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THE SECOND GAP OF THE MARKOFF SPECTRUM OF Q(i)

DISSertation

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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* * * * * *

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ACKNOWLEDGMENTS

First and foremost, I wish to thank my father and mother for their continual support and confidence. I am indebted to them not only for providing for me a fine undergraduate education at the University of Rochester, but also for their dedicated guidance and encouragement which can not be measured.

In my graduate studies I must especially thank Professor Arnold Ross for his inspiration and teaching, Professor Kurt Mahler for his excellent lectures and influence on my studies, and Professor Alan Woods for his insights and dedication both as an adviser and as a teacher.

And most importantly, I thank my wife Nan for her love, patience, and encouragement at home.
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CHAPTER I

INTRODUCTION

1. The Markoff Spectrum of an Order.

Let $m$ be a square-free positive integer, and let $\mathbb{Q}_m = \mathbb{Q}(\text{im}^{1/2})$ be an imaginary quadratic extension of the field of rational numbers $\mathbb{Q}$. An order $\mathcal{O}$ of $\mathbb{Q}_m$ is a $\mathbb{Z}$-module of degree 2 in $\mathbb{Q}_m$ which is also a ring containing 1. As usual, $\mathbb{Z}$ denotes the ring of rational integers. It is a standard theorem from algebraic number theory that there is one order, $\mathcal{O}_m$, called the maximal order, containing all the others. Furthermore, $\mathcal{O}_m$ is just the integral closure of the ring $\mathbb{Z}$ in the field $\mathbb{Q}_m$, and the elements of $\mathcal{O}_m$ are called integers of $\mathbb{Q}_m$.

Since $\mathbb{Q}_m$ is a separable extension of $\mathbb{Q}$, there are precisely two distinct isomorphisms $\sigma_1, \sigma_2$ of $\mathbb{Q}_m$ into the field $\mathbb{C}$ of complex numbers. If $(\omega_1, \omega_2)$ is a module basis for an order $\mathcal{O}$, then the square of the determinant

$$D(\omega_1, \omega_2) = [\det(\sigma_1(\omega_j))]^2$$

is called the discriminant of the order $\mathcal{O}$. This is well-defined, for if $(\omega'_1, \omega'_2)$ is another module basis of $\mathcal{O}$, then the transition matrix between the two bases has determinant $+1$, and hence $D(\omega_1, \omega_2) = D(\omega'_1, \omega'_2)$.

The discriminant of the maximal order $\mathcal{O}_m$ is called the discriminant of the field $\mathbb{Q}_m$.

As a basis for $\mathbb{Q}_m$ we may take

$$\omega_1 = 1, \quad \omega_2 = \text{im}^{1/2} \quad \text{if} \quad m \equiv 1, 2 \pmod{4},$$

$$\omega_1 = 1, \quad \omega_2 = \frac{1}{2}(1 + \text{im}^{1/2}) \quad \text{if} \quad m \equiv 3 \pmod{4}.$$ 

Hence $\mathbb{Q}_m$ has discriminant
Denote by $-D$ this discriminant of $Q_m$.

Next, let $\mathbb{R}$ denote the field of real numbers, and let $a_1, a_2$ be linearly independent vectors in $\mathbb{R}^2$. The set $\Lambda$ of points

$$u_1a_1 + u_2a_2, \quad u_1, u_2 \in \mathbb{Z},$$

is called a lattice in $\mathbb{R}^2$ with basis $a_1, a_2$. If $a'_1, a'_2$ is another basis of $\Lambda$, then

$$a'_i = \sum v_{ij}a_j \quad (1 \leq i, j \leq 2)$$

where $v_{ij} \in \mathbb{Z}$ and $\det(v_{ij}) = \pm 1$. The converse is also true. Hence the value

$$d(\Lambda) = |\det(a_1, a_2)|$$

is independent of the choice of basis, and we call $d(\Lambda)$ the determinant of $\Lambda$. Geometrically, $d(\Lambda)$ is the area of the parallelogram with vertices $0, a_1, a_2, a_1 + a_2$.

The maximal order, now denoted by $O_D$, of an imaginary quadratic field, henceforth denoted by $Q_D = Q(\sqrt{d^{1/2}})$, can be imbedded in the plane $\mathbb{R}^2$ in a natural fashion, namely

$$u_1w_1 + u_2w_2 + (u_1, u_2^{m^{1/2}}) \quad \text{if} \quad m = 1, 2 \pmod{4},$$

$$u_1w_1 + u_2w_2 + (u_1 + \frac{1}{2}u_2, \frac{1}{2}u_2^{m^{1/2}}) \quad \text{if} \quad m = 3 \pmod{4}.$$ 

Hence, the image of $O_D$ is a lattice $\Lambda_D$ in $\mathbb{R}^2$ with basis

$$(1,0), \quad (0, m^{1/2}) \quad \text{if} \quad m = 1, 2 \pmod{4},$$

$$(1,0), \quad (\frac{1}{2}, \frac{1}{2} m^{1/2}) \quad \text{if} \quad m = 3 \pmod{4}.$$ 

This lattice has determinant

$$D(\Lambda_D) = \frac{1}{2} d^{1/2}.$$ 

We note that an arbitrary order $Q$, being a submodule of $O_D$ containing 1, is mapped into a sublattice of $\Lambda_D$ containing $(1,0)$.

A quadratic form of degree $n$ over $\mathbb{C}$ is a homogeneous polynomial in $n$ variables, of degree 2, and with coefficients in the field of
complex numbers. Such a quadratic form can be written as

\[ f = \sum_{i,j=1}^{n} a_{ij} x_i x_j, \]

where \( a_{ij} = a_{ji} \). The determinant of the coefficient matrix, \( d(f) = \det(a_{ij}) \), is called the determinant of the quadratic form \( f \). \( f \) is called singular if \( d(f) = 0 \), and nonsingular otherwise. The discriminant of \( f \) can be defined by \( D(f) = 2^n d(f) \).

For a nonsingular quadratic form \( f(x_1, \ldots , x_n) \) over \( \mathbb{C} \), and for an order \( 0 \) of \( \mathbb{Q}_D \), define the minimum \( M(f) \) of \( f \) over \( 0 \) by

\[ M(f) = \inf |f(x_1, \ldots , x_n)| \]

where the infimum is taken over the set of all \( n \)-tuples \((x_1, \ldots , x_n) \neq (0, \ldots , 0), x_1 \in 0.\)

Next, define the Markoff constant \( \mu(f) \) of \( f \) by

\[ \mu(f) = \frac{|D(f)|^{1/n}}{M(f)} \quad \text{if} \quad M(f) \neq 0, \]

\[ \mu(f) = \infty \quad \text{if} \quad M(f) = 0. \]

Finally, define the \( n \)-dimensional Markoff spectrum of the order \( 0 \) to be the set of all constants \( \mu(f) \) as \( f \) runs over all nonsingular quadratic forms \( f(x_1, \ldots , x_n) \) over \( \mathbb{C} \).

By the Markoff spectrum of \( \mathbb{Q}_D \) we shall mean the \( 2 \)-dimensional Markoff spectrum of the maximal order \( 0_D \).

2. The Approximation Spectrum of an Order.

Given a complex number \( x \notin \mathbb{Q}_D \), and an order \( 0 \) of \( \mathbb{Q}_D \), define the approximation constant \( C(x) \) of \( x \) with respect to the order \( 0 \) by

\[ C(x) = \lim \sup (|q| : |qx - p|)^{-1} \quad (1) \]

where the \( \lim \sup \) is taken over all \( p, q \in 0, q \neq 0 \). The set of all such approximation constants \( C(x), x \in \mathbb{Q}_D, \) is called the approximation spectrum of \( 0 \).
In order to compare the Markoff and approximation spectra of these imaginary quadratic fields, it would help to briefly look at the theory of rational approximations. The approximation spectrum of \( \mathbb{Q} \) is the set of all approximation constants \( C(x) \), where \( C(x) \) is defined for \( x \notin \mathbb{Q} \) again by (1), this time the lim sup being taken over all \( p, q \in \mathbb{Z} \), \( q \neq 0 \). Similarly, the Markoff spectrum of \( \mathbb{Q} \) is the set of all values \( \mu(f) = \left| D(f) \right|^{1/2} M(f)^{-1} \) as \( f \) runs over all real nonsingular indefinite forms. As usual, the discriminant \( D(f) \) and the minimum \( M(f) \) are defined respectively by \( D(f) = B^2 - 4AC \) and \( M(f) = \inf |f(x,y)| \). Here \( f(x,y) = Ax^2 + Bxy + Cy^2 \), and the infimum is taken over all pairs of rational integers \((x,y)\) not both zero.

Cassels [1] gave a proof that the approximation spectrum of \( \mathbb{Q} \) is contained in the Markoff spectrum. The generation of the Markoff chain of forms, i.e., forms with \( \mu(f) < 3 \) originally done by Markoff [10], with later proofs by Frobenius [5], Remak [15], and Cassels [1], show that the two spectra do indeed coincide on the interval \((-\infty, 3)\).

In Section 2 of Chapter II we shall give a proof that the Markoff spectrum of \( \mathbb{Q}_D \) contains the approximation spectrum when \( \mathbb{Q}_D \) has unique factorization.

The question as to whether or not this inclusion is proper in the rational case was decided by G. A. Freiman [4] in 1968, who constructed a number \( \alpha \) satisfying the following three properties:

A. The number \( \alpha \) lies in the Markoff, but not in the approximation spectrum of \( \mathbb{Q} \).

B. The number \( \alpha \) is the sole number of the Markoff spectrum in the interval \((\alpha - 10^{-35}, \alpha + 10^{-35})\).

C. In the interval \((\alpha - 10^{-10}, \alpha + 10^{-10})\), there exist a countable number of isolated numbers of the Markoff spectrum which do not lie in the approximation spectrum.

This number \( \alpha \) is equal to the sum of the two continued fractions

\[
\alpha_1 = [2,2,2,2,1,1,2,2,1, \ldots ],
\]

\[
\alpha_2 = [0,1,2,2,1,2,2,2,2,1,1,2,2,1,1,2,2,1,1,2,2,1, \ldots ].
\]
To eleven decimal places,

\[ \alpha = \alpha_1 + \alpha_2 = 3.11812017816... \]

Thus, it is most probable that there is also strict inclusion over the algebraic number fields \( \mathbb{Q}(iD^{1/2}) \).

3. History and statement of the problem.

The first work on complex approximation problems was a generalization of Hurwitz's theorem on rational approximations to the approximation of complex numbers by Gaussian integers, i.e., integers of \( \mathbb{Q}(i) \).

Hurwitz [9] proved that there exist, for each irrational number \( x \), infinitely many rational fractions \( \frac{p}{q} \) such that

\[ |x - \frac{p}{q}| < 5^{1/2} q^{-2}. \]

The constant \( 5^{1/2} \) is called "the best possible" because for \( c > 5^{1/2} \) there is some number \( x \) such that the inequality

\[ |x - \frac{p}{q}| < c^{1/2} q^{-2} \]

has only finitely many solutions. The number \( 5^{1/2} \) is called the Hurwitz constant of the field \( \mathbb{Q} \).

However, more can be said. Two real numbers \( x, y \) are called equivalent if there exist rational integers \( r, s, t, u \) such that \( ru - ts = \pm 1 \) and

\[ x = \frac{ry + s}{ty + u}. \]

It is easily verified that this is an equivalence relation.

Now, if \( x \) is equivalent to a root of \( x^2 + x - 1 \), then the constant \( 5^{1/2} \) cannot be improved. If \( x \) is not equivalent to this root, however, then the constant \( 5^{1/2} \) can be improved to give infinitely many solutions to

\[ |x - \frac{p}{q}| < 8^{1/2} q^{-2}. \]
Again, if $x$ is equivalent to a root of $x^2 + 2x - 1$, then the constant $\delta^{1/2}$ is best possible, but if $x$ is not equivalent to one of these roots, then there are infinitely many solutions to

$$|x - \frac{p}{q}| < \left(\frac{221}{25}\right)^{-1/2} q^{-2}.$$  

This process can be continued to give a sequence of numbers of the approximation spectrum of $Q$,

$$5^{1/2}, 8^{1/2}, (\frac{221}{25})^{1/2}, (\frac{1517}{169})^{1/2}, \ldots,$$

these numbers converging to 3.

A number of investigations have been carried out to determine the Hurwitz and further approximation constants for imaginary number fields. Analogous to $Q$, the Hurwitz constant for $Q_D$ is the lim sup of all positive constants $c$ such that the inequality

$$|x - \frac{p}{q}| < c^{-1} q^{-2}$$

has infinitely many solutions in integers $p, q \in \mathbb{Q}_D$. This constant, denoted by $C_D$, is the smallest element of the approximation spectrum of $Q_D$.


Hofreiter [7] in 1935 extended a theorem of Furtwängler [6] to establish that for the imaginary quadratic fields of class number 1,

$$C_D \leq |\Delta|^{1/2},$$

where $\Delta$ is, among the discriminants of quadratic extension fields of $Q_D$, that of minimum modulus. In the cases $D = 3, 4, 7, 8$, the equality actually holds, so it was conjectured that $C_{11} = 5^{1/4}$.

However, in 1953, Poitou [14] proved that $C_{11} = \frac{1}{2} 5^{1/2}$, disproving the conjecture. Moreover, he proved that the value $C_{11}$ is isolated in the approximation spectrum, and that the next value is greater than 1.21. We say that a number $\alpha$ is isolated in a spectrum if there is an open interval $(\alpha - \delta, \alpha + \delta)$, $\delta > 0$, in which $\alpha$ is the only
element of the spectrum. An open interval \((\alpha, \beta)\) is said to be a gap in the spectrum if the intersection of the interval with the spectrum is empty. Thus \((\frac{1}{2}, \frac{1}{2})\) is a gap for \(Q_1\).

Poitou also proved that the values 2 and \((\frac{32}{13})^{1/2}, \frac{1}{4}\) are the next values in the approximation spectrum spectrum of \(Q(1, 1)\), and that all other values are greater than 2.070068. Furthermore, the number \((\frac{28 + 16}{13}^{3/2}) = 2.070169\ldots\) was shown to be an accumulation point of the spectrum, although it is not known whether it is the first such point of accumulation.

Finally, in the same paper, Poitou proved that \(C_{19} = 1\) and \(C_{43} < (\frac{5}{11})^{1/2}\).

Using a generalization of Farey sequences, A. L. Schmidt [16] in 1967 was able to determine all approximation constants \(C(x) < c_D\) for the fields \(Q, D = 3, 4, 7, 8\), where

\[
c_3 = 1.90, \quad c_4 = 1.80, \quad c_7 = 1.75, \quad c_8 = 1.733.
\]

The results were respectively

\(13^{1/4}, \quad 3^{1/2}, \quad 8^{1/4}\) and \(3^{1/2}, \quad 2^{1/2}\) and \(3^{1/2}\).

The major importance of this was the determination of the second gaps for \(Q_7\) and \(Q_8\).

All of the approximation constants listed in this section are also elements of the Markoff spectra of their respective fields. Cassels [2] in 1952 proved an isolation theorem for the first value of the Markoff spectrum of \(Q(1)\). Namely, he proved

**Theorem A.** There is a constant \(M_1 > 3^{1/2} = \mu(x^2 + xy + y^2)\) such that if \(\mu(f) < M_1\) for some nonsingular binary quadratic form \(f\), then \(f\) is equivalent to a multiple of \(f_1(x,y) = x^2 + xy + y^2\).

As usual, two forms \(f(x,y)\) and \(g(x,y)\) are said to be equivalent if there exist integers \(a, b, c, d \in \mathbb{Z}, \quad |ad - bc| = 1\), such that

\[f(ax + cy, bx + dy) = g(x,y).\]

In Chapter II of this study, we shall determine the first and second gaps of the Markoff spectrum of \(Q(1)\), and in Chapter III we
shall prove that the forms corresponding to the second gap are isolated. That is, we shall prove

**Theorem B.** There is a constant $M_2 > \left(\frac{3}{2}\right)^{1/2} 41^{1/4} = 1.96007...$ such that

if $\sqrt{3} < \mu(f) < M_2$, then $f$ is equivalent to a multiple of either

\[ f_2(x,y) = (1+2i)x^2 + xy + (2-i)y^2 \]

or its conjugate

\[ \overline{f}_2(x,y) = (1-2i)x^2 + xy + (2+i)y^2. \]

**Note:** In March, 1973, I discovered a paper by L. Ya. Vulakh [17], who constructed a chain of forms $f(x,y)$, the Markoff constants of which converge to 2. This chain does not include the form $f_2(x,y)$ of Theorem B or its conjugate $\overline{f}_2(x,y)$, and isolation is not proved.

On May 14, 1973, a new paper by Vulakh [18] appeared in the mathematics library, claiming to prove that any form satisfying the condition $\mu(f) < 2$ is equivalent to a multiple of either one of the forms in his earlier constructed chain, or to one of the two forms $f_2(x,y)$, $\overline{f}_2(x,y)$ in Theorem B.
CHAPTER IX

THE FIRST AND SECOND GAPS OF THE FIELD $\mathbb{Q}(i)$

1. Nonsingular Binary Quadratic Forms.

Let $f(x,y) = Ax^2 + Bxy + Cy^2$, $A,B,C$ complex numbers, be a binary quadratic form with discriminant $D(f) = B^2 - 4AC$ and minimum $M(f) = \inf |f(x,y)|$, where the infimum is taken over the ring of integers $\mathcal{O}_D$ of the field $\mathbb{Q}_D = \mathbb{Q}(iD^{1/2})$. Define

$$\mu(f) = |D(f)|^{1/2}M(f)^{-1} \quad \text{if} \quad M(f) \neq 0,$$

$$\mu(f) = \infty \quad \text{if} \quad M(f) = 0.$$

Lemma 1. If $r,u,s,t$ are integers in $\mathcal{O}_D$ with $|ru-st| = 1$, and if $g(x,y) = f(rx+sy, tx+uy)$, then

(i) $|D(g)| = |D(f)|$,

(ii) $M(g) = M(f)$.

Further, if $\lambda \neq 0$ is a complex number, then

(iii) $D(\lambda f) = \lambda^2 D(f)$,

(iv) $M(\lambda f) = \lambda M(f)$.

Thus

$$\mu(\lambda f) = \mu(f), \quad \mu(g) = \mu(f).$$

Proof: By simple computation,

$$D(g) = (ru-st)^2 D(f),$$

proving (i). Next, we note that

$$g(ux-sy, -tx+ry) = f((ru-st)x, (ru-st)y) = \lambda^2 f(x,y).$$

Hence $g$ and $f$ take the same values, so that $M(g) = M(f)$, proving (ii).

Statements (iii) and (iv) are obvious since

$$(\lambda B)^2 - 4(\lambda A)(\lambda C) = \lambda^2 (B^2 - 4AC) \quad \text{and} \quad (\lambda f)(x,y) = \lambda f(x,y).$$
Hence \( \mu(f) \) is invariant under scalar multiplication and unimodular transformations.

**Lemma 2.** Let \( \mathbb{O}_D \) have class number 1. Let \( r, s \) be relatively prime integers in \( \mathbb{O}_D \), and suppose, say, \( f(r, s) = A' \neq 0 \). Then there is a constant \( L_D > 0 \) depending only on \( D \), and integers \( t, u \in \mathbb{O}_D \) such that

\[
ru - st = 1,
\]

\[
f(rx + ty, sx + uy) = A'x^2 + B'xy + C'y^2,
\]

and

\[
|B'| \leq L_D |A'|.
\]

**Proof:** Since \( \mathbb{O}_D \) is a unique factorization domain and \( r, s \) are relatively prime, there are integers \( u', t' \in \mathbb{O}_D \) such that \( ru' - st' = 1 \). Then

\[
f(rx + t'y, sx + u'y) = A'x^2 + B''xy + C''y^2,
\]

where

\[
B'' = 2Ar't' + B(ru' + st') + 2Cs'u'.
\]

It is geometrically clear from the lattice \( \Lambda_D \) that we can choose \( L_D \) large enough, say \( L_D = (1+D)^{1/2} \), so that there always exists an integer \( d \in \mathbb{O}_D \) such that

\[
\left| \frac{B''}{2A}, + d \right| \leq \frac{L_D}{2}, \quad \text{i.e.,} \quad |B'' + 2dA'| \leq L_D |A'|.
\]

Now setting \( t = t' + dr, u = u' + ds \), it follows that \( ru - st = 1 \), and moreover that

\[
f(rx + ty, sx + uy) = A'x^2 + (B'' + 2dA')xy + C'y^2,
\]

proving the lemma.

