H(c)R(r)E^{p-1}(h_{i+1}) = 0 \). If \( R(r) = F^{p-1} \), then by Lemma 1.15 \( E^{p}H(c)F^{p-1}E^{p-1}(h_{i+1}) = 0 \). We have finished showing that \( f(c)E^{p-1}(h_{i+1}) = 0 \)

for all \( c \in (Z_p)^1 \) and all \( i=1,2,...,1 \).

Let \( c = (-1,-1,...,-1) \in (Z_p)^1 \). By Lemma 1.14, \( f(c) \) is spanned by elements of the form \( E^{p-1}H(c)R(r) \), \( R(r) = R((r_1,r_2,...,r_m)) = e_{r_1}e_{r_2}...e_{r_m}, \ 0 \leq r_i \leq p-1 \).

If \( R(r) \neq 1 \), then \( \varepsilon(R(r)) < 0 \), by Proposition 1.10, \( \varepsilon(R(r)(h_{i+1})F^{p-1}) = \varepsilon(R(r)) + \varepsilon(F^{p-1}) = \varepsilon(F^{p-1}) \), but \( F^{p-1} \) is of minimal extent, so we have \( R(r)(h_{i+1})F^{p-1} = 0 \). Hence \( E^{p-1}H(c)R(r)(h_{i+1})F^{p-1} = 0 \). If \( R(r) = 1 \), then \( E^{p-1}H(c)R(r)(h_{i+1})F^{p-1} = E^{p-1}H(c)(h_{i+1})F^{p-1} = 0 \), because \( H(c)(h_{i+1}) = (-1+1)H(c) = 0 \). Next let \( c = (c_1,c_2,...,c_1) \in (Z_p)^1 \) such that \( c_i \neq -1 \) for some \( i \).
By Theorem I.12, \( j(c) = E_{p^{-1}}^1 H(c) P_{p^{-1}}^1 U \), hence \( j(c) \) is spanned by elements of the form \( E_{p^{-1}}^1 H(c) E_{p^{-1}}^1 P(p) Q(q) R(r) \).

If \( R(r) \neq 1 \), then by Proposition I.10 \( \varepsilon(R(r)(h_i+1)P_{p^{-1}}^1) = \varepsilon(R(r))+\varepsilon(P_{p^{-1}}^1) \leq \varepsilon(P_{p^{-1}}^1) \), but \( P_{p^{-1}}^1 \) is of minimal extent, so we have \( R(r)(h_i+1)P_{p^{-1}}^1 = 0 \). Hence
\[
E_{p^{-1}}^1 H(c) E_{p^{-1}}^1 P(p) Q(q) R(r)(h_i+1)P_{p^{-1}}^1 = 0.
\]

If \( R(r) = 1 \), then \( E_{p^{-1}}^1 H(c) E_{p^{-1}}^1 P(p) Q(q) R(r) = E_{p^{-1}}^1 H(c) E_{p^{-1}}^1 P(p) Q(q) \).

Since \( (E_{p^{-1}}^1 H(c) E_{p^{-1}}^1 P(p) Q(q))(h_i+1)P_{p^{-1}}^1 = (E_{p^{-1}}^1 H(c) E_{p^{-1}}^1 P(p)) Q(q)(h_i+1)P_{p^{-1}}^1 = E_{p^{-1}}^1 H(c)(E_{p^{-1}}^1 P(p) Q(q)(h_i+1)P_{p^{-1}}^1) \) and either \( P(p) = E_{p^{-1}}^1 \) or \( P(p) \neq E_{p^{-1}}^1 \): if \( P(p) = E_{p^{-1}}^1 \), then by Lemma I.15 \( E_{p^{-1}}^1 H(c) E_{p^{-1}}^1 P(p) = E_{p^{-1}}^1 H(c) P_{p^{-1}}^1 E_{p^{-1}}^1 = 0 \); if \( P(p) \neq E_{p^{-1}}^1 \), then \( \varepsilon(P(p)) < \varepsilon(E_{p^{-1}}^1) = -\varepsilon(E_{p^{-1}}^1) \), by Proposition I.10 \( \varepsilon(E_{p^{-1}}^1 P(p) Q(q)(h_i+1)P_{p^{-1}}^1) = \varepsilon(E_{p^{-1}}^1)+\varepsilon(P(p))+\varepsilon(E_{p^{-1}}^1) \leq \varepsilon(E_{p^{-1}}^1) \), but \( E_{p^{-1}}^1 \) is of minimal extent, hence \( P_{p^{-1}}^1 P(p) Q(q)(h_i+1)P_{p^{-1}}^1 = 0 \), and \( E_{p^{-1}}^1 H(c) E_{p^{-1}}^1 P(p) Q(q)(h_i+1)P_{p^{-1}}^1 = 0 \). We have finished showing that \( j(c)(h_i+1)P_{p^{-1}}^1 = 0 \) for all \( c \in \mathbb{Z}_p^1 \) and all \( i=1,2,\ldots, l \). This completes the proof of the theorem.

