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DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By
Seetharama Lakshmi Narasimhan, B.E., M.S.

The Ohio State University
1973

Reading Committee:
Dr. A. R. Bishop
Dr. R. A. Miller
Dr. W. T. Morris

Approved By

Albert B. Bishop
Adviser
Department of Industrial
and Systems Engineering
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S. L. Narasimhan

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VITA

June 1, 1936  Born - Mohanur, India

1959  B.S.M.E., P.S.G. College of Technology, University of Madras, Coimbatore, India

1960 - 1962  Instructor, Workshops Superintendent and Head of Mechanical Engineering, Nachimuthu Polytechnic, Pollachi, India

1963  M.S.I.E., The University of Tennessee, Knoxville, Tennessee


1967 - 1972  Industrial Engineer, Western Electric Company, Columbus, Ohio

1973 -  Member of Research Staff, Engineering Research Center, Western Electric Company, Princeton, New Jersey

FIELDS OF STUDY

Major Field:  Industrial and Systems Engineering


Studies in Optimization Theory and Mathematical Programing.  Professors Albert B. Bishop and Richard L. Francis

Studies in Management Decision Theory.  Professor William T. Morris

Studies in Experimental Design and Data Analysis.  Professor John B. Neuhardt
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CHAPTER I
INTRODUCTION

Production Functions

Production functions have been extensively used by economists in the theory of firms. The production function provides mathematical expression to the relationship between the quantities of inputs and the quantity of output. The production functions cover a range of activities such as fabrication and assembly, servicing a bank, publishing a book and several other input-output situations.

The theory of the firm consists of an analysis of how the entrepreneur combines various inputs efficiently to produce a stipulated output, given the existing technology or the state of the art. Aggregate production functions are assumed to represent the firm regardless of the size and number of products manufactured. Production functions do not explicitly account for the subgoals such as (1) the selection of the most profitable product or the product mix to be produced, (2) the selection of the best day-to-day operating procedure including the consumption rates of operating inputs to produce the desired outputs and (3) the investment in inventory (Smith, 1961).

Smith (1961) has analyzed some production planning models to guide the decision maker on the product mix and the quantity of inputs.
Subject of the Study

Optimization problems associated with the design of certain types of production systems constitute the subject of this dissertation. The primary objective of this research is to bring quantitative analysis to bear on some production system problems which include research and development as an integral part of planning and have been neglected due to the lack of data and/or the complexity of the problem. The second objective is to assist the production manager in decisions concerning the product mix. The third objective is to develop solution procedures for some complex problems involving research and development.

Description of the Problem

The research problem undertaken in this dissertation may be described by the following situation. A production system has to be designed for manufacturing products. Orders are received from the distributors for various products which may consume different quantities of available resources. Decisions must be made concerning the factors of production, scarce resources, and the number of each variety of product in order to achieve some objectives specified by the management. A familiar example of this type of problem is the Holt, Modigliani, Muth and Simon model for production and employment scheduling. The model determines the monthly production rate and workforce level for the given monthly demands so as to minimize certain costs. However, the model does not include the possibility of increasing the production rate by research and development activities of the firm during the planning
period. Chapter III examines these aspects in detail.

We will build several models portraying the integrated R and D, production and inventory planning situations. First we will consider a single product static case. Labor, capital and engineering are inputs to the model. The one product case is extended to a dynamic model in which engineering input has a cumulative effect over time in increasing the productivity of the product. Linear and nonlinear production functions are used as constraints.

We will then consider similar situations where the manufacturer is producing two products sharing a common technology. The amount of engineering effort spent on one product will have the effect of increasing the productivity of the other. An interaction element is introduced to represent this effect.

In addition, we will also consider some multiproduct models. They include several inputs in addition to labor, capital and engineering. These models treat the increase in productivity differently. This will be explained in Chapter III. We will discuss various solution procedures for solving these models and obtain solutions where ever possible. We point out the difficulties involved when it is not possible to obtain an analytical solution.

**Importance of the Topic**

Production programming problems are important in the allocation of resources for a firm due to the large amount of capital involved. The specific topic addressed in this dissertation is important to the
multiproduct manufacturing firm's decision analysis. Even though the literature is flooded with publications on scheduling, production planning, and R and D activities, there appears to be no single source of literature which deals with the types of integrated production systems planning as we have addressed in our research. Hopefully, our research will guide the decision maker to choose the right technique or the model to solve the problem encountered. Fortunately, recent advances in computational techniques make solution of more problems possible today than a decade ago.

Elements of the Objective Function

Production functions are employed in this study to relate production inputs and outputs. The objective function to be included is also considered. The objective function normally would comprise either a profit maximization or a cost minimization as indicated by the decision maker. Although any utility function is used in the decision-theory literature we assume the utility function is linear for the purpose of our study. The elements of cost considered in this research are the costs of inputs and the storage or the shortage cost of the product. The planning period, which is assumed to be known, will be expressed as continuous or discrete depending upon the type of solution procedure suggested for each particular model.

Types and Elements of the Production Functions

Most of the empirical studies made in the area of production
functions consider two types of functions. First is the nonlinear Cobb-Douglas production function expressed as

\[ Y = a L^\alpha K^\beta, \]

and the second is the linear Leontief input-output model expressed as

\[ Y = aL + bK, \]

where \( Y \) is the desired output and \( L \) and \( K \) are the inputs of labor and capital, respectively. The other elements in the functions are constants. Static and dynamic versions of these two production functions will be utilized in our analysis. The elements labor and capital are often referred to as the factors of production. Details of inputs and outputs are discussed in detail in Chapter III.

**Scope and Limitations of the Study**

Our production functions will be taken from the economics literature and engineering publications. No attempt will be made to estimate the parameters and constants of the production functions in our study since much has been written on this subject elsewhere. Our research is aimed at the optimization of production planning aspects of a firm with production functions as constraints. A few important models will be built and solution techniques will be discussed for each. The scope of this dissertation does not permit any extensive analysis of the computational aspects; however, few simple problems portraying production planning situations will be solved.
Summary of Dissertation

A review of recent literature is presented in Chapter II. Modeling aspects and problem formulation are the subjects of Chapter III. Chapter IV solves certain types of static and dynamic problems involving one and two products. The general model and its modifications in Chapter V extend the analysis to include several inputs in addition to labor, capital, and development expenditures. Large scale optimization techniques are utilized for obtaining solutions to problems involving multiproduct models. Finally, Chapter VI summarizes the results of the research and suggests recommendations for further study.
CHAPTER II
LITERATURE REVIEW

Introduction

This chapter begins with a review of literature on production functions. This is followed by a discussion of production functions in the theory of firm, manufacturing, and production planning. Finally, the effects of technical progress, and R and D on production are discussed.

Review of Production Functions

The theory of firm in the microeconomic literature discusses the utility of production functions in decision making. The theory assumes the prices of inputs and outputs are known quantities. In addition, the demand function for the particular product is assumed to be known. For the given technology, suppose the following function portrays the most efficient schedule of the input-output relationships:

\[ q = f(X) \]

where \( X = (x_1, x_2, \ldots, x_n) \) the vector of inputs and \( q \) the output. The relationship \( q = f(X) \) is called the production function. For every point in the input space \( I \) there exists a corresponding unique
non-negative real number called the maximum output (Ferguson, 1969).

Production functions are assumed to be continuously differentiable and to fulfill the following two axioms (Intriligator, 1971).

The Axioms

**Axiom 1.** A subset of the input space called the economic region is assumed to exist. In this region, increasing any input does not decrease the output. If a point \( X^1 > X^2 \) is in the economic region then it implies \( f(X^1) > f(X^2) \) and their marginal products (MP)

\[
\frac{\partial f(X)}{\partial x_i} > 0 ; \ i = 1, 2, \ldots, n.
\]

Thus the economic region is a subset of the input space \((I)\), that is, \((X \in I | MP(X) > 0)\), where \( MP(X) \) represents the marginal product of \( X \).

**Axiom 2.** There exists a region \( R \), a convex subset of the economic region for which the matrix of second partial derivatives, \( H \), of the production function is negative definite. The fact that

\[
H = H(X) = \frac{\partial^2 f(X)}{\partial x^2}
\]

is negative definite implies the production function in the region

\((X \in I | f(X) \geq q^0)\)
is concave for every non-negative number $q^0$. A continuous increase of any one input eventually decreases the marginal product of the input.

The axiom implies that we operate the firm in the region $R$ which lies in the positive quadrant since all inputs are non-negative. The axiom also implies that we operate at the level specified by the stationary point since any continuous increase of an input does not increase the marginal product of input beyond this point. The properties of the production functions are discussed next.

The Properties of Production Functions

**Returns to Scale.** Returns to scale describes the output response when all the inputs are changed by the same proportion. Suppose a certain point in the input space $X = (x_1, x_2, \ldots, x_n)$ is multiplied by a scalar factor $a$, where $a > 1$, to get

$$aX = (ax_1, ax_2, \ldots, ax_n).$$

Then the production function is said to exhibit constant returns to scale since the output increases by the same proportion as all inputs:

$$f(aX) = af(X)$$

If $f(aX) > af(X)$, then the function is said to exhibit increasing returns to scale, that is, the output increases by a larger proportion than all inputs. Similarly if $f(aX) < af(X)$, it is said to exhibit decreasing returns to scale, that is, the output increases by a
smaller proportion than all inputs. It is possible for a given production function to exhibit constant returns to scale at some regions of the input space and increasing returns to scale at other regions of the input space. The returns to scale provide a good insight of the behavior of the production function at various regions.

**Elasticity of Production**

The elasticity of production for any input is equal to its marginal product (MP) divided by its average product (AP), where

\[ MP_j(X) = \frac{\partial f(X)}{\partial x_j} ; \ j = 1, 2, \ldots, n \]

\[ AP_j(X) = \frac{f(X)}{x_j} ; \ j = 1, 2, \ldots, n \]

Therefore the elasticity of production for the input \( X_1 \) is

\[ e_1(X) = \frac{MP_1(X)}{AP_1(X)} = \frac{\partial f(X)}{\partial x_1} \frac{x_1}{f(X)} \]

and the elasticity of the production at any point \( X \) in the region \( X \in I | f(X) \geq q^0 \) is the sum of all elasticities of output with respect to the various inputs at this point:

\[ e(X) = \sum_{j=1}^{n} e_j(X) \]

The elasticity of production provides a local measure of returns to scale defined at a point in input space.
Elasticity of Substitution

Ferguson (1969) states: "At the heart of neoclassical theory is the elasticity of substitution, a concept introduced by Hicks in 1932."

They characterize the production function by various combinations of inputs producing the same level of output. Suppose we consider two inputs, labor $X_1$ and capital $X_k$. The elasticity of substitution $\sigma(X)$ is given by the following formula:

$$\sigma(X) = \frac{\Delta \left( \frac{X_k}{X_1} \right)}{\frac{X_k}{X_1}} + \frac{\Delta \left( \frac{w}{r} \right)}{\frac{w}{r}}$$

where $w$ and $r$ are the prices of labor and capital respectively. The formula shows a proportional change in the capital-labor ratio produced by a given proportional change in the inputs price ratio. As the input $X_1$ becomes relatively more expensive, the entrepreneur substitutes the input $X_k$ and vice versa to the extent permitted by the production function. The substitution possibilities provide a measure to the entrepreneur when optimizing the cost of total inputs to obtain a given production rate.

Production functions have been used by economists to identify the national productivity trends, productivity differences among nations and industries. The applications of production functions in the theory of the firm are dealt with in detail by Ferguson (1969) and its applications in investment and returns are described in length by Arrow and Kurz (1971). Solow has pioneered the application of production functions in growth theory (1959). Since our interest lies mainly in
the application of production functions in manufacturing planning, the literature review focuses on this area. A brief overview of the theory of the firm follows next.

Theory of the Firm

Cross-section studies

In the theory of the firm it is assumed that the entrepreneur is interested in maximizing profits given the production function, the input prices, and the output prices. The theory deals with the equilibrium characteristics of the interactions arising among firms due to a change in input or output price. For example, a firm may analyze the effect of increasing the production of a particular commodity on its price level in the market.

Dynamic studies

In any economy, choices must be made on the proportion of consumption and savings for any given output. An increase in present consumption will decrease the capital accumulation and hence the future investments will be decreased. Similarly a decrease in present consumption will increase the possibility of future capital investments. Therefore the choices must be made over time between consumption and capital accumulation, such that optimal economic growth is obtained. Neoclassical growth models address the various aspects of the economy using production functions. They extensively deal with cross section studies as well as dynamic studies (Ferguson, 1969).
Production Functions in Manufacturing

Cross section studies were pioneered by Zvi Griliches (1964) in agricultural production. This study was followed by others in industry. Most of these studies use a census of manufacturer's data and estimate the production function parameters (Eisner) for the manufacturing industries.

Production Functions in Production Planning

Samuelson says "go to any machine plant, pick up any engineering catalogue, study the books of physics and the history of industrial processes and you will see a variety of ways of doing things." Chenery and Leontief have pioneered the use of production functions in economic analysis of production planning problems. The following example by Chenery (1949) analyzes the transmission of gas between two places and determines the most economical diameter of the pipe and the compressor capacity.

Example 1. The Gas Transmission Example

In Figure 2.1, the relationship between the output quantity \( y \), the diameter of the pipeline \( D \) for the given pressures at the end points \( p_1 \) and \( p_2 \) is provided by the formula:

\[
\begin{align*}
y &= KD^{9/3} \left( p_1^2 - p_2^2 \right)^{1/2}
\end{align*}
\]

where \( K \) is a constant. A given value of output \( y \) can be obtained by several combinations of diameters of the pipe and the pressure difference between two points \( p_1 - p_2 \). Given the atmospheric pressure \( p_2 \),
Figure 2.1

Gas Transmission Example
the output is a trade off between the diameter of the pipe D and the pressure $p_1$. The pressure $p_1$ at the point of the source can be increased by a bigger capacity of the compressor. The bigger the capacity of the compressor, the greater is the fuel consumption. It means there is a substitution possibility between the pipe diameter and the compressor capacity to obtain the desired output economically. Therefore the formula exhibits the property of substitution and hence the relationship between the output $y$ and the inputs $D$, and $(p_2 - p_1)$ are expressed as a production function:

$$y = f(D, (p_2 - p_1)) \text{ or } y = f(D, X)$$

where $X$ is the fuel consumption rate or the capacity of the compressor.

**Example 2. The Automobile Body Manufacturing Problem**

Figure 2.2 portrays the situation of an automobile body manufacturer. The inputs are steel, labor and machinery and obviously the output is the finished good, the automobile body.

The amount of steel required for manufacturing a body depends on the design specifications. The labor requirements are determined by the type of machines used and the decision on the type of machine normally depends upon the demand for the product. Therefore, there is a trade off between the labor requirements and capital requirements of the machinery. Once the machinery is bought and installed the capacity is fixed even though the utilization could be varied. The capacity may be increased only by installing additional machines.

Therefore for a given demand the determination of the relationship between inputs (steel, labor and the capital for the machinery) and
Labor \( (X_1) \)

Steel \( (X_2) \)

MACHINERY \( (X_3) \)

Automobile body output \( (y) \)

Figure 2.2

Example of Production Function

Automobile Body Manufacturing
the output (the automobile body) given by the production function

\[ y = f(X_1, X_2, X_3) \]

is a technical problem.

However, the selection of the best input combination for the particular output level depends upon the input and output prices and it is the subject of examination of this research. The decision analysis for a single-period problem or for a dynamic model with a constant demand is portrayed by the following situation.

Decision Analysis

Let \( W_1 \) and \( W_2 \) be the prices of labor input \( X_1 \) and the capital input \( X_2 \), respectively, and \( r \) be the fraction of the capital input consumed due to depreciation and obsolescence for the same period. The object of the decision analysis is to optimize the total cost per period:

\[ C = W_1 X_1 + r W_2 X_2 \]

subject to the production function constraints

\[ y = f(X_1, X_2) \]

The solution can be obtained by the Lagrange Multiplier technique. Rewriting the equations,

\[ \text{Min } \phi = W_1 X_1 + r W_2 X_2 - \lambda [f(X_1, X_2) - y] \]

where \( \lambda \) is the lagrange multiplier. The necessary conditions in addition
to the production function are given by Smith (1961) as:

\[
W_1 - \lambda \frac{\partial f}{\partial X_1} = 0
\]

\[
W_2 - \lambda \frac{\partial f}{\partial X_2} = 0
\]

Solving, the optimal values of \(X_1, X_2,\) and \(\lambda\) are:

\[
X_1^0 = d_1(W_1, rW_2, y)
\]

\[
X_2^0 = d_2(W_1, rW_2, y)
\]

\[
\lambda^0 = \lambda(W_1, rW_2, y)
\]

The labor input is called the variable input and the capital input is called the fixed input. The decision analysis can be carried out to include more than one period and for different demands. It is the subject of the next example.

**Example 3. Dynamic Problems in Production-Inventory Planning**

The previous problem assumed the demand rate was constant and hence a constant optimum production rate was calculated. In the real world, it is very possible for the production rate \(y(t)\) to vary over the time \(t \in [0, T]\). Suppose the items manufactured have to be sold at the same time, for some reason. Then the decision maker has a choice of how much of the demand he wants to meet so as to optimize his cost of inputs. The following example by Smith clarifies the problem on
the assumptions that output cannot be produced without some positive inputs and it is constrained by the production function of the form:

\[ y(t) = f(X_1(t), X_2) \]

Therefore the problem is to minimize the total cost

\[ Z = \int_0^T [W_1 X_1(t) + W_2 X_2] dt \quad (1) \]

s.t. \[ y(t) = f[X_1(t), X_2] \quad (2) \]

where \( X_1 \) is the variable input (labor) which can be varied during consecutive periods and \( X_2 \) is the fixed input which cannot be changed once the investment decision is made. Smith (1961) has solved this problem using the calculus of variations technique. The problem may be rewritten as:

\[ \text{Min } F = \int_0^T [W_1 X_1(t) - \lambda \{ f(X_1(t), X_2) - y(t) \}] dt + W_2 X_2 \quad (3) \]

with respect to the optimal \( X_1(t) \) and scalar variable \( X_2 \). The \( \lambda \) is a function of \( W_1, W_2, T \) and \( y^T(t) \) to be determined. The \( \lambda \) may be thought of as a vector of Lagrange multipliers. The necessary conditions are given by:

\[ \frac{\partial F}{\partial X_1} = W_1 - \lambda f_1(X_1(t), X_2) = 0 \quad (4) \]

\[ \frac{\partial F}{\partial X_2} = -\int_0^T \lambda f_2(X_1(t), X_2) dt + W_2 = 0 \quad (5) \]

where \( f_1 \) and \( f_2 \) refer to the derivatives of the production function \( f \).
with respect to variables $X_1$ and $X_2$, respectively. The production function along with the necessary conditions yield the optimal path of inputs $X_1^0(t)$ and $X_2^0$. Equation (4) yields

$$
\lambda^0 = \frac{W_1}{f_1(X_1(t), X_2)}
$$

Substituting the value of $\lambda^0$ in equation (5) we obtain

$$
W_2 = \int_0^T \frac{W_1 f_2(X_1(t), X_2)}{f_1(X_1(t), X_2)} \, dt
$$

A solution can be obtained if the specific relationship for the production function is given. However, the optimal solution may be specified in the functional form as follows:

$$
\bar{X}_1(t) = h_1[W_1, W_2, T, y_0^T(t)]
$$

$$
\bar{X}_2 = h_2[W_1, W_2, T, y_0^T(t)]
$$

$$
\bar{X}(t) = \lambda[W_1, W_2, T, y_0^T(t)]
$$

The sufficient conditions verify whether the point actually represents a minimum.

**Example 4. Dynamic Models to Include Inventory and Sales**

The previous problem assumed that the product manufactured should be sold at the same time. The following example by Smith (1961) analyzes the effect of inventory on production rate. Let

$$
H(t) = \text{the amount of product inventory held up to time} \ t
$$
The cumulative cost over the time period \((0, T)\) can be written

\[
Z = \int_0^T [W_1 X_1(t) + k(R(0) + P(t) - S(t))] dt + W_2 X_2
\]

Then the problem is to choose \(X_1(t), X_2\), and \(P(t)\) so as to minimize the cost \(Z\) subject to the production function constraint

\[
y(t) = \dot{P}(t) = f(X_1(t), X_2)
\]

If we assume the demand should be met at all times, then the inventory at any time \(t\) may be written as:

\[
H(t) = [H(0) + P(t) - S(t)] \geq 0
\]

The cumulative production

\[
P(T) = \int_0^T f(X_1(t), X_2) dt = S(T) - H(0)
\]

The necessary conditions for \(Z\) to be a minimum are given by:

\[
W_1 - \lambda f_1(X_1(t), X_2) = 0
\]

\[
- \int_0^T \lambda f_2(X_1(t), X_2) dt + W_2 = 0
\]

and the Euler equation:

\[
k - \frac{d}{dt} (\lambda) \geq 0
\]
where $f_1$ and $f_2$ refer to the partial derivatives of the production function with respect to $X_1$ and $X_2$.

Integrating the equation (7) over the region $0 < t$ to $t + \theta < T$, the following relationship may be established:

$$k\theta + \lambda t \geq \lambda(t + \theta)$$

The L.H.S. of equation (8) refers to the marginal cost of producing a unit at time $t$ and holding it until time $t + \theta$. The R.H.S. refers to the cost of producing a unit at time $t + \theta$. Therefore the equation says the marginal cost of producing a unit at given time must not exceed the cost of producing it at an earlier time and holding it until given time. In any given problem the inequality of equation (7) may hold some or the entire planning period. The effect of constraint (7) in obtaining solutions is discussed next.

1) The inequality holds for some interval of time

We know it costs more to produce it earlier and hold it until given time. Therefore, the policy should be to carry minimum or no inventory. Ideally the production rate $P(t)$ should equal the demand $S(t)$. Then the inventory term in the objective function equals zero and hence the problem becomes equivalent to example 3 during this interval.

2) The equality holds for some interval of time

Integrating the equation (7) we obtain

$$\lambda(t) = \lambda(T) + k(T - t)$$ (9)
which says the marginal product cost is independent of production and demand. In fact, it is a function of time only.

3) The equality holds for the entire planning horizon

For this case to hold the equation (9) should be compatible with other necessary conditions. Therefore, the following equations should be simultaneously solved to obtain solutions:

\[ Y(t) = \int_0^t f[X_1(t), X_2]dt \]

\[ Y(T) = \int_0^T f[X_1(t), X_2]dt = S(T) - H(0) \]

\[ \lambda(t) = k(T - t) + \frac{W_1}{f_1[X_1(t), X_2]} \]

and \[ W_2 = \int_0^T \lambda f[X_1(t), X_2]dt \]

We have reviewed some examples pertinent to production planning. Next a discussion on the effect of technical progress on production functions follows.