**Note:** We may take \( L_D = 2^{1/2} \).

Since we are only concerned with forms \( f \) satisfying \( \mu(f) < \infty \), it may be assumed from here on that \( M(f) > 0 \).

A form \( f \) is said to attain its minimum \( M(f) \) is there is a pair of integers \( r, s \in \mathbb{O}_D \) for which \( |f(r, s)| = M(f) \). Otherwise \( f \) is said not to attain its minimum.
Lemma 3. If \( f \) does not attain its minimum \( M(f) > 0 \) over \( O_D \), then there is a form \( f^* \) attaining its minimum at \((1,0)\), i.e., \( f^*(1,0) = M(f^*) \), and satisfying the conditions \( |D(f^*)| = |D(f)|, M(f^*) = M(f) \).

Thus \( \mu(f^*) = \mu(f) \).

Proof: By definition of the minimum, there exists for each \( n = 1,2,3, \ldots \) a pair of relatively prime integers \( r_n, s_n \in O_D \) such that
\[
M(f) < |f(r_n, s_n)| < M(f) + \frac{1}{n} \quad (n = 1,2,3, \ldots).
\]

Upon setting \( A_n = f(r_n, s_n) \) and applying Lemmas 1 and 2, we generate a sequence of equivalent forms
\[
f_n(x,y) = A_n x^2 + B_n xy + C_n y^2 \quad (n = 1,2,3, \ldots)
\]
satisfying
\[
0 < M(f) < |A_n| < M(f) + \frac{1}{n}, \quad (2)
\]
\[
|B_n| \leq L_D |A_n|,
\]
\[
|D(f_n)| = |D(f)| \quad \text{and} \quad M(f_n) = M(f).
\]

Since \( B_n^2 - 4A_n C_n = |D(f)| \), it follows that \( |C_n| \) is bounded by
\[
|C_n| \leq \frac{|B_n|^2 + |D(f)|}{4|A_n|} \leq \frac{L_D^2 (|A_n| + 1)}{4} + \frac{|D(f)|}{4 M(f)}.
\]

The infinite sequence of triples \((A_n, B_n, C_n)\) thus lies in a compact subset of three-dimensional complex space, and so contains a subsequence convergent to, say, \((A^*, B^*, C^*)\). We may assume then that the given sequence actually converges to this point.

The new form \( f^*(x,y) = A^* x^2 + B^* xy + C^* y^2 \) has discriminant \( D(f^*) = \lim D(f_n) \), so that \( |D(f^*)| = |D(f)| \).

By the bounds (2) on \( |A_n| \), it follows that
\[
|A^*| = M(f).
\]

Furthermore, since for \((x,s) \neq (0,0)\),
\[
|f^*(x,s)| = \lim |f_n(x,s)| \geq \lim M(f_n) = M(f) = |A^*|,
\]
and
\[
f^*(1,0) = A^* \;
\]

it follows that
\[ M(f^*) = |\lambda^*| = M(f). \]

Therefore \( \mu(f^*) = \mu(f) \) and \( f^* \) attains its minimum at \((1,0)\), proving the lemma.

In order, therefore, to determine the Markoff spectrum of \( Q_\mathfrak{D} \), it suffices to consider only those forms attaining non-zero minima at \((1,0)\). This choice of forms can be narrowed down even further by dividing by the leading coefficient since \( \mu(\lambda f) = \mu(f) \). Thus we shall consider only forms of the type

\[ f(x,y) = (x-ay)(x-by) \]

satisfying the conditions

\[ |D(f)| = |a - b|^2 > 0, \]

\[ |x-ay| |x-by| \geq 1 \quad \text{for every } (x,y) \neq (0,0), x,y \in \mathbb{O}_\mathfrak{D}. \]

2. Relationship between the Markoff and Approximation Spectra of \( Q_\mathfrak{D} \).

**Theorem 1.** Let \( Q_\mathfrak{D} \) be an imaginary quadratic field of class number 1 and discriminant \(-\mathfrak{D}\), let \( \xi \notin Q_\mathfrak{D} \) be a complex number, and put

\[ v = \lim \inf |q||q\xi - p|, \]

the \( \lim \inf \) being taken over all \( p, q \in \mathbb{O}_\mathfrak{D}, q \neq 0 \). Then there is a form \( f_\mathfrak{D}(x,y) \) such that

\[ \mu(f_\mathfrak{D}) = v^{\sim 1}. \]

**Proof:** The proof is analogous to that given by Cassels [1] in the rational case. Set

\[ f(x,y) = x(\xi x - y). \]

By the definition of \( \lim \inf \), given any \( \epsilon > 0 \), there are only finitely many pairs \( q,p, q \neq 0 \), of integers in \( \mathbb{O}_\mathfrak{D} \) such that

\[ |f(q,p)| \leq v - \epsilon. \]

Hence there is a positive constant \( X_\mathfrak{D} = X_\mathfrak{D}(\epsilon) \) such that

\[ |f(x,y)| > v - \epsilon \quad \text{whenever} \quad |x| > X_\mathfrak{D}. \]
Equivalently, there is a positive constant $Y_0 = Y_0(\epsilon)$ such that
\[ |f(x,y)| > \nu - \epsilon \quad \text{whenever} \quad |x-y| < Y_0. \]  
(7)

Note that $|x-y| \neq 0$ since $x \notin Q_D$.

Again, by the properties of the lim inf, there exists a sequence of relatively prime pairs of integers $(q_n, p_n)$ in $Q_D$ such that
\[ |f(q_n, p_n)| > \nu, \quad |q_n| > \omega, \quad |s_n - p_n r_n| > 0. \]  
(8)

Then by Lemma 2, there is a unimodular transformation
\[ x_n = q_n x + r_n y, \quad y_n = p_n x + s_n y, \quad q_n s_n - p_n r_n = 1, \]  
(9)
such that the form $f_n$ defined by
\[ f_n(x,y) = f(x_n, y_n) = A_n x^2 + B_n xy + C_n y^2 \]  
(10)
satisfies the conditions
\[ f(q_n, p_n) = A_n \quad |A_n| > \nu \]  
(11)
and
\[ |B_n| \leq L_0 |A_n|, \quad B_n^2 - 4A_n C_n = D(f_n) = D(f) = 1. \]  
(12)

Taking a subsequence if necessary, we may assume as in Lemma 3 that
\[ A_n \to A_0 \quad \text{where} \quad |A_0| = \nu, \quad B_n \to B_0, \quad C_n \to C_0. \]  
(13)

Hence
\[ B_0^2 - 4A_0 C_0 = \lim (B_n^2 - 4A_n C_n) = 1. \]  
(14)

Now set
\[ f_0(x,y) = A_0 x^2 + B_0 xy + C_0 y^2 = A_0 (x-\xi_0 y)(x-\varphi_0 y). \]  
(15)

Also put
\[ \xi_n = (-s_n + s_n^2)/(s_n - p_n), \quad \varphi_n = -x_n/s_n. \]

Then by (9),
\[ x_n = q_n (x - \varphi_n y) \]  
(16)
and
\[ \xi_n - y_n = (\xi x_n - y_n) = (\xi p_n - p_n) \] 
\[ x_n - \xi_n y_n = (\xi q_n - p_n)(x-\xi_n y_n). \] 

(17)

Hence by (10), (6), (16), (17), and (11),

\[ f_n(x,y) = f(x_n,y_n) = x_n(\xi x_n - y_n) = q_n(x-\xi_n y_n)(\xi q_n - p_n)(x-\xi_n y_n) = \]
\[ A_n(x-\xi_n y_n)(x-\xi_n y_n). \] 

(18)

It now follows from (13) and (18), that after interchanging \( \xi_n, \phi_n \) if necessary, and possibly taking a subsequence, that

\[ \xi_n \to \xi, \quad \phi_n \to \phi. \] 

(19)

If \( x,y \in O \) are fixed, and \( x_n,y_n \) are defined by (9), then

\[ \lim_{n \to \infty} |\xi x_n - y_n| = \lim_{n \to \infty} |x-\xi_n y_n| |\xi q_n - p_n| = 0. \]

The first equality is immediate from (17), and the second from (8) since (19) implies that \( |x-\xi_n y_n| \) is bounded. So it follows from (7), by taking a sequence of \( \epsilon \to 0 \), that

\[ |f_0(x,y)| = \lim |f_n(x,y)| = \lim |f(x_n,y_n)| \geq v. \]

Moreover,

\[ |f_0(1,0)| = |A_0| = v, \]

so that

\[ M(f_0) = v. \]

Together with (12), we have

\[ \mu(f_0) = v^{-1}, \]

proving the lemma.

Noting that

\[ v^{-1} = \lim \sup (|q| |q^5 - p|)^{-1}, \]

we have proved that given an element \( v^{-1} \) in the approximation spectrum, it also lies in the Markoff spectrum. Thus over imaginary quadratic fields of class number 1 we have the desired containment of the approximation in the Markoff spectrum.

The remainder of this study will investigate the Markoff spectrum of the Gaussian field $Q(i)$. As shown in Section 1 of this chapter, only forms of the type

$$f(x,y) = (x-ay)(x-by)$$

satisfying (4) and (5) need be considered. Since we are mainly interested in the first and second gaps, we shall also assume that

$$|a-b| = |D(f)|^{1/2} = \mu(f) < 2.$$  \hspace{1cm} (20)

Next, it may be assumed that the line joining the roots $a$ and $b$ has non-negative slope. For if not, the equivalent form $f(x,iy)$ has roots $ia$, $ib$ with this property. Thus, we may assume, interchanging $a$ and $b$ if necessary, that

$$\text{Re}(a) \leq \text{Re}(b), \quad \text{Im}(a) < \text{Im}(b).$$ \hspace{1cm} (21)

Now, if $c$ is any Gaussian integer, then the form

$$f_c(x,y) = f(x-cy, y) = (x-(a+c)y)(x-(b+c)y)$$

is equivalent to $f$ and has roots $a+c$, $b+c$. By choosing

$$c = -[\text{Re}(a)+1] - i[\text{Im}(a)+1]$$

where $[x]$ is the greatest integer in $x$, we may therefore assume that the root $a$ lies in the square $X = [0, -1, -1, -1]$.

The two conditions (20) and (21) then force $b$ into one of the nine squares $X$, $Y$, $Z$, $U$, $V$, $W$, $Y'$, $Z'$, $V'$ shown below in Figure 1.

If $b$ lies in one of the squares $Y'$, $Z'$, $U'$, then the form

$$h(x,y) = (x-i\bar{a}y)(x-i\bar{b}y)$$

has roots $i\bar{a} \in X$ and $i\bar{b} \in Y,Z,U$, respectively. Furthermore,

$$|D(h)| = |\bar{b}-\bar{a}|^2 = |b-a|^2 = |D(f)|,$$

and

$$M(h) = M(f) \quad \text{since} \quad h(i\bar{x},\overline{y}) = -f(x,y).$$
Thus

\[ \mu(h) = \mu(f), \]

and we may finally assume that \( a \in X \), and that \( b \) lies in one of the six squares \( X, Y, Z, U, V, W \).

These six cases will be investigated successively.

The method of proof used to obtain the first and second gaps of the Markoff spectrum of \( \mathbb{Q}(i) \) involves the repeated application of inequalities (5) and (20). Inequality (5) can be replaced by the two separate inequalities

\[ \left| \frac{x}{y} - a \right| \geq \frac{1}{|y|^2|\frac{x}{y} - b|} \]

(221)

\[ \left| \frac{x}{y} - b \right| \geq \frac{1}{|y|^2|\frac{x}{y} - a|} \]

(222)

In each of the six cases to be considered, fixed Gaussian
integers \( x, y \) will be chosen such that G.C.D. \((x, y) = 1\), \( \frac{x}{y} \in X \) when (22.) is applied, and \( \frac{x}{y} \) in the square of \( b \) when (22.) is applied.

For example, fix two relatively prime Gaussian integers \( x_0, y_0 \) such that \( \frac{x_0}{y_0} \in X \). Since \( b \) is limited to a certain unit square, the value \( \left| \frac{x_0}{y_0} - a \right| \) is bounded above by some positive number \( K \). But then, by (22.),

\[
\left| \frac{x_0}{y_0} - a \right| \geq \frac{1}{K|y_0|^2}.
\]

Hence the root \( a \) cannot lie inside the circle with center \( \frac{x_0}{y_0} \) and radius \( \frac{1}{K|y_0|^2} \).

The root \( a \), already restricted to lie in the square \( X \), is thus further limited by having to lie outside this circle. Similar restraints will be put on the root \( b \) by applying (22.).

The object of the proof is to cover the two squares in each case with enough circles so that either one square is completely covered, meaning that no forms satisfying (20) and (22) arise in this case, or at least the two roots are forced into very small regions. Once this is accomplished, isolation techniques similar to those used by Cassels [2] will then be used to give the gaps and isolate the forms.

4. The First Gap.

Case I: \( a \in X, b \notin X \). Since both \( a, b \in X \), we have

\[
|x-a| \leq 2^{1/2}, \quad |x-b| \leq 2^{1/2} \quad \text{for} \quad x = 0, -1, -1 - i, -1.
\]

Setting \( y = 1 \), and substituting these above inequalities into (22), it follows that

\[
\frac{1}{|x-a|} \geq \frac{1}{|x-b|} \geq \frac{1}{2^{1/2}},
\]

\[
\frac{1}{|x-b|} \geq \frac{1}{|x-a|} \geq \frac{1}{2^{1/2}}
\]

for \( x = 0, -1, -1 - i, -1 \). The whole square \( X \) is then covered by these
circles with centers at the vertices and radii $2^{-1/2}$, except for the midpoint $-\frac{1-i}{2}$. But since both $a$ and $b$ must lie outside or on these circles, it follows that

$$a = b = -\frac{1-i}{2}.$$  

This contradicts (4), and hence no form arises in this case.

**Case II: $a \in X, b \in Y$.** For convenience, we shall translate each root by $i$ so that $a \in Y'$, $b \in U$. The geometric picture of the following computations, which cover most of the two squares with circles, is shown in Figure 2 below.

Applying (22) with $(x,y) = (1,1)$ gives

$$|1-b| \geq \frac{1}{|1-a|} \geq \frac{1}{|1-(1-i)|} = \frac{1}{\sqrt{2}}.$$  

But then the distance from the point $-1+i$ to the root $b$ is at most the distance from $-1+i$ to the intersection of the circle $(x-1)^2 + y^2 = \frac{1}{2}$ with the line $x = 1$. Thus

$$|(-1+i)-a| \geq \frac{1}{|(-1+i)-b|} \geq \frac{1}{|(-1+i)-(3+\sqrt{5})|} > \frac{1}{2.08} > 0.480.$$  

Symmetry then gives

$$|x-b| > 0.480 \text{ for } x = 1,1+i, \quad (23_1)$$

$$|x-a| > 0.480 \text{ for } x = -1,-1+i. \quad (23_2)$$

These inequalities immediately above imply that the distance from $\frac{2+i}{2}$ to $a$ is at most

$$\left|\frac{2+i}{2} - a\right| \leq \left|\frac{2+i}{2} - (-1+1.48i)\right| = \left(2^2 + (.02)^2\right)^{1/2} < 2.01.$$  

Hence

$$\left|\frac{2+i}{2} - b\right| \geq \frac{1}{4\left|\frac{2+i}{2} - a\right|} > \frac{1}{4(2.01)} > 0.124. \quad (24)$$

(23_1) and (24) now imply that $\left|\frac{2+i}{2} - b\right|$ is at most the distance between $\frac{2+i}{2}$ and the intersection of the two circles

$$(x-1)^2 + y^2 = (.480)^2 \quad \text{and} \quad (x-1)^2 + \left(y - \frac{1}{2}\right)^2 = (.124)^2.$$  

But the point $(0.89) + i(0.46)$ lies inside both of these circles, thus
Figure 2.

The First Gap
being further away from $\frac{-2+i}{2}$ than the intersection of the two circles, and hence

$$\left|\frac{-2+i}{2} - a\right| > \frac{1}{4}\left|\frac{-2+i}{2} - b\right| > \frac{1}{4}\left|\frac{-2+i}{2} - (0.89+1(0.46))\right| > \frac{1}{4}(1.9) > .133.$$  \hfill (25.1)

By symmetry it follows that also

$$\left|\frac{2+i}{2} - b\right| > .133 .$$  \hfill (25.2)

From (23.2) and (25.1), $|1-a|$ is now at most the distance from the intersection of the circle

$$(x+1)^2 + (y-1)^2 = (0.480)^2$$  \hfill (26)

with either the line $y = 1$ or the circle

$$(x+1)^2 + (y-\frac{1}{2})^2 = (0.133)^2 .$$  \hfill (27)

But (26) and the line $y = 1$ intersect at the point $-0.52 + i$, whose distance to the point 1 is less than 1.85. On the other hand, the point $-0.85 + 0.51$ lies outside (26), and has real component greater than any point on the smaller circle (27).

Thus the distance from the intersection of the two circles (26), (27) to the point 1 is greater than the distance

$$\left|(-0.85 + 0.51) - 1\right| > 1.9 .$$

Hence $|1-a|$ is at most the distance between 1 and the intersection of the two circles (26) and (27).

The point $-0.88 + 0.54 i$ lies inside both (26), (27), however, and hence lies further from the point 1 than the intersection of (26) and (27). Thus

$$|1-b| \geq \frac{1}{|1-a|} > \left|1 - (-0.88+0.54i)\right| > \frac{1}{1.96} > 0.510 .$$

Again by symmetry

$$|x-b| > 0.510 \text{ for } x = 1,1+i, \quad (28.1)$$

$$|x-a| > 0.510 \text{ for } x = -1+i,-1. \quad (28.2)$$

By (28) and (25.2), $|1-b|$ is at most the distance from 1 to the intersection of the circle (28.1) about 1,

$$(x-1)^2 + y^2 = (0.510)^2$$  \hfill (29)
with either the x-axis or with the circle
\[(x-1)^2 + (y-\frac{1}{2})^2 = (0.133)^2.\] (30)

But the distance from \(i\) to the intersection of the circle (29) with the x-axis is greater than \(1.1\), and the distance from \(i\) to the intersection of the two circles (29), (30) is less than
\[|i - (0.9 \pm 0.451)| < 1.06.\]

Thus \(|i-b|\) is at most the distance from \(i\) to the intersection of the circle \((x-1)^2 + y^2 = (0.510)^2\) with the x-axis, so that
\[|i-a| > \frac{1}{|i-b|} > \frac{1}{|i - 0.49|} > \frac{1}{1.12} > 0.892.\]

With this above added restraint on \(|i-a|\), the value \(|a|\) is then by (28) and (30) at most the distance from the origin to the intersection of the two circles
\[(x+1)^2 + (y-1)^2 = (0.510)^2\] and \((x+1)^2 + (y-\frac{1}{2})^2 = (0.133)^2.\] (31)

But the point \(-0.87 + 0.511\) lies inside both of these circles, so that
\[|b| > \frac{1}{|a|} > \frac{1}{|-0.87 + 0.511|} > \frac{1}{1.01} > 0.99.\] (32)

By symmetry, also
\[|a|, |i-a|, |i-b| > 0.99.\] (322)

Finally, the distance between \(\frac{1}{1-i} = \frac{1+i}{2}\) and \(a\) is at most the distance between \(\frac{1+i}{2}\) and the intersection of the two circles (23), so that
\[\left|\frac{1}{1-i} - b\right| > \frac{1}{2\left|\frac{1}{1-i} - a\right|} > \frac{1}{2\left|\frac{1+i}{2} - (-0.87+0.511)\right|} > \frac{1}{2(1.38)} > 0.362.\] (33)

Thus also
\[\left|\frac{1+i}{2} - a\right| > 0.362.\] (332)

The set of all these restraints on \(a\) and \(b\), as seen in Figure 2, cover the two squares \(Y'\) and \(U\) except for two small regions \(\mathcal{A} \subseteq Y'\) and \(\mathcal{B} \subseteq U\). Moreover, \(\frac{\sqrt{3}}{2} + \frac{1}{2} \in \mathcal{A}\), and \(\frac{\sqrt{3}}{2} + \frac{1}{2} \in \mathcal{B}\), since these points lie outside all circles constructed above.

