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THE LEARNING OF MULTIPLICATION AND OTHER MATHEMATICAL CONCEPTS AND SKILLS BY FOUR CHILDREN IN A FOURTH GRADE OPEN CLASSROOM: A CASE STUDY

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

by

Charles Stanley Thompson, B.A., M.A.T.

* * * * *

The Ohio State University 1973

Reading Committee:
Charlotte S. Huck
Richard J. Shumway
Harold C. Trimble

Approved by

Advisor
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VITA

July 21, 1943

Born, Halstead, Kansas

1965

B.A., Mathematics, University of Kansas, Lawrence, Kansas

1967

M.A.T., Mathematics and Education, Wesleyan University, Middletown, Connecticut

Spring, 1967

Mathematics teacher, Staples High School, Westport, Connecticut

1967-1969

Mathematics teacher, Shawnee Mission South High School, Shawnee Mission, Kansas

1969-1970

Mathematics teacher; Pembroke Country Day School, Kansas City, Missouri

1970-1973

Teaching Associate, Department of Mathematics Education, Ohio State University, Columbus, Ohio

1973-

Assistant Professor, Department of Education and Department of Mathematics, University of Wyoming, Laramie, Wyoming
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CHAPTER I

INTRODUCTION

Introduction and Need for the Study

Open education is a reality in at least one-third of the infant schools in England (Plowden, 1967) and is becoming evident in many elementary schools in the United States (Footlick, 1971; Silberman, 1970; Walberg and Thomas, 1972). Despite the many descriptions of open classrooms (e.g. Blackie, 1967; Featherstone, 1971; Plowden, 1967; Rogers, 1970; Silberman, 1970) and descriptions of the theories used to justify open classroom practices (Brearley, 1970; Copeland, 1970; Dienes, 1963, 1964, 1967; Flavell, 1963; Isaacs; Lovell, 1971; Piaget, 1965; Piaget and Inhelder, 1956 and 1960; Silberman, 1970; Weber, 1971), "... there really has been little intensive evaluation of open classrooms (Perrone, 1972, p. 26)."

The need for evaluation is further pointed out by Ronald W. Henderson, Director of the Arizona Center for Early Childhood Education:

If open education programs are to receive a true test, and if they are to be offered as a realistic option to American educational consumers, then careful attention must be given to the development of new techniques for the assessment of programs (1972, p. 2).

Walberg and Thomas suggest a specific focus for such assessment:

Before it [open education] is expanded from the limited number of extant experimental settings in this country, administrators, teachers, and parents quite properly should know if it leads to more learning ... (1972, p. 207).

Bussis and Chittenden also address the assessment of open education and
suggest "... further research on learning as it occurs in an open classroom ... (1970, p. 60)" as one of five areas for needed study.

The focus of the current research was the mathematics learning of four nine-year-old children in one open classroom. The small number of subjects is recommended by Bussis and Chittenden. "Intensive study of a limited sample of children in a few open classrooms appears to us an urgently needed research endeavor (1970, p. 77)." The research techniques followed the suggestion of the same authors:

Observation of children, paired with some form of semi-systematic interviewing by a participant-observer, within the classroom setting, would seem to be an excellent way of finding out much more about children's learning in the open context (Chittenden and Bussis, 1971, p. 9).

The children were observed and occasionally interviewed for eleven weeks. The researcher identified the mathematical concepts and skills the children dealt with, the processes by which the children dealt with the concepts and skills, and the children's developmental understanding of multiplication and its properties. The purpose was to generate hypotheses about mathematics learning in open classrooms which may serve as reference points for future research.

Statement of the Problem

The focus was the mathematics learning of four nine-year-old children in one open classroom. The following questions expressed the major goals.

1 i. What mathematical concepts and skills do children deal with?

ii. Does one child deal with the same concepts and skills as
the other children?

iii. Do children spend different amounts of time dealing with mathematical concepts and skills?

2. What are the immediate sources of the mathematical concepts and skills? That is, do the concepts and skills evolve from the children's interactions with classmates, materials, the teacher, or from some other source?

3. The questions below refer to the children as they deal with mathematical concepts and skills.

i. What activities do they engage in?

ii. Who else is involved in the children's activities?

iii. What role does the teacher have?

iv. With what concrete materials do they work?

v. How much time is spent working with concrete and semi-concrete materials?

4. The questions below refer to the children as they deal with multiplication and its properties over an extended period of time.

i. How much time do they spend?

ii. What activities do they engage in?

iii. Who else is involved in the children's activities?

iv. What role does the teacher have for the children?

v. Do they work with concrete or semi-concrete materials?

vi. How does their understanding of multiplication develop?

**Design of the Study**

From November 15, 1972, until December 15, 1972, a pilot study
was carried out in an open classroom in the same geographic and socio-economic area as the classroom used for the study. Eleven half-days were spent observing eight children, talking to them about their mathematical activities, and recording the observations in written form. As a result of the pilot, observation techniques were modified and refined.

From December 12, 1972, until December 19, 1972, the researcher spent four half-days in the classroom used for the study to become acquainted with the children and the activities. Following the visitation period, two boys and two girls were chosen randomly as subjects.

Observations were made for eleven weeks, from January 3, 1973, until March 15, 1973. The researcher spent forty-nine half-days observing and occasionally interviewing the subjects. The observational records consisted primarily of narrative descriptions of the children's activities and drawings depicting the children's work.

Definition of Terms

1. **Concept**: A structure in the mind that

   
   ... exists whenever two or more distinguishable objects or events have been grouped or classified together and set apart from other objects on the basis of some common feature or property characteristic of each (Bourne, 1970, p. 1).

2. **Concrete Materials**: Physical objects in the classroom which can be manipulated in the process of learning.

3. **Infant Schools**: The schools in England for children who are 5 to 7 years old.

4. **Mathematical Concept**: A concept which the individual has encountered in the study of systematic patterns and relationships in space or
among numbers.

5. **Mathematical Skill**: A procedure or mental ability performed largely from memory which yields the answer to a mathematical problem, e.g. the division algorithm.

6. **Mathematics**: For the purposes of this study, mathematics will refer to relationships in space (geometry), and numbers (arithmetic).

7. **Mathematics Learning**: The formation of mathematical concepts or the acquisition of mathematical skills.

8. **Open Education**: An educational practice which emphasizes a high degree of teacher and child participation and involvement in classroom activities, a teacher-child relationship of trust and understanding, a child-child relationship based on mutual respect, the provision of concrete materials for learning, and learning that is student-initiated, activity-oriented, and individualized relative to pace, methods, and outcomes.

**Basic Assumptions**

1. To generate hypotheses about mathematical learning in an open classroom, intensive observation of student activity for eleven weeks coupled with informal interviewing represents a valid research technique.

2. The conceptual development of one person is similar to that of other persons. What was learned about conceptual development of the four subjects in this study yields valuable clues about the conceptual development of other children in similar learning environments.
3. The subjects' basic conceptual development during the eleven weeks was not altered by the direct observation and occasional interviewing done by the researcher.

Limitations

1. The study was carried out in an already existent classroom – a "natural setting." Therefore, the researcher could not control confounding variables.

2. Since the study was carried out for eleven weeks, the children's behavior may have been a function of maturation more than their experiences in the classroom.

3. The children's behavior may have been affected by their own "personal history" outside the classroom.

4. The observation, recording, and the interpretation of student activity was carried out by a single researcher and was subject to his biases and personal judgments.

5. The subjects of the study were four nine-year-old children from an upper-middle-class community.

6. The subjects were part of a unique classroom environment. Their activities were influenced by the characteristics of that environment – the materials, the other children, etc.

7. Since the subjects had prior experiences in non-open classrooms, the findings of this study may not be applicable for students whose schooling has been exclusively in the open context.

8. The subjects were observed for eleven consecutive weeks of the 1972-73 school year.
Significance

One responsibility of educators is the thorough assessment and evaluation of new educational practices. The research described is a beginning for needed evaluation of open education. The results will serve as groundwork for future research. The hypotheses generated warrant testing in other open classroom settings. The results of that testing may be combined with other findings to provide an in-depth view of mathematics learning in open classrooms.

The results of the current study may also be combined with other open-education studies to yield a comprehensive view of open education as it is practiced and to provide material for comparative studies of open education practices and other educational practices.

Beyond open education, the results of the current study may serve as reference points for researchers investigating mathematics learning in any setting.
CHAPTER II
LITERATURE REVIEW

The literature is reviewed in four sections: literature defining open education, literature describing mathematics learning in open classrooms, literature about the assessment of children's understanding of the multiplication process, and literature having implications for the methodology of the current study.

Definition of Open Education

Many open education practices are similar to practices common in good progressive schools of the 1930's. For example, open educators and progressive educators such as Dewey emphasize the importance of the child's experience as the basis for learning. However, open education differs from progressive education in several important respects. For progressive educators the socialization of the child was much more important than it is for open educators. In progressive schools the children worked more in group activities than do children in open education schools. In progressive schools if one child needed help in making change, the teacher might suggest that he set up a play store and would involve the entire class in the operation of the store. In open education, however, children have more choice about which activities they participate in. Another difference between the two educational approaches lies in the degree of integration required in the daily activities of the children. In progressive
education more integration among daily activities was required by the teacher. Two other differences exist between the progressive education movement and the current movement toward open education. First, whereas in progressive education the theory preceded the practice, in open education the practices were well established before theories were sought which justified the practices. Second, whereas progressive education received no official government endorsement, open education has been endorsed in England (Plowden, 1967), Canada (Provincial Committee on Aims and Objectives of Education in the Schools of Ontario, 1968), and in the United States (Vermont Department of Education, 1968).

The first definitions of open education were descriptions of open classroom practices in England by observers such as Featherstone (1967), Blackie (1967), Plowden (1967), and Rogers (1970). Initial systematic attempts at definition were made by Barth (1972), Rathbone (1971), and Bussis and Chittenden (1970). Those initial attempts became the groundwork for an experimentally tested operational definition formulated by Walberg and Thomas (1972).

As a first step toward definition Barth hypothesized "... a number of covert assumptions about children's learning and the nature of knowledge which underlie the practices and statements of open educators (1972, p. 17)." His assumptions were based primarily upon contemporary writings. (The assumptions are included in Appendix A.) Barth "... 'tested' these assumptions with over a dozen British primary teachers, headmasters, and inspectors ... and with a number
of American proponents of open education . . . (1972, p. 18)." None of them disagreed with a single assumption on his list. He did not "test" the assumptions with other educators.

Rathbone (1971) also formulated some assumptions underlying open education practices. His assumptions evolved from "... observing open education teachers in action and by reading the books and articles they refer to most often . . . (Rathbone, 1971, p. 99)." In addition to statements about the nature of learning and the nature of knowledge (which complement those presented by Barth), he formulated statements regarding the purpose of schooling, the teacher's role, the psycho-emotional climate of the classroom, and the moral context of open education.

The first comprehensive systematic attempt at defining open education was made by Bussis and Chittenden (1970). They attempted to characterize open education in a study of Project Follow Through of the Education Development Center. They identified ten "themes" based on participation in workshops, essays written by Follow Through directors, observations of classes, and conversations with teachers. Bussis and Chittenden focused on the teacher's behavior. (The "themes" are given in Appendix B.)

Walberg and Thomas (1972) made use of the works of Barth, Rathbone, and Bussis and Chittenden. Bussis and Chittenden's "themes" were used as the framework for an operational definition of open education. Walberg and Thomas reviewed the writing of Barth, Rathbone, and sixteen other open education writers. They transformed and reduced the
Bussis and Chittenden list of "themes" from ten to eight:

1. Provisioning for Learning,
2. Humaneness, Respect, Openness, and Warmth,
3. Diagnosis of Learning Events,
4. Instruction, Guidance, and Extension of Learning,
5. Evaluation of Diagnostic Information,
6. Seeking Opportunities for Professional Growth,
7. Self-Perception of Teacher, and
8. Assumptions about Children and Learning Process
(Walberg and Thomas, 1972, pp. 200-201).

Walberg and Thomas compiled a classroom observation instrument consisting of statements which focused on each of the "themes." (The instrument is found in Appendix C.) Trained observers visited twenty-one open classrooms and twenty-one traditional classrooms in the United States, and twenty open classrooms in Great Britain.

The sites for observations were selected on basis of reputation and also on the personal knowledge of the investigators. The sample was by no means random but represented urban and suburban public and private schools with administrators and teachers cooperative enough to permit intrusive observers. An effort was made to gain access to both Open and Traditional classes with teachers regarded as excellent by outside experts and their principals, and the sample was further restricted to classes of five to seven year old children in their first three years of school. (Walberg and Thomas, 1972, p. 202).

The results indicated that the instrument differentiated (p < .001) between open and traditional classrooms and that seven of the eight subscales differentiated (p < .001 for each subscale) between open and traditional classrooms. (The items representing "Seeking Opportunities for Professional Growth" did not differentiate.)

Mathematics in Open Classrooms

This section is divided into two parts. The first part consists of ideas from the theories of Piaget and Dienes which support
current mathematical practices in open classrooms. The second part consists of descriptions of mathematical practices in open classrooms.

Piaget and Dienes. According to Piaget, children construct their own private mental worlds by acting upon the environment. The type of interaction children have with the environment and what is learned depends, in part, on their chronological age. Piaget says that until about the age of seven children need to physically manipulate objects to learn from them. Thereafter the child can acquire knowledge by manipulating objects mentally. For example, a child may solve problems like $5 - 2 = \_\_\_$ by imagining a set of five objects and mentally removing two of them. The child has acquired conservation of number and can learn the four basic mathematical operations and their properties (Copeland, 1970, p. 116). However, the concepts that can be acquired are limited to ones which can be derived from concrete experiences. After age eleven or twelve Piaget claims that the child can acquire concepts which are not based on concrete experiences, such as proportion. For the first time the child can learn mathematics through verbalization only.

However, Piaget does not state that concrete materials and concrete experiences are not needed by children older than eleven or twelve. As Adler states, concrete experiences are used by all children ... past the age of seven, but until the age of eleven they are, in general, the most advanced operations of which the child is capable. Moreover in the development of new concepts at all stages in learning, it is necessary to proceed from the concrete to the abstract (1966, p. 580).

Piaget also asserts that the knowledge a child acquires
depends on the amount of "social transmission" (interaction) he experiences (Piaget, 1964, p. 10). Since a child tends to view experiences from his own perspective, the amount of verbal intercourse he has affects his perception of reality. The opportunity to openly exchange ideas and to discuss and evaluate his own ideas and the ideas of others allows the child to progress from having a subjective outlook to having a more objective one.

Dienes' theory of mathematics learning (1963, 1964, 1967; Reys and Post, 1973) also emphasizes the active nature of the learner. Dienes hypothesizes that a child proceeds to understand a concept by completing three temporally ordered stages. The three stages are components of Dienes' "Dynamic Principle" and are presented here as they would occur in a school learning situation. First, the child is placed in a prepared environment and allowed to play. The play is not random but is only loosely structured by the materials available. The second stage is characterized by more structured activities. The child is given experiences which are structurally similar to the concept he is learning. The child reaches the third stage when evidence of the mathematical concept emerges. The child is provided numerous opportunities to recognize and apply the concept in new situations, and only when he can do both does Dienes claim that the child understands the concept.

Central to Dienes' theory are concrete experiences by the learner, experiences which frequently involve concrete materials. Dienes emphasizes that the child should have experiences with a number
of concrete materials which embody the same concept. Dienes contends that the learner will abstract the concept and the concept will not be associated with particular materials. Because of such experiences he feels that the child will be better prepared to recognize and apply the concept in new situations.

Dienes asserts that mathematics should be learned because of its intrinsic interest and not because of its possible utilitarian function (1964, p. 7). He feels that the learning of mathematical ideas and structures should supersede the acquisition of skills. Dienes states, "It is certainly being found that exploration of the "difficult" underlying mathematical structures, in place of the customary rote-learning of rules, delights rather than repels children (1964, p. 7)."

**Mathematical Practices in Open Classrooms.** Detailed descriptions of mathematical practices in open classrooms have been presented in works by Biggs (1971; Biggs and Maclean, 1969; Schools Council, 1969) in the Plowden Report (Plowden, 1967), and in the booklets of the Nuffield Mathematics Teaching Project (1967). The following summary is compiled from those sources.

Biggs and MacLean state the aims of learning mathematics in open classrooms:

1. to free students, however young or old, to think for themselves.
2. to provide opportunities for them to discover the order, patterns and relations which are the very essence of mathematics, not only in the man-made world, but in the natural world as well.
3. to train students in the necessary skills (1969, p. 3).
To attain the goal of freeing students to think for themselves, a learning sequence such as the following is common. First, the child is introduced to a prepared environment and allowed to experiment freely, as Dienes recommends. Within that context

... the role of the teacher is to help children to acquire acute powers of observation and to assess the possibilities that lie within the most commonplace objects and events (Nuffield Mathematics Teaching Project, 1967, Vol. I, p. 16).

During the experimentation questions often arise for the child which he poses for the teacher. Biggs and MacLean suggest that "... tossing a child's question back at him is sometimes an effective way of encouraging him to think it out for himself (1969, p. 56)." Discussion between the student and the teacher often ensues, and frequently issues are raised which cause the child to modify his perspectives. (Piaget points out the necessity of such discussions.) The teacher may raise issues by asking open-ended questions which require thought by the student. Care is taken not to provide too much help for the student, however. During the discussion the child is given the responsibility of identifying a problem for subsequent study. The importance of the foregoing sequence of events is pointed out in the teachers' guides of the Nuffield Mathematics Teaching Project,

The confidence to assess situations, formulate questions, and attempt to determine solutions only grows through experience, yet it is something that all children urgently need today (1967, Vol. I, p. 14).

To provide children the opportunity to discover mathematics (the second goal identified by Biggs and MacLean) teachers often follow a learning sequence like the one just described. After the child, perhaps with help from the teacher, has identified a problem to
investigate,

It is the role of the teacher to suggest a certain arrangement of the data, or to ask a question demanding some evaluation of the data in order to lead the child to the discovery of the relationship (Nuffield Mathematics Teaching Project, 1967, Vol. I, p. 13).

As the child works, the teacher may occasionally observe him and ask an appropriate open-ended question if it appears that the child is not making progress. The child may also be asked to record his investigation in writing or in graphical form. Finally, the teacher and the child discuss in detail what was learned. The emphasis throughout the investigation just described is on the active involvement of the child, a prerequisite for effective learning according to both Piaget and Dienes.

The students' investigations often focus on the structures of mathematics, such as the relationships among various types of numbers or among various geometric shapes. (Dienes stressed the importance of such investigations.) However, teachers do not rely solely upon discovery methods for mathematics instruction. Biggs states, "There are some facts which children cannot discover, and these the teacher needs to tell them (1961, p. 8)."

The materials with which children work during discovery learning include both natural and man-made materials. Children make extensive use of common objects such as beans, sand, rubber balls, buttons, plastic containers, and washers. Classrooms also contain many commercially prepared materials such as Dienes' Multibase Blocks, Unifix cubes, Cuisenaire rods, geoboards, and abaci. The relationship
between the use of commercial materials and common objects is pointed out by Weber:

Structural aids (Cuisenaire, Unifix, Stern, Dienes, etc.) are used to clarify and emphasize what a child has already begun to understand through firsthand experience in ordinary life situations. The possibilities for mathematical exploration in a child's use of the ordinary materials of the environment are always stressed . . . (1971, p. 119).

In addition to working in the classroom, children use the out-of-doors for their investigations. Activities common in the schoolyard include measuring distances and timing the running of classmates. Beyond the school grounds children often investigate geometrical patterns in nature and in architecture. For children in open classrooms the total environment is used for mathematics learning.

In attempting to attain the third goal identified by Biggs and MacLean (training students in the necessary skills) the focus is on child involvement also. The skills arise from the child's experiences. For example, finding the number of tiles in a floor or estimating the number of beans in a one-pound bag may lead to practice with multiplication. Because the skills arise in practical activities, the children understand and appreciate the need for polishing and expanding the skills (Biggs and MacLean, 1969, p. 13). The approach to the learning of skills is not based on memorization but rather on the processes involved in the child's activity. If a child does not understand a skill he is practicing, he can re-create the activity to recall the mathematical processes involved. Students often use manipulative aids in conjunction with skill practice. For example, a student might regroup Dienes' Multibase Blocks when
practicing subtraction exercises involving two-digit numerals. Open classroom teachers seek to maintain a close correlation between the child's activity and his skill practice.

Assessing Children's Understanding of the Multiplication Process

A review of research related to multiplication revealed that most studies measured children's understanding of multiplication by achievement tests. Subjects were usually asked to respond to simple multiplication combinations or to word problems, and answers were scored as correct or incorrect. Distinctions were not made among individuals who obtained identical answers to a multiplication problem, although their solution processes may have been quite different. Brownell (Brownell and Watson, 1936; Brownell and Moser, 1949) and Weaver (1955) contend that attention must be given to the children's processes for solving multiplication tasks to ascertain a complete picture of their understanding of the multiplication process. Accordingly, each study reviewed attempted to describe children's conceptual processes for solving multiplication problems. The results served as a framework for interpretation in the current study.

Brownell and Carper (1943, Chapter 4) reported a study which investigated how children thought about multiplication. Children responded orally to multiplication combinations, each involving two non-zero one-digit numerals. The students' methods of solution were sorted into eleven categories ranging from "guessing" to "memory" to "counting" to "meaningful habituation." The results of the interviews indicated that: 1) the subjects' use of a mature method of solution,
"meaningful habituation," increased steadily from thirty-nine percent at the end of grade three to about eighty-eight percent at the end of grade five; 2) "meaningful habituation" was used more frequently to solve "easy" combinations than "difficult" ones; and 3) individual children tended not to employ the same method of solution for all multiplication combinations. The profile of children's learning of multiplication combinations that resulted is summarized by Brownell and Carper:

... these children did not learn the facts all at once - at one jump, as it were. Instead, they seemed to learn by a series of jumps. This series carried them from undesirable, or inefficient processes, or immature processes through other processes intermediate in desirability, efficiency, and maturity, to the final stage of meaningful habituation (1943, Chapter 4, p. 70).