**Theorem III.4.** Let \( \mathcal{L} \) be a classical Lie algebra of rank 1, with an ordered Chevalley basis \( \{e_1, \ldots, e_m, h_1, \ldots, h_1, e_{-1}, \ldots, e_{-m}\} \) over an algebraically closed field \( K \) of characteristic \( p \neq 7 \), and let \( U \) be the
u-algebra of $\mathfrak{L}$. If $\varphi$ is a completely reducible restricted representation of $\mathfrak{L}$, then

$$(\varphi(e_1))^{p-1}(\varphi(e_2))^{p-1}\ldots(\varphi(e_m))^{p-1}\varphi(h_i)$$

$$= -(\varphi(e_1))^{p-1}(\varphi(e_2))^{p-1}\ldots(\varphi(e_m))^{p-1},$$

$$\varphi(h_i)(\varphi(e_1))^{p-1}(\varphi(e_2))^{p-1}\ldots(\varphi(e_m))^{p-1}$$

$$= -(\varphi(e_1))^{p-1}(\varphi(e_2))^{p-1}\ldots(\varphi(e_m))^{p-1},$$

for $i=1,2,\ldots,1$.

**Proof.** Let $\varphi$ be a completely reducible restricted representation of $\mathfrak{L}$. Then by Proposition I.18 $\varphi$ vanishes on the radical $\mathfrak{R}$ of $\mathfrak{L}$, and by Theorem III.3 $E^{p-1}(h_i+1)$ and $(h_i+1)^{p-1}$ belong to $\mathfrak{R}$, we have $0 = \varphi(E^{p-1}(h_i+1)) = (\varphi(e_1))^{p-1}\ldots(\varphi(e_m))^{p-1}(\varphi(h_i)+I)$, and

$$0 = \varphi((h_i+1)^{p-1}) = (\varphi(h_i)+I)(\varphi(e_1))^{p-1}\ldots(\varphi(e_m))^{p-1},$$

where $I$ is the identity linear transformation. Hence

$$(\varphi(e_1))^{p-1}\ldots(\varphi(e_m))^{p-1}\varphi(h_i) = -(\varphi(e_1))^{p-1}\ldots(\varphi(e_m))^{p-1}$$

and

$$\varphi(h_i)(\varphi(e_1))^{p-1}\ldots(\varphi(e_m))^{p-1} = -(\varphi(e_1))^{p-1}\ldots(\varphi(e_m))^{p-1}.$$

**Remark.** If $\mathfrak{L}$ is of rank 1, then the necessary conditions in Theorem III.4 become $(\varphi(e_1))^{p-1}\varphi(h_i) = -(\varphi(e_1))^{p-1}$ and $\varphi(h_i)(\varphi(e_1))^{p-1} = -(\varphi(e_1))^{p-1}$, which are also sufficient conditions for $\varphi$ to be completely reducible by Corollary II.19. However when $\mathfrak{L}$ is of
rank 1 \geq 2$, these conditions are no longer sufficient, as we shall see from the following:

Example III.5. To show that the conditions
\[
\begin{align*}
&= -(\varphi(e_1))^{p-1}(\varphi(e_2))^{p-1}(\varphi(e_3))^{p-1} \varphi(h_1) \\
&= -(\varphi(e_1))^{p-1}(\varphi(e_2))^{p-1}(\varphi(e_3))^{p-1} \\
&= -(\varphi(e_1))^{p-1}(\varphi(e_2))^{p-1}(\varphi(e_3))^{p-1}, \quad i = 1, 2,
\end{align*}
\]
are not sufficient for complete reducibility of a restricted representation $\varphi$ of a classical Lie algebra $\mathcal{L}$ of type $A_2$ over an algebraically closed field $K$ of characteristic $p = 13$. Let $\mathcal{L} = \mathfrak{h}, \mathfrak{g}, K; \{e_1, e_2, e_3, h_1, h_2, e_{-1}, e_{-2}, e_{-3}\}$, $\mathcal{L}_c = \mathfrak{h}_c; \{e_1, e_2, e_3, h_1, h_2, e_{-1}, e_{-2}, e_{-3}\}$, $L(6,6)$ and $\overline{L(6,6)}$ be defined as in Example III.2. Hence $\mathcal{L}$ is a classical Lie algebra of type $A_2$ over an algebraically closed field $K$ of characteristic 13, and $\overline{L(6,6)}$ is a restricted $\mathcal{L}$-module which is not completely reducible by Theorem 2 of [2 p. 16] and Theorem 2 of [2 p. 54]. By examining the weight diagram of $L(6,6)$ (Figure 3), we see that $e_1, e_2, e_3$ sends every vector in any given weight space of $L(6,6)$ down 12 steps along the $\alpha_1$ direction, then up 12 steps along the $\alpha_2$ direction and then horizontally to the right 12 steps. Hence every vector in any non-zero weight space of $L(6,6)$ is sent outside the
weight diagram. We have \( L(6, 6) e_1^6 e_2^6 e_3^6 \) \( h_1^i - 12 \) = 0 , \\
i=1, 2 . \( e_1^6 e_2^6 e_3^6 \) sends every vector in any given weight space of \( L(6, 6) \) up 12 steps along the \(-\alpha_1\) direction, then down 12 steps along the \(-\alpha_2\) direction and then horizontally to the left 12 steps. Hence every vector in any non-zero weight space of \( L(6, 6) \) is sent outside the weight diagram. We have \( L(6, 6) e_1^6 e_2^6 e_3^6 (h_1^i) = 0 , \)
\[ i=1, 2 . \] Reducing modulo 13 we have \( L(6, 6) e_1^6 e_2^6 e_3^6 (h_1^i+1) \)
\[ = L(6, 6) (h_1^i+1) e_1^6 e_2^6 e_3^6 = 0 , i=1, 2 . \] Now let \( \psi \) be the restricted representation of \( \mathcal{L} \) associated with the module \( \overline{L}(6, 6) \). Then \( (\psi(e_1))^12 (\psi(e_2))^12 (\psi(e_3))^12 \psi(h_1) \)
\[ = -(\psi(e_1))^12 (\psi(e_2))^12 (\psi(e_3))^12 , \]
\( \psi(h_1) (\psi(e_1))^12 (\psi(e_2))^12 (\psi(e_3))^12 \)
\[ = -(\psi(e_1))^12 (\psi(e_2))^12 (\psi(e_3))^12 . \] But \( \psi \) is not completely reducible because \( \overline{L}(6, 6) \) is not completely reducible.