Now \(\mathcal{A}\) is bounded to the left by the circle (27)
\[(x+1)^2 + (y- \frac{1}{2})^2 = (0.133)^2,\]

to the right by the circle \[(33_2)\]
\[(x+ \frac{1}{2})^2 + (y- \frac{1}{2})^2 = (0.362)^2,\]

and above and below by circles \[(28_2)\]
\[(x+1)^2 + (y-1)^2 = (0.510)^2,\]
\[(x+1)^2 + y^2 = (0.510)^2,\]

respectively. The circles \[(28_2)\] and \[(33_2)\] intersect at a point \(x_0 + iy_0\) where \(x_0 < -0.85\) and \(y_0 < 0.52\). Likewise, the circles \[(28_2)\] about \(-1+i\) and \[(27)\] intersect at a point \(x_1 + iy_1\) where \(x_1 > -0.89\) and \(y_1 < 0.52\). Thus, by symmetry and by the concavity of the four sides of \(\mathcal{A}\) and \(\mathcal{B}\), it follows that we can write

\[a = (-\frac{\sqrt{3}}{2} + \epsilon_1) + i(\frac{1}{2} + \epsilon_2),\]  \(\text{(34}_1)\)

\[b = (\frac{\sqrt{3}}{2} + \epsilon_3) + i(\frac{1}{2} + \epsilon_4),\]  \(\text{(34}_2)\)

where the \(\epsilon_i\) are real, and

\[|\epsilon_i| < 0.04\]  \(i = 1,2,3,4).\] \(\text{(35)}\)

It will next be shown that actually \(\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0.\)

The law of arithmetic and geometric means, applied to \((5),\)
gives the inequalities

\[\gamma|x-ay|^2 + \frac{1}{\gamma}|x-by|^2 \geq 2,\] \(\text{(36)}\)

where \(x,y\) are fixed Gaussian integers not both zero, and \(\gamma > 0\) is any positive constant. Upon the substitution of the eight pairs of Gaussian integers

\[\text{(0,1), (i,1), (1,1), (1+i,1), (2+i,2), (-2+i,2), (-1+i,1), (-1,1)}\]

for \((x,y)\) in \((36),\) along with the choice

\[\gamma = |x - (\frac{\sqrt{3}}{2} + \frac{1}{2})|^2,\]

and \(a\) and \(b\) given by \(\text{(34)},\) eight inequalities of the type

\[A\epsilon_1 + B\epsilon_2 + C\epsilon_3 + D\epsilon_4 + E(\epsilon_1^2 + \epsilon_2^2) + F(\epsilon_3^2 + \epsilon_4^2) \geq 0\]

are obtained. The forms are listed below in Table 1.
Only the coefficients will be given. Thus, for \((x,y) = (0,1)\), \(\gamma = 1\), the resulting form
\[-3 \epsilon_1 + \epsilon_2 + \sqrt{3} \epsilon_3 + \epsilon_4 + (\epsilon_1^2 + \epsilon_2^2) + (\epsilon_3^2 + \epsilon_4^2) \geq 0\]
will be listed simply as
\((-\sqrt{3}, 1, \sqrt{3}, 1; 1, 1)\).

<table>
<thead>
<tr>
<th>((x,y))</th>
<th>(\gamma)</th>
<th>(\epsilon_1)</th>
<th>(\epsilon_2)</th>
<th>(\epsilon_3)</th>
<th>(\epsilon_4)</th>
<th>(\epsilon_1^2+\epsilon_2^2)</th>
<th>(\epsilon_3^2+\epsilon_4^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,1))</td>
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<td>-\sqrt{3}</td>
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<td>\sqrt{3}</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>-\sqrt{3}</td>
<td>-1</td>
<td>-\sqrt{3}</td>
<td>-1</td>
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<td>1</td>
</tr>
<tr>
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<td>2-\sqrt{3}</td>
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<td>2-\sqrt{3}</td>
<td>2+\sqrt{3}</td>
</tr>
<tr>
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<td>1</td>
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<td>2+\sqrt{3}</td>
<td>2-\sqrt{3}</td>
</tr>
</tbody>
</table>

When these inequalities (37)-(44) are multiplied by scalars and added in the proper ways, bounds are obtained for each of the \(\epsilon_i\), \((i = 1,2,3,4)\). These inequalities are given in Table 2, again as in Table 1 with only the coefficients showing. The first column gives the linear combination of inequalities (37)-(44) necessary to obtain the inequalities.
Taking the largest coefficients for the quadratic terms, 
\( \varepsilon_1^2 + \varepsilon_2^2 \) and \( \varepsilon_3^2 + \varepsilon_4^2 \), the eight inequalities of Table 2 give
\[
\sum_{i=1}^{4} |\varepsilon_i| + (3+\sqrt{3}) \sum_{i=1}^{4} \varepsilon_i^2 \geq 0.
\]

Since \( 3+\sqrt{3} < 10 \), then certainly
\[
\sum_{i=1}^{4} (10|\varepsilon_i|^2 - |\varepsilon_i|) \geq 0. \tag{45}
\]

If some term of (45) were positive, say
\[
|\varepsilon_i|(10|\varepsilon_i| - 1) > 0,
\]
then \( |\varepsilon_i| > 0.1 \), contradicting (35). Thus, each term in (45) is equal to zero.

If some \( |\varepsilon_i| \) were positive, then \( |\varepsilon_i| = 0.1 \), again contradicting (35). Therefore
and hence we have proved the following

Lemma 4. If \( f(x-ay)(x-by) \) has roots \( a \in Y', b \in U \), and if \( a, b \) satisfy conditions (4) and (5), then

\[ f(x,y) = (x - \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)y)(x - \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\right)y) = x^2 - ixy + y^2. \]

When the investigation of the later cases is completed, it will be seen that this is the only form, up to multiples of equivalent forms, which attains its minimum and satisfies

\[ 0 < \mu(f) < \left(\frac{3}{2}\right)^{1/2} \cdot (\lambda_1)^{1/4}. \]

Thus \( f_1(x,y) = x^2 - ixy + y^2 \) gives the first gap of the Markoff spectrum of \( Q(1) \).


Case III: \( a \in X, b \in U \). By symmetry, we may even assume that the root \( b \) lies in the triangle with vertices \( 0, i, l+i \), henceforth denoted by \( \Delta(0, i, l+i) \). For if \( b \in \Delta(0, l+i, l) \), then the form

\[ h(x,y) = (x - i\bar{a}y)(x - i\bar{b}y) \]

has roots \( i\bar{a} \in X \) and \( i\bar{b} \in \Delta(0, i, l+i) \), discriminant

\[ D(h) = |i\bar{a} - i\bar{b}|^2 = |a-b|^2 = D(f), \]

and minimum \( M(h) = M(f) \) since

\[ h(i\bar{x}, \bar{y}) = x \bar{x} \]

Thus \( \mu(h) = \mu(f) \).

The same methods as in the last section will be used to cover the square \( X \) and the triangle \( \Delta(0, i, l+i) \) with circles. To simplify the locations of the centers \( \frac{x}{y} \) of the circles, the inequality (22_1), say, will be written
whenever \( y \) is not purely real. Also, whenever clear, the notation will often be shortened, so that instead of, say,

\[
|\frac{1+2i}{2-3i} - a| \geq \frac{1}{13|\frac{1+2i}{2-3i} - b|} \geq \frac{1}{13(2.03)} > 0.037,
\]

we shall simply write

\[
|\frac{1+7i}{13} - a| \geq \frac{1}{13(2.03)} > 0.037.
\]

The following set of basic calculations is shown geometrically in Figure 3. We get as first restraints on the roots \( a, b, \)

\[
|(\frac{-1-i}a)| \geq \frac{1}{\sqrt{1+(\frac{-1-i}{a})^2}} \geq \frac{1}{\sqrt{1-(\frac{-1}{\sqrt{8}})^2}} > 0.353.
\]

But then the point in \( X \) furthest away from \( 1+i \) and lying outside the circle \( (x+1)^2 + (y+1)^2 = \frac{1}{8} \) is the point \( 1+(1-\frac{1}{\sqrt{8}})i \), so that

\[
|(1+i) - b| \geq \frac{1}{|(1+i) - a|} \geq \frac{1}{|(1+i)-(1-\frac{1}{\sqrt{8}}i)|} > \frac{1}{2.6} > 0.384.
\]

Thus by symmetry we get

\[
|(1+i) - b|, \quad |(\frac{-1-i}{a})| > 0.384.
\]  \hspace{1cm} (46)

Now, given a fixed point \( z_o \in X \), the point of \( \Delta(0,1,1+i) \) furthest from \( z_o \) and lying outside or on the circle

\[
(x-1)^2 + (y-1)^2 = (0.384)^2
\]

is the point \( 0.616 + i \). Similarly, given a fixed point \( z_1 \) in \( \Delta(0,1,1+i) \), the point of \( X \) lying furthest from \( z_1 \) and outside the circle

\[
(x+1)^2 + (y+1)^2 = (0.384)^2
\]

is the point \( -0.616 - i \).

Using these two points to give upper bounds on the distances \( |\frac{X}{Y} - a| \) and \( |\frac{X}{Y} - b| \), the following sequence of restraints on \( a \) and \( b \) are obtained:
\[ |a| > \frac{1}{1.18} > 0.847, \quad |b| > 0.847; \quad (47) \]
\[ |-1-a| > \frac{1}{2.1} > 0.476, \quad |1-b| > 0.476; \quad (48) \]
\[ |-1-a| > \frac{1}{1.91} > 0.523; \quad (49) \]
\[ |\frac{1+1}{2} - a| > \frac{1}{2(1.88)} > 0.266, \quad |\frac{1+1}{2} - b| > 0.266; \quad (50) \]
\[ |\frac{1-21}{2} - a| > \frac{1}{4(2.3)} > 0.108, \quad |\frac{1+21}{2} - b| > 0.108; \quad (51) \]
\[ |\frac{1-1}{2} - a| > \frac{1}{4(2.22)} > 0.112; \quad (52) \]
\[ |\frac{2+41}{5} - a| > \frac{1}{5(2.08)} > 0.096, \quad |\frac{2+41}{5} - b| > 0.096; \quad (53) \]
\[ |\frac{3+41}{5} - a| > \frac{1}{5(2.18)} > 0.091, \quad |\frac{3+41}{5} - b| > 0.091; \quad (54) \]
\[ |\frac{4-21}{5} - a| > \frac{1}{5(2)} = 0.100; \quad (55) \]
\[ |\frac{4-31}{5} - a| > \frac{1}{5(2.15)} > 0.093. \quad (56) \]

The circles corresponding to inequalities (47)-(56) cover the square \( X \) and the triangle \( \Delta(0,1,1+i) \) with the exception of five small connected regions, which are seen in Figure 3. One region is contained in the union of the three triangles

\[ \Delta_1 = \Delta(\frac{3+41}{5}, \frac{1+21}{2}, 1+i), \]
\[ \Delta_2 = \Delta(\frac{2+41}{5}, \frac{1+21}{2}, \frac{3+41}{5}), \]
\[ \Delta_3 = \Delta(\frac{1+1}{2}, \frac{2+41}{5}, \frac{3+41}{5}), \]

the second region in

\[ \Delta_4 = \Delta(\frac{1+1}{2}, \frac{3+41}{5}, 1+i), \]

the third in the union of

\[ \Delta_5 = \Delta(\frac{-1-i}{2}, \frac{-1-31}{5}, \frac{-1-21}{5}), \]
\[ \Delta_6 = \Delta(\frac{-4-31}{5}, \frac{-2-1}{2}, \frac{-4-21}{5}), \]
\[ \Delta_7 = \Delta(-1-i, \frac{-2-1}{2}, \frac{-4-31}{5}), \]

the fourth in the union of

\[ \Delta_8 = \Delta(-1-i, \frac{-4-31}{5}, \frac{-1-1}{2}), \]
\[ \Delta_9 = \Delta(-1-i, \frac{-1-1}{2}, \frac{-3+41}{5}). \]
and the fifth in the union of the three triangles
\[ \Delta_{10} = \Delta(-1-i, \frac{-3+4i}{5}, \frac{-1-2i}{2}) \],
\[ \Delta_{11} = \Delta(-\frac{-3+4i}{5}, \frac{-2+4i}{5}, \frac{-1-2i}{2}) \],
\[ \Delta_{12} = \Delta(-\frac{-3+4i}{5}, \frac{-1+4i}{2}, \frac{-2+4i}{5}) \].

Therefore \( b \) lies in one the four triangles \( \Delta_i \) \((i = 1,2,3,4)\), and \( a \) lies in one of the eight triangles \( \Delta_j \) \((j = 5,6,7,8,9,10,11,12)\).

One such form that does arise in this case is
\[ g_2(x,y) = \frac{1}{1+2i} f_2(x,y) = x^2 + \left(\frac{1}{5} - \frac{21}{5}\right)xy - iy^2. \] (57)

This form \( g_2 \) has roots
\[ a_2 = -(0.78208...) - (0.50373...)i \in \Delta_5, \] (58)
\[ b_2 = (0.58208...) + (0.90373...)i \in \Delta_1. \] (59)

By calculation,
\[ D(f_2) = -3(5+4i), \]
and it will be proved later in Lemma 8 that
\[ M(f_2) = |1+2i| = \sqrt{5}. \]

Therefore
\[ \mu(g_2) = \mu(g_2) = \frac{|D(f_2)|^{1/2}}{M(f_2)} = \left(\frac{3}{5}\right)^{1/2} \cdot \sqrt{\frac{\sqrt{5}}{4}} = 1.96007... . \] (60)

Under the symmetry \((a,b) \to (-ib, -ia)\), another form
\[ g_{22}(x,y) = (x - (-ib_2)y)(x - (-ia_2)y) = g_2(-x, iy) \]
also arises. By (58) and (59), \( g_{22}(x,y) \) has roots
\[ -ib_2 = -(0.90373...) - (0.58208...)i \in \Delta_7, \]
\[ -ia_2 = (0.50373...) + (0.78208)i \in \Delta_3. \]

Furthermore, \( \mu(g_{22}) = \mu(g_2) \), since
\[ D(g_{22}) = |(-ib_2) - (-ia_2)|^2 = |b_2 - a_2|^2 = D(g_2) \]
and
M(g_{22}) = M(g_2),
the latter equality following from
\[ g_{22}(-ix, y) = -g_2(x, y). \]

After some extensive covering calculations, it will be proved that
\[ g_2(x, y), g_2(-x, y), g_{22}(x, y), g_{22}(-x, y) \] are the only forms arising in
Case III with roots satisfying
\[ |a - b| < 1.961. \] 
(61)

In order to obtain the second gap, we shall assume henceforth that the
roots a and b satisfy inequality (61).

6. The Diagonal Case, Continued.

The determination of the second gap will be carried out by splitting
the computations of Case III into four subcases, namely \( b \in \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \).

**Subcase (i):** \( b \in \Lambda_1 \).

Since the point \( b \in \Lambda_1 \) satisfies the inequality \((5\frac{1}{2})\), it follows
that the point of \( \Lambda_1 \) closest to the eight triangles \( \Lambda_j \), \( (j=5, \ldots, 12) \),
and satisfying \((5\frac{1}{2})\), is the intersection of the circle
\[ (x - \frac{3}{5})^2 + (y - \frac{4}{5})^2 = (0.091)^2 \] 
(62)
with the line
\[ y = -2x + 2 \] 
(63)
through \( \frac{3+4i}{5} \) and \( \frac{1+2i}{2} \).

The point \((0.56) + (0.88)i\) lies on the line (63) and is inside
the circle (62). It is closer to the triangles \( \Lambda_j \subset \mathbb{X} \) than any point
in \( \Lambda_1 \) also satisfying \((5\frac{1}{2})\). For this reason it will be used a number
of times in this subcase to obtain bounds on \(|a-b|\).

First, if \( a \in \Lambda_j \) \( (j = 7, 8, 9, 10, 11) \), then
\[ |a-b| > |a - [(0.56) + (0.88)i]| > 1.97, \]
contradicting (61). Thus if \( b \in \Lambda_1 \), then we may assume that
\[ a \in \Lambda_5, \Lambda_6, \text{ or } \Lambda_{12}. \]
Suppose that $a \in \Delta_{12}$. The point of $\Delta_{12}$ satisfying $(50_1)$, $(53_1)$, $(54_1)$ and closest to $\Delta_1$ is the point of intersection of the circle $(50_1)$ about $-\frac{1-1}{2}$ with the circle $(54_1)$ about $-\frac{3-41}{5}$. This intersection point may be approximated by the point

$$- (0.485) - (0.765)i,$$

which, lying inside both $(50_1)$ and $(54_1)$, has real and imaginary components greater than those of the intersection point. Hence

$$\left| [-0.485-0.765i] - b \right| < |a-b| < 1.961. \quad (64)$$

Now, the circle $(64)$ intersects both the circle $(54_1)$ and the line $(62)$. The approximation for the intersection of the two circles $(64)$ and $(54_1)$ can taken as $(0.565) + (0.89)i$, which lies inside both circles. And the approximation for the intersection of $(64)$ and $(62)$ can be taken as $(0.549) + (0.902)i$, as this point lies on the line $(62)$ and outside the circle $(64)$. Since $b$ is assumed at one of these two points, we get after comparison that

$$|b| < |0.549 + 0.902i| < 1.056. \quad (65)$$

Hence

$$|a| > \frac{1}{|b|} > \frac{1}{1.056} > 0.9469. \quad (66)$$

Next, the circle $(66)$ intersects both the line $y = -\frac{1}{5}$ and the circle $(54_1)$ about $-\frac{3-41}{5}$. The point $-0.51 - 0.79i$ lies inside both $(66)$ and $(54_1)$, so that the imaginary component of the intersection point is less than $-0.79$. Similarly, the point $-0.505 - 0.81i$ lies on the line $y = -\frac{1}{5}$ and inside $(66)$, so that the real component of this intersection is greater than $-0.505$. Hence by $(66)$,

$$|a-b| > |[-0.505-0.791] - [0.56+0.881]| > 1.98.$$

But this contradicts $(61)$, so there is no form arising with roots $a \in \Delta_{12}$, $b \in \Delta_1$, and satisfying $(46)$-$\Delta$. In the next subcase, suppose that $a \in \Delta_6$, $b \in \Delta_1$.

The root $a$ lies in a region of $\Delta_6$ bounded by the circles defined by $(49)$, $(52)$, $(55)$, $(56)$. The point of this region lying closest to
$\Delta_1$ is the intersection of the circle

$$(x + \frac{4}{5})^2 + (y + \frac{2}{5})^2 = \left(\frac{1}{10}\right)^2$$

with the line $x = -\frac{4}{5}$. This point is clearly $-0.8 - 0.5i$. By (61), the root $b \in \Delta_1$ must therefore satisfy the inequality

$$|b - [-0.8-0.5i]| < 1.961.$$  \hspace{1cm} (67)

The root $b$ also lies outside or on the circles about $\frac{1+2i}{2}$ and $\frac{3+4i}{5}$ defined by (51) and (54), respectively. The circles (67) and (51$_2$) intersect at a point $x_1 + iy_1$, where

$$x_1 < 0.573, \quad y_1 < 0.92,$$

and the two circles (67) and (54$_2$) intersect at a point $x_2 + iy_2$ where

$$x_2 < 0.585, \quad y_2 < 0.891.$$  

Now $0.585 + 0.891i$ is further from the point $-1$ than any point of $\Delta_1$ satisfying (67), (51), (54), and likewise $0.573 + 0.92i$ is furthest from $\frac{4-3i}{5}$. Hence we get

$$|-1-a| \geq \left|\frac{1}{1} - (0.585+0.891i)\right| > \frac{1}{1.82} \geq 0.549,$$  \hspace{1cm} (69)

and

$$\left|\frac{4-3i}{5} - a\right| \geq \left|\frac{4-3i}{5} - (0.573+0.92i)\right| > \frac{1}{5(2.05)} > 0.097.$$  \hspace{1cm} (70)

We shall now investigate the region covered by these two circles.

First, the point $-0.8 - 0.503i$ lies on the circle defined by (70), and inside the circle defined by (69), so the entire segment between $\frac{4-3i}{5}$ and $\frac{4-2i}{5}$ is covered. These two circles intersect at a point $x_3 + iy_3 \in \Delta_6$, where the components satisfy

$$x_3 < -0.85, \quad y_3 < -0.52,$$

and the point $-0.85 - 0.52i$ lies inside both circles. Since the point $a$ must lie outside circles, it must be further from $\Delta_1$ than $-0.85-0.52i$, and hence

$$|a-b| > \left|\frac{-0.85-0.52i}{-0.56+0.88i}\right| > 1.98.$$  

But this contradicts (61), so no form arises in the subcase $a \in \Delta_6$. 