*Solow's Investment and Technical Progress*

So far we have discussed some models which included the production functions as constraints in manufacturing planning. The static as well as dynamic models minimized the total cost of inputs. The production functions assumed the input-output relationship remained the same during the period of time \((0, T)\) considered. The production functions do not account for the possibility of increased output due to technical
progress in the time period. Mansfield (1968) defines technical progress as:

"Technical progress is advance in knowledge. Technical change often occurs as a result of inventions that do not depend on new scientific principles."

Solow (1959) expresses that technical progress should include the following:

1) Improvement in skill and quantity of labor force,
2) Returns to improvement in research and education,
3) Improvement in techniques within industries, and
4) Changes in industrial composition of input and output.

Solow suggested the following production function model to include technical change

\[ y(t) = e^{at} L(t)^a K(t)^b \]

where

- \( y(t) \) = the production rate at time \( t \)
- \( L(t) \) = Labor utilization rate at time \( t \)
- \( K(t) \) = capital utilization rate at time \( t \)
- \( e^{at} \) = factor reflecting the technical change
- \( a, b \) = Constants determined from past data

Solow describes this phenomenon as the aggregate production function shifting through time.

---

**Leontief's Input-output Model**

A well known linear production function is Leontief's input-output model. It has found wide use in economic planning and forecasting. Most of this work was done by government agencies. For example,
the model could be employed to study employment levels and gross production necessary to meet various bills of goods (Hadley, 1963). It can also be used to describe how the increase in wage rate of automobile workers may change prices of the entire economy assuming the companies pay the increased wages by charging the customers more for the cars.

Leontief assumes that the economy consists of a number of interacting industries. To produce goods one industry uses another's output. Each industry should produce enough goods so that the overall demand is satisfied. The static model may be described for \( n \) industries. Let,

- \( y_{ij} \) = the demand of goods from industry \( i \) needed by industry \( j \)
- \( b_i \) = the exogenous demand for good \( i \)
- \( x_i \) = the total amount that industry \( i \) must produce to meet the demand exactly

\[
x_i = \sum_{j=1}^{n} y_{ij} + b_i, \quad i = 1, 2, \ldots, n
\]

The inputs \( y_{ij} \) to industry \( j \) were provided by another industry \( i \). The output of industry \( i \) was expressed by a simple linear expression.

\[
y_{ij} = a_{ij} x_j, \quad \text{all } i, j
\]

which in fact is in the form of a production function for industry \( j \). The static function does not include technological improvement or any other factor which will change \( a_{ij} \).
Dynamic Leontief Model

If we express the inputs and outputs as functions of time then the dynamic Leontief's model may be expressed as:

\[ y_{ij}(t) = a_{ij}x_j(t) \]

Wagner (1957) has illustrated the applications of dynamic models to industries producing products. These models consider the total production of industries (including the change in productivity due to technical change) and minimize their total costs during the period \((0,T)\). Other dynamic Leontief models which were developed in the 1960's include more detailed versions of technical change. Stone (1963) indicates that four types of technical changes are important:

1. Substitution among energy resources such as shifts from coal to petroleum or electricity
2. Substitution of manufactured raw material such as shifts from natural rubber to synthetic
3. General increase in the use of manufactured and service inputs due to mechanization, education, and R & D expenditures
4. Long run effect (trends) of the type of industry such as manufacturing, construction and others.

Using the dynamic Leontief model and taking into account the various changes Totemota et al. (1967) have done a study on the Japanese economy. For details on this and other studies on the applications of the dynamic Leontief model the interested reader is referred to Carter (1970).
Research and Innovation

The production functions discussed so far included general technical progress. In addition to general technical change, the federal government and its agencies, many nonprofit organizations, and industrial giants pour billions of dollars into specific research and development projects. Table 2.1 shows the total expenditures on R and D (Mansfield et al., 1971).

Many private industries received defense contracts from the federal government for doing research in aircraft, instruments, and electrical equipment. The R & D expenditures have an accelerating effect on technical progress of specific industries. The meaning of the terms R & D, invention, and innovation are often confused. An attempt is made to clarify their meanings before exploring their effects on manufacturing planning. The following definitions are from National Science Foundation publications as quoted in Mansfield (1968) and Mansfield et al. (1971).

Basic Research

Basic research includes: "research projects which represent original investigation for the advancement of scientific knowledge and which do not have specific commercial objectives although it may be in the fields of potential interest to the reporting company."

Applied Research

Applied research includes: "research projects which represent investigation directed to discovery of new scientific knowledge and
TABLE 2.1
Research and Development Expenditures

<table>
<thead>
<tr>
<th>Year</th>
<th>Total R &amp; D (USA) Expenditures in Billions of dollars</th>
</tr>
</thead>
<tbody>
<tr>
<td>1965</td>
<td>20.449</td>
</tr>
<tr>
<td>1966</td>
<td>22.285</td>
</tr>
<tr>
<td>1967</td>
<td>23.680</td>
</tr>
<tr>
<td>1968(^a)</td>
<td>25.330</td>
</tr>
<tr>
<td>1969(^a)</td>
<td>26.250</td>
</tr>
<tr>
<td>1970(^a)</td>
<td>27.250</td>
</tr>
</tbody>
</table>

\(^a\) Estimated.
which have specific commercial objectives with respect to either products or processes." For example, an applied research project may be carried out to find out whether a new type of material is superior to an old material in a specific application.

**Development**

Development projects attempt to reduce research findings to practice. "This often entails the making of various types of experiments, the design and development of prototypes and construction of pilot plants."

Many firms carry out research but relatively smaller proportion of the budget is spent on basic research. There is normally an overlap between basic research and applied research. Mansfield indicates that there is considerable overlap between activities of research and the activities of development. It would be very difficult to estimate exactly how much is spent on each activity.

The characteristics of R & D are similar to those of any other economic activity which is surrounded by profitability and uncertainty. It is extremely difficult to evaluate the returns from an investment made in R & D. The returns generally depend on the size of the firm, type and size of the project, utility, marketability and rate of imitation by others.

In spite of the confusion surrounding the semantics on R & D and the uncertainty of its economic outcome, many firms spend considerable amount of resources in R & D activity. Most of the industrial research and development projects carried out by the firms are relatively
safe from a technical viewpoint (Mansfield, 1971). The average probability of technical completion and range among laboratories are given in Table 2.2.

The evidence also indicates that sixty percent of the technically incomplete projects were caused by their poor commercial prospects rather than insolvable technical problems. These facts lead us to believe that most firms aim at relatively moderate development projects and their technical completion is relatively high.

Production Functions in R & D

Model 1

A study made by Mansfield (1968) indicates that the cumulative R & D expenditures may be included as a part of the model to establish the relationship between the inputs and output. Let

\[ y(t) = \text{the output rate at time } t \]
\[ L(t) = \text{the labor input at time } t \]
\[ K(t) = \text{the stock of capital employed at time } t \]
\[ R(g) = \text{the rate of expenditure on R & D by the firm at time } g \]

Then the production function is:

\[ y(t) = A e^{at} \int_{0}^{t} e^{-\lambda t} R(g) \, dg^b L(t)^{a} K(t)^{1-a} \]

where \( \lambda \) is the rate of depreciation of an investment in R & D, and \( A, a, b, \) and \( a \) the parameters to be established by analyzing the past data. Mansfield has indicated that the formula should be used with caution for particular applications. He has also illustrated how the returns
TABLE 2.2
Mean and Range of the Technical Completion
of R & D Projects in U. S. Industries (1963-65)

<table>
<thead>
<tr>
<th>Industry</th>
<th>Probability of Technical Completion</th>
<th>Average</th>
<th>Range among Laboratories</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>0.56</td>
<td>0.10 - 0.99</td>
<td></td>
</tr>
<tr>
<td>Chemical</td>
<td>0.70</td>
<td>0.37 - 0.99</td>
<td></td>
</tr>
<tr>
<td>Drug</td>
<td>0.32</td>
<td>0.12 - 0.62</td>
<td></td>
</tr>
<tr>
<td>Electronics</td>
<td>0.73</td>
<td>0.20 - 0.99</td>
<td></td>
</tr>
<tr>
<td>Petroleum</td>
<td>0.50</td>
<td>0.10 - 0.74</td>
<td></td>
</tr>
</tbody>
</table>
from R & D may be calculated from this model.

**Model 2**

A study conducted by Lele and O'Leary (1971) on a particular product indicates the following relationship holds:

\[ y(t) = L(t)^\alpha K(t)^\beta E(t)^\gamma, \]

where

- \( y(t) \) = the production rate at time \( t \)
- \( L(t) \) = the labor input at time \( t \)
- \( K(t) \) = the stock of capital employed at time \( t \)
- \( E(t) \) = cumulative engineering expenditure spent on the product up to time \( t \).

It can be noticed this model is a special case of Mansfield's model.

**Returns from R & D**

Another study conducted by Mansfield *et al.* (1971) has established a relationship among sales, R & D expenditures and effectiveness of R & D.

\[ E = a_0 + a_1 R + a_2 S + Z \]

where

- \( E \) = Effectiveness or output per dollar of R & D program
- \( R \) = Total R & D expenditure of the firm
- \( S \) = Size of the firm measured in terms of dollars
- \( Z \) = Random error
The relationship was verified in chemical and drug industries and found to be useful.

**Conclusion**

We have reviewed the roles of production functions in the theory of the firm, in overall manufacturing, in product planning, and in research and development. We noticed that technical progress and research and development activities have an accelerating effect on the production rate. Taking advantage of the knowledge of the production functions gained from the theory of the firm and the research and development, in the next chapter we build some models to include the effects of these factors on a firm's production planning decisions.
CHAPTER III
PROBLEM FORMULATION

Introduction

A review of production functions and their applications indicate that empirical studies have been conducted in testing the validity of technical change. Specific models have also been tested in industries to include the effect of R and D in the production function. However, most of the production planning literature addresses the optimum allocation of resources assuming the capacity of the plant is fixed in spite of the fact that the capacity can be varied through engineering research and development work. To the author's knowledge there appears to be no analytical study published in the area of production planning which accounts for the potential of increasing the capacity through R and D and its benefits of interaction among products.

The lack of research in this area may partly be explained by the complex nature of the effect of R & D on production. An example would help us to understand the nature of the problem confronted by the decision maker.

When a firm manufactures a single item, the decision problem involves optimizing the available resources including the R & D expenditure for that particular item. When the firm produces two items
which possess similar technology the amount of R & D spent on one product may have an effect on the productivity of the other product. This may be called an interaction effect. The interaction effect increases enormously as the number of products which have common technology increase. The purpose of this chapter is to develop analytical models for production planning which would include the effect of engineering R & D. Before we get into the details, a brief review is made on the art of model building.

**Model Building**

The usefulness of models in the analysis of systems has been recognized widely. Bishop (1969) defines: "...the term model refers simply to an abstraction of the real system. It is an abstraction, moreover, which embodies those features of the actual system which must be treated explicitly in any analysis or synthesis of the system. They show the basic elements of their respective systems and indicate the interconnections among them." In our analysis an attempt will be made to include the essential features of production planning in building the desired model.

The art of model building has been discussed in detail. Morris (1968) states that the process of model building is intuitive. Specifying any set of rules for obtaining models would have only very limited use. Morris suggests some basic guidelines for model builders, in general. Suggestions relevant to our situation are discussed next.

1. Factor the system problem into simpler problems.
The simpler problems could be easily modeled and subsequently they may be integrated into complex system model. Solving the factored system yields an approximate or suboptimal solution from the viewpoint of the system model. It will become evident later how exactly this explanation portrays the decomposition technique used to solve large problems in Chapter V.

2. Establish a clear statement of the deductive objectives.

One should specify, as clearly as possible, whether the objective of the model is to optimize the entire system or to predict the consequences of various policies. Our objective is to obtain an optimal policy although modifications to the model would answer the latter.

3. If a tractable model is obtained, enrich it. Otherwise simplify.

In general the models may be simplified by eliminating some variables, using linear relations, adding stronger assumptions and restrictions, and suppressing randomness. Many of these suggestions will be adapted later in our analysis.

It would be ideal to be able to build an elaborate model to contain all the fine features of the real system. However, it may not yield a computationally feasible and economically acceptable solution. Therefore, our research objective will be to build some models and then solve certain special cases. Optner (1960) suggests the following
building blocks for model builders.

1. Describe the input variables
2. Describe the output variables
3. Describe the relationship between the input and output variables by identifying the activities associated with the process
4. Describe the objective function

The description of these building blocks is the subject of our next examination.

The Input Variables

The input variables, which are also referred to as the independent variables, embrace all goods and services which are necessary to produce a product or products. The goods include all items such as raw materials, machinery, buildings and other supporting facilities. Typically, the services would include direct labor and indirect labor which embraces control, supervision and maintenance. Engineering support and managerial effort would be mandatory for efficient operation of a firm. The inputs are also referred as the factors of production.

The Output Variables

The output variables which are also known as the dependent variables, refer to the end products. An end product may contain several sub-assemblies and components. The details of the intermediate
operations will not be considered explicitly as this would necessitate
the formulation of a waiting line problem. However, if the intermediate operations form an end product in addition to being a part of
another end product, then they can be regarded as two independent prod-
ucts.

Relationships between Inputs and Outputs

The relationships between the inputs and the outputs are specified by the production functions. The input and output variables could
be in time units, cubic feet of volume, pounds of weight, or value in
dollars as appropriate. The empirical studies made of production
functions use dollar values for establishing the relationships. As
discussed earlier, the establishment of such input output relationships
constitutes a technological study. The existence of such functions
were already discussed in detail by Mansfield (1968) and hence no at-
tempt will be made in the present research to establish them.

Another important relationship is the inventory status at any
given time.

$$I = y_t - d_t,$$

where I represents the change in inventory at time t, $y_t$ the produc-
tion rate at time t, and $d_t$ is the demand for the item at time t. All
inputs such as raw materials, machinery, buildings, supplies and labor
force are assumed to be consumed at time t to produce necessary prod-
ucts. The only exception would be the effort spent on research and
development which, in general, would have a cumulative effect on
production rate. The effect of research and development on production rate has already been discussed in detail. These relationships will be used in establishing the models portraying the desired system.

The Elements of the Objective Function

The overall objective of the entrepreneur is to allocate resources so as to benefit the firm as a whole. The analyst should be able to transform the management goals into mathematical models. It is presumed that the management will be able to state its objectives as clearly as possible regardless of whether it is a short run or a long run decision. The analyst should be able to convert decisions which involve gambles or risks such as a small but certain profit and a large but highly uncertain profit into mathematical terms. It would be preferable if the model can find the effect of each and every action on the firm as a whole. An ideal model should be able to accomplish almost everything.

However, the nature of our research restricts our objectives to certain commonly acceptable special structures which would permit us to obtain tractable solutions. The author's view is the same as Morris' which he calls the systems approach:

"If he (the analyst) can discover or create a mathematical structure which reasonably reflects the management decision, then he is in a position to use the mathematical structure or model to predict the results of various managerial choices."

Associated with each input, output and inventory there exist several cost elements. The following assumptions will be made in what we consider an input, output and associated costs.
Input Costs

The inputs include, labor, supervision, management, capital, engineering, and other necessary items to manufacture the products. The cost per unit of labor, supervision and management is assumed to remain constant during the planning horizon. The cost of hiring and terminating an employee will not be considered explicitly. However, a penalty cost will be imposed on any change in level of employment during consecutive periods (in some models). In real world complexities such as union regulations, division of labor and other control systems exist. For a detailed discussion on this subject see Holt et al. (1963). In our model it is assumed that an aggregate unit of labor consisting of all necessary manpower to produce an item (excludes engineering) exists and the cost per unit is given. This assumption will be relaxed when presenting the general model in Chapter V. Incremental costs of overtime, parttime or multiple shifts are not considered.

The unit cost of capital inputs such as machinery, buildings, material handling and other support equipment is assumed to be given. The capital inputs are also assumed to be expendable during the period which was employed to manufacture products. Several problems exist in data collection and manipulation for obtaining a unit cost of capital since factors such as obsolescence and depreciation exist. These problems are discussed in detail by Walters (1963). The unit cost of engineering research and development is assumed to be given. In contrast to labor and capital inputs, where the effects of the inputs are expended during the same period, the R and D effort is assumed to have a cumulative effect on production rate.
Output and Inventory Costs

The outputs or the products manufactured will be sold to satisfy the demand or stored to meet future demands. In some cases back-ordering will also be considered. The cost of storage includes rent, deterioration, spoilage and opportunity costs. The back-order cost includes bookkeeping, goodwill of customers and other associated costs. Back orders will be fulfilled before satisfying the present demand so that all customers are treated in the FIFO system (Hadley, 1963).

Uncertainty

There are several uncertainties associated with the demand for the products; costs of inputs and outputs, profit, and functional relationships. Factors such as competition, strikes and other unpredictable aspects also interact with the real system. The possible uncertainties will be suppressed in our model.

Taxes & Interest

Although it is possible to include factors such as taxes and interest rates in dynamic models, we will ignore them for the sake of simplicity.

The Production Planning Models

Consider a plant which combines all the inputs $X_i, i = 1, 2, \ldots, n$ to produce end products at time $t$. The end products are either sold at time $t$ or stored in warehouses for use in subsequent periods. We make the following assumptions in our modeling of production planning problems:
Assumption 1

The system is able to overcome all delays in processing of the product and hence there would be no in-process goods.

Assumption 2

No defective goods are manufactured. Otherwise, we can assume that suitable allowances have been added to the inputs to produce the demand rate.

Assumption 3

The required amounts of inputs are delivered in time at the proper place.

Assumption 4

The inventory is not damaged or stolen and does not become obsolete. Otherwise, allowances should be made to compensate for them.

Relaxing any one of these assumptions would constitute a major research topic by itself. Since our objective is to build integrated system models we cannot be concerned with many intricate details.

So far we have discussed the aspects of model building and objectives of management. Now we are in a position to build models representing many production planning situations. We will be concerned with both static and dynamic models. Models containing one product will be described first followed by models containing two products.

Then these can be extended to general models portraying multi-product manufacturing. These models will be unique in the sense they include engineering research and development as an integral part of planning.
The models are presented in the following order.

1. Static model containing one product
2. Static model containing two products
3. Dynamic model containing one product
4. Dynamic model containing two products
5. General model and its extensions

The Models

PI Static Model Containing One Product

As in any static system we will be concerned with the overall performance of the system over some period of time or at a point of time (Bishop, 1969). The performance of the system is measured by a cost function known as the objective function. Let $X_1$ represent the total amount of each input necessary to produce a certain product in order to meet the demand. Then it is required to determine the quantity of each $X_1$ such that the total cost of the inputs of the system is minimized for the period under consideration.

The system is portrayed by Figure 3.1. Inputs and outputs are labeled. Let $X_1$ be the total units of labor, $X_2$ be the total units of capital, and $X_3$ be the total units of engineering necessary to produce $Y_1$ units of the product. Assume unlimited quantities of inputs are available and that the demand must be met. Then the problem may be stated as:

$$\text{Min. } Z = c_1X_1 + c_2X_2 + c_3X_3$$ (1)
Figure 3.1
One Product Static Model
S. to.

\[ Y_1 = f(X_1, X_2, X_3) \]  

(2)

and

\[ X_1, X_2, X_3 \geq 0 \]  

(3)

where \( c_1, c_2, \) and \( c_3 \) are cost coefficients. Solving the problem represented by (1), (2), and (3) would yield the optimal input quantities.

In addition to the lower bounds imposed on the input variables by equation (3) upper bounds may also be included.

**P2 Static Model Containing Two Products**

The model P1 considered a system containing one product. Imagine an entrepreneur producing two items. One of the following two cases may exist.

**Case 1**

The two products may be different in that they may have nothing in common as portrayed by Figure 3.2. Let \( X_1, X_2 \) and \( X_3 \) represent the total units of labor, capital and engineering R & D respectively necessary to manufacture \( Y_1 \) units of product 1. Similarly, \( X_4, X_5, \) and \( X_6 \) represent the corresponding total units of inputs required to manufacture \( Y_2 \) units of product 2.

If the entrepreneur wanted to meet the demand of both products, then the quantities produced will equal the demand. Then the system of equations for product 1 and product 2 can be written as:

**Product 1**

\[ \text{Min. } Z_1 = c_1X_1 + c_2X_2 + c_3X_3 \]  

(4)
Figure 3.2
Static Model Containing Two Products
\[ Y_1 = f_1(X_1, X_2, X_3) \]  
\[ \text{and } 0 \leq X_i \leq b_i, \quad i = 1, 2, 3 \]  

**Product 2**  
\[ \text{Min. } Z_2 = c_4X_4 + c_5X_5 + c_6X_6 \]  
\[ \text{S.t.o. } Y_2 = f_2(X_4, X_5, X_6) \]  
\[ \text{and } 0 \leq X_i \leq b_i, \quad i = 4, 5, 6 \]

where \( c_i, b_i; \ i = 1, 2, \ldots, 6 \) represent the cost of inputs and the upper bounds on the availability of inputs (raw materials) \( X_i, \ i = 1, 2, \ldots, 6 \) respectively.

Optimizing individually for products 1 and 2 would also optimize the entire system.

In addition to case 1, the following may also exist.

**Case 2**

The two products may use different raw materials but share the same technology. It was discussed earlier that the inputs \( X_1 \) and \( X_2 \) are utilized in manufacturing the product whereas \( X_3 \), the engineering input is used to accelerate production in general. Since products 1 and 2 share common technology the quantity of \( X_3 \) spent on product 1 would also benefit product 2 and the same fact holds for input \( X_6 \) on product 1. The effects are represented in Figure 3.3.

Suppose we indicate the effect of \( X_3 \) on product 2 as equivalent
Figure 3.3

Two Product Static Model

with Interaction Effect
to $U_1$. If it is possible to quantify $U_1$ as a function of $X_3$, the following relationship holds.

$$U_1 = f(X_3)$$

Similarly,

$$U_2 = f(X_6)$$

The relationship may be linear or nonlinear. The systems of equations for product 1 and product 2 may be specified as:

**Product 1**

$$\text{Min } Z_1 = c_1X_1 + c_2X_2 + c_3X_3$$

S. to.

$$Y_1 = f_1(X_1, X_2, X_3, U_2)$$

and

$$0 \leq X_i \leq b_i, \quad i = 1, 2, 3$$

**Product 2**

$$\text{Min } Z_2 = c_4X_4 + c_5X_5 + c_6X_6$$

S. to.

$$Y_2 = f_2(X_4, X_5, X_6, U_1)$$

and

$$0 \leq X_i \leq b_i, \quad i = 4, 5, 6$$

If the products 1 and 2 were produced by departments A and B, A would want to increase $U_2$ as much as possible because B is bearing the expenses and A is benefiting to a certain extent. Similarly, B would want $U_1$ to be increased. But the entrepreneur has the problem
of optimizing both the systems. He would ideally want to minimize the
total cost of the entire system, i.e.,

$$\text{Min } Z = Z_1 + Z_2 = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 +$$
$$c_5X_5 + c_6X_6$$  
(18)

S. to.

$$Y_1 = f_1(X_1, X_2, X_3, U_2)$$  
(19)

$$Y_2 = f_2(X_4, X_5, X_6, U_1)$$  
(20)

$$U_2 = f_3(X_6)$$  
(21)

$$U_1 = f_4(X_3)$$  
(22)

and

$$0 \leq X_i \leq b_i, \ i = 1, 2, \ldots, 6$$  
(23)

The solution procedure to solve this problem is illustrated in
Chapter IV.

