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DISSERTATION
Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Robert William Betts, M.S., B.A.A.E.

The Ohio State University
1973

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FIELDS OF STUDY

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Studies in Real Gas Flows. Professor Stuart L. Petrie

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<td>$C_A$</td>
<td>axial force coefficient, $C_A = C_{AF} + C_{AP}$</td>
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<td>axial force coefficient due to skin friction, $C_{AF} = \frac{F_{AXF}}{q_\infty S_b}$</td>
</tr>
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<td>$C_{AP}$</td>
<td>axial force coefficient due to surface pressure, $C_{AP} = \frac{F_{AXP}}{q_\infty S_b}$</td>
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<tr>
<td>$C_F$</td>
<td>local skin friction coefficient, $C_F = \frac{C_F}{\rho_\infty u_\infty^2}$</td>
</tr>
<tr>
<td>$C_N$</td>
<td>normal force coefficient, $C_N = \frac{F_N}{q_\infty S_b}$</td>
</tr>
<tr>
<td>$C_P$</td>
<td>local pressure coefficient, $\frac{P - P_\infty}{\rho_\infty}$; specific heat at constant pressure</td>
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<td>$C_{po}, C_{pw}$</td>
<td>local pressure coefficients, windward and leeward sides respectively</td>
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<td>$C_M$</td>
<td>pitching moment coefficient, $C_M = \frac{P_M}{q_\infty S_b L}$</td>
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<td>$F$</td>
<td>dimensionless variable gas composition parameter, $F = F'/F_{AIR}$</td>
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<td>$F'$</td>
<td>variable gas composition parameter (see Equation 33)</td>
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<td>$F_{AXF}$</td>
<td>axial force due to skin friction</td>
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<td>$F_{AXP}$</td>
<td>axial force due to surface pressure</td>
</tr>
<tr>
<td>$F_N$</td>
<td>normal force</td>
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<tr>
<td>$G$</td>
<td>dimensionless variable gas composition parameter, $G = G'/G_{AIR}'$</td>
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I
L
L_s
m
M_∞
N
P
PM
q_∞
R_i R_l
Re_{L_∞}
Re_χ
Re_{χ^*_c}
Re_θ
S_b
T
\overline{T}
u
χ
x
x^*_c

see Equations 39 and 44
cone height
cone slant height
mass flow rate
mach number
exponent in the pressure law
pressure
pitching moment
free-stream dynamic pressure
specific gas constant of the mixture or species i respectively
free-stream Reynolds number based on the cone height
local Reynolds number based on χ
local Reynolds number based on χ^*_c
local Reynolds number based on the local momentum thickness
cone base area
temperature
Eckert reference temperature
χ component of velocity
distance from cone tip measured along the surface
stretched χ coordinate; see Equation 20
\( \alpha \) angle of attack
\( \beta \) argon mole fraction
\( \gamma \) ratio of specific heats
\( \delta^* \) displacement thickness
\( \theta \) momentum thickness or effective local surface angle of a yawed cone \( \theta = \theta_c + \alpha \cos \phi + \frac{\delta^*}{\delta^*} \)
\( \bar{\theta} \) \( \bar{\theta} = \theta_c + \alpha \cos \phi \)
\( \mu \) viscosity
\( \epsilon \) non-dimensional axial coordinate, \( \xi = \xi/L_s \)
\( \rho \) density
\( \tau \) skin friction
\( \phi \) circumferential angle measured from the windward cone generator

**Subscripts**

AIR evaluated for air properties
\( b \) cone base
\( c \) cone conditions
\( e \) boundary layer edge properties
\( i \) species \( i \)
\( o \) stagnation condition
\( w \) wall
\( \infty \) free stream

**Superscript**

\(^-\) evaluated at the Eckert reference temperature
I. INTRODUCTION

The results presented herein represent one phase of an extensive similitude study of slender bodies in low density non-equilibrium flows. The overall study is involved with the acquisition and correlation of heat transfer and pressure data for flat plates and sharp cones, as well as force data for sharp cones. The body flow fields encompassed observable regions of both weak and strong viscous interaction. The type of data correlation desired would allow accurate, yet simple comparison of heat transfer, pressure, and force data from a variety of test facilities. Such an undertaking is by no means new, but if successful, it would provide a common ground for comparison of a wide source of information.

Hayes and Probstein (Reference 1) established the foundation for the fundamental viscous interaction parameters in usage today. For some years since this original work a search has been underway to define more general similarity parameters which would correlate experimental data from a variety of test facilities. Although several empirical forms
of similarity parameters have been defined (Reference 2) all seem, in one way or another, to maintain the original form and restrictions of Hayes and Probstein.

An essential ingredient in extending the applicable range of similarity parameters is the proper choice of variables from which the parameters are constructed. For example, when attempting to correlate non-equilibrium data, Mach number is difficult to define. The analytical expression for the speed of sound is quite complex when the internal energy modes of a gas have different characteristic temperatures. On the other hand, there are variables such as free stream pressure, which are less sensitive to non-equilibrium effects. However, the flow field may be desirably sensitive to these variables at supersonic speeds but undesirably insensitive to these variables at hypersonic speeds. It was this type of reasoning that guided the present study. It was essential to maintain the salient features of viscous interaction on hypersonic slender bodies, but at the same time maintain a simplicity of form which would cast the similarity parameters in terms of easily obtainable, yet meaningful flow properties. In addition, it was the goal of this study to identify similarity parameters whose ease of usage would be common to free-flight tests as well as perfect gas and non-equilibrium test facilities.
The specific purpose of the phase of the study presented here was to perform both experimental and theoretical investigations of the forces and moments on slender bodies. The flow field was restricted to low density with variable ratio of specific heats flows for which the presence of viscous interaction was not negligible. The range of the variation of the ratio of specific heats, $\gamma$, was specifically chosen to duplicate the $\gamma$-variation of air undergoing complete oxygen dissociation. The results of this study are to be used as a foundation for future tests of the static stability of cones in an arc-heated wind tunnel. This particular wind tunnel characteristically has chemically frozen flow with a constant $\gamma$ downstream of the nozzle throat.

The predicted aerodynamic loads encountered by a 1 inch base diameter sharp cone subject to low density viscous interaction are of the order of ounces. In order to obtain the necessary measurement accuracy under these conditions, test and calibration equipment must be chosen to accommodate the low load levels. With this specific purpose in mind, a balance was developed and a rather unique calibration test rig was designed.

The test model was a sharp $10^\circ$ half-angle cone. To circumvent the mechanical difficulties associated with water cooling, a "quick dip" injection technique was used. The force model was "quick dipped" into the flow field for
time sufficiently long to obtain stable data, but sufficiently short to prevent a large temperature rise.
II. TEST PROGRAM

A. EQUIPMENT

1. Facility

The static stability tests of the cone were performed in the twelve-inch hypersonic wind tunnel at the Aeronautical and Astronautical Research Laboratory using discrete combinations of air and argon. The mole fraction range of argon used was chosen to simulate that of 0% to 100% oxygen dissociation in air. The wind tunnel used has a Mach number range of 6 to 14 which is achieved by the proper matching of several interchangeable throat sections and supersonic contoured nozzle sections. The high pressure air is supplied by two four-stage compressors and stored in two cylindrical tanks of 1500 ft³ total volume. Maximum operational pressure of the tanks is 2650 psia. To retard the onset of liquefaction, the air is dried to less than one part water vapor per million parts of air by a silica-gel dryer before entering the storage tanks. Air from the tanks flows to a manifold where the argon is introduced. This partially diffused mixture then enters the
downstream end of the heater through several radially positioned ports. The heater is enclosed in a thick walled high pressure cylindrical tank which serves as the stagnation chamber. The high pressure-electric resistance element heater is capable of dissipating 650 KW of power at air temperatures up to 2500°R at stagnation pressures up to 2600 psia. The tunnel normally operates with an open-jet, approximately three nozzle diameters in length. The proper diffuser back-pressure is maintained by a three-stage vacuum pumping system. A rather complete description of the facility can be found in References 3 and 4.

Preliminary test indicates the existence of thermal loads to the uncooled balance due to flow spillage during the start up and shut down phase of the tunnel operation. To alleviate this problem a constant area diffuser extension was placed between the nozzle and the diffuser. The only cut-outs provided in the extension were for the pitot pressure probe and the force model. In addition, the model was encased in a removable shroud.

No problems were encountered in the use of air-argon mixtures in the twelve-inch tunnel. In fact, the basic design of the heater makes it such an ideal mixing chamber that premixing of gases is not required. The heater is composed of 5" x 5" elements formed by looping 0.114-inch diameter ferro-chromium wire (Haskins alloy 875) over aluminum rods. The elements are then stacked in an alternating
pattern of vertically and horizontally looped elements. The entire array is then insulated along its length at the outer surface and placed within the stagnation chamber. An air gap is provided between the heater and the inner walls of the chamber. Cold gases entering the heater flow over the outer surface of the heater absorbing any heat which penetrates the insulator and at the same time cool the inner walls of the stagnation chamber. Upon reaching the upstream end of the heater the gases enter the heater proper and then pass through the maze of criss-crossed elements. It is this network of grids which supports the mixing process which extends over the full 6-foot length of the heater.