\( b \in \Delta_1 \) satisfying (61).

Therefore, it has been shown that if the roots \( a \) and \( b \) of a form \( f \) satisfies inequality (61), and if \( b \in \Delta_1 \), then \( a \) must lie in the triangle \( \Delta_f \). From the symmetry
\[
(a, b) \rightarrow (-ib, -ia),
\]
(71)
it follows that if \( a \in \Delta_7 \), then \( b \in \Delta_3 \). And from the symmetry
\[
(a, b) \rightarrow (-b, -a),
\]
(72)
it is seen that if \( a \in \Delta_{10} \), then \( b \in \Delta(0,1+1,1) \).

From here on, we may therefore assume that
\[
a \notin \Delta_7, \quad a \notin \Delta_{10}.
\]
(73)

**Subcase (ii):** \( b \in \Delta_3 \).

Since the root \( b \) must satisfy the three inequalities (50), (53), (54), it follows from (54) that \( b \) is bounded above by the modulus of the intersection point of the line \( y = \frac{3}{2} \) and the circle
\[
(x - \frac{3}{2})^2 + (y - \frac{1}{2})^2 = (0.091)^2.
\]
Thus
\[
|b| \leq |0.509 + 0.8i| < 0.95,
\]
and hence
\[
|a| \geq \frac{1}{|b|} > \frac{1}{0.95} > 1.052 .
\]
(74)

Therefore (74) and (73) immediately eliminate all possibilities for the location of the root \( a \) except for
\[
a \in \Delta_6, \quad a \in \Delta_{11}.
\]

Now
\[
|-1-a| \geq \frac{1}{|-1-b|} \geq \frac{1}{|-1-(0.509+0.8i)|} > \frac{1}{1.72} > 0.581,
\]
(75)
and the circles defined by (75) and (56) completely cover \( \Delta_6 \). This is checked easily by noting that the points \(-0.88 - 0.52i\) and \(-0.8 - 0.52i\) both lie on the boundary of \( \Delta_6 \) and both are contained in the interior of the two covering circles. Therefore no form arises with roots \( a \in \Delta_6, b \in \Delta_3 \).
Next, by again using the point $0.509 + 0.8i$ to get an upper bound on $|\frac{x}{y} - b|$, it follows that
\[
\left|\frac{-1-2i}{2} - a\right| \geq \frac{1}{\frac{-1-2i}{2} - (0.509+0.8i)} > \frac{1}{(0.2094)} > 0.121. \tag*{(76)}
\]

The circles defined by (76) and (74) then cover the triangle $\Delta_{11}$, this covering being represented in Figure 4. This can be checked by noting that the point

\[-0.553 - 0.894i\]

lies on the line segment joining $\frac{-3-4i}{5}$ to $\frac{-1-2i}{2}$, and also lies inside the two circles defined by (76), (74). This forces the two circles to intersect outside of $\Delta_{11}$. Thus no form arises with roots $a \in \Delta_{11}$, $b \in \Delta_3$.

The result of these subcases is that if a form $f$ has roots $a,b$ satisfying (61) and if $b \in \Delta_3$, then $a \in \Delta_7$.

Furthermore, by the symmetries (71) and (72), we may from here on also assume that

\[a \notin \Delta_5, \quad a \notin \Delta_{12}. \tag*{(77)}\]

Subcase (iii); $b \in \Delta_4$.

After (73) and (77), there only exist the possibilities $a \in \Delta_6$, $\Delta_8$, $\Delta_9$, $\Delta_{11}$ for the location of the root $a$. Each one of these will be shown to be impossible by the application of (61).

First we shall consider the possibility that $a \in \Delta_8 \cup \Delta_9$.

The region in $\Delta_4$ to be investigated is bounded by the circles about $\frac{1+i}{2}$, $\frac{3+i}{5}$, $1+i$ defined by (50), (54), (46), respectively, and the line $y = x$. The point in this region furthest from $\frac{-1-i}{2}$ is the intersection $x_4 + iy_4$ of the two circles (54), (46). Here

\[x_4 < 0.69, \quad y_4 < 0.77.\]

Thus
\[
\left|\frac{-1-i}{2} - a\right| \geq \frac{1}{2\left|1-i\right| - (0.69+0.771)} > \frac{1}{(1.741)} > 0.287. \tag*{(78)}
\]

Since the radius of the circle (56) about $\frac{-1-3i}{5}$ is larger than the
Figure 4.

The Covering of $\Delta_{11}$.
radius of (54) about \( \frac{-3-4i}{5} \), it follows by symmetry that also
\[
|\frac{-1-1}{2} - b| > 0.287.
\] 
(79)

Hence the root a must lie inside the region bounded by the relevant circles of (46), (54), (56), (78), and the other root b lies in the region bounded by the line \( y = x \) and the circles (79), (54), (46). These regions are shown in Figure 5.

Now, the points a and b, each in their respectively regions, which give the minimal distance \( |a-b| \), are respectively the intersection \( x_5 + iy_5 \) of the circles (78), (54), and the intersection \( x_6 + iy_6 \) of the circles (79), (54).

Moreover,
\[
x_5 < -0.73, \quad y_5 < -0.665,
\]
and
\[
x_6 > 0.66, \quad y_6 > 0.73,
\]
so that
\[
|a-b| \geq |(0.73-0.665i) - (0.66+0.73i)| > 1.969.
\]
But this contradicts (61), so that no form satisfying (61) occurs with roots \( a \in \Delta_8 \cup \Delta_9 \) and \( b \in \Delta_4 \).

Next, suppose that \( a \in \Delta_6, b \in \Delta_{14} \). The two circles (46), (54) have been seen to intersect at a point \( x_4 + iy_4 \) whose components satisfy
\[
x_4 < 0.69, \quad y_4 < 0.77.
\]
Also, the circle (46) and the line \( y = x \) intersect at the point
\[
[1 - \frac{\sqrt{2}}{2}(0.384)](1+i),
\]
and we can approximate
\[
1 - \frac{\sqrt{2}}{2}(0.384) < 0.729.
\]
The point \( (0.729)(1+i) \) gives an upper bound for the values \( |a-b| \) for the points considered, so that
\[
|1-a| > \frac{1}{1.729} > 0.532,
\] 
(80)
The point \( x_7 + iy_7 \) of the region in \( A_4 \) containing \( b \) and closest to \( A_6 \) is the intersection of the two circles (50), (54), and
\[
x_7 > 0.64, \quad y_7 > 0.72.
\]
So by (61), the root \( a \) must also satisfy the inequality
\[
|a - (0.64 + 0.72i)| < |a - b| < 1.961. \quad (83)
\]
Now the circles (83) and (82) intersect at a point \( x_8 + iy_8 \) where
\[
x_8 > -0.875, \quad y_8 > -0.535.
\]
And circles (83), (80) intersect at a point \( x_9 + iy_9 \) where
\[
x_9 > -0.88, \quad y_9 > -0.53.
\]
Since an upper bound on \( |X \overline{Y} - a| \) is given by one of these two points for the vertices \( X = \overline{Y} = a \) of \( A_4 \), we get
\[
\left| \frac{1+i}{2} - b \right| > \frac{1}{2(1.732)} > 0.2901, \quad (84)
\]
\[
\left| (1+i) - b \right| > \frac{1}{2.424} > 0.4125, \quad (85)
\]
\[
\left| \frac{3+i}{5} - b \right| > \frac{1}{5(1.99)} > 0.1005. \quad (86)
\]
The region defined by (80)-(86) is shown in Figure 6.

The intersection of (85) and (86) can be approximated by
\[
0.685 + 0.74i,
\]
and the intersection of (85) with the line \( y = x \) can be approximated by
\[
(0.709)(1+i).
\]
These approximations give
\[
\left| -1 - a \right| > \frac{1}{1.851} > 0.5402, \quad (87)
\]
\[
\left| \frac{1+i}{5} - a \right| > \frac{1}{5(1.875)} > 0.1066, \quad (88)
\]
\[
\left| \frac{3+i}{5} - a \right| > \frac{1}{5(2)} = 0.1. \quad (89)
\]
These last three inequalities are plotted in Figure 7, from which it is clear that the point of $\Delta_6$ closest to $\Delta_4$ and satisfying (87)-(89) is the intersection of (87) and (89). This point can be approximated by

$$-0.86 - 0.52i,$$

which lies inside (87) and on (89).

Likewise, the point of $\Delta_4$, closest to $\Delta_6$ and satisfying (84) - (86) is the intersection of (84) and (86). This point can be approximated by

$$0.67 + 0.73i.$$

Thus,

$$|a - b| > |(-0.86-0.52i) - (0.67+0.73i)| > 1.975,$$

contradicting (61). Hence no form satisfying (61) occurs with roots $a \in \Delta_6$, $b \in \Delta_4$.

Finally, suppose that $a \in \Delta_1$, $b \in \Delta_4$. The point of the region in $\Delta_4$ bounded by (45), (50), (54) furthest from $\Delta_1$ is the intersection of (46) and (54), and this point can be approximated by

$$0.69 + 0.77i.$$

Therefore

$$|a| \geq \frac{1}{|b|} > \frac{1}{0.69 + 0.77i} > \frac{1}{1.034} > 0.9671,$$

(90)

$$|-i - a| > \frac{1}{1.9} > 0.5263,$$

(91)

$$|\frac{-3\sqrt{3} + 1}{5} - a| > \frac{1}{5(2.032)} > 0.0984.$$

(92)

Also, the point of $\Delta_4$ satisfying (46), (50), (54) and closest to $\Delta_1$ is the intersection of the circle (50) and $\frac{1+i}{2}$ and the line $y = x$. This intersection is the point

$$\left[\frac{1}{2} + \frac{\sqrt{2}}{2}(0.266)\right](1+i),$$

and for an approximation we may take

$$\frac{1}{2} + \frac{\sqrt{2}}{2}(0.266) > 0.6880.$$

Hence the root $a$ must satisfy
Figure 7.
\[ |a - (0.6880)(1+i)| < |a-b| < 1.961. \]  

(93)

Now the point \( a \in \Delta_1 \) satisfying (93), (90), (91), (92) and furthest from \( \frac{1+i}{2} \) and \( \frac{3+i}{5} \) is the intersection of circles (91), (93), and the point furthest from \( 1+i \) is the intersection of circles (92), (93). Relevant approximations for these points are

\[-0.51 - 0.875i\]

and

\[-0.53 - 0.86i,\]

respectively.

Therefore

\[ |\frac{1+i}{2} - b| > \frac{1}{2(1.707)} > 0.2929, \]  

(94)

\[ |\frac{3+i}{5} - b| > \frac{1}{5(2.01)} > 0.0995, \]  

(95)

\[ |(1+i) - b| > \frac{1}{2.41} > 0.4149. \]  

(96)

These circles are shown in Figure 8.

But now the intersection of (96), (95) gives the point of \( \Delta_4 \) furthest from \( \Delta_{11} \) and satisfying (94) - (96). An approximation for this point is

\[ 0.685 + 0.751i, \]

this point lying inside both circles (95) and (96).

This in turn gives the bounds

\[ |a| \geq \frac{1}{|b|} > \frac{1}{1.016} > 0.9842, \]  

(97)

\[ |-i-a| > \frac{1}{1.88} > 0.5319. \]  

(98)

The intersections of (97), (98), and of (94) with the line \( y = x \) give the points \( a,b \) with minimal distance \( |a-b| \) and satisfying all the required bounds. Taking as approximations

\[-0.5 - 0.84i \quad \text{and} \quad (0.707)(1+i)\]

we get

\[ |a-b| > \left| [-0.5-0.84i] - (0.707)(1+i) \right| > 1.962, \]

contradicting (61). Therefore no form arises with roots in \( \Delta_{11} \) and \( \Delta_4 \), and satisfying (61).
Since all possibilities have been eliminated, the results of this subcase can be summarized in saying that no form arises with roots \( a \in X \) and \( b \in \Delta_1 \), also satisfying (61). Once more applying the symmetries (71) and (72), we may thus also assume from here on that

\[
a \notin \Delta_8, \quad a \notin \Delta_9.
\]

Subcase (iv): \( b \in \Delta_2 \).

If \( b \in \Delta_2 \), then from assumptions (73), (77), (99), there are only two possibilities for the location of the root \( a \), namely in \( \Delta_6 \) or \( \Delta_{11} \).

First, suppose that \( a \in \Delta_6 \). The root \( b \) in \( \Delta_2 \) lies outside the circles about \( i, \frac{1+2i}{2}, \frac{2+4i}{5}, \frac{3+4i}{5} \) defined by (48), (51), (53), (54), respectively. The point \( b \) satisfying these inequalities and having maximal distance from the points \(-1\) and \( \frac{1-3i}{5} \) is the intersection of the circle (51) with the line \( y = -2x+2 \). This intersection point can be approximated by

\[
0.55 + 0.91i,
\]

which lies inside the circle (51), and in the triangle \( \Delta_1 \).

Then

\[
|\overline{-1-a}| \geq \frac{1}{|\overline{-1-b}|} > \frac{1}{1.8} > 0.5555, \quad (100)
\]

\[
|\overline{-\frac{4-3i}{5} - a}| > \frac{1}{5(2.026)} > 0.0987. \quad (101)
\]

By symmetry about the line \( y = -x \), also

\[
|\overline{i-\overline{b}}| > 0.5555, \quad (102)
\]

\[
|\overline{\frac{3+4i}{5} - \overline{b}}| > 0.0987. \quad (103)
\]

Now the point in \( \Delta_6 \) satisfying (100), (52), (55), (101), and closest to the triangle \( \Delta_1 \), is the intersection of (100) and (101), the circles about \(-1\) and \( \frac{1-3i}{5} \), respectively. These circles are shown in Figure 9.

An approximation for this intersection can be taken to be

\[-0.875 - 0.541,\]
which lies inside both circles (100) and (101), and is closer to $\Delta_2$

than the intersection point.

By symmetry, we can take

$$0.54 + 0.875i$$

as approximating the intersection of (102), (103).

Then

$$|a-b| > |(-0.875-0.54i) - (0.54+0.875i)| > 2,$$

contradicting (61). Hence no form arises with roots $a \in \Delta_6', b \in \Delta_2'$.

The only remaining possibility is $a \in \Delta_{11}, b \in \Delta_2'$. Since this

case is symmetric about the origin, every calculation with respect to

one triangle also holds for the other.

The point of $\Delta_{11}$ satisfying (48), (51), (53), (54) and closest to

$\Delta_4$ is the intersection of the circle (53) about $-\frac{2-8i}{5}$ with the line

$y = -\frac{4}{5}$, that is, the point

$$-0.496 - 0.8i.$$  

Hence the root $b$ must satisfy

$$|b - (-0.496-0.8i)| < |a-b| < 1.961, \quad \text{(104)}$$

and by symmetry

$$|a - (0.496+0.8i)| < 1.961. \quad \text{(105)}$$

The circle (104) intersects both the circles (54_2) and (51_2). That (104) and (51) intersect can be checked by noting that both

$$0.48 + 0.9i \quad \text{and} \quad 0.5 + 0.9i \quad \text{(106)}$$

lie inside (51), but the first lies inside (104), and the latter

outside. Thus the latter point of (106) may be taken as an approx-

imation to this intersection.

Likewise, the point $0.54 + 0.87i$ lies outside both (104) and (54),

and so may be taken as that approximation point.

Thus, using the point giving the maximum distance, namely

$0.5 + 0.9i$, we get

$$|a| > \frac{1}{|0.5+0.9i|} > \frac{1}{1.03} > 0.9708, \quad |b| > 0.9708; \quad \text{(107)}$$
\[ |1-a| > \frac{1}{1.965} > 0.5089, \quad |1-b| > 0.5089; \quad (108) \]
\[ |\frac{-2^2 - 1}{5} - a| > \frac{1}{4(2.148)} > 0.1163, \quad |\frac{1+2a}{2} - b| > 0.1163; \quad (109) \]
\[ |\frac{-2^5 - 1}{5} - a| > \frac{1}{5(1.925)} > 0.1038, \quad |\frac{2+4a}{5} - b| > 0.1038; \quad (110) \]
\[ |\frac{-2^5 - 1}{5} - a| > \frac{1}{5(2.025)} > 0.0987, \quad |\frac{3+4a}{5} - b| > 0.0987 . \quad (111) \]

The circles corresponding to the above calculations are shown in Figure 10.

Now the points of \( \Delta_{11} \) satisfying (107)-(111) which are closest to \( \Delta_4 \) lie along the arc of the circle (107) between its intersections with circles (110) and (111).

The point \(-0.505 - 0.825i\) lies inside both (107) and (111), and the point \(0.49 - 0.841\) lies outside (107) and inside (110). Thus the point \(-0.49 - 0.825i\), obtained by taking the imaginary and real components of (112), (113), respectively, lies closer to \( \Delta_2 \) than any point on the above-mentioned arc of (107). Therefore the root \( b \) satisfies

\[ |b - (-0.49-0.825i)| < |a-b| < 1.961 . \quad (114) \]

The maximum value of \( b \) is taken at the intersection of (104) with either the circle (108) about \( i \), or (111) about \( \frac{3+4a}{5} \). The point \(0.49 + 0.875i\) lies outside (114) and (108), and the point \(0.52+0.856i\) lies outside (114) and inside (111).

Hence

\[ |a| \geq \frac{1}{|b|} > \frac{1}{|0.49+0.875i|} > \frac{1}{1.006} > 0.994 , \quad (115) \]

and by symmetry

\[ |b| > 0.994 . \quad (116) \]

But then the shortest distance between the respective regions in \( \Delta_{11} \) and \( \Delta_2 \) is bounded below by the distance between the intersections of (115), (108) in \( \Delta_{11} \) and (116), (111) in \( \Delta_2 \). The intersection point in \( \Delta_{11} \) can be easily approximated by \(-0.48 - 0.861\), and the point in \( \Delta_2 \) by \(0.51 + 0.841\). Hence
Figure 10.
\[ |a-b| > |(-0.48-0.861) - (0.510+0.8401)| > 1.965, \]
contra dic ting (61).

Combining the results of Subcases (i)-(iv), we have proved the following

**Lemma 5.** If \( f(x,y) = (x-ay)(x-by) \) has roots \( a \in X, b \in U \), and if \( a, b \) satisfy conditions (4), (5), and (61), then up to the symmetries

\( (a,b) \rightarrow (-i\bar{b}, -i\bar{a}) \) and \( (a,b) \rightarrow (-\bar{b}, -a) \),

\( a \) and \( b \) must lie in the triangles \( \Delta_5 \) and \( \Delta_1 \), respectively.

7. Final Restriction of the Roots.

By Lemma 5, we may now assume that \( a \in \Delta_5 \) and \( b \in \Delta_1 \). The following computations are shown geometrically in Figure 11.

The point in \( \Delta_5 \) satisfying (50), (55), (56), and furthest from any fixed point of \( \Delta_1 \) is the intersection \(-0.8 - 0.507i\) of the circle (56) with the line \( x = -0.8 \). Hence

\[ |(1+i) - b| \geq \frac{1}{|1+i - (-0.8-0.507i)|} > \frac{1}{2.348} > 0.4258, \] (117)

\[ \frac{|1+\frac{2i}{5} - b|}{4(1.99)} > 0.1256, \] (118)

\[ \frac{|3+\frac{4i}{5} - b|}{5(1.916)} > 0.1043. \] (119)

Now, the point in \( \Delta_1 \) satisfying (117)-(119), and furthest from any fixed point of \( \Delta_5 \) is the intersection of the circles (118) about \( \frac{1+2i}{2} \) and (119) about \( \frac{3+4i}{5} \). The point \( 0.584 + 0.911i \) lies inside both circles (118) and (119), and is thus further away from \( \Delta_5 \) than the intersection. Hence

\[ \frac{|-\frac{1-i}{2} - a|}{2|\frac{-1-i}{2} - (0.584+0.911i)|} > \frac{1}{2(1.779)} > 0.2810, \] (120)

\[ \frac{|\frac{4-2i}{5} - a|}{5(1.906)} > 0.1049, \] (121)

\[ \frac{|\frac{4-2i}{5} - a|}{5(2.049)} > 0.0976. \] (122)
Using the additional points
\[
\frac{-3-21}{4}, \frac{5+1}{5+21} = \frac{-23-151}{29},
\]
we also get
\[
\left|\frac{-3-21}{4} - a\right| > \frac{1}{16(1.942)} > 0.03218, \quad (123)
\]
\[
\left|\frac{-23-151}{29} - a\right| > \frac{1}{29(1.944)} > 0.01738. \quad (124)
\]

Triangle $\Lambda_5$ and the circles $(121)-(124)$ are shown in Figure 12. These circles, along with $(120)$, cover all of $\Lambda_5$ with the exception of the small region $\mathcal{A}$ bounded by the four arcs of $(121)-(124)$.