Example III.6. To show that the conditions
\[ \psi(e_1)^{p-1} \psi(e_2)^{p-1} \psi(e_3)^{p-1} \psi(h_1)^{p-1} \]
\[ = -(\psi(e_1)^{p-1} \psi(e_2)^{p-1} \psi(e_3)^{p-1} \psi(h_1)^{p-1} , \]
\( \psi(h_1)^{p-1} \psi(e_1)^{p-1} \psi(e_2)^{p-1} \psi(e_3)^{p-1} \psi(h_1)^{p-1} \)
\[ = -(\psi(e_1)^{p-1} \psi(e_2)^{p-1} \psi(e_3)^{p-1} \psi(h_1)^{p-1} , \] for
\[ i=1, 2 \] are not sufficient for complete reducibility of a restricted representation \( \psi \) of a classical Lie algebra \( \mathcal{L} \) of type \( B_2 \) over an algebraically closed field of characteristic \( p \geq 7 \). Braden [3 p.485] gave
a counter-example to the following conjecture of Jacobson: Complete reducibility holds for any representation $\psi$ of a classical Lie algebra $\mathcal{L}$ over a field of characteristic $p \neq 0$ in which $(\psi(e_\alpha))^{p-1} = 0$ for all roots $\alpha \neq 0$ of $\mathcal{L}$. The module being $L(r,s)$, $r > 0$, $s > 0$; $2r+s = p-2$, which was obtained by the same method as in Example III.2 from the irreducible $\mathcal{L}_g$-module $L(r,s)$, save that now $\mathcal{L}_g$ is a complex classical Lie algebra of type $B_2$ with a Chevalley basis $\{e_1^i, e_2^i, e_3^i, e_4^i, h_1, h_2, e_{-1}, e_{-2}, e_{-3}, e_{-4}\}$, and $\mathcal{L}$ is classical of type $B_2$ with a Chevalley basis $\{e_1, e_2, e_3, e_4, h_1, h_2, e_{-1}, e_{-2}, e_{-3}, e_{-4}\}$. Braden has shown that $L(r,s)$ is an indecomposable reducible restricted $\mathcal{L}$-module and is annihilated by $e_{+i}^{p-1}$, $i=1,2,3,4$, a fortiori annihilated by $e_1^{p-1}e_2^{p-1}e_3^{p-1}e_4^{p-1}$ and $e_{-1}^{p-1}e_{-2}^{p-1}e_{-3}^{p-1}e_{-4}^{p-1}$. Let $\psi$ be the restricted representation of $\mathcal{L}$ associated with the module $L(r,s)$, then $\psi$ is not completely reducible and yet

$$(\psi(e_1))^{p-1}(\psi(e_2))^{p-1}(\psi(e_3))^{p-1}(\psi(e_4))^{p-1}\psi(h_1)$$

$= -(\psi(e_1))^{p-1}(\psi(e_2))^{p-1}(\psi(e_3))^{p-1}(\psi(e_4))^{p-1} = 0$,

and $\psi(h_1)(\psi(e_{-1}))^{p-1}(\psi(e_{-2}))^{p-1}(\psi(e_{-3}))^{p-1}(\psi(e_{-4}))^{p-1}$

$= -(\psi(e_{-1}))^{p-1}(\psi(e_{-2}))^{p-1}(\psi(e_{-3}))^{p-1}(\psi(e_{-4}))^{p-1} = 0$,

for $i=1,2$. 


From Proposition I.18 and Example III.5 we have:

**Remark III.7.** Let $\mathcal{L}$ be a classical Lie algebra of type $A_2$, with a Chevalley basis $\{e_1, e_2, e_3, h_1, h_2, e_{-1}, e_{-2}, e_{-3}\}$ over an algebraically closed field $K$ of characteristic $p = 13$. Let $\mathfrak{U}$ be the $\mathfrak{u}$-algebra of $\mathcal{L}$, then the two-sided ideal in $\mathfrak{U}$ generated by the 4 elements $e_1^{p-1} e_2^{p-1} e_3^{p-1} (h_1+1), (h_1+1) e_1^{p-1} e_2^{-1} e_3^{p-1}, i=1,2$, is properly contained in the radical $\mathfrak{R}$ of $\mathfrak{U}$.