Two separate control systems were provided for the air and argon. In both cases the control valves operated in a choked configuration and the total pressure upstream of the control valve was maintained at an essentially constant value. Due to the large volume of air stored and the small mass flow rate of air required for tunnel operating \( (\dot{m} \sim 5 \times 10^{-3} \text{ slugs/sec}) \), once the air control valve is set manually the mass flow rate remains constant. On the other hand, because the available bottled argon volume was much smaller than that of the air, a bottle pressure drop of 700 psia is typical. In order to maintain a constant total pressure upstream of the argon control valve a Tescom pressure regulator was placed between the source of argon
and the control valve. To obtain the proper mixture ratio the following method was used.

The tunnel was first brought up to operating stagnation temperature on air at a reduced pressure. Leaving the air control valve fixed, the stagnation temperature was set to automatic control and a sufficient amount of argon was added to increase the total pressure to the desired operating level. The mole fraction of argon introduced could then be determined as a function of the reduced air pressure and the final operating pressure. At the conclusion of each test, the argon was turned off and the resulting air pressure level was compared to its initial value. At no time was any discernible change noticed.

2. The Models

a. The Cone-Balance Model

The cone-balance model was a sharp 10° half-angle cone with a 1 inch base diameter. As shown in Figure 1, the model consisted of three detachable sections: a removable sharp-conical nose tip which screwed into the balance locking taper; a truncated conical section which is locked in position by the combined effect of the balance taper, and the nose tip; and an adjustable locking collar which is screwed onto the sting, aft of the instrumented portion of the balance, and held in position by a lock ring. All three sections are constructed of Armco 17-4PH stainless steel.
The locking collar is designed with an external taper which matched the aft internal taper of the truncated conical section. When properly mounted, only the rear cylindrical portion of the collar remained external to the conical body. The overall appearance was that of a sharp cone with a short $3/4''$ diameter by $29/32''$ long cylindrical base section followed by a $3/8''$ diameter sting (Figure 1).

The locking taper served two purposes: first, it protected the strain gage bridges from the direct effect of air circulation in the base region, and secondly, it acted as a stop to deflections in the axial and normal force directions in case of balance overloads. An air gap, approximately 0.001 inches wide, was maintained between the locking collar and the cone during testing.

b. The Base Pressure Model

Due to the lack of water-cooling, the base pressure of the force model could not be obtained simultaneously with the force measurements without sustaining damage to the balance. Therefore, in order to account for the effects of base pressure on the measured axial force, a water-cooled brass model was constructed which was identical to the force model in external features. Since it was the pressure in the cavity between the locking collar and the conical body that acted on the base of the cone, particular care was taken to make the cavities of the two models equivalent. The base pressure was measured with a 0.03 psia full-scale
Pace Engineering Company variable reluctance pressure transducer.

c. Model Insertion

Due to the commonality of design of the external features of the force and base pressure models, the same insertion mechanism was used for both. The model was injected vertically downward into the flow field from the top of the test cabin. Angle of attack was obtained by rotation about the support strut which corresponded to yawing in the horizontal plane. This particular configuration was chosen to insure that the model's weight would not interact with the balance as an angle of attack dependent tare load. In addition, the balance was structurally strongest to the insertion process for this configuration.

The angle of attack was determined by an angular position potentiometer mounted through a system of gears to the support strut. The potentiometer was calibrated using the tunnel centerline as the zero angle of attack position. The indicated angle of attack accuracy was $\pm 0.05^\circ$. The true wind angle of attack was determined from the force data.

3. The Balance

The static stability tests were performed with an uncooled, three-component strain gage balance (Figures 2 and 3) particularly suited for internal mounting within small slender bodies. The balance and sting are of one-piece
construction approximately 4 inches long. The force sensitive portion of the balance is approximately 1 1/2 inches long and 1/2 inch in diameter at the base. The pitch sections, which form the forward part of the balance, are machined from a slender rod which, in turn, can be positioned deep within the conical body. The axial section, which has a diameter nearly 2 1/2 times that of the pitch sections, is easily mounted toward the base of the cone. The cone and balance were designed as an internal unit specifically to permit the pitch sections to be mounted such that the expected center of pressure was midway between the two pitch sections. This configuration was particularly desirable since it maximized the difference in moments sensed at these two bridge locations, and at the same time gave a low design maximum pitching moment, permitting the pitch sections to be designed for maximum sensitivity.

The design load limit of the balance was one pound axial force, one pound normal force, and one half inch-pound pitching moment where the reference point for the pitching moment was taken to be midway between the two pitch sections. It should be noted that for the tests, the cone tip was used as the moment reference point.

The balance design is a modification of the Naval Ordnance Laboratory water-cooled three-component strain gage balance described in References 5 and 6. The basic modifications were the elimination of the water cooling system and
several minor changes to make the balance compatible with the test plan.

The balance was constructed of 7075-T6 Alcoa aluminum. The choice of aluminum over a comparably high quality stainless steel was prompted primarily by two factors. The maximum stress under the strain gages was limited to 1/10 the material yield strength. With this design criteria, the critical areas of the sensing flexures for 7075-T6 aluminum were approximately four times as thick as those of Armco 17-4PH stainless steel. By choosing aluminum as the structural material, the construction was possible with conventional tooling methods. Second, because the balance was uncooled, it was also necessary to use a material which would minimize the temperature gradients in the balance due to the aerodynamic heating load. Non-uniform heating leads to thermally induced strain, an effect which is difficult to separate from the strain due to the applied aerodynamic loads. The use of aluminum minimized the thermal gradients.

Interaction between balance bridges is always a serious problem to be considered in the design of balances. The modified NOL balance design results in a very low level of interaction (see Section II-B-"Calibration Results"). Each pitch section has three main structural members. Two of these members are eccentrically loaded columns to which four strain gages are mounted and wired to form a four arm active bridge. The third member is a much thicker flexure
whose mass is concentrated about the neutral axis of the pitch section. In this manner, the moment of inertia of the thicker flexure has a negligible effect in resisting applied pitching loads to the instrumented flexures. On the other hand, this third member is very resistant to pure axial loads. Thus, the interaction of axial force on pitching moment is minimized. The axial section also has three main structural members, but only the center one is instrumented. Because of the orientation of the pitch sections with respect to the axial section, pitching loads result in a twisting of the axial flexures as opposed to bending which results from an axial load. Due to the presence of the three flexures, the center flexure lies at the center of rotation, thus the twisting load is carried predominantly by the fore and aft dummy flexures. For axial loading, because the dummy flexures are 75% as thick as the instrumented flexure, only 54% of the applied moment due to axial force is carried by the center flexure.

Micro-Measurement 1/32" by 1/32" foil gages, trimmed to remove all excess foil, were used for all three bridges. They have a nominal gage factor of 2.1 and an internal resistance of 120 ohms. The gages were bounded to the balance with Eastman 910 cement and cured at 250° for 12 hours. This particular bonding method insured the integrity of the cement for loads up to twice the design loads. After the curing process was completed a thin
silicone rubber coating was applied to each bridge to protect against moisture. The lead tab configuration of the strain gages was chosen to prevent any direct interference of the tabs with the flexure deflection. The pitch section strain gage tabs were positioned across the flexure, perpendicular to the bending axis. The axial section strain gage tabs were located in the non-deflecting fillet at the base of the instrumented flexure. 39 gage lead wire was used between the bridges and the terminal strips. The lead wire was carefully routed to avoid any direct interference with the deflection of the instrumented flexures. The combined thickness of the foil gage, silicone coating, and cement (approximately 0.001") was negligible in comparison to the thickness of the flexures (0.016" minimum); thus, hysteresis effects were found to be negligible.

The balance temperature was monitored by a copper-constantan thermocouple which was attached with Duco cement to the forward portion of the axial section on a non-deflecting surface.

4. The Calibration Test Rig and Calibration Procedure

Since balance calibration is by no means a new area of engineering interest, it is not at all surprising to find several rather standardized calibration techniques currently in existence. Nearly all of these techniques, however, employ a system of pulleys and cables to load the balance as it would be loaded during a wind tunnel test.
The force of a hanging calibration weight is redirected by the pulley and cable system to the proper loading direction and position on the balance. As the calibration weight becomes smaller, a point is reached where the internal friction of the pulleys is no longer negligible. It is this particular problem which led to the development of the rather unique calibration test rig used in this study. Although designed with the specific intent of calibrating low load balances, in principle, the basic features of the design are by no means restricted to low load levels.

The unique feature of the test rig is the employment of a pendulum to apply horizontal loads in both the normal and axial directions. This loading configuration corresponds to the loading orientation of the balance during the wind tunnel testing. The calibration test rig is shown schematically in Figures 4 and 5. A photograph of the entire system is shown in Figure 6.

An optical system composed of two mirrors mounted on a vertical stand in front of the table and a cathetometer (a highly accurate short focal length-small field of view transit) located behind the table is used to align the calibration test rig. The cathetometer, which has a cross-hair eyepiece, is also leveled and positioned such that the two reference points on the table are aligned with the vertical cross-hair. The two reference points correspond to a scribed cross at the front of the table and the second to
a needle mounted at the pivot point of the cone-balance angle of attack assembly. The cone tip is rotated above the second reference point until it is aligned with the vertical cross-hair. Thus, the cone tip and the two reference points are co-linear. Because the two reference points have been chosen to lie along the mid-span line of the test rig, when the cone tip is aligned with the reference points the balance model axis is co-linear with the mid-span line.