Circles (121) and (124) intersect at a point $x_1 + iy_1$ where
\[
x_1 > -0.7835.
\]
This is clear since $-0.7835 - 0.503i$ lies inside both (121) and (124).

Circles (121) and (123) intersect at a point $x_2 + iy_2$ where
\[
y_2 < -0.525.
\]
This is clear since the point $-0.7815 - 0.525i$ lies inside both (121) and (123).

Circles (122) and (123) intersect at a point $x_3 + iy_3$ where
\[
x_3 < -0.7815, \quad y_3 > -0.5042.
\]
This is clear since the point $-0.7815 - 0.5042i$ lies inside both (122) and (123).

Letting the intersection of (122) and (124) be denoted by $x_4 + iy_4$, it is clear from Figure 10 that
\[
-0.7835 < x_1 < x_4 < x_2 < x_3 < -0.7815 \quad (125)
\]
and
\[
-0.5042 < y_3 < y_4 < y_1 < y_2 < -0.5025. \quad (126)
\]
Thus the region $\mathcal{A}$ lies inside a square with vertices
\[
-0.7835-0.5042i, -0.7835-0.5025i, -0.7815-0.5025i, -0.7815-0.5042i. \quad (127)
\]
Figure 12.

Restricting the Root a.
The form \( g_2(x,y) \) defined by (57) has a root \( a_2 \) in \( \Delta_2 \), and it follows from (58), (125), (126) that a root \( a \in \mathcal{A} \) must have the representation

\[
a = a_2 + (s_1 + is_2)
\]

where \( s_1, s_2 \) are real, and

\[
|s_1| < 0.0015, \quad |s_2| < 0.0013.
\]  \hfill (129)

Similar bounds can be obtained for the root \( b \). Already we have (117)-(119). By (127), the point \(-0.7835 - 0.5042i\) can be used for additional bounds on \( b \). Namely, for

\[
\frac{4}{2-31} = \frac{8+12i}{13}, \quad -\frac{5+21}{-1+5i} = \frac{15+23i}{26},
\]

we get

\[
|\frac{8+12i}{13} - b| > \frac{1}{13(1.999)} > 0.0348,
\]  \hfill (130)

\[
|\frac{15+23i}{26} - b| > \frac{1}{26(1.945)} > 0.01977.
\]  \hfill (131)

Circles (118), (119), (130), (131) are shown in Figure 13. With the addition of (117), the triangle \( \Delta_1 \) is completely covered with the exception of a small region \( \mathcal{B} \) bounded by arcs of (118), (130), (131).

Now, circles (118) and (130) intersect at a point \( x_5 + iy_5 \) where \( y_5 < 0.905 \). This follows from the observation that \( 0.5818 + 0.9051 \) lies inside both (118) and (130).

Circles (118) and (131) intersect at a point \( x_6 + iy_6 \) where \( x_6 > 0.5808 \). This follows from the observation that \( 0.5808 + 0.9039 \) lies inside both (118) and (131).

Circles (130) and (131) intersect at a point \( x_7 + iy_7 \) where \( x_7 < 0.5824, \quad y_7 > 0.9033 \). This follows from the fact that \( 0.5824 + 0.9033 \) lies inside both (130)
Figure 13.

Restriction of the Root b.
and (131).

Thus it is clear from Figure 13 and the above bounds that

\[ 0.5808 < x_6 < x_5 < x_7 < 0.5824, \]
\[ 0.9033 < y_7 < y_6 < y_5 < 0.905. \]

The region \( \mathcal{B} \) is therefore contained in a square with vertices

\[ 0.5808 + 0.9033i, 0.5808 + 0.9051i, 0.5824 + 0.9051i, 0.5824 + 0.9033i. \] (132)

Hence \( b \) can be written as

\[ b = b_2 + (\delta_3 + i\delta_4) \] (133)

where \( b_2 \) is the root of \( g_2 \) given by (59). Furthermore, from (59) and the coordinates of the square (132), it follows that

\[ |\delta_3| < 0.0013, \quad |\delta_4| < 0.0013. \] (134)

These bounds allow us to complete the proof of the following

**Lemma 6.** Let \( g(x,y) = (x-ay)(x-by) \) have roots \( a \in \Delta_5, \ b \in \Delta_1 \), satisfying (4) and (5). Then

\[ g(x,y) = x^2 + [(\frac{1}{2}+\epsilon_1) - (\frac{1}{2}+\epsilon_2)i]xy - [\epsilon_3 + (1+\epsilon_4)i]y^2 \] (135)

where

\[ |\epsilon_1| < 0.0004 \quad \text{for} \quad i = 1, 2, 3, 4. \] (136)

**Proof:** It has been shown above that the roots \( a \) and \( b \) can be written as in (128) and (133), respectively, and where the \( \delta_1 \) (i=1,2,3,4) satisfy (129) and (134). Then

\[ g(x,y) = (x-ay)(x-by) = x^2 + (a+b)xy + aby^2. \] (137)

Writing

\[ a_2 = a_1 + i\alpha_2, \quad b_2 = b_1 + i\beta_2, \] (138)

it follows from (57) that

\[ a_2 + b_2 = -\frac{1}{5} + \frac{2}{5}i, \quad a_2b_2 = -1. \] (139)

Hence from (128), (133), (138), (139), we get

\[ a + b = (-\frac{1}{5} + \delta_1 + \delta_3) + (\frac{2}{5} + \delta_2 + \delta_4)i \]
and
\[ ab = (\alpha_1 \delta_3 + \alpha_2 \delta_4 + \beta_1 \delta_1 + \beta_2 \delta_2 + \delta_3 + \delta_4) \]
\[ - i (1 + (\alpha_1 \delta_4 + \alpha_2 \delta_3 + \beta_1 \delta_2 + \beta_2 \delta_1 + \delta_3 + \delta_4)). \]

Upon setting
\[ \epsilon_1 = \delta_1 - \delta_3, \]
\[ \epsilon_2 = \delta_2 + \delta_4, \]
\[ \epsilon_3 = \alpha_1 \delta_3 - \alpha_2 \delta_4 - \beta_1 \delta_1 + \beta_2 \delta_2 - \delta_3 + \delta_4, \]
\[ \epsilon_4 = \alpha_1 \delta_4 - \alpha_2 \delta_3 + \beta_1 \delta_2 + \beta_2 \delta_1 + \delta_3 + \delta_4, \]
it follows that \( g(x, y) \) can be expressed as in (135).

Finally, by using decimal approximations (58), (59) for \( \alpha_1, \alpha_2, \beta_1, \beta_2, \) and bounds (129), (134), we get from the triangle inequality
\[ |\epsilon_1| < 0.0028, \quad |\epsilon_2| < 0.0026, \]
\[ |\epsilon_3| < 0.0038, \quad |\epsilon_4| < 0.0038. \]

This proves the Lemma.

8. Isolation of the Second Gap in the Diagonal Case.

The isolation of the second gap is obtained by a procedure similar to that used in Section 4 of this chapter. However, the arithmetic and geometric mean theorem is not used.

Suitable pairs of Gaussian integers \((x, y)\) will be substituted into the inequality
\[ |g(x, y)|^2 \geq 1, \]
where \( g(x, y) \) is given by (135). This yields, for each pair of integers, and after multiplication by a positive integer to eliminate denominators, an inequality
\[ A\epsilon_1 + B\epsilon_2 + C\epsilon_3 + D\epsilon_4 + E(\epsilon_1^2 + \epsilon_2^2) + F(\epsilon_3^2 + \epsilon_4^2) + \]
\[ + G(\epsilon_1 \epsilon_3 - \epsilon_2 \epsilon_4) + H(\epsilon_2 \epsilon_3 + \epsilon_1 \epsilon_4) \geq 0. \]
This inequality will be denoted simply by

\[(A, B, C, D; E, F; G, H).\]

The pairs of integers and resulting forms are listed in Table 3 below.

**TABLE 3**

<table>
<thead>
<tr>
<th>((x,y))</th>
<th>(\epsilon_1)</th>
<th>(\epsilon_2)</th>
<th>(\epsilon_3)</th>
<th>(\epsilon_4)</th>
<th>(\epsilon_1+\epsilon_2)</th>
<th>(\epsilon_3+\epsilon_4)</th>
<th>(\epsilon_1+\epsilon_3+\epsilon_2+\epsilon_4)</th>
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<tbody>
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<td>((0,1))</td>
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<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((1,1))</td>
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<td>-6</td>
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<td>8</td>
<td>5</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>((-1,1))</td>
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<td>-8</td>
<td>5</td>
<td>5</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>((1+1,1))</td>
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<td>-2</td>
<td>-6</td>
<td>-8</td>
<td>10</td>
<td>5</td>
<td>-10</td>
</tr>
<tr>
<td>((-1,1+1))</td>
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<td>-2</td>
<td>12</td>
<td>16</td>
<td>10</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>((1+21,2))</td>
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<td>-2</td>
<td>2</td>
<td>0</td>
<td>5</td>
<td>4</td>
<td>-4</td>
</tr>
<tr>
<td>((2,-2+1))</td>
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<td>-8</td>
<td>-8</td>
<td>6</td>
<td>20</td>
<td>25</td>
<td>40</td>
</tr>
<tr>
<td>((-1+21,1+21))</td>
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<td>10</td>
<td>-8</td>
<td>6</td>
<td>25</td>
<td>25</td>
<td>-30</td>
</tr>
<tr>
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<td>10</td>
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<td>25</td>
<td>25</td>
<td>40</td>
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<tr>
<td>((41,3+21))</td>
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<td>32</td>
<td>-126</td>
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<td>-1040</td>
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<td>((54,-5+21))</td>
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<td>34</td>
<td>-288</td>
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<td>4205</td>
<td>6670</td>
</tr>
</tbody>
</table>

Linear combinations of inequalities \((140)-(150)\) can now be taken to single out each \(\epsilon_i\).

\[
\frac{1}{7}[3(147)+4(148)] + (0,10,0,-2; 25,25; 10,0).
\]

\[(151)+(140) \rightarrow (0,10,0,0; 25,26; 10,0). \]  \[(152)\]

\[
2(143) + (144) \rightarrow (42,-6,0,0; 30,20; 0,0).
\]

\[(153)\]

\[
6(152) + 10(153) \rightarrow (420,0,0,0; 450,356; 60,0).
\]

\[(154)\]

\[
420(145) + 84(152) + (154) \rightarrow (0,0,840,0; 4650,4220; -780,-3360).
\]

\[(155)\]

Dividing by the leading coefficients, we obtain from \((154),(152),\)
\((155),(140),\) respectively,
(1,0,0,0; 1.0715, 0.8477; 0.1428, 0).
(0,1,0,0; 1.5, 2.6; 1.0).
(0,0,1,0; 5.5358, 5.0239; -0.9286, -4).
(0,0,0,1; 0, 0.5; 0, 0).

This same process will also give inequalities for the $-e_1$.

\[4(146)+3(148) \rightarrow (-16,-2,-14,0; 155,175; 280,170).\]  
(160)

\[60(160)+12(152)+(155) \rightarrow (-960,0,0,0; 14250,15032; 16140,6840).\]  
(161)

Dividing (161) by 960 gives

\[(-1,0,0,0; 14.8438, 15.6584; 16.8125, 7.125).\]  
(162)

Continuing,

\[(141)+(143) \rightarrow (6,-8,0,0; 15,10; -10,-20).\]  
(163)

\[160(163)+(162) \rightarrow (0,-1280,0,0; 16650,16632; 14540,3640).\]  
(164)

Dividing (164) by 1280, we get

\[(0,-1,0,0; 13.0079, 12.9938; 11.3594, 2.8438).\]  
(165)

Continuing again,

\[17(142)+(139) \rightarrow (0,-150,-168,-24; 1125,930; -1040,-1570).\]  
(166)

\[(166)+(15(152)+(12(140) \rightarrow (0,0,-168,0; 1500,1332; -890,-1570).\]  
(167)

Dividing (167) by 168 gives

\[(0,0,-1,0; 8.9286, 7.9286; -5.2977, -9.3453).\]  
(168)

And finally,

\[49(150)+122(156)+246(165)+34(168) \rightarrow \]
\[(0,0,0,-288; 7100.6664, 7504.8942; 9481.834, 5049.5748).\]  
(169)

Dividing by 288, this last inequality becomes

\[(0,0,0,-1; 24.6551, 26.0587; 32.9231, 17.5333).\]  
(170)
Now, applying the inequalities

\[ \varepsilon_1^4 - \varepsilon_2^4 \leq \frac{1}{2} \sum_{i=1}^{4} \varepsilon_i^2, \]
\[ \varepsilon_2^4 + \varepsilon_1^4 \leq \frac{1}{2} \sum_{i=1}^{4} \varepsilon_i^2 \]

to \((A, B, C, D; E, F, G, H)\) gives

\((A, B, C, D; E + \frac{1}{2}(G + H), F + \frac{1}{2}(G + H); 0, 0)\).

The quadratic part of the last inequality is positive definite.

This inequality, applied to \((156)-(159), (162), (165), (168), (170)\) gives the eight inequalities of Table 4.

**TABLE 4**

<table>
<thead>
<tr>
<th>$\varepsilon_1$</th>
<th>$\varepsilon_2$</th>
<th>$\varepsilon_3$</th>
<th>$\varepsilon_4$</th>
<th>$\varepsilon_1^2 + \varepsilon_2^2$</th>
<th>$\varepsilon_3^2 + \varepsilon_4^2$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
<td>1.1429</td>
<td>0.9191</td>
</tr>
<tr>
<td>-1</td>
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<td>0</td>
<td>0</td>
<td>26.8126</td>
<td>27.6271</td>
</tr>
<tr>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>3.0</td>
<td>3.1</td>
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<td>0</td>
<td>0</td>
<td>20.1095</td>
<td>20.0954</td>
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<td>1</td>
<td>0</td>
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<tr>
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<td>0</td>
<td>16.2501</td>
<td>15.2501</td>
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<tr>
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<td>0</td>
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<td>0.0</td>
<td>0.5</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>-1</td>
<td>49.8833</td>
<td>51.2869</td>
</tr>
</tbody>
</table>

These above eight inequalities of Table 4 imply that

\[-\sum_{i=1}^{4} |\varepsilon_i| + 113.0555(\varepsilon_1^2 + \varepsilon_2^2) + 114.2595(\varepsilon_3^2 + \varepsilon_4^2) \geq 0,\]

that is,

\[-\sum_{i=1}^{4} (-|\varepsilon_i| + 113.0555 \varepsilon_i^2) + \sum_{i=3}^{4} (-|\varepsilon_i| + 114.2595 \varepsilon_i^2) \geq 0. \quad (171)\]

If the $i^{th}$ term of (171) were positive, then certainly

\[|\varepsilon_i| (115|\varepsilon_i| - 1) > 0,\]
forcing

\[ |\epsilon_1| > \frac{1}{115} > 0.0086. \]

But this contradicts (136). Therefore all the terms of (171) are zero, and \( \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0 \). We have therefore proved

**Lemma 7.** Let \( g(x,y) = (x-ay)(x-by) \) have roots \( a \in \mathbb{X}, b \in \mathbb{U} \) satisfying (4), (5), (61). Let \( a_2 \) and \( b_2 \) be the roots of

\[ g_2(x,y) = x^2 + (\frac{1}{5} - \frac{2}{5}i)xy - iy^2. \]

Then \( (a, b) \) must be one of the four pairs

\[ (a_2, b_2), (-b_2, -a_2), (i\overline{a_2}, -i\overline{b_2}), (-i\overline{a_2}, -i\overline{b_2}). \]  

(172)

Here \( \overline{a_2}, \overline{b_2} \) are the complex conjugates of \( a_2, b_2 \), and are therefore roots of

\[ g_2(x,y) = x^2 + (\frac{1}{5} + \frac{2}{5}i)xy + iy^2. \]

9. **The Minimum of** \( f_2(x,y). \)

As promised in Section 5, we shall now show that \( M(f_2) = \sqrt{5} \).

This result was stated by Cassels in his paper [2], and Vulakh in his paper [17].

**Lemma 8.** \( M(f_2) = M((1+2i)x^2 + xy + (2-i)y^2) = \sqrt{5} \).

**Proof:** It must be shown that \( f_2 \) cannot take any of the values

\[ i^g, (1+i)i^g, 2i^g \quad (g = 1, 2, 3, 4). \]

It is easier, however, to prove this for a form equivalent to \( f_2 \).

This, of course, does not alter the conclusion of the Lemma, since equivalent forms take the same values.

Applying first the substitution \((x,y) \rightarrow (ix+y, x)\), and then to the resulting form the further substitution \((x,y) \rightarrow (x+y, y)\), it follows that \( f_2 \) is equivalent to

\[ f_2^*(x,y) = (1-2i)x^2 - (1+2i)xy - (1-2i)y^2. \]
To simplify the proof, we shall use an identity from Vulakh [17],

\[ f_2^*(1+1)x+(2+1)y, (2+1)x+3y) = -1f_2^*(x,y). \]

It will suffice to show only that the congruences

\[
\begin{align*}
&f_2^*(x,y) \equiv -1 \pmod{3}, \\
&f_2^*(x,y) \equiv 0 \pmod{(1+1)}
\end{align*}
\]

have no non-trivial solutions.

Set \( x = x_1 + ix_2, y = y_1 + iy_2 \), and substitute into the first congruence above. This gives two real congruences, one for the real, the other for the imaginary component. Namely,

\[
\begin{align*}
&x_1^2 - x_2^2 - y_1^2 + y_2^2 + x_1x_2 - y_1y_2 - x_1y_1 + x_2y_2 - x_1y_2 - x_2y_1 \equiv -1 \pmod{3}, \\
&x_1^2 - x_2^2 - y_1^2 + y_2^2 - x_1x_2 + y_1y_2 + x_1y_1 - x_2y_2 - x_1y_2 + x_2y_1 \equiv 0 \pmod{3}.
\end{align*}
\]

Upon adding these two congruences, we get

\[
-x_1^2 + x_2^2 + y_1^2 - y_2^2 + x_1y_2 + x_2y_1 \equiv -1 \pmod{3},
\]

that is,

\[
(x_1 + y_2)^2 - (x_2 - y_1)^2 \equiv 1 \pmod{3}.
\]

Since 1 is the only non-zero square modulo 3, it follows that

\[
(x_1 + y_2)^2 \equiv 1 \pmod{3}, \quad (x_2 - y_1)^2 \equiv 0 \pmod{3},
\]

so that

\[
x_2 \equiv y_1 \pmod{3}.
\]

But now, after subtracting the above two congruences previously added, and then substituting \( x_2 \) for \( y_1 \), we get

\[
1 \equiv x_1x_2 - y_1y_2 - x_1y_1 + x_2y_2 - x_1x_2 - x_2y_2 - x_1x_2 + x_2y_2 \equiv 0 \pmod{3},
\]

which is absurd. Hence

\[
f_2^*(x,y) \equiv -1 \pmod{3},
\]

has no solution, so that

\[
f_2^*(x,y) = -1, 2
\]

and therefore also
have no solutions. Thus
\[ |f^*_2(x,y)| > 1 \text{ for all } (x,y) \neq (0,0). \]
Similarly, since \(1+2i = 1-2i = 1 \text{ (mod } (1+i))\), then
\[ f^*_2(x,y) = x^2 + xy + y^2 = 0 \text{ (mod } (1+i)). \]
Thus
\[ x = (1+i)x', \quad y = (1+i)y', \]
so that
\[ f^*_2(x,y) = 2if^*_2(x',y'). \]
But since we have shown that \( |f^*_2(x,y)| > 1 \) for non-trivial \((x,y)\), then
\[ |f^*_2(x,y)| > |2i| = 2 \text{ when } (x,y) \neq (0,0). \]
Therefore \( M(f^*_2) = |1+2i| = \sqrt{5} \), proving the lemma.