Now we formulate a dynamic model containing one product.

P3 Continuous Time Model Containing One Product

The static model included the R and D expenditure as an input
similar to labor and capital. We discussed in Chapter II that R and D
expenditures have a cumulative effect over time on production rate.
This fact has to be recognized especially when the firm is planning
for medium range or long range resource allocation situations. To
meet this situation a dynamic model is formulated next.

The performance of a dynamic system is measured by a cost func­
tion which is a function of time. The dynamic model may be either dis­
crete or continuous. In the discrete system the measurements of inputs
is constrained to specific points or intervals such as weeks or months whereas in a continuous system the inputs and outputs can be measured at any point of time t in the interval. The inputs \( X_1 \) are assumed to be instantaneous rates of expenditure of resources and the output \( Y_1 \) to be the instantaneous rate of production.

We will be concerned with a continuous time model where it is required to determine the quantity of each \( X_i \) as a function of time such that the total cost of the system is minimized over the entire period of interest.

In the dynamic model provision is also made for storing inventory and we assume the cost of inventory is proportional to the inventory or backorder at time \( t \). Therefore, the problem is to minimize:

\[
Z = \int_0^T [c_1 X_1(t) + c_2 X_2(t) + c_3 X_3(t) + c_4 g[I(t)]] dt
\]

S. t.

\[
Y_1(t) = f[X_1(t), X_2(t), \int_0^t X_3(\tau) d\tau]
\]

\[
I(t) = I(0) + P_1(t) - D_1(t)
\]

\[
X_i(t) \geq 0; \quad X_i(0) = X_i^0; \quad i = 1, 2, 3
\]

where \( I(0) \) denotes the initial inventory and \( P_1(t) \) and \( D_1(t) \) represent the cumulative production and demand respectively through time \( t \). The coefficients \( c_1, c_2 \) and \( c_3 \) represent the similar factors as described in \( P_1 \) and \( c_4 \) is the cost coefficient for inventory, and \( g[I(t)] \) is some function of \( I(t) \).

Equation (25) represents a production function constraint as discussed in Chapter II. The production function reflects that \( R \) and
D expenditure $X_3$ cumulatively contributes towards the production rate $Y_1$. If we assume the production process is in operation, then we know the initial status $X_4^0$ of the system. Next we explore some additional constraints which could be included as a part of the problem just described.

**Additional Constraints**

The equation (26) imposes non-negativity constraints on the inputs. The availability of resources may also be limited. This would add upper bounds on the inputs. The limitations on the availability of resources may be represented by the equation:

$$X_i(t) \leq b_i, \ i = 1, 2, 3$$  \hspace{1cm} (27)

In addition, if it were required that the items manufactured could not be stored for the succeeding periods then the inventory carried will be zero and hence the objective function would not contain any inventory term. Then the problem is to optimize the cost of inputs while exactly meeting the demand for all $t \in [0, T]$.

In some instances the entrepreneur may be able to keep inventory but may not want to leave the customer orders unfulfilled. This would impose a non-negativity constraint on the inventory for all time $t$.

$$I(t) = I(0) + P(t) - D(t) \geq 0$$  \hspace{1cm} (28)

Inventory levels could also be bounded from above and below. For example,

$$0 \leq I(t) = I(0) + P(t) - D(t) \leq R$$  \hspace{1cm} (29)
In many instances backorders are allowed which leads to the following equality constraint

\[ I(t) = I(0) + P(t) - D(t) \]  

(30)

Next we explore the effect of these constraints.

**Effect of Inventory Constraints on the Objective Function**

The objective function represents the total costs of inputs and the inventories. For \( g[I(t)] = I(t) \), positive values of inputs, and non-negative inventories represented by equation (28) or (29) the objective function portrays many real-world production-inventory situations. However, when we remove the non-negativity constraint, we see the objective function is minimized for the most backorders. Assuming the total demand should be met by the end of the planning horizon, the solution would be to wait until the last possible moment and start the production at a time such that the total demand is met.

To alleviate this problem, the cost of inventory term in the objective function should be modified such that a positive cost is involved either for any inventory or backorder incurred. Assuming the cost of inventory equal the cost of backorder, either of the following procedures may be followed:

a) Let the absolute value of \( I(t) \) enter the objective function, that is,

\[ g[I(t)] = |I(t)| = |I(0) + P(t) - D(t)| \]  

(31)

b) Let the cost of inventory be proportional to the square of the inventory or backorder, that is,
\[ g[I(t)] = I(t)^2 \]  \hspace{1cm} (32)

The square is an approximation to equation (31) and is mathematically more tractable. The cost \( c_4 \) has to be carefully chosen such that the function \( c_4 I(t)^2 \) realistically represents the total cost of inventory at time \( t \).

Thus the objective function can be changed to contain a positive inventory cost regardless of inventory or backorder incurred in the system for constraints (28), (29), (31) or (32). The constraint (29) may be considered a more restricted case of (28); and terms involving absolute values in the objective functions are mathematically difficult to solve. Therefore, we will be using constraints (28) and (32) in solving dynamic problems. Next, we proceed to formulate a dynamic model containing two products.

**P4 Continuous Time Model Containing Two Products**

The static model P2 portrayed the situation of an entrepreneur manufacturing two products. The model P4 extends the analysis to a dynamic situation with time again considered as continuous. Let \( X_1(t), X_2(t) \) and \( X_3(t) \) be the instantaneous rate of consumption of resources, \( \text{i.e.,} \) labor, capital and engineering, respectively, for product 1. Similarly \( X_4(t), X_5(t) \) and \( X_6(t) \) represent corresponding inputs for product 2. Then the problem is to choose the instantaneous rates of inputs such that the cost of overall system is minimized. Then the system of equations for the continuous time model containing two products may be specified as:
Min \( Z = \int_0^T \left[ c_1 X_1(t) + c_2 X_2(t) + c_3 X_3(t) + c_4 X_4(t) +
\right.
\left. c_5 X_5(t) + c_6 X_6(t) + c_7 g_1[I_1(t)] +
\right.
\left. c_8 g_2[I_2(t)] \right] dt \)  

s. to.

\( Y_1(t) = f_1[X_1(t), X_2(t), \int_0^t x_3(\sigma) \, d\sigma, \int_0^t v_2(\sigma) \, d\sigma] \)  
\( Y_2(t) = f_2[X_4(t), X_5(t), \int_0^t x_6(\sigma) \, d\sigma, \int_0^t v_1(\sigma) \, d\sigma] \)  

\( I_1(t) = I_1(0) + P_1(t) - D_1(t) \)  
\( I_2(t) = I_2(0) + P_2(t) - D_2(t) \)  

\( V_2(t) = f_3[X_6(t)] \)  
\( V_1(t) = f_4[X_3(t)] \)  

and

\( 0 \leq X_i(t) \leq b_i(t), \, i = 1, 2, \ldots, 6 \)  

where \( I_i(0) \) denotes the initial inventories and \( P_i(t) \) and \( D_i(t) \) represent the cumulative production rates and cumulative demand rates, respectively, up to time \( t \) for products \( i = 1, 2 \). Additional bounds on inventory may also be imposed analogous to equations (28) through (32) of P3.

The entire discussion on 'additional elements' and the 'effect of inventory constraints on the objective function' made in P3 also apply to P4. Hence they will not be repeated here.

The equations (38) and (39) represent the effects of
interactions as a function of time. They are similar to equations (21) and (22) respectively of the static model. Equations (34) and (35) are production functions. For example, \( Y_1(t) \) is a function of the instantaneous rates of expenditures \( X_1(t) \) and \( X_2(t) \) and cumulative expenditures \( X_3(t) \) and \( V_2(t) \), respectively.

In general the dynamic models are more difficult to solve compared to static models. The complexity increases enormously as the interaction elements \( V_j(t) \) enter the picture. The problem becomes even more difficult to solve when engineering efforts and interaction efforts become cumulative over time. The complexity of the solution procedure also depends upon the nature of the production functions present in the model.

Discussion of Models

So far we have modeled static and dynamic situations involving one and two products. The same concept of modeling can be extended to systems manufacturing three or more products. However, several additional elements enter the decision analysis and the modeling process becomes correspondingly complex.

As an illustration, the firm might conduct research on individual processes or raw materials. Specifically, improvement in soldering technique would increase the productivity of the product wherever the new technique is adapted. Nylon bushings may replace metal bearings which might save time of assembly or the cost of the product wherever it is used. The employee training might increase the productivity of a workman regardless of where he is working.
In a huge firm where several products are manufactured often many products consume the same type of raw materials. The R and D is spent on individual processes, raw materials and equipment. Many products are manufactured using the same facilities and hence it is a very complex task to allocate R and D expenditures to individual products. Therefore it is more appropriate to consider the effect of R and D on individual processes, raw materials, improvement in techniques and other inputs rather than on corresponding products. This requires describing all inputs in detail and keeping track of all expenditures on every input instead of products. To accommodate these factors a general model is formulated.

General Model

Consider the total engineering effort spent. Some of it may be expended on routine maintenance of the firm such as planning production, quality control, and facilities maintenance; and the rest may be directed towards R and D for increasing the production rate. Assume the routine engineering as an expendable input similar to labor and capital whereas the engineering spent on R and D increases the production rate cumulatively as discussed earlier. Then it is possible to categorize the total production rate the following way:

1) Production rate obtained from utilizing the factors of production such as labor, capital, raw materials and routine engineering.

2) The increase in production rate due to R and D projects which is cumulative over time. We will not
explicitly define the effect of R and D on individual inputs. However, we will define a function relating the R and D inputs and the increase in production rate. This is explained next.

Suppose the relationship between inputs $X_1$ and $X_2$ and the output $Y$, as represented by Figure 3.4, is given by

$$a_1X_1 + a_2X_2 = Y$$

If the R and D inputs $U_1$ and $U_2$ were spent on $X_1$ and $X_2$ respectively, then they will have an effect of increasing the productivity resulting from $X_1$ and $X_2$ or of increasing the production rate in general.

The increase in effectiveness from the R and D inputs may be expressed in two ways:

1) They increase the effectiveness of inputs such that

$$\frac{\delta Y}{\delta X_1} = \frac{\delta f}{\delta X_1} = h_1(X_1, U_1) > 0$$

That is, the change in the effectiveness of the input may be expressed as a function of input $X_1$ and R & D input $U_1$. Then the production rate may be expressed as:

$$Y = g(X)$$

S. to,

$$\dot{Y} = h(X, U)$$

Models containing these equations may be solved utilizing differential equation theory.
2) They increase the production rate, in general, such that the new production rate

\[ Y_R = f(X) + g(U) \]

That is, the total production is a function of inputs \( X \), and R and D inputs \( U \). We follow the latter version since the latter is mathematically easier to manipulate to obtain solutions.

Therefore we can express the new production rate \( Y_R \) as represented by the Figure 3.5, as:

\[ a_1X_1 + a_2X_2 + b_1U_1 + b_2U_2 = Y_R \]

where \( b_1U_1 \) and \( b_2U_2 \) represent the increases in production rate due to R and D inputs \( U_1 \) and \( U_2 \) on the inputs \( X_1 \) and \( X_2 \) respectively. The same concept may be extended to any number of inputs and outputs.

In the future we will refer to these relationships as a function of time. The basic production rate is given by

\[ w(t) = a_1X_1(t) + a_2X_2(t) \]

and the increase in production rate due to R and D inputs by

\[ q(t) = b_1U_1(t) + b_2U_2(t). \]

The increase in production rate is cumulative over time and hence the total production rate at time \( t \) is

\[ y(t) = w(t) + \int_0^t q(\sigma) \, d\sigma. \]
Figure 3.4
Relationship between Inputs and Output

Figure 3.5
Relationship between Inputs, R & D Inputs and Output
Next, we proceed to formulate the general model.

**Formulation of the General Model**

Define the following. Let

1) The $n$ vector $\overline{X}$ be the inputs of the system excluding the R and D inputs

2) The $n$ vector $\overline{U}$ be the associated R and D inputs corresponding to inputs $\overline{X}$.

3) The $m$ vector $\overline{w}$ be the output of the system or the number of units of products made using the inputs $\overline{X}$.

4) The $m$ vector $\overline{q}$ be the instantaneous increase in output produced by the R and D inputs $\overline{U}$.

5) The $m$ vector $\overline{v}$ be the cumulative increase in output due to the R and D expenditure.

6) The $m$ vector $\overline{y}$ be the total production

7) The $m$ vector $\overline{I}$ be the inventory of products

8) The $m$ vectors $\overline{Y}$, $\overline{D}$ be the cumulative production and demand respectively

9) The $n$ vector $\overline{c}_1$ be the unit cost of inputs $\overline{X}$

10) The $n$ vector $\overline{c}_2$ be the unit cost of R and D inputs $\overline{U}$

11) The $m$ vector $\overline{c}_3$ be the unit cost of backorder or the cost of holding the inventory for one unit of time.

Utilizing these definitions either a continuous or a discrete time model can be formulated. Also, a continuous time model may be transformed to a discrete time model by segmenting the total period of time into intervals and vice versa.

A continuous time model may be formulated if we assume all the
parameters describe the situation of the process at time \( t \). The production rate \( \overline{w}(t) \) is some function of the input \( \overline{x} \)

\[
\overline{w}(t) = f(\overline{x}, t)
\]

The increase in production rate is some function of R and D inputs \( \overline{U} \)

\[
\overline{q}(t) = g(\overline{U}, t)
\]

The cumulative increase in production rate may be represented by (for R and D inputs):

\[
\overline{v}(t) = \int_0^t \overline{q}(\sigma) \, d\sigma = \int_0^t g(\overline{U}(\sigma)) \, d\sigma
\]

The total production rate \( \overline{y}(t) \) may be represented by the following identity:

\[
\overline{y}(t) = \overline{w}(t) + \overline{v}(t)
\]

The inventory at time \( t \) may be represented by

\[
\overline{I}(t) = \overline{I}(0) + \overline{Y}(t) - \overline{D}(t)
\]

Therefore, a continuous time model may formally be stated as:

\[
\text{Min } Z = \int_0^T [c_1 f(\overline{x}, t) + c_2 g(\overline{U}, t) + c_3 \overline{I}(t)] \, dt
\]

S. to.

\[
\begin{align*}
\overline{w}(t) &= f(\overline{x}, t) \\
\overline{v}(t) &= \int_0^t g(\overline{U}, \sigma) \, d\sigma \\
\overline{y}(t) &= \overline{w}(t) + \overline{v}(t) \\
\overline{I}(t) &= \int_0^t \overline{y}(\sigma) \, d\sigma
\end{align*}
\]
and

\[ \bar{I}(t) = \bar{I}(0) + \bar{Y}(t) - \bar{D}(t) \]
\[ \bar{V}(0) = 0 \text{ and } \bar{Y}(0) = 0. \]

where the objective function includes the cost of inputs used for manufacturing products, the cost of R and D inputs and the inventory holding costs. In addition, bounds may be imposed on the variables.

Addtional Considerations

The system models P1 through P4 assumed the existence of labor, capital, and engineering inputs. The general model portrayed a multi-product planning situation with several inputs. It was tacitly assumed in the general model that research is performed on individual inputs or processes. In all cases the engineering/R-and-D inputs increased the production rate.

In many instances, situations arise where groups of products belonging to the same technology are manufactured in different locations. The engineers learn from the experience of each other which has the effect of increasing the production rate. For example, a manufacturer may produce two groups of products:

1) Several models of black and white television sets and
2) Several models of color television sets.

or,

1) Several models of televisions and radios and
2) Associated communication equipment.

In each case, they share some common technology. The products
may be either manufactured in the same location or at different plants. Nevertheless, they learn from each other. The learning effect may be incorporated in a model containing groups of products. The phenomenon is very common in giant industries, a model of which is presented in Chapter V.

Although it is possible to solve the general model and then adapt it to solve other cases, such a course will not be pursued here. Instead an attempt will be made to analyze different solution techniques suitable for solving specific problems and their extensions. The one product and two product models are considered in Chapter IV and the general model and its extensions are discussed in Chapter V.
CHAPTER IV
ONE PRODUCT AND TWO PRODUCT MODELS

Introduction

Several static and dynamic models portraying production planning were formulated in Chapter III. The simpler the functions used in the model the easier to obtain results. Probably, the easiest model to solve would be the static case with a linear objective function and a linear production constraint. The most difficult problem would be the dynamic model with nonlinear objective function and a nonlinear production function constraint. The rest of this chapter is devoted to solving problems of the type formulated in Chapter III. Whenever the problems are not directly solvable, the difficulties associated with solving them will be explained.

Solution to Model PI

The model PI represents a one product static situation. Two special examples of this model will be considered. Example 1 contains a linear objective function and a linear constraint and Example 2 is comprised of a linear objective function and a nonlinear Cobb-Douglas type constraint.
EXAMPLE 1

The model PI considers a process using three inputs $X_1$, $X_2$ and $X_3$ and producing an output $Y$. If $a_1$, $a_2$ and $a_3$ are the coefficients of inputs $X_1$, $X_2$ and $X_3$, respectively, in the linear production function and $c_1$, $c_2$ and $c_3$ are costs per unit of $X_1$, $X_2$ and $X_3$, respectively, then the problem is to choose the inputs $X_1$, $X_2$ and $X_3$ such that the production $Y$ satisfies the demand $d$, and the total cost of production is minimized. In addition to the non-negativity constraints, suppose we impose the upper bounds $b_1$ on resources $X_i$, $i = 1, 2, 3$, then the problem may be formally stated as:

$$\text{Min } Z = c_1X_1 + c_2X_2 + c_3X_3$$

s.t.

$$Y = a_1X_1 + a_2X_2 + a_3X_3$$

and

$$0 \leq X_i \leq b_i, \ i = 1, 2, 3$$

The trivial solution $X_1 = X_2 = X_3 = 0$ does not meet the demand. Therefore a feasible solution, if one exists, will be sought. We know that

$$X_i = \text{number of units of resource } i, \ i = 1, 2, 3$$

$$c_i = \text{cost per unit of resource } i$$

$$a_i = \text{units of product obtained per unit of resource } i \text{ expended.}$$

If $\lambda_ia_i = c_i$
then

\[ \lambda_i = \frac{\text{cost per unit of resource } i}{a_i} \]

The vector \( \bar{\lambda} \) consists of shadow prices or the measure of effectiveness of utilizing the resources in minimizing the cost of the product \( Y \). Therefore, we list the resources in ascending order of \( \lambda \).

The decision rule is to use that resource \( i \) which costs the least per unit of product, first and when this resource is exhausted select the next most efficient element in the resource vector. It is very likely in the situation we are considering that lower bounds for the resources may also be specified. For example a firm may be forced to certain minimum quantities \( l_i \geq 0 \) \((i = 1, 2, 3)\) of resources such that

\[ l_i \leq x_i \leq b_i \]

If the demand was met by utilizing the lower bounds on resources, optimization would not be necessary. However, if the demand was not satisfied, then the strategy would be to choose the additional resource requirements using the decision rule already specified.

In addition to the strategy specified here, linear programming may also be applied to solve this problem. Numerical solutions can be obtained by using simplex or the special bounded variable algorithms.
EXAMPLE 2

In the following problem, the modified Cobb-Douglas function is used as a nonlinear constraint along with a linear objective function.

\[ \text{Min } Z = c_1 X_1 + c_2 X_2 + c_3 X_3 \]
\[ \text{S. to } \]
\[ Y = X_1^{\alpha X_2} X_3^\gamma \]
\[ \text{and } \]
\[ X_i > 0, \ i = 1,2,3 \]

The problem may be solved using the Lagrange multiplier technique (Teichroew, 1969). Introducing the Lagrange Multiplier \( \lambda \) for the production constraint, the function may be written as:

\[ L = c_1 X_1 + c_2 X_2 + c_3 X_3 - \lambda [X_1^{\alpha X_2} X_3^\gamma - Y] \]  \hspace{1cm} (1)

The non-negativity constraints will be considered later. Setting the partial derivatives equal to zero, we obtain the necessary conditions for an extremum:

\[ \frac{\partial L}{\partial X_1} = c_1 - \alpha \lambda X_1^{\alpha - 1} X_2 X_3^\gamma = 0 \]  \hspace{1cm} (2)

\[ \frac{\partial L}{\partial X_2} = c_2 - \beta \lambda X_1^\alpha X_2^{\beta - 1} X_3^\gamma = 0 \]  \hspace{1cm} (3)

\[ \frac{\partial L}{\partial X_3} = c_3 - \gamma \lambda X_1^\alpha X_2^\beta X_3^{\gamma - 1} = 0 \]  \hspace{1cm} (4)
Substituting $Y = X_1^a X_2^b X_3^y$ in equations (2), (3) and (4) we obtain

$$c_1 = \frac{\lambda a Y}{X_1}$$  \hspace{1cm}  (6)  \\
$$c_2 = \frac{\lambda b Y}{X_2}$$  \hspace{1cm}  (7)  \\
$$c_3 = \frac{\lambda c Y}{X_3}$$  \hspace{1cm}  (8)  \\

from which the variables $X_1$, $X_2$ and $X_3$ can be expressed in terms of $\lambda$.

$$X_1 = \frac{\lambda a Y}{c_1}$$  \hspace{1cm}  (9)  \\
$$X_2 = \frac{\lambda b Y}{c_2}$$  \hspace{1cm}  (10)  \\
$$X_3 = \frac{\lambda c Y}{c_3}$$  \hspace{1cm}  (11)  \\

The equations (9), (10) and (11) indicate that each $X_i$ is inversely proportional to its associated $c_i$, for $i = 1, 2, 3$. Now substituting the values of $X_1$, $X_2$ and $X_3$ in the production function equation we obtain:

$$Y = \left(\frac{\lambda a Y}{c_1}\right)^a \left(\frac{\lambda b Y}{c_2}\right)^b \left(\frac{\lambda c Y}{c_3}\right)^y$$  \hspace{1cm}  (12)  \\

$$= \left(\frac{\alpha}{c_1}\right)^a \left(\frac{\beta}{c_2}\right)^b \left(\frac{\gamma}{c_3}\right)^y (\lambda Y)^{\alpha+\beta+\gamma}$$  \hspace{1cm}  (13)
and

\[ \lambda = \left( \frac{a}{c_1} \right)^{\frac{\alpha}{\alpha + \beta + \gamma}} \left( \frac{b}{c_2} \right)^{\frac{\beta}{\alpha + \beta + \gamma}} \left( \frac{c}{c_3} \right)^{\frac{\gamma}{\alpha + \beta + \gamma}} \times \frac{1-(\alpha + \beta + \gamma)}{\alpha + \beta + \gamma} \]  \tag{14}

Substituting the values of \( \lambda \) in the equations for \( X_1, X_2 \) and \( X_3 \) yields

\[ X_1 = \left( \frac{a}{c_1} \right)^{\frac{\alpha}{\alpha + \beta + \gamma}} \left( \frac{b}{c_2} \right)^{\frac{\beta}{\alpha + \beta + \gamma}} \left( \frac{c}{c_3} \right)^{\frac{\gamma}{\alpha + \beta + \gamma}} \times \frac{1}{\alpha + \beta + \gamma} \]  \tag{15}

\[ X_2 = \left( \frac{a}{c_1} \right)^{\frac{\alpha}{\alpha + \beta + \gamma}} \left( \frac{b}{c_2} \right)^{\frac{\beta}{\alpha + \beta + \gamma}} \left( \frac{c}{c_3} \right)^{\frac{\gamma}{\alpha + \beta + \gamma}} \times \frac{1}{\alpha + \beta + \gamma} \]  \tag{16}

\[ X_3 = \left( \frac{a}{c_1} \right)^{\frac{\alpha}{\alpha + \beta + \gamma}} \left( \frac{b}{c_2} \right)^{\frac{\beta}{\alpha + \beta + \gamma}} \left( \frac{c}{c_3} \right)^{\frac{\gamma}{\alpha + \beta + \gamma}} \times \frac{1}{\alpha + \beta + \gamma} \]  \tag{17}

If \( c + \beta + \gamma = 1 \), we said the production function exhibits constant returns.