For normal force and pitching moment loading, the pivot knife-edge (Figures 7 and 8) on which the pendulum loader rests, is brought into the field of view of the transit. The knife-edge is rotated until its edge lies along the same vertical as the two reference points and the cone tip (Figure 8). The groove in the pendulum loader and the smaller loading knife-edge (Figure 7) have been constructed to be perpendicular; thus, in this configuration, the loading knife-edge is oriented 90° to the balance-model axis. The pendulum loader is then placed over the pivot knife-edge and balanced to a leveled position using the balancing weight. A bubble-level is mounted atop the pendulum to determine the leveled position. The pendulum loader and pivot knife-edge assembly are mounted in a double track whose motion has two degrees of freedom. The track can move both parallel to and perpendicular to the mid-span line of the test rig which is co-linear with the cone-model axis to
a preselected location for the normal force and pitching moment loading. The assembly is then positioned so as to come in contact with the model surface without causing a load to be applied to the balance (Figure 9). To determine when contact is made the loader and model are electrically insulated from one another, thus an ohm meter can be used to determine when contact is made. By moving the loading weight from its balanced position, a normal force and pitching moment are applied to the balance. The balance is then rotated until the outputs of the two pitch bridges are a maximum, which corresponds to the load acting directly through these two bridges. Having established the proper balance orientation, the range of calibration loads is applied by preselected incremental movements of the loading weight. The loader is then unloaded (returned to its balanced state) and then moved to the next axial location where the loading process is repeated.

For axial force loading, the pivot knife-edge is rotated 90° from the normal force loading position. Next, the loading knife-edge is removed and the pendulum loader is re-balanced. The pendulum loader and pivot knife-edge assembly are positioned so that the flat side surface of the pendulum loader just makes contact with the cone tip (Figures 10 and 11). Again, the full range of calibration loads are applied by preselected incremental movements of the loading weight.
The results of the calibration are presented in Section II-B.

B. BALANCE CALIBRATION RESULTS

For normal force and pitching moment loading, the balance was calibrated at three axial positions with the cone nose as the reference point for pitching moment. The positions corresponded approximately to the electrical centers of the fore pitch section and the aft pitch section, and a location midway between the two electrical centers (Figure 1). The data scatter for repeated application for a fixed load at a fixed axial location amounted to ± 1% of the full-scale bridge outputs. An averaging technique was used to resolve the data scatter. Each loading was repeated five times and the highest and lowest outputs for each bridge were discarded. A least squares linear fit of the retained outputs versus the loading weight was then obtained. The linear fits for each axial location are shown in Figures 12 through 14. Although the axial force loading data scatter was considerably less than that for normal force loading, the same averaging technique was used. The axial force loading results are presented in Figure 15.

The data scatter is attributed to a very slight freedom of movement which exists between the pendulum loader groove and the pivot knife-edge, which allows angular as well as axial variation in the positioning of the loading
knife-edge against the model's surface. The axial variation in the positioning of the loading knife-edge is less than \( \pm 0.002^\circ \) and the angular variation is less than \( \pm 0.15^\circ \). Due to the extreme sensitivity of the balance, even these slight misalignments are detectable.

The calibration results presented in Figures 12 through 15 appropriately illustrate the low degree of interaction which exists among the three bridges.

In order to obtain the interaction constants, which relate the applied loads as functions of the bridge outputs, the method outlined in Appendix I was used in conjunction with the linear least squares fit of the calibration data. As a check of the accuracy of the resulting interaction constants, the bridge output data was substituted into interaction equations and the resulting calculated applied loads were compared with the applied calibration loads. The maximum absolute error spread was less than 1.21%.

As a further check of the accuracy of the results of Appendix I, the interaction equations were programmed for the OSU-AARL analog computer which is used for on-line data reduction during wind tunnel operation. The bridge outputs constituted the input information for the analog computer and normal force, the pitching moment, and the axial force outputs were displayed on plotters. The balance was loaded in three different configurations: (1)
at a fixed axial position with a linearly varying normal force; (2) at varying axial locations with a fixed normal force and a linearly varying pitching moment; and (3) at the cone tip with a linearly varying axial load. In each case the displayed outputs varied according to the applied loads with the unloaded force or moment displaying an essentially zero value. Some scatter was again noted and, as before, it was substantiated that it was due to slight misalignments between the pendulum loader and the pivot knife-edge. Unlike the calibration data, the scatter in this case was slightly higher. The maximum deviation however was less than 2%.

The electrical centers of the two pitch sections were determined from the interaction constants. The calculated electrical center of the fore pitch section is 1.651 ± 0.001 inches aft of the cone tip as compared with the geometric electrical center of 1.543 inches. The calculated electrical center of the aft pitch sections is 2.217 ± 0.001 inches aft of the cone tip as compared with the geometric electrical center of 2.247 inches. The slight differences between the calculated and the geometric electrical centers were due primarily to slight imperfections in the construction of the balance, to errors in the position of mounting the strain gages on the balance, and to the error inherent in the method of Appendix I.
III. THEORETICAL ANALYSIS

A. GENERAL DISCUSSION

Only a limited quantity of experimental data is available for hypersonic viscous interaction of yawed cones. The disparity of test conditions and models precludes any decisive data correlation. However, from reliable high Reynolds number-hypersonic flow data, some rather general and indicative conclusions can be drawn as to expected viscous interaction effects.

For $\alpha < 0.75^\circ$ the circumferential pressure distribution is independent of Reynolds number (References 7, 8, and 9). Including $\delta^*$ effects, the effective cross-sectional cone shape remains essentially circular (Reference 7). As $\alpha$ is increased, a pressure minimum occurs on the leeward side which causes a leeward side "hump" in the effective cross-sectional shape. The "hump" formation is coupled with the cross-flow of low energy boundary layer fluid to the leeward side. This phenomena is not separation in the
two-dimensional sense, because the fluid doesn't detach from the surface until a much higher angle of attack. The "hump" spreads radially and circumferentially with increasing angle of attack until it eventually assumes a fixed area of influence approximately 30° (Reference 7) to 50° (Reference 10) either side of the leeward side. The resulting effect on the pressure distribution is the development of a pressure plateau on the leeward side which spreads circumferentially in a manner which conforms to the "hump" development.

The circumferential pressure coefficient distribution, for angles of attack less than approximately 75% of the cone half-angle, is identical in shape to the theoretical distribution derived from Newtonian theory. The absolute pressure coefficient level, however, is not predicted by Newtonian theory (References 7 and 10). Frieberg (Reference 11) showed that excellent agreement between theory and experiment could be obtained by matching the Newtonian form for the pressure coefficients at the windward and leeward meridian to the corresponding experimental pressure coefficients.

The theoretical model for a yawed cone subject to viscous interaction used in this study incorporates the above features. First a closed-form expression for the axial pressure distribution and skin friction distribution
about an unyawed sharp cone are obtained by an approxima-
tion to the results of Harney (References 12 and 13). Harney obtained iterative expressions for the skin 
friction and pressure distribution based on the assump-
tions of hypersonic small perturbation theory as applied 
to a reference property momentum integral development. The unyawed results are then perturbed for angle of attack 
under the assumptions of tangent-cone theory to give an 
expression for the axial variation of the pressure 
coefficients on the windward and leeward sides of the cone. The angular variation in pressure coefficient at each 
axial location is obtained by matching the above mentioned 
pressure coefficients to the Newtonian expression for the 
circumferential pressure coefficient. The same technique 
is applied to obtain the circumferential skin friction 
distribution since the theoretical skin friction variation 
is shown to be pressure dependent.

B. CIRCUMFERENTIAL PRESSURE DISTRIBUTION

Based on the coordinate system shown in Figure 
16, the Newtonian expression for the local pressure coef-
ficient for a yawed cone can be written as

\[ C_p = 2 \left( \sin \phi \cos \alpha + \cos \beta \sin \alpha \cos \phi \right)^2 \]  

(1)
The pressure coefficients for the windward (\( \phi = 0 \)) and 
leeward (\( \phi = \pi \)) sides are
Defining
\[ a = \sin \theta \cos \alpha \]
and \[ b = \cos \theta \sin \alpha \]

Equation (2) and (3) can be solved for \( a \) and \( b \) as functions of \( C_{\text{po}} \) and \( C_{\text{pn}} \). Substituting \( a \) and \( b \) in Equation (1) gives

\[ C_p = \frac{1}{2} \left[ \left( \frac{C_{\text{po}}}{a} \right)^2 (1 + \cos \gamma) + \left( \frac{C_{\text{pn}}}{a} \right)^2 (1 - \cos \gamma) \right] \]  \hspace{1cm} (4)

This is the modified form for the local pressure coefficient which is used to define the force and moment coefficients.

C. FORCE AND MOMENT COEFFICIENTS

In terms of the nomenclature of Figure 16, the expressions for the force and moment coefficients are stated below. The cone base area is used as the reference area and the cone slant length is used as the reference length.