Jerman (1970) investigated the strategies children used in solving one-digit multiplication combinations. He noted that in Brownell and Carper's study (1943, Chapter 4) sixty percent of the students' processes were classified simply as either "memory" or "meaningful habituation," and he felt that processes children use to solve simple multiplication combinations should be further identified. Using the computer, he compared student response times for solving simple multiplication combinations to times predicted according to each of eleven strategies. The strategy for which the predicted times agreed best overall with the subjects' actual times was, the smaller factor is added repeatedly as many times as the numerical value of the other factor (Jerman, 1970, p. 99). (That is, 2 x 3 equals 2 + 2 + 2.) That strategy fit the times best overall.
at each grade level, three through six. Further analysis of the data yielded conclusions which supported those of Brownell and Carper (1943, Chapter 4): 1) children apparently used different strategies with different combinations, and 2) the strategies third grade children apparently used were also used in seventy-two percent of the cases by sixth grade students in solving the same multiplication combinations.

One additional study investigated children's strategies in solving multiplication problems. Gray (1965) found that subjects who had received multiplication instruction stressing the distributive property used the distributive property significantly more often than subjects who were receiving multiplication instruction but who had not been taught about the distributive property.

Whereas Brownell and Carper (1943, Chapter 4) and Jerman (1970) investigated children's strategies in solving simple multiplication combinations, Hervey (1966) investigated children's conceptualizations of two different interpretations of multiplication. Second graders who had not received multiplication instruction responded orally to word problems involving multiplication interpreted as 1) equal addends and as 2) Cartesian products. "It was assumed that the method a subject used in solving a problem indicated how he conceptualized the problem (Hervey, 1966, p. 290)." Subjects were also asked to select a "way to think about" each problem from five figural representations shown to them. Analysis of the data yielded results including the following: 1) subjects were able to correctly conceptualize seventy-one percent of the equal addends problems but
only thirty-five percent of the Cartesian product problems: 2) it was significantly less difficult to choose a "way to think about" equal addends problems as compared to Cartesian product problems: and 3) subjects gave correct final answers to sixty-four percent of the equal addends problems but only to thirty-two percent of the Cartesian product problems.

The next two studies reviewed investigated strategies used (to solve problems involving multiplication) by children who had not received multiplication instruction. In a study by Carper (Brownell and Carper, 1943, Chapter 8), subjects responded orally to some problems involving the simple combination of equal groups (e.g. "How many are two 2's?"). The children's solution processes were sorted into twelve categories. The categories were essentially breakdowns of five of the categories used by Brownell and Carper in the previously reviewed study (1943, Chapter 4). About sixty-five percent of the students' responses involved processes "closely allied to multiplication"; that is, they involved the use of a multiple of the basic group identified in the original problem. Another result was that twice as many students responses involved processes "closely allied to multiplication" when the number of groups in the original problem was three, as compared to when the number of groups in the original problem was two. In a study reported by Gunderson (1955) children responded orally to word problems involving multiplication, again interpreted as combining equal groups. If the student responses are sorted into the twelve categories used by Carper (Brownell and Carper,
1943, Chapter 8), the result is that about forty percent involve processes "closely allied to multiplication." Possible explanations for the different results include: 1) the problems used by Gunderson involved larger numbers, and 2) the problems used by Brownell and Carper were more explicit and more directive.

In a number of studies (Gray, 1965; Hall, 1967; Schell, 1965; Schrankler, 1967; Tietz, 1969) subjects received instruction emphasizing certain representations or properties of multiplication (e.g. arrays, distributive property) and subsequently took achievement tests, transfer tests, or "understanding" tests. However, since the processes which the subjects used to solve the multiplication problems were not recorded or measured, the studies were not considered further in this review.

**Literature Having Implications for the Methodology of the Current Study**

Characteristics of the current study included observations for eleven weeks, a focus on four subjects, and systematic interviewing. Support for such methodology in open classroom research is given by Bussis and Chittenden in two articles focusing on research and assessment strategies (1970, 1971). Components of the methodology of the current study have also been used successfully in various settings by Wright (1967), Smith and Geoffrey (1968), Jackson (1968), Dienes (1963), Piaget (e.g. 1951), Navarra (1955), Brownell and Carper (1943), and Hervey (1966).

The primary method of data collection in the current study was by direct observation. Wright used direct observation in investigating
the behavior of children in natural settings (1967; Barker and Wright, 1955) and pointed out the need for such research:

Scientific knowledge of what persons can do if their life situations are thus and so, notwithstanding its great importance, leaves an unfilled need for research to describe what persons actually do . . . (1967, p. 2).

One possible difficulty with direct observation is the influence of the observer. Since it is probable that such influence cannot be eliminated, one alternative is to minimize it, describe it, and keep it as nearly constant as possible. Accordingly, the approach taken in the current study generally followed Wright's guidelines for an observer:

. . . he can get to know and be known by his subjects and their associates, and build for himself the role of a friendly, nonevaluating, nondirective, nonparticipating person with interest in what people do. . . . most children 'get used to' the presence of onlookers who, though friendly and not entirely anonymous or unresponsive, let what the child is doing take its course and rarely or never bother him. This holds especially if observations are long enough to permit adaptation to any new or uncertain elements of the situation . . . . A child cannot stop being himself for long, if at all, because someone is watching him . . . (1967, p. 42).

The observations were recorded concurrently in written narrative form. Wright suggested: "The observer should take notes on the scene of the observed behavior. Sufficiently detailed and accurate reporting of events in their true order is otherwise impossible (1967, pp. 45-46)."

Two additional studies affirm the value of direct observation in school settings. Smith and Geoffrey (1968) used direct observation by a nonparticipant-observer in a daily, semester-long study of one urban classroom. Jackson (1968) also used direct observation
techniques in a year-long investigation concerning the nature of life in several classrooms. In both studies observational records consisted of narrative descriptions of classroom occurrences.

The current research focused on a small number of subjects to obtain an in-depth perspective on the mathematics learning in one open classroom. In Dienes' experimental study of mathematics learning (1963) he and three other researchers focused on individual children during six weeks of observation. Piaget's work is replete with findings based on study of small numbers of children (e.g. 1951, 1952, 1954). Piaget observed his own three children extensively to investigate the origins of children's spontaneous mental growth. Navarra observed his own child for about three years to investigate the development of the child's scientific concepts (1955). The fact that the studies by Dienes, Piaget, and Navarra yielded a number of significant hypotheses regarding the nature of children's learning supports the focus on a small number of subjects in the current study.

The researcher administered individual interviews to the subjects systematically to gain additional information about their level of understanding of multiplication. Brownell used interview techniques in a number of research studies (1939; Brownell and Chazal, 1935; Brownell and Moser, 1949) including one which investigated children's understanding of multiplication (Brownell and Carper, 1943). In writing about the value of interviews Brownell and Moser stated:

... the interview enables the investigator to penetrate further (than by group testing) into the behavior of his subjects and to note significant aspects of that behavior which otherwise would almost certainly elude him. In
the absence of these intimate and comprehensive measures of performance, evaluation must necessarily be incomplete (1949, p. 36).

In a study designed to compare the merits of interviewing with pencil and paper testing, Brownell and Watson (1936) concluded that interviewing was both more reliable and more valid in determining the processes used and the individual difficulties which children experienced (in adding fractions). Weaver (1955) also pointed out the advantages of interviewing. He compared the results of hypothetical interviews with four children about multiplication facts with results that would have been obtained if only the students' final answers had been considered. He concluded that although the four students' final answers were identical, the students "... differed in their level of understanding of the multiplication process, and in their level of ability to use other mathematical concepts and relationships (Weaver, 1955, p. 43)."

The interviews administered in the current study involved the use of concrete and semi-concrete materials. The subjects could either physically manipulate objects or touch diagrams as they solved the tasks presented. Piaget made extensive use of concrete materials with individual interviews in his studies of children's geometric concepts (1960) and their number concepts (1965). Hervey (1966) also used semi-concrete materials along with interviews in her study of children's conceptualizations of multiplication. Dienes pointed out, from a nonparticipant-observer's perspective, the advantage of having subjects work with materials:
The observer assigned to each subject could record his subject's attempts with the physical material - an added experimental advantage as it would thus be easier to extrapolate these subjects' underlying thought processes than if one were simply observing a subject silently trying to solve a problem (1963, p. xix).

The advantage in an interview situation are even greater. Following a child's manipulation of materials the interviewer can ask the child about the manipulations made. Piaget states:

... conversation with the child is much more reliable and more fruitful when it is related to experiments made with adequate material, and when the child, instead of thinking in the void, is talking about actions he has performed (1965, p. vii).

Summary

The most significant body of research concerning open education has attempted to define open education. An instrument developed by Walberg and Thomas (1972) has differentiated (p < .001) between classrooms thought to be open and classrooms thought to be traditional. Although little research was found which investigated the learning of mathematics in open classrooms, several authors (e.g. Biggs, 1971; School Council, 1969; Plowden, 1967; Nuffield Mathematics Teaching Project, 1967) have described how mathematics is learned in some open classrooms. The learning theories of Piaget and Dienes have been used to justify the methods employed for learning mathematics in open classrooms.

Although few studies have investigated the learning of mathematics in open classrooms, the research methods used in the current study (e.g. extensive observation, interviewing, focusing on a small
number of subjects) have been used successfully in other settings. Wright (1967) outlined procedures for observation and his suggestions were appropriate for classroom research. The interviewing techniques used for assessing children's understanding of multiplication were based on studies by Brownell and Carper (1943) and Hervey (1966), and employed the use of concrete and semi-concrete materials as did the tasks given to children by Piaget (e.g. 1965). Finally, studies by Piaget (e.g. 1951, 1952, 1954) and Navarra (1955) indicated that focus on a small number of subjects could be profitable for the investigation of children's learning.
CHAPTER III
PROCEDURES

Introduction

The purpose of this study was to generate hypotheses about mathematics learning in open classrooms. Four nine-year-old children in one open classroom were observed and occasionally interviewed for eleven weeks. Observational records consisted primarily of narrative descriptions. The researcher identified the mathematical concepts and skills the children dealt with, the processes by which the children dealt with the concepts and skills, and the children's developmental understanding of multiplication and its properties.

The Pilot

The pilot was carried out in a combination second-third grade open classroom in an upper-middle-class suburban community; there were six second-graders and sixteen third-graders in the class. The classroom was one of three open classrooms in the school and one of fifteen in the seven elementary schools in the community. The fifteen open classrooms were so designated because the teachers had participated in a summer workshop on open education and met weekly to discuss common teaching concerns. Both the pilot classroom and the classroom used for the study were located in the same geographic area and both groups of students were approximately the same age.

The pilot was carried out from November 15, 1972, until
December 15, 1972. The purposes were:

1. To determine an appropriate experimental climate for the researcher to maintain with the teacher and the students,
2. To test and refine observation techniques and recording procedures, and
3. To determine questions about children's mathematical learning appropriate for the study.

The teacher selected one boy and one girl for observation, and three boys and three girls were randomly selected. All students selected were third graders. The teacher and the researcher discussed the observations of the two teacher-selected students and one additional boy and one girl. The observations of the four remaining students were not discussed. The teacher and the researcher conversed frequently about general classroom happenings during the pilot, and in those discussions the researcher was supportive of the teacher's regular classroom practices.

Eleven half-days, mostly afternoons, were spent observing and talking with the eight subjects and other students and recording the observations in written form. The students were permitted to read the recorded observations and their initial apprehension about the note-taking diminished. The subjects were observed primarily when involved in mathematical activity. The researcher also observed and talked with other students engaged in various activities. When two subjects were involved in mathematical activity, the researcher observed the one for whom fewer observations had been made. The observational records
consisted primarily of narrative descriptions of the subjects' activities and diagrams depicting their work. The observations focused on identifying and categorizing:

1. the mathematical concepts dealt with by the subject,
2. the area of the classroom in which the subject was located,
3. the activities the subject engaged in,
4. the concrete and semi-concrete materials used,
5. the subject's immediate source of the mathematical concepts he dealt with,
6. the people involved with the subject, and
7. the role of the teacher for the subject.

The researcher utilized the categorized data and the narrative comments to determine the processes by which the subjects dealt with mathematical concepts. He also attempted to determine the development of the subjects' mathematical concepts.

As a result of the pilot, the following changes were made in the procedures for the study:

1. The teacher was not told the identity of the subjects.
2. The teacher was not given information about the subjects' mathematical activities.
3. The subjects were observed during four mornings and one afternoon each week.
4. The researcher eliminated the recording of the areas of the room in which the subjects worked.
5. The researcher recorded the subjects' activities with
mathematical skills in addition to their activities with mathematical concepts.

6. The researcher focused on multiplication and its properties in analyzing the children's development of mathematical concepts.

7. The researcher presented each subject with three sets of multiplication tasks at spaced intervals during the study to more accurately determine his developmental understanding of multiplication.

The Study

Sample

The study was carried out in a fourth grade open classroom in an upper-middle-class community. The classroom was one of five open classrooms in the school and one of fifteen open classrooms in the seven elementary schools in the community. The classroom was used for the study because the teacher had taught for two years in open classrooms.

The Teacher

The teacher was female and 30 years old. She had taught for five years in California and three years in Ohio; two of the three years in Ohio were in open classrooms. The teacher had attended several workshops on open education and was finishing a Master's Degree program in Early and Middle Childhood Education.

The relationship between the researcher and the teacher was one of friendship and mutual respect. The researcher was supportive of the
teacher's efforts and ideas. Likewise, the teacher felt the research was well planned and cooperated eagerly.

The researcher always felt welcome in the classroom. The researcher and the teacher conversed frequently during the eleven weeks about general issues related to the classroom activities; however, the researcher avoided discussing issues which might have affected the teacher's interactions with the subjects.

The teacher was provided with a good deal of information about the study before it began. The teacher knew the purpose was to generate hypotheses about how children learn mathematics in open classrooms and that four students would be selected randomly for close observation. Initially the teacher did not know the identity of the four subjects. After four weeks she tentatively identified one subject and after nine weeks another, but the researcher neither confirmed nor denied her identifications. At the conclusion the teacher was uncertain about the identity of the other two subjects. The teacher knew that the researcher would be taking notes and talking with the children. The teacher did not know that the subjects would be given a series of tasks designed to measure their development of the concept of multiplication and at the conclusion of the study she was not aware that the tasks had been given.

The teacher was encouraged not to change any classroom practices because of the research. At the conclusion she indicated that if the research had not been in progress she might have given the students a few more mathematics worksheets; the teacher said that the worksheets
were helpful in "sparking interest." Otherwise, the teacher felt that the research did not affect what occurred in class. The teacher also indicated that because of other commitments she did not contribute new ideas and materials for math projects during the last few weeks of the study.

The Students

There were thirteen girls and ten boys in the class. For most of the students this was their second consecutive year in an open classroom. For about half this was the second consecutive year with the same teacher. The students generally came from upper-middle-class professional families.

The students often gave no evidence of being watched and accepted the researcher's observations as normal classroom activity. The teacher said that the researcher was "good at being neutral" and that the students never complained about the researcher being in the classroom. A feeling of friendship existed between the students and the researcher. The researcher generally accepted and supported student behavior to build trust between the students and himself. The teacher said that the researcher established a "good rapport" with the children.

The students were not told the purpose of the research; they were told that the researcher was seeing what children did in that classroom. The teacher said that the students were accustomed to visitors and would not be affected by having another one. During the same eleven week period an education student from a local university
participated in the classroom all day each Tuesday and Thursday.

**The Subjects**

The subjects were two boys and two girls chosen randomly from the twenty-three children. The relationship between the subjects and the researcher was essentially the same as between the other students and the researcher, ranging from acceptance to friendship. The researcher came to know more about the subjects than about many non-subjects, but he knew as much about several non-subjects as he did about the subjects. The subjects knew no more about the study than did the non-subjects.

The subjects' reactions to the multiplication tasks were mixed; one of the subjects thoroughly enjoyed the tasks and asked the researcher to give her some others, two enjoyed the tasks but with less apparent enthusiasm than the first subject, and the fourth performed the tasks but would rather have done something else.

**NR**  
NR was a nine-year-old boy; his father was a professor. He had been in the same school for four years. In grades one and two he was in non-open classrooms; in grade three he was in the open classroom of his current teacher. He was slight and missed several days of school because of illness. He was very verbal and self-confident; he did not hesitate to ask for assistance from the teacher or the researcher. The teacher remarked that he was very creative. He was sometimes impatient when people did not respond immediately to his wishes. He often did not put away materials after he used them. NR read avidly and
occasionally preferred to read rather than go outside for recess. His particular reading interest was Indians and he made an Indian village replica from art materials. He was also very interested in puppets and marionettes and made several in class. He wrote a play in which his marionette had the lead part. NR concentrated on reading, social studies, and art activities in class and participated in mathematics or science infrequently.

RS  RS was a nine-year-old girl; her father was a professor. She had spent the first three grades in another school and this was her first year in an open classroom. She was heavy and uncoordinated; in gym class she was the last student chosen for the basketball teams. RS was quiet in class. She had one good friend with whom she spent most of her time. Her friend was outgoing and had many ideas and RS relied a great deal on her for ideas; RS occasionally became an observer in their activities. She liked art and reading. She worked methodically, often looking around at the other happenings. The teacher frequently suggested activities for her to do. RS sometimes talked with the researcher about her family or what she was doing. Once, she read him a story she had written.

EZ  EZ was a nine-year-old boy, the son of a physician. He had attended a parochial school for the first grade; for the second grade he was in a non-open classroom at his current school. For grade three he was in the open classroom with his current teacher. He was stocky and self-conscious about his size. Once during the study he was ill
for several days. He was shy and very quiet. Even though he was quiet, he was self-confident about his abilities. The teacher remarked that he was becoming more outgoing. EZ was polite and neat; he always put away materials that he used. His particular interests were reading, writing, art, and math. He was content to sit as a desk and work for substantial periods of time. EZ was enthusiastic about sports and often talked with his best friend about the latest sporting event.

YD YD was the nine-year-old daughter of a dentist. She had attended her present school for the first four grades; grades one and two were in non-open classrooms and grade three was in an open classroom but with a teacher other than her current teacher. YD was quiet, friendly, and well-liked. She was meticulous and very careful about keeping her materials in order. She liked reading and math. She read a book every day or two and concentrated on activities for long periods of time. YD had one good friend but during the day participated in activities with other children as well. She occasionally asked the researcher for ideas about projects she could do.

The Classroom

I. Physical Arrangements

The classroom was the same size as other fourth grade classrooms in the school. The teacher had arranged the furniture to provide for small group activities in different parts of the room and had supplemented the available materials with many of her own and with some that had been donated. For example, the teacher supplied a phonograph,
woodworking tools, children's books, maps, games, art materials, an
iron, math measuring and counting apparatus, and science equipment.
Donated articles included a typewriter, cooking utensils, wallpaper
samples, and much scrap wood and "junk" suitable for carpentry or art
work. Figure 1 illustrates the general layout of the room.

The following materials were found in the math area:
1. metric balance, fishing spring scale, diet scale
2. odometer
3. yardsticks, metersticks, foot rulers, cloth tape measures
4. stop watch, egg timer, kitchen timers
5. volume relationship set - cylinder, cone, sphere
6. bottles
7. compasses and protractors, 20" protractor
8. french curves
9. road maps, airline maps
10. systems of measurement chart
11. jars with beans, pennies, toothpicks, rice, corn, marbles
    washers, split peas, straws, smooth rocks, golf tees, macaroni, metal screws, metal rings
12. 1" counting blocks
13. place value strips
14. abacus
15. play money
16. Cuisenaire rods and cards
17. Quizmo, cross-number puzzles, factor game, dice, dominoes
FIGURE 1

D = Student Desk

Display Table

T. Desk

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Music Area

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18. flash cards
19. multiplication practice machine
20. fraction pieces, fraction chart, "Fractions are as Easy as Pie"
21. 6 - 5 x 5 geoboards, 1 - 19 x 19 geoboard, rubber bands
22. solid and cardboard 2-D and 3-D geometric figures, geoblocks and perspective drawings
23. 1" and 2" squares
24. mirror cards set
25. kaleidoscope
26. Attribute blocks
27. teacher-written math project cards
28. teacher-written computation cards and cards on measurement and estimating
29. graph paper, textbooks.

The Classroom

II. Procedural Arrangements

What the students did each day was a function of (1) the school schedule and (2) the structuring of activities by the teacher and students.

The school schedule for the class is contained in Figure 2. The art, gym, and music periods were taught by teachers other than the classroom teacher. The classroom teacher and the librarian both worked with the children during the library period.
All other activities for the day were structured by the teacher and the students. The degree of structuring by the teacher varied depending on the activity and on the particular children involved. For example, with arithmetic computation the teacher often structured the children's work. The teacher made separate sets of problems on addition, subtraction, multiplication, and division and about twice a week at the beginning of the day asked each child to choose a set of problems to do. Thus, for 15 or 20 minutes everyone practiced computation. The teacher structured the learning to a lesser degree with other math activities. Each day the children were encouraged to do some math; they could do some type of a project (e.g., measuring, estimating, graphing), play a mathematical game, or practice the basic number facts. For these activities the teacher allowed the students to decide what they wanted to do and when they wanted to do it.
The teacher also provided different degrees of structure for different children. For example, if a certain child tended to do only routine math activities such as computation, the teacher might talk with him in detail about different math projects he could do and help him begin one. With a student who frequently did math projects the teacher might simply observe what the student had done, think of some logical extensions, and ask the student if he was interested in doing one of them.

Though the degree of structuring of learning activities varied, there were several general procedures within which the children worked.

1. The children worked individually or in small groups of two to four students eighty to ninety percent of the time. They only met as an entire group for organizational purposes at the beginning of the day, for the classes indicated in the weekly schedule, for teacher-read stories, for sharing time, and for occasional class meetings.

2. Except for group work each child determined the order in which he did his activities.

3. The children were encouraged to do some math, writing, and reading each day.

4. Twice a week for fifteen minutes at the beginning of the day the children did math computation.

5. The children voted to have a fifteen minute silent reading period three times a week.

6. Each week the children were encouraged to do some work in
art, science, geography, and social studies.

7. Each week the teacher brought in some new materials or new suggestions for activities.

8. Approximately every other week the teacher asked the children to do a project on a particular topic. For example, the children were once asked to read a folk tale and another time to do a project on Indians.

9. Each day the children kept a written record of their activities.

10. Each week the children evaluated in writing what they had done and decided what specific changes they wanted to make the next week.