Therefore, for a production function with constant returns the equations (15) through (17) can be written as:
\[
x_1 = \left( \frac{c_1}{\alpha} \right)^{\alpha-1} \left( \frac{c_2}{\beta} \right)^{\beta} \left( \frac{c_3}{\gamma} \right)^{\gamma} \tag{18}
\]
\[
x_2 = \left( \frac{c_1}{\alpha} \right)^{\alpha} \left( \frac{c_2}{\beta} \right)^{\beta-1} \left( \frac{c_3}{\gamma} \right)^{\gamma} \tag{19}
\]
\[
x_3 = \left( \frac{c_1}{\alpha} \right)^{\alpha} \left( \frac{c_2}{\beta} \right)^{\beta} \left( \frac{c_3}{\gamma} \right)^{\gamma-1} \tag{20}
\]

It is interesting to note all \(X\)'s are directly (linearly) proportional to production \(Y\), when \(\alpha + \beta + \gamma = 1\). If \(\alpha + \beta + \gamma < 1\), then the value of all \(X\)'s will increase with respect to those given in (18), (19) and (20), for the same production \(Y\) which is true for production functions exhibiting decreasing returns. If \(\alpha + \beta + \gamma > 1\), then the value of all \(X\)'s will decrease for the same production \(Y\), which is true for all production functions exhibiting increasing returns. These properties were discussed in Chapter II.

**Non-negativity Constraint on Inputs**

For positive values of \(\alpha, \beta, \gamma, c_1, c_2, c_3\) and \(Y\), the values of the \(X\)'s in equations (18) through (20) are positive and hence the non-negativity constraint is automatically satisfied. Therefore, it is not necessary to include the non-negativity constraints explicitly in the Lagrange function.

The optimum values of \(X_1, X_2\) and \(X_3\) are given by equations (18), (19) and (20) respectively. To ascertain whether they actually represent the minimum of \(Z\), the sufficient conditions must be checked.
Sufficient Conditions

The sufficient conditions for a local minimum are stated in terms of the principal minors $\Delta_i$ of the Hessian matrix of the Lagrangian function (Equation 1). The stationary points will represent the minimum if the principal minors $\Delta_3$ and $\Delta_4$ of the Hessian matrix $H$ are less than zero (Teichroew, 1963 p. 553). Let $g = [X_1^a, X_2^b, X_3^\gamma - Y] = 0$. Then the matrix $H$, can be written the following way:

$$
H = \begin{bmatrix}
0 & -g_{X_1} & -g_{X_2} & -g_{X_3} \\
-g_{X_1} & Z_{X_1X_1} - \lambda g_{X_1X_1} & Z_{X_1X_2} - \lambda g_{X_1X_2} & Z_{X_1X_3} - \lambda g_{X_1X_3} \\
-g_{X_2} & Z_{X_2X_1} - \lambda g_{X_2X_1} & Z_{X_2X_2} - \lambda g_{X_2X_2} & Z_{X_2X_3} - \lambda g_{X_2X_3} \\
-g_{X_3} & Z_{X_3X_1} - \lambda g_{X_3X_1} & Z_{X_3X_2} - \lambda g_{X_3X_2} & Z_{X_3X_3} - \lambda g_{X_3X_3}
\end{bmatrix}
$$

where the subscripts refer to the partial derivatives of the functions $Z$ and $g$. The objective function $Z$ does not have any quadratic or cross product terms of $X_i$, $i = 1, 2, 3$ and hence all the second partials of $Z$ with respect to $X_i$ will be zero. We will be left with various partial derivatives of the function $g$ only in the $H$ matrix. Taking necessary partial derivatives and substituting in $H$ we obtain,

$$
H = \begin{bmatrix}
0 & -\frac{\alpha}{X_1} & -\frac{\beta}{X_2} & -\frac{Y}{X_3} \\
-\frac{\alpha}{X_1} & -\frac{\lambda\alpha(\alpha-1)}{X_1^2} & -\frac{\lambda\alpha\beta}{X_1X_2} & -\frac{\lambda\alpha\gamma}{X_1X_3} \\
-\frac{\beta}{X_2} & -\frac{\lambda\alpha\beta}{X_1X_2} & -\frac{\lambda\beta(\beta-1)}{X_2^2} & -\frac{\lambda\beta\gamma}{X_2X_3} \\
-\frac{Y}{X_3} & -\frac{\lambda\alpha\gamma}{X_1X_3} & -\frac{\lambda\beta\gamma}{X_2X_3} & -\frac{\lambda\gamma(\gamma-1)}{X_3^2}
\end{bmatrix}
$$
Evaluating we obtain (see Appendix B)

\[
\Delta_3 = - \frac{\lambda \alpha \beta (\alpha + \beta) Y}{X_1^2 X_2^2} \\
\Delta_4 = - \frac{\lambda^2 \alpha \beta Y}{X_1^2 X_2^2 X_3^2} (\alpha + \beta + \gamma)
\]

We know from equation (14) that \( \lambda \) is positive and we also know that all other constants, parameters and variables are positive. Therefore \( \Delta_3 \) and \( \Delta_4 \) are negative for all values of inputs and outputs. Therefore, we can say the equations (18) through (20) provide minimum values of \( X_1, X_2 \) and \( X_3 \).

**Upper Bounds**

If the solution was found not to violate available resources then the problem is solved. Otherwise, bounds need to be included in the problem and solved again. Next we proceed to solve model P2.

**Solution to Model P2**

In Chapter III, we formulated two different situations involving a static model containing two products. We mentioned that the two different products in case 1 do not have anything in common and hence optimizing individually for the products 1 and 2 would also optimize the entire system. Optimizing individual products involves applying the solution technique described in P1 to product 1 and product 2 separately and hence it will not be necessary to pursue it here.

The interactive case of problem P2 requires a two product
static model where the effects of interaction between research and development projects in increasing the productivity of each of the individual products are accounted for. We present two examples of this interactive situation. Example 1 consists of a linear objective function and a linear constraint and Example 2 contains a linear objective function and a Cobb-Douglas type production function constraint.

EXAMPLE 1

Refer to equations (18) through (23) of P2, Chapter III. Assuming the functions \( f_1, f_2, f_3 \) and \( f_4 \) are linear, this example containing a linear objective function, linear production function constraints and linear interaction constraints can be written as:

\[
\begin{align*}
\text{Min } Z & = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + c_6X_6 \quad (21) \\
\text{S. to } \quad Y_1 & = a_1X_1 + a_2X_2 + a_3X_3 + a_4U_2 \\
Y_2 & = a_5X_4 + a_6X_5 + a_7X_6 + a_8U_1 \\
U_2 & = a_9X_6 \quad (24) \\
U_1 & = a_{10}X_3 \quad (25)
\end{align*}
\]

and

\[
\bar{0} \leq X_i \leq \bar{b}_i, \quad i = 1,2,\ldots,6 \quad (26)
\]

where \( \bar{0} \) and \( \bar{b}_i \) represent the lower and upper bounds on inputs \( X_i \); and \( U_1 \) and \( U_2 \) represent the interaction effects as indicated by the Figure 3.3. We can substitute the equations (24) and (25) in (22) and (23),
respectively, and rewrite:

\[ Y_1 = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_6 \]  \hspace{1cm} (27)

\[ Y_2 = a_5 x_4 + a_6 x_5 + a_7 x_6 + a_8 x_3 \]  \hspace{1cm} (28)

where

\[ a_4 = a_4 a_9 \text{ and } a_8 = a_8 a_{10} \]

Therefore, we have the problem with a linear objective function and two linear constraints along with bounds on the inputs. Such problems can be solved utilizing linear programming with upper-bound techniques.

**EXAMPLE 2**

We presented the previous example assuming the production function constraints are linear. Unfortunately, no data or literature exist on the form or the existence of the Cobb-Douglas type production function which would account for the interaction effects. However, in the interest of getting some insight we have assumed the following form

\[ Y_1 = x_1^\alpha x_2^\beta x_3^\gamma y_2^\delta \]

\[ Y_2 = x_4^m x_5^n x_6^p y_1^q \]

The reason for assuming the form stems from the fact that all inputs have posinomial form. We also assume linear interaction constraints, i.e.,

\[ U_2 = k_1 x_6 \]

\[ U_1 = k_2 x_3 \]
We now attempt to solve this problem. The problem is to:

\[
\text{Min } Z = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_6
\]

S. to,

\[
y_1' = x_1^{\alpha}x_2^\beta x_3^\gamma y_2^\delta
\] (32)

\[
y_2' = x_4^{m}x_5^n x_6^p y_1^q
\] (33)

\[
u_2 = k_1x_6
\] (34)

\[
u_1 = k_2x_3
\] (35)

Substituting the values of \(u_2\) and \(u_1\) from equations (34) and (35), respectively, in (32) and (33) we can write:

\[
y_1 = x_1^{\alpha}x_2^\beta x_3^\gamma x_6^\delta
\] (36)

\[
y_2 = x_4^{m}x_5^n x_6^p x_3^q
\] (37)

where \(y_1 = y_1'/k^\delta\) and \(y_2 = y_2'/k_2^q\). Therefore we have reduced an example containing four constraints to another example with two constraints. To solve this problem, form the Lagrangian, take partial derivatives, and set them equal to zero and obtain the necessary conditions for an extremum. The Lagrangian is,

\[
L = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_6 - \\
\lambda_1[x_1^{\alpha}x_2^\beta x_3^\gamma x_6^\delta - y_1] - \lambda_2[x_4^{m}x_5^n x_6^p x_3^q - y_2]
\] (38)

and the necessary conditions are:

\[
\frac{\partial L}{\partial x_1} = c_1 - \lambda_1x_1^{\alpha-1}x_2^\beta x_3^\gamma x_6^\delta = 0
\] (39)
\[
\frac{\partial L}{\partial x_2} = c_2 - \lambda_1 \beta x_1 \alpha x_2 ^{\beta - 1} x_3 ^{\gamma x_6} \delta = 0 \tag{40}
\]

\[
\frac{\partial L}{\partial x_3} = c_3 - \lambda_1 \gamma x_1 ^{\alpha x_2} \beta x_3 ^{\gamma - 1} x_6 ^{\delta} - \lambda_2 q x_4 ^{m x_5} n x_6 ^{p x_3 ^{q-1}} = 0 \tag{41}
\]

\[
\frac{\partial L}{\partial x_4} = c_4 - \lambda_2 m x_4 ^{m-1} x_5 ^{n x_6 ^{p x_3 ^q}} = 0 \tag{42}
\]

\[
\frac{\partial L}{\partial x_5} = c_5 - \lambda_2 n x_4 ^{m x_5} n^{-1} x_6 ^{p x_3 ^q} = 0 \tag{43}
\]

\[
\frac{\partial L}{\partial x_6} = c_6 - \lambda_1 \delta x_1 ^{\alpha x_2} \beta x_3 ^{\gamma x_6} \delta^{-1} - \lambda_2 p x_4 ^{m x_5} n x_6 ^{p-1} x_3 ^q = 0 \tag{44}
\]

\[
\frac{\partial L}{\partial \lambda_1} = x_1 ^{\alpha x_2} \beta x_3 ^{\gamma x_6} \delta - y_1 = 0 \tag{45}
\]

\[
\frac{\partial L}{\partial \lambda_2} = x_4 ^{m x_5} n x_6 ^{p x_3 ^q} - y_2 = 0 \tag{46}
\]

Substituting equations (45) and (46) in equations (39) through (44) we obtain:

\[
x_1 = \frac{\lambda_1 \alpha y_1}{c_1} \tag{47}
\]

\[
x_2 = \frac{\lambda_1 \beta y_1}{c_2} \tag{48}
\]

\[
x_3 = \frac{\lambda_1 \gamma y_1 + \lambda_2 q y_2}{c_3} \tag{49}
\]

\[
x_4 = \frac{\lambda_2 m y_2}{c_4} \tag{50}
\]

\[
x_5 = \frac{\lambda_2 n y_2}{c_5} \tag{51}
\]
Substituting the values of $X_1$ in the production function we obtain:

\[
X_6 = \frac{\lambda_1 \delta Y_1 + \lambda_2 p Y_2}{c_6}
\]  

(52)

Now we seek to solve (53) and (54) for $\lambda_1$ and $\lambda_2$ (as we solved P2, case 2, equation 14 for $\lambda$). However, we are unable to obtain analytical solutions. Therefore, a numerical procedure is explained below.

The suggested procedure is:

1) Substitute all values of constants, parameters and production rates $Y_1$ and $Y_2$ in equations (53) and (54). This will result in two equations as a function of $\lambda_1$ and $\lambda_2$.

2) Try several sets of $\lambda_1$ and $\lambda_2$ and list all sets which satisfy the equations (53) and (54).

3) Substitute these sets of values in equations (47) through (52) and list the corresponding values of $X_1$, $i = 1,2,\ldots, 6$ for each set of $\lambda_1$ and $\lambda_2$. 

\[
Y_1 = \left(\frac{\lambda_1 a Y_1}{c_1}\right)^\alpha \left(\frac{\lambda_1 b Y_1}{c_2}\right)^\beta \left(\frac{\lambda_1 q Y_1 + \lambda_2 q Y_2}{c_3}\right)^\gamma \left(\frac{\lambda_1 \delta Y_1 + \lambda_2 p Y_2}{c_6}\right)^\delta
\]

(53)

and

\[
Y_2 = \left(\frac{\lambda_2 m Y_2}{c_4}\right)^m \left(\frac{\lambda_2 n Y_2}{c_5}\right)^n \left(\frac{\lambda_1 \delta Y_1 + \lambda_2 p Y_2}{c_6}\right)^p \left(\frac{\lambda_1 q Y_1 + \lambda_2 q Y_2}{c_3}\right)^q
\]

(54)
4) Substitute each set of $\lambda$'s and $X$'s in the Lagrangian function (38) and obtain the cost.

5) The optimal solution is given by the set of $\lambda$'s and $X$'s which provided the minimum of the costs provided by the objective functional.

The problem may also be solved utilizing the generalized Lagrange multiplier technique. The technique is explained in detail by Everett (1963). Solution to Model P3 follows.

**SOLUTION TO MODEL P3**

The model P3 represents a dynamic continuous time single product manufacturing planning problem. Refer to equations (24) to (26) of P3, Chapter III. Assume the costs per unit of inputs $X_1, X_2, X_3$ and cost of storing one unit of inventory are given. We make some modifications to the model, the reason for which will become obvious soon.

Let the production rate equal $Y(t)$. Then the cumulative production can be represented by $Y(t)$. The inventory level at time $t$, $I(t)$, can be represented by $I(0) + Y(t) - D(t)$, and the rate of inventory change $I(t)$ can be represented by the following equation:

$$\dot{I}(t) = Y(t) - D(t)$$

substituting for $Y(t)$ we can write

$$\dot{I}(t) = a_1X_1(t) + a_2X_2(t) + a_3X_3(t) - D(t)$$
Let \( X_3(t) = v \), the engineering R & D input rate. Then \( X_3(t) \) will represent the cumulative engineering R & D input through time \( t \). In addition, we can also include bounds on input levels and on inventory. Next we illustrate five different examples of P3.

**EXAMPLE 1**

Incorporating these changes in the model and assuming that the cost of inventory is proportional to the square of inventory or back order the problem may be rewritten as:

\[
\begin{align*}
\text{Min } Z &= \int_0^T \left( c_1 x_1(t) + c_2 x_2(t) + c_3 v(t) + c_4 I^2(t) \right) dt \quad (61) \\
\text{S. to} \\
\dot{I} &= a_1 X_1 + a_2 X_2 + a_3 X_3 - D \quad (62) \\
\dot{X}_3 &= v \quad (63) \\
\text{and} \\
\underline{b}_1 &\leq X_1(t) \leq \overline{b}_1 \quad (64) \\
\underline{b}_2 &\leq X_2(t) \leq \overline{b}_2 \quad (65) \\
\underline{b}_3 &\leq v(t) \leq \overline{b}_3 \quad (66)
\end{align*}
\]

for the given initial conditions

\[
\begin{align*}
I(0) &= I^0 \\
X_3(0) &= X_3^0
\end{align*}
\]

where \( \underline{b}_i \) and \( \overline{b}_i \), \( i = 1, 2, 3 \) represent the lower and upper bounds on the inputs.

Problems of this structure may be solved by Pontryagin's
maximum principle technique (Pontryagin, 1962). The maximum principle provides necessary conditions for the control to be optimal. In other words, the technique provides a general method of obtaining control functions $X_1(t)$, $X_2(t)$ and $v(t)$ in such a way that the trajectory $I(t)$ and $X_3(t)$ defined by the constraining equations (62), (63), (67) and (68) is optimal with respect to the objective function (61). One condition for optimality is that the Hamiltonian $H$ be maximized. The Hamiltonian may be written as:

$$H = -c_1 X_1 - c_2 X_2 - c_3 v - c_4 I^2 + \psi_1 [a_1 X_1 + a_2 X_2 + a_3 X_3 - D] + \psi_2 v$$

Grouping the control variables which are linear in the Hamiltonian equation, we can write:

$$H = (-c_1 + a_1 \psi_1)X_1 + (-c_2 + a_2 \psi_1)X_2 + (-c_3 + \psi_2)v +$$
$$\psi_1 [a_3 X_3 - D] - c_4 I^2$$

(69)

The necessary conditions for the Hamiltonian to attain a maximum are (Pontryagin, 1962):

$$X_1 = \begin{cases} \overline{b}_1 & \text{if } \psi_1 > c_1/a_1 \\ \underline{b}_1 & \text{if } \psi_1 < c_1/a_1 \\ \text{Not specified} & \text{if } \psi_1 = 0 \end{cases}$$

(70)

$$X_2 = \begin{cases} \overline{b}_2 & \text{if } \psi_1 > c_2/a_2 \\ \underline{b}_2 & \text{if } \psi_1 < c_2/a_2 \\ \text{Not specified} & \text{if } \psi_2 = 0 \end{cases}$$

(71)
where $\Psi_1$ and $\Psi_2$ are calculated from the following additional necessary conditions, that is,

$$
\Psi_1 = \frac{-3H}{\delta I} = 2c_4 I ; \quad \Psi_1(T) = 0
$$

$$
\Psi_2 = \frac{-3H}{\delta X_3} = -a_3 \Psi_1 ; \quad \Psi_2(T) = 0
$$

The equations (62), (63), (70) through (74) should be solved together. An attempt has been made to solve the problem in Appendix A, utilizing Riccati equation approach. Unfortunately, it can only be solved numerically. Example 2 follows.

**EXAMPLE 2**

Example 1 contained a quadratic term in the objective function for the inventory cost. In this example we will have a linear term instead and impose a non-negativity constraint on the inventory level as we discussed in Chapter III, equation 28. Letting $X_3 = v$ and $I = a_1 X_1 + a_2 X_2 + a_3 X_3 - D$ we can state the problem as:

$$
\text{Min } Z = \int_0^T (c_1 X_1(t) + c_2 X_2(t) + c_3 v(t) + c_4 I(t)) dt
$$

S. to

$$
\dot{I}(t) = a_1 X_1(t) + a_2 X_2(t) + a_3 X_3(t) - D(t)
$$

$$
X_3(t) = v(t)
$$
and

\[ I(t) \geq 0 \quad \text{for} \quad t, 0 \leq t \leq T \]  
\[ I(0) = I^0, \quad X_3(0) = X_3^0 \]
\[ 0 \leq b_j \leq X_j(t) \leq \bar{b}_j, \quad j = 1, 2 \]
\[ 0 \leq b_3 \leq v(t) \leq \bar{b}_3 \]

The problem is to choose the level of inputs \( X_1, X_2 \) and \( v \) such that the objective functional (75) is minimized. The Hamiltonian \( H \) may be written as:

\[
H = (a_1 \psi_1 - c_1)X_1 + (a_2 \psi_1 - c_2)X_2 + (\psi_2 - c_3)v + \\
\psi_1(a_3X_3 - D) - c_4I
\]

(82)

Since we have imposed bounds on the state variable \( I(t) \), this is a bounded-state-variable problem. We need to consider three cases for this bounded-state-variable problem (Pontryagin, 1962): 1) state variables lying inside the boundary \( \forall t \in (0, T) \), 2) state variables lying on the boundary, and 3) jump conditions. Next we examine these problems in this same order.

**Case 1. State Variables Lying Inside the Boundary**

The necessary conditions for the problem with state variables lying inside the boundary can be written as:

\[
X_1 = \begin{cases} 
\bar{b}_1 & \text{if } \psi_1 > c_1/a_1 \\
\underline{b}_1 & \text{if } \psi_1 < c_1/a_1 \\
\text{Not specified} & \text{if } \psi_1 = c_1/a_1
\end{cases}
\]

(83)
and

\[
\begin{align*}
\psi_1 &= \frac{-3\Phi}{3I} = c_4, \quad \psi_1(T) = 0 \\
\psi_2 &= \frac{3X_3}{3a_3} = -a_3\psi_1, \quad \psi_2(T) = 0
\end{align*}
\]

Solving,

\[
\begin{align*}
\psi_1 &= -c_4(T - t) \\
\psi_2 &= -a_3c_4(T - t)^2
\end{align*}
\]

We notice \( \psi_1 \) and \( \psi_2 \) are negative for all \( t \) whereas \( c_1/a_1, c_2/a_2 \) and \( c_3 \) are always positive. Therefore, the optimum decision rule is to utilize the resources at the lower boundary, i.e., the minimum level. The decision rule is logical when the initial inventory \( I(0) \) and the minimum production rate is sufficient to satisfy the demand, as shown in Figure 4.1a.