**Corollary:** \( \mu(x^*_2) = (\frac{3}{5})^{1/2} \cdot 41^{1/4}. \)

10. The Horizontal Case.

**Case IV:** \( a \in X, \ b \in Z. \) For the sake of convenience, we may assume that, after translating the roots by \( i \), that \( a \in X', \ b \in V. \)

Under the unimodular transformation \((x,y) \to (ix+y,x)\), the form \( f^*_2(x,y) \) is transformed into
\[ f'_2(x,y) = (1-2i)x^2 + (-3+2i)xy + (1+2i)y^2. \quad (173) \]
Moreover, the roots \( a'_2, \ b'_2 \) of \( f^*_2 \) are transformed into
\[ a'_2 = \frac{1}{a_2 - i}, \quad b'_2 = \frac{1}{b_2 - i} \quad (174) \]
Or, if one wishes, the roots can be computed directly from (173) to be
\[ \frac{7+14i}{10} \pm \frac{(1+2i)}{10} \cdot \frac{(15-12i)^{1/2}}{10}. \]

Hence
\[ a_2^* = -0.27231... + i(0.52343...), \quad (175) \]
\[ b_2^* = 1.67231... + i(0.27656...). \quad (176) \]

We shall show, that up to symmetry, there is no other form with roots in \( X^* \) and \( V \) satisfying (4), (5), (61).

First, we may assume that the root \( a \) lies in the rectangle \( (-\frac{1}{2}, -\frac{1}{2} + i, i, 0) \). For suppose \( a \) lies in \((-1, -1 + i, \frac{1}{2} + i, \frac{1}{2})\).

Then (61) forces \( b \) into \((1, 1 + i, \frac{3}{2} + i, \frac{3}{2})\), and the equivalence relation

\[
(a, b) \rightarrow (1+i-b, 1+i-a) \quad (177)
\]
transforms \( a \) into \((\frac{1}{2}, \frac{1}{2} + i, i, 0)\) and leaves \( b \) in \( V \).

In the same way, the symmetry

\[
(a, b) \rightarrow (1-\bar{b}, 1-\bar{a}) \quad (178)
\]
about the line \( y = \frac{1}{2} \) allows us to assume that \( a \) lies in the upper square \((\frac{-1+i}{2}, \frac{1}{2} + i, i, \frac{1}{2})\).

This square is cut into two triangles by the line \( y = x + 1 \). It shall now be shown that a root \( a \) cannot lie in the upper triangle \( \Delta^* = (\frac{-1+i}{2}, \frac{1}{2} + i, i) \).

Here we only assume that

\[ |a-b| < 2. \quad (179) \]

These calculations are shown geometrically in Figure 14.

First noting that \( i \) is closer to \( V \) than any other point of \( \Delta^* \), it follows from (179) that

\[ |i-a| \geq \frac{1}{|1-b|} \geq \frac{1}{2}. \quad (180) \]

This circle intersects the line \( y = x + 1 \) at the point

\[ x_1 + iy_1 = -\frac{\sqrt{2}}{4} + (1 - \frac{\sqrt{2}}{4})i \quad (181) \]

Further, this point (181) is the point of \( \Delta^* \) closest to \( V \) and also satisfying (180). Hence \( b \) must satisfy the inequality

\[ |b - (x_1 + iy_1)| \leq |b-a| < 2. \quad (182) \]
The point of $V$ furthest from $i$ and satisfying (182) is the intersection of (182) with the $x$-axis; the point furthest from $\frac{-1+i}{2}$ and satisfying (182) is the intersection of (182) with the line $y = 1$. The first intersection may be approximated by $1.54$, the second by $1.62+i$, since both of these approximations lie outside (182) and on the boundary of $V$. Thus

$$|1-a| \geq \frac{1}{|1-1.54|} > \frac{1}{1.837} > 0.5443, \quad (183)$$

$$\left|\frac{-1+i}{2} - a\right| \geq \frac{1}{2\left|\frac{-1+i}{2} - (1.62+i)\right|} > \frac{1}{2(2.115)} > 0.2364. \quad (184)$$

The point of $\Delta^*$ furthest from $1$ and $\frac{3+i}{2}$ and also satisfying (183), (184) is the intersection of (183) with the line $x = -\frac{1}{2}$. The point $\frac{1}{2} + \frac{4}{5}i$ lies on this line and inside (183), and hence may be taken as an approximation point. Thus

$$|1-b| \geq \frac{1}{|1-(\frac{-1+i}{2})|} = \frac{1}{1.7} > 0.5882, \quad (185)$$

$$\left|\frac{3+i}{2} - b\right| \geq \frac{1}{2\left|\frac{3+i}{2} - (\frac{-1+i}{2})\right|} > \frac{1}{2(2.023)} > 0.2471. \quad (186)$$

Similarly, the point of $\Delta^*$ furthest from $1+i$ and satisfying (183), (184) is the intersection of (183) with the line $x = -\frac{1}{2}$, that is, the point $-0.5 + 0.7364i$. Hence

$$|(1+i) - b| \geq \frac{1}{|(1+i) - (-0.5+0.7364i)|} > \frac{1}{1.523} > 0.6565. \quad (187)$$

The point of $\Delta^*$ closest to $V$ is the intersection of the circles (183) and (184). This intersection can be approximated by $-0.46 + 0.7251$, which lies inside both of the circles, and hence is closer to $V$ than the intersection.

The point of $V$ closest to $\Delta^*$ is either the intersection of (185) and (186), approximated by $1.51 + 0.271$, or the intersection of (186) and (187), approximated by $1.59 + 0.72i$. In either case,

$$|a-b| > |-0.46+0.7251| - (1.51+0.271)| > 2,$$

or
\[ |a-b| > |(-0.46+0.725i) - (1.59+0.72i)| > 2, \]

so that no form exists having roots \( a \in \Delta^*, b \in V \), and satisfying (4), (5), (61).

Hence we may assume that \( a \) lies in the lower triangle \( \Delta^* = (-\frac{1+i}{2}, i, \frac{i}{2}) \).

To limit the possible locations of the roots \( a \) and \( b \), we first get

\[
|\frac{-1+i}{2} - a| \geq \frac{1}{2|\frac{-1+i}{2} - (2+i)|} > \frac{1}{2(2.55)} > 0.1960. \tag{188}
\]

Now the point of \( \Delta^* \) satisfying (188) and furthest from any fixed point in \( V \) is the intersection of (188) with the line \( y = x+1 \). This may be approximated by \( -0.362 + 0.638i \), which lies on the diagonal and inside (188). Then we compute

\[
|2-b| > \frac{1}{2-(-0.362+0.638i)} > \frac{1}{2.447} > 0.4086, \tag{189}
\]
\[
|(2+i) - b| > \frac{1}{2.39} > 0.4184, \tag{190}
\]
\[
|1-b| > \frac{1}{1.505} > 0.6644, \tag{191}
\]
\[
|(1+i) - b| > \frac{1}{1.41} > 0.7092, \tag{192}
\]
\[
|\frac{2+3i}{5} - b| > \frac{1}{5(2.163)} > 0.0924, \tag{193}
\]
\[
|\frac{2+2i}{5} - b| > \frac{1}{5(2.176)} > 0.0919. \tag{194}
\]

Now, the point of \( V \) furthest from any fixed point of \( \Delta^* \) is the intersection of (189) with the vertical line \( x = 2 \), i.e., the point \( 2 + 0.4086i \). Hence

\[
|i-a| > \frac{1}{1 - (2+0.4086i)}| > \frac{1}{2.085} > 0.4793, \tag{195}
\]
\[
|\frac{1}{2} - a| > \frac{1}{4(2.0021)} > 0.1248. \tag{196}
\]

The point of \( \Delta^* \) satisfying (195), (196) and closest to any fixed point of \( V \) is the intersection of (195), (196). This may be approximated by \( -0.118+0.54i \), lying inside both (195), (196). Thus, from (61)

\[
|b - (-0.118 + 0.54i)| \leq |a-b| < 1.961. \tag{197}
\]
Next, the point of $V$ furthest from any fixed point of $A^{**}$ and satisfying (189)-(194), (197) is the intersection of (194) and (197). This intersection can be approximated by $1.843 + 0.485i$, which lies outside both intersecting circles. Then

$$|1-a| > \frac{1}{\frac{1}{4} - (1.843 + 0.4851)} > \frac{1}{1.914} > 0.5224, \quad (198)$$

$$\frac{1}{2} - a| > \frac{1}{4(1.8431)} > 0.1356, \quad (199)$$

$$\frac{1}{2} - a| > \frac{1}{4(2.3431)} > 0.2133, \quad (200)$$

$$\frac{1}{5} - a| > \frac{1}{5(2.047)} > 0.0977, \quad (201)$$

$$\frac{1}{5} - a| > \frac{1}{13(2.154)} > 0.0357, \quad (202)$$

$$\frac{1}{4} - a| > \frac{1}{16(2.094)} > 0.0298. \quad (203)$$

A root $a$ must therefore be restricted to one of the regions $R_1$, $R_2$, $R_3$ bounded by circles (198)-(203), the line $y = \frac{1}{2}$, and labelled in Figure 15.

The point of $A^{**}$ satisfying (198)-(203) and furthest from $V$ is either the intersection of (198) and (200), or that of (200) and the line $y = \frac{1}{2}$. The first intersection may be approximated by $-0.305 + 0.581$, lying inside both (198), (200). The second intersection is given by $-0.2867 + 0.51$. Thus

$$\frac{3+1}{2} - b| > \frac{1}{2|\frac{3+1}{2} - (-0.305+0.581)|} > \frac{1}{1.807} > 0.2767, \quad (204)$$

$$\frac{8+1}{5} - b| > \frac{1}{5|\frac{8+1}{5} - (0.305+0.581)|} > \frac{1}{5(1.918)} > 0.1042, \quad (205)$$

$$\frac{8+1}{5} - b| > \frac{1}{5|\frac{8+1}{5} - (-0.305+0.581)|} > \frac{1}{5(1.943)} > 0.1029. \quad (206)$$

A root $b$ must therefore be restricted to one of the three regions $S_1$, $S_2$, $S_3$ bounded by (189)-(194), (197), (204)-(206), and labelled in Figure 15.

It will next be shown that (61) will force $a$ and $b$ into $R_2$ and $S_3$, respectively.
Figure 15.

The Regions $R_1, R_2, R_3, S_1, S_2, S_3$. 
11. Three subcases.

**Subcase (i):** $b \in S_1$. We get, using our last approximations,

$$| (2+i) - b | > \left| \frac{1}{(2+i) - (-0.305+0.581)} \right| > \frac{1}{2.343} > 0.4268. \quad (207)$$

The point of $S_1$ furthest from $\Delta^{**}$ is the intersection of (207) with (193). This can be approximated by $1.74 + 0.67i$, which lies inside both (207) and (193). Then

$$|i-a| > \left| \frac{1}{i - (1.74+0.67i)} \right| > \frac{1}{1.771} > 0.5646. \quad (208)$$

But (208) covers both $R_1$ and $R_3$, leaving only $R_2$ as a possibility for the root $a$.

The point of $R_2$ closest to $S_1$ is the intersection of (208) and (203), which may be approximated by $-0.275 + 0.51i$, lying inside both circles. Thus $b$ satisfies

$$|b - (-0.275 + 0.51i)| < |a-b| < 1.961. \quad (209)$$

Finally, the point of $S_1$ furthest from $\Delta^{**}$ is the intersection of (209) with (204). This may be approximated by $1.676 + 0.713i$, lying inside (204) and outside (209). Then

$$|i-a| > \left| \frac{1}{i - (1.676+0.713i)} \right| > \frac{1}{1.701} > 0.5878. \quad (210)$$

But the point $-0.3 + 0.5i$ lies inside (210), inside (200), and on the line $y = \frac{1}{2}$, and therefore $R_2$ is covered by (200), (210). Therefore no form arises in this subcase satisfying (61).

**Subcase (ii):** $b \in S_2$. In this subcase, it follows from (193), (194), (204) that $\text{Re}(b) > 1.75$. Inequality (61) then forces

$$\text{Re}(a) > -0.211,$$

eliminating the possibilities $a \in R_1, R_2$. Hence we may assume that $a \in R_3$.

The point of $S_2$ closest to $R_3$ is the intersection of (204) with either (193) or (194). These intersections may be approximated by $1.77 + 0.51i$ and $1.77 + 0.49i$, respectively, these points lying outside the respective circles. So the root $a$ must satisfy
\[ |a - (1.77 + 0.511)| < |a-b| < 1.961, \]  
\[ |a - (1.77 + 0.491)| < |a-b| < 1.961. \]  

The point \(-0.1911 + 0.511\) lies outside both (211), (212), inside (201), and is further from \(S_2\) than any point of \(R_3\). Thus
\[
|\frac{3}{2} + b| > \frac{1}{2|\frac{3}{2} - (-0.1911+0.511)|} > \frac{1}{2(1.6912)} > 0.2956. \]  

The region \(S_2\) is enlarged in Figure 16.

The point 1.795 is closer to \(R_3\) than any point of \(S_2\). This is checked by noting that \(\text{Re}(b) > 1.795\) for every \(b \in S_2\), and the point of \(R_3\) closest to \(S_2\) is the intersection of (198) with the line \(y = \frac{1}{2}\). Thus \(a\) must satisfy
\[
|a - (1.795 + \frac{1}{2})| < |a-b| < 1.961. \]  

Hence the point of \(R_3\) furthest from \(S_2\) is the intersection of (214) with either the line \(y = \frac{1}{2}\) or the circle (198). In either case, we may approximate by \(-0.166 + 0.511\). Then
\[
|\frac{2+2i}{3} - b| > \frac{1}{5(1.97)} > 0.1015, \]  
\[
|\frac{2+3i}{5} - b| > \frac{1}{5(1.97)} > 0.1015, \]  
\[
|\frac{3+i}{2} - b| > \frac{1}{2(1.667)} > 0.2999. \]  

Circles (215)-(217) cover all of \(S_2\) with real component less than 1.8. This follows from the observation that the point \(1.7998 + \frac{1}{2}\) lies in all of (215)-(217). Thus the point of \(S_2\) closest to \(R_3\) is the intersection of (215) and (216), approximated by \(1.817 + \frac{1}{2}\), lying inside both circles.

Finally, the point of \(R_3\) closest to \(S_2\) is the intersection of (198) with the line \(y = \frac{1}{2}\), and this can be approximated by \(-0.151 + \frac{1}{2}\), lying inside (192) and on the line. Thus
\[
|a-b| > |(-0.151 + \frac{1}{2}) - (1.817 + \frac{1}{2})| = 1.968 > 1.961, \]  
contradicting (61).

Thus no form arises with roots \(a \in \Delta^{**}\) and \(b \in S_2\) satisfying (61).
Subcase (iii): \( b \in S_3 \). \( S_3 \) is divided into two subregions by the diagonal line \( y = -x + 2 \). Denote the subregion on the side of the origin by \( S_3^* \), the other subregion by \( S_3^{**} \).

From a previous approximation,
\[
\left| \frac{3^{1/4}}{3} - b \right| > \frac{1}{9 \left( \frac{3^{1/4}}{3} - (-0.305 + 0.581) \right)} > \frac{1}{9(1.989)} > 0.0558. \quad (218)
\]

So the point of \( S_3^{**} \) closest to \( \Delta^{**} \) is the intersection of \((218)\) with the line \( y = -x + 2 \). This may be approximated by \( 1.705 + 0.2951i \), lying inside \((218)\) and on the diagonal line. Hence if \( b \in S_3^{**} \), then the root \( a \) must satisfy
\[
|a - (1.705 + 0.2951)| < |a-b| < 1.961. \quad (219)
\]

This eliminates both of the possibilities \( a \in R_1 \cup R_2 \). For the point of \( R_1 \cup R_2 \) closest to \( S_3^{**} \) is the intersection of circles \((201)\) and \((203)\) with least real component. This intersection can be approximated by \( -0.25 + 0.525i \), lying inside both \((201),(203)\), and hence closer to \( S_3^{**} \) than the intersection. So for \( a \in R_1 \cup R_2 \), \( b \in S_3^{**} \), we have
\[
|a-b| > |(-0.25+0.5251) - (1.705+0.2951)| > 1.968.
\]

But this contradicts \((61)\), so we may assume that \( a \in R_3 \).

Now the point of \( R_3 \) furthest from \( S_3^{**} \) is the intersection of \((201)\) and \((203)\) with greatest real component. This can be approximated by \( -0.225 + 0.511i \), lying inside both \((201)\) and \((203)\). Hence
\[
|2-b| > \frac{1}{2.283} > 0.4380,
\]
\[
|\frac{3^{1/4}}{2} - b| > \frac{1}{2(1.7251)} > 0.2898.
\]

That the circles defined by these two inequalities cover \( S_3^{**} \) follows from the observation that there is a point \( 1.74 + 0.343i \) lying inside both of these two circles, and also inside \((194)\) about \( \frac{a+21}{5} \). This is shown in Figure 17.

Since there is no form with roots \( a \in \Delta^{**}, b \in S_3^{**} \), the root \( b \) must lie in \( S_3^* \).

The point of \( S_3^* \) furthest from \( \Delta^{**} \) is the intersection of the
circle (189) with the line \( y = -x + 2 \). This intersection may be approximated by \( 1.717 + 0.2831 \), lying on the diagonal line and inside the circle. Then, first, \( R_1 \) is covered by (200), (202) and the circle

\[
|1-a| \geq \left| \frac{1}{1 - (1.717 + 0.2831)} \right| > \frac{1}{1.861} > 0.5373. \tag{220}
\]

The covering follows from the observation that the point \(-0.303 + 0.571\) lies in all three circles (200), (202), (220).

Similarly, \( R_3 \) is covered by the circles (201), (203), and

\[
|a| \geq \left| \frac{1}{b} \right| > \left| \frac{1}{1.717 + 0.2831} \right| > \frac{1}{1.741} > 0.5743. \tag{221}
\]

For \(-0.225 + 0.511\) lies in all three circles.

Therefore we have proven

**Lemma 2:** Let \( g(x,y) = (x-ay)(x-by) \) satisfy (4), (5), (61) and have roots \( a \in Y', b \in V \). Then up to symmetry about the lines \( x = \frac{1}{2} \) and \( y = \frac{1}{2} \), we have \( a \in R_2, b \in S_3 \). Here \( R_2 \) is bounded by circles (200)-(203) and the line \( y = \frac{1}{2} \), and \( S_3 \) by (189), (204), (206), (218) and the line \( y = -x + 2 \).

12. Final Restriction of the Roots in the Horizontal Case.

The following calculations are described in Figures 18-21.

The point of \( R_2 \) furthest from \( S_3 \) is the intersection of circles (190) and (192). This intersection can be approximated by \(-0.29 + 0.511\), which lies inside both circles. Then

\[
|2-b| > \left| 2 - (-0.29 + 0.511) \right| > \frac{1}{2.347} > 0.4260, \tag{222}
\]

\[
\left| \frac{5+1}{3} - b \right| > \frac{1}{9(1.964)} > 0.0565. \tag{223}
\]

The point of \( S_3 \) now furthest from \( R_2 \) is the intersection of (222) and (223). The point \( 1.678 + 0.2768i \) lies inside both of these circles so that

\[
|1-a| > \left| 1 - (1.678 + 0.2768i) \right| > \frac{1}{1.827} > 0.5473, \tag{224}
\]

\[
\left| \frac{1+1}{2} - a \right| > \frac{1}{2(2.19)} > 0.2283, \tag{225}
\]
The region is enlarged in Figure 19.

Suppose \( a \in R_2 \). The point \(-0.271 + 0.52351i\) lies in both (226) and (228), and hence

\[
\Re(a) < -0.271. \tag{229}
\]

Similarly, the point \(-0.2727 + 0.52271i\) lies in both (227) and (228), so that

\[
\Re(a) > -0.2727, \quad \Im(a) > 0.5227. \tag{230}
\]

Finally, the point \(-0.272 + 0.52451i\) lies in both (226), (227), so

\[
\Im(a) < 0.5245. \tag{231}
\]

It follows from (229)-(231) that

\[
-0.2727 < \Re(a) < -0.271, \quad 0.5227 < \Im(a) < 0.5245. \tag{232}
\]

We can therefore write

\[
a = a_2^* + (s_1 + i s_2) \tag{233}
\]

where \( a_2^* \) is the root of \( f_2^* \) given by (175), the \( s_1 \) are real, and

\[
|s_1| < 0.0014, \quad |s_2| < 0.0011. \tag{234}
\]

To get similar bounds, we note that the point of \( R_2 \) now furthest from \( S_3 \) is the intersection of (227), (228). By (230), this can be approximated by \(-0.2727 + 0.52271i\). Thus

\[
|2-b| \geq \frac{1}{|2 - (0.2727 + 0.52271i)|} \geq \frac{1}{2.333} > 0.4286, \tag{235}
\]
Figure 18.
Restriction of the Root a.
Figure 19.