Observations

I. Pre-Study

Prior to the beginning of the collection of data on math learning (January 3, 1973) the researcher spent four mornings (December 12, 13, 14, and 19, 1972) becoming acquainted with the children and the classroom procedures. The teacher told the class, "Today we have a visitor, Mr. Thompson, so he'll be wandering around." When asked by several students why he was there, the researcher responded that he was "trying to see what they did in that class." The researcher learned the names of the students and observed most of them as they were involved in learning activities. For example, the researcher played some estimating games with a small group of girls and went to gym with the entire class. To establish rapport with the children the researcher
talked with them, supported their ideas, and became interested in their projects.

II. Schedule

The researcher spent forty-nine half-days in the classroom during eleven weeks, from January 3, 1973 until March 15, 1973. The researcher usually observed from 8:30 a.m. until 11:20 a.m. on Mondays, Tuesdays, Thursdays, and Fridays, and from 12:30 p.m. until 3:00 p.m. on Wednesdays. Occasionally observations were made at other times.

Figure 3 gives the calendar of observations during the eleven weeks. The number indicates the date of the month on which the observation was made. An "a" indicates an observation was made in the morning, a "p" indicates an observation was made in the afternoon.

FIGURE 3
CALENDAR OF OBSERVATIONS

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<td>1a</td>
<td>2a</td>
<td></td>
</tr>
<tr>
<td>5a</td>
<td>6a</td>
<td>7p</td>
<td>8a</td>
<td>9a</td>
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</tr>
<tr>
<td>12a</td>
<td>13a</td>
<td>15a</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
III. Techniques

1. Two boys and two girls were chosen randomly as subjects for the observations.

2. The subjects were observed primarily when they were dealing with mathematics; occasionally they were observed when they were not dealing with mathematics.

(A more detailed description of the observations is found in section IV, Recording and Analysis.

3. The subjects were occasionally questioned by the researcher during the observations.

4. The recording of the observations and questioning was done in writing by the researcher concurrent with his observations. Only the observations dealing with mathematics were recorded.

5. The observational records consisted primarily of narrative descriptions of the children's activities and diagrams of the children's work.

6. The children were permitted to look at the observation records if they wished.

7. The researcher also observed and questioned other children as they dealt with mathematics and as they engaged in other activities, but the observations were not recorded.

8. The observation of a subject began when the researcher felt the subject's activity might involve mathematics.

9. When it appeared that two or more subjects might deal with mathematics concurrently, the researcher observed the one
for whom fewer observations had been made.

10. The researcher located himself near a subject when observing him, but did not disrupt the subject's activity. Usually the researcher was four to six feet from a subject.

11. The researcher concluded an observation of a subject when the subject no longer dealt with mathematics.

12. The researcher examined subjects' work and recorded as much of it as possible when he was unable to observe a subject directly.

13. Each evening the researcher reread the day's observations and added details when they had been omitted. Data were compiled to form a cumulative on-going record of each subject's mathematical activity.

14. The researcher kept daily logs of his activities and of important classroom happenings.

IV. Recording and Analysis

This section consists of questions which expressed the major goals of the study. Following each question is an explanation indicating what observational data were gathered to answer the question and how they were analyzed.

1 i. What mathematical concepts and skills do the children deal with?

ii. Does one child deal with the same concepts and skills as the other children?
iii. Do children spend different amounts of time dealing with mathematical concepts and skills?

The specific questions will be considered separately.

li. What mathematical concepts and skills do the children deal with?

When observing the subjects, the researcher made note of any behavior which might conceivably indicate which concepts or skills the subjects dealt with. In many instances what a subject wrote indicated what skill he dealt with. For example, when a subject wrote a long division problem it was clear that he was dealing with the long division algorithm. At other times the child's actions indicated what concept he was dealing with. One time a subject was practicing mentally the multiplication facts and was using the square grid with the indicated multiplications as cues. When the subject came to $4 \times 9$ she moved her finger over to $9 \times 4$, whispered 36, looked back at $4 \times 9$, and then went to $5 \times 9$. In this case it was assumed that the subject was dealing with commutativity of multiplication.

Sometimes it was not obvious what concepts or skills the subjects dealt with. One student constructed an octahedron using straws as edges and pipe cleaners at the vertices. The student first constructed a two dimensional figure consisting of adjacent congruent equilateral triangles by looking at a similar pattern in a booklet. Then he folded it to make the three-dimensional figure. It is clear that the student
dealt with the concept of equilateral triangle; however, it is not clear what other concepts he dealt with. In such instances the researcher often re-enacted the student's activity in an attempt to determine what concepts must be dealt with. In the above instance the researcher concluded that the subject also dealt with the concept of congruent triangles, and a multi-faceted concept called "the three-dimensional structure of an octahedron."

The concepts and skills identified were subsequently grouped into seven types: (1) classes, (2) relations, (3) number, (4) space, (5) probability, (6) mass, weight, and volume, and (7) time, movement and velocity. The times spent in dealing with the concepts and skills in each category were compared and percentages were calculated which indicated the types of concepts and skills with which the children primarily dealt.

III. Does one child deal with the same concepts and skills as the other children?

The concepts and skills dealt with were paired with the subject, or subjects, who dealt with them. The pairing yielded lists of concepts and skills dealt with by four subjects, three subjects, two subjects, and only one subject. The researcher then calculated the percentage of time spent on the concepts and skills dealt with by all four subjects, the percentage of time spent on the concepts and skills dealt with by three subjects, two subjects, and only one subject. Finally, a list of the concepts and skills dealt with by all four subjects, together
with the times spent on those concepts and skills, was constructed.

Do children spend different amounts of time dealing with mathematical concepts and skills?

During the observations the researcher recorded the amount of time subjects spent dealing with mathematical concepts and skills. Differences between the times, and ratios between the times, were calculated to compare the time spent by one subject with the times spent by other subjects.

2. What are the immediate sources of the mathematical concepts and skills? That is, do the concepts and skills evolve from the children's interaction with classmates, materials, the teacher, or from some other source?

The guideline used by the researcher to decide on the source of the concepts and skills was, "What caused the subject to deal with the concept or skill?" Occasionally the teacher would ask a subject to do some practice with subtraction, for example, and would write down a numerical problem to be done. In that instance the teacher would have been judged as the source of the skill involved, subtraction. If a subject decided he wanted to do a math project and used one of the activity cards from the math area, then the activity card would have been the source. Subsequently, the child may have chosen to do particular activities within the guidelines set by the activity card; in
that case the child himself would have been considered the source of
the concepts and skills. Sometimes a subject interacted with materials
and those materials became the source. For example, if a subject
played dominoes, then the dominoes were judged as the source of the
number pattern recognition involved.

In many instances more than one source was identified. Fre­
quently subjects interacted with classmates, and ideas from both
participants caused the subject to deal with certain concepts. Both
participants were then judged as sources.

Each time a subject dealt with a concept or skill a source was
identified. For the analysis the researcher counted the number of
times the following potential sources actually became sources for each
subject: (1) the subject himself, (2) the teacher, (3) concrete mater­
ials, (4) dittos, and (5) activity cards or non-evaluative dittos.
Percentages were calculated which indicated the relative frequencies
of the five sources identified. (For example, the subjects themselves
were involved as sources for 37 percent of the concepts and skills they
dealt with.)

3. The questions below refer to the children as they deal with
mathematical concepts and skills.

i. What activities do they engage in?

ii. Who else is involved in the children's activities?

iii. What role does the teacher have?

iv. With what concrete materials do they work?

v. How much time is spent working with concrete and semi­
concrete materials?
The specific questions will be considered separately.

31. What activities do they engage in?

The subjects' activities were classified in the following categories:

(a) WGCP - working given computation problems
(b) WOCP - working own computation problems
(c) PMG - playing mathematical games
(d) MGF - making geometric figures
(e) DoMProb - doing mathematical problems - non-computational
(f) DiMProb - discussing mathematical problems
(g) DoMProj - doing mathematical project
(h) DiMProj - discussing mathematical project
(i) E - explaining
(j) L - listening
(k) Wr - writing
(l) Di - discussing
(m) W - watching

For analysis the categories were combined somewhat. Time spent on the following five activities accounted for 98 percent of the children's work with mathematics: (1) working computation problems, (2) working non-computational problems, (3) doing mathematical projects (includes discussing mathematical projects), (4) playing mathematical games and, (5) making geometric figures. Ratios and differences between the times spent in each activity were calculated and compared to determine if
the times spent appeared to differ across subjects. Percentages were also calculated for each subject indicating the relative amounts of time spent in each of the five activities. Simple observations of the percentages were made to determine if the four subjects appeared to spend different percentages of time in each activity.

3i1. Who else is involved in the children's activities?

As the subjects dealt with mathematical concepts and skills, the researcher noted all persons involved in the subjects' activities. For analysis the times, and percentages of time, were calculated when each subject (1) was Alone, (2) was Alone or Alone Part-Time, (3) was Involved with a Classmate, and (4) was Involved with the Teacher. Percentages were also calculated which indicated the portion of time spent alone or spent with other persons by the group of four subjects. For example, the four subjects spent about half (56) percent of their time alone. The times that individual subjects spent alone and with other persons were compared by finding ratios and differences between the times. Individual subjects' percentages were compared by simple observation to determine if the subjects apparently spent different percentages of time alone and with other persons.

3i1i. What role does the teacher have?

The role of the teacher for the subject while the subject was involved in mathematical activity was classified in one of four
categories:

(a) Not involved - the teacher was not involved in the subject's activity
(b) Guide - the teacher aided or directed the subject
(c) Starter - the teacher initiated the subject's mathematical activity, or
(d) Evaluator - the teacher evaluated the subject's mathematical activity.

On several occasions the teacher served a combination of roles for subjects.

Percentages were calculated which indicated the proportion of the subjects' time that the teacher served in each of the roles identified. For example, for 35 percent of the time that the subjects dealt with mathematics, the teacher was Not Involved with the subjects.

Times, and percentages of time, were calculated which indicated the time and percentage of time that the teacher served in each role for individual subjects. Those times and percentages were compared to determine if the teacher appeared to spend different amounts of time, and different percentages of time in each role across subjects.

3iv. With what concrete materials do they work?

The researcher noted all concrete materials with which the subjects worked. Two types of analysis were carried out. First, the materials used were grouped into four categories: (1) materials used by all four subjects, (2) materials used by three subjects, (3) mater-
ials used by two subjects, and (4) materials used by only one subject. Second, the researcher determined which materials came from the math center. A percentage was calculated indicating what portion of the materials used in mathematical activity were located in the math center.

3v. How much time is spent working with concrete and semi-concrete materials?

The researcher recorded the time spent by each subject while working with concrete and semi-concrete materials. Percentages were calculated which represented the portion of time spent by each subject, and the group of subjects, using concrete and semi-concrete materials. Comparisons of individual percentages were made by simple observation to determine if the four subjects appeared to spend different percentages of time using materials while doing mathematics.

4. The questions below refer to the children as they deal with multiplication and its properties.

   i. How much time do they spend?
   ii. What activities do they engage in?
   iii. Who else is involved in the children's activities?
   iv. What role does the teacher have for the children?
   v. Do they work with concrete or semi-concrete materials?
   vi. How does their understanding of multiplication develop?
Each part of question 4 will be considered separately.

4i. How much time do they spend?

The amount of time spent on multiplication and its properties was determined from the observational records. The times, and the percentages of time, were compared to determine if the subjects appeared to spend different amounts of time, and different percentages of time, dealing with multiplication and its properties.

Calculations were also made to determine the percent of time spent on multiplication by the group of four subjects.

4ii. What activities do they engage in?

4iii. Who else is involved in the children's activities?

4iv. What role does the teacher have for the children?

For each of the questions, 4ii, 4iii, and 4iv, the data were analyzed exactly as they were for the corresponding questions in number 3. (The data, of course, were the times spent while dealing with multiplication and its properties.)

4v. Do they work with concrete or semi-concrete materials?

A list was made of the concrete and semi-concrete materials
used by each subject. Proportions were calculated which indicated the percentages of time that subjects used materials while dealing with multiplication and its properties.

4vi. How does their understanding of multiplication develop?

In addition to writing narrative records of each subject's multiplication-related activities, the researcher presented each subject with three sets of multiplication tasks at spaced intervals to more accurately assess his developmental understanding of multiplication. The researcher decided that, for purposes of the study, a subject understood multiplication if when given a counting task which lent itself to multiplication the subject used multiplication to perform the task. The tasks were given individually; the first set during the fifth week, the second set during the eighth week, and the third set during the eleventh week. Each set of tasks consisted of a counting problem interpreted as (1) equal addends, one interpreted as (2) an array, and one interpreted as (3) a Cartesian product. The tasks involved the use of concrete and semi-concrete materials. (The tasks are explained in Chapter 4 and in Appendix E and the records of the subjects work are given in Tables 21, 22, 23, and 24 in Chapter 4.)

The tasks were given to individual subjects when the subjects were not involved in other activities. The researcher simply told the subject that he (the researcher) had "a couple of problems" that the subject might enjoy working with, or the researcher asked the subject if he wanted to do "a couple of problems." The tasks were given at a location in the classroom where it was convenient and where other
students were not working. The subjects spent from five to fifteen minutes on each set of three tasks.

The procedure for an individual task was as follows:

1) The subject was presented with the materials and the counting task was described orally; then the subject was asked to perform the task. Paper and pencil were available if the subject wished to use them.

ii) The researcher observed the subject as he performed the task.

iii) The subject told his answer.

iv) The researcher asked the subject to explain how he got his answer; sometimes the researcher challenged the subject's answer or gave an absurd method of solution to motivate the subject to explain his solution.

v) The researcher accepted the subject's answer and method of solution and gave no indication of correctness.

Immediately following the three tasks, the researcher recorded in written form all relevant details regarding the subject's solutions.

The researcher examined both the narrative records and the results of the multiplication tasks to determine the children's development relative to each of the three interpretations of multiplication. Also, each subject's understanding relative to one interpretation was compared to his understanding relative to each of the other interpretations of multiplication. Finally, the students' performances on the multiplication tasks were interpreted in light of their performances of three basic multiplication facts practice sheets which the teacher asked them to complete at spaced intervals during the eleven weeks of the study.
CHAPTER IV
DATA ANALYSIS

Observations were made for eleven weeks and were recorded concurrently in written form. Records consisted primarily of narrative descriptions and drawings depicting the children's work. The data presented are based on examination and analysis of the written records.

Each question posed in the problem statement is discussed separately.

1 i. What mathematical concepts and skills do children deal with?

ii. Does one child deal with the same concepts and skills as the other children?

iii. Do children spend different amounts of time dealing with mathematical concepts and skills? Do they spend different amounts of time dealing with the same types of concepts and skills?

The concepts and skills were classified in seven categories: 1) classes, 2) relations, 3) numbers, 4) space, 5) probability, 6) mass, weight, and volume, and 7) time, movement, and velocity. The number concepts and skills (category 3) were further classified into six subgroups: i) operations and equations, ii) counting, iii) numeration and place value, iv) fractions, v) number properties, and vi) miscel-
lanes. The space concepts and skills (category 4) were divided into two subgroups: i) measurement, and ii) other geometric concepts and skills.

Table 1 lists, within each of the seven categories, under a) the concepts and skills dealt with by all four subjects, under b) the ones dealt with by three subjects, under c) the ones dealt with by two subjects, and under d) the ones dealt with by only one subject. Listed after each concepts or skill is the number of minutes spent by each subject who dealt with the concept or skill.

The concepts and skills listed in Table 1 represent units dealt with by subjects. For example, even though the addition of two three-digit numbers involves several subskills, such as the addition of single digits and regrouping ones and tens, the general skill was not divided into subskills unless the subject seemed to focus on them.

Table 2 summarizes the numbers of concepts and skills dealt with by the subjects in each of the seven categories and the times spent dealing with the concepts and skills. The total number of different concepts and skills dealt with by the subjects was 95. The total amount of time spent dealing with mathematical concepts and skills was 2322 minutes.

Obviously, the subjects dealt primarily with concepts and skills related to number and to space. The 71 different concepts and skills dealt with in those two categories represent 74.7 percent of the total number of concepts and skills dealt with. The same emphasis on dealing with number and spatial concepts and skills is reflected in the time spent by subjects dealing with concepts and skills involving number and
TABLE 1

CONCEPTS AND SKILLS DEALT WITH AND THE TIMES (IN MINUTES) SPENT DEALING WITH THOSE CONCEPTS AND SKILLS

<table>
<thead>
<tr>
<th>Category and concept or skill</th>
<th>NR</th>
<th>Subjects</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>RS</td>
</tr>
<tr>
<td>I. Classes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a) None</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b) Classification</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>c) None</td>
<td></td>
<td></td>
</tr>
<tr>
<td>d) Set</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Subset</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Set inclusion</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>II. Relations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a) Writing cents as dollars</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Greater than, less than</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Reading bar graphs</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>b) One-to-one correspondence</td>
<td>2</td>
<td>25</td>
</tr>
<tr>
<td>Seriation</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>c) Equivalent fraction</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>representations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>d) Similarity</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Multiple representations</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Reading scale drawings</td>
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<td></td>
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<tr>
<td>Transitive property of equality</td>
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<td>Congruency</td>
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<td>Multiple comparisons</td>
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<td>III. Number: (i) operations and equations</td>
<td>70</td>
<td>204</td>
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<td>a) Multiplication</td>
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<td>76</td>
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<td>Addition</td>
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<td>Subtraction</td>
<td>43</td>
<td>23</td>
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<tr>
<td>Division</td>
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<td></td>
</tr>
<tr>
<td>b) None</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>c) Equation</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>d) Factor</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Number: (ii) counting</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a) None</td>
<td>5</td>
<td>27</td>
</tr>
<tr>
<td>b) Counting by 1's</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>c) None</td>
<td></td>
<td></td>
</tr>
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TABLE 1 (Continued)

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<thead>
<tr>
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<th>Subjects</th>
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<tbody>
<tr>
<td></td>
<td>NR</td>
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<tr>
<td>d) Arithmetic progression</td>
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<tr>
<td>Infinite set notation</td>
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</tr>
<tr>
<td>Exponential growth</td>
<td>1</td>
</tr>
<tr>
<td>Estimation</td>
<td>5</td>
</tr>
<tr>
<td>Counting by 10's</td>
<td>2</td>
</tr>
<tr>
<td>Counting by 2's and 3's</td>
<td>4</td>
</tr>
<tr>
<td>Order of natural numbers</td>
<td>6</td>
</tr>
<tr>
<td>Number: (iii) numeration and place value</td>
<td></td>
</tr>
<tr>
<td>a) Place value</td>
<td>1</td>
</tr>
<tr>
<td>Writing numbers</td>
<td>3</td>
</tr>
<tr>
<td>b) None</td>
<td></td>
</tr>
<tr>
<td>c) Regrouping in subtraction</td>
<td>12</td>
</tr>
<tr>
<td>d) Reading numbers</td>
<td>1</td>
</tr>
<tr>
<td>Expanded numeral notation</td>
<td>1</td>
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<tr>
<td>Number: (iv) fractions</td>
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</tr>
<tr>
<td>a) Half</td>
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<tr>
<td>Fraction, basic concept</td>
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<tr>
<td>b) None</td>
<td></td>
</tr>
<tr>
<td>c) None</td>
<td></td>
</tr>
<tr>
<td>d) 1/4, 1 1/2</td>
<td>4</td>
</tr>
<tr>
<td>1/4, 1/8</td>
<td>1</td>
</tr>
<tr>
<td>Equivalent fractions</td>
<td>10</td>
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<tr>
<td>Subtraction of fractions</td>
<td>20</td>
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<tr>
<td>Order of fractions</td>
<td>4</td>
</tr>
<tr>
<td>Number: (v) number properties</td>
<td></td>
</tr>
<tr>
<td>a) None</td>
<td></td>
</tr>
<tr>
<td>b) Distributivity</td>
<td>1</td>
</tr>
<tr>
<td>Commutativity of multiplication</td>
<td>1</td>
</tr>
<tr>
<td>c) Associativity of addition</td>
<td>2</td>
</tr>
<tr>
<td>d) Commutativity of addition</td>
<td>1</td>
</tr>
<tr>
<td>Multiplicative property of 1</td>
<td>2</td>
</tr>
<tr>
<td>Number: (vi) miscellaneous</td>
<td></td>
</tr>
<tr>
<td>a) None</td>
<td></td>
</tr>
<tr>
<td>b) None</td>
<td></td>
</tr>
<tr>
<td>c) Average</td>
<td>11</td>
</tr>
<tr>
<td>d) Even and odd numbers</td>
<td>2</td>
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<tr>
<td>Sets of five</td>
<td>2</td>
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<td>Sets of four</td>
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(Continued on next page)
TABLE 1 (Continued)

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<tr>
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<th>NR</th>
<th>RS</th>
<th>EZ</th>
<th>YD</th>
</tr>
</thead>
<tbody>
<tr>
<td>IV. Space: (i) measurement</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>a) Linear measurement</td>
<td>15</td>
<td>1</td>
<td>40</td>
<td>44</td>
</tr>
<tr>
<td>b) None</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c) Translation of axes</td>
<td>2</td>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>d) Middle</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Triangle inequality</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Height</td>
<td>1</td>
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</tr>
<tr>
<td>Length</td>
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<td></td>
</tr>
<tr>
<td>Circumference</td>
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<td></td>
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<tr>
<td>Units of measure</td>
<td>16</td>
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<td></td>
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</tr>
</tbody>
</table>

Space: (ii) other geometric concepts and skills

a) None
b) None
c) Circle
   - Rectangle 2 1
   - Structure of right triangular prism
     - Line 1 1
     - Radial symmetry 1 1
     - Square 15 1
d) Horizontal line
   - Vertical line 1
   - Intersecting lines 1
   - Three-dimensional perspective 8
   - Parallel lines 1
   - Line segment 1
   - Angle 3
   - Structure of oblique triangular prism 5

Area
   - Diagonal line
   - Triangle
   - Equilateral triangle 15
   - Equilateral triangle patterns 5
   - Spiral 15
   - Base of solid figure 2
   - Structure of cube 5
   - Structure of octahedron 3
   - Structure of icosahedron 30