The normal force coefficient:

\[ C_N = \frac{F_N}{g \infty S_b} = \frac{1}{2 \tan \theta} \int_{\theta=0}^{\theta=1} \left[ \frac{C_{\text{po}}}{a} - \frac{C_{\text{pn}}}{a} \right] \xi d\xi \]  \hspace{1cm} (5)
The pitch moment coefficient:

\[ C_{PM} = \frac{P_M}{q_\infty S_b L_s} = \frac{1}{\sin \alpha} \int_{\phi=0}^{\phi=1} \left[ \frac{C_{P_0}}{\frac{a}{2}} - \frac{C_{Pn}}{\frac{a}{2}} \right] \xi^2 d\xi \]  

(6)

The axial coefficient due to pressure:

\[ C_{AP} = \frac{F_{AxP}}{q_\infty S_b} = \int_{\phi=0}^{\phi=1} \left[ \frac{3}{4} C_{P_0} + \frac{3}{4} C_{Pn} + \left( \frac{C_{eg}}{\frac{a}{2}} \right)^{\frac{1}{2}} \left( \frac{C_{Pn}}{\frac{a}{2}} \right)^{\frac{1}{2}} \right] \xi^2 d\xi \]  

(7)

The axial coefficient due to skin friction:

\[ C_{AF} = \frac{F_{AxF}}{q_\infty S_b} = \frac{1}{\pi \tan \alpha} \int_{\phi=0}^{\phi=1} 2 \left[ \int_{\phi=0}^{\phi=1} \frac{\tau_{wz}}{q_\infty} d\phi \right] \xi^2 d\xi \]  

(8)

D. AXIAL PRESSURE DISTRIBUTION

1. Inviscid Flow Field

The inviscid flow field is assumed to be described by the usual hypersonic small disturbance theory for slender bodies:

\[ M_\infty >> 1 \]

\[ \sin \theta << 1 \]  

(9)

\[ M_\infty \sin \theta > 1 \]
where the angle $\theta$ is the effective local surface angle for a yawed cone as defined by

$$\theta = \theta_c + \frac{d}{dx} + \alpha \cos \phi$$  \hspace{1cm} (10)

The cone leading edge is assumed to be of sufficient sharpness to preclude any bluntness effects.

The inviscid cone surface pressure is defined by an empirically obtained expression. The form of the cone surface pressure equation was chosen to be identical in form to that for a sharp wedge obtained by matching exact oblique shock pressure to the linearized pressure equation of Linnel (Reference 14). The coefficients for the cone surface pressure equation were matched to the pressure results of exact Taylor-Maccoll theory over the range $1 < M \sin \theta < \infty$ and $1.2 \leq \phi \leq 1.6$ with an error less than 1%. The wedge and cone pressures are given by Harney (Reference 13) as

**Wedge**

$$\frac{P}{\rho_{\infty}} = 1.015 (1 + \phi) \sin^2 \theta + \frac{2.286 \phi + 1}{\phi + 1} \frac{P_0}{\rho_{\infty}}$$  \hspace{1cm} (11)

**Cone**

$$\frac{P}{\rho_{\infty}} = 1.8 (1 + 0.12 \phi) \sin^2 \theta + \frac{1.37 \phi + 1}{\phi + 1} \frac{P_0}{\rho_{\infty}}$$  \hspace{1cm} (12)

Note that the $\phi'$-dependency of the cone surface pressure appears explicitly in Equation (12).
2. Viscous Flow Field

Equation (10) provides the coupling between the inviscid flow field and the boundary layer displacement effect which characterizes viscous interaction. To determine the axial variation of \( \frac{d\delta}{dx} \), the momentum integral representation of the boundary layer equations is introduced. Since the factors of prime interest to the determination of the force and moment coefficients are the displacement thickness and the wall shear stress rather than heat transfer effects, a reference temperature is defined to account for the effects of energy dissipation within the boundary layer. The no-slip condition is imposed at the cone surface. With the exception of displacement thickness, all second and higher order effects such as the entropy layer, longitudinal and transverse curvature, and vorticity interaction are neglected. It is further assumed that the pressure gradient term which appears in the momentum integral equation can be neglected on the basis that the local effect of the pressure gradient is small. On the other hand, the integrated effect of the pressure gradient on the flow within the boundary layer cannot be neglected. A pressure dependent transformed axial coordinate is introduced to account for the effects of the pressure history on the boundary layer development.
With the above assumptions, the compressible momentum integral equation for an unyawed cone is

$$\frac{d \Theta}{d \chi} + \frac{\Theta}{\chi} = \frac{\tau_u}{\rho_e u_e^2} = \frac{C_f}{2} \tag{13}$$

Under the assumptions that \( \frac{d \rho}{d \chi} \approx 0 \) and that \( \rho \mu = \bar{\rho} \bar{\mu} = \) constant, the Mangler transformation can be used to relate the wall shear stresses of a cone and a flat plate.

$$\frac{\tau_u}{\rho_e u_e^2} = \frac{\sqrt{3} \, \frac{\tau_u}{\rho_e u_e^2}}{\frac{\rho_e u_e^2}{\mu_e}} = \frac{0.332 \sqrt{3}}{\sqrt{\frac{\rho_e u_e^2}{\mu_e}}} \tag{14}$$

Therefore,

$$\frac{d \Theta}{d \chi} + \frac{\Theta}{\chi} = \frac{0.332 \sqrt{3}}{\sqrt{Re_x}} \tag{15}$$

Solving Equation (15) for \( \Theta \) with the boundary condition that \( \Theta = 0 \) at \( x = 0 \) results in

$$\Theta = \frac{\pi}{\sqrt{3}} \left( \frac{0.332 \sqrt{x}}{\sqrt{\frac{\rho_e u_e^2}{\mu_e}}} \right) \tag{16}$$

Equation (15) can now be rewritten in terms of a Reynolds number based on the momentum thickness.
Noting that \( \frac{d\theta}{dx} + \frac{\theta}{x} = \frac{1}{x} \frac{d(\theta x)}{dx} \), Equation (17) can be integrated to obtain

\[
\theta x = 0.664 \left( \int_0^x \frac{x^2 dx}{\frac{\rho_0 u_c}{\mu_0}} \right)^{\frac{1}{2}}
\]

It is at this point that the integrated effect of the pressure history on the boundary layer is accounted for by relaxing the assumption of zero pressure gradient. The local pressure, which is contained implicitly in the integrand of Equation (18), is now allowed to vary with axial position. It is noted that

\[
\frac{\rho_0 u_c}{\mu_0} = \frac{\rho_0}{RT_0} \left( \frac{u_c}{\mu_0} \right) = \frac{\rho_0}{g_{\infty}} \left( \frac{1}{2} \rho_\infty u_\infty^2 \right) \frac{u_c}{\mu_0 RT_0}
\]

Under hypersonic small disturbance theory \( u_\infty \approx u_c \) so that

\[
\frac{\rho_0 u_c}{\mu_0} \approx \left( \frac{1}{2} \rho_\infty \right) \frac{u_c^3}{R T_\infty} \approx \frac{u_c^3}{\mu_0 T_\infty}
\]

Making use of the high temperature approximation to the Sutherland viscosity law, \( \mu_0 \propto T_\infty^{-\frac{1}{2}} \), we obtain
\[
\frac{u_e^3}{\mu_e T_e} \propto \frac{u_e^3}{T_e^{\frac{3}{2}}} = \left( \frac{u_e}{T_e} \right)^{\frac{3}{2}}
\]

From the energy equation

\[C_p T_0 = C_p T_e + \frac{1}{2} u_e^2\]

the form

\[\frac{T_0}{u_e} = \frac{T_0}{u_e} - \frac{1}{2C_p} \propto \frac{T_0}{u_e} - \frac{1}{2C_p} = \text{constant}\]

is obtained which results in

\[\frac{\rho a u_e}{\mu_e} \propto C \rho e\]

where \(C\) is a constant. (19)

Substituting Equation (19) into Equation (18) gives

\[\theta = 0.044\left[ \frac{1}{\frac{\rho a u_e}{\mu_e}} \frac{\rho e}{\kappa} \int_0^\kappa \frac{\kappa^2}{\rho e} d\kappa \right]^{\frac{1}{2}}\]

Now, the transformed axial coordinate is defined as

\[\kappa_c = \frac{\rho e}{\kappa^2} \int_0^\kappa \frac{\kappa^2}{\rho e} d\kappa\] (20)
Thus, the momentum thickness and momentum integral equation in the transformed coordinates become

\[
\theta = \frac{(0.664)^* \kappa^*}{\sqrt{\frac{\rho_a u_a \kappa^*}{\mu_a}}}
\]

and

\[
\frac{de}{dx} = \frac{0.332}{\sqrt{\frac{\rho_a u_a \kappa^*}{\mu_a}}} \left( 1 - \frac{2 \kappa^*}{\kappa} \right)
\]

Rather than \( \frac{de}{dx} \), it is \( \frac{df^*}{dx} \) which is of importance here. For hypersonic boundary layers about slender bodies, Vaglio-Laurin (Reference 15) makes note of the existence of an inner and outer viscous layer. The flow within the inner layer, which is the thicker of the two layers, resembles a constant density-linear velocity gradient couette type flow. The outer layer defines a region of rapid flow property changes, with strong normal gradients in both density and velocity. In order to establish an approximate relationship between the momentum thickness and the displacement thickness it is assumed that the boundary layer can be approximated by a linear velocity profile and a constant density equal to the reference property density, (the assumed boundary layer is like the inner layer of Reference 15). Substituting the approximate velocity and density profiles into the integral
representation of $\delta^*$ and $\theta$ yields

$$\frac{\delta^*}{\theta} = \left[ 2 \frac{T}{T_a} - 1 \right] (\delta^*)_{\text{incompressible}} \quad (23)$$

Again the Mangler transformation can be applied, this time for incompressible flow.

$$\left( \frac{\delta^*}{\theta} \right)_{\text{cone}} = \left( \frac{\delta^*}{\theta} \right)_{\text{flat plate}} = 2.61 \quad (24)$$