From Proposition I.18 and Example III.6 we have:

**Remark III.8.** Let $\mathcal{L}$ be a classical Lie algebra of type $B_2$, with a Chevalley basis $\{e_i, h_j \mid i=1,2,3,4; j=1,2\}$ over an algebraically closed field $K$ of characteristic $p > 7$. Let $\mathfrak{U}$ be the $\mathfrak{u}$-algebra of $\mathcal{L}$. Then the two-sided ideal in $\mathfrak{U}$ generated by the 4 elements $e_1^{p-1} e_2^{p-1} e_3^{p-1} e_4^{p-1} (h_1+1), (h_1+1) e_1^{p-1} e_2^{-1} e_3^{-1} e_4^{p-1}, i=1,2$, is properly contained in the radical $\mathfrak{R}$ of $\mathfrak{U}$.

So far for a classical Lie algebra $\mathcal{L}$ of rank $l \geq 2$, we have found by Theorem III.3 a two-sided ideal $\mathcal{N}$ in the $\mathfrak{u}$-algebra $\mathfrak{U}$ of $\mathcal{L}$, and $\mathcal{N} \subseteq \mathfrak{R}$ = the radical of $\mathfrak{U}$. Since $\mathfrak{R}$ is the unique maximal nilpotent ideal in $\mathfrak{U}$ and by Theorem I.13 the $p^{l-1}$ Nielsen's minimal right ideals $J(c); c \neq (-1, -1, \ldots, -1)$, are nilpotent;
so \[ \sum_{c \in \mathfrak{c} \cap (-1,-1, \ldots, -1)} j(c) \subseteq \mathcal{R} \]. Since \( \mathcal{R} \) is a two-sided ideal, the sum \[ \sum_{c \in \mathfrak{c} \cap (-1,-1, \ldots, -1)} u j(c) = \sum_{c \in \mathfrak{c} \cap (-1,-1, \ldots, -1)} u E^{p-1} H(c) F^{p-1} u \]
is also contained in \( \mathcal{R} \). We would like to investigate the relation between the nilpotent two-sided ideal

\[ \sum_{c \in \mathfrak{c} \cap (-1,-1, \ldots, -1)} u E^{p-1} H(c) F^{p-1} u \]
and \( \mathcal{N} \) in the following: Theorem III.9. Let \( \mathcal{L} \) be a classical Lie algebra of rank \( 1 \geq 2 \), with an ordered Chevalley basis \( \{ e_1, \ldots, e_m, h_1, \ldots, h_1, a_1, \ldots, e_m \} \) over an algebraically closed field \( \mathbb{K} \) of characteristic \( p > 7 \), let \( \mathcal{U} \) be the \( u \)-algebra of \( \mathcal{L} \). For each \( c = (c_1, c_2, \ldots, c_l) \in (\mathbb{Z}/p)^{l-1-\{(-1,-1, \ldots, -1)} \), let \( H(c) = \prod_{i=1}^{p-1} H(h_i, c_i) \),

where \( H(h_i, c_i) = \sum_{j=1}^{1} (h_i/c_i)^j \) if \( c_i \neq 0 \) and \( H(h_i, 0) = 1-h_i^{p-1} \). Let \( E^{p-1} = e_1^{p-1} \ldots e_m^{p-1} \), \( F^{p-1} = e_1^{p-1} \ldots e_m^{p-1} \) and let \( \mathcal{N} \) be the two-sided ideal in \( \mathcal{U} \) generated by the 21 elements \( E^{p-1}(h_i+1) \), \( (h_i+1)F^{p-1} \), \( i=1, \ldots, l \).

Then the sum \[ \sum_{c \in \mathfrak{c} \cap (-1,-1, \ldots, -1)} u E^{p-1} H(c) F^{p-1} u \]
of two-sided ideals generated by \( E^{p-1} H(c) F^{p-1} \), \( c \in (\mathbb{Z}/p)^{l-\{(-1,-1, \ldots, -1)} \) is properly contained in \( \mathcal{N} \).
Proof. We first show that

\[ \sum_{c \neq (-1, \ldots, -1)} u E_{c} E_{C} \subseteq N. \]

For each \( c = (c_1, \ldots, c_1) \in (\mathbb{Z}_p)^{1-\{(-1, \ldots, -1)\}} \), by Theorem 1.13