However, if the initial inventory \( I(0) \) and the cumulative production \( P(t) \) did not satisfy the cumulative demand \( D(t) \) for all \( t \in (0,T) \) as portrayed by the Figure 4.1b we may be limited by the upper level of some resources so that \( I(t) \geq 0 \) for all \( t \in (0,T) \). To derive necessary conditions for this situation we consider case 2.
(a) $I(t) > 0$, for all $t \in (0,T)$

(b) $I(t) > 0$, for some $t \in (0,T)$

Figure 4.1
Case 2. State Variable Lies on the Boundary

If the initial inventory $I(0)$ and the minimum production rate is not sufficient to meet the demand during some period of time, say for $t \in (t_1, t_2)$, then we impose a condition $I(t) = 0$ for $t_1 \leq t \leq t_2$. Integrating the equation (76) we can write $t \in (t_1, t_2)$

$$I(t) = I(0) + t_1 \int_{t_1}^{t} (a_1 x_1(\sigma) + a_2 x_2(\sigma) + a_3 x_3(\sigma)) d\sigma - D(t - t_1)$$

$$= I(0) + a_1 k_1(t - t_1) + a_2 k_2(t - t_1) + a_3 x_3(t_1) + \frac{k_3}{2} (t^2 - t_1^2) - t_1 \int_{t_1}^{t} D(t) dt$$

where $k_1$, $k_2$, and $k_3$ are constants representing the lower or the upper bound of inputs as specified by equations (83), (84) and (85), and $D$ represent the demand during the period.

The expression is of the form

$$a_0 + a_1 t + a_2 t^2 + f(t) = 0 \quad \forall t \in [t_1, t_2]$$

which cannot hold over finite interval of time for most demand functions. Therefore we explore the applicability of jump conditions to solve this problem. Case 3, the jump condition, is presented next.

Case 3. Jump Conditions

The inventory will not necessarily stay on the boundary. The other possible solutions result when the inventory becomes zero at some time $t_1$ and the inventory is positive for all $t \in (0, T)$ except at $t_1$, as exhibited by Figure 4.2. Then we can consider the interval of time $(0, t_1)$ as subproblem 1 with terminal condition $I(t_1) = 0$ and
Figure 4.2

Possible Inventory Pattern for Jump Condition
the interval of time \((t_1, T)\) as subproblem 2 without any additional terminal conditions on inventory. Then the jump conditions provide additional requirements which must be satisfied in addition to the necessary conditions provided by subproblem (1) and (2).

The jump conditions for this problem are (Bryson, 1969):

\[
\psi_1(t_1^-) = \psi_1(t_1^+) + \Pi \tag{90}
\]

and the Hamiltonian

\[
H(t_1^-) = H(t_1^+) \tag{91}
\]

where \(\Pi\) is a constant to be determined, and \(t_1^-\) and \(t_1^+\) represent times just before and just after \(t_1\).

The Hamiltonian \(H\) and the necessary conditions (83) through (85) remain the same. However, (86) and (87) vary for subproblem (1) and (2). We can derive from the Hamiltonian that

\[
\dot{\psi}_1 = \frac{\partial H}{\partial \psi_1} = c_4 \tag{92}
\]

and

\[
\dot{\psi}_2 = \frac{\partial H}{\partial \psi_2} = -a_3 \psi_1 \tag{93}
\]

Next we compute \(\psi_1(t)\) and \(\psi_2(t)\) for each of the subproblems.

Subproblem 2

Solving

\[
\psi_1(t) = c_4, \quad \psi(T) = 0
\]

for the fixed time problem with free end conditions on the state
variables, we obtain

\[ \psi_1(t) = -c_4(T - t) \quad t_1 < t \leq T \quad (94) \]

Substituting for \( \psi_1(t) \) in the following equation

\[ \psi_2(t) = -a_3 \psi_1, \quad \psi_2(T) = 0 \]

and evaluating similarly we obtain,

\[ \psi_2(t) = \frac{-a_3 c_4}{2} (T - t)^2 \quad t_1 < t \leq T \quad (95) \]

Subproblem 1

Solving

\[ \psi_1(t) = c_4, \quad \psi_1(t_1) \neq 0 \text{ and } I(t_1) = 0 \]

for the fixed time problem with constrained end condition on the state variable, we obtain

\[ \psi_1(t) = \psi_1(0) + c_4 t \]

We can evaluate \( \psi_1(0) \) by substituting the values for \( \psi_1(t_1^-) \) and \( \psi_1(t_1^+) \) in equation (90), that is,

\[ \psi_1(0) + c_4 t_1 = -c_4(T - t_1) + \Pi \]

Therefore,

\[ \psi_1(0) = \Pi - c_4 T \]

and

\[ \psi_1(t) = \Pi - c_4(T - t) \quad 0 \leq t \leq t_1 \quad (96) \]
Similarly, substituting for $\psi_1(t)$ in the equation

$$\psi_2(t) = -a_3 \psi_1$$

and integrating we obtain

$$\psi_2(t) = \psi_2(0) - \int_{t_1}^{t} a_3 [\Pi - c_4(T - t)] dt$$

$$= \psi_2(0) - a_3 [\Pi t - \frac{c_4}{2}(T - t)^2]$$

Evaluating at $t = t_1$ we obtain

$$\psi_2(t_1) = \psi_2(0) - a_3 [\Pi t_1 - \frac{c_4}{2}(T - t_1)^2 + \frac{c_4}{2} T^2] 0 \leq t \leq t_1$$

For the unbounded $v$, the engineering input, $\psi_2(t)$ is continuous, and hence

$$\psi_2(t^-) = \psi_2(t^+)$$

and from equation (95) that

$$\psi_2(t_1) = \frac{-a_3 c_4}{2} (T - t_1)^2.$$ 

Evaluating

$$\psi_2(0) = -a_3 c_4 (T - t_1)^2 + a_3 \Pi t + \frac{a_3 c_4}{2} T^2$$

and

$$\psi_2(t) = -a_3 c_4 (T - t_1)^2 + a_3 \Pi (t_1 - t) + \frac{a_3 c_4}{2} (T - t)^2 \quad (97)$$

We notice that $\psi_1$ and $\psi_2$ are always negative in the interval of time $t \in [t_1, T]$ and hence we would use the minimum level of resources as specified by equations (83), (84), and (85). However the
values of \( \psi_1 \) and \( \psi_2 \) during \((0, t_1)\) depend on the value of \( \Pi \). Therefore if we compute for \( \Pi \) and \( t_1 \) such that equations (90), (91) and \( I(t) \geq 0 \) in addition to (83), (84) and (85) are simultaneously satisfied, we would also obtain the optimal level of inputs. However, computing \( \Pi \) and \( t_1 \), simultaneously satisfying all conditions, is a difficult task since several integrals appear when we attempt to solve the problem. An algorithm has been developed for solving this problem iteratively. The algorithm will be described after presenting a numerical example. The numerical example will provide an insight into the difficulties involved in obtaining analytical solutions for this problem.

Numerical Example 2A

Suppose the following were given in a situation

\[
\begin{align*}
a_1 &= 2, \quad a_2 = 3, \quad a_3 = 2.5 \\
c_1 &= 3, \quad c_2 = 2, \quad c_3 = 5, \quad c_4 = 1 \\
\overline{b}_1 &= 20, \quad \overline{b}_2 = 15, \quad \overline{b}_3 = 6 \\
\underline{b}_1 &= 15, \quad \underline{b}_2 = 10, \quad \underline{b}_3 = 4, \quad I(0) = 0, \quad X_3(0) = 0
\end{align*}
\]

\( D = 80 \) \( t \in [0, T] \) and \( T = 5 \). Compute the optimum rate of inputs.

Case 1

We find after a quick check that the inventory \( I(t) \) is not positive for all \( t \in [0, T] \) and hence we need not consider case 1.

Case 2

In case 2 the inventory \( I(t) \) will not stay on the boundary for extended periods of time. Therefore we assume that \( I(t) = 0 \) at time
t = t_1 and proceed to case 3.

**Case 3**

First we assume \( X_1 = \overline{b}_1, X_2 = \overline{b}_2 \) and \( v = \overline{b}_3 \) at \( t = t_1^- \). We know \( X_1 = b_1, X_2 = b_2 \) and \( v = b_3 \) at \( t = t_1^+ \). Substituting these values in equation (91) we obtain

\[
-c_1 \overline{b}_1 - c_2 \overline{b}_2 - c_3 \overline{b}_3 + [\Pi - c_4(T - t)]\left[a_1 \overline{b}_1 + a_2 \overline{b}_2 + a_3 \overline{b}_3 t_1 - D\right] + [-a_3 \Pi(t_1 - t) + a_3 c_4(T - t)^2 - a_3 c_4(T - t_1)^2] \overline{b}_3 = -c_1 b_1 - c_2 b_2 - c_3 b_3 + [-c_4(T - t_1)]\left[a_1 b_1 + a_2 b_2 + a_3 \overline{b}_3 t_1 + a_3 b_3(t - t_1) - a_3 c_4 \left(\frac{T - t}{2}\right) \overline{b}_3 \right]
\]

Simplifying,

\[
c_1(\overline{b}_1 - b_1) + c_2(\overline{b}_2 - b_2) + c_3(\overline{b}_3 - b_3) + \frac{a_3 c_4}{2}(T - t_1)^2(\overline{b}_3 - b_3) + c_4(T - t_1)[a_1(\overline{b}_1 - b_1) + a_2(\overline{b}_2 - b_2) + a_3(\overline{b}_3 - b_3)t_1 - \Pi[a_1 \overline{b}_1 + a_2 \overline{b}_2 + a_3 \overline{b}_3 t_1 - D] = 0
\]

Substituting the given numerical values, we obtain:

\[
3.75t_1^2 + 50t_1 + 15\Pi t_1 + 5\Pi - 408.75 = 0
\]

which is a function of \( \Pi \) and \( t_1 \). By arbitrarily fixing \( \Pi \), we can calculate \( t_1 \) and check whether conditions (83) through (85) and (94) through (97) are met for subproblem 1. We assumed in equation (I) that the inputs are at the upper bound. We observe from equation
(96) that any \( \Pi > 5.5 \) will force \( X_1 \) and \( X_2 \) to its upper bound at \( t_1^- \), even if \( t_1 = 5 \). Therefore, we choose \( \Pi = 6 \) arbitrarily for which we obtain \( t_1 = 1.6 \) from equation (II). The decision rule follows.

\[
\psi_1(t) = 6 - 5 + t = 1 + t
\]

\[
\psi_2(t) = -2.5(5 - 1.6)^2 + 2.5 \times 6 \times (1.6 - t) + \frac{2.5}{2} (5 - t)^2
\]

\[
= 1.25t^2 - 27.5t + 25.75
\]

We know from (83) through (85) that

\[
X_1 = \begin{cases} 
\overline{b}_1 & \text{if } 1 + t_{s1} > 3/2 \text{ i.e., } t_{s1} > 1/2 \\
\underline{b}_1 & \text{if } 1 + t_{s1} < 3/2 \text{ i.e., } t_{s1} < 1/2 
\end{cases}
\]

\[
X_2 = \begin{cases} 
\overline{b}_2 & \text{if } 1 + t_{s2} > 2/3 \text{ i.e., } t_{s2} > -1/3 \\
\underline{b}_2 & \text{if } 1 + t_{s2} < 2/3 \text{ i.e., } t_{s2} < -1/3 
\end{cases}
\]

\[
X_3 = \begin{cases} 
\overline{b}_3 & \text{if } \psi_2(t_{s3}) > 5 \text{ i.e., } t_{s3} < .8 \\
\underline{b}_3 & \text{if } \psi_2(t_{s3}) < 5 \text{ i.e., } t_{s3} > .8 
\end{cases}
\]

where \( t_{s1}, t_{s2} \) and \( t_{s3} \) represent times at which the resources move from one bound to another. Using the above decision rules, we calculate the inventory at time \( t_1 = 1.6 \), that is,

\[
I(t_1) = \int_{0^+}^{0.5} a_1\overline{b}_1 \, dt + \int_{0.5}^{1.6} a_1\overline{b}_1 \, dt + \int_{0.5}^{1.6} a_2\underline{b}_2 \, dt + \int_{0^+}^{3/4} a_3\overline{b}_3 \, dt + \int_{3/4}^{1.6} a_3X_3 \, dt - Dt_1
\]

where

\[
X_3 = X_3(.8) + b_3 t
\]

\[
I(t_1) = 150 - 128 = 22
\]

We notice the inventory at time \( t_1 \) is not zero, which may be explained
the following way. In equation (I) we substituted \( X_3 = \overline{b}_3 t \). However
the decision rule was to use \( v = \overline{b}_3 \) for \( t < .8 \) and \( v = \overline{b}_3 \) for \( t > .8 \).
Therefore we can substitute the actual value of \( X_3 = \overline{b}_3 t \cdot 8 + \int T \overline{b}_3 \) in
equation (I) and repeat the calculations to check whether \( I(t_1) = 0 \),
that is,

\[
-c_1 \overline{b}_1 - c_2 \overline{b}_2 - c_3 \overline{b}_3 + (\Pi - T + t_1) [a_1 \overline{b}_1 + a_2 \overline{b}_2 + a_3 \overline{b}_3 \cdot 8 +
\frac{a_3 \overline{b}_3 (t_1 \cdot 8) - D}{2} - \frac{a_3 (T - t_1)^2}{2} - a (T - t_1)^2] \overline{b}_3
\]

\[= -c_1 \overline{b}_1 - c_2 \overline{b}_2 - c_3 \overline{b}_3 - (T - t_1) [a_1 \overline{b}_1 + a_2 \overline{b}_2 + a_3 \overline{b}_3 \cdot 8 +
\frac{a_3 \overline{b}_3 (t_1 \cdot 8) - D}{2} - \frac{a_3 (T - t_1)^2}{2} \overline{b}_3
\]

Substituting the given numerical values and simplifying we obtain

\[
t_1 = \frac{150 - 9\Pi}{10\Pi + 25}
\]

We also know decreasing the value of \( \Pi \) will decrease the duration of
time between \( t_{s1} \) and \( t_1 \), etc., which would decrease the total produc-
tion. For \( \Pi = 5.5 \) we obtain \( t_1 = 1.25 \) for which the decision rules are:

\[
X_1 = \overline{b}_1 \text{ if } \Pi - 5 + t_{s1} > 3/2, \text{ i.e., } t_{s1} > 1.00
\]

\[
X_2 = \overline{b}_2 \text{ if } \Pi - 5 + t_{s2} > 2/3, \text{ i.e., } t_{s2} > 1/6
\]

\[
v = \overline{b}_3 \text{ if } v_2 > 5 \text{ i.e., } t_{s3} < .52
\]
Calculating the inventory at time \( t_1 = 1.25 \), we find

\[
I(t_1) = 0 \int a_1 b_1 dt + \int_{1.25}^{1} a_1 b_1 dt + 0 \int a_2 b_2 dt + 0.16 \int_{1.25}^{1.52} a_2 b_2 dt + 0.52 \int_{1}^{1.25} [X_3(.52) + b_3 dt] dt - 0.16 \int_{1.25}^{1} b_2 dt + 0.52 \int_{1}^{1.25} b_3 dt = 106.28 - 100 = 6.28
\]

We note all the necessary conditions are satisfied except \( I(t_1) = 0 \) for subproblem 1. Since the production is slightly more than required, we reduce the value of \( \Pi \) to equal 5.2 for which we find \( t_1 = 1.34 \).

The decision rule follows.

\[
X_1 = \begin{cases} 
\overline{b}_1 & \text{if } t_{s1} > 1.3 \\
\underline{b}_1 & \text{if } t_{s1} < 1.3 
\end{cases} \quad \text{(VII)}
\]

\[
X_2 = \begin{cases} 
\overline{b}_2 & \text{if } t_{s2} > .46 \\
\underline{b}_2 & \text{if } t_{s2} < .46 
\end{cases} \quad \text{(VIII)}
\]

\[
v = \begin{cases} 
\overline{b}_3 & \text{if } t_{s3} < .46 \\
\underline{b}_3 & \text{if } t_{s3} > .46 
\end{cases} \quad \text{(IX)}
\]

Calculating the inventory at time \( t_1 = 1.34 \) we find

\[
I(t_1) = .38 \% 0
\]

A computer program was written to check whether \( I(t) \geq 0, \forall t \in (0,t_1) \). Unfortunately the condition was violated. Since moving \( v \) to upper bound did not satisfy all conditions, it was held at the lower bound and \( X_1 \) and \( X_2 \) were moved to upper bounds instead.
The given numerical values were substituted in equation (1) and the following relationship was established.

\[ 150 - 25t_1 - 5\pi - 10\pi t_1 = 0 \]

or

\[ t_1 = \frac{150 - 5\pi}{10\pi + 25} \]

Any value of \( \pi \) greater than 6.5 will provide \( X_1 = \bar{b}_1 \) and \( X_2 = \bar{b}_2 \) for all \( t > 0 \). For several values of \( \pi \), \( t_{s1}, t_{s2} \) and \( t_{s3} \) were tabulated (see Table 4.1) where \( t_{s1}, t_{s2} \) and \( t_{s3} \) represent the times as described in equations VII, VIII and IX. A computer program provides (see Table 4.2) the inventory pattern and the cost of implementing the program. Therefore, optimum decision rule, as given by Table 4.1, is:

\[
X_1 = \begin{cases} 
\bar{b}_1 & \text{for } 0 < t_{s1} < 0.8 \\
\bar{b}_2 & \text{for } 0.8 < t_{s1} < 5
\end{cases}
\]

\[
X_2 = \begin{cases} 
\bar{b}_2 & \text{for } 0 < t_{s2} < 0.8 \\
\bar{b}_3 & \text{for } 0.8 < t_{s2} < 5
\end{cases}
\]

\[ v = b_3 \quad t > 0 \]

The algorithm is presented next.

THE ALGORITHM

An algorithm has been developed for solving this problem. It contains the following steps.
TABLE 4.1
Cost of Program for Various \( n \)'s

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Step 1

Assume certain levels of inputs lie on the upper bound at $t_1^-$. Substitute all available numerical values in equation I. Simplify and obtain equation II.

Step 2

From the knowledge of the process parameters and equation (96) arbitrarily select a value of $\Pi$. Calculate $t_1$ using equation II for the selected value of $\Pi$.

Step 3

Compute decision rules similar to III, IV and V for the $t_1$ computed in step 2.

Step 4

If the decision rules computed in step 3 coincide with the values of $X_1$, $X_2$ and $X_3$ substituted in equation I and if $I(t_1) = 0$, then the solution is provided by III, IV and V. Otherwise go to step 5.

Step 5

If all the assumptions made in equation I are satisfied, and $I(t_1) \neq 0$, go to step 6. Otherwise go to step 7.

Step 6

If the terminal inventory $I(t_1)$ is negative (positive), then increase (decrease) the value of $\Pi$ since increasing (decreasing) $\Pi$ will increase (decrease) the duration of time $t_{e1}$ to $t_1$ which has the effect
of increasing (decreasing) the production rate. For the new \( \Pi \) compute \( t_1 \) and go to step 3.

**Step 7**

If the decision rules computed in step 3 contradict the values of \( X_1, X_2 \) and \( X_3 \) substituted in equation I, then substitute the new values of \( X_1, X_2 \) and \( X_3 \) in equation I. Simplify the equation I, obtain equation II and proceed to step 2.

END
EXAMPLE 3

Suppose the entrepreneur is interested in satisfying the demand for a product at the end of some period instead of satisfying the demand continuously at all times as described in Example 2. Then the entrepreneur would really want to schedule the production such that the total demand is met at the end of some period. In our models, the engineering cumulatively contributes towards production rate and hence the production rate increases, for the same quantity of inputs, as time passes. Therefore, it is conceivable that backorders accumulate initially and they may be satisfied at a later time as the production rate increases.

In fact, three different possibilities exist. 1) The initial inventory is big enough to satisfy the entire demand of the planning horizon as illustrated by Figure 4.1a. Optimization is not necessary in this case. 2) The system has a positive initial inventory and the inventory is depleted to zero at time $t_1$. Then the inventory becomes positive or zero as approximately illustrated by Figure 4.2. The inventory could even become negative before the system builds up positive inventory. We can state that once the system encounters positive inventory again, it will stay positive for the rest of the period. Also this situation can occur only when the system is in operation consuming the lower level of resources. If the system were in operation consuming some or all resources at its upper bound, we can always switch them to the lower bound at some appropriate time such that the demand is met exactly at the end of the planning horizon. Therefore,
we can state that optimization procedure is not necessary when the sys-
tem consumes minimum level of resources and maintains non-negative in-
ventory for all \( t \in (0,T) \). 3) As a third case, the system may start
with any initial inventory (negative, zero, or positive). It is re-
quired to schedule the production such that the demand is met exactly
at the end of the planning period.

Next we present a model describing case 2 and prove it becomes
case 3 if a feasible solution exists. We assume for simplicity that
the initial inventory is zero.

Assume the inventory is negative until time \( t_1 \) and then it is
non-negative until time \( T \). Also assume, in such a system, that back
orders are permitted without any penalty and carrying inventory would
involve certain costs. Then the problem may be stated as:

Min \( Z = \int_0^{t_1} [c_1x_1(t) + c_2x_2(t) + c_3v(t)] dt + \)
\( \int_{t_1}^T [c_1x_1(t) + c_2x_2(t) + c_3v(t) + c_4I(t)] dt \) \( (100) \)

S. to

\( \dot{x}_1(t) = a_1x_1(t) + a_2x_2(t) + a_3x_3(t) - d(t) \) \( (101) \)

\( \dot{x}_3(t) = v(t) \quad t \in (0,T) \) \( (102) \)

\( I(T) \geq 0 \) \( (103) \)

and

\( 0 \leq b_j \leq x_j(t) \leq b_j \quad , \quad j = 1,2 \) \( (104) \)

\( 0 \leq b_3 \leq v \leq b_3 \) \( (105) \)

\( x_3(0) = x_3^0 \) and \( I(0) = I^0 = 0 \) \( (106) \)
Therefore, the problem is to choose $X_1$, $X_2$ and $v$ such that the objective functional (100) is minimized and the total demand is met at the end of the planning horizon. The Hamiltonian is similar to Example 2, except we will have two Hamiltonian functions in this example.

(a) One for the time $t \in (0,t_1)$ where back orders are permitted at no cost, and (b) another for the time $t \in (t_1,T)$ where an inventory cost is incurred for carrying inventory.