Enlargement of Figure 18.
The region \( S_3 \) is now bounded by circles (238), (239), (240). This is shown in Figures 20 and 21. It follows from the observation that the point \( 1.6729 + 0.2715i \) lies in all three circles (238), (239), (240), and that \( 1.685 + 0.2831 \) is contained in (240), (239), (235).

Suppose \( b \in S_3 \). The point \( 1.6715 + 0.2771 \) lies inside both (238), (239) so that

\[
\text{Re}(b) > 1.6715. \tag{241}
\]

Similarly, \( 1.6725 + 0.2771 \) lies inside both (238), (240) so that

\[
\text{Im}(b) < 0.277. \tag{242}
\]

Finally, the point \( 1.6721 + 0.2755i \) lies inside both (239), (240), so that

\[
\text{Re}(b) < 1.6725, \quad \text{Im}(b) > 0.2755. \tag{243}
\]

It follows from (241)-(243) that

\[
1.6715 < \text{Re}(b) < 1.6725, \quad 0.2755 < \text{Im}(b) < 0.277. \tag{244}
\]

We can therefore write

\[
b = b_2 + (s_3 + 184), \tag{245}
\]

where \( b_2 \) is the root of \( f_2 \) given by (176), the \( s_4 \) are real, and

\[
|s_3| < 0.001, \quad |s_4| < 0.0011. \tag{246}
\]

We have therefore proved the following
Figure 20.

Restriction of the Root b.
Figure 21,
Enlargement of Figure 20.
Lemma 10. Let \( g(x,y) = (x-ay)(x-by) \) have roots \( a \in \Lambda^* = \Lambda((-1/2,1/2)^2) \) and \( b \in \mathbb{V} \). If \( g \) satisfies (4), (5), (61), then \( g \) can be written

\[
ge^*(x,y) = x^2 - [(\frac{7}{5} + \varepsilon_1) + (\frac{4}{5} + \varepsilon_2)]xy + [(\frac{3}{5} + \varepsilon_3) + (\frac{4}{5} + \varepsilon_4)]y^2,
\]

where

\[
|\varepsilon_1| < 0.0035 \quad (i = 1,2,3,4).
\]

Proof: Dividing (173) by the leading coefficient \( l+2i \), we obtain

\[
a'_2 + b'2 = \frac{7+4i}{5}, \quad a'_2b'_2 = \frac{-3+4i}{5}.
\]

As in Lemma 6, set

\[
a'_2 = \alpha'_1 + i\alpha'_2, \quad b'_2 = \beta'_1 + i\beta'_2.
\]

Then from (233), (245), (249), we get

\[
a + b = (\frac{7}{5} + \delta_1 + \delta_3) + (\frac{4}{5} + \delta_2 + \delta_4)i,
ab = (\frac{3}{5} + \alpha_1\delta_3 + \beta_1\delta_1 - \alpha_2\delta_4 - \beta_2\delta_2 + \delta_1\delta_3 - \delta_2\delta_4) + \delta_1(\frac{4}{5} + \alpha_1\delta_4 + \alpha_2\delta_3 + \beta_1\delta_2 + \beta_2\delta_1 + \delta_1\delta_4 + \delta_2\delta_3).
\]

Upon setting

\[
\varepsilon_1 = \delta_1 + \delta_3,
\varepsilon_2 = \delta_2 + \delta_4,
\varepsilon_3 = \alpha_1\delta_3 + \beta_1\delta_1 - \alpha_2\delta_4 - \beta_2\delta_2 + \delta_1\delta_3 - \delta_2\delta_4,
\varepsilon_4 = \alpha_1\delta_4 + \beta_1\delta_2 + \alpha_2\delta_3 + \beta_2\delta_1 + \delta_1\delta_4 + \delta_2\delta_3,
\]

\( g(x,y) \) takes the required shape (247).

Finally, by using the decimal approximations (175), (176), (234), (246), it follows from the triangle inequality that

\[
|\varepsilon_1| < 0.0024, \quad |\varepsilon_2| < 0.0022, \quad |\varepsilon_3| < 0.0035, \quad |\varepsilon_4| < 0.0031.
\]

This satisfies (248), and hence proves the lemma.

13. Isolation of the Second Gap in the Horizontal Case.

As in Section 8, substitution of suitable Gaussian integers into
the inequality
\[ |g'(x,y)|^2 \geq 1 \]
will allow us to prove that in fact
\[ \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0 \]
in (247). Once again, the inequality
\[
A\epsilon_1 + B\epsilon_2 + C\epsilon_3 + D\epsilon_4 + E(\epsilon_1^2 + \epsilon_2^2) + F(\epsilon_3^2 + \epsilon_4^2) + G(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4) + H(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) \geq 0
\]
will be simply denoted by
\[(A,B,C,D; E,F; G,H).\]

The pairs of integers and resulting forms are listed below in Table 5.

<table>
<thead>
<tr>
<th>(x,y)</th>
<th>(\epsilon_1)</th>
<th>(\epsilon_2)</th>
<th>(\epsilon_3)</th>
<th>(\epsilon_4)</th>
<th>(2\epsilon_1 + 2\epsilon_2)</th>
<th>(2\epsilon_3 + 2\epsilon_4)</th>
<th>(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4)</th>
<th>(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,1)</td>
<td>0</td>
<td>0</td>
<td>-6</td>
<td>8</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(1,1)</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>(2,1)</td>
<td>-12</td>
<td>16</td>
<td>6</td>
<td>-8</td>
<td>20</td>
<td>5</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>(1,-1)</td>
<td>6</td>
<td>-8</td>
<td>-8</td>
<td>-6</td>
<td>5</td>
<td>5</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>(-1,1)</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>(1+2i,1+i)</td>
<td>10</td>
<td>-30</td>
<td>-12</td>
<td>16</td>
<td>10</td>
<td>4</td>
<td>-4</td>
<td>-12</td>
</tr>
<tr>
<td>(-2+1,3+2i)</td>
<td>70</td>
<td>40</td>
<td>112</td>
<td>-66</td>
<td>325</td>
<td>845</td>
<td>-910</td>
<td>520</td>
</tr>
<tr>
<td>(-1+5i,3i)</td>
<td>-2</td>
<td>-10</td>
<td>0</td>
<td>6</td>
<td>78</td>
<td>27</td>
<td>-90</td>
<td>-18</td>
</tr>
<tr>
<td>(-4,3+6i)</td>
<td>8</td>
<td>-16</td>
<td>-18</td>
<td>-24</td>
<td>240</td>
<td>675</td>
<td>-720</td>
<td>360</td>
</tr>
</tbody>
</table>

As before, linear combinations of (250)-(258) can be taken to single out the \(\epsilon_4\) terms.
(252)+2(253) + (0,0,-10,-20; 30,15; -20,-20).

\[ \frac{1}{2}[(259)+5(250)] + (0,0,-10,0; 0,11; -8,-8). \]

\[ \frac{1}{2}[(250)+(252)] + (-6,8,0,0; 10,5; 0,-10). \]

(251)+(254) + (3,1,0,0; 2,3; -2,0).

(261)+2(262) + (0,10,0,0; 14,11; -4,-10).

3(252)+4(257) + (-44,8,18,0; 372,123; -72,-420).

4(257)+(258) + (0,0,-18,0; 552,783; -792,0).

\[ \frac{1}{2}[(264)+(265)] + (-22,-24,0,0; 462,453; -432,-210). \]

\[ \frac{1}{4}[3(261)+(266)] + (-10,0,0,0; 123,117; -108,-60). \]

4(264)+(253) + (10,-4,0,-6; 9,13; -18,8).

5(268)+5(267)+2(263) + (0,0,0,-30; 688,672; -638,-280).

2(254)+(257) + (0,-8,4,6; 80,31; -22,-86).

3(270)+2(250) + (0,-24,0,34; 274,103; -66,258).

5(271)+12(263) + (0,0,0,170; 1368,647; -378,-1410).

8(254)+(253) + (14,0,8,-6; 13,23; -26,16).

(273)+8(260)+6(272) + (14,0,0,0; 70.88236,54.63534; -42.54118,36.96472).

\[ \frac{22}{24}(274)+(266) + (0,-24,0,0; 573.38666, 538.85566; 498.85052,268.08748). \]

4(255)+3(256) + (250,0,288,-134; 1015,2551; -2746,1512).

25(267)+13(272)+(275) + (0,0,288,0; 5168.806,5985.989; -5793.953,1099.412).

Dividing (274), (267), (263), (275), (277), (260), (272), (269),
by their leading coefficients, we get, respectively, the forms

(1,0,0,0; 5.06303,3.90253; -3.03866, -2.64034).

(-1,0,0,0; 12.3,11.7; -10.8,-6).
Using the inequalities
\[ \varepsilon_i \varepsilon_j \leq \frac{1}{2}(\varepsilon_i^2 + \varepsilon_j^2) \]
we obtain as before
\[
-|\varepsilon_1| + 20.7(\varepsilon_1^2 + \varepsilon_2^2) + 20.1(\varepsilon_3^2 + \varepsilon_4^2) \geq 0,
-|\varepsilon_2| + 39.869(\varepsilon_1^2 + \varepsilon_2^2) + 38.4303(\varepsilon_3^2 + \varepsilon_4^2) \geq 0,
-|\varepsilon_3| + 29.8264(\varepsilon_1^2 + \varepsilon_2^2) + 32.6656(\varepsilon_3^2 + \varepsilon_4^2) \geq 0,
-|\varepsilon_4| + 38.2334(\varepsilon_1^2 + \varepsilon_2^2) + 37.7(\varepsilon_3^2 + \varepsilon_4^2) \geq 0.
\]
So certainly,
\[
\sum_{i=1}^{4} (-|\varepsilon_i| + 130\varepsilon_i^2) \geq 0. \tag{278}
\]

If some term of (278) were negative, then some other term, say the \( i \)th, must be positive. But then
\[ |\varepsilon_i| > \frac{1}{130} > 0.007, \]
contradicting (248). Hence all the terms of (278) must be zero, forcing either all \( \varepsilon_i = 0 \) (\( i = 1,2,3,4 \)), or some \( \varepsilon_i = \frac{1}{130} \). But the latter choice again contradicts (248), so that
\[ \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0. \]

From Lemma 9, it then follows that the only form having roots \( a \in \Delta^{**}, b \in V \), and satisfying (4), (5), (61) is
\[ g_2'(x,y) = x^2 - \left( \frac{1+41}{2} \right) xy + \left( \frac{-3+41}{5} \right)y^2. \]

Therefore we have proved
Lemma 11. Let \( g(x,y) = (x-ay)(x-by) \) have roots \( a \in Y', b \in V \) satisfying (4), (5), (61). Let \( a'_2, b'_2 \) be roots of \( g'_2(x,y) \) defined immediately above. Then \( (a,b) \) must be one of the four pairs

\[
(a'_2, b'_2), \quad (1+i-a'_2, 1+i-b'_2), \\
(1-a'_2, 1-b'_2), \quad (1+a'_2, 1+b'_2).
\]

(279) (280)

Proof: The only allowable symmetries in the reduction are (177) and (178), which are simply products of reflections about the lines \( y = \frac{1}{2} \) and \( x = \frac{1}{2} \). If \( a \) lies in the square \( (\frac{1+1}{2}, \frac{1+2}{2}, i, \frac{1}{2}) \), then it has been shown that \( b \) lies in the square \( (\frac{3+1}{2}, \frac{1+1}{2}, \frac{1}{2}, 2) \). This gives only the four pairs \( (a,b) \) stated in the lemma, completing the proof.

14. The Cases V and VI.

Case V: \( a \in X', b \in V \). It will be shown that there is no form arising in this case.

The work is divided into three subcases.

Subcase (i): \( a \in \Delta_1 = \Delta(0,-1,-1-1), \ b \in \Delta_2 = \Delta(1,2,2+i); \)

Subcase (ii): \( a \in \Delta_3 = \Delta(0,-i,-1-1), \ b \in \Delta_2; \)

Subcase (iii): \( a \in \Delta_3, \ b \in \Delta_4 = \Delta(1,1+i,2+1). \)

The remaining possibility \( a \in \Delta_1, b \in \Delta_4 \) is equivalent to Subcase (ii) under the transformation

\[ (a,b) \rightarrow (1-b, 1-a). \]

Subcase (i): \( a \in \Delta_1, b \in \Delta_2 \). (See Figure 22).

Here, from (61), we get

\[
|a| \geq \frac{1}{|b|} \geq \frac{1}{|b-\alpha|} \geq \frac{1}{1.961} > 0.5099,
\]

(281) (281)

and by symmetry

\[
|1-b| > 0.5099.
\]

(282)
The point of \( \Lambda_1 \) closest to \( \Lambda_2 \) is the intersection of (281) with the line \( y = x \), the intersection given by \( \frac{0.5009}{\sqrt{2}}(-1-i) \). Hence

\[
|0.3605(-1-i) - b| < |a-b| < 1.961. \tag{282}
\]

The point of \( \Lambda_2 \) satisfying (282) and furthest from the origin is the intersection of (282) with the real axis. The point 1.575 lies on the axis and outside (282), so that

\[
|a| \geq \frac{1}{|b|} > \frac{1}{1.575} > 0.6349. \tag{283}
\]

By symmetry, also

\[
|1-b| > 0.6349. \tag{284}
\]

But now, the points of \( \Lambda_1 \) and \( \Lambda_2 \) with minimal distance between each other are the intersections of (283), (284) with the lines \( y=x \), \( y=x-1 \), respectively. These intersections are respectively

\[
\frac{0.6349}{\sqrt{2}}(-1-i), \quad 1 + \frac{0.6349}{\sqrt{2}}(1+i).
\]

Hence

\[
|a-b| > |(0.4489(-1-i) - (1 + 0.4489(1+i))| > 2.09,
\]

contradicting (61). Therefore no form satisfying (4), (5), (61) arises in this subcase.

**Subcase (ii):** \( a \in \Lambda_2, b \in \Lambda_2 \). (See Figure 23).

Once again, as in Subcase (i),

\[
|a| \geq \frac{1}{|b|} > \frac{1}{|b-a|} > \frac{1}{1.961} > 0.5099, \tag{285}
\]

and similarly

\[
|1-b| > 0.5099. \tag{286}
\]

The point of \( \Lambda_3 \) closest to \( \Lambda_2 \) is the intersection of (285) with the y-axis, so that

\[
\left|\frac{1}{2} - a\right| > \frac{1}{4(1.961)} > 0.1274. \tag{287}
\]

now the point of \( \Lambda_3 \) closest to \( \Lambda_2 \) is the intersection \(-0.6274i\) of (287) with the y-axis. Hence \( b \) must satisfy
\[ |b - (-0.62741)| < |b-a| < 1.961. \]  

The point of \( \Lambda_2 \) furthest from \(-i\) and satisfying (288) is the intersection of (288) with the line \( y = x-1 \). Since \( 1.565 + 0.5651 \) lies on the line and outside (288), we get

\[ \frac{1}{\sqrt{2}} > \frac{1}{2.215} > 0.4514. \]  

The two points of \( \Lambda_2 \) closest to \( \Lambda_3 \) are the intersections \( 1.5099 \) and \( 1 + \frac{0.5099}{\sqrt{2}}(1+i) \) of (286) with the \( x\)-axis and the line \( y = x-1 \), respectively. Hence \( a \) must satisfy \textbf{at least} one of the two inequalities

\[ |a - 1.5099| < |a-b| < 1.961, \quad (290) \]
\[ |a - (1 + 0.3605(1+i))| < |a-b| < 1.961. \quad (291) \]

But now the point of \( \Lambda_3 \) furthest from the point 2 is the intersection of (291) with line \( y = x \), and this can be approximated by \( 0.435(-1-i) \), lying on the line and outside (291). Hence

\[ \frac{1}{\sqrt{2} - 0.435(-1-i)} > \frac{1}{2.474} > 0.4042. \]  

Next, the point of \( \Lambda_3 \) furthest from the point 1 is the intersection of (290) with (289), approximated by \(-0.335 - 0.691\), lying outside both circles. Hence

\[ \frac{1}{\sqrt{2} - 0.335(-1-i)} > \frac{1}{1.503} > 0.6653. \]  

The point of \( \Lambda_3 \) closest to \( \Lambda_2 \) is the intersection of (287) with (285). This intersection can be approximated by \(-0.125 - 0.491\), lying inside both circles. Hence

\[ |b - (-0.125-0.491)| < |b-a| < 1.961. \quad (294) \]

Now, the point of \( \Lambda_2 \) furthest from the origin is the intersection of (294) and (292). The point \( 1.69 + 0.2591 \) lies inside (292) and outside (294), so that

\[ |a| > \sqrt{\frac{1}{1.69 + 0.2591}} > \frac{1}{1.71} > 0.5847. \]  

Similarly, using the same point of approximation,

\[ \frac{1}{\sqrt{\frac{1}{5(2.077)}}} > 0.0962. \quad (296) \]
The point $-0.125 - 0.568i$ lies inside all three circles (296), (295), (289), so that the point of $\Delta_3$ closest to $\Delta_2$ is the intersection of (295), (296). This may be approximated by $-0.25 - 0.5271$, which lies inside both circles. Hence

$$|b - (-0.25 - 0.5271)| < |a - b| < 1.961.$$  (297)

But the point of $b$ closest to $\Delta_2$ is the intersection of (293) with the line $y = x - 1$, i.e., $l + \frac{0.6653}{\sqrt{2}}(1 + i)$. But then

$$|a - b| > |[1 + \frac{0.6653}{\sqrt{2}}(1 + i)] - [-0.25 - 0.5271]| > 1.98,$$

contradicting (61). Hence no form satisfying (4), (5), (61) arises in this subcase.

Subcase (iii): $a \in \Delta_3, b \in \Delta_4$. (See Figure 24).

These two triangles are symmetric to each other under reflections about the $x$-axis and $x = \frac{1}{2}$. Hence, every inequality true for $\frac{x}{y} \in \Delta_3$ is also true for $1 - \frac{x}{y} \in \Delta_4$, and conversely.

As in the last two subcases, we begin with

$$|a| > 0.5099, \quad |1 - b| > 0.5099.$$  (298)

The point of $\Delta_4$ closest to $\Delta_3$ is the intersection $l + 0.5099i$ of (298) with the line $x = 1$, so that

$$|\frac{2+i}{2} - b| > \frac{1}{4(1.961)} > 0.1274), \quad |\frac{1}{2} - a| > 0.1274.$$  (299)

The points of $\Delta_4$ closest to $\Delta_3$ are the intersections of (299) with the lines $x = 1$ and (298), respectively. The first intersection is the point $1 + 0.6274i$, and the second can be approximated by $1.125 + 0.49i$, lying inside both (299), (298). Thus $a$ must satisfy at least one of the two inequalities

$$|a - (1 + 0.6274i)| < |a - b| < 1.961, \quad (300)$$

$$|a - (1.125 + 0.49i)| < |a - b| < 1.961.$$  (301)

Thus the point of $\Delta_3$ furthest from the point $l$ is the intersection of (300) and the line $y = x$. Since $0.561(-1 - i)$ lies outside (300), we get

$$|1 - b| > \frac{1}{|1 - 0.561(-1 - i)|} > \frac{1}{1.659} > 0.6027, \quad |a| > 0.6027.$$  (302)
Similarly, the point of \( \Lambda_3 \) furthest from \( 1+i \) is now the intersection of (300) with the line \( y = -1 \). Since \(-0.1 - 1\) lies outside (300), we get

\[
| (1+i) - b | > \frac{1}{| (1+i) - (-0.1-1) |} > \frac{1}{2.283} > 0.4380, \quad (303_1)
\]

\[
| -1-a | > 0.4380. \quad (303_2)
\]

The point of \( \Lambda_3 \) furthest from \( \frac{6+3i}{5} \) is the intersection of (300) with the line \( y = x \), and this has been approximated by \( 0.561(-1-i) \).