For hypersonic flow $\frac{2T}{T_a} >> 1$, Equation (23) becomes

$$\frac{\delta^*}{\theta} \approx 5.22 \left( \frac{T}{T_a} \right) \quad (25)$$

For hypersonic small disturbance flow the right-hand side of Equation (25) is constant. Thus, the expression for the rate of growth of the displacement thickness is

$$\frac{d\delta^*}{dx} = 5.22 \frac{T}{T_a} \frac{(0.332)}{\sqrt{\frac{\rho u_a \kappa}{\mu_e}}} \left( 1 - \frac{2 \kappa_e}{\kappa} \right) \quad (26)$$

The Reynolds number can be written in terms of reference properties. Recalling that $\rho A = \bar{\rho} \bar{A}$
\[
\frac{\rho e u_e}{\mu e} = \rho e \left( \frac{u_e}{\rho e \mu e} \right) = \rho e \left( \frac{u_e}{\rho \mu} \right) = \frac{\rho e u_e}{\rho \mu} \\
\frac{\rho e u_e}{\mu e} = \left( \frac{T}{T_0} \right)^{\alpha} \left( \frac{\rho u_e}{\mu} \right)
\]

Therefore,

\[
\frac{d \delta^*}{d x} = \frac{1.73}{\sqrt{Re_{\kappa}}} \left( 1 - 2 \frac{\kappa_e^*}{\kappa} \right) \quad (27)
\]

The boundary layer displacement effect, as indicated in Equation (27), is controlled by the local reference property Reynolds number. At the same time the Reynolds number is implicitly dependent upon the local pressure coefficient, \( \rho e/\gamma \). In order to display the pressure influence explicitly, the reference property Reynolds number is rewritten as

\[
Re_{\kappa_e^*} = \left( \frac{\rho}{\gamma} \right) \left( \frac{u_e}{\sqrt{\frac{\mu}{\rho}}} \right) (\rho \gamma \kappa_e^*) \quad (28)
\]

The Eckert reference temperature is defined as

\[
\overline{T} = 0.045 \frac{u_e^2}{C_p} + 0.5 (T_e + T_w) \quad (29)
\]

The Sutherland viscosity relationship for the reference temperature \( \overline{T} \),
\[ \sqrt{\bar{\mathcal{M}}} = 2.27 \times 10^{-8} \sqrt{\frac{1}{T}} \left[ \frac{1}{1 + \frac{200}{T}} \right] \]

is approximated by

\[ \sqrt{\bar{\mathcal{M}}} \approx 2.27 \times 10^{-8} \sqrt{\frac{1}{T}} \]

where the hypersonic approximation that \( \frac{200}{T} \ll 1 \) has been imposed. Making the additional hypersonic small disturbance assumption that \( \mathbf{u}_a^a = \mathbf{u}_b^a \approx 2 \mathbf{k} = 2 c_p T_0 \)

for the velocity term in Equation (29) and then substituting into Equation (28) gives

\[ \frac{\bar{R} e \kappa_c}{(R e_\infty \kappa_c)} = \left( \frac{R e}{R e_\infty} \right) \left( \frac{1}{2 R} \right) \left[ 2.27 \times 10^{-8} \left( \frac{0.095}{c_p} \right)^{3/2} \left( 1 + 2 \varepsilon \frac{T_0 + T_w}{T_0} \right)^{3/2} \right]^{-1} \left( \rho_\infty \kappa_c^* \right) \]  

(31)

For the typical test conditions of this study, \( T_e \approx 100^\circ R \) and \( T_w \approx 570^\circ R \). Thus, assuming that \( T_e \) is negligible compared to \( T_w \), the final form for the reference property Reynolds number becomes

\[ \frac{\bar{R} e \kappa_c}{(R e_\infty \kappa_c)} = \left( \frac{R e}{R e_\infty} \right) \left( \frac{1}{2 R} \right) \left[ 2.27 \times 10^{-8} \left( \frac{0.095}{c_p} \right)^{3/2} \left( 1 + 2 \varepsilon \frac{T_w}{T_0} \right)^{3/2} \right]^{-1} \left( \rho_\infty \kappa_c^* \right) \]

(32)
To expand the application of Equation (32) to gases other than air, a non-dimensional parameter, $F$, is introduced which contains all of the gas dependent thermodynamic and transport properties. Hence,

$$ F = \frac{F'}{F'_{\text{AIR}}} \quad \text{where} \quad F' = \left[ \frac{R \mu}{C_{p} A} \right]^\frac{1}{2} $$

Thus, the final form for the displacement thickness becomes

$$ \frac{dS}{d\chi} \sqrt{\frac{p_{e}}{q_{\infty}}} = 3.84 \times 10^{-6} F \left( 1 + 2 \frac{\mu}{\eta_{0}} \right)^{\frac{3}{4}} \left( \rho_{\infty} \chi_{e}^* \right)^{-\frac{1}{2}} \left( 1 - \frac{2 \chi_{e}^*}{\chi} \right) $$

The constants of Equation (34) correspond to the units feet, second, slug, and °R. Equations (34), (20) and (12) constitute the set of equations needed to completely define the pressure field. Once the function $\chi_{e}^* = \chi_{e}^*(\chi, \rho(\chi))$ is known, the exact solution of $p_{e}/q_{\infty}$ for small $\theta$ reduces to the solution of a quartic equation in $(dS/d\chi)^{\frac{1}{2}}$.

The procedure for obtaining $\chi_{e}^*$ is contained in Appendix II.

E. AXIAL PRESSURE DISTRIBUTION—APPROXIMATE SOLUTION

For small values of $\theta$, which characterizes slender bodies at low angles of attack, Equation (12) for the cone surface pressure can be rewritten as
\[ \frac{\tau_e}{\tau_{\infty}} = 1.8 \left[1 + 0.12 \beta^r \left( \bar{\alpha} + \frac{d \bar{f}^*}{d x} \right)^2 \right] + \frac{1.37 \beta^r + 1}{\beta^r + 1} \frac{P_{\infty}}{\tau_{\infty}} \]  

(35)

where \( \bar{\alpha} = \alpha + \alpha \cos \gamma \)

The inviscid surface pressure for a yawed cone at high Reynolds number (i.e. \( \frac{d \bar{f}^*}{d x} \ll \bar{\alpha} \)) is given by

\[ \left( \frac{\tau_e}{\tau_{\infty}} \right)_{xNV} = 1.8 \left[1 + 0.12 \beta^r \right] (\bar{\alpha})^2 + \frac{1.37 \beta^r + 1}{\beta^r + 1} \frac{P_{\infty}}{\tau_{\infty}} \]  

(36)

Equation (35) thus becomes

\[ \left( \frac{\tau_e}{\tau_{\infty}} \right) = 1.8 \left[1 + 0.12 \beta^r \right] \left( 2 \bar{\alpha} \frac{d \bar{f}^*}{d x} + \left( \frac{d \bar{f}^*}{d x} \right)^2 \right) + \left( \frac{\tau_e}{\tau_{\infty}} \right)_{xNV} \]  

(37)

A second and somewhat weaker approximation is required for a closed-form solution of Equation (34). For \( M_\infty \to \infty \) the second term in Equation (35) becomes negligible, thus

\[ \frac{\tau_e}{\tau_{\infty}} \approx 1.8 \left(1 + 0.12 \beta^r \right) \left( \bar{\alpha} + \frac{d \bar{f}^*}{d x} \right)^2 \]  

(38)

Substituting this approximation into Equation (34)

\[ \frac{d \bar{f}^*}{d x} \left( \bar{\alpha} + \frac{d \bar{f}^*}{d x} \right) = \frac{3.84 \times 10^{-6} \mathcal{F}}{\left[1.8(1 + 0.12 \beta^r) \right]^{\frac{1}{2}}} \left[1 + 2.6 \frac{T_{\infty}}{T_0} \right]^{\frac{3}{2}} (\rho_{\infty} x_c^*)^{-\frac{1}{2}} \] 

\[ \cdot \left(1 - 2 \frac{x_c^*}{x} \right) = 0 \]  

(39)
Solving the quadric in \( \frac{d\ell^*}{dx} \) with the boundary condition that \( \frac{d\ell^*}{dx} \to 0 \) as \( \kappa \to \infty \) gives

\[
\frac{d\ell^*}{dx} = \frac{\theta}{2} \left( \sqrt{1 + \frac{4 \tau}{\theta x^2}} - 1 \right)
\]  

(40)

Substituting this into Equation (37) yields the approximate closed-form solution for the cone surface pressure.

\[
\left( \frac{P_c}{q_{\infty}} \right) = \left( \frac{P_c}{q_{\infty}} \right)_{\text{INV}} + 1.8 (1 + 0.12 \delta) \left( \theta - \frac{\theta^2}{2} + \frac{\theta^3}{3} \right) \left( \sqrt{1 + \frac{4 \tau}{\theta x^2}} \right)
\]  

(41)

In order to apply the above result directly to the determination of the force and moment coefficients \( \kappa_c \) must be determined.

It has been observed by several authors (References 1 and 16) that the induced pressure varies axially according to a power law in \( \kappa \). That is

\[
P_c \propto \kappa^{-N}
\]  

(42)

where \( N = 0 \) characterizes weak viscous interaction and \( N = 1/2 \) characterizes strong interaction. Substituting this assumed pressure form into Equation (20) for \( \kappa_c^\star \) yields

\[
\frac{\kappa_c^\star}{\kappa} = \frac{1}{N + 3}
\]  

(43)
From an exact solution for \( \kappa^* \), as proposed in Section D-2, an average \( N \) can be defined which characterizes the entire flow field. Substituting the approximate form for \( \kappa^*/\kappa \) into the expression for \( I \) results in

\[
I = \frac{N+1}{\sqrt{N+3}} \frac{3.84 \times 10^{-6}}{[1.8(1+0.12\#)]^{1/2}} F \left( 1 + 2.0 \frac{T_w}{T_0} \right)^{3/4} \left( \rho_{\infty} \kappa \right)^{-1/2} \quad (44)
\]

Thus, Equations (41) and (44) constitute an approximate solution for the pressure distribution over a slightly yawed cone.