**Necessary Conditions for the Time $t \in (0,t_1)$**

The Hamiltonian for all $t \in (0,t_1)$ can be written as:

$$H_1 = (a_1 \psi_1 - c_1)X_1 + (a_2 \psi_1 - c_2)X_2 + (a_3 \psi_1 - c_3)v + \psi_1(a_3X_3 - D)$$  \hspace{1cm} (107)

and the necessary conditions are:

$$X_i = \begin{cases} \bar{b}_1 & \text{if } \psi_1 > \frac{c_1}{a_1} \\ \frac{b_1}{a_1} & \text{if } \psi_1 < \frac{c_1}{a_1} \\ \text{Not specified} & \text{if } \psi_1 = 0 \end{cases} \quad i = 1, 2$$  \hspace{1cm} (108)

$$v = \begin{cases} \bar{b}_3 & \text{if } \psi_2 > c_3 \\ \frac{b_3}{c_3} & \text{if } \psi_2 < c_3 \\ \text{Not specified} & \text{if } \psi_2 = 0 \end{cases}$$  \hspace{1cm} (109)

and

$$\psi_1 = \frac{-\partial H_1}{\partial t} = 0$$  \hspace{1cm} (110)

$$\psi_2 = \frac{-\partial H_1}{\partial X_3} = -a_3 \psi_1, \psi_2(t_1) = 0$$  \hspace{1cm} (111)
since the state variable $X_3(t_1)$ representing the cumulative engineering is not constrained. We imposed a constraint on the state variable $I(t)$, that is, $I(t_1) = 0$, and hence an additional necessary condition is:

$$I(t_1)\Psi_1(t_1) = 0$$  \hspace{1cm} (112)

where $\Psi_1(t_1)$ is unconstrained.

Integrating we obtain for all $t \in (0, t_1)$

$$\Psi_1(t) = k$$  \hspace{1cm} (114)

where $k$ is some constant. Substituting (114) in equation (111) and integrating we obtain

$$\Psi_2(t) = a_3 k(t_1 - t)$$  \hspace{1cm} (115)

since $\Psi_2(t_1) = 0$

Necessary Conditions for the Time $t \in (t_1, T)$

The Hamiltonian function for all $t \in (t_1, T)$ includes cost of inventory which may be written as

$$H_2 = (a_1 \Psi_1 - c_1)X_1 + (a_2 \Psi_1 - c_2)X_2 + (\Psi_2 - c_3)v +$$

$$\Psi_1(a_3 X_3 - \dot{B}) - c_4 I$$  \hspace{1cm} (116)

and the necessary conditions are

$$X_1 = \begin{cases} \overline{b_1} & \text{if } \Psi_1 > c_4/a_4 \\ b_1 & \text{if } \Psi_1 < c_4/a_4 \\ \text{Not specified} & \text{if } \Psi_1 = 0 \end{cases}$$  \hspace{1cm} (117)
for $i = 1, 2$

$$v = \begin{cases} 
\frac{b_3}{v_2 - c_3} & \text{if } v_2 > c_3 \\
\frac{b_3}{v_2 - c_3} & \text{if } v_2 < c_3 \\
\text{Not specified} & \text{if } v_2 = 0
\end{cases} \quad (118)$$

and

$$\dot{\psi}_1 = \frac{\partial H_2}{\partial \psi_1} = c_4 \quad (119)$$

$$\dot{\psi}_2 = -\psi_2' = -a_3\psi_1, \quad \psi_2(T) = 0 \quad (120)$$

We stated $I(T) > 0$ and we also know the cost is minimized for $I(T) = 0$. Hence, an additional necessary condition is:

$$I(T)\psi_1(T) = 0 \quad (121)$$

and $\psi_1(T)$ is unconstrained.

We mentioned earlier that situations with $I(T) > 0$ cannot occur in our type of problems and if it occurred we cannot prevent it from happening. Hence we will not discuss problems involving constraints $I(T) > 0$. Solving equations (119) through (121) for all $t \in (t_1, T)$ we obtain

$$\dot{\psi}_1(t) = \alpha - c_4(T - t) \quad (122)$$

and

$$\dot{\psi}_2(t) = a_3\alpha(T - t) - a_3c_4 \frac{(T - t)^2}{2} \quad (123)$$

where $\alpha$ is a constant similar to $k$ in equations (114) and (115). We
have derived the necessary conditions for $t \in (0, t_1)$ and $t \in (t_1, T)$ separately. However, the following additional conditions should be satisfied for the integrated problem with planning period $(0, T)$ to hold true. The conditions as given by Bryson (1969) are:

$$\psi_1(t_1^-) = \frac{\partial \phi}{\partial I(t_1^-)} = \Pi$$  \hfill (124)  

$$\psi_1(t_1^+) = -\frac{\partial \phi}{\partial I(t_1^+)} = -\Pi$$  \hfill (125)  

$$\psi_2(t_1^-) = \frac{\partial \phi}{\partial X_3(t_1^-)} = 0$$  \hfill (126)  

$$\psi_2(t_1^+) = -\frac{\partial \phi}{\partial X_3(t_1^+)} = 0$$  \hfill (127)  

and

$$\frac{\partial \phi}{\partial t_1} + H_1(t_1^-) - H_2(t_1^+) = 0$$  \hfill (128)  

where $\phi = \Pi g(I)$. The function $g(I)$ represent the constraint on the state variable $I(t)$, and $\Pi$ is a Lagrange Multiplier. In our case $g(I) = I$ and hence $\phi = \Pi I$. Substituting these values in equations (114), (115), (122) and (123) we conclude the following:

The equation (122) states that $\psi_1$ is a function of time and hence

$$\psi_1(t_1^+) = a - c_q(T - t_1^+)$$

However, equation (125) states $\psi_1(t_1^+)$ is a constant. The equations (122) and (125) cannot simultaneously hold unless $T = t_1$ in which case the integral for time $t_1$ to $T$ in equation (100) vanishes and hence the
necessary conditions are given by equations (108) through (115). Therefore, we can conclude that the optimal solution, if a feasible solution exists, will contain back orders only. It is logical since carrying inventory incurs cost whereas back orders do not penalize the system. Therefore, the optimum solution will contain only back orders, in our model. Problems involving more complicated inventory and demand patterns may also be modeled and solved utilizing the same technique.

Next, we present two numerical examples. The Examples 3A and 3B portray situations represented by case 2 and case 3, respectively.

**Numerical Example 3A**

Utilizing the same data provided in numerical Example 2A find the optimum decision rule if the entrepreneur adopted the criterion suggested by Example 2.

\[
\begin{align*}
a_1 &= 2 & a_2 &= 3 & a_3 &= 2.5 \\
c_1 &= 3 & c_2 &= 2 & c_3 &= 5 \\
b_1^l &= 20 & b_2^l &= 15 & b_3^l &= 6 \\
b_1^r &= 15 & b_2^r &= 10 & b_3^r &= 4 \\
I(0) &= 0 & X_3(0) &= 0 & D &= 80 & \text{and } T = 5
\end{align*}
\]

The problem is to choose \( k \) such that all necessary conditions are satisfied while the demand is met by the end of the planning horizon.

Let \( k = 0 \) for which \( \psi_1 < \frac{c_1}{a_1} \) and \( \frac{c_2}{a_2} \); and \( \psi_2 < c_3 \). Hence the decision rule is
\[ X_1 = b_1, X_2 = b_2 \text{ and } v = b_3 \]

Substituting the numerical values we find the terminal inventory

\[
I(T) = \int_0^5 a_1 b_1 \, dt + \int_0^5 a_2 b_2 \, dt + \int_0^5 a_3 b_3 \, dt - \int_0^5 D \, dt
\]

\[ = 425 - 400 = 25 \]

Since we have already used only minimum possible inputs, we cannot reduce the production any further and hence the optimum decision rule is to utilize the minimum level of resources for all \( t \in (0,5) \), as described in case 2. A computer program provides the level of inventory and cost of implementing the program. See Table 4.3. The cost of implementing this program is 439.20.

Comparing Tables 4.2 and 4.3, we find that it costs 13 percent more for the entrepreneur if he had to satisfy the customer's demand continually, as opposed to the policy of Example 3 where back orders are permitted.

In this example, we were able to satisfy the demand by utilizing the minimum level of resources. It may not always be possible to do so. In those cases, we might have to use the upper level of some or all resources as illustrated by case 3. A numerical example is presented next to illustrate this situation.

**Numerical Example 3B**

Compute the optimum decision rule for the following data if the entrepreneur adopted the criterion suggested by Example 3.
### Table 4.3

Inventory Pattern and Cost of Implementing the Optimal Solution for Numerical Example 3A

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\[
\begin{align*}
a_1 &= 2 & a_2 &= 3 & a_3 &= 2.5 \\
c_1 &= 3 & c_2 &= 2 & c_3 &= 6 \\
\bar{b}_1 &= 20 & \bar{b}_2 &= 15 & \bar{b}_3 &= 6 \\
b_1 &= 10 & b_2 &= 5 & b_3 &= 3 \\
I(0) &= 0 & X_3(0) &= 0 & D &= 100, & T &= 5
\end{align*}
\]

The problem is to choose \( k \) such that the necessary conditions (108), (109), (114) and (115) are satisfied and the demand is met by the end of the planning horizon.

**Trial 1**

Let \( k = 0 \) for which \( \Psi_1 = 0 \) and \( \Psi_2 = 0 \). Since \( \Psi_1 < c_1/a_1 \) and \( c_2/a_2 \); and \( \Psi_2 < c_3 \) the decision rule is to use

\[
\begin{align*}
X_1 &= b_1 \\
X_2 &= b_2 \\
v &= b_3
\end{align*}
\]

Substituting the given numerical values we find the terminal inventory

\[
I(T) = \int_0^5 a_1 b_1 dt + \int_0^5 a_2 b_2 dt + \int_0^5 a_3 b_3 t dt - \int_0^5 D dt
\]

\[
= 268.75 - 500 = -231.75.
\]

Since the demand is not satisfied, we increase the value of \( k \) to one and check whether all conditions are satisfied.

For any constant \( k \) the inputs \( X_1 \) and \( X_2 \) remain either at its upper bound or at its lower bound. The input \( v \) will stay at its upper bound if \( \Psi_2 > 6 \). Otherwise, it will move to its lower bound. Trial 2 calculations follow.
Trial 2

Let $k = 1$, $\Psi_1 = 1$ and $\Psi_2 > 6$ for $t < 2.5$

$< 6$ for $t > 2.5$

The decision rule is

$$X_1 = \overline{b}_1$$
$$X_2 = \overline{b}_2$$
$$v = \overline{b}_3 \text{ for } t < 2.5$$
$$\overline{b}_3 \text{ for } t > 2.5$$

Substituting these numerical values we find the terminal inventory

$$I(T) = \int_1^{2.5} a_1 b_1 dt + \int_2^{2.5} a_2 \overline{b}_2 dt + \int_2^{2.5} a_3 \overline{b}_3 dt + \int_2^{2.5} a_2 \overline{b}_3 dt$$

$$= 535 - 500 = 35$$

Since a positive inventory is incurred, we decrease the value of $k$ to .67 and check whether the conditions are satisfied.

Trial 3

Let $k = .67$, $\Psi_1 = .67$ and $\Psi_2 > 6$ for $t < 1.36$

$\Psi_2 < 6$ for $t > 1.36$

The decision rule is

$$X_1 = \overline{b}_1$$
$$X_2 = \overline{b}_2$$
$$v = \begin{cases} 
\overline{b}_3 \text{ for } t < 1.36 \\
\overline{b}_3 \text{ for } t > 1.36 
\end{cases}$$
Substituting these numerical values we find the terminal inventory

\[ I(T) = \int_0^5 a_1 b_1 \, dt + \int_0^5 a_2 b_2 \, dt + \int_0^{1.36} a_3 b_3 \, dt + 1.36 \int_0^5 a_3 (1.36 + b_3 t) \, dt - \int_0^5 \delta \, dt \]

\[ = 499.95 - 500 = -0.05 \approx 0. \]

and hence all necessary conditions are satisfied.

We indicated earlier we might have to use upper levels of some inputs for some time. We see the input \( X_1 \) is at its lower level, the input \( X_2 \) is at its upper level and the engineering input \( v \) is at its upper level for \( t < 1.36 \) and its lower level for \( t > 1.36 \). A computer program provides the inventory level and the cost of implementing this program. See Table 4.4. The cost of implementing this policy is 414.48. We also notice that the inventory is always negative as we concluded earlier.

The solution seems logical since the engineering expenditure \( v \) which is cumulative over time is at its upper level during some early period and is at its lower level during the rest of the period. The labor input \( X_1 \) always stays at its lower bound and the capital input \( X_2 \) stays at its upper bound throughout the planning horizon.

In Example 1, we assumed the cost of inventory is proportional to the square of the inventory or back order. In Example 2, we assumed the demand should be met at all times and in Example 3, we permitted back orders. In all three examples, we allowed the input levels to charge without any penalty. In the real world, however, different policies may be pursued. Example 4 is presented next to describe such a situation.
TABLE 4.4

Inventory Pattern and Cost of Implementing

the Optimal Solution for Numerical Example 3B

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<tr>
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<tr>
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<td>0.00</td>
<td>398.88</td>
<td>398.88</td>
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<tr>
<td>5.00</td>
<td>-0.06</td>
<td>0.00</td>
<td>414.48</td>
<td>414.48</td>
</tr>
</tbody>
</table>
EXAMPLE 4

In Example 2, we assumed a linear cost function. The resulting optimum decision rule was to utilize the inputs either at the lower bound or at the upper bound. In the real world, there is often a penalty associated with changing the inputs (Holt, 1963). For illustrative purposes we assume the cost associated with this penalty is proportional to the square of the change in the value of the inputs $X_1$ and $X_2$. We also assume any amount of engineering services can be obtained at time $t$ and hence no penalty term is included for the changes in the engineering input in the objective function. A question may be raised on the feasibility of such an assumption. The answer might be that a firm might have several products and hence it should direct its resources to benefit other products. Also if the entrepreneur were hiring engineering services from consulting firms, the solution is very appropriate. Making all the assumptions of Example 1, we can state the problem as:

$$
\begin{align*}
\text{Min } Z &= \int_0^T \left( c_1 X_1(t) + c_{11} \dot{X}_1^2(t) + c_2 X_2(t) + c_{21} \dot{X}_2^2(t) + c_3 X_3(t) + c_4 I^2(t) \right) dt \\
\text{S. to } & \\
I &= a_1 X_1 + a_2 X_2 + a_3 X_3 - D
\end{align*}
$$

for the given bounds and initial conditions. Let

$$
\begin{align*}
\dot{X}_1(t) &= v_1(t) \\
\dot{X}_2(t) &= v_2(t)
\end{align*}
$$
\[ x_3(t) = v_3(t) \]

Then the Hamiltonian for this problem may be written as:

\[
H = \psi_1 \left( a_1 x_1 + a_2 x_2 + a_3 x_3 - D \right) + \psi_2 v_1 + \psi_3 v_2 + \psi_4 v_3 \\
- c_1 x_1 - c_{11} v_1^2 - c_2 x_2 - c_{21} v_2^2 - c_3 v_3 - c_4 t^2
\]

The necessary conditions for the control variables to be optimum are:

\[
\dot{\psi}_1 = -\frac{\partial H}{\partial I} = 2c_4 I
\]

\[
\dot{\psi}_2 = -\frac{\partial H}{\partial x_1} = -\psi_1 a_1 + c_1
\]

\[
\dot{\psi}_3 = -\frac{\partial H}{\partial x_2} = -\psi_1 a_2 + c_2
\]

\[
\dot{\psi}_4 = -\frac{\partial H}{\partial x_3} = -\psi_1 a_3
\]

\[
\frac{\partial H}{\partial \psi_1} = \psi_2 - 2c_{11} v_1 = 0
\]

\[
\frac{\partial H}{\partial \psi_2} = \psi_3 - 2c_{21} v_2 = 0
\]

Next we solve for \( \psi_1(t) \) and \( \psi_2(t) \).

\[
\psi_1(t) = \psi_1(0) + \int_0^t 2c_4 I(t) \, dt
\]

\[
= -2c_4 \int_0^t I(\tau) \, d\tau
\]

\[
\psi_2(t) = \psi_2(0) + \int_0^t [c_1 - a_1 \psi_1(\tau)] \, d\tau
\]
\begin{align*}
\psi_2(0) &= -\int_{\tau=0}^{T}[c_1 - a_1\psi_1(\tau)]d\tau \\
\psi_2(t) &= -\int_{\tau=t}^{T}[c_1 - a_1\psi_1(\tau)]d\tau \\
&= -c_1(T - t) + \int_{\tau=t}^{T}a_1[\int_{\rho=\tau}^{T}2c_4I(\rho)d\rho]d\tau
\end{align*}

Similarly
\begin{align*}
\psi_3(t) &= -c_2(T - t) + \int_{\tau=t}^{T}a_2[\int_{\rho=\tau}^{T}2c_4I(\rho)d\rho]d\tau \\
\psi_4(t) &= \int_{\tau=t}^{T}a_3[\int_{\rho=\tau}^{T}2c_4I(\rho)d\rho]d\tau
\end{align*}

and
\begin{align*}
\psi_1 &= \frac{\psi_2}{2c_{11}} \\
\psi_2 &= \frac{\psi_3}{2c_{21}}
\end{align*}

Since \( \psi_3 \) is linear in the objective function, the Hamiltonian is maximized for
\[
\psi_3 = \begin{cases} 
\bar{b}_1 & \text{if } \psi_4 > c_3 \\
\underline{b}_1 & \text{if } \psi_4 < c_3 \\
\text{Unspecified} & \text{if } \psi_4 = 0
\end{cases}
\]

We see \( \psi_1, \psi_2 \) and \( \psi_3 \) are functions of \( \psi_1 \) and time and \( \psi_1 \)'s are functions of inventory and the inventory is a function of inputs \( X_1 \) and in turn the inputs are functions of \( \psi_1 \). Therefore the necessary
conditions along with initial conditions may be solved simultaneously to obtain solutions (utilizing numerical methods).

**Example 4 with a Modified Objective Function**

We assumed in Example 4 that the cost of changing engineering input level can be ignored. In many instances, firms maintain their own engineering departments. The firms desire to keep the engineering force as constant as possible. In these instances, we might consider a penalty for changing the engineering level instead of using the cost of engineering input as a part of the objective functional. For example, we might assume that the cost associated with this penalty is proportional to the square of the change in the engineering input.

Under this assumption, we will have only quadratic terms involving state and control variables and linear functions of state variables in the objective function. Such problems may be more easily solved than problems with objective functions containing linear functions of control variables also. Making these changes, we can rewrite the problem as:

\[
\text{Min } Z = \int_0^T \left( c_1 X_1(t) + c_{11} v_1^2 + c_2 X_2(t) + c_{21} v_2^2 + c_3 v_3^3 + c_4 I^2 \right) dt
\]

(131)

**S. to**

\[
\begin{align*}
\dot{I} &= a_1 X_1 + a_2 X_2 + a_3 X_3 - D \\
\dot{X}_1 &= v_1 \\
\dot{X}_2 &= v_2 \\
\dot{X}_3 &= X_4 \\
\dot{X}_4 &= v_3 
\end{align*}
\]

(132)
\[ x_i(0) = x_i^0, \quad i = 1,2,3,4 \]  
\[ I(0) = I^0 \]

where \( b_j, \bar{b}_j \), \( X(0) \) and \( I(0) \) represent similar definitions as described in earlier problems. The problem is to choose \( v_1, v_2 \) and \( v_3 \) such that the objective functional is minimized. The Hamiltonian \( H \) may be written as:
\[
H = -c_1 x_1 - c_2 x_2 - c_{11} v_1^2 - c_{21} v_2^2 - c_{33} v_3^2 - c_4 I^2 + \psi_1 [a_1 x_1 + a_2 x_2 + a_3 x_3 - D] + \psi_2 v_1 + \psi_3 v_2 + \psi_4 x_4 + \psi_5 v_3 \]  
(134)

The necessary conditions for the Hamiltonian \( H \) to attain a maximum are:

\[
\frac{\partial H}{\partial v_1} = \psi_2 - 2c_{11} v_1 = 0
\]
\[
\frac{\partial H}{\partial v_2} = \psi_3 - 2c_{21} v_2 = 0
\]  
(135)
\[
\frac{\partial H}{\partial v_3} = \psi_5 - 2c_{33} v_3 = 0
\]

\[ \psi_1 = \frac{-\partial H}{\partial I} = -2c_4 I \]
\[ \psi_2 = \frac{-\partial H}{\partial x_1} = -a_1 \psi_1 \]
\[ \psi_3 = \frac{-\partial H}{\partial x_2} = -c_2 - a_2 \psi_1 \]  
(136)
The necessary conditions may be written in matrix form as

\[ \bar{\psi} = A\bar{X} + B\bar{u} - \bar{D} \]

That is

\[
\begin{bmatrix}
\dot{X}_1 \\
\dot{X}_2 \\
\dot{X}_3 \\
\dot{X}_4
\end{bmatrix} =
\begin{bmatrix}
0 & a_1 & a_2 & a_3 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4
\end{bmatrix} +
\begin{bmatrix}
I \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4 \\
\psi_5
\end{bmatrix}
-
\begin{bmatrix}
D \\
0 \\
0 \\
0
\end{bmatrix}
\]

and

\[ \bar{\psi} = -A^T\bar{\psi} - Q\bar{X} - \bar{C} \]

That is

\[
\begin{bmatrix}
\dot{\psi}_1 \\
\dot{\psi}_2 \\
\dot{\psi}_3 \\
\dot{\psi}_4 \\
\dot{\psi}_5
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4 \\
\psi_5
\end{bmatrix}
-\begin{bmatrix}
c_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
I \\
X_1 \\
X_2 \\
X_3 \\
X_4
\end{bmatrix}
-\begin{bmatrix}
0 \\
c_1 \\
c_2 \\
0 \\
0
\end{bmatrix}
\]

Equations of this structure may be solved using a Riccati
transformation (Athans, 1966). Performing manipulations similar to those of Example 1 (see Appendix B), we can write

$$\ddot{X} = A\dot{X} - BR^{-1}B^T\dot{X} + BR^{-1}B^T\ddot{X} - D$$

(139)

The value of the matrix $K$ and the vector $\ddot{g}$ are obtained by solving the following differential equations

$$\dot{K} = -A^T_K - KA + KBR^{-1}B^T_K - Q; \quad K(T) = 0$$

(140)

$$\dot{\ddot{g}} = -A^T_{\ddot{g}} + KBR^{-1}B^T_{\ddot{g}} + KD - \dddot{c}; \quad \dddot{g}(T) = 0$$

(141)

where

$$R = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

and all other terms were already defined. Programs (Melsa, 1970) are available to solve the Riccati equation (140). Once the matrix $K$ has been obtained, the equation (141) can be solved. Substituting these values in (139) the trajectory of $\dddot{X}$ can be solved numerically. Thus we see problems of this structure approximate the objectives of the entrepreneur and also provide easier solutions compared to the original problem.