(Continued on next page)
<table>
<thead>
<tr>
<th>Category and concept or skill</th>
<th>Subjects</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>NR</td>
</tr>
<tr>
<td>Paper folding - 1/3, 1/2</td>
<td></td>
</tr>
<tr>
<td>Sphere</td>
<td></td>
</tr>
<tr>
<td>Structure of square pyramid</td>
<td></td>
</tr>
<tr>
<td>V. Probability</td>
<td></td>
</tr>
<tr>
<td>a) None</td>
<td></td>
</tr>
<tr>
<td>b) None</td>
<td></td>
</tr>
<tr>
<td>c) Probability, basic concept</td>
<td></td>
</tr>
<tr>
<td>d) None</td>
<td></td>
</tr>
<tr>
<td>VI. Mass, weight, and volume</td>
<td></td>
</tr>
<tr>
<td>a) None</td>
<td></td>
</tr>
<tr>
<td>b) None</td>
<td></td>
</tr>
<tr>
<td>c) None</td>
<td></td>
</tr>
<tr>
<td>d) Center of balance</td>
<td></td>
</tr>
<tr>
<td>Principle of number balance</td>
<td></td>
</tr>
<tr>
<td>Liquid measure equivalences</td>
<td></td>
</tr>
<tr>
<td>Weight</td>
<td></td>
</tr>
<tr>
<td>Weighing</td>
<td></td>
</tr>
<tr>
<td>VII. Time, movement, and velocity</td>
<td></td>
</tr>
<tr>
<td>a) None</td>
<td></td>
</tr>
<tr>
<td>b) None</td>
<td></td>
</tr>
<tr>
<td>c) Length of one minute</td>
<td></td>
</tr>
<tr>
<td>Sundial</td>
<td></td>
</tr>
<tr>
<td>d) None</td>
<td></td>
</tr>
</tbody>
</table>
TABLE 2

NUMBERS OF CONCEPTS AND SKILLS DEALT WITH (IN EACH OF SEVEN CATEGORIES) AND TIMES SPENT DEALING WITH THOSE CONCEPTS AND SKILLS

<table>
<thead>
<tr>
<th>Category</th>
<th>Subject</th>
<th>NR</th>
<th>T</th>
<th>RS</th>
<th>T</th>
<th>EZ</th>
<th>T</th>
<th>YD</th>
<th>T</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Classes</td>
<td>4</td>
<td>34</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>40</td>
<td>1</td>
<td>30</td>
<td>4</td>
<td>104</td>
</tr>
<tr>
<td>2. Relations</td>
<td>7</td>
<td>23</td>
<td>4</td>
<td>12</td>
<td>8</td>
<td>67</td>
<td>7</td>
<td>44</td>
<td>12</td>
<td>146</td>
</tr>
<tr>
<td>3i. Number: Operations and Equations</td>
<td>6</td>
<td>243</td>
<td>4</td>
<td>352</td>
<td>5</td>
<td>394</td>
<td>4</td>
<td>376</td>
<td>6</td>
<td>1365</td>
</tr>
<tr>
<td>3ii. Number: Counting</td>
<td>4</td>
<td>12</td>
<td>4</td>
<td>10</td>
<td>3</td>
<td>35</td>
<td>2</td>
<td>36</td>
<td>9</td>
<td>93</td>
</tr>
<tr>
<td>3iii. Number: Numeration and Place Value</td>
<td>4</td>
<td>6</td>
<td>3</td>
<td>19</td>
<td>3</td>
<td>20</td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>51</td>
</tr>
<tr>
<td>3iv. Number: Fractions</td>
<td>3</td>
<td>13</td>
<td>2</td>
<td>14</td>
<td>5</td>
<td>43</td>
<td>3</td>
<td>7</td>
<td>7</td>
<td>77</td>
</tr>
<tr>
<td>3v. Number: Number Properties</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>8</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>5</td>
<td>19</td>
</tr>
<tr>
<td>3vi. Number: Miscellaneous</td>
<td>2</td>
<td>13</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>15</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>31</td>
</tr>
<tr>
<td>3. Number (Totals)</td>
<td>22</td>
<td>291</td>
<td>17</td>
<td>405</td>
<td>18</td>
<td>508</td>
<td>15</td>
<td>432</td>
<td>36</td>
<td>1636</td>
</tr>
<tr>
<td>4i. Space: Measurement</td>
<td>3</td>
<td>18</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>40</td>
<td>6</td>
<td>66</td>
<td>8</td>
<td>126</td>
</tr>
<tr>
<td>4ii. Space: Other Geometric Concepts and Skills</td>
<td>13</td>
<td>51</td>
<td>1</td>
<td>30</td>
<td>14</td>
<td>108</td>
<td>5</td>
<td>10</td>
<td>27</td>
<td>199</td>
</tr>
<tr>
<td>4. Space (Totals)</td>
<td>16</td>
<td>69</td>
<td>3</td>
<td>32</td>
<td>15</td>
<td>148</td>
<td>11</td>
<td>76</td>
<td>35</td>
<td>325</td>
</tr>
<tr>
<td>5. Probability</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>6. Mass, Weight, Volume</td>
<td>2</td>
<td>26</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>30</td>
<td>2</td>
<td>18</td>
<td>5</td>
<td>74</td>
</tr>
<tr>
<td>7. Time, Movement, Velocity</td>
<td>1</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>25</td>
<td>2</td>
<td>31</td>
</tr>
</tbody>
</table>

Totals | 52 | 448 | 24 | 449 | 45 | 799 | 39 | 626 | 95 | 2322 |

N = Number of concepts and skills dealt with.
T = Time spent (in minutes).
N* = Number of different concepts and skills dealt with by the four subjects.
space (Table 2). The subjects spent 1961 minutes out of a total of 2322 minutes (84.5 percent) dealing with number and spatial concepts and skills.

The total number of concepts for NR in Table 2 is somewhat misleading. Although NR dealt with 52 different concepts and skills, 11 of them were dealt with only once. He completed a diagnostic worksheet prepared by the teacher which the other subjects did not complete. A more valid indication of the extent of NR's mathematical activity is the total amount of time he spent dealing with mathematical concepts and skills (Table 2). The times indicate that although NR dealt with more concepts and skills than the other subjects he spent the least amount of time.

lili. Does one child deal with the same concepts and skills as the other children?

Examination of Table 1 reveals that of the 95 different concepts and skills, 12 were dealt with by all four subjects, 7 were dealt with by three subjects, 15 by two subjects, and 61 were dealt with by only one subject. However, a very different result is obtained if the times spent are considered (Table 3).

Thus, even though only 12 of the 95 concepts (12.6 percent) were dealt with by all four subjects, the time spent on those 12 concepts represented more than two-thirds (68.1 percent) of the total time spent on mathematics. Table 4 lists those 12 concepts along with the time spent on each.
TABLE 3
NUMBERS OF CONCEPTS AND SKILLS DEALT WITH BY VARIOUS NUMBERS OF SUBJECTS, THE TIMES SPENT, AND THE PERCENTAGES OF TIME SPENT

<table>
<thead>
<tr>
<th>Concepts and Skills Dealt With by All Four Subjects</th>
<th>Number</th>
<th>Time Spent</th>
<th>Percent of Time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>12</td>
<td>1581</td>
<td>68.1</td>
</tr>
<tr>
<td>Concepts and Skills Dealt With by Three Subjects</td>
<td>7</td>
<td>236</td>
<td>10.2</td>
</tr>
<tr>
<td>Concepts and Skills Dealt With by Two Subjects</td>
<td>15</td>
<td>144</td>
<td>6.2</td>
</tr>
<tr>
<td>Concepts and Skills Dealt With by only One Subject</td>
<td>61</td>
<td>361</td>
<td>15.5</td>
</tr>
</tbody>
</table>

¹Times given are in minutes.
²Percent of total time spent in mathematics

Further examination of the data in Table 4 reveals that 58.4 percent of the time spent in mathematics concerned the four basic operations: multiplication, addition, subtraction, and division.

liii. Do children spend different amounts of time dealing with mathematical concepts and skills?

The times spent by NR, RS, EZ, and YD were 448, 449, 626, and 799 minutes respectively (Table 2). The differences in time spent are
<table>
<thead>
<tr>
<th>Category</th>
<th>Concept</th>
<th>Time (in Minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>II. Relations</td>
<td>Writing cents as dollars</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Greater than, less than</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Reading bar graphs</td>
<td>37</td>
</tr>
<tr>
<td>III. Number</td>
<td>Multiplication</td>
<td>550</td>
</tr>
<tr>
<td></td>
<td>Addition</td>
<td>297</td>
</tr>
<tr>
<td></td>
<td>Subtraction</td>
<td>231</td>
</tr>
<tr>
<td></td>
<td>Division</td>
<td>279</td>
</tr>
<tr>
<td></td>
<td>Place value</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>Writing numbers</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Half</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>Fraction</td>
<td>11</td>
</tr>
<tr>
<td>IV. Space</td>
<td>Linear Measurement</td>
<td>100</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>1581</td>
</tr>
</tbody>
</table>

prominent. For example, EZ spent almost six more hours on mathematics (1.8 times as much) than did NR or RS, and almost 3 hours more than YD (1.3 times as much).

2. What are the immediate sources of the mathematical concepts and skills? That is, do the concepts and skills evolve from the children's interaction with classmates, materials, the teacher, or from some other source?

Table 5 indicates the frequencies of the various sources for the concepts and skills dealt with by subjects. An overall profile of
### TABLE 5

Frequencies of the various sources for the concepts and skills dealt with by subjects and the percentages that potential sources were actually sources for concepts and skills dealt with.

<table>
<thead>
<tr>
<th>Subject</th>
<th>Number of Concepts and Skills Dealt With&lt;sup&gt;1&lt;/sup&gt;</th>
<th>Subject Involved</th>
<th>Teacher Involved</th>
<th>Concrete Materials Involved</th>
<th>Ditto Involved</th>
<th>A. C. and N.-E. Dittos Involved&lt;sup&gt;2&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>NR</td>
<td>107&lt;sup&gt;3&lt;/sup&gt;</td>
<td>37</td>
<td>17</td>
<td>19</td>
<td>51</td>
<td>10</td>
</tr>
<tr>
<td>RS</td>
<td>81</td>
<td>25</td>
<td>17</td>
<td>8</td>
<td>32</td>
<td>9</td>
</tr>
<tr>
<td>EZ</td>
<td>131</td>
<td>45</td>
<td>22</td>
<td>38</td>
<td>38</td>
<td>21</td>
</tr>
<tr>
<td>YD</td>
<td>105</td>
<td>51</td>
<td>21</td>
<td>9</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>Totals</td>
<td>424</td>
<td>158</td>
<td>77</td>
<td>74</td>
<td>151</td>
<td>50</td>
</tr>
</tbody>
</table>

**Percentages**<sup>4</sup>

|                  | 37 | 18 | 17 | 36 | 12 |

---

<sup>1</sup>Includes multiple dealings with the same concept.

<sup>2</sup>Activity Cards and Non-Evaluative Dittos Involved.

<sup>3</sup>The sum of the frequencies of sources for any one subject does not equal the total number of concepts and skills dealt with by that subject because for many concepts and skills two or more sources were identified.

<sup>4</sup>Calculated on a basis of 424 concepts and skills dealt with.
the sources for the concepts and skills dealt with is given by the percentages in Table 5. One interesting result is that subjects were involved as sources for 37 percent of the concepts and skills dealt with, whereas the teacher was involved as a source for only 18 percent.

The high percentage (36 percent) on concepts and skills for which dittos served as sources is a reflection of the teacher's occasional practice of diagnosing children's understanding by using dittoed worksheets. The extent to which dittos served as sources of the subjects' mathematical activity is more accurately reflected in the percent of time spent by the teacher in evaluation activities (17.7 percent), of which evaluation with dittos was only a part. (See question 3iii.)

3. The questions below refer to the children as they are dealing with mathematical concepts and skills.

i. What activities do they engage in?

ii. Who else is involved in the children's activities?

iii. What role does the teacher have?

iv. With what concrete materials do they work?

v. How much time is spent working with concrete and semi-concrete materials?

Each part of question 3 will be considered separately.

3i. What activities do they engage in?
Table 6 gives the number of minutes spent in various activities by each subject and the percentages of time spent in each activity. The percent of time spent doing computation problems (48 percent) partially reflected the teacher's practice of having the students spend 15 minutes two or three times a week doing computation practice.

The summary percentages in Table 6 indicate that subjects spent about half their time (48 percent) doing computation problems, about a fourth (25 percent) doing math projects, and about an eighth (12 percent) doing math problems.

**TABLE 6**

TIME SPENT (IN MINUTES) IN VARIOUS MATHEMATICAL ACTIVITIES AND PERCENT OF TIME SPENT IN VARIOUS MATHEMATICAL ACTIVITIES

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T</td>
<td>%</td>
<td>T</td>
<td>%</td>
<td>T</td>
</tr>
<tr>
<td>NR</td>
<td>174</td>
<td>39</td>
<td>104</td>
<td>26</td>
<td>65</td>
</tr>
<tr>
<td>RS</td>
<td>295</td>
<td>66</td>
<td>42</td>
<td>9</td>
<td>113</td>
</tr>
<tr>
<td>EZ</td>
<td>325</td>
<td>41</td>
<td>165</td>
<td>21</td>
<td>74</td>
</tr>
<tr>
<td>YD</td>
<td>330</td>
<td>53</td>
<td>268</td>
<td>43</td>
<td>21</td>
</tr>
<tr>
<td>Totals</td>
<td>1124</td>
<td>579</td>
<td>273</td>
<td>12</td>
<td>228</td>
</tr>
<tr>
<td>Percentages(^1)</td>
<td>48</td>
<td>25</td>
<td>12</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

\(^1\)The percentages total more than 100 percent because occasionally subjects participated in two activities simultaneously.
The times spent in any one activity seem to vary extensively across subjects. For example, YD spent about twice as much time (2 1/2 hours more) Doing Computation Problems than did NR. EZ spent twice as much time Making Geometric Figures as did NR and ten times as much as did YD. The overall impression is that subjects spent different amounts of time in each of the activities.

One could expect the subjects to have spent different amounts of time in each activity, since the subjects spent different amounts of time dealing with mathematics. Comparing percentages of time spent in each activity eliminates that possible distortion. The results, however, are analogous. The subjects spent different percentages of time involved in each activity. The most striking example is YD spent about 43 percent of her time Doing Math Projects as compared to 26 percent for NR, 21 percent for EZ and only 9 percent for RS.

3ii. Who else is involved in the children's activities?

The data in Table 7 indicates the number of minutes that each subject worked alone while dealing with mathematics and the number of minutes other persons were involved with him. Table 7 also gives the percentages of time spent alone and with other persons.

An overall indication of the degree to which other persons were involved in subjects' activities is obtained by considering the summary percentages in Table 7. Two percentages are prominent. First, the subjects were alone for a total of 56.3 percent of the time. Second, the teacher was involved in the subjects' activities only 17.3
TABLE 7
TIMES (IN MINUTES) SPENT, AND PERCENTAGES OF TIME SPENT, BY SUBJECTS WORKING ALONE OR WITH OTHERS

<table>
<thead>
<tr>
<th>Subject</th>
<th>Persons Involved with Subject</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Alone</td>
<td>Alone</td>
<td>Alone</td>
<td>Classmates</td>
<td>Teacher</td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>%</td>
<td>T</td>
<td>%</td>
<td>T</td>
</tr>
<tr>
<td>NR</td>
<td>254</td>
<td>57</td>
<td>314</td>
<td>70</td>
<td>61</td>
</tr>
<tr>
<td>RS</td>
<td>281</td>
<td>63</td>
<td>337</td>
<td>75</td>
<td>89</td>
</tr>
<tr>
<td>EZ</td>
<td>520</td>
<td>55</td>
<td>555</td>
<td>70</td>
<td>136</td>
</tr>
<tr>
<td>YD</td>
<td>253</td>
<td>41</td>
<td>276</td>
<td>44</td>
<td>343</td>
</tr>
<tr>
<td>Totals</td>
<td>1308</td>
<td>1482</td>
<td>629</td>
<td>401</td>
<td></td>
</tr>
<tr>
<td>%</td>
<td>56</td>
<td>64</td>
<td>27</td>
<td>17</td>
<td></td>
</tr>
</tbody>
</table>

percent of the time.

Individual subjects apparently spent different amounts of time, and different percentages of time Alone, with Classmates, and with the Teacher. Time-wise EZ spent about twice as much time alone (4 hours more) than did NR or RS. However, percentage-wise the three EZ, NR, and RS spent about the same proportions of time Alone. Percentage differences were prominent when comparing YD to the other three subjects. She spent a much smaller portion of her time Alone than did the other three and a much greater portion of her time involved with Classmates.
3iii. What role does the teacher have?

As each subject was involved in mathematical activity, the researcher noted the role the teacher served for the subject. Table 8 summarizes the amount of time spent, and the percentage of time spent, by the teacher serving various roles for the four subjects.

The percentage data reveal that the teacher served different roles to various extents. For example, the teacher served as an Evaluator only 18 percent of the time whereas she was Not Involved in the subject's activity 35 percent of the time.

<table>
<thead>
<tr>
<th>TABLE 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>TIME SPENT, AND PERCENTAGE OF SUBJECT'S TIME SPENT, BY TEACHER IN VARIOUS ROLES</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Teacher Role</th>
<th>Subject</th>
<th>Not Involved</th>
<th>Guide</th>
<th>Evaluator</th>
<th>Starter</th>
<th>Guide and Evaluator</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>T</td>
<td>%</td>
<td>T</td>
<td>%</td>
<td>T</td>
</tr>
<tr>
<td>NR</td>
<td>131</td>
<td>29</td>
<td>42</td>
<td>9</td>
<td>118</td>
<td>26</td>
</tr>
<tr>
<td>RS</td>
<td>77</td>
<td>17</td>
<td>118</td>
<td>36</td>
<td>98</td>
<td>22</td>
</tr>
<tr>
<td>EZ</td>
<td>268</td>
<td>34</td>
<td>77</td>
<td>10</td>
<td>106</td>
<td>13</td>
</tr>
<tr>
<td>YD</td>
<td>338</td>
<td>54</td>
<td>42</td>
<td>7</td>
<td>89</td>
<td>14</td>
</tr>
<tr>
<td>Totals</td>
<td>814</td>
<td>279</td>
<td>441</td>
<td>85</td>
<td>709</td>
<td>31</td>
</tr>
</tbody>
</table>

% 35 12 18 31 4
The data in Table 8 also suggest that the teacher spent different amounts of time (and different percentages of time) in each role for the four subjects. The most striking differences result from comparing RS's times and percentages with the times and percentages of the other three subjects. The teacher was much more involved with RS when RS was doing mathematics than she was for the other subjects. For example, the teacher was a Guide for RS during 36 percent of the time that RS worked with mathematics, as compared percentages of 9, 10, and 7 for NR, EZ, and YD, respectively.

3iv. With what concrete materials do they work? Do the subjects work with the same concrete materials? What percentage of the materials with which children work when dealing with mathematical concepts and skills are found in the mathematics center?

The data used to answer the three questions is presented in five tables. Table 9 consists of the concrete materials dealt with by four subjects, three subjects, and two subjects. To the right of each item an "x" is placed below the name of a subject if he worked with that material. Table 10 consists of the materials dealt with only by NR, Table 11 consists of the materials dealt with by RS, Table 12 consists of materials dealt with by EZ, and Table 13 consists of materials used by YD. An asterisk before an item indicates that the material was not found in the mathematics center.
## TABLE 9

MATERIALS USED BY FOUR, THREE, OR TWO SUBJECTS

<table>
<thead>
<tr>
<th>Concrete Material</th>
<th>NR</th>
<th>RS</th>
<th>EZ</th>
<th>YD</th>
</tr>
</thead>
<tbody>
<tr>
<td>12&quot; Ruler</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Pennies</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>*Scissors</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Factor game</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>Wooden block</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
</tr>
<tr>
<td>*Pipe cleaners</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>*Straws</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>*Scrap wood</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Yardstick</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>*Construction paper</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Three-dimensional models</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Graph paper</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Place value strips</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beans</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>*Classmate's watch</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>.100 gram balance</td>
<td></td>
<td>x</td>
<td></td>
<td></td>
</tr>
<tr>
<td>*Pencil (not used for writing)</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Stopwatch</td>
<td></td>
<td></td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Meterstick</td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>Wooden sectors of circle</td>
<td></td>
<td></td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>*Xylophone</td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>*Crayons</td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>Compass</td>
<td>x</td>
<td></td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>Dominoes</td>
<td>x</td>
<td></td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>Two dimensional figures</td>
<td>x</td>
<td></td>
<td>x</td>
<td></td>
</tr>
</tbody>
</table>

* = Material was not found in the mathematics center.
### TABLE 10
MATERIALS USED ONLY BY NR

<table>
<thead>
<tr>
<th>Material</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geoblocks</td>
<td></td>
</tr>
<tr>
<td>Perspective drawings</td>
<td></td>
</tr>
<tr>
<td>*Marionette</td>
<td></td>
</tr>
<tr>
<td>*Classroom wall</td>
<td></td>
</tr>
<tr>
<td>*Paint</td>
<td></td>
</tr>
<tr>
<td>*Lazy Susan</td>
<td>1&quot; cubes</td>
</tr>
<tr>
<td>*Buttons</td>
<td>*Masking tape</td>
</tr>
</tbody>
</table>

* = Material was not found in the mathematics center.

### TABLE 11
MATERIALS USED ONLY BY RS

<table>
<thead>
<tr>
<th>Material</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Airlines map</td>
<td>4&quot; x 6&quot; mirror</td>
</tr>
<tr>
<td>Road map</td>
<td>*Desk top</td>
</tr>
<tr>
<td>*Desk top</td>
<td>Flash cards</td>
</tr>
<tr>
<td>Jar</td>
<td>Cross-number puzzles</td>
</tr>
<tr>
<td>*3&quot; x 5&quot; cards</td>
<td></td>
</tr>
<tr>
<td>1&quot; square</td>
<td></td>
</tr>
</tbody>
</table>

* = Material was not found in the mathematics center.
TABLE 12

MATERIALS USED ONLY BY EZ

<table>
<thead>
<tr>
<th>Material Used Only by EZ</th>
</tr>
</thead>
<tbody>
<tr>
<td>♦ Basketball scorebook</td>
</tr>
<tr>
<td>♦ Saw</td>
</tr>
<tr>
<td>♦ Quizmo</td>
</tr>
<tr>
<td>♦ Water</td>
</tr>
<tr>
<td>♦ Liquid measuring containers</td>
</tr>
<tr>
<td>♦ Cottage cheese cartons</td>
</tr>
<tr>
<td>♦ Marbles</td>
</tr>
<tr>
<td>&quot;Fractions Are as Easy as Pie&quot;</td>
</tr>
<tr>
<td>♦ Cloth</td>
</tr>
<tr>
<td>♦ String</td>
</tr>
<tr>
<td>♦ Needle</td>
</tr>
<tr>
<td>♦ Baseball cards</td>
</tr>
<tr>
<td>♦ Ohio Almanac</td>
</tr>
<tr>
<td>♦ Wire ties</td>
</tr>
<tr>
<td>♦ Dictionary</td>
</tr>
<tr>
<td>♦ Record player</td>
</tr>
<tr>
<td>♦ Newspaper</td>
</tr>
<tr>
<td>♦ 2&quot; cardboard squares</td>
</tr>
<tr>
<td>♦ Dice</td>
</tr>
<tr>
<td>♦ Pendulum</td>
</tr>
</tbody>
</table>

* = Material was not found in the mathematics center.