\( J(c) = E_{c} E_{C} \subseteq \mathcal{R} \) is a nilpotent right ideal, and \( \mathcal{R} \) is the unique maximal nilpotent right ideal, hence \( J(c) \subseteq \mathcal{R} \) and \( E_{c} E_{C} \subseteq \mathcal{R} \). By Proposition 1.19

\( J(d) E_{c} E_{C} = 0 \) for all \( d = (d_1, \ldots, d_1) \in (\mathbb{Z}_p)^{1} \), in particular, when \( d = (-1, -1, \ldots, -1) \),

\( J((-1, -1, \ldots, -1)) E_{c} E_{C} = 0 \). Since \( J((-1, \ldots, -1)) = E_{c} E_{C}((-1, \ldots, -1)) \),

\[ 0 = E_{c} E_{C}((-1, \ldots, -1)) \subseteq E_{c} E_{C}. \]

Nielsen has shown [11 pp.22-23] that \( J((-1, \ldots, -1)) = E_{c} E_{C}((-1, \ldots, -1)) \mathcal{U} \) and that for some \( 0 \neq k \in K \),

\( k E_{c} E_{C}((-1, \ldots, -1)) \mathcal{U} \) is a non-zero idempotent in \( J((-1, \ldots, -1)) \), hence \( E_{c} E_{C}((-1, \ldots, -1)) \mathcal{U} \neq 0 \).

\( E_{c} E_{C}((-1, \ldots, -1)) \mathcal{U} \) and \( E_{c} E_{C}((-1, \ldots, -1)) \mathcal{U} \) both are non-zero and belong to the 1-dimensional maximal weight space of \( J((-1, \ldots, -1)) \) (viewed as an irreducible \( \mathcal{U} \)-module), hence \( E_{c} E_{C}((-1, \ldots, -1)) \mathcal{U} \neq 0 \).

\[ 0 = E_{c} E_{C}((-1, \ldots, -1)) \mathcal{U} E_{c} E_{C}((-1, \ldots, -1)) \mathcal{U} \]

for some \( 0 \neq \delta \in K \). We now have:

\[ 0 = E_{c} E_{C}((-1, \ldots, -1)) \mathcal{U} E_{c} E_{C}((-1, \ldots, -1)) \mathcal{U} \]

\[ = \delta E_{c} E_{C}((-1, \ldots, -1)) \mathcal{U} E_{c} E_{C}((-1, \ldots, -1)) \mathcal{U} \]
= \delta \varepsilon^{P-1} H((-1, \ldots, -1)) \left( \prod_{i=1}^{r} \Pi (h_i, c_i) \right)^{P-1} \\
= \delta \varepsilon^{P-1} \left( \prod_{i=1}^{r} \Pi (h(-1, c_i)) \right) H((-1, \ldots, -1))^{P-1}

by Proposition I.11

= \delta \left( \prod_{i=1}^{r} \Pi (h(-1, c_i)) \right) \varepsilon^{P-1} H((-1, \ldots, -1))^{P-1}.

Since \delta \varepsilon^{P-1} H((-1, \ldots, -1))^{P-1} \neq 0, H(-1, c_{i_0}) = 0 for some \ i_0. As H(h_{i_0}, c_{i_0}) is a polynomial in h_{i_0} with coefficients in K, we have H(h_{i_0}, c_{i_0}) = (h_{i_0} + 1)G(h_{i_0}), where G(h_{i_0}) is some polynomial in h_{i_0} over K. Hence H(c) = (h_{i_0} + 1)G(h_{i_0}) \left( \prod_{i \neq i_0} \Pi (h_i, c_i) \right), and \varepsilon^{P-1} H(c) = \varepsilon^{P-1} (h_{i_0} + 1)G(h_{i_0}) \left( \prod_{i \neq i_0} \Pi (h_i, c_i) \right)^{P-1} \in \mathcal{N}. Hence \varepsilon\varepsilon^{P-1} H(c)^{P-1} \subseteq u\mathcal{N}u \subseteq \mathcal{N} for each c \in (Z_p)^{r-1}\{-(-1, \ldots, -1)\}, and \sum_{c \not\in (-1, \ldots, -1)} \varepsilon^{P-1} H(c)^{P-1} u \subseteq \mathcal{N}.