No bounds were imposed on the control variables in this case. Problems with bounds on control variables may also be solved in the same manner.
EXAMPLE 5

This example contains a Cobb-Douglas type production function constraint. We assume that the cost of inventory is proportional to the square of the inventory. This problem is the same as Example 1 with a nonlinear constraint instead of a linear constraint. Therefore the problem is to

\[
\begin{align*}
\text{Min } Z &= \int_0^T \left( c_1 X_1(t) + c_2 X_2(t) + c_3 v(t) + c_4 I_1^2(t) \right) dt \\
\text{S. to } \quad &\dot{I}(t) = X_1^\alpha(t) X_2^\beta(t) X_3^\gamma(t) - D(t) \\
&\dot{X}_3(t) = v(t) \\
&0 < b_3 < v(t) < b_3
\end{align*}
\]

and for the given initial conditions

\[
\begin{align*}
I(0) &= I^0 \\
X_3(0) &= X_3^0
\end{align*}
\]

The maximum principle technique provides necessary conditions for the control variables to be optimum. The Hamiltonian may be written as:

\[
H = \psi_1 \left[ X_1^\alpha X_2^\beta X_3^\gamma - D \right] + \psi_2 v - c_1 X_1 - c_2 X_2 - c_3 v - c_4 I^2
\]

and the necessary conditions are:

\[
\psi_1 = -\frac{2H}{dI} = 2c_4 I
\]
Since \( v \) is linear in the objective function, the Hamiltonian will be maximized for:

\[
\begin{cases}
\bar{b}_3 & \text{if } \psi_2 > c_3 \\
\underline{b}_3 & \text{if } \psi_2 < c_3 \\
\text{Not specified} & \text{if } \psi_2 = c_3
\end{cases}
\]  

(152)

Next we solve for \( \psi_1(t) \) and \( \psi_2(t) \).

\[
\dot{\psi}_1(t) = 2c_4 I(t)
\]

(153)

\[
\psi_1(t) = -\int_0^t 2c_4 I(\sigma)d\sigma
\]

\[
\dot{\psi}_2(t) = \gamma \psi_1 x_1^\alpha x_2^\beta x_3^\gamma - 1
\]

\[
\psi_2(t) = \int_0^t \psi_1(\tau)x_1^\alpha(\tau)x_2^\beta(\tau)x_3^\gamma - 1(\tau)d\tau
\]

Substituting for \( \psi_1 \) we obtain

\[
\psi_2(t) = -2c_4 \gamma \int_0^t x_1^\alpha(\tau)x_2^\beta(\tau)x_3^\gamma - 1(\tau)\left[ \int_\rho^t (\rho)dp \right]d\tau
\]

(154)

Although we have derived the necessary conditions we will not be able to obtain \( x_1, x_2 \) and \( v \) in terms of \( \psi_1, x_3 \) and \( I \) alone, since the equations (148) through (152) cannot be analytically solved. A
numerical search technique, however, would provide optimal solution to the problem.

**SOLUTION TO MODEL P4**

We mentioned earlier that the problem P4 could be difficult to solve depending upon the type of functions present in the model. We assume all functions described by equations (34) through (37) of Chapter III are linear and let

\[ \dot{X}_3 = v_1, \dot{X}_6 = v_2 \]

Now the problem may be written as:

\[ \min Z = \int_0^T (c_1 X_1(t) + c_2 X_2(t) + c_3 v_3(t) + c_4 X_4(t) + \]
\[ c_5 X_5(t) + c_6 v_6(t) + c_7 I_1(t) + c_8 I_2(t)) dt \]  
(155)

\[ I_1(t) = a_1 X_1(t) + a_2 X_2(t) + a_3 X_3(t) + a_4 X_4(t) - D_1 \]  
(156)

\[ I_2(t) = a_5 X_5(t) + a_6 X_6(t) + a_7 X_7(t) + a_8 X_8(t) - D_2 \]  
(157)

\[ \dot{X}_3(t) = v_3 \]  
(158)

\[ \dot{X}_6(t) = v_6 \]  
(159)

and

\[ 0 < b_j \leq X_j(t) \leq \bar{b}_j, \quad j = 1, 2, 4, 5 \]  
(160)

\[ 0 < b_j \leq v_j(t) \leq \bar{b}_j, \quad j = 3, 6 \]  
(161)

\[ I_1(T) \geq 0, \quad I_1(0) = I_1^0, \quad i = 1, 2 \]
The equations (156) and (157) are similar to equations (76) of P3 Example 2. Therefore, we have a problem with a linear objective function and four linear constraints. The problem is also similar to Example 2 of P3. The objective function contains costs for both products and the production functions contain an extra term representing the interaction effect. However, the problem is more complicated than Example 2, P3 but can be solved in the same manner. Numerical examples illustrating solutions of P4 will not be pursued here.

CONCLUSION

In this chapter we have discussed solution procedures for one product and two product static and dynamic situations. The one product static model yielded analytic solutions. The two product static model provided analytic solution when linear functions were involved. Two product static models containing nonlinear Cobb-Douglas type production functions are solvable by numerical methods only.

The dynamic continuous time models containing one product provided analytic solutions when linear objective and linear production functions were present in the examples. The problems involving complicated nonlinear objective functions or production functions may be solved only numerically. The complexity would be increased as the number of variables in the model increase.

In many instances, continuous time models with linear objective
functional, linear constraints and bounds may be approximated with
discrete time models. The discrete time models may be more easily
solved utilizing programming techniques which is explained in Chapter
V for multiproduct planning situations.
CHAPTER V
MULTI-PRODUCT MODEL

Introduction

The models P1 and P3 portrayed static and dynamic situations of a one production problem. Models P2 and P4 discussed the static and dynamic versions of a two product situation involving interaction effects. These models described the engineering as an input for individual products. In Chapter III, we discussed the importance of specifying the R & D expenditures on individual inputs rather than on the products and then formulated a general model.

The general model formulated in Chapter III may be solved utilizing the maximum principle technique. However, obtaining solutions to the general model will be more difficult than the one product or the two product models. Therefore, an alternate solution technique is pursued.

This solution technique involves the modification of the continuous time model to a discrete time model by segmenting the planning horizon into intervals. Bishop (1969) points out that interpretation of discrete models is more straightforward than the concept of a continuous time model in certain instances. This will become obvious in our case as we proceed to build the discrete time model. We shall also see later that efficient computerized computational techniques
are available to solve discrete models. In fact it is a prime reason for discrete models gaining more prominence in recent years as opposed to continuous time models.

**Formulation of Discrete Time Model**

In the multiproduct continuous time model described in Chapter III, we assumed that the vector of inputs $\overline{X}$ (the factors of production) yield the vector of products $\overline{w}$ and the vector of R & D inputs $\overline{u}$ yield an increase in output $\overline{v}$ (the products). The total output of the system was defined as the sum of the two vectors $\overline{w} + \overline{v}$. The relationships between $\overline{X}$ and $\overline{w}$, and $\overline{u}$ and $\overline{v}$ were expressed by functions $f$ and $g$, respectively, in Chapter III, p.62.

First, we assume the functions $f$ and $g$ are linear. Then we convert the planning horizon (0-T) into T discrete time periods. Now we can formulate a linear programming problem with T periods. If the demand for every period is known, then the problem is to choose the factors of production $\overline{X}$, R & D inputs $\overline{u}$ and the inventory levels $\overline{s}$ for each period of time $t$ such that the total demand is met and the total cost of the system is minimized.

**The General Model Linear Programming Formulation**

Based on assumptions made in the previous paragraph we redefine the variables present in the continuous time model to suit the formulation of the linear programming model. Let
\( \overline{x}(t) \) n-vector of factors of production (inputs) used during period \( t \)

\( \overline{u}(t) \) n-vector of the unused portion of inputs during period \( t \)

\( A(t) \) n\times m matrix of coefficients necessary to convert the input \( \overline{x}(t) \) to products \( \overline{w}(t) \). \( A(t) \) is constant over time

Therefore

\[ \overline{w}(t) = A \overline{x}(t). \]

\( \overline{u}(t) \) n-vector of R & D inputs used corresponding to factors of production \( \overline{x}(t) \) during period \( t \)

\( \overline{un}(t) \) n-vector of the unused portion of R & D inputs during period \( t \)

\( B(t) \) m\times n matrix of coefficients necessary to transform the R & D inputs to increases in production rate \( \overline{q}(t) \).

\( B(t) \) is constant over time

\( \overline{v}(t) \) m-vector of the sum of the increases in production rate corresponding to R & D inputs \( \overline{u}(t) \) used through period \( t \).

Therefore

\[ \overline{q}(t) = B \overline{u}(t) \]

\[ \overline{v}(t) = \sum_{\tau=1}^{t} \overline{q}(\tau) = B \sum_{\tau=1}^{t} \overline{u}(\tau) \]

The total production rate \( \overline{y}(t) \) during any period \( t \) is the sum of the production rate due to the factors of production and the cumulative
increases in production rate due to R & D inputs, that is,

\[ \bar{y}(t) = \bar{w}(t) + \bar{v}(t) \]

In addition let us define the following.

- \( \bar{s}(t) \) m-vector quantity of products stored after meeting the required demand during period \( t \)
- \( \bar{s}(0) \) m-vector quantity of initial stock of products
- \( \bar{d}(t) \) m-vector demand for products during period \( t \) assumed to be known
- \( \bar{u}s(t) \) m vector of unused warehouse space in number of units
- \( \bar{c}_1 \) n-vector of costs per unit of resource \( x \) used
- \( \bar{c}_2 \) n-vector of costs per unit of unused resource \( u_x \)
- \( \bar{c}_3 \) n-vector of costs per unit of R & D input \( u \) used
- \( \bar{c}_4 \) n-vector of costs per unit of unused R & D input \( u_n \)
- \( \bar{c}_5 \) m vector of costs per unit of product stored \( \bar{s} \)
- \( \bar{c}_6 \) m vector of costs per unit of unused storage space \( u_s \)

The problem is to minimize the total cost of the system for periods \( t = 1, 2, \ldots, T \) subject to the constraints described in the next section. Assuming the cost vectors do not change over time the total cost of the system during period \( t \) is given by:

\[ Z(t) = \bar{c}_1 \bar{x}_1(t) + \bar{c}_2 \bar{u}x(t) + \bar{c}_3 \bar{u}(t) + \bar{c}_4 \bar{u}_n(t) + \bar{c}_5 \bar{s}(t) + \bar{c}_6 \bar{u}s(t) \]

The following material balance equations serve as constraints of our system. They are:
\[ y(t) - s(t) = d(t) - s(t-1) \]

Substituting for \( y(t) \) we obtain,

\[ w(t) + v(t) - s(t) = d(t) - s(t-1), \]

or

\[ w(t) + \sum_{\tau=1}^{t} q(\tau) - s(t) = d(t) - s(t-1) \]

Making proper substitution for \( w(t) \) and \( q(t) \) we obtain:

\[ A \bar{x}(t) + B \sum_{\tau=1}^{t} u(\tau) - s(t) = d(t) - s(t-1) \]  \hspace{1cm} (1)

\[ \bar{x}(t) + ux(t) = \bar{b}(t) \]  \hspace{1cm} (2)

\[ \bar{u}(t) + \bar{u}n(t) = \bar{r}(t) \]  \hspace{1cm} (3)

\[ \bar{s}(t) + \bar{u}s(t) = \bar{k}(t) \]  \hspace{1cm} (4)

\[ \bar{x}(t), \bar{u}(t) \geq 0 \]  \hspace{1cm} (5)

where \( b(t), r(t)\) and \( k(t) \) are upper bounds on the factors of production, R & D inputs and storage availability.

The problem has been illustrated in the Tableau 5.1 for three periods.

The presence of a summation in constraint (1) makes it different from the usual linear constraint. Row 1 of the Tableau represents the constraint (1) for the period (1) which is obtained by letting \( t = 1 \), that is,

\[ Ax(1) + Bu(1) - s(1) = d(1) - s(0) \]

Indicating the time with subscripts we obtain the row 1 of the tableau as:
TABLEAU 5.1.

Tableau for a Three-Period Linear Model

<table>
<thead>
<tr>
<th>Row</th>
<th>Period 1</th>
<th>Period 2</th>
<th>Period 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x_1$</td>
<td>$ux_1$</td>
<td>$u_1$</td>
</tr>
<tr>
<td>1</td>
<td>A</td>
<td>B</td>
<td>-I</td>
</tr>
<tr>
<td>2</td>
<td>I</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>I</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>I</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>B</td>
<td>I</td>
<td>A</td>
</tr>
<tr>
<td>6</td>
<td>I</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>I</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>B</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>B</td>
<td>I</td>
<td>A</td>
</tr>
<tr>
<td>10</td>
<td>I</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>I</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$c_3$</td>
</tr>
</tbody>
</table>
\[ Ax_1 + B\ul_1 - \ul_1 = \ul_1 - \ul_0 \]

where \( \ul_1 \) is the vector of products manufactured during period one to be utilized during period 2. The constraints 2, 3 and 4 are represented by rows 2, 3 and 4 respectively.

Similarly, row 5 represents constraint 1 for period 2, that is,

\[ A\ul(2) + B\ul(1) + B\ul(2) - \ul(2) = \ul(2) - \ul(1) \]

Indicating the time with subscripts and rearranging we obtain the row 5 of the tableau as:

\[ B\ul_1 + \ul_1 + A\ul_2 + B\ul_2 - \ul_2 = \ul_2 \]

In the same manner other rows can be represented.

We have formulated a problem representing a multiproduct situation for three periods and the model can be extended to any number of periods. The problem can theoretically be solved using the linear programming technique. The non-negativity constraints insure against negative inputs and negative production. Linear programming algorithms automatically take care of the non-negativity constraints.

Computational Considerations

A multiperiod model will contain a big matrix with thousands of elements. For example, consider a problem with 100 inputs and 30 outputs. The matrices \( A \) and \( B \) will each have 100 columns and the inventory matrix will have 30 columns. Totally, there will be 4600 columns in a ten period model including the slack vectors. Even
though the problem may theoretically be solved it requires large computers and they are expensive. Therefore, we explore the avenue of simplifying computations.

According to Lasdon (1970, p. 104), linear programming problems with 4095 rows have been successfully solved (and problems with many more rows may be solved) using the revised simplex method as implemented in the IBM mathematical programming system MPS/360. Dantzig (1965) indicates linear programming problems with $10^6$ equations and $10^6$ variables have been successfully solved. However, special algorithms for exploiting the structure of the system model had to be constructed.

To take advantage of a particular structure which has proven useful in the efficient solution of large linear programming problems, rearrange rows 9 and 5 of Tableau 5.1 as shown in Tableau 5.2. Thus, the former row 9 is now row 1 and former row 5 is now row 2. The problem is now in what called block angular form and such problems may be solved utilizing the Dantzig-Wolfe decomposition algorithm.

The rows 1, 2 and 3 are called the coupling constraints and each diagonal block comprise a subproblem in the Dantzig-Wolfe Decomposition algorithm.

Dantzig (1963) indicates that one should exploit the special structures of subproblems, if any, to reduce the computational requirements. We notice the rows of all subproblems contain only values of unity. Problems containing coupling constraints and subproblem rows containing unity are called group problems. The group problem treats all the subproblems as one subproblem and they are solved with the
### TABLEAU 5.2

Tableau for a Three-Period Linear Model Using Dantzig-Wolfe Decomposition Algorithm

| Row | \( \bar{x}_1 \) | \( \bar{u}_x \) | \( \bar{u}_1 \) | \( \bar{u}_n \) | \( \bar{s}_1 \) | \( \bar{s}_u \) | \( \bar{x}_2 \) | \( \bar{u}_x \) | \( \bar{u}_2 \) | \( \bar{u}_n \) | \( \bar{s}_2 \) | \( \bar{s}_u \) | \( \bar{x}_3 \) | \( \bar{u}_x \) | \( \bar{u}_3 \) | \( \bar{u}_n \) | \( \bar{s}_3 \) | \( \bar{s}_u \) |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 9   | B              | B              | I              | A              | B              | -I             | \( \bar{d}_1 - s_0 \) |
| 5   | B              | I              | A              | B              | -I             |                 | \( \bar{d}_2 \)   |
| 1   | A              | B              | -I             |                |                |                 | \( \bar{d}_3 \)   |
| 2   | I              | I              |                |                |                | \( \bar{b}_1 \) |                 |
| 3   | I              | I              |                |                |                | \( \bar{r}_1 \) |                 |
| 4   | I              | I              |                |                |                | \( \bar{k}_1 \) |                 |
| 6   | I              | I              |                |                |                | \( \bar{b}_2 \) |                 |
| 7   | I              | I              |                |                |                | \( \bar{r}_2 \) |                 |
| 8   | I              | I              |                |                |                | \( \bar{k}_2 \) |                 |
| 10  | I              | I              |                |                |                | \( \bar{b}_3 \) |                 |
| 11  | I              | I              |                |                |                | \( \bar{r}_3 \) |                 |
| 12  | I              | I              |                |                |                | \( \bar{k}_3 \) |                 |
|     | \( c_1 \)      | \( c_2 \)      | \( c_3 \)      | \( c_4 \)      | \( c_5 \)      | \( c_6 \)      | \( c_1 \)      | \( c_2 \)      | \( c_3 \)      | \( c_4 \)      | \( c_5 \)      | \( c_6 \)      | \( c_1 \)      | \( c_2 \)      | \( c_3 \)      | \( c_4 \)      | \( c_5 \)      | \( c_6 \)      | Min Z |

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generalized upper-bound technique more efficiently (Orchard-Hays, 1968).

In many instances, it is also possible to simplify the problem by modifying the assumptions. This is discussed next.

Upper-Bounding Techniques

We have assumed the vectors $\bar{u}_x$, $\bar{u}_n$ and $\bar{u}_s$ represent the unused portions of resources $\bar{x}$, $\bar{u}$ and storage $\bar{s}$ respectively. In the real world, it is often possible to obtain resources $\bar{x}$ and $\bar{u}$ to meet the requirements exactly. We have already assumed that the unused portion of the warehouse does not cost anything. Under this assumption, the vectors $\bar{u}_x$, $\bar{u}_n$ and the cost vector $\bar{c}_6$ will be zeros. Therefore the vectors $\bar{u}_x$, $\bar{u}_n$ and $\bar{u}_s$ can be eliminated from the tableaus. Now the problem can be solved using upper bounding techniques. Special algorithms for reducing the computations of problems with block angular structure are also available (R. N. Kaul, 1965). Commercial computer programs are written for handling upper bounds as a part of a linear programming problem (IBM, MPS/360).

The general model just discussed contains a linear objective function and linear constraints. If convex objective functions are considered then various convex programming algorithms may be used for obtaining solutions. Next we present a numerical example for the linear programming problem and then discuss a solution procedure for solving problems with a nonlinear objective function and nonlinear constraints.
Numerical Example D

Let

\[
A = \begin{bmatrix}
2 & 6 & 3 \\
3 & 3 & 5 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
.5 & .6 & .8 \\
.5 & .9 & .7 \\
\end{bmatrix}
\]

Cost of factors of production \( x_1, x_2 \) and \( x_3 \) be

\[
c_1 = 1.5/\text{unit}
\]

\[
c_2 = 3.0/\text{unit}
\]

\[
c_3 = 5.0/\text{unit}, \text{ respectively}
\]

The demand for products (1) and (2) be

<table>
<thead>
<tr>
<th>Period</th>
<th>Product 1</th>
<th>Product 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>140</td>
<td>150</td>
</tr>
<tr>
<td>2</td>
<td>200</td>
<td>200</td>
</tr>
<tr>
<td>3</td>
<td>200</td>
<td>170</td>
</tr>
<tr>
<td>4</td>
<td>225</td>
<td>200</td>
</tr>
<tr>
<td>5</td>
<td>210</td>
<td>180</td>
</tr>
</tbody>
</table>

In addition limits were also imposed on the availability of inputs and R & D inputs. The cost of not using any input was assumed to be zero.

The data were analyzed utilizing the IBM MPS/360 mathematical programming system. Several runs were made using different costs for R
& D inputs. In some cases lower bounds were imposed on R & D inputs and in some cases the lower bounds were not imposed. The results are tabulated in Table 5.3.

The results indicate that the R & D inputs, $u_i$, were fully utilized during period one. Some or part of the R & D inputs, depending on the price, were utilized during period two and three. None of the R & D inputs were utilized during period four and five when lower bounds were not present on the R & D inputs (refer to Table 5.3).

The results appear to be reasonable. Since the R & D inputs cumulatively increase the production rate they are consumed only if the cumulative increase in production justifies the R & D input costs. Therefore we can state, in our example, that they are profitable during early periods and not so during later periods. When lower bounds were imposed on the R & D inputs, the total cost of products also increased accordingly.

We have assumed a five period model in our example. However if we segmented the total planning horizon into ten periods, more accurate results may have been obtained. The number of periods, for the given planning horizon, would normally depend on the accuracy needed, and the availability of resources for planning purposes.

**Nonlinear Objective Function and Nonlinear Constraints**

The Dantzig-Wolfe Decomposition algorithm has been extended to minimize nonlinear functions. The nonlinear version also uses the same strategy for entering new variables in the solution. Unfortunately, the dual does not provide a solution until an optimal solution is
### TABLE 5.3
Solution to Numerical Example D

<table>
<thead>
<tr>
<th>Period</th>
<th>Input</th>
<th>Data Set I</th>
<th>Data Set II</th>
<th>Data Set III</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Unit Cost</td>
<td># of Units Required</td>
<td>Unit Cost</td>
</tr>
<tr>
<td>1</td>
<td>X</td>
<td>1.5</td>
<td>24.6</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>3.0</td>
<td>10.0</td>
<td>3.0</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>1.0</td>
<td>10.0</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>2.0</td>
<td>10.0</td>
<td>1.6</td>
</tr>
<tr>
<td>2</td>
<td>X</td>
<td>1.5</td>
<td>30.0</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>3.0</td>
<td>14.3</td>
<td>3.0</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>1.0</td>
<td>10.0</td>
<td>1.8</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>1.5</td>
<td>10.0</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>2.0</td>
<td>10.0</td>
<td>1.6</td>
</tr>
<tr>
<td>3</td>
<td>X</td>
<td>1.5</td>
<td>13.5</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>3.0</td>
<td>19.2</td>
<td>3.0</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>1.0</td>
<td>10.0</td>
<td>1.8</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>1.5</td>
<td>-</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>2.0</td>
<td>-</td>
<td>1.6</td>
</tr>
<tr>
<td>4</td>
<td>X</td>
<td>1.5</td>
<td>23.5</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>3.0</td>
<td>20.0</td>
<td>3.0</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>1.0</td>
<td>-</td>
<td>1.8</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>1.5</td>
<td>-</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>2.0</td>
<td>-</td>
<td>1.6</td>
</tr>
<tr>
<td>5</td>
<td>X</td>
<td>1.5</td>
<td>16.0</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>3.0</td>
<td>20.0</td>
<td>3.0</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>1.0</td>
<td>-</td>
<td>1.8</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>1.5</td>
<td>-</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>2.0</td>
<td>-</td>
<td>1.6</td>
</tr>
</tbody>
</table>

Total Cost: 637, 635, 657
found. The convergence problems in nonlinear cases make it often infeasible to obtain optimal solutions. This fact led to the exploration of new decomposition-based solution procedures for solving primal nonlinear problems by Geoffrion (1970), Silverman (1972) and others.

We now discuss Geoffrion's procedure.