TABLE 13

MATERIALS USED ONLY BY YD

<table>
<thead>
<tr>
<th>Material Used Only by YD</th>
</tr>
</thead>
<tbody>
<tr>
<td>♦ Jigsaw puzzle</td>
</tr>
<tr>
<td>♦ Seashells</td>
</tr>
<tr>
<td>♦ Macaroni</td>
</tr>
<tr>
<td>♦ Large rock</td>
</tr>
<tr>
<td>♦ Sand</td>
</tr>
<tr>
<td>♦ Glue</td>
</tr>
<tr>
<td>♦ Rice</td>
</tr>
<tr>
<td>♦ Plastic sphere</td>
</tr>
<tr>
<td>♦ Checkerboard</td>
</tr>
<tr>
<td>♦ Rubber bands</td>
</tr>
<tr>
<td>♦ Chair</td>
</tr>
<tr>
<td>♦ Cloth measuring tape</td>
</tr>
<tr>
<td>♦ Shoes</td>
</tr>
<tr>
<td>♦ Violin</td>
</tr>
<tr>
<td>♦ Classmate</td>
</tr>
<tr>
<td>♦ Metal ring</td>
</tr>
<tr>
<td>♦ Washer</td>
</tr>
<tr>
<td>♦ Screw</td>
</tr>
<tr>
<td>♦ Multiplication practice machine</td>
</tr>
<tr>
<td>♦ Hourglass</td>
</tr>
<tr>
<td>♦ Paper plate</td>
</tr>
<tr>
<td>♦ Floor</td>
</tr>
</tbody>
</table>

* = Material was not found in the mathematics center.
The tables indicate that NR used 28 different materials, RS used 15, EZ used 38, and YD used 44. The small number of materials used by RS may reflect the high percentage of time (66 percent) spent doing computation problems.

The lists also reveal that of the 85 concrete materials dealt with, only 25 (29 percent) were used by more than one subject. Furthermore, only three items were used by all four subjects.

Enumeration of the materials not found in the mathematics center yields a total of 43. Thus, about half (51 percent) of the concrete materials used by the subjects were not found in the mathematics center.

3v. How much time is spent working with concrete and semi-concrete materials?

Table 14 gives the times spent by subjects while working with concrete and semi-concrete materials.

Table 14

<table>
<thead>
<tr>
<th>Subject</th>
<th>NR</th>
<th>RS</th>
<th>EZ</th>
<th>YD</th>
<th>Total</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time Spent</td>
<td>247</td>
<td>161</td>
<td>443</td>
<td>334</td>
<td>1185</td>
<td>51</td>
</tr>
<tr>
<td>Percentages</td>
<td>55</td>
<td>36</td>
<td>55</td>
<td>53</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The data indicate that the four subjects spent about half of their time (51 percent) working with concrete and semi-concrete materials. Three of the subjects NR, EZ, and YD spent more time, and a bigger percentage of time, working with materials than did RS.

4. The questions below refer to the children as they deal with multiplication and its properties.

i. How much time do they spend?

ii. What activities do they engage in?

iii. Who else is involved in the children's activities?

iv. What role does the teacher have for the children?

v. Do they work with concrete or semi-concrete materials?

vi. How does their understanding of multiplication develop?

Each question will be considered separately.

4i. How much time do they spend?

Table 15 gives the time spent by each subject on multiplication and its properties, the percent of time spent on multiplication and its properties. The data suggest that RS spent more time, and a higher percentage of time, dealing with multiplication than did the other three subjects.

The total time spent by the subjects in dealing with multiplication and its properties (552 minutes) represented 24 percent of the
TABLE 15

TIME SPENT (AND PERCENTAGE OF TOTAL TIME SPENT IN MATHEMATICS) BY SUBJECTS WHILE DEALING WITH MULTIPLICATION

<table>
<thead>
<tr>
<th>Subject</th>
<th>NR</th>
<th>RS</th>
<th>EZ</th>
<th>YD</th>
<th>Total</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>68</td>
<td>210</td>
<td>152</td>
<td>122</td>
<td>522</td>
<td>24</td>
</tr>
<tr>
<td>Percent</td>
<td>15</td>
<td>47</td>
<td>19</td>
<td>20</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

total time spent (2322 minutes) in dealing with mathematical concepts and skills. The concept dealt with the next largest amount of time was addition (13 percent).

4ii. What activities do they engage in?

Table 16 gives data used to answer question 4ii.

The data in Table 16 indicates that when subjects dealt with multiplication they were primarily doing computational problems. It is also interesting to note that the subjects dealt with multiplication in the context of math projects (11 percent) and while doing non-computational math problems (10 percent).

The data in Table 16 also suggest that subjects, when dealing with multiplication, spent different amounts of time, and different percentages of time, in each of several activities. For example, NR spent much less time, and a smaller percentage of time, Doing Computational Problems than did the other three subjects.
**TABLE 16**

**TIME SPENT (AND PERCENTAGE OF TIME SPENT IN DEALING WITH MULTIPLICATION) BY SUBJECTS WHILE INVOLVED IN VARIOUS ACTIVITIES IN LEARNING MULTIPLICATION**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>NR</td>
<td>38 T 56 %</td>
<td>5 T 7 %</td>
<td>22 T 32 %</td>
<td>3 T 4 %</td>
<td>0 T 0 %</td>
</tr>
<tr>
<td>RS</td>
<td>172 T 82 %</td>
<td>9 T 4 %</td>
<td>17 T 8 %</td>
<td>0 T 0 %</td>
<td>23 T 10 %</td>
</tr>
<tr>
<td>EZ</td>
<td>113 T 74 %</td>
<td>1 T 1 %</td>
<td>16 T 11 %</td>
<td>0 T 0 %</td>
<td>23 T 13 %</td>
</tr>
<tr>
<td>YD</td>
<td>89 T 73 %</td>
<td>43 T 35 %</td>
<td>0 T 0 %</td>
<td>1 T 1 %</td>
<td>0 T 0 %</td>
</tr>
</tbody>
</table>

**Totals** | 412 T 58 % | 55 T 10 % | 4 T 7 % | 40 T 11 %

% = Percentage of total time spent in dealing with multiplication

**4i.ii. Who else is involved in the children's activities?**

While the subjects dealt with multiplication and its properties, the researcher noted if other persons were involved with the subject. Table 17 gives the amounts of time that subjects were alone and the amounts of time that other persons were involved with the subjects.

The percentages given in Table 17 indicate that subjects were predominantly alone while dealing with multiplication. It is significant to note that the teacher was directly involved with the subjects only 12 percent of the time.
TABLE 17
AMOUNT OF TIME SPENT ALONE AND WITH OTHER PERSONS BY SUBJECTS WHILE DEALING WITH MULTIPLICATION AND ITS PROPERTIES

<table>
<thead>
<tr>
<th>Subject</th>
<th>Alone</th>
<th>Alone Part-time</th>
<th>Classmates</th>
<th>Teacher</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T</td>
<td>%</td>
<td>T</td>
<td>%</td>
</tr>
<tr>
<td>NR</td>
<td>254</td>
<td>57</td>
<td>314</td>
<td>70</td>
</tr>
<tr>
<td>RS</td>
<td>281</td>
<td>63</td>
<td>337</td>
<td>75</td>
</tr>
<tr>
<td>EZ</td>
<td>520</td>
<td>65</td>
<td>555</td>
<td>70</td>
</tr>
<tr>
<td>YD</td>
<td>253</td>
<td>41</td>
<td>276</td>
<td>44</td>
</tr>
<tr>
<td>Totals</td>
<td>1308</td>
<td></td>
<td>1482</td>
<td></td>
</tr>
<tr>
<td>%</td>
<td>56</td>
<td></td>
<td>64</td>
<td></td>
</tr>
</tbody>
</table>

The data in Table 17 reflect the same way of working with multiplication as did Table 7 for working with mathematics generally. YD spent her time working with classmates. The other three subjects worked alone.

4iv. What role does the teacher have for the children?

As a subject dealt with multiplication and its properties, the researcher noted the role that the teacher served for that subject. Table 18 summarizes the amount of time spent by the teacher in various roles for each subject.
TABLE 18
TIME SPENT BY THE TEACHER IN VARIOUS ROLES FOR EACH SUBJECT, AND PERCENTAGE OF SUBJECT'S TIME SPENT BY THE TEACHER, WHILE THE SUBJECT DEALT WITH MULTIPLICATION AND ITS PROPERTIES

<table>
<thead>
<tr>
<th>Subject</th>
<th>Not Involved T</th>
<th>Not Involved %</th>
<th>Guide T</th>
<th>Guide %</th>
<th>Evaluator T</th>
<th>Evaluator %</th>
<th>Starter T</th>
<th>Starter %</th>
<th>Guide and Evaluator T</th>
<th>Guide and Evaluator %</th>
</tr>
</thead>
<tbody>
<tr>
<td>NR</td>
<td>9</td>
<td>13</td>
<td>3</td>
<td>4</td>
<td>28</td>
<td>41</td>
<td>12</td>
<td>18</td>
<td>16</td>
<td>24</td>
</tr>
<tr>
<td>RS</td>
<td>29</td>
<td>14</td>
<td>24</td>
<td>11</td>
<td>47</td>
<td>22</td>
<td>110</td>
<td>52</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>EZ</td>
<td>34</td>
<td>22</td>
<td>10</td>
<td>7</td>
<td>55</td>
<td>36</td>
<td>35</td>
<td>35</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>YD</td>
<td>45</td>
<td>37</td>
<td>25</td>
<td>21</td>
<td>41</td>
<td>34</td>
<td>11</td>
<td>9</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Totals</td>
<td>117</td>
<td>62</td>
<td>171</td>
<td>31</td>
<td>186</td>
<td>34</td>
<td>16</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>%</td>
<td>22</td>
<td>11</td>
<td>31</td>
<td>34</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One statistic in Table 18 stands out more than any other. For RS the teacher served as a Starter for 110 minutes of the total time spent on multiplication (210 minutes) by RS. That 110 minutes represents 52 percent of the time spent dealing with multiplication by RS.

The data in Table 18 again suggests that the teacher served in each role for different amounts of time across subjects, and for different percentages of each subject's time.

4v. Do they work with concrete or semi-concrete materials?

As subjects dealt with multiplication and its properties, the
researcher noted the materials with which subjects worked. Table 19 gives the materials (other than pencil and paper) used by subjects while dealing with multiplication and its properties.

**TABLE 19**

**CONCRETE AND SEMI-CONCRETE MATERIALS USED BY SUBJECTS WHILE DEALING WITH MULTIPLICATION AND ITS PROPERTIES.**

<table>
<thead>
<tr>
<th>NR</th>
<th>RS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yardstick</td>
<td>Road map</td>
</tr>
<tr>
<td>Yardstick-Meterstick</td>
<td>Place-value strips</td>
</tr>
<tr>
<td>Wooden block</td>
<td>Factor game</td>
</tr>
<tr>
<td>Factor Game</td>
<td>Pennies</td>
</tr>
<tr>
<td>Graph paper</td>
<td>1&quot; squares</td>
</tr>
<tr>
<td>Picture</td>
<td>3&quot; x 5&quot; cards</td>
</tr>
<tr>
<td></td>
<td>Flash cards</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>EZ</th>
<th>YD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yardstick</td>
<td>Multiplication practice</td>
</tr>
<tr>
<td>Pennies</td>
<td>machine</td>
</tr>
<tr>
<td>Wooden block</td>
<td>Rulers</td>
</tr>
<tr>
<td>Number line grid</td>
<td>Wood</td>
</tr>
<tr>
<td>Factor game</td>
<td></td>
</tr>
</tbody>
</table>

The amount of time and percentage of time each of the subjects used concrete or semi-concrete materials while dealing with multiplication and its properties is given in Table 20.

The sum of the times spent by subjects while using concrete and semi-concrete materials and dealing with multiplication, 99 minutes, represents 17 percent of the total time spent dealing with multiplication.
4vi. How does their understanding of multiplication develop?

Three sources of information were used to assess the subjects' developing understanding of multiplication: 1) basic facts exercises given by the teacher, 2) multiplication tasks given by the researcher, and 3) observational records kept by the researcher. First, the details of the basic facts practice sheets and the multiplication tasks will be given. Then, each subject's developing understanding of multiplication will be described and some general trends presented.

**Basic Facts Practice Sheets**

Independent of the research, the teacher gave the class two
practice sheets involving 100 multiplication combinations. The practice sheets were given the first and seventh weeks of the study. At the researcher's request the teacher gave the students the practice sheet again during the eleventh week. Each time students timed themselves and were told that they could skip the "twelves" if they did not know them. The practice sheets were used for practice and diagnosis, and students sometimes made flash cards for combinations which they missed. Appendix F-2 gives NR's responses to the practice sheets, Appendix F-3 gives RS's responses, Appendix F-4 gives EZ's responses, and Appendix F-5 gives YD's responses.

**Multiplication Tasks**

The researcher gave three sets of multiplication tasks to individual subjects. The first set was given during the fifth week, the second set during the eighth week, and the third set during the eleventh week. Each set of tasks consisted of a counting problem interpreted as (1) equal addends, one interpreted as (2) an array, and one interpreted as (3) a Cartesian product. The tasks involved the use of concrete or semi-concrete materials. The tasks are explained in Appendix E. NR's responses are given in Table 21, RS's in Table 22, EZ's in Table 23, and YD's in Table 24.
Descriptions of Multiplication Tasks

1. Equal Addends Tasks

1. The first equal addends task, given the fifth week, involved determining the number of disks on a Yoder abacus. The subject was given the abacus with an arrangement of disks (either 15, 9, 9, 9 or 13, 8, 8, 8) on the strands. The researcher asked the subject to forget that it was an abacus and asked how many disks there were.

2. In the second equal addends task, given the eighth week, the subject was asked to work with 30 red and 1 brown Cuisenaire rods and a stick. The researcher showed the subject the stick which had five equally-spaced pencil marks on it. He told the subject that he had previously measured the stick with a brown Cuisenaire rod and had made a mark on it for each length of the rod that it took to cover the stick. He showed the subject that a brown Cuisenaire rod fit between the end of the stick and the first mark. The researcher asked the subject how many brown Cuisenaire rods it had taken to cover the stick. After the subject's response, the researcher asked how many red rods it would take to cover the stick.

3. The third equal addends task, given the eleventh week, made use of 4 one-dozen egg cartons. The researcher gave the subject the egg cartons and asked how many eggs it would take to fill them.
II. Arrays

1. In the first set of tasks, the fifth week, the researcher used a pegboard for the array problem. He placed a rubber band around a 12 x 6 rectangle of pegs and asked the subject how many pegs were inside the rubber band.

2. The array task for the eighth week made use of a rectangular grid composed of eleven vertical lines and seven horizontal lines approximately 3/8" apart with dots at the intersections of the lines. The grid looked like graph paper. The subject was told that the researcher had made a dot at each place where the lines crossed and that the problem was to figure out how many dots there were altogether.

3. The third array task was given the eleventh week. An array of 1/4" diamond-shaped pieces of blue construction paper was glued onto a sheet of white paper. There were seven rows, four rows of six diamonds alternated with three rows of five diamonds. The arrangement looked like the star arrangement of the United States flag. The array was approximately 5" x 4" in size. The researcher gave the array to the subject and asked him how many diamonds there were.

III. Cartesian Products

1. The Cartesian product problem in the
first set of tasks, given the fifth week, was different for the male subjects and the female subjects in an effort to stimulate interest in solving the problem. Each girl was given paper cut-outs of three different colored blouses and four different colored skirts. She was to determine how many different outfits she could create, using any combination of one blouse and one skirt. The problem for the male subjects was similar. Each was given paper cut-outs of three tops of cars and four bodies of cars and asked how many cars there could be if every possible color combination was used.

2. The Cartesian task for the eighth week was a card game. Eight large cards (approximately 1 1/2" x 1") were made out of different colored construction paper. The researcher told the subject that in this card game a "hand" consisted of one large card and one small card and asked the subject how many different hands he could get.

3. The third Cartesian product task, given the eleventh week, involved a pictorial map. The map depicted a school at the top and a house at the bottom labeled "Home," with two rows of other places in the middle. The first row included a library, a swimming pool, a playground, an ice cream store, a house labeled "Friend's house"; the second row included a drug store, a lake, and a church. The researcher told the subject that this was his school and his house and on his way home he could stop at one place in the first row and one in the second row.
The researcher asked him how many different ways he could go home using this pattern.
Analyses of the Subjects' Developing Understanding of Multiplication

A study of each subject's developing understanding of multiplication is presented in this section. Each subject's responses to the multiplication tasks were analyzed and supplemented by information from the basic facts practice sheets and from the observational records.

NR - Equal Addends

NR's understanding of multiplication interpreted as equal addends appeared to be good. He did not use the more efficient multiplication, 4 x 9, with the abacus problem but did perceive the use of equal groups in all three problems. Furthermore, in his classwork NR once wrote a column of eight 8's and added to solve 8 x 8.

NR's response to the second problem with the rods might suggest confusion about which number, 4 or 6, is to be used repeatedly. In the third problem he again used the "wrong" number, 4, repeatedly. However, it is more likely that his verbal responses are an indication of his awareness of the commutative property of multiplication. His response to the first Cartesian product multiplication indicates that he had knowledge of that property.

NR's response to the third equal addends problem is a clear indication of a practical knowledge of the distributive property \((12 \times 4 = (11 + 1) \times 4 = (11 \times 4) + (1 \times 4))\). In his classwork NR also used the distributive property. Once when finding half of 39 3/8 inches on the reverse side of a meterstick, he first found half of 39 mentally
<table>
<thead>
<tr>
<th>Equal Addends</th>
<th>Arrays</th>
<th>Cartesian Products</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>First Set</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(Fifth Week)</strong></td>
<td>Problem: ((3 \times 9) + 15) or ((4 \times 9) + 6)</td>
<td>Problem: (12 \times 6) or (6 \times 12)</td>
</tr>
<tr>
<td><strong>Context:</strong> Abacus</td>
<td>Context: Pegboard</td>
<td><strong>Context:</strong> Cars</td>
</tr>
<tr>
<td>15, 9, 9, and 9 discs were placed on separate strands. NR did (3 \times 9 = 27), &quot;28, 29, ..., 42.&quot;</td>
<td>The rubber band was placed around a (12 \times 6) rectangle of pegs. NR counted 11 pegs, purposely skipped the twelfth peg, and counted 5 more pegs at right angle to the 12. Then he did (11 \times 5 = 55), (+1 = 56).</td>
<td>NR, given the car tops and 4 car bodies, thought a few seconds and said, &quot;12, each of the tops would go with 4 bottoms and (4 \times 3) or (3 \times 4), either way, that's 12.&quot;</td>
</tr>
<tr>
<td><strong>Second Set</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(Eighth Week)</strong></td>
<td>Problem: (6 \times 4)</td>
<td>Problem: (11 \times 7) or (7 \times 11)</td>
</tr>
<tr>
<td><strong>Context:</strong> Cuisenaire Rods</td>
<td>Context: Line Intersections</td>
<td><strong>Context:</strong> Card game</td>
</tr>
<tr>
<td>NR took the red rods and placed 4 of them on the brown rod and said &quot;24.&quot; NR said, &quot;If you put the brown rod along the stick you get 1 times 6, and so if you use the red rods, that's 4 times 6.&quot; NR moved his hand up the long stick in six separate steps.</td>
<td>NR responded &quot;77&quot; then &quot;61.&quot; (He obtained 61 by counting 10 along one edge, skipping the eleventh one, and counting 6 in the other direction. Then he multiplied (10 \times 6 = 60), (+1 = 61).) Then NR said, &quot;There are 11 in each column and (11 \times 7 = 77.&quot; He counted again and got 67 (a (10 \times 6) array (+7)). Finally he said, &quot;I still think it's 77.&quot;</td>
<td>NR said, &quot;48. There are 8 of these (large ones) for each of these (small ones), so for 2 small ones that's (2 \times 8) and for all these (small ones) that would be (6 \times 8), and that's 48.&quot;</td>
</tr>
<tr>
<td><strong>Third Set</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(Eleventh Week)</strong></td>
<td>Problem: (4 \times 12)</td>
<td>Problem: ((4 \times 6) + (3 \times 5))</td>
</tr>
<tr>
<td><strong>Context:</strong> Egg Cartons</td>
<td>Context: Diamonds</td>
<td><strong>Context:</strong> Paths</td>
</tr>
<tr>
<td>NR said, &quot;I don't know my twelves, but 11 times 4 is 44 and one more 4, that's 48.&quot;</td>
<td>NR counted the number of diamonds in the longer row (6) and the number of longer rows (4), and said &quot;24.&quot;</td>
<td>NR immediately said &quot;15. You could have each of the five first places with one of the second places and so you multiply.&quot;</td>
</tr>
</tbody>
</table>
### Table 22