Next we show that \sum_{c \not\in (-1, \ldots, -1)} \varepsilon^{P-1} H(c)^{P-1} u \not\subseteq \mathcal{N}. \varepsilon^{P-1} H(c)^{P-1} u is spanned over K by elements of the form \varepsilon\varepsilon^{P-1} H(c)^{P-1} v where u = P(p)Q(q)R(r) and v = P(p')Q(q')R(r') are standard monomials in \mathcal{U}. Furthermore, if \varepsilon(u) > 0, then by Proposition I.10.
\[ \varepsilon(uE_{p-1}) = \varepsilon(u) + \varepsilon(E_{p-1}) = \varepsilon(E_{p-1}) = \text{maximal extent}, \]
hence \( uE_{p-1} = 0 \). If \( \varepsilon(v) = 0 \), then by Proposition I.10 \( \varepsilon(F_{p-1}v) = \varepsilon(F_{p-1}) + \varepsilon(v) = \varepsilon(F_{p-1}) = \text{minimal extent} \), hence \( F_{p-1}v = 0 \), and if \( \varepsilon(v) = \varepsilon(E_{p-1}) \), then \( v = E_{p-1}Q(q) \), by Lemma I.15. \( E_{p-1}H(c)F_{p-1}v = E_{p-1}H(c)F_{p-1}E_{p-1}Q(q) = 0 \), where \( c \neq (-1, \ldots, -1) \).

Therefore for each \( c = (c_1, \ldots, c_l) \in (Z_p)^l - \{(-1, \ldots, -1)\} \), \( uE_{p-1}H(c)F_{p-1}u \) is spanned over \( K \) by elements of the form \( uE_{p-1}H(c)F_{p-1}v \) where \( u \) and \( v \) are standard monomials in \( U \) such that \( \varepsilon(u) \leq 0 \), and \( 0 \leq \varepsilon(v) \leq \varepsilon(E_{p-1}) \). If \( uE_{p-1}H(c)F_{p-1}v \neq 0 \) is in the set which spans \( uE_{p-1}H(c)F_{p-1}u \), then by Definition I.9, Proposition I.10, and the fact that \( \varepsilon(E_{p-1}H(c)F_{p-1}) = 0 \), we have \( \varepsilon(uE_{p-1}H(c)F_{p-1}v) = \varepsilon(u) + \varepsilon(E_{p-1}H(c)F_{p-1}) + \varepsilon(v) = \varepsilon(u) + \varepsilon(v) \leq \varepsilon(E_{p-1}) \). The generator \( E_{p-1}(h_{i+1}) \) of \( N \) is of extent \( \varepsilon(E_{p-1}) \), hence \( E_{p-1}(h_{i+1}) \) is not in any \( uE_{p-1}H(c)F_{p-1}u \) where \( c \in (Z_p)^l - \{(-1, \ldots, -1)\} \), hence \( E_{p-1}(h_{i+1}) \notin \sum_{c \neq (-1, \ldots, -1)} uE_{p-1}H(c)F_{p-1}u \). We have finished proving that \( \sum_{c \neq (-1, \ldots, -1)} uE_{p-1}H(c)F_{p-1}u \notin N \). The proof of our theorem is completed.
Possible Directions for Further Research

Let \( \mathfrak{L} \) be a classical Lie algebra of type \( A_2 \) over an algebraically closed field \( K \) of characteristic \( p > 7 \), with a Chevalley basis \( \{e_1, e_2, e_3, h_1, h_2, e_{-1}, e_{-2}, e_{-3}\} \). Following the same method and notation described in Example III.2, for each finite dimensional irreducible \( \mathfrak{L} \)-module \( m_\Lambda \) of maximal weight \( \Lambda \), we obtain a finite dimensional irreducible \( \mathfrak{L} \)-module \( L(r,s) \) such that \( m_\Lambda \) is a homomorphic image of \( L(r,s) \). We identify \( e_{\pm i} \) and \( h_i \) with \( e_{\mp i} \) and \( h_i \), respectively.