Hence

\[
\left| \frac{6+3i}{5} - b \right| > \frac{1}{5(2.11)} > 0.0947, \quad \left| \frac{-1-3i}{5} - a \right| > 0.0947. \quad (304)
\]

Similarly, using the the same approximation point,

\[
\left| \frac{3+i}{2} - b \right| > \frac{1}{2(2.32)} > 0.2155, \quad \left| -1-2i - a \right| > 0.2155. \quad (305)
\]

Finally, the point of \( \Lambda_4 \) closest to \( \Lambda_3 \) is the intersection of (304) with (302). The point \( 1.26 + 0.54i \) lies inside both circles, so that \( a \) must satisfy

\[
| a - (1.26 + 0.54i) | < | a - b | < 1.961. \quad (306)
\]

Thus the point of \( \Lambda_3 \) furthest from \( 1 \) is the intersection of (306) with (305). The point \(-0.33 - 0.62i\) lies inside (305) and outside (306). Hence

\[
\left| -1-b \right| > \frac{1}{1 - (0.33-0.62i)} > \frac{1}{1.47} > 0.6802, \quad |a| > 0.6802. \quad (307)
\]

The point of \( \Lambda_4 \) closest to \( \Lambda_3 \) is the intersection of (307) with (304). This can be approximated by \( 1.285 + 0.61i \), lying inside both circles. By symmetry, \(-0.285-0.61i\) is the point of \( \Lambda_3 \) closest to \( \Lambda_4 \). Therefore

\[
| a-b | > \left| (-0.285-0.61i) - (1.285+0.61i) \right| > 1.98,
\]

contradicting (61). Hence no form occurs in this case satisfying (4), (5), (61).

Actually, it can be proved that no forms satisfying (4), (5), and

\[
| a-b | < 2 \quad (308)
\]

arise in Case V.
Case VI: \( a \in X, b \in W \). (See Figure 25).

Here, we shall assume that (4), (5), (308) hold for a form \( g(x,y) = (x-ay)(x-by) \). Then from (308),

\[
\left| \frac{1}{a} \right| = \frac{1}{|b|} > \frac{1}{|a-b|} > \frac{1}{2},
\]

and by symmetry

\[
|1+i - b| > \frac{1}{2}.
\]

The points \( a \in X \) and \( b \in W \) with minimal distance are then \( \frac{1}{2} \) and \( \frac{2+3i}{2} \), so that

\[
|b-a| \geq \frac{2+3i}{2} - (\frac{1}{2}) > 1.1,
\]

contradicting (308). Therefore no forms arise in this case.

15. Summary of Results.

At this point we are able to prove the following

**Theorem 2.** Let \( g(x,y) = (x-ay)(x-by) \) be a quadratic form form satisfying the conditions

\[
M(g) = g(1,0) = 1,
\]

\[
0 < \mu(g) = \sqrt{D(g)} = \frac{|a-b|}{M(g)} < 1.961.
\]

Then \( g(x,y) \) is equivalent to a multiple of either

\[
f_1(x,y) = x^2 - 1xy + y^2,
\]

\[
f_2(x,y) = (1+2i)x^2 + xy + (2-1)y^2,
\]

or its conjugate

\[
\overline{f_2}(x,y) = (1-2i)x^2 + xy + (2+1)y^2.
\]

**Proof:** Let \( g(x,y) \) satisfy the hypotheses of the theorem. To summarize the reduction of Section 3, the use of the equivalence relations

\[
(x,y) + (x,iy), \quad (x,y) + (y,x), \quad (x,y) + (x-cy,y)
\]
Figure 25.
where $c$ is a suitably chosen Gaussian integer, it can be assumed that the root $a$ lies in the square $X = (0,-1,-1,-1)$ and the root $b$ satisfies the conditions

$$\text{Re}(b) \geq \text{Re}(a), \quad \text{Im}(b) > \text{Im}(a).$$

Next, by imposing the inequality

$$|a-b| < 1.961,$$

the root $b$ is forced into one of the nine squares labelled $X,Y,Z,Y',U,V,Z',V,W$ in Figure 1.

To this point, the reduced form is equivalent to the original.

Next, if $b$ lies in one of the three squares $Y',Z',V'$, then the unimodular substitution

$$(x,y) \to (x,-iy)$$

and subsequent conjugation transforms $(x-ay)(x-by)$ into

$$(x-ia y)(x-ib y).$$

This new form is then equivalent to the conjugate of the original form, and has one root in $X$, and the other in one of $Y,Z,V$.

So up to equivalence and conjugation, we can assume that $a \in X$, and $b \in X,Y,Z,U,V$, or $W$.

The possibilities $b \in X,Y,W$ were eliminated in Sections 4 and 12.

If $a \in Y'$, $b \in U$, which is equivalent to $a \in X$, $b \in Y$ under the translation of the roots by $i$, it was shown in Section 4 that $g(x,y)$ must be the form

$$f_1(x,y) = x^2 - ixy + y^2.$$

In this case, the conjugate form

$$\overline{f_1}(x,y) = x^2 + ixy + y^2$$

is equivalent to $f_1(x,y)$ under the substitution

$$(x,y) \to (x+iy,y).$$

Since the only other forms $g$ arising in the last two cases $b \in Z,U$ satisfy $\mu(g) = \mu(f_2) = 1.96007...$, it follows that if $g(x,y)$ satisfies
the hypotheses of the theorem and $\mu(g) < \mu(f_2)$, then $g$ is equivalent to a multiple of $f_1(x,y)$.

If $b \in U$, then by Lemma 7, the only forms are multiples of

$$f_2(x,y) = (1+2i)x^2 + xy + (2-i)y^2,$$
$$f_{21}(x,y) = (1+2i)x^2 - xy + (2+i)y^2,$$
$$f_{22}(x,y) = (1-2i)x^2 + ixy - (2+i)y^2,$$
$$f_{23}(x,y) = (1-2i)x^2 - ixy - (2+i)y^2.$$

Now $f_2$ and $f_{21}$ are equivalent, as are $f_{22}$ and $f_{23}$, under

$$(x,y) + (-x,y).$$

Also, $f_2$ is equivalent to

$$f_2^*(x,y) = (1+2i)x^2 - xy - (2-i)y^2$$

under the substitution

$$(x,y) + (-x, iy)$$

Since

$$f_{22}(x,y) = f_2^*(x,y)$$

$f_{22}$ is equivalent to $f_2^*(x,y)$.

Similarly, if $b \in Z$, then after translating the roots by $i$, giving an equivalent form, we may assume that $a \in Y'$, $b \in V$. By Lemma 10, the only forms arising are

$$f_{21}^1(x,y) = (1-2i)x^2 - (3-2i)xy + (3+4i)y^2 = (1-2i)(x-a_2^1y)(x-b_2^1y),$$
$$f_{22}^1(x,y) = (1-2i)(x - (1+i-a_2^1)y)(x - (1+i-b_2^1)y),$$
$$f_{22}^1(x,y) = (1+2i)(x - (1-a_2^1)y)(x - (1-b_2^1)y),$$
$$f_{23}^1(x,y) = (1+2i)(x - (i-a_2^1)y)(x - (i+b_2^1)y),$$

where $a_2^1$ and $b_2^1$ are the roots of $f_{21}^1(x,y)$.

Again, under the substitution

$$(x,y) + (-x+(1+i)y, y),$$

$f_2$ and $f_{21}^1$ are equivalent, as are $f_{22}^1$ and $f_{23}^1$.

Under the substitution
(x,y) → (-x+y,y),

\(f_2'\) is equivalent to

\[f_2'^*(x,y) = (x-(1-a_2')y)(x-(1-b_2')y).\]

Since

\[f_2'^{22}(x,y) = f_2'^*(x,y),\]

\(f_2'^{22}\) is equivalent to \(f_2'^{22}\).

Furthermore, \(f_2(x,y)\) and \(f_2'(x,y)\) are equivalent under \((x,y) → (ix+y,x)\).

Therefore, the set of forms satisfying the hypotheses of the theorem and the stronger condition

\[\sqrt{3} < \mu(f) < 1.961\]

are partitioned into two classes, those equivalent to a multiple of \(f_2\), and those equivalent to a multiple of \(f_2'\). This proves the theorem.

**Corollary:** The interval \((\mu(f_1), \mu(f_2))\) is the second gap of the Markoff spectrum of \(Q(i)\). Moreover, \(\mu(f_2)\) is isolated in this spectrum.

**Proof:** Let \(f\) be any form satisfying \(0 < \mu(f) < 1.961\). By Lemma 3, we may assume that \(f(x,y)\) attains its minimum at \((1,0)\), so the corollary follows directly from the theorem.
CHAPTER III

ISOLATION OF THE FORMS \( f_2(x,y) \) AND \( \overline{f}_2(x,y) \).

1. The Isolation Lemma.

To complete the proof of Theorem B, it suffices to show by Theorem 2 that if \( \mu(f) < 1.961 \), then \( f \) attains its minimum. The method is analogous to Cassels [2] in his proof of Theorem A.

Lemma 12. Let \( g(x,y) = x^2 + \alpha xy + \beta y^2 \) where

\[
\alpha = \left( \frac{1}{2} + a \right) - \left( \frac{2}{5} + b \right)i, \quad \beta = -(c + (l+d)i). \tag{309}
\]

Suppose that

\[
|g(x,y)|^2 \geq 1 - 2\varepsilon \tag{310}
\]

for all non-zero pairs of Gaussian integers \((x,y)\), and that \( \varepsilon \) is chosen small enough. Then there exist two absolute constants \( \delta_1, \delta_2 \) (independent of \( a,b,c,d,e \)) such that for

\[
X = \max(|a|, |b|, |c|, |d|),
\]

either

\[
X \leq \delta_1 \varepsilon \quad \text{or} \quad X \geq \delta_2.
\]

Proof: The substitution of the eleven pairs of Gaussian integers (140)-(151) of Table 3 into (310) gives eleven inequalities

\[
\theta_{1i}a + \theta_{12}b + \theta_{13}c + \theta_{14}d + Q_i(a,b,c,d) \geq -2\varepsilon \quad (1 \leq i \leq 11), \tag{311}
\]

where the quadratic forms \( Q_i \) and the coefficients \( \theta_{ij} \) are obtained from Table. Upon taking the linear combinations as in Section 8, four new inequalities...
\[ |a|,|b|,|c|,|d| \leq Q(a,b,c,d)+O(e) \]

are obtained, and the new quadratic forms \( Q \) are taken from Table 4 and are positive definite.

Hence there exist two absolute constants \( A_1, A_2 \) (independent of \( a,b,c,d,e \)) such that

\[ X \leq A_2 x^2 + A_1 e. \]

Thus, the quadratic formula implies that either

\[ X \leq \frac{1}{2A_2} + \frac{1}{2A_2} \sqrt{1-4A_1 A_2 e} \quad (312) \]

or

\[ X \leq \frac{1}{2A_2} - \frac{1}{2A_2} \sqrt{1-4A_1 A_2 e}, \quad (313) \]

where \( e \) must be chosen small enough, say

\[ 1 - 4A_1 A_2 e \geq 0. \]

Then since \( A_1, A_2, e \) are positive, we get successively

\[ 1 - 4A_1 A_2 e \leq 1, \]

\[ 1 - 4A_1 A_2 e \leq \sqrt{1-4A_1 A_2 e} \]

\[ \frac{1}{2A_2} = \frac{1}{2A_2} \sqrt{1-4A_1 A_2 e} \leq 2A_1 e. \quad (314) \]

Therefore, from (312)-(314) it follows that either

\[ X \geq \frac{1}{2A_2} \quad \text{or} \quad X \leq 2A_1 e. \]

This proves the lemma.

**Corollary:** Lemma 12 is also true for the conjugate form

\[ g(x,y) = x^2 + \overline{c} xy + \overline{y}^2. \]

**Proof:** The substitution of the Gaussian integers \( \overline{x}, \overline{y} \) into

\[ |\overline{g}(x,y)| \geq 1 - 2e \]

gives the same inequalities as those in the lemma, so the proofs are identical for the form \( g \) and its conjugate.
Next, let $X_0 + iY_0$ be a fixed complex number, and let $\Delta x = 0(\epsilon)$, $\Delta y = 0(\epsilon)$ be real variables. Then

$$
| (X_0 + \Delta x) + i(Y_0 + \Delta y) | = \sqrt{X_o^2 + Y_o^2} + 
$$

$$
+ [\sqrt{X_o^2 + Y_o^2 + 2X_0 \Delta x + 2Y_0 \Delta y + (\Delta x)^2 + (\Delta y)^2} - \sqrt{X_o^2 + Y_o^2}] =
$$

$$
= \sqrt{X_o^2 + Y_o^2} + \sqrt{X_o^2 + Y_o^2} \left[ \sqrt{1 + Z} - 1 \right] 
$$

(315)

where

$$
Z = \frac{2X_0 \Delta x + 2Y_0 \Delta y + (\Delta x)^2 + (\Delta y)^2}{\sqrt{X_o^2 + Y_o^2}}.
$$

(316)

Expanding $\sqrt{1 + Z}$ about $Z = 0$, we obtain

$$
\sqrt{1 + Z} = 1 + \frac{1}{2} Z - \frac{1}{8} Z^2 + \frac{1}{16} Z^3 - \ldots.
$$

Substituting (316) into this expansion, and making use of the hypothesis that both $\Delta x$, $\Delta y = 0(\epsilon)$, we get

$$
\sqrt{1 + Z} - 1 = \frac{1}{X_o^2 + Y_o^2} (X_0 \Delta x + Y_0 \Delta y) + 0(\epsilon^2).
$$

(317)

From (315), (317) follows the equation

$$
| (X_0 + \Delta x) + i(Y_0 + \Delta y) | = |X_0 + iY_0| + \frac{X_0 \Delta x}{|X_0 + iY_0|} + \frac{Y_0 \Delta y}{|X_0 + iY_0|} + 0(\epsilon^2).
$$

(318)

With this expression can be proved

Lemma 13. (The Isolation Lemma.) If $|\alpha^2 - 4\beta| \leq \frac{3}{4\sqrt{1}} (1-2\epsilon)$, $X < a_2$, $\epsilon < \epsilon_0$, where $\epsilon_0$ does not depend on $a, b, c, d$, then

$$
\alpha = \frac{1-2\epsilon}{5}, \quad \beta = -1, \quad \epsilon = 0.
$$

Proof: From (309),

$$
|\alpha^2 - 4\beta| = |(X_0 + \Delta x) + i(Y_0 + \Delta y)|
$$

where

$$
X_0 = -\frac{3}{25}, \quad Y_0 = \frac{96}{25},
$$

$$
x = \frac{2}{5}a - \frac{4}{5}b + 4c + a^2 - b^2,
$$

$$
y = -\frac{4}{5}a - \frac{2}{5}b + 4d - 2ab.
$$
Further, $X < \varepsilon_2$ implies from Lemma 12 that all of $a, b, c, d = 0(\varepsilon)$.

Thus, from (318),

$$\frac{3}{5}\sqrt{41}(1-2\varepsilon) \geq \left| x^2 - 4\beta \right| = \frac{3}{5}\sqrt{41} + \frac{1}{5\sqrt{41}}(\frac{2}{5}a - \frac{1}{5}b + 4c) +$$

$$+ \frac{32}{5\sqrt{41}} \left( -\frac{4}{5}a - \frac{2}{5}b + 4d \right) + o(\varepsilon^2). \quad (319)$$

It will be shown below that by taking proper linear combinations of the inequalities (311),

$$- \frac{2}{5}a + \frac{4}{5}b - 4c \geq Q(a,b,c,d) - 5\varepsilon, \quad (320)$$

$$- \frac{4}{5}a - \frac{2}{5}b + 4d \geq Q - 3.8\varepsilon. \quad (321)$$

Since $Q = 0(\varepsilon^2)$, it will then follow from (319), (320), (321) that

$$(1-2\varepsilon)\frac{3}{5}\sqrt{41} \geq \frac{3}{5}\sqrt{41} + \frac{1}{5\sqrt{41}}(Q-5\varepsilon) + \frac{32}{5\sqrt{41}}(Q-3.8\varepsilon) + o(\varepsilon^2) =$$

$$= \frac{3}{5}\sqrt{41} - \frac{1}{\sqrt{41}} \varepsilon - \frac{24}{41} \varepsilon + o(\varepsilon^2)$$

$$\geq \frac{3}{5}\sqrt{41} - 4\varepsilon + o(\varepsilon^2).$$

Thus

$$(\frac{6}{5}\sqrt{41} - 4)\varepsilon + o(\varepsilon^2) < 0.$$
1. Linear combination

Now the linear combination
\[ \frac{2}{11} (322) + 4(323) + 2(324) + 31(325) + 4(326) + 12(327) \]
gives
\[ \frac{2}{5}a + \frac{4}{5}b - 4c \geq Q - 4.64e, \]
certainly giving (320).

Similarly, the linear combination
\[ \frac{1}{25} (76(322) + 15(324) + 4(325) + 6(326)) \]
gives
\[ -\frac{4}{5}a - \frac{2}{5}b + 4d \geq Q - 3.68e, \]
certainly giving (321), and proving the lemma.

Corollary: If \(|\alpha^2 - 4\beta| \leq \frac{3}{5} \sqrt{41} (1-2e)\), \(X < \delta_2\), \(e < \epsilon_0\), where \(\epsilon_0\)
does not depend on \(a, b, c, d\), then
\[ \alpha = \frac{1+24}{5}, \quad \beta = 1, \quad \epsilon = 0. \]

2. Proof of the Main Theorem.

Proof of Theorem B: Let \(f(x, y)\) be any form satisfying the inequality
\[ \sqrt{3} < \mu(f) > 1.961. \]

By Lemma 3 and by multiplication by a scalar, there is a form \(f^*\) with
\[ M(f^*) = f(1,0) = 1, \quad \mu(f^*) = \mu(f). \]

Hence by Theorem 2,
\[ \mu(f) = \mu(f^*) = \mu(f_2) = (\frac{3}{5})^{1/2} (1^{1/4}). \]

After multiplication by a scalar, we may assume that \(M(f) = 1.\)
Hence to each $e'$, $0 < e' < \frac{1}{2}$, there are Gaussian integers $x_0, y_0$ with
\[ f(x_0, y_0) = (1-2e)^{-1/2}, \quad 0 < e < e'. \]
By Lemma 2, after a unimodular transformation and multiplication by a complex unit, we may assume further that
\[ f(1,0) = (1-2e)^{-1/2}, \]
and that the coefficient of the $xy$ term is bounded by $\frac{\sqrt{2}}{\sqrt{1-2e}}$. Then
\[ f_{e}(x,y) = \sqrt{1-2e} f(x,y) \]
has minimum and discriminant
\[ M(f_{e}) = \sqrt{1-2e}, \quad D(f_{e}) = (1-2e)D(f), \quad (328) \]
respectively. Furthermore $f_{e}(1,0) = 1$.

Let $f_{e}(x,y)$, $e' \to 0$, be a sequence of such forms. By applying the reduction procedure of Lemma 3, we may assume, by taking a subsequence if necessary, that this sequence converges to a form
\[ F(x,y) = x^2 + Rxy + Sy^2 \]
satisfying the conditions
\[ M(F) = F(1,0) = 1, \quad |D(F)| = \frac{3}{2\sqrt{11}}. \]

But by Theorem 2, $F$ must be equivalent to a multiple of either $f_2(x,y)$ or $\bar{f}_2(x,y)$. So after the proper equivalence transformations and multiplications we may assume that the $f_{e}$ converge to either
\[ g_2(x,y) = (1+2i)^{-1}f_2(x,y) \quad \text{or} \quad \bar{g}_2(x,y). \]
Hence the $f_{e}$ can be written either as
\[ f_{e}(x,y) = x^2 + [(\frac{1}{2} + a) - (\frac{2}{5} + b)i]xy - [c + (1+d)i]y^2 \]
or $\bar{f}_e$. By the convergence of the sequence $f_{e}$, there exists a constant $\varepsilon > 0$ such that for all $e' < \varepsilon$,
\[ x = \max(|a|, |b|, |c|, |d|) < \delta_2. \]
But then Lemma 13 and its corollary imply that $e = 0$. Therefore
\[ f_{e}(x,y) = g_2(x,y) \quad \text{or} \quad \bar{g}_2(x,y), \]
proving Theorem B.
REFERENCES