**RS's Responses to Three Types of Multiplication Tasks Given the Fifth, Eighth, and Eleventh Weeks**

<table>
<thead>
<tr>
<th></th>
<th>Equal Addends</th>
<th>Arrays</th>
<th>Cartesian Products</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>First Set</strong> (Fifth Week)</td>
<td><strong>Problem:</strong> ((3 \times 8) + 13) or ((4 \times 8) + 5) <strong>Context:</strong> Abacus</td>
<td><strong>Problem:</strong> (12 \times 6) or (6 \times 12) <strong>Context:</strong> Pegboard</td>
<td><strong>Problem:</strong> (4 \times 3) or (3 \times 4) <strong>Context:</strong> Outfits</td>
</tr>
<tr>
<td></td>
<td>13, 8, 8, and 8 discs were placed on separate strands. RS counted to 8, then did (3 \times 8 + 24, 25, 26, \ldots, 37).</td>
<td>RS separated the (12 \times 6) array into two (6 \times 5) arrays and a (2 \times 6) array. RS said, &quot;6 times 5 is 30, so this is 30,&quot; and counted the last (2 \times 6) array 1, 2, \ldots, 11 and said, &quot;71.&quot;</td>
<td>RS moved the skirts and blouses around, matching them up, orderly for the first six pairs, then haphazardly, finally getting a total of 9.</td>
</tr>
<tr>
<td><strong>Second Set</strong> (Eighth Week)</td>
<td><strong>Problem:</strong> (6 \times 4) <strong>Context:</strong> Cuisenaire Rods</td>
<td><strong>Problem:</strong> (11 \times 7) or (7 \times 11) <strong>Context:</strong> Line Intersections</td>
<td><strong>Problem:</strong> (8 \times 6) or (6 \times 8) <strong>Context:</strong> Card game</td>
</tr>
<tr>
<td></td>
<td>RS lined red rods all along the stick (the first four lined up with the first mark on the stick) and counted 1, 2, 3, \ldots, 24.</td>
<td>RS counted 1, 2, \ldots, 11 and 1, 2, \ldots, 7 (along each edge) and said, &quot;So that's 77.&quot;</td>
<td>RS matched up 1 small card with each of the large ones and made a mark for each; then RS did the same for the second small card, etc. RS counted the marks and got 48, but there were only 47. RS failed to make a mark for the eighteenth pair.</td>
</tr>
<tr>
<td><strong>Third Set</strong> (Eleventh Week)</td>
<td><strong>Problem:</strong> (4 \times 12) <strong>Context:</strong> Egg Cartons</td>
<td><strong>Problem:</strong> ((4 \times 6) + (3 \times 5)) <strong>Context:</strong> Diamonds</td>
<td><strong>Problem:</strong> (3 \times 5) or (5 \times 3) <strong>Context:</strong> Paths</td>
</tr>
<tr>
<td></td>
<td>RS did &quot;12 + 12 = 24 here, and 24, that makes 48.&quot;</td>
<td>RS counted 6 and 4 and said there were 24 diamonds. RS said &quot;You multiply 6 times 4 because it's 6 along one side and 4 along the other.&quot;</td>
<td>RS counted the paths rather systematically especially the last 6 getting a total of 12. RS said there were probably more, &quot;20 at least.&quot;</td>
</tr>
</tbody>
</table>
TABLE 23

EZ'S RESPONSES TO THREE TYPES OF MULTIPLICATION TASKS GIVEN THE FIFTH, EIGHTH, AND ELEVENTH WEEKS

<table>
<thead>
<tr>
<th>Equal Addends</th>
<th>Arrays</th>
<th>Cartesian Products</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>First Set</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(Fifth Week)</strong></td>
<td>Problem: ((3 \times 9) + 14) or ((4 \times 9) + 5)</td>
<td>Problem: (12 \times 6) or (6 \times 12)</td>
</tr>
<tr>
<td>Context: Abacus</td>
<td>Context: Pegboard</td>
<td>Context: Cars</td>
</tr>
<tr>
<td>9, 9, 9 and 14 discs were placed on separate strands. EZ responded, 10,004&quot; - using the place values of the abacus. After an explanation to use the instrument not as an abacus but only to determine the total number of discs, EZ responded, &quot;10,004&quot; again.</td>
<td>EZ counted the pegs along one edge (12) and then the pegs along the edge at a right angle to the first pegs (6) and wrote 12</td>
<td>EZ placed each of the 3 tops with one bottom and said, &quot;12.&quot; EZ said, &quot;three (tops) go with this one (a body) and 4 times 3 is 12.</td>
</tr>
<tr>
<td><strong>Second Set</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(Eighth Week)</strong></td>
<td>Problem: (6 \times 4)</td>
<td>Problem: (11 \times 7) or (7 \times 11)</td>
</tr>
<tr>
<td>Context: Cuisenaire Rods</td>
<td>Context: Line intersections</td>
<td>Context: Card Game</td>
</tr>
</tbody>
</table>
| EZ placed four red rods beside the brown rod and said "24." EZ said, "4 of these (red rods) make one brown one, and 6 times 4 is 24." | EZ said, "77, it's 11 across this way and 7 across this way, and so it's 77." EZ said he multiplied because "it was easier." | EZ thought for thirty seconds or so and said "48." He said, "you could have 6 (small ones) with each large one and you multiply to get 48."
| **Third Set** |        |                    |
| **(Eleventh Week)** | Problem: \(4 \times 12\) | Problem: \((4 \times 6) + (3 \times 5)\) | Problem: \(3 \times 5\) or \(5 \times 3\) |
| Context: Egg Cartons | Context: Diamonds | Context: Paths |
| EZ responded immediately, "48." EZ said that "You take 4 times 12." | Without pointing to the diamonds, EZ counted the six in the long row and did \(6 + 6 + 6 + 6 = 24\). Then he counted the five diamonds in the short row and did \(5 + 5 + 5 = 15\) and \(24 + 15 = 39\). | EZ responded immediately "15," He said there were three second paths for each of the 5 first choices, and \(5 + 5 + 5 = 15\). |
TABLE 24

YD'S RESPONSES TO THREE TYPES OF MULTIPLICATION TASKS GIVEN THE FIFTH, EIGHTH AND ELEVENTH WEEKS

<table>
<thead>
<tr>
<th>Equal Addends</th>
<th>Arrays</th>
<th>Cartesian Products</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>First Set</strong></td>
<td><strong>Problem:</strong> ((3 \times 8) + 13) or ((4 \times 8) + 5)</td>
<td><strong>Problem:</strong> (12 \times 6) or (6 \times 12)</td>
</tr>
<tr>
<td><strong>Context:</strong> Abacus</td>
<td><strong>Context:</strong> Pegboard</td>
<td><strong>Context:</strong> Outfits</td>
</tr>
<tr>
<td><strong>(Fifth Week)</strong></td>
<td>YD counted 12 pegs along one edge, 6 along the other, and said &quot;its 12 times 6, but I don't know my twelves.&quot; She said 6 times 6 is 36 and wrote 36</td>
<td>YD moved the skirts and blouses around rather orderly and wrote an ordered pair for each combination. She wrote ((p, b)) for the combination of pink blouse and blue skirt. YD counted pairs, got 11, and said, &quot;There must be a trick.&quot;</td>
</tr>
<tr>
<td>13, 8, 8, and 8 discs were placed on separate strands. YD counted by ones, 1, 2, . . . , 37.</td>
<td><strong>Then YD counted by 1's to 72 to check.</strong></td>
<td><strong>Problem:</strong> (8 \times 6) or (6 \times 8)</td>
</tr>
<tr>
<td><strong>Second Set</strong></td>
<td><strong>Problem:</strong> (6 \times 4)</td>
<td><strong>Context:</strong> Line Intersections</td>
</tr>
<tr>
<td><strong>(Eighth Week)</strong></td>
<td><strong>Context:</strong> Cuisenaire Rods</td>
<td>YD counted 11 one way and 7 the other way and said she &quot;times'd it.&quot; When asked if she was sure, she counted by ones, got 78 and said, &quot;I must have miscounted.&quot;</td>
</tr>
<tr>
<td>YD lined up 4 of the red ones beside the brown rod and said &quot;24, I times'd it.&quot; To check her answer YD lined up 24 red rods on the stick and counted 1, 2, . . . , 24.</td>
<td><strong>Problem:</strong> (11 \times 7) or (7 \times 11)</td>
<td><strong>Problem:</strong> (8 \times 6) or (6 \times 8)</td>
</tr>
<tr>
<td><strong>Third Set</strong></td>
<td><strong>Context:</strong> Egg Cartons</td>
<td><strong>Context:</strong> Line Intersections</td>
</tr>
<tr>
<td><strong>(Eleventh Week)</strong></td>
<td><strong>Problem:</strong> (4 \times 12)</td>
<td><strong>Context:</strong> Diamonds</td>
</tr>
<tr>
<td>YD tried to figure the number of diamonds by counting short rows of 3 and rows of 4, but got confused. Then she combined rows of 3 and 4 into groups of 7, but got confused again. Then YD turned the paper vertically, combined rows of 6 and 5 and did (11 + 11 + 11 = 33), and 6 is 39.</td>
<td><strong>Problem:</strong> ((4 \times 6) + (3 \times 5))</td>
<td><strong>Problem:</strong> (3 \times 5) or (5 \times 3)</td>
</tr>
<tr>
<td>First YD did ((4 \times 10) + (4 \times 2)) is 48 and asked if that was right. Then she asked if 3 times 12 was 36. Then YD said, &quot;Yes, 48 is right, since 2 times 12 is 24 and 24 + 24 is 48.&quot;</td>
<td><strong>Context:</strong> Diamonds</td>
<td><strong>Context:</strong> Paths</td>
</tr>
<tr>
<td><strong>Then YD indicated the next five paths with her hands and went</strong></td>
<td><strong>YD tried to count the paths mentally and by using her hands but lost track. Then she drew</strong></td>
<td><strong>YD tried to count the paths mentally and by using her hands but lost track. Then she drew</strong></td>
</tr>
<tr>
<td><strong>5 + 5 + 5 = 15.</strong></td>
<td><strong>(\circ \circ \circ \circ \circ \circ)</strong></td>
<td><strong>(\circ \circ \circ \circ \circ)</strong></td>
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</tbody>
</table>
(19 1/2) and put his finger at that point. Then he said half of 3 (eighths) is 1 1/2 (eighths) and moved his finger 1 1/2 units (eighths) more from 19 1/2 to 19 1/2 + 1 1/2 (eighths). \( \frac{1}{2} \times 39 \frac{3}{8} = \left( \frac{1}{2} \times 39 \right) + \left( \frac{1}{2} \times \frac{3}{8} \right) = 19 \frac{1}{2} + 1 \frac{1}{2} \) (eighths).

NR's response to the third problem raises some questions about his work on the basic facts practice sheets (See Appendix F-2). He did not respond to the problem 4 x 12 (or to 3 x 12 or to 12 x 3) on any of the three practice sheets, but he was able to find the product in a matter of seconds in the task situation and did not use the egg cartons in his solution.

**NR - Arrays**

NR's responses indicate a confusion about how multiplication should be applied to find the number of points in an array. In the first two problems he was confused about the point at the intersection of the two sets of points along adjacent sides of an array. Apparently he thought that that point would be counted twice if included in the dimensions of both sides of the array. Consequently, he "solved" the problem by omitting the point initially and then adding it in at the end. In the second problem there is an indication that he was resolving the problem regarding the double counting of the point at the corner of an array. However, he was still not sure about the correct procedure. Between the times that the second and third array problems were given, during the ninth week, the teacher asked NR to solve a 7 x 5 array problem and he did it correctly. He counted the points along one
edge, then the points along an adjacent edge - including the corner point, and multiplied the results together. It appeared that he had resolved the problem about double counting the corner point.

With the third array problem another difficulty appeared. NR ignored or did not perceive the shorter rows of diamonds. He appeared to have generalized that, to find the number of points in any "rectangular looking array," you count the number of points along one side and multiply that result by the number of points along an adjacent edge.

NR - Cartesian Products

NR's responses to all three problems indicate an awareness of how multiplication can be used to solve such problems. In fact, he apparently saw the equal addends interpretation in the Cartesian product context.

As mentioned earlier, his response to the first problem indicates an awareness that multiplication is a commutative operation.

NR - Summary

NR appears to have developed good understanding of multiplication interpreted as equal addends and as Cartesian products. Support for his clear understanding of equal addends problems is given by his response to another problem which the teacher posed. She asked him what he would have to do with 4 to make 6 and he said, "Take 1 1/2 times it." (He successfully used the addend (4) one-and one-half times.)
NR's understanding of multiplication interpreted as arrays was apparently in the process of development. It appeared that he resolved the problem of how multiplication applies to rectangular arrays but had not developed an understanding of why it applies.

Another indication of his understanding of multiplication is his work on a multiplication and division project. In that context he said, "21 equals 3 times 7 and 21 divided by 3 equals 7."

**RS - Equal Addends**

RS's responses to the first and third problems indicate a perception of the use of equal groups within the problem contexts. Her classwork also indicated that she interpreted the product of two numbers as an equal addends problem. Once when solving 4 x 7, she added together 14 and 14 to obtain the answer.

RS's response to the problem involving Cuisenaire Rods indicates that she apparently did not perceive it as an equal addends problem. She counted by 1's. An alternative explanation might be that she did not know the product of 6 and 4 (See Basic Facts Practice Sheets, Appendix F-3).

However, she did find the correct product of 6 and 4 in the third set of multiplication tasks involving arrays. It seems that if she had perceived the problem as one of equal groups she would have used a more mature method of solution than counting by 1's. (For example, she might have used a procedure similar to the one she used with the egg cartons.)
RS's responses to the first and third problems raise questions about her responses on the Basic Facts Practice Sheets (See Appendix F-3). Although she was able to solve $4 \times 12$ in a few seconds in the task situation (and did not use the egg cartons) she only responded to that combination on one of the three practice sheets. Moreover, she responded immediately, "24" to the problem $3 \times 8$ in the task situation but only responded once to that combination on the practice sheets.

**RS - Arrays**

RS's responses to the first two problems indicate that she knew how to find the number of points in a rectangular array. Furthermore, in the first problem she understood that she could subdivide the original array into smaller arrays and find the number of points in the original by adding together the numbers of points in the smaller arrays.

RS's response in the third problem indicates that perhaps she had generalized her method of solution for rectangular arrays to include "rectangular-looking arrays." (Of course, it could be that she did not perceive that the array presented was not rectangular.) Perhaps her response is an indication that she did not understand why the product of the number of points along one side and the number of points along an adjacent side yielded the number of points in a rectangular array.

**RS - Cartesian Products**

RS's responses to the three problems indicate some development
toward understanding Cartesian products problems of that type. Whereas in the first problem her pairing up of skirts and blouses was orderly initially and then haphazard, in the second problem she was very orderly throughout. Her method of solution for the third problem was systematic, but she was unable to determine the correct total number of paths. (One wonders what she would have done if the third problem had involved manipulative materials.) However, it is clear that RS did not see Cartesian product problems as multiplication problems.

**RS - Summary**

From RS's responses to the multiplication tasks it appears that when the use of equal groups was easily perceived (e.g. with the abacus and the egg cartons) she used those groups in solving the problems. However, apparently she had difficulty perceiving the presence of equal groups (e.g. with Cuisenaire Rods and with the Cartesian product tasks).

One situation in class emphasizes that she understood that multiplication was used for the addition of equal groups. After it was determined that five people were going to a play and tickets cost $3 apiece she said, "Let's see, 3 times 5, that's $15."

A discussion between the teacher and RS indicated that she had a practical understanding of the distributive property. When faced with the problem $6 \times 4$, RS solved it by "using a lower one." RS said that $5 \times 4 = 20$ so $6 \times 4 = (5 + 1) \times 4 = (5 \times 4) + (1 \times 4) = 20 + 4 = 24$.

It is not clear whether or not RS understood why multiplication gives the number of points in a rectangular array. It is certain that
she did not perceive Cartesian product problems as multiplication problems.

**EZ - Equal Addends**

EZ's responses to the last two problems indicate a good understanding of multiplication interpreted as equal addends. He responded immediately in those contexts and in each case used the appropriate number repeatedly. EZ's classwork substantiated his performance on the tasks. Early in the study he did a group of word problems from a practice card which involved each of the four basic arithmetic operations. Four of those problems had equal addend interpretations of multiplication and he did them correctly. However, he did misinterpret one measurement division problem as an equal addends multiplication problem.

EZ's response to the first problem was clearly a problem of not being able to see the abacus as a disc-holder instead of a place-value system.

EZ's skill in the basic multiplication facts (See Appendix F-4) separates his response to the third equal addends problem from the responses of the other subjects. He was the only subject who responded immediately "48" to the product 4 x 12.

**EZ - Arrays**

EZ's work with the three array problems, especially the third one, indicates that he realized that the method that he used for the
rectangular arrays did not apply for the "rectangular-looking array."

However, it is somewhat surprising that, although he perceived two sets of equal groups in the third problem he apparently did not use multiplication. In a similar instance occurring in class with an array which was distorted somewhat he counted by ones. Apparently he was not certain that the same number of objects were in each row.

Another instance in class substantiated his work with the rectangular arrays in the multiplication tasks. When faced with the problem of determining the number of tiles on the ceiling he found that he could count the number of tiles along one edge and multiply that result by the number of tiles in an adjacent edge to obtain the desired answer.

**EZ - Cartesian Products**

It is evident that EZ perceived the equal addends aspects of multiplication in problems interpreted as Cartesian products. He seemed to have a good understanding of how multiplication solves Cartesian product problems, even though he used addition in the third problem of the set.

**EZ - Summary**

EZ's responses to all three interpretations of multiplication reveal his understanding of the relevance of multiplication in several settings. Apparently, none of the three types of problems posed
significant difficulties for him. His understanding of multiplication was also evident in his classwork. Early in the study the teacher noted that EZ used multiplication to check a long division problem, apparently without any suggestion from her to do so.

**YD - Equal Addends**

The responses by YD to the equal addends problems reveal continual progress toward an understanding of the relevance of multiplication in that setting. In the first problem she did not use equal groups but counted by 1's. In the second problem she used equal groups and multiplied, but checked her answer by counting by 1's. In the third problem she not only used equal groups and multiplied but also did not count by 1's either to solve the problem or to check her answer. In fact, she checked her answer in several ways that indicated a good understanding of the equal addends aspect of the problem 4 times 12.

In class YD solved a problem much like the second equal addends problem. She and a classmate were using a yardstick to measure the width of the room. They found that the room was 8 yards across and then wanted to find out how many inches it would be. YD's classmate wanted to measure the room, but YD thought she could simply multiply 8 times 36 to get the desired answer and proceeded to carry out the multiplication.

YD's responses to the third equal addends problem and the first array problem again raise the question about the absence of certain
responses on the basic facts practice sheets (See Appendix F-5). She was able to solve $4 \times 12$ and $12 \times 6$ in the task situation with a little difficulty but answered very few of the combinations involving 12's on the practice sheets.

**YD - Arrays**

YD showed evidence of progression toward a better understanding of arrays on the sequence of three problems. On each of the first two she counted by 1's to check her answer, although on the second problem she essentially disregarded her count. On the third problem she did not count by 1's at all to check her results.

Her different method of solving the third problem, as compared to the method used for the first two, indicates that she realized it was a different type of a problem. Nevertheless, she used equal groups to solve the problem. In fact, she tried several different methods involving equal groups before she found one that worked successfully.

In light of her reliance on counting by 1's in earlier problems, one might have expected her to resort to counting when several initial attempts to use equal groups failed.

Overall, it is clear that YD understood that the method she used to solve the first two problems did not apply to the third one. However, it is not clear whether or not she understood why the method used for the first two gave the correct answer.
YD - Cartesian Products

YD's responses, especially her method of solution for the first problem as compared to her methods for the other two, indicate definite progress. Even though she was very clever in the first problem, writing ordered pairs, she was not systematic in her combining of skirts and blouses and, thus, did not find all of the combinations. On second problem, however, she was more systematic, found all of the card combinations, and discovered that multiplication helped solve that Cartesian product problem. YD's response to the third problem was again systematic, and once again very clever. She drew a model of the situation and once again realized that equal groups were involved in the problem. Moreover, she realized that the starting point and ending point of the paths was irrelevant and did not draw them in her model. One wonders whether she would have multiplied if there had been more than 3 groups of 5.

YD - Summary

On each of the three sets of multiplication tasks YD's responses suggest an increasing understanding of the three interpretations of multiplication. First, she relied less on counting in the later problem sets. And secondly, she used equal groups more frequently in successive problems than in initial problems.

YD also understood other aspects of multiplication. First, she was obviously aware of the commutative property. Twice in classwork
she was practicing some basic multiplication facts and came to a combination she could not recall. Each time she looked at the combination with the digits in reverse order, verbalized the product, looked back at the original combination, and went on.

On another occasion YD used the distributive property in a measurement problem. She was trying to find half of 6 5/8 inches on a ruler. First she found half of 6, then half of 1/2, and then half of 1/8. That is $\frac{1}{2} \times 6 \frac{5}{8} = \frac{1}{2} \times (6 + \frac{1}{2} + \frac{1}{8}) = (\frac{1}{2} \times 6) + (\frac{1}{2} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{8})$.

**Multiplication Tasks - General Trends**

Trends in the subjects' responses to individual tasks will be discussed, followed by an assessment of the subjects' understanding of each interpretation of multiplication. Then comparisons among the subjects' understandings of each interpretation will be given.

I. **Equal Addends**

On the problem with the disks and the abacus none of the subjects used the basic group of disks (either 8 or 9) as an addend four times. The basic group was used as an addend only three times. It would seem natural for a child to place his hand at the top of the basic group of disks and move it across the four strands of disks and determine that the fourth strand of disks contained within it a group the same size as the basic group. Such
a realization would diminish the size of the "extra" part that the child would need to count separately.

The responses to the task involving the egg cartons revealed several significant issues. First, each of the subjects used a different method for calculating $4 \times 12$, and all obtained the same final result. One subject used three different methods. If the subjects had not been interviewed and only their final answers had been recorded, the only conclusion that could have been drawn was that each could calculate $4 \times 12$ correctly. By interviewing, however, it was also determined that the subjects differed with respect to their levels of understanding of multiplication interpreted as equal addends. RS used addition exclusively, YD used multiplication combined with addition in practical applications of the distributive principle, NR used the commutative principle along with the distributive principle, and EZ knew the combination immediately. Clearly, more information about the subjects' thinking processes was acquired by interviewing them.

Secondly, although only one subject, EZ, responded to the combination $4 \times 12$ on the three basic facts practice sheets (Appendices F-1 to F-5), all four of the subjects were able to calculate the product correctly in the task situation. Even though three of the subjects did not know the product $4 \times 12$ immediately, they were able to use other multiplication facts and properties of multiplication to calculate $4 \times 12$. The fact that the subjects responded to the combination $4 \times 12$ in the task situation might indicate that the subjects were motivated to solve the problem in the context of a
practical situation but not in a practice situation involving numbers only. Or, perhaps the subjects responded to $4 \times 12$ because they had a concrete representation of the problem. It is true that the subjects were timed on the practice sheets and that it took the three subjects a few seconds, 5 to 20, to calculate $4 \times 12$. Maybe they did not want to spend the extra time on the practice sheets. Nonetheless, it appeared that the major difference between the practice situation and the task situation was the practical nature of the problems in the task situation.