THE PROBLEM

Suppose the problem is to minimize the total cost of resources allocated to subsystems \((k = 1, 2, \ldots, p)\) subject to production function and boundary conditions. Then the problem may formally be stated as:

\[
\begin{align*}
\text{Min } & \sum_{k=1}^{p} f_k(x_k) \\
\text{S. to } & \sum_{k=1}^{p} g_k(x_k) \leq b \\
& x_k \in s_k
\end{align*}
\]

where \(x_k\), \(f_k\) and \(g_k\) are associated with the subsystem \(k\) and \(x_k\) has \(n\) components in the subsystem \(k\). Also \(x_k\) is contained in the set \(s_k\). \(b\) is a vector of resources available. In addition, production function and inventory constraints could be present but are omitted for ease in presentation.

Suppose the allocation to subsystem \(k\) is \(y_k\) then

\[
\sum_{k=1}^{p} y_k \leq b \quad (2)
\]
assuming feasible allocations were made to the subsystems. The subsystem would optimize their criterion function by utilizing \( y_k \). Therefore the system of equations for the subsystem may be written as:

\[
\begin{align*}
\text{Min } & f_k(x_k) \\
\text{S.t.} & \quad g_k(x_k) \leq \bar{y}_k \\
& \quad x_k \in s_k
\end{align*}
\]  

(3)

If we choose \( y_k \in \nu_k = \{y_k|x_k \in s_k \text{ and } g_k(x_k) \leq y_k\} \) then the subproblem objective function may be written as:

\[
v_k(y_k) = \min \{f_k(x_k)|g_k(x_k) \leq y_k, x_k \in s_k\}
\]  

(4)

To minimize (1) we must choose \( y_k \) to minimize the sum of these functions subject to equations (2) and (4). Substituting (4), the primal problem (1) can be rewritten as the master problem:

\[
\begin{align*}
\text{Min } & v(y) = \sum_{k=1}^{p} v_k(y_k) \\
\text{S. to} & \quad \sum_{k=1}^{p} y_k = \bar{b} \\
& \quad y_k \in \nu_k, k = 1, 2, \ldots, p
\end{align*}
\]

(5)

The primal problem may be thought of as allocating the resource \( \bar{b} \) optimally to the subsystems. Geoffrion (1972) states that the initial allocation is tested for optimality. If it was not optimal the solution is improved by some strategy. He further indicates that
Feasible Directions Strategy is well suited for solving the master problem (5). The technique generates an improving sequence of feasible points. At any stage say \(\bar{y}^o = (y_1^o, ..., y_p^o)\) which is not optimal. A step is taken towards the feasible improving direction 's' which is determined by the previous solution.

If an optimal solution \(\bar{y}^o = (y_1^o, y_2^o, ..., y_p^o)\) solves the master and \(\bar{x}_k^o\) solve subproblems (3) then \(\bar{x}^o\) solves the primal. Conversely, if \(\bar{x}^o\) solves the primal then \(g_k(\bar{x}_k^o), k = 1, 2, ..., p\) solves the master. A detailed discussion of this technique is found in Geoffrion (1972).

The discrete time model we addressed is a multiperiod model. A one period model would be similar to the static model represented by \(P_1\). In addition, the continuous time models \(P_3\) and \(P_4\) can be transformed to discrete time models and solved using the techniques described in this chapter. Next we proceed to formulate a model for groups of products.

**Model for Groups of Products**

The general model discussed earlier portrayed a multiproduct manufacturing planning situation. We stated that the matrix \(A\) consists of the necessary coefficients to convert the inputs to products and matrix \(B\) consists of coefficients necessary to convert R & D inputs to increase the production rate. We discussed in Chapter III, that situations arise where learning takes place across product groups. This in turn increases the production rate in addition to the increased productivity obtained through the R & D expenditure spent within the
product groups. We called the effect of learning across product
groups as the interaction effect. Define matrix GI where GI consists
of coefficients for converting the learning effect to increase the
production rate of product group I. Define the following:

\[ A_1, A_2 \] - represent the matrix A for product group 1 and
product group 2, respectively

\[ B_1, B_2 \] - represent the matrix B for product group 1 and
product group 2, respectively

\[ G_1, G_2 \] - represent the matrix GI for product group 1 and
product group 2, respectively

\[ \bar{x}_{ij}, \bar{u}_{ij}, \bar{g}_{ij} \] - represent the vectors of inputs, R & D inputs,
and storage requirements for product i during
period j

\[ c_{ik} \] - represent the cost vectors k similar to the
vectors described in the general model for
products 1 and 2

Comparison of Tableaus

Compare rows (1), (2), (3) and (4) of the Tableau 5.1 and 5.4.
They are similar except the matrix G2 is present in the first row of
the tableau 5.4. The matrix G2 represents the interaction effect on
product 2 of the R & D expenditure originally spent on product 1.

The rows (5), (6), (7) and (8) of the Tableau 5.4 represent
similar equations for product 2 during period 1. The rows (9) through
(16) of the Tableau 5.4 are similar to rows (5) through (8) of the
Tableau 5.1. The interaction effect is influenced by the R & D
expenditures and hence the interaction cumulatively increases the production rate of products during succeeding periods. Therefore, Gl will appear in every row and as many times as B1 appears. The same reasoning holds good for B2 and G2.

The Tableau 5.4 is a two period model for two groups of products and the example may easily be extended to any number of periods and to include any number of product groups.

We assumed that the two product groups learn from each other. This assumption leads to the introduction of the interaction matrices G1 and G2. In the event, the two product groups had nothing in common or nothing to learn from each other, then the elements of the interaction matrices will be zero. Then the two groups of products can be independently optimized which will be similar to two general models. This becomes evident when we remove the matrices G1 and G2 from the Tableau 5.4 since the product groups 1 and 2 are independent of other in absence of any coupling equations or matrices.

**Computational Considerations**

Once again, we see the Tableau 5.4 is similar to 5.1 in structure even though the former is bigger. We can rearrange the rows of the Tableau 5.4 and utilize similar techniques suggested for solving the general model. The entire discussion on the computational consideration of the general model also apply to the multiproduct multifacility model (group of products).

The solutions obtained from the linear programming techniques need not be integers. If for some reason the entrepreneur wanted
### Tableau 5.4

**Tableau for a Two Product Group Model**

<table>
<thead>
<tr>
<th>Row</th>
<th>Product 1, Period 1</th>
<th>Product 2, Period 1</th>
<th>Product 1, Period 2</th>
<th>Product 2, Period 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A1</td>
<td>B1</td>
<td>-I</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>I</td>
<td>I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>I</td>
<td>I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>I</td>
<td>I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>G2</td>
<td>A2</td>
<td>B2</td>
<td>-I</td>
</tr>
<tr>
<td>6</td>
<td>I</td>
<td>I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>I</td>
<td>I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>B1</td>
<td>I</td>
<td>G1</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>I</td>
<td>I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>I</td>
<td>I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>I</td>
<td>I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>G2</td>
<td>B2</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>C_{11}C_{21}C_{31}C_{41}C_{51}C_{61}</td>
<td>C_{12}C_{22}C_{32}C_{42}C_{52}C_{62}</td>
<td>C_{11}C_{21}C_{31}C_{41}C_{51}C_{61}</td>
<td>C_{12}C_{22}C_{32}C_{42}C_{52}C_{62}</td>
</tr>
</tbody>
</table>
them to be all integer then integer programming techniques can be used to obtain integer solutions. Computer programs utilizing several algorithms are available. The interested reader is referred to Geoffrion (1972) for details.

CONCLUSION

In this chapter, we have formulated the general model and its extensions. We have also discussed some solution procedures for solving them. We find that commercial computer programs are available for solving large linear discrete time models which is a definite advantage over the continuous time linear models.
CHAPTER VI
CONCLUSIONS AND RECOMMENDATIONS

Introduction

In out models, the production functions acted as constraints that determined the set of production possibilities as a function of the quantities of various inputs. We pointed out earlier that a knowledge of these production functions is the basis for our decision analysis. The models point out the variables on which to operate and indicate the results of various combinations of inputs. Therefore, the more complete and correct the production functions, the more accurate the reliable results of the analysis.

Uncertainty in the Elements of the Analysis

The problem of estimation and uncertainty of relationships in production functions is significant. For example, Johansen (1972) writes that econometric research in production functions is growing very sophisticated as far as functional forms and statistical methodology is concerned. But the results of the research have produced diverging conclusions.

Earlier we pointed out that technological advances played a major role in the economy of the United States during the recent decade as a result of government spending and private sector spending.
The benefit derived by an individual firm also depends upon the government policies on research and development spending in the related industries. Considerable uncertainty clouds the amount of future growth of technology since the future policy of the government and the private agencies regarding research and development is not known. The exact relationships derived for the production function may change in the future.

In addition to the uncertainty present in the future government policy, tax credit policies, patent benefits, tax rate on profits, subsidies, allowable depreciation schedules and interest rates which have an impact on the profitability of R & D in increasing the productivity are very uncertain.

We indicated earlier that labor training may increase the productivity by decreasing the number of mistakes and increasing the understanding, environment and information processing capacity of people. Such assumptions are complicated by transfers, promotions, bumping and turnovers in employment. These factors cast some uncertainty on the assumptions underlying the analysis.

We assumed the demand is given. The demand may change in the dynamic analysis due to competition, price, and other market conditions.

We have assumed in our decision analysis that utility is linear and can be approximated with a cost function. The society is constantly changing. Issues such as environment, the firm's prestige, security and other factors surrounding the goals of the firm. Therefore we are in a weak position to assess the impact of the nature of the technical advances and their effects on production in the future. To alleviate
these and other uncertain factors not mentioned here all basic data should be updated periodically.

**Summary of Dissertation**

Based on the knowledge derived from the production functions in the theory of the firm and R & D literature, we developed some models portraying manufacturing planning situations. Our analysis focused on single product, two product and multiple product firms.

The discussions on the single product firm may be compared to the analysis by Smith (1961) that we discussed in Chapter II, except we have an additional element included in the model. That is, the R & D input has a cumulative effect on the production.

We have extended the analysis to two product and multiple product models sharing the common technology. In addition to the cumulative effects of R & D, the productivity increase due to interaction among products were also modeled.

We found the continuous time dynamic analysis, in general, was more complicated than static analysis. Single product dynamic models containing nonlinear functions can be solved only numerically. In addition, we have demonstrated the usefulness of the linear programming technique in portraying situations involving discrete time dynamic analysis.

We have discussed various techniques such as Dantzig-Wolfe decomposition and the upper-bound algorithm reducing computations while solving large linear programs. We have also described a
technique for solving large nonlinear programs with nonlinear constraints.

The value and utility of these models and techniques of solving problems depend on the real world situations. The emphasis of the present study was on modeling and solving certain simple problems. However, it is hoped that various types of analysis and suggestions of new approaches will lead to more new approaches to empirical work on production functions.

Recommendations for Further Research

In our research, we have developed several deterministic models portraying static and dynamic production planning situations. However, in the real world several uncertain elements exist. For example, we assumed that the technological coefficients are constants and the R & D expenditures cumulatively increases the production rate. There is always an uncertain element attached to the outcome of R & D expenditure. Therefore, as an extension of this research uncertainty may be included in the models.

We mentioned earlier that the major emphasis has been on modeling and solving certain simple problems. As an alternative one may also start with a simple model and develop more complicated models portraying the same situation. Also, the effect of various objective functions may be analyzed for a given situation.

We have assumed that demand is a given element. In reality, forecasting the demand is not a trivial matter. The effect of various
demand functions on various inputs and especially the R & D expenditure may be explored in detail.

We have solved certain simple problems. As an extension of our research one may evaluate the additional benefits derived by including the various uncertain elements discussed in this chapter as opposed to the increased computational requirements.

Conclusion

We have discussed the usefulness of models for portraying production planning problems. Some models may be superior to others in particular situations. The theory developed in this research provides a starting point for evaluating various policies of the firm. More complicated models may be built by the inclusion of factors such as tax structures, interest, expected government aid and other system variables such that the model may be manipulated for alternate policies.

The theory and models will provide helpful guidelines on the specific role of variables in the firm's growth. The integrated policy-making technique will hopefully allow the firm to project their future more accurately. The economy and amount of effort necessary to design these systems should be explored.

The effectiveness of the models in portraying a particular situation needs verification to real world data. Finally validation of dynamic analysis to real world decision making is important. Continued effort should be made to enrich these analytical models to include various factors suggested in this chapter. The models may
become more complicated. In those instances, as an alternate approach, the use of simulation techniques to portray and solve these models may be explored.
The Appendix A evaluates the principal minors A, for the problem P1, Example 2, Chapter IV. The stationary points derived will represent the minimum if the last two principal minors of the following Hessian matrix H are less than zero (Teichroew, 1964). Let us represent $g = x_1^a x_2^b x_3^c - Y = 0$. Then the Hessian matrix H, can be written the following way:

$$\begin{pmatrix}
0 & -g_{x_1} & -g_{x_2} & -g_{x_3} \\
-g_{x_1} & z_{x_1} x_1 - \lambda g_{x_1} x_1 & z_{x_1} x_2 - \lambda g_{x_1} x_2 & z_{x_1} x_3 - \lambda g_{x_1} x_3 \\
-g_{x_2} & z_{x_2} x_1 - \lambda g_{x_2} x_1 & z_{x_2} x_2 - \lambda g_{x_2} x_2 & z_{x_2} x_3 - \lambda g_{x_2} x_3 \\
-g_{x_3} & z_{x_3} x_1 - \lambda g_{x_3} x_1 & z_{x_3} x_2 - \lambda g_{x_3} x_2 & z_{x_3} x_3 - \lambda g_{x_3} x_3
\end{pmatrix}$$

where the subscripts refer to the partial derivatives of the functions Z and g. The objective function Z does not have any quadratic or cross product terms $x_i, i = 1,2,3$ and hence all the second partials of Z will be zero. We will be left with various partial derivatives of the function g only. Evaluating the partials and substituting in the Hessian matrix, we obtain

$$\begin{pmatrix}
0 & -\frac{\alpha}{x_1} & -\frac{\beta}{x_2} & -\frac{\gamma}{x_3} \\
-\frac{\alpha}{x_1} & -\frac{\lambda \alpha (\alpha - 1)}{x_1^2} & -\frac{\lambda \alpha \beta}{x_1 x_2} & -\frac{\lambda \alpha \gamma}{x_1 x_3} \\
-\frac{\beta}{x_2} & -\frac{\lambda \alpha \beta}{x_1 x_2} & -\frac{\lambda \beta (\beta - 1)}{x_2^2} & -\frac{\lambda \beta \gamma}{x_2 x_3} \\
-\frac{\gamma}{x_3} & -\frac{\lambda \alpha \gamma}{x_1 x_3} & -\frac{\lambda \beta \gamma}{x_2 x_3}
\end{pmatrix}$$
The principal minor $A_3$ was evaluated, that is,

$$
\begin{vmatrix}
-\gamma & -\lambda \alpha \gamma & -\lambda \beta \gamma \\
X_3 & X_1 X_3 & X_2 X_3 \\
X_3 & X_1 X_3 & X_2 X_3 \\
\end{vmatrix}
$$

The principal minor $A_3$ was evaluated using the cofactor method of finding the determinants (Hadley, 1962)

$$
A_3 = \frac{-\lambda \alpha \beta (\alpha + \beta) \gamma}{X_1^2 X_2^2 X_3^2}
$$

The principal minor $A_4$ was evaluated using the cofactor method of finding the determinants (Hadley, 1962)

$$
A_4 = -\frac{\lambda^2 \alpha \beta \gamma}{X_1^2 X_2^2 X_3^2} (\alpha + \beta + \gamma) \gamma
$$

We know $\lambda$ is positive. The parameters $\alpha$, $\beta$ and $\gamma$, and $X_1$, $X_2$, $X_3$ and $Y$ are positive. Therefore, $A_3$ and $A_4$ are negative, indicating the stationary point, represents the minimum.
Here we attempt to solve for \( \Psi_1 \) and \( \Psi_2 \) and explore the possibility of obtaining a closed form solution to P3, Example 1. The necessary conditions, as derived earlier, are:

\[
\begin{align*}
\dot{I}(t) &= a_1 X_1(t) + a_2 X_2(t) + a_3 X_3(t) - D t \\
\dot{X}_3(t) &= v(t) \\
\Psi_1(t) &= 2c_4 I(t) \\
\Psi_2(t) &= -a_3 \Psi_1(t)
\end{align*}
\]

\[
X_1 = \begin{cases} 
  \overline{b}_1 & \text{if } \Psi_1 > c_1/a_1 \\
  b_1 & \text{if } \Psi_1 < c_1/a_1 \\
  \text{Not specified} & \text{if } \Psi_1 = c_1/a_1 
\end{cases} \quad i = 1, 2
\]

\[
v = \begin{cases} 
  \overline{b}_3 & \text{if } \Psi_2 > c_3 \\
  b_3 & \text{if } \Psi_2 < c_3 \\
  \text{Not specified} & \text{if } \Psi_2 = c_3
\end{cases}
\]

Rewriting the necessary conditions in the matrix form we obtain

* My thanks to Dr. R. A. Miller for introducing me the technique and assisting me in carrying out the analysis further.
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_3
\end{bmatrix} = \begin{bmatrix}
a_1 & a_2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 & a_3 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
v \\
x_3
\end{bmatrix} - \begin{bmatrix}
d \\
0
\end{bmatrix} \tag{1}
\]

and
\[
\begin{bmatrix}
\dot{\psi}_1 \\
\dot{\psi}_2
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
-a_3 & 0
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} + \begin{bmatrix}
2c_4 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
x_3
\end{bmatrix}, \quad \psi(T) = 0 \tag{2}
\]

Now define
\[
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} = \begin{bmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_3
\end{bmatrix} + \begin{bmatrix}
g_1 \\
g_2
\end{bmatrix} \tag{3}
\]

then
\[
\bar{\psi} = k_{11} + k_{21} + g \tag{4}
\]

Substituting for \(\bar{\psi}\) and \(X\) in (4) we obtain,
\[
\begin{bmatrix}
0 & 0 \\
-a_3 & 0
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} + \begin{bmatrix}
2c_4 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
I \\
x_3
\end{bmatrix} = k
\begin{bmatrix}
I \\
x_3
\end{bmatrix} + k
\begin{bmatrix}
0 & a_3 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
I \\
x_3
\end{bmatrix}
\]

\[
k
\begin{bmatrix}
a_1 & a_2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} - k
\begin{bmatrix}
d \\
0
\end{bmatrix} + \bar{\psi} \tag{5}
\]

Rearranging we obtain,
\[
\begin{bmatrix}
0 & a_3 \\
0 & 0
\end{bmatrix}
k
\begin{bmatrix}
I \\
x_3
\end{bmatrix} - \begin{bmatrix}
0 & a_3 \\
0 & 0
\end{bmatrix}g + \begin{bmatrix}
2c_4 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
I \\
x_3
\end{bmatrix} = k +
\[
\begin{align*}
K \begin{bmatrix} 0 & 0 \\ -a_3 & 0 \end{bmatrix} \begin{bmatrix} I \\ X_3 \end{bmatrix} + K \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ v \end{bmatrix} - K \begin{bmatrix} D \\ 0 \end{bmatrix} + \bar{g} \\
\end{align*}
\]
or
\[
\begin{align*}
K + K \begin{bmatrix} 0 & 0 \\ -a_3 & 0 \end{bmatrix} &= \begin{bmatrix} 2c_4 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & a_3 \\ 0 & 0 \end{bmatrix} K \\
\end{align*}
\]
and
\[
\bar{g} = - \begin{bmatrix} 0 & a_3 \\ 0 & 0 \end{bmatrix} \bar{g} - K \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{v} \end{bmatrix} + K \begin{bmatrix} \ddot{D} \\ 0 \end{bmatrix}
\]

\[
K(T) = 0 \\
\bar{g}(T) = 0
\]

Integrating we obtain,

\[
\begin{align*}
\bar{g}(t) &= \phi(t, t_0) \bar{g}(t_0) + t_0 \int_{t_0}^{t} - \phi(t, \tau) KBu(\tau) d\tau \\
&\quad + \int_{t_0}^{t} \dot{\phi}(t, \tau) KDd\tau \\
&= -[t_0 \int_{t_0}^{T} - \phi(t_0, \tau) KBu(\tau) d\tau + t_0 \int_{t_0}^{T} \dot{\phi}(t_0, \tau) KDd\tau]
\end{align*}
\]

\[
\begin{align*}
\bar{g}(t) &= -t \int_{t_0}^{T} \phi(t_0, \tau) KBu(\tau) d\tau - t \int_{t_0}^{T} \dot{\phi}(t_0, \tau) KDd\tau \\
&= -t \int_{t_0}^{T} \phi(t, \tau) K[Bu(\tau) - D(\tau)] d\tau
\end{align*}
\]

\[
\begin{align*}
KBu &= \begin{bmatrix} K_{11}(a_1X_1 + a_2X_2) + K_{12}v \\ K_{21}(a_1X_1 + a_2X_2) + K_{22}v \end{bmatrix}
\end{align*}
\]
\[ \phi = \begin{bmatrix} 0 & 0 \\ -a_3 & 0 \end{bmatrix} \phi \]

\[ \phi = \begin{bmatrix} I & 0 \\ -a_3t & I \end{bmatrix}, \quad \phi(t, \tau) = \begin{bmatrix} I & 0 \\ -a_3(t - \tau) & I \end{bmatrix} \]

and

\[ \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} 0 & a_3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_3^T & 0 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \]

\[ = \begin{bmatrix} 2c_4 & 0 \\ 0 & 0 \end{bmatrix} = 0 \]

Therefore

\[ \dot{K}_{11} - 2c_4 = 0 \]

\[ \dot{K}_{12} + K_{11}a_3 = 0 \]

\[ \dot{K}_{22} + K_{21}a_3 + a_3^T K_{12} = 0 \]

Solving

\[ K = \begin{bmatrix} -2c_4(T - t) & -a_3c_4(T - t)^2 \\ -a_3c_4(T - t)^2 & -2a_3^2c_4(T - t)^3 \end{bmatrix} \quad (7) \]

and
\[
\overline{g}(t) = \int_t^T \begin{bmatrix} -2a_1c_4(T - \tau) & -2a_2c_4(T - \tau) & -a_3c_4(T - \tau)^2 \\ a_1^2a_3c_4(T - \tau)^2 & a_2^2a_3c_4(T - \tau)^2 & \frac{c_4a_3^2(T - \tau)^3}{3} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ v \end{bmatrix} \, d\tau - t^T \begin{bmatrix} -a_3c_4(T - \tau)^2 \\ \frac{c_4a_3^2(T - \tau)^3}{3} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ v \end{bmatrix} \, d\tau
\]

Substituting the matrix \(K\) and the vector \(g\) in the equation (3), we can obtain \(\psi_1\) and \(\psi_2\). Since the values \(X_1, X_2\) and \(v\) are dependent on \(\psi_1\) and \(\psi_2\), it will not be possible for us to obtain any closed form solutions to this problem. However, the problem may be solved numerically to obtain optimal solutions.
BIBLIOGRAPHY