F. SKIN FRICTION DISTRIBUTION

The skin friction distribution can be obtained from the results of Section III-D. The momentum integral equation is

\[
\frac{d\theta}{d\kappa} + \frac{\theta}{\kappa} = \frac{\tau_w}{\rho_0 u_e^2} \quad (45)
\]

Substituting Equations (21) and (22) into Equation (17)

\[
\frac{\tau_w}{\rho_0 u_e^2} = \frac{0.332}{\sqrt{Re \kappa^*_e}} = \frac{\overline{P}}{\rho_0} \frac{0.332}{\sqrt{Re \kappa^*_e}} \quad (46)
\]

or

\[
\frac{\tau_w}{\rho_{\infty} u_e^2} = \frac{\overline{P}}{\rho_{\infty}} \frac{0.332}{\sqrt{Re \kappa^*_e}} \quad (47)
\]

Making the hypersonic approximation that \( u_e \approx u_{\infty} \)

Equation (47) becomes
The local reference property Reynolds number of Equation (31) is now substituted into Equation (48) giving

\[
\frac{\tau w}{\rho_\infty u_\infty^2} \simeq \frac{\overline{\rho}}{\rho_\infty} \frac{0.332}{\sqrt{Re \kappa_c^2}}
\]  

(48)

The term \( \overline{\rho}/\rho_\infty \) can be rewritten as

\[
\frac{\overline{\rho}}{\rho_\infty} = \frac{\rho_e}{R \overline{\rho}} \left( \frac{1}{\rho_\infty} \right) = \left( \frac{\rho_e}{\rho_\infty} \right) \left( \frac{g_\infty}{R \overline{T} \rho_\infty} \right)
\]  

(50)

Again making use of the hypersonic small disturbance approximation \( u_e \simeq u_\infty \), the reference temperature of Equation becomes

\[
\overline{T} \simeq \frac{0.095 u_\infty^2}{C_P} \left[ 1 + 2.6 \left( \frac{T_e + T_w}{T_0} \right) \right]
\]  

(51)

Substituting Equations (43), (50) and (51) into Equation (49)
\[
\frac{\left( \frac{\tau_{\infty}}{q_{\infty}} \right)}{\sqrt{\frac{P_c}{q_{\infty}}}} = 2.64 \times 10^{-5} G \left( 1 + 2 \frac{\alpha}{T_e} \right)^{-\frac{3}{4}} \left( 1 + 2 \frac{\alpha (T_e + T_{in})}{T_c} \right) \cdot \left( \rho_{\infty} \frac{\alpha}{T_c} \right)^{-\frac{1}{2}} (N + 3)^{\frac{1}{2}}
\]

where 
\[G = \frac{G'}{G_{AIR}} \quad \text{and} \quad G' = \left( \frac{\mu C}{R} \right) \frac{1}{\alpha} \]

(52)

The non-dimension factor \( G \), like \( F \), extends the usage of Equation (52) to other gases.

If it is now assumed that \( T_e \) is much less than \( T_{in} \), the final form for the skin friction can be written as

\[
\frac{\left( \frac{\tau_{\infty}}{q_{\infty}} \right)}{\sqrt{\frac{P_c}{q_{\infty}}}} = 2.64 \times 10^{-5} G \left( 1 + 2 \frac{\alpha}{T_e} \right)^{-\frac{1}{4}} \left( \rho_{\infty} \frac{\alpha}{T_c} \right)^{-\frac{1}{2}} (N + 3)^{\frac{1}{2}}
\]

(53)

G. AXIAL FORCE COEFFICIENT DUE TO SKIN FRICTION

The axial force coefficient due to skin friction is defined as

\[
C_{AF} = \frac{F_{AF}}{q_{\infty} S_b} = \frac{1}{\pi \tan \theta_c} \int_{\xi = 0}^{\xi = 1} \left[ 2 \int_{\varphi = 0}^{\varphi = \pi} \frac{\tau_{\varphi \xi}}{q_{\infty}} \, d \varphi \right] \xi \, d \xi
\]

(54)

The modified Newtonian expression for the circumferential pressure coefficient is

\[
C_p = \frac{1}{2} \left[ \left( \frac{C_{p_m}}{2} \right)^{\frac{1}{2}} (1 + \cos \varphi) + \left( \frac{C_{p_m}}{2} \right)^{\frac{1}{2}} (1 - \cos \varphi) \right]^2
\]

Making the assumption that \( P_c/q_{\infty} \gg P_{in}/q_{\infty} \)
Substituting Equations (53) and (55) into Equation (54) gives

\[
\left( \frac{P_e}{Q_{\infty}} \right)^{\frac{1}{2}} \approx (C_p)^{\frac{1}{2}} = \frac{1}{\sqrt{\beta}} \left[ \left( \frac{C_{p_0}}{\sqrt{\beta}} \right)^{\frac{1}{2}} (1 + \cos \rho) + \left( \frac{C_{p_n}}{\sqrt{\beta}} \right)^{\frac{1}{2}} (1 - \cos \rho) \right]
\]  

(55)

H. THERMODYNAMIC AND TRANSPORT PROPERTIES

The mixture ratio of argon to air is given in terms of the atomic mole fraction, \( \beta \), where \( \beta \) is obtained from the absolute reduced and final pressures in the stagnation chamber as described in Section II. Since both gases possess perfect gas behavior for the specified wind tunnel operating conditions, \( \beta \) can be determined on a basis of a mixture of atomic and diatomic perfect gases. Hence,

\[
\beta' = \frac{3.5 - \beta}{2.5 - \beta}
\]  

(57)
Likewise, the specific heat at constant pressure is given by

$$C_P = R(3.5 - \beta)$$ (58)

where $R$ is the specific gas constant for the mixture which is related to the individual species specific gas constant by

$$R = \sum \frac{P_i}{\rho} R_i$$ (59)

The viscosity temperature relationship of the individual gases was assumed to follow that of Sutherland. Standard temperature and pressure values of the pure species viscosity were used as reference values for the Sutherland relation. The viscosity of the mixture was calculated using the semi-empirical equation of Wilke (Reference 16).

For an in depth treatment of the thermodynamic and transport properties of air-argon mixtures see Reference 13.
A summary of the wind tunnel test conditions for the force and base pressure models is presented in Table 1. In order to apply the theoretical analysis of Section III, a value of $N$ which is representative of the entire cone flow field must be obtained from the exact solution for $X^*$. A typical set of wind tunnel operating conditions for air were used to obtain the axial distribution of $\chi_c^*/\chi$ and the results are shown in Figure 17. Neglecting the immediate neighborhood of the cone tip, the $\chi_c^*/\chi$ distribution is closely representative of weak viscous interaction ($\chi_c^*/\chi + 1/3$) as opposed to strong viscous interaction ($\chi_c^*/\chi + 2/7$). Hence, referring to Equation 43, which relates $\chi_c^*/\chi$ and $N$, a value of $N = 0$ was selected for the theoretical computations.

The theoretical analysis of Section III assumed a cone base pressure equal to the free-stream pressure. However, the wind tunnel force model may be subject to quite a different value of base pressure. Since the effect of the base pressure is to augment the measured axial force,
In order to directly compare the experimental force data with the theoretical predictions, all experimental axial force data were referenced to a base pressure equal to the free-stream static pressure. Base pressure corrections were limited to $\beta = 0$ and $\beta = 0.121$ test data (Figure 18) due to the large volume of argon required at the higher $\beta$ values to maintain constant test conditions during the base pressure measurements.

For the test conditions of Table 1, the application of the Modified Tangent Cone (MTC) theory is limited to a maximum angle of attack of approximately $5^\circ$ by the hypersonic small disturbance restriction that $M_\infty \sin \theta$ be greater than or equal to 1. Figures 19 through 24 present a comparison of experimental data and MTC theory for each value of $\beta$. Figures 20 and 21 contain data obtained from consecutive injections at the same nominal angle of attack. The error of repeatability is less than 0.5%. MTC theory overpredicts the linear variation of $C_N$ and $C_M$ versus angle of attack by a constant error of approximately 18% and 13%, respectively, over the range of mole fraction and angle of attack investigated. On the other hand, the error in the prediction of $C_A$ increases with angle of attack. As shown in Figures 19 and 20, for angles of attack less than $5^\circ$, MTC theory underpredicts the base pressure corrected data by approximately 6%.
increases the axial force coefficient by 28%. The increase in the zero angle of attack axial force coefficient is predicted within 0.8% by MTC theory (with a base pressure correction being applied to the MTC predicted value).