In assessing the subjects' overall understanding of multiplication interpreted as equal addends (and later as arrays and Cartesian products), reference will be made to the data in Table 25. Included are the numbers of instances that subjects 1) used equal groups, 2) used multiplication, 3) obtained correct final answers, and 4) used different solution processes in responding to the multiplication tasks.

Carper (Brownell and Carper, 1943, Chapter 8) designated the repeated use of equal groups as a criterion for assessing the levels of children's understanding of multiplication processes. If that criterion is applied to the subjects' responses to the equal addends tasks, the conclusion is that the subjects had a reasonably good understanding. Nine of the twelve responses involved repeated use of the basic group identified in the problem. A further indication of the subjects' understanding is that they used multiplication to combine the equal groups in eight of the nine instances.

The subjects' responses to the tasks and their work in class
TABLE 25

NUMBER OF INSTANCES THAT SUBJECTS 1) USED EQUAL GROUPS, 2) USED MULTIPLICATION, 3) OBTAINED CORRECT FINAL ANSWERS, AND 4) USED DIFFERENT SOLUTION PROCESSES FOR EACH OF THE MULTIPLICATION TASKS

<table>
<thead>
<tr>
<th></th>
<th>First Set</th>
<th>Second Set</th>
<th>Third Set</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equal Addends</td>
<td>EG 2</td>
<td>3</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>M 2</td>
<td>3</td>
<td>3</td>
<td>8</td>
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<tr>
<td></td>
<td>CA 3</td>
<td>4</td>
<td>4</td>
<td>11</td>
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<tr>
<td></td>
<td>DP 2</td>
<td>3</td>
<td>4</td>
<td>9</td>
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<tr>
<td>Arrays</td>
<td>EG 0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>M 4</td>
<td>4</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>CA 2</td>
<td>4</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>DP 4</td>
<td>2</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>Cartesian Products</td>
<td>EG 2</td>
<td>3</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>M 2</td>
<td>3</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>CA 2</td>
<td>4</td>
<td>3</td>
<td>9</td>
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<tr>
<td></td>
<td>DP 3</td>
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</tr>
<tr>
<td>Totals</td>
<td>EG 4</td>
<td>7</td>
<td>9</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>M 8</td>
<td>10</td>
<td>6</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>CA 7</td>
<td>12</td>
<td>9</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>DP 9</td>
<td>8</td>
<td>10</td>
<td>27</td>
</tr>
</tbody>
</table>

EG = Equal Groups
M = Multiplication
CA = Correct Answers
DP = Different Processes

seemed to indicate that they viewed all multiplication as equal addends. When faced with number combinations in both situations, subjects found the products by repeated addition of one of the factors rather than by sketching an array or a diagram representing the Cartesian product.
It is the researcher's opinion that the task involving the use of the stick and the Cuisenaire rods was the most valid for assessing the subjects' understanding of multiplication interpreted as equal addends. The counting task lent itself to multiplication and it was tedious and time-consuming not to use multiplication. Furthermore, the researcher found it easier to determine which processes subjects used with the Cuisenaire rods task than with the other tasks.

II. Arrays

The responses to the first array task again emphasize the variety of processes used by the four subjects to solve the same problem. All four subjects used different procedures.

It is surprising to compare the subjects' responses to the first two array problems with their responses to the third problem. On the first two problems three of the subjects determined the number of points in the rectangular arrays by counting the number of points in each of two adjacent edges and multiplying the resulting two numbers. In the third problem, which involved a non-rectangular array, two of the four subjects applied the same procedure that had been used successfully for the rectangular arrays. Such a procedure clearly does not give the correct result because there are not identical numbers of points in each row of the non-rectangular
array. One can only speculate about the reasons for the subjects' responses. Did they not perceive that there were different numbers of points in adjacent rows in the third array problem? Did they over-generalize about the procedure used to find the number of points in rectangular arrays? Had they not seen any non-rectangular arrays when they were apparently learning procedures for determining the number of points in rectangular arrays?

It is difficult to assess the subjects' understanding of multiplication interpreted as arrays. Although it is clear that the subjects knew how to find the number of points in a rectangular array, it is not clear whether they understood why their procedure was a correct one. Their responses do not indicate that they knew why multiplication of the two dimensions of the array gave the correct total number of points in the array. Furthermore, it was not clear whether or not they used equal groups more than the one time that NR used them. The subjects' responses to the third problem reinforce the possibility that they did not know how multiplication is applied to determine the number of points in arrays. The two subjects who used multiplication applied it incorrectly. The two subjects who did not use multiplication did use equal groups correctly. Apparently they realized that the procedure involving multiplication which worked for rectangular arrays did not work for non-rectangular ones.

III. Cartesian Products
The responses to the first Cartesian product task ranged from haphazard to very systematic. RS made pairs haphazardly, YD was more orderly, EZ and NR paired only the objects from one set with one object from the other set. The range in responses may have occurred because the subjects had not confronted Cartesian product problems prior to the task situations. By contrast, the subjects' responses to the second and third Cartesian product problems were not nearly as diverse. On those two problems, three of the subjects paired only the objects from one set with one of the objects from the other set before reaching a conclusion. The fourth subject, RS, was much more systematic than in the first task, but still attempted to pair up all objects from one set with all the objects in the other set.

The subjects' understanding of multiplication interpreted as Cartesian products appeared to be fairly good. Applying Carper's criterion, eight of the twelve responses involved the repeated use of equal groups. Six of those eight responses involved multiplication to combine the equal groups. It is possible that two additional responses to the third task would have involved multiplication if the cardinality of both sets involved had been 5 or larger. It was almost as easy to use $5 + 5 + 5$ as to use $3 \times 5$.

Another indication of the subjects understanding of Cartesian products is that they all used correct processes for solving the second
Cartesian product task. The second task theoretically should have been the most difficult because the two sets involved both contained greater than five elements.

Comparisons Among the Subjects' Understandings of the Three Interpretations of Multiplication

The subjects' responses to the three sets of multiplication tasks and to problems which arose in class indicate that they understood the equal addends interpretation better than either the arrays interpretation or the Cartesian products interpretation. The data in Table 25 provide grounds for comparison of the responses to the multiplication tasks. First, subjects definitely used equal groups in the solving of nine equal addends problems as compared to three for arrays and eight for Cartesian product problems. It is possible that the subjects mentally used equal groups in solving more of the arrays problems. However, that conclusion cannot be ascertained from their responses to the tasks. Secondly, subjects used multiplication eight times for equal addends problems, ten times for arrays, and six times for Cartesian products. The fact that subjects used multiplication ten times for array problems is misleading. Three of the times, subjects multiplied wrong factors together in solving array problems. Thirdly, the subjects obtained eleven correct answers for the equal addends tasks versus eight for the arrays and nine for the Cartesian products.

The most valid indications of the subjects' superior understanding of equal addends, however, came not only from their responses to the equal addends tasks but also from their responses to the other
two types of tasks and to problems that arose in class. On the equal addends tasks there was never any uncertainty by the subjects about the processes they used. On the other types of tasks there was a good deal of uncertainty, however. For example, students took longer to work array and Cartesian product tasks than equal addends tasks. Also, throughout the subjects' classwork, they interpreted multiplication problems as equal addends problems. Whether solving array problems or Cartesian product problems, they repeatedly added equal groups.

One can only speculate that the bulk of multiplication instruction received by the subjects was in terms of equal addends. Multiplication instruction occurred only infrequently during the study. In those instances there was work both with arrays and with equal addends. No instruction concerning Cartesian products was observed.

The subjects' responses to the equal addends problems and to the Cartesian product problems seem to differ with the results of Hervey's study (1966). Whereas Hervey found that children were able to accurately solve about half as many Cartesian product problems as equal addends problems, in the current study there appeared to be no differences between the number of equal addends tasks solved correctly (11) and the number of Cartesian product tasks solved correctly (9). Hervey also found that children were able to accurately conceptualize about half as many Cartesian product problems as equal addends problems. However, inspection of the responses in the current study revealed that subjects conceptualized at least nine of the twelve Cartesian product
tasks correctly as compared to eleven of the twelve equal addends tasks. Again, the number of Cartesian products conceptualized correctly did not differ from the number of equal addends tasks conceptualized correctly.

One might expect the differences which Hervey reported with second-grade students to be more pronounced with fourth-grade students. Whereas her second-graders had not received multiplication instruction, the fourth graders in the current study had received instruction, probably stressing the equal addends interpretations of multiplication with no mention of the Cartesian product interpretation. Yet, the fourth-graders appeared to do as well on Cartesian product tasks as on the equal addends tasks. Of course, the possibility exists that the two years difference in the subjects' ages accounts for the apparent differences in the results.

One of the most striking results of the multiplication tasks was the number of different processes used to solve the tasks. For each task posed there was an average of three different processes used to solve it. The variability in processes used amplifies the results reported by Brownell and Carper (1943). They stated that individual children tended not to employ the same method for solving different multiplication combinations. The results of this study suggest further that different children used different processes to solve the same combinations.

Discussion of the Use of Multiplication Tasks in the Current Study

The purpose of the multiplication tasks was to assess each
subject's developmental understanding of multiplication. The results suggest that the sequence of tasks did elicit gradual changes in the methods of solution used by individual subjects. Individual subjects' responses changed with respect to the:

i. identification of equal groups,

ii. dependence on counting by 1's,

iii. systematization of combinations made in Cartesian product problems, and

iv. "double-counting" the corner point in an array.

Furthermore, it appeared that these factors were related to a subject's developing understanding of multiplication.

The multiplication tasks also revealed different methods of solution for different subjects. The subjects' methods of solution differed with respect to each of the four factors just identified plus v) the identification of non-rectangular arrays. It appeared that the fifth factor also was associated with the subjects' understanding of multiplication.

The use of the multiplication tasks appeared to have at least three advantages over presenting the subjects with simple multiplication combinations. First, the tasks elicited responses to three different interpretations of multiplication: equal addends, arrays, and Cartesian products. Second, the tasks appeared to motivate the subjects to solve the problems posed. The problems were designed to be of interest to children. Third, the tasks involved the use of concrete and semi-concrete materials which the subjects could manipulate to solve the problems presented.
The experiences of this study suggest that changes could be made in the tasks, and that those changes would provide for better assessment of the subjects' methods of solution of multiplication problems. The proposed changes are:

i. Use larger numbers as factors, especially in the Cartesian product problems.

ii. Use more concrete materials which subjects can manipulate. Use fewer semi-concrete materials.

iii. Ask questions of subjects working with arrays which better assertain their methods of solution.

iv. Use additional non-rectangular arrays which require unconventional methods of solution.

Summary of Results

The purpose of the study was to identify 1) the mathematical concepts and skills the subjects dealt with, 2) the processes by which subjects dealt with the concepts and skills, and to assess 3) the subjects' developmental understanding of multiplication and its properties. The results will be grouped according to these divisions.

1. Identification of the mathematical concepts and skills the subjects dealt with.

The subjects spent most of their time dealing with number concepts and skills (71 percent) and with spatial concepts and skills (14 percent). There was a great variance in the specific concepts and skills dealt with across subjects. Of the 95 concepts and skills
dealt with, only twelve of them were dealt with by all four subjects. However, about 68 percent of the time spent in mathematics was spent in dealing with those twelve concepts. Furthermore, about 58 percent of the subjects' time was spent in dealing with the four basic arithmetic operations: addition, subtraction, multiplication, and division.

Part of identifying the mathematical concepts and skills dealt with involved determining the subjects' immediate sources of the concepts and skills. The two primary sources were the subjects themselves (37 percent) and teacher-constructed dittos (36 percent).

2. Identification of the processes by which subjects dealt with mathematical concepts and skills.

When subjects dealt with mathematical concepts and skills, about half of the time (48 percent) they worked computation problems and about a fourth of the time (25 percent) they worked with mathematical projects. About half of the time (51 percent) was spent working with concrete and semi-concrete materials. They used a total of 85 concrete and semi-concrete materials, 43 of which were not from the mathematics center. About half of their time (56 percent) was spent alone and about a fourth (24 percent) was spent with classmates. As the subjects dealt with mathematics the teacher was frequently Not Involved (35 percent of their time), although she spent a sizeable portion of their time (31 percent) as a Starter for their activities.

Comparisons of the processes by which individual subjects worked with mathematics revealed that they dealt with mathematics
differently, with respect to:

i. the activities in which they participated,

ii. whether they worked alone or with other people, and

iii. the role the teacher served for the subjects as they dealt with mathematics.

3. Assessment of the subjects' developmental understanding of multiplication.

The subjects were observed as they dealt with multiplication in class and they were given a sequence of three sets of multiplication tasks by the researcher. In class, the subjects spent about 24 percent of their time (spent in dealing with mathematics) dealing with multiplication. They dealt with multiplication in basically the same way as they dealt with mathematics generally. The only major difference was that the subjects spent about 71 percent of their time working multiplication computation problems (as compared to 48 percent for mathematical computation generally).

The subjects' responses to the multiplication tasks and their classwork indicated that they had a better understanding of multiplication interpreted as equal addends than either as arrays or Cartesian products. In fact, the subjects apparently considered any multiplication problem to be an equal addends problem.

In comparing the subjects' responses to the equal addends tasks only with their responses to the Cartesian product tasks only, one does not obtain the same conclusions that Hervey (1966) reached. Whereas Hervey concluded that subjects correctly solved (and accurately
conceptualized) only half as many Cartesian product problems as equal addends problems, the results of the current study suggest that no differences existed between the subjects' abilities to solve (or conceptualize) Cartesian product tasks and equal addends tasks.

The variability in the processes used in the current study amplifies the results reported by Brownell and Carper (1943). They stated that individual children tended not to employ the same method for solving different multiplication combinations. This study supports that finding and also suggests that different subjects used different methods for solving the same combinations.
The focus of the current research was the mathematics learning of four nine-year-old children in one open classroom. The children were observed and occasionally interviewed for eleven weeks. The researcher identified the mathematical concepts and skills the children dealt with, the processes by which the children dealt with the concepts and skills, and the children's developmental understanding of multiplication and its properties. The purpose was to generate hypotheses about mathematics learning in open classrooms which may serve as reference points for future research.

**Conclusions**

The first five conclusions describe the four subjects' mathematics learning. Conclusions 6 and 7 focus on the counting tasks which were used to assess the subjects' understanding of multiplication.

1. The four subjects appeared to have a major responsibility for their own mathematics learning. The subjects themselves were involved as sources for the concepts and skills they dealt with (37 percent), spent a sizeable portion of their time alone (56 percent), and for much of their mathematical activity (35 percent) the teacher was not directly involved.

2. The subjects appeared to deal with mathematics differently, with respect to:
1. the activities in which they participated,
   ii. the number of people with whom they worked, and
   iii. the role the teacher served for the subjects when they dealt with mathematics.

3. The subjects spent about half their time (58 percent) dealing with the four basic arithmetic operations and about half their time (42 percent) dealing with other types of concepts and skills (e.g. relations, classes, geometry).

4. The subjects spent about half of their time (51 percent) working with concrete and semi-concrete materials.

5. The subjects' mathematical experiences were not restricted to experiences which involved the use of materials in the math center. About half (51 percent) of the materials children used while dealing with mathematics were not found in the math center.

6. The multiplication tasks elicited responses which revealed differences in subjects' methods of solution. The different methods of solution seemed to relate to the subjects' understanding of multiplication. The subjects' methods of solution differed with respect to:

   i. identification of equal groups
   ii. dependence on counting by 1's
   iii. systematization of combinations made in Cartesian product problems
   iv. identification of non-rectangular arrays
   v. "double-counting" the corner point in an array.
7. The multiplication tasks elicited responses from individual subjects (over six weeks time) which revealed gradual changes in methods of solution with respect to certain factors identified in conclusion 6.

**Implications**

The major purpose of the current study was to generate hypotheses about mathematics learning in open classrooms. The following hypotheses represent logical implications of each conclusion in the previous section.

1. Children can be given a great deal of the responsibility for their own mathematics learning in an open classroom situation. They can be expected to deal with mathematical concepts and skills if materials are available and the teacher has a good understanding of open education and can implement that understanding.

2. When children are given a major responsibility for their own mathematics learning, they differ with respect to the concepts and skills they deal with and the processes by which they deal with them. Therefore, it seems unreasonable to expect a classroom of children to learn identical concepts and skills through identical processes. It seems more likely that children will learn mathematics better and enjoy it more if they deal with concepts and skills in which they are interested and through processes they choose to participate in.

3. A classroom can exist in which there is a balance between an emphasis on the basic arithmetic operations and an emphasis on other types of concepts and skills (e.g. classes, relations, linear
4. Children need to use concrete and semi-concrete materials while in the process of learning mathematics.

5. Children in open classrooms are more likely to confront mathematics in a variety of settings. Consequently, they are also more likely to see mathematics as being relevant to everyday experiences and to see mathematics as more than computational arithmetic.

6a. The use of counting tasks involving concrete and semi-concrete materials represents a valid means of assessing children's understanding of multiplication.

6b. Learning experiences in multiplication should be different for different children, depending on their understanding of multiplication. If children differ in their solution processes, then instruction should be individualized to deal with each child's particular solution process.

7. The use of counting tasks involving concrete and semi-concrete materials represents a valid means of assessing individual children's developmental understanding of multiplication.

Recommendations for Further Study

The hypotheses mentioned in the previous section represent a first step toward the evaluation of open education. They deserve to be tested in other open classrooms and to serve as reference points for future research.

Other studies would also seem profitable.

1. In the current study the researcher had difficulty in
identifying the sources of the mathematical concepts and skills. It is the researcher's contention that differences exist across subjects with respect to the sources of the concepts and skills they deal with, and that those differences can be detected if a different, more reliable method of identifying sources can be developed.

2. The next logical step, after testing the six hypotheses in open classrooms, would be to test them in non-open classrooms and to make the resulting comparisons.

3. Another important task would be to determine which of the variables (across which subjects differed in dealing with mathematical concepts and skills, e.g. time spent in each activity) are related to subjects' learning.

An important part of the current research was the development and testing of counting tasks to assess children's understanding of multiplication. The following suggestions for research deal with counting tasks.

4. Student responses in the current study indicate an apparent lack of understanding of why an appropriate multiplication yields the total number of points in a rectangular array. It appeared that subjects were simply applying a rule. One study which seems necessary is one which probes children's understanding about arrays, rectangular and non-rectangular, through further use of counting tasks.

5. Research with tasks could attempt to determine the correlation between the factors identified in Conclusion 6 and an understand-
6. Finally, tasks need to be developed which assess children's understanding of other basic arithmetic processes.

This study is a beginning contribution to a body of research assessing open education practices. The conclusions suggest that open education fosters independent and individualized learning. Children had a major responsibility for their own mathematics learning and learned mathematics in a variety of ways. They used a wide range of concrete and semi-concrete materials. In this classroom some of the theoretical goals of open education had become a reality.
APPENDICES
APPENDIX A

BARTH'S ASSUMPTIONS ABOUT CHILDREN'S LEARNING AND KNOWLEDGE
(Barth, 1972, pp. 18-47)
BARTH'S ASSUMPTIONS ABOUT CHILDREN'S LEARNING AND KNOWLEDGE
(Barth, 1972, pp. 18-47)

I. ASSUMPTIONS ABOUT CHILDREN'S LEARNING

Motivation

1. Children are innately curious and will explore their environment without adult intervention.

2. Exploratory behavior is self-perpetuating.

Conditions for Learning

3. The child will display natural exploratory behavior if he is not threatened.

4. Confidence in self is highly related to capacity for learning and for making important choices affecting one's learning.

5. Active exploration in a rich environment, offering a wide array of manipulative materials, facilitates children's learning.

6. Play is not distinguished from work as the predominant mode of learning in early childhood.

7. Children have both the competence and the right to make significant decisions concerning their own learning.

8. Children will be likely to learn if they are given considerable choice in the selection of the materials they wish to work with and in the choice of questions they wish to pursue with respect to those materials.

9. Given the opportunity, children will choose to engage in activities which will be of high interest to them.

10. If a child is fully involved in and is having fun with an activity, learning is taking place.

Social Learning

11. When two or more children are interested in exploring the same problem or the same materials, they will often choose to collaborate in some way.

12. When a child learns something which is important to him, he will wish to share it with others.

Intellectual Development

13. Concept formation proceeds very slowly.
14. Children learn and develop intellectually at their own rate and in their own style.

15. Children pass through similar stages of intellectual development - each in his own way, and at his own rate and in his own time.

16. Intellectual growth and development take place through a sequence of concrete experiences followed by abstractions.

17. Verbal abstractions should follow direct experience with objects and ideas, not precede them or substitute for them.

Evaluation

18. The preferred source of verification for a child's solution to a problem comes through the materials he is working with.

19. Errors are necessarily a part of the learning process; they are to be expected and even desired, for they contain information essential to further learning.

20. Those qualities of a person's learning which can be carefully measured are not necessarily the most important.

21. Objective measures of performance may have a negative effect upon learning.

22. Evidence of learning is best assessed intuitively, by direct observation.

23. The best way of evaluating the effect of the school experience on the child is to observe him over a long period of time.

24. The best measure of a child's work is his work.

II. ASSUMPTIONS ABOUT KNOWLEDGE

25. The quality of being is more important than the quality of knowing; knowledge is a means of education, not its end. The final test of an education is what a man is, not what he knows.

26. Knowledge is a function of one's personal integration of experience and therefore does not fall into neatly separate categories or "disciplines."

27. The structure of knowledge is personal and idiosyncratic, and is a function of the synthesis of each individual's experience with the world.

28. There is no minimum body of knowledge which is essential for everyone to know.
29. It is possible, even likely, that an individual may learn and possess knowledge of a phenomenon and yet be unable to display it publicly. Knowledge resides with the knower, not in its public expression.
APPENDIX B

BUSSIS AND CHITTENDEN'S TEN "THEMES" TENTATIVELY PROPOSED AS DEFINING CHARACTERISTICS OF THE "OPEN TEACHER"
(Bussis and Chittenden, 1970, p. 31)
Bussis and Chittenden's Ten "Themes" Tentatively Proposed as Defining Characteristics of the "Open Teacher"
(Bussis and Chittenden, 1970, p. 31)

Teacher's Internal Frame of Reference

1. Ideas related to children and to the process of learning, including:
   a. knowledge, beliefs, attitudes
   b. trust in ideas
   c. valuing processes.