From Remark III.7, we know that there are elements in \( \mathfrak{N} \), the radical of the \( \mathfrak{U} \)-algebra \( \mathfrak{U} \) of \( \mathfrak{L} \), and not in the two-sided ideal \( \mathfrak{N} \) generated by the four elements

\[
e_1^{p-1} e_2^{p-1} e_3^{p-1}(h_i+1) \quad (h_i+1)e_1^{p-1} e_2^{p-1} e_3^{p-1}, \quad i=1,2.
\]

One way to obtain some more elements in \( \mathfrak{N} \) is to find elements of \( \mathfrak{U} \) which send \( L(r,s) \) into its maximal submodule upon module multiplication, for all \( 0 \leq r,s \leq p-1 \). Unfortunately, almost nothing of such maximal submodules is known. Hence this approach naturally suggests first the study of the maximal submodules of \( L(r,s) \); \( 0 \leq r,s \leq p-1 \). Another way of finding elements in \( \mathfrak{N} \) is to find elements of the universal associative algebra of \( \mathfrak{L} \) which annihilate \( L(r,s) \) for all \( 0 \leq r,s \leq p-1 \); i.e. to find elements of the set...
Ker L(r,s), where Ker L(r,s) is the kernel of the representation of the universal associative algebra of \( \mathfrak{g} \) associated with L(r,s). This can be done by examining the weight diagrams of L(r,s), 0 ≤ r ≤ p-1; 0 ≤ s ≤ p-1, as was done in Example III.2. By reducing modulo p and using the identities \( e_i^p = 0, h_i^p = h_i \), we then obtain elements in \( \mathcal{K} \). This latter method amounts to first finding the intersection of the kernels of all the representations associated with the finite dimensional irreducible \( \mathfrak{g} \)-module L(r,s), 0 ≤ r,s ≤ p-1. Such intersection is a two-sided ideal in the universal associative algebra of \( \mathfrak{g} \). If a generating set of such ideal is found and consists of expressions in terms of \( e_i, h_i \), then by reducing modulo p and replacing \( e_i^p \) and \( h_i^p \) by 0 and \( h_i \) respectively, we obtain a generating set of a two-sided ideal of \( \mathcal{U} \) contained in \( \mathcal{K} \). The procedure in finding such a generating set when \( p > 7 \) involves some quite complicated arithmetic computation. In case \( p = 3 \), we found that the two-sided ideal \( \bigcap_{0 \leq r,s \leq 2} \ker L(r,s) \) in the universal associative algebra of \( \mathfrak{g} \) has a generating set consisting of the four elements \( x_1, x_2, y_1, y_2 \).
\[ x_i = (h_3+4) \prod_{j=0}^{6} (h_3-4+j) \]

\[ + e_1 \left( \frac{7h_1+2}{4} h_3 \right) (h_3+3) \left( \prod_{j=0}^{5} (h_3-4+j) \right) f_j \]

\[ + e_1^2 \left( \frac{7h_1-14}{2} h_3 \right) (h_3+2) \left( \prod_{j=0}^{4} (h_3-4+j) \right) f_1^2 \]

and

\[ y_i = \prod_{j=0}^{7} (h_3-4+j) - 4e_1 \left( \prod_{j=0}^{6} (h_3-4+j) \right) f_1 \]

\[ i=1,2. \]

Although these four elements reduce to 0, yet the element \( e_1^2 e_2^2 e_3^2 (h_1-2) \) is in \( \bigcup_{0 \leq r \leq 2} \bigcup_{0 \leq s \leq 2} \text{Ker} \ L(r,s) \) and reduces to \( e_1^2 e_2^2 e_3^2 (h_1+1) \) which is a non-zero element in \( K \).

This shows that the study of \( \bigcup_{0 \leq r \leq p-1} \bigcup_{0 \leq s \leq p-1} \text{Ker} \ L(r,s) \) would help us find some more elements of \( K \).
BIBLIOGRAPHY