The variation of the transport property functions $F$ and $G$ with mole fraction, $\beta$, is shown in Figure 26. The effect of gas property variation over the range of mole fraction considered causes a 15% variation in $F$, which is directly coupled to the pressure distribution, and a 4% variation in $G$ which, on the other hand, is directly coupled to the skin friction distribution. A summary of all of the experimental data is presented in Figures 27 through 29. The net effect of the transport property change on the static stability coefficients presented in Figures 27 through 29 is seen to be negligible. In addition, experimental normal force and pitching moment data of Harrington and Wilkinson (Reference 17) for both a 6.5° and a 9.0° sharp cone for $\beta = 0$ are also included in Figures 27 and 28. As can be seen the agreement with the present data is excellent. Thus, due to the insensitivity to transport property changes, static stability data for perfect diatomic gases could be directly applied to frozen non-equilibrium flows with weak interaction without any need of a detailed knowledge of the flow structure.

As previously stated, MTC theory leads to a more accurate prediction of $C_A$ than Newtonian theory. However,
Newtonian theory has been a long accepted standard for static stability coefficients at high Reynolds numbers. A comparison of MTC and Newtonian theory for a stagnation pressure of 74.3 psia and a mole fraction of 0. is given in Figure 24 which shows that Newtonian theory does indeed predict \( C_N \) and \( C_M \) more accurately than MTC. The resulting error is approximately 6% and 4% for \( C_N \) and \( C_M \), respectively. However, a comparison of the two theories (Figures 24 and 25) for the inviscid axial pressure coefficient for zero angle of attack shows an error of 41% for Newtonian theory whereas the error for MTC is less than 1%. Thus, although improvement over MTC theory can be made in the prediction of \( C_N \) and \( C_M \) through the use of Newtonian theory, it is accomplished at the sacrifice of an increased error in the predicted value of \( C_A \). Since the main purpose of this study is the investigation of the effects of viscous interaction on static stability, the small inaccuracy sacrificed in the predicted values of \( C_N \) and \( C_M \) by use of MTC theory is accepted in order to account for the predominant influence of viscous interaction on \( C_A \).

To determine the effect of Reynolds number on the axial force coefficient and the ability of MTC to predict this effect, a comparison was made of the \( p_o = 74.3 \) psia and \( p_o = 39.3 \) psia experimental data at a stagnation temperature of 1860°R at \( \phi = 0 \). The results shown in Figure 25, indicate that approximately halving the Reynolds number
the form of $C_A$ as obtained from MTC theory is rather cumbersome to apply as a similarity relationship. The following observation was made in order to simplify the form of $C_A$: due to the weak explicit effect of the integrated induced pressure on the stability coefficients the implicit influence of the induced pressure on skin friction was found to be the dominant effect and thus dictated the choice of similarity parameter. The similarity parameter, 
\[
\left( 1 + \frac{\alpha \cdot L}{\rho} \right)^{\frac{1}{4}} \cdot \left( \rho_{\infty} L \right)^{-\frac{1}{2}},
\]
was suggested by the form of the axial force coefficient due to skin friction in Equation 56. Additional data was sought in order to fully assess the applicability of the similarity parameter for $C_A$.

Dayman (Reference 18) has compiled a rather extensive collection of free flight data for $10^\circ$ half angle sharp cones with test conditions which cover the full range of viscous interaction (from weak interaction to fully merged flow fields). Dayman's data were all referenced to free flight base pressures. Thus a correction was applied to reference the data to a free-stream base pressure. The dominant effect of the base pressure correction is on the relative displacement between theory and data. Thus, an average base pressure coefficient was selected from the experimental data presented in Reference 19 which best represented the range of Reynolds number of interest. It should again be emphasized that the choice of $C_{PB} = -0.04$ is at best a qualitative indication of the effect of such
a correction. Even with the disagreement between theory and data, a rather important conclusion can be drawn from the correlation information of Figure 30. Over a range of Reynolds number based on cone height \(2800 \leq \text{Re}_{L_\infty} \leq 33,500\) encompassing both strong and weak viscous interaction the effect of wall temperature on the drag of a slender cone is correlated by the similarity parameter \((\rho_\infty L_\infty)^{\frac{1}{2}}(1 + 2.6 \frac{T_w}{T_0})^{\frac{1}{4}}\) over the range \(0.45 \leq \frac{T_w}{T_0} \leq 0.90\).
V. SUMMARY AND CONCLUSION

Force and moment coefficients were obtained experimentally for a 10° half-angle cone at hypersonic low density conditions for which viscous interaction was important. By the introduction of select amounts of argon to air it was possible to simulate the range of specific heat ratio, $\gamma'$, variation for 0 to 100% oxygen dissociation in air. In order to account for the influence of transport properties on the static stability coefficients, two parameters, $F$ and $G$, were defined. $F$ and $G$ are a measure of the deviation of the transport properties of the air-argon mixture from air transport properties. (The parameters $F$ and $G$ are not unique to an air-argon mixture, but are applicable, for example, to dissociation and ionization of air as well.) Both experimental and theoretical results indicated no appreciable effect of Reynolds number on $C_N$ or $C_M$. However, the axial force coefficient, $C_A$, increases with decreasing Reynolds number by an amount predictable by MTC theory.
Although the errors associated with MTC's prediction of $C_N$ and $C_M$ are reasonably small, Newtonian theory offers simpler and slightly more accurate similarity forms of $C_N$ and $C_M$ for weak viscous interaction of a chemically frozen flow. On the other hand MTC is far superior to Newtonian theory for predicting $C_A$. Of particular note, a similarity parameter suggested by the MTC form of $C_{AF}$ was defined which correlates zero angle of attack drag data over a wide range of viscous interaction and wall temperature ratio for a fixed cone angle. Not only is this similarity parameter unique in this respect but its simplicity lends itself to application in both free flight and perfect gas wind tunnel data.

It remains to be shown whether this similarity parameter is also applicable to non-equilibrium conditions as well as to moderate angle of attack data where viscous interaction is an important consideration. Such future studies could not only serve to add a greater dimension to the above approach to viscous hypersonic interaction but also could lead to new multiplicative factors to account for cone half-angle and non-equilibrium effects.
APPENDIX I

METHOD OF OBTAINING INTERACTION CONSTANTS FROM THE CALIBRATION

A balance calibration rig is a device designed to apply mechanical loads to a balance which duplicates the range of expected aerodynamic loads. Specifically, in this study, the calibration rig loads the balance with an axial force, and a normal force and pitching moment at a given loading. The output of the three strain gage bridges can be related to the applied load through a Taylor series expansion.

\[
F = \left( \frac{\partial F}{\partial O_1} O_1 + \frac{1}{2} \frac{\partial^2 F}{\partial O_1^2} O_1^2 + \frac{1}{2} \frac{\partial^2 F}{\partial O_1 \partial O_2} O_1 O_2 + \frac{1}{2} \frac{\partial^3 F}{\partial O_1 \partial O_2 \partial O_3} O_1 O_2 O_3 + \ldots \right) \\
+ \left( \frac{\partial F}{\partial O_2} O_2 + \frac{1}{2} \frac{\partial^2 F}{\partial O_2^2} O_2^2 + \frac{1}{2} \frac{\partial^2 F}{\partial O_2 \partial O_1} O_2 O_1 + \frac{1}{2} \frac{\partial^3 F}{\partial O_2 \partial O_1 \partial O_3} O_2 O_1 O_3 + \ldots \right) \\
+ \left( \frac{\partial F}{\partial O_3} O_3 + \frac{1}{2} \frac{\partial^2 F}{\partial O_3^2} O_3^2 + \frac{1}{2} \frac{\partial^2 F}{\partial O_3 \partial O_1} O_3 O_1 + \frac{1}{2} \frac{\partial^3 F}{\partial O_3 \partial O_1 \partial O_2} O_3 O_1 O_2 + \ldots \right)
\]

where \( F \) is the applied load and \( O_1, O_2, \) and \( O_3 \) are the outputs of the strain gage bridges, and the partial derivatives are the interaction factors.

In applying the above equation to the balance
calibration data it is assumed that the outputs of the strain gage bridges and the applied loads are related linearly. This reduces equation (I-1) to the form

\[ F = \frac{\partial F}{\partial a_1} a_1 + \frac{\partial F}{\partial a_2} a_2 + \frac{\partial F}{\partial a_3} a_3 \]  

(I-2)

where

a) \( \frac{\partial F}{\partial a_1}, \frac{\partial F}{\partial a_2}, \frac{\partial F}{\partial a_3} \) are constant; i.e. these are the interaction constants and

b) all higher order partial derivatives are zero; i.e. non-linear effects such as second order interactions are negligible.

Although great amount of care is taken to design balances to minimize non-linear effects, the validity of the above assumption must be substantiated aposteriori.

Making the following definitions:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_F P_1 )</td>
<td>output of the forward pitch strain gage bridge for the ( i^{th} ) loading</td>
</tr>
<tr>
<td>( O_A P_1 )</td>
<td>output of the aft pitch strain gage bridge for the ( i^{th} ) loading</td>
</tr>
<tr>
<td>( O_A X_1 )</td>
<td>output of the axial strain gage bridge for the ( i^{th} ) loading</td>
</tr>
<tr>
<td>( F_N i )</td>
<td>applied normal force for the ( i^{th} ) loading</td>
</tr>
</tbody>
</table>
XiFNi applied pitching moment for the ith loading; 
Xi is measured from a fixed reference point 
( )c calculated output of a particular strain gage bridge 
Vi the difference between the measured and the 
calculated output for the ith loading 
S  
the linear relationship between the applied loads and the 
strain gage bridge outputs can be expressed as: 

\[ (\text{OFF}i)_c = \alpha FN_i + b x_i FN_i + c FAX_i \]  \hspace{1cm} (I-3) 
\[ (\text{OAP}i)_c = \epsilon FN_i + f x_i FN_i + g FAX_i \]  \hspace{1cm} (I-4) 
\[ (\text{OAX}i)_c = p FN_i + q x_i FN_i + r FAX_i \]  \hspace{1cm} (I-5) 

where \( \{\alpha, b, \ldots, g, r\} \) are the interaction constants.