2. Ideas related to the perception of self, including:
   a. a "beyond the classroom" self
   b. responsibility
   c. decision-maker
   d. continual learner.

Activities when Children Are NOT Present


4. Reflective evaluation of diagnostic information.

5. Seeking activity to promote personal growth.

Interactive Behaviors With Children

6. Diagnosis of learning events.

7. The guidance and extension of learning

8. Honesty of encounters.


10. Warmth.
APPENDIX C

CLASSROOM OBSERVATION
RATING SCALE

developed for
The Pilot Communities Program
Education Development Center
Newton, Massachusetts
by
TDR Associates, Inc.
Newton, Massachusetts

under U.S. Office of Education Contract
Number OEC-1-7-062805-3963
Amendment #10

March 1971
1. Texts and materials are supplied in class sets so that all children may have their own. 1 2 3 4

2. Each child has a space for his personal storage and the major part of the classroom is organized for common use. 1 2 3 4

3. Materials are kept out of the way until they are distributed or used under the teacher's direction. 1 2 3 4

4. Many different activities go on simultaneously. 1 2 3 4

5. Children are expected to do their own work without getting help from other children. 1 2 3 4

6. Manipulative materials are supplied in great diversity and range, with little replication. 1 2 3 4

7. Day is divided into large blocks of time within which children, with the teacher's help, determine their own routine. 1 2 3 4

8. Children work individually and in small groups at various activities. 1 2 3 4

9. Books are supplied in diversity and profusion (including reference, children's literature). 1 2 3 4
10. Children are not supposed to move about the room without asking permission.

11. Desks are arranged so that every child can see the blackboard or teacher from his desk.

12. The environment includes materials developed by the teacher.

13. Common environmental materials are provided.

14. Children may voluntarily make use of other areas of the building and school yard as part of their school time.

15. The program includes use of the neighborhood.

16. Children use "books" written by their classmates as part of their reading and reference materials.

17. Teacher prefers that children not talk when they are supposed to be working.

18. Children voluntarily group and regroup themselves.

19. The environment includes materials developed or supplied by the children.

20. Teacher plans and schedules the children's activities through the day.

21. Teacher makes sure children use materials only as instructed.
22. Teacher groups children for lessons directed at specific needs.

23. Children work directly with manipulative materials.

24. Materials are readily accessible to children.

25. Teacher promotes a purposeful atmosphere by expecting and enabling children to use time productively and to value their work and learning.

26. Teacher uses test results to group children for reading and/or math.

27. Children expect the teacher to correct all their work.

28. Teacher bases her instruction on each individual child and his interaction with materials and equipment.

29. Teacher gives children tests to find out what they know.

30. The emotional climate is warm and accepting.

31. The work children do is divided into subject matter areas.

32. The teacher's lessons and assignments are given to the class as a whole.

33. To obtain diagnostic information, the teacher closely observes the specific work or concern of a child and asks immediate, experience-based questions.
34. Teacher bases her instruction on curriculum
guides or text books for the grade level she
teaches.  

35. Teacher keeps notes and writes individual
histories of each child's intellectual, emotional,
physical development.  

36. Teacher has children for a period of just one
year.  

37. The class operates within clear guidelines
made explicit.  

38. Teacher takes care of dealing with conflicts
and disruptive behavior without involving the group.  

39. Children's activities, products, and ideas are
reflected abundantly about the classroom.  

40. The teacher is in charge.  

41. Before suggesting any extension or redirection
of activity, teacher gives diagnostic attention to
the particular child and his particular activity.  

42. The children spontaneously look at and discuss
each other's work.  

43. Teacher uses tests to evaluate children and rate
them in comparison to their peers.  

44. Teacher uses the assistance of someone in a
supportive, advisory capacity.  

45. Teacher tries to keep all children within her
sight so that she can make sure they are doing what
they are supposed to do.
46. Teacher has helpful colleagues with whom she discusses teaching.

47. Teacher keeps a collection of each child's work for use in evaluating his development.

48. Teacher views evaluation as information to guide her instruction and provisioning for the classroom.

49. Academic achievement is the teacher's top priority for the children.

50. Children are deeply involved in what they are doing.
### SCORING KEY

WITH WEIGHTED ITEM SCORES

FOR CLASSROOM OBSERVATION RATING SCALE AND TEACHER QUESTIONNAIRE

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APPENDIX D

SAMPLE OBSERVATIONS
NR 10:40 NR classifying buttons by the number of holes – first those that had 0 (zero) holes, then 2 holes. He went through all of the buttons and took out those which had 2 holes.

He put the buttons he had classified onto the "sticky" side of masking tape to keep them separated.

Next he took out those buttons which had 1 hole (e.g. □). All of the buttons that were left had 4 holes.

He put all the buttons on tape and recorded his project as follows.

All of the buttons were plastic. There were many small ones and many large buttons. I asked him if there were more plastic ones or more small ones. Without hesitation he said "plastic, because even the small ones are plastic."

End 11:10

RS 9:25 The teacher asked RS to estimate the number of pennies it would take to cover the surface of a 4" x 6" mirror. RS estimated 20. Then the teacher asked her to cover it. RS did it, but some pennies overlapped and some hanged over the edge. The teacher
suggested, and they rearranged the pennies in neat rows (8 x 5). The teacher asked how many there were. RS counted 8 along one edge, 5 along an adjacent edge, and said 45. The teacher asked her how she got it. RS said "multiply," but almost inaudibly. The teacher asked RS to count them. RS stared at the pennies. The teacher said, "How could you count them?" RS said, "By 2's." The teacher said, "or by 5's?" RS counted by 5's as the teacher pointed out the rows, "5, 10, 15, 20, . . ., 50. The teacher asked RS again what she had done with the 8 and the 5. RS said louder, "Multiply." The teacher told RS that multiplying was correct but they agreed that RS had just done it incorrectly. The teacher asked RS what was wrong with the pennies. RS said that they were round and didn't cover all of the space. The teacher asked, "So what could you use?" RS responded, "You could try 1" squares." RS did and got a 4 x 6 array. The teacher asked RS how many she had. RS said, "You'd take 4 times 6, but I don't know it." The teacher reminded her that she could do it as (2 x 6) + (2 x 6). RS did and figured that it was 24. The teacher asked RS which worked better, the pennies or the squares. RS said, "The squares."

End 9:45

EZ 10:45
EZ is doing a project with the newspaper - trying to find out how many times each letter of the alphabet appears in a 2" square. He took a 3" x 5" card and cut out a 2" square. He put the card on the paper and drew a line on the newspaper around the edge of the 2" square. Then he started making a graph of his results. Each time he counted a letter on the newspaper he crossed it out and filled in a box on graph paper. So far he has these results.

End 11:10

**YD** 12:48

YD and a friend NF, are measuring the width of the room across the front wall. They used a yardstick, marking with their fingers where each length of the yardstick ended up. They got 8 yards, 12 inches. Then they wanted to find the width in inches. YD suggested 36 times 8. NH shook her head no and started measuring by herself, but she had difficulty. YD asked me if 36 times 8 was right. I said that I wouldn't tell. So she went ahead and multiplied:

\[
\begin{array}{c}
36 \\
8 \\
\hline
48 \\
240 \\
\hline
288 \\
12 \\
\hline
2910
\end{array}
\]

YD asked me if that was right. I said that I wouldn't tell. They asked the teacher if 36 times 8 would give the right answer. The teacher agreed that it would. Another friend, NJ said the answer was 300. On the blackboard NH did,

\[
\begin{array}{c}
36 \\
8 \\
\hline
48 \\
240 \\
\hline
288 \\
12 \\
\hline
300
\end{array}
\]

and then showed the teacher. The teacher nodded that that was right. Then YD and NH wanted to find out how many feet it was across the room. YD said, "We'd have to take twelves out of 300," YD tried

\[
\begin{array}{c}
12 \quad \boxed{300} \\
\div 12 \\
\hline
2105
\end{array}
\]

but erased it and decided to use beans. They counted out 300 by 1's and then took away groups of 12. They got 24' 9" and wrote it down.

End 1:22
APPENDIX E

DESCRIPTIONS OF MULTIPLICATION TASKS
1. Equal Addends Tasks

1. The first equal addends task, given the fifth week, involved determining the number of disks on a Yoder abacus. The subject was given the abacus with an arrangement of disks (either 15,9,9,9 or 13,8,8,8) on the strands. The researcher asked the subject to forget that it was an abacus and asked how many disks there were.

2. In the second equal addends task, given the eighth week, the subject was asked to work with 30 red and 1 brown Cuisenaire rods and a stick. The researcher showed the subject the stick which had five equally-spaced pencil marks on it. He told the subject that he had previously measured the stick with a brown Cuisenaire rod and had made a mark on it for each length of the rod that it took to cover the stick. He showed the subject that a brown Cuisenaire rod fit between the end of the stick and the first mark. The researcher asked the subject how many brown Cuisenaire rods it had taken to cover the stick. After the subject's response, the researcher asked how many red rods it would take to cover the stick.

3. The third equal addends task, given the eleventh week, made use of 4 one-dozen egg cartons. The researcher gave the subject the egg cartons and asked how many eggs it would take to fill them.
II. Arrays

1. In the first set of tasks, the fifth week, the researcher used a pegboard for the array problem. He placed a rubber band around a 12 x 6 rectangle of pegs and asked the subject how many pegs were inside the rubber band.

2. The array task for the eighth week made use of a rectangular grid composed of eleven vertical lines and seven horizontal lines approximately 3/8" apart with dots at the intersections of the lines. The grid looked like graph paper. The subject was told that the researcher had made a dot at each place where the lines crossed and that the problem was to figure out how many dots there were altogether.

3. The third array task was given the eleventh week. An array of 1/4" diamond-shaped pieces of blue construction paper was glued onto a sheet of white paper. There were seven rows, four rows of six diamonds alternated with three rows of five diamonds. The arrangement looked like the star arrangement of the United States flag. The array was approximately 5" x 4" in size. The researcher gave the array to the subject and asked him how many diamonds there were.

III. Cartesian Products

1. The Cartesian product problem in the
first set of tasks, given the fifth week, was different for the male subjects and the female subjects in an effort to stimulate interest in solving the problem. Each girl was given paper cut-outs of three different colored blouses and four different colored skirts. She was to determine how many different outfits she could create, using any combination of one blouse and one skirt. The problem for the male subjects was similar. Each was given paper cut-outs of three tops of cars and four bodies of cars and asked how many cars there could be if every possible color combination was used.

2. The Cartesian task for the eighth week was a card game. Eight large cards (approximately 1 1/2" x 1") were made out of different colored construction paper. The researcher told the subject that in this card game a "hand" consisted of one large card and one small card and asked the subject how many different hands he could get.

3. The third Cartesian product task, given the eleventh week, involved a pictorial map. The map depicted a school at the top and a house at the bottom labeled "Home," with two rows of other places in the middle. The first row included a library, a swimming pool, a playground, an ice cream store, a house labeled "Friend's house"; the second row included a drug store, a lake, and a church. The researcher told the subject that this was his school and his house and on his way home he could stop at one place in the first row and one in the second row.
The researcher asked him how many different ways he could go home using this pattern.
## APPENDIX F-1
### BASIC FACTS PRACTICE SHEET

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</table>

My score in 7_ 6_ 5_ 4_ 3_ 2_ 1_ minutes.
NR'S RESPONSES TO BASIC MULTIPLICATION FACTS PRACTICE SHEETS

Each multiplication combination is followed by an ordered triplet: the first number is the subject's response the first week, the second number is the subject's response the seventh week, and the third number is the subject's response the eleventh week. (The actual instrument is found in Appendix F-2)

1X 1 = 1, 1, 1
1X 2 = 2, 2, 2
1X 3 = 3, 3, 3
1X 4 = 4, 4, 4
1X 5 = 5, 5, 5
1X 6 = 6, 6, 6
1X 7 = 7, 7, 7
1X 8 = 8, 8, 8
1X 9 = 9, 9, 9
1X 10 = 10, 10, 10
1X 11 = 11, 11, 11
1X 12 = 12, 12, 12

2X 2 = 4, 4, 4
2X 3 = 6, 6, 6
2X 4 = 8, 8, 8
2X 5 = 10, 10, 10
2X 6 = 12, 12, 12
2X 7 = 14, 14, 14
2X 8 = 16, 16, 16
2X 9 = 18, 18, 18
2X 10 = 20, 20, 20
2X 11 = 22, 22, 22
2X 12 = 24, 24, 24

3X 3 = 9, 9, 9
3X 4 = 12, 12, 12
3X 5 = 15, 15, 15
3X 6 = 18, 18, 18
3X 7 = 21, 21, 21
3X 8 = 24, 24, 24
3X 9 = 27, 27, 27
3X 10 = 30, 30, 30
3X 11 = 33, 33, 33
3X 12 = x, x, x

4X 4 = 16, 16, 16
4X 5 = 20, 20, 20
4X 6 = 24, 24, 24
4X 7 = 28, 28, 28
4X 8 = 32, 32, 32
4X 9 = 36, 36, 36
4X 10 = 40, 40, 40
4X 11 = 44, 44, 44
4X 12 = x, x, x

5X 3 = 15, 15, 15
5X 5 = 25, 25, 25
5X 7 = 35, 35, 35
5X 8 = 40, 40, 40
5X 9 = 45, 45, 45
5X 10 = 50, 50, 50
5X 11 = 55, 55, 55
5X 12 = x, 60, 60

6X 5 = 30, 30, 30
6X 6 = 36, x, 36
6X 7 = 38, x, x
6X 8 = 48, 48, 48
6X 9 = 44, x, x
6X 10 = 60, 60, 60
6X 11 = 66, 66, 66
6X 12 = x, x, x

7X 4 = 28, 28, 28
7X 7 = x, x, x
7X 8 = x, x, x
7X 9 = x, x, x
7X 10 = 70, 70, 70
7X 11 = 77, 77, 77
7X 12 = x, x, x

8X 4 = 32, 32, 32
8X 5 = 40, 40, 40
8X 6 = 48, 48, 48
8X 7 = x, x, x
8X 8 = x, x, x
8X 9 = x, x, x
8X 10 = 80, 80, 80
8X 11 = 80, x, 88
8X 12 = x, x, x

9X 5 = 45, 45, 45
9X 6 = x, x, x
9X 7 = x, x, x
9X 8 = x, x, x
9X 9 = x, x, x
9X 10 = 90, 90, 90
9X 11 = 99, 99, 99
9X 12 = x, x, x

10X 10 = 100, 100, 100
10X 11 = 110, 110, 110
10X 12 = x, 120, 120

11X 8 = 88, 88, 88
11X 9 = 99, 99, 99
11X 10 = 110, 110, 110
11X 11 = 102, 111, 111
11X 12 = x, 121, 132

12X 3 = x, x, x
12X 4 = x, x, x
12X 5 = x, x, x
12X 6 = x, x, x
12X 7 = x, x, x
12X 8 = x, x, x
12X 9 = x, x, x
12X 10 = x, 120, 120
12X 11 = x, 121, 132

x indicates no response
APPENDIX F-3

RS’S RESPONSES TO BASIC MULTIPLICATION FACTS PRACTICE SHEETS

Each multiplication combination is followed by an ordered triple: the first number is the subject’s response the first week, the second number is the subject’s response the seventh week, the third number is the subject’s response the eleventh week. (The actual instrument is found in Appendix F-1)

1 X 1 = 1, 1, 2
1 X 3 = 3, 3, 3
1 X 5 = 5, 5, 5
1 X 6 = 6, 6, 6
1 X 7 = 7, 7, 7
1 X 8 = 8, 8, 8
1 X 9 = 9, 9, 9
1 X 10 = 10, 10, 10
1 X 12 = 12, 12, 12

4 X 4 = 16, 16, 16
4 X 5 = 20, 20, 20
4 X 6 = x, x, x
4 X 7 = 24, x, x
4 X 8 = 34, x, x
4 X 9 = 38, x, x
4 X 10 = 40,40,40
4 X 11 = 44,44,44
4 X 12 = x, 48, x

5 X 3 = x, 15,15
5 X 5 = 25,25,25
5 X 6 = 30,30,30
5 X 7 = x, x, 35
5 X 8 = 32,44,40
5 X 9 = 49,49, x
5 X 10 = 50,50,50
5 X 11 = 55,55,55
5 X 12 = x, 56, x

6 X 4 = 24, x, x
6 X 5 = x, x, 30
6 X 6 = 30,30,30
6 X 7 = x, x, x
6 X 8 = x, x, x
6 X 9 = 36, x, x
6 X 10 = 60,60,60
6 X 11 = 66,66,66
6 X 12 = x, x, 72

7 X 4 = 24, x, x
7 X 5 = x, x, x
7 X 6 = x, 18,18
7 X 7 = x, x, x
7 X 8 = 18, x, x
7 X 9 = x, x, x
7 X 10 = 70,70,70
7 X 11 = 77,77,77
7 X 12 = x, 78, x

8 X 3 = x, x, x
8 X 4 = x, x, x
8 X 5 = x, x, x
8 X 6 = x, x, x
8 X 7 = x, x, x
8 X 8 = x, 19, x
8 X 9 = x, x, x
8 X 10 = 80, 80, 80
8 X 11 = x, 88, 88
8 X 12 = x, 89, x

9 X 4 = x, 45, 45
9 X 5 = x, x, x
9 X 6 = 36, x, x
9 X 7 = x, 63, x
9 X 8 = x, x, x
9 X 9 = 118, x, x
9 X 10 = 90, 90, 90
9 X 11 = x, 99, 99
9 X 12 = x, x, x

10 X 4 = x, x, x
10 X 5 = 100,100,100
10 X 6 = 100,100,100
10 X 7 = x, 210,101
10 X 8 = x, x, x
10 X 9 = x, x, x
10 X 10 = 120,120,120
10 X 11 = x, x, x
10 X 12 = x, x, x

11 X 3 = 36,360, x
11 X 4 = x, 99, 99
11 X 5 = x, 101,101
11 X 6 = x, 11,202
11 X 7 = x, x, x
11 X 8 = x, x, x
11 X 9 = x, 120, x
11 X 10 = x, 120, x
11 X 11 = x, 120, x
11 X 12 = x, 120, x

x indicates no response
EZ's RESPONSES TO BASIC MULTIPLICATION FACTS PRACTICE SHEETS

Each multiplication combination is followed by an ordered triple: the first number is the subject's response the first week, the second number is the subject's response the seventh week, the third number is the subject's response the eleventh week. (The actual instrument is found in Appendix F-1)

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<td>3X12 = 36,36,36</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

x indicates no response
Each multiplication combination is followed by an ordered triple: the first number is the subject's response the first week, the second number is the subject's response the seventh week, the third number is the subject's response the eleventh week. (The actual instrument is found in Appendix F-1)

| 1x1 | 1, 1, 1 | 4x4 | 16,16,16 | 8x4 | 32, 32, 32 |
| 1x3 | 3, 3, 3 | 4x5 | 20,20,20 | 8x5 | 40, 40, 40 |
| 1x5 | 5, 5, 5 | 4x6 | 24,24,24 | 8x6 | 46, 46, 46 |
| 1x7 | 7, 7, 7 | 4x8 | 32,32,32 | 8x7 | 56, 56, 56 |
| 1x9 | 9, 9, 9 | 4x9 | 36,36,36 | 8x8 | x, 64, 64 |
| 1x10 | 10,10,10 | 4x10 | 40,40,40 | 8x9 | 72, 72, 72 |
| 1x12 | 12,12,12 | 4x12 | x, x, x | 8x10 | 80, 80, 80 |
| 2x2 | 4, 4, 4 | 5x3 | 15,15,15 | 9x5 | 45, 45, 45 |
| 2x2 | 4, 4, 4 | 5x5 | 25,25,25 | 9x6 | 54, 54, 54 |
| 2x3 | 6, 6, 6 | 5x6 | 30,30,30 | 9x7 | 63, 63, 63 |
| 2x4 | 8, 8, 8 | 5x7 | 35,35,35 | 9x8 | 72, 72, 72 |
| 2x5 | 10,10,10 | 5x9 | 45,45,45 | 9x9 | 81, 81, 81 |
| 2x6 | 12,12,12 | 5x10 | 50,50,50 | 9x10 | 90, 90, 90 |
| 2x7 | 14,14,14 | 5x11 | 55,55,55 | 9x11 | 99, 99, 99 |
| 2x8 | 16,16,16 | 5x12 | x, x, x | 9x12 | x, x, x |
| 2x9 | 18,18,18 | 6x5 | 30,30,30 | 10x10 | 100,100,100 |
| 2x10 | 20,20,20 | 6x6 | 36,36,36 | 10x11 | x, x, x |
| 2x11 | 22,22,22 | 6x7 | 42,42,42 | 10x12 | x, x, x |
| 2x12 | 24,24,24 | 6x8 | 54,54,54 | 11x8 | 88, 88, 88 |
| 2x12 | 24,24,24 | 6x9 | 54,54,54 | 11x9 | 99, 99, 99 |
| 3x3 | 9, 9, 9 | 7x4 | 27,27,27 | 11x10 | x, x, x |
| 3x4 | 12,12,12 | 7x5 | 28,28,28 | 11x11 | x, x, x |
| 3x5 | 15,15,15 | 7x6 | 49,49,49 | 11x12 | x, x, x |
| 3x6 | 18,18,18 | 7x7 | 56,56,56 | 12x3 | x, x, x |
| 3x7 | 21,21,21 | 7x8 | 56,56,56 | 12x3 | x, x, x |
| 3x8 | 24,24,24 | 7x9 | 63,63,63 | 12x10 | x, x, x |
| 3x9 | 27,27,27 | 7x10 | 70,70,70 | 12x10 | x, x, x |
| 3x10 | 30,30,30 | 7x11 | 77,77,77 | 12x11 | x, x, x |
| 3x11 | 33,33,33 | 7x12 | x, x, x | 12x11 | x, x, x |
| 3x12 | x, x, x | 7x12 | x, x, x | 12x12 | x, x, x |

x indicates no response
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