As with almost any calibration procedure, some data scatter is expected. What is needed, therefore, is a method of choosing the unknown interaction constants which will minimize the difference between the actual value of the outputs and the values calculated from equations (I-3) through (I-5).

The technique which is used is the method of least squares. This method defines the interaction constants so as to minimize the sum of the squares of the differences between the actual and calculated values of the applied loads. This can be expressed as:

\[ \frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} = \ldots = \frac{\partial S}{\partial q} = \frac{\partial S}{\partial r} = 0 \]  \hspace{1cm} (I-6)
For the forward pitch output

\[ S = \sum_{i} V_i^2 = \sum_{i} \left( a F_{Ni} + b x_i F_{Ni} + c F_{AXi} - OFP_i \right)^2 \]

In order for \( S \) to be a minimum with respect to \( a, b, \) and \( c \)

\[ \frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} = \frac{\partial S}{\partial c} = 0 \]

Therefore,

\[ a \sum_{i} F_{Ni}^2 + b \sum_{i} x_i F_{Ni}^2 + c \sum_{i} F_{AXi} \cdot F_{Ni} = \sum_{i} OFP_i \cdot F_{Ni} \]

\[ a \sum_{i} x_i F_{Ni}^2 + b \sum_{i} x_i^2 F_{Ni} + c \sum_{i} F_{AXi} \cdot x_i \cdot F_{Ni} = \sum_{i} OFP_i \cdot x_i \cdot F_{Ni} \]

\[ a \sum_{i} F_{Ni} \cdot F_{AXi} + b \sum_{i} x_i \cdot F_{Ni} \cdot F_{AXi} + c \sum_{i} F_{AXi}^2 = \sum_{i} OFP_i \cdot F_{AXi} \]

Similar results are obtained for the aft pitch and axial bridge outputs. Resorting to matrix notation the results may be written:

\[
\begin{pmatrix}
\sum_{i} F_{Ni}^2 & \sum_{i} x_i \cdot F_{Ni}^2 & \sum_{i} F_{AXi} \cdot F_{Ni} \\
\sum_{i} x_i \cdot F_{Ni}^2 & \sum_{i} x_i^2 \cdot F_{Ni} & \sum_{i} F_{AXi} \cdot x_i \cdot F_{Ni} \\
\sum_{i} F_{Ni} \cdot F_{AXi} & \sum_{i} x_i \cdot F_{Ni} \cdot F_{AXi} & \sum_{i} F_{AXi}^2
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
= 
\begin{pmatrix}
\sum_{i} OFP_i \cdot F_{Ni} \\
\sum_{i} OFP_i \cdot x_i \cdot F_{Ni} \\
\sum_{i} OFP_i \cdot F_{AXi}
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
\begin{pmatrix}
\sum_{i} OFP_i \cdot x_i \cdot F_{Ni} \\
\sum_{i} OFP_i \cdot x_i \cdot F_{Ni} \\
\sum_{i} F_{AXi} \cdot F_{AXi}
\end{pmatrix}
Now, solving for the interaction constants

\[
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}
= 
\begin{pmatrix}
a & b & c \\
\alpha & \beta & \gamma \\
p & q & r
\end{pmatrix}
\]

\[
\begin{pmatrix}
\sum_i \text{OFP}_i \cdot \text{FN}_i \\
\sum_i \text{OAP}_i \cdot \text{FN}_i \\
\sum_i \text{OAX}_i \cdot \text{FN}_i \\
\sum_i \text{OF}_{i} \cdot \text{FA}_{i} \\
\sum_i \text{OAP}_{i} \cdot \text{FA}_{i} \\
\sum_i \text{OAX}_{i} \cdot \text{FA}_{i}
\end{pmatrix}
= 
\begin{pmatrix}
\sum_i \text{OFP}_i \cdot \text{FN}_i \\
\sum_i \text{OAP}_i \cdot \text{FN}_i \\
\sum_i \text{OAX}_i \cdot \text{FN}_i \\
\sum_i \text{OF}_{i} \cdot \text{FA}_{i} \\
\sum_i \text{OAP}_{i} \cdot \text{FA}_{i} \\
\sum_i \text{OAX}_{i} \cdot \text{FA}_{i}
\end{pmatrix}
\]

(I-7)
The applied loads can be obtained by inverting the matrix of interaction constants.

\[
\begin{pmatrix}
F_{Ni} \\
\kappa_{i} F_{Ni} \\
F_{AX_{i}}
\end{pmatrix}
= 
\begin{pmatrix}
a & b & c \\
e & f & g \\
p & q & r
\end{pmatrix}
\begin{pmatrix}
o_{FP_{i}} \\
o_{AP_{i}} \\
o_{AX_{i}}
\end{pmatrix}
\]

(I-8)

Note, in the above, that the calculated value of output has been replaced by the measured output. This is a legitimate step since the method of least squares minimizes the difference between the calculated and measured output.
APPENDIX II

SOLUTION FOR $x_c^*$

The expression for $x_c^*$

$$x_c^* = \frac{P_c}{K} \int_0^\kappa \frac{K_c^2}{P_c} \, dx$$  \hspace{1cm} (II-1)

can be rewritten as the differential equation

$$\frac{d x_c^*}{dx} = 1 - 2 \frac{x_c^*}{\kappa} + \frac{x_c^*}{P_c/q} \frac{d(P_c/q)}{dx}$$  \hspace{1cm} (II-2)

Assuming a power law variation of pressure with axial location, i.e. $P_c \propto x^{-N}$, the boundary conditions of Equation (II-2) can be specified by Equation (II-1).

As $x \to \infty$, $P_c \to \text{constant}$; $\frac{x_c^*}{\kappa} \to \frac{1}{3}$  \hspace{1cm} (II-3)

As $x \to 0$, $P_c \to x^{-\frac{1}{2}}$; $\frac{x_c^*}{\kappa} \to \frac{2}{7}$  \hspace{1cm} (II-3)

Using the small approximation for $P_c/q$, Equation (II-2) can be written as

$$\frac{d x_c^*}{dx} = \frac{1 + \Theta - 2 x_c^*/\kappa}{(1 + H)}$$  \hspace{1cm} (II-4)
where

\[
C = 3.84 \times 10^{-6} F \left( 1 + 2.6 \frac{n_U}{n_0} \right)^{3/2} \left( \rho_{\infty} l_s \right)^{-1/2}
\]

(II-5)

\[
K = C \kappa_c^{-1/2} \left( 1 - 2 \frac{\kappa_c}{\kappa} \right)
\]

(II-6)

\[
D = \frac{1.8 \left( 1 + 0.12 n_U \right) \left[ 2 \Theta_c \left( \frac{P_c}{\rho_{\infty}} \right)^{-1/2} + 2 K \left( \frac{P_c}{\rho_{\infty}} \right)^{-1} \right]}{1 + 1.8 \left( 1 + 0.12 n_U \right) \left[ K^2 \left( \frac{P_c}{\rho_{\infty}} \right)^{-3/2} + \Theta_c \left( \frac{P_c}{\rho_{\infty}} \right)^{-1} \right]}
\]

(II-7)

\[
H = \kappa_c C \cdot D \left[ \frac{1}{2} \kappa_c^{-3/2} \left( 1 - 2 \frac{\kappa_c}{\kappa} \right) + 2 \kappa_c^{-1/2} \right]
\]

(II-8)

and \[
G = 2 \frac{C \cdot D}{\left( \frac{P_c}{\rho_{\infty}} \right)} \frac{\kappa_c^{3/2}}{\kappa}
\]

(II-9)

The above system was solved numerically with a fourth-order Runge-Kutta integration method.
REFERENCES


<table>
<thead>
<tr>
<th>$M_\infty$</th>
<th>$p_o$(psia)</th>
<th>$T_o$($^\circ$R)</th>
<th>$\beta$</th>
<th>$P_o \times 10^6$(slug/ft$^3$)</th>
<th>Re/ft</th>
<th>$\gamma$</th>
<th>$T_w/T_o$</th>
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$p_0 = 74.3$ psia; $T_0 = 1200^\circ$F; $\beta = 0$. 
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- $P_0 = 39.3$ psia, $\beta = 0.0$
- $P_0 = 74.3$ psia, $\beta = 0.0$
- $P_0 = 74.3$ psia, $\beta = 0.121$
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$C_A$ (UNCORRECTED FOR BASE PRESSURE)

$\sim \beta = 0.0$
$\sim \beta = 0.229$
$\sim \beta = 0.350$
$\sim \beta = 0.121$
Figure 30: $C_D$ versus Similarity Parameter

$$C_D \propto \left( \frac{T_w}{T_0} \right)^{0.90}$$

DAYMAN (Ref. 18)

$m, n = 0.45 \left( \frac{T_w}{T_0} \right)^{0.45} C_{pb} = -0.04$

$$C_D \propto \left( \frac{T_w}{T_0} \right)^{0.90} \left( 1 + 2.6 \left( \frac{T_w}{T_0} \right)^{0.45} \right)^{1/2}$$

ft/slug$^{1/2}$

Figure 30: $C_D$ versus Similarity Parameter