INFORMATION TO USERS

This material was produced from a microfilm copy of the original document. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the original submitted.

The following explanation of techniques is provided to help you understand markings or patterns which may appear on this reproduction.

1. The sign or “target” for pages apparently lacking from the document photographed is “Missing Page(s)”. If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting thru an image and duplicating adjacent pages to insure you complete continuity.

2. When an image on the film is obliterated with a large round black mark, it is an indication that the photographer suspected that the copy may have moved during exposure and thus cause a blurred image. You will find a good image of the page in the adjacent frame.

3. When a map, drawing or chart, etc., was part of the material being photographed the photographer followed a definite method in “sectioning” the material. It is customary to begin photoing at the upper left hand corner of a large sheet and to continue photoing from left to right in equal sections with a small overlap. If necessary, sectioning is continued again – beginning below the first row and continuing on until complete.

4. The majority of users indicate that the textual content is of greatest value, however, a somewhat higher quality reproduction could be made from “photographs” if essential to the understanding of the dissertation. Silver prints of “photographs” may be ordered at additional charge by writing the Order Department, giving the catalog number, title, author and specific pages you wish reproduced.

5. PLEASE NOTE: Some pages may have indistinct print. Filmed as received.

Xerox University Microfilms
300 North Zeeb Road
Ann Arbor, Michigan 48106
GLANGES, Theodore Constantine, 1937-
TIME-DEPENDENT LEARNING AND THE DYNAMIC DEMAND
OF THE COMPETITIVE FIRM FOR VARIABLE FACTOR.

The Ohio State University, Ph.D., 1973
Economics, theory

University Microfilms, A XEROX Company, Ann Arbor, Michigan

© Copyright by
Theodore Constantine Glanges
1973
TIME-DEPENDENT LEARNING AND THE DYNAMIC DEMAND
OF THE COMPETITIVE FIRM FOR VARIABLE FACTOR

DISERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Theodore Constantine Glanges

The Ohio State University
1973

Approved by

[Signature]
Adviser
Department of Economics
ACKNOWLEDGMENTS

My profound thanks go to my three advisors Professors Tetsunori Koizumi, Kenneth J. Kopecky, and Wilford L. L'Esperance for the time, patience, and especially their palpable suggestions. But I single out Professor Koizumi who as supervisor bore the heaviest part of the counseling burden.

Needless to say, I alone am responsible for any remaining errors in the text.

I feel compelled, in addition, to give recognition to my typist, Mrs. Dolores Kacsor, who transformed the rough manuscript to the typewritten page with incredible equanimity and control.
VITA

January 25, 1937......Born - Fort Worth, Texas


1965..................JD., University of Michigan.

1968..................M.A., University of Michigan.

1968-1971............Teaching Assistant, Department of
Economics, The Ohio State University, Columbus, Ohio.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>VITA</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF PLATES</td>
<td>x</td>
</tr>
<tr>
<td>LIST OF PROPOSITIONS</td>
<td>xi</td>
</tr>
<tr>
<td>Chapter</td>
<td></td>
</tr>
<tr>
<td>I  INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II MODELS OF LEARNING AND THE FIRM'S EMPLOYMENT DECISION</td>
<td>5</td>
</tr>
<tr>
<td>A. Statement of the General Model.</td>
<td>5</td>
</tr>
<tr>
<td>B. Choice of Strategy A</td>
<td>13</td>
</tr>
<tr>
<td>1. Concavity of Objective Function.</td>
<td>13</td>
</tr>
<tr>
<td>2. The Necessary Conditions</td>
<td>16</td>
</tr>
<tr>
<td>3. Analysis of the First Necessary Condition</td>
<td>20</td>
</tr>
<tr>
<td>4. Analysis of the Second Necessary Condition</td>
<td>24</td>
</tr>
<tr>
<td>5. Determination of Strategy A</td>
<td>29</td>
</tr>
<tr>
<td>a. Submodel: ( m(x) = \bar{m}(x) = m )</td>
<td>29</td>
</tr>
<tr>
<td>b. Submodel: ( \theta = 0 )</td>
<td>36</td>
</tr>
<tr>
<td>C. Choice of Strategy B</td>
<td>44</td>
</tr>
<tr>
<td>1. Concavity of Objective Function.</td>
<td>45</td>
</tr>
<tr>
<td>2. The Necessary Conditions</td>
<td>48</td>
</tr>
<tr>
<td>3. Analysis of the First Necessary Condition</td>
<td>53</td>
</tr>
<tr>
<td>4. Analysis of the Second Necessary Condition</td>
<td>58</td>
</tr>
<tr>
<td>5. Determination of Strategy B</td>
<td>63</td>
</tr>
</tbody>
</table>
Table of Contents (Continued)

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>D. Synthesis</td>
<td>70</td>
</tr>
<tr>
<td>1. Preliminaries</td>
<td>70</td>
</tr>
<tr>
<td>2. The Submodel: ( m(\cdot) = \bar{m}(\cdot) = \text{constant} )</td>
<td>73</td>
</tr>
<tr>
<td>3. The Submodel: ( \theta = 0 )</td>
<td>79</td>
</tr>
<tr>
<td>E. Example</td>
<td>82</td>
</tr>
</tbody>
</table>

### III

COMPARISON OF A SIMPLE LEARNING MODEL WITH A CONVENTIONAL FACTOR-AUGMENTING MODEL | 89

| A. Conventional Factor-Augmenting Model | 89 |
| B. Example of the Conventional Model Using a Cobb-Douglas Production Function | 93 |
| C. The Price-Sensitivity of the Firm's Employment Plans in the Conventional Model | 95 |
| D. Price Sensitivity of the Firm's Employment Plans for the Case \( \theta = 0, m = \text{constant} \) | 97 |

### IV

EXPECTED CHANGING PRICES AND WAGES IN A LEARNING MODEL | 116

| A. Statement of the Model | 117 |
| B. Choice of Strategy A | 120 |
| 1. Concavity of Objective Function | 120 |
| 2. The Necessary Conditions | 121 |
| 3. Analysis of the First Necessary Condition | 124 |
| 4. Analysis of the Second Necessary Condition | 130 |
| 5. Simultaneous Solution of the Necessary Conditions | 132 |
Table of Contents (Continued)

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>C.</td>
<td></td>
</tr>
<tr>
<td>1.</td>
<td>147</td>
</tr>
<tr>
<td>2.</td>
<td>148</td>
</tr>
<tr>
<td>3.</td>
<td>149</td>
</tr>
<tr>
<td>4.</td>
<td>152</td>
</tr>
<tr>
<td>5.</td>
<td>157</td>
</tr>
<tr>
<td></td>
<td>159</td>
</tr>
<tr>
<td>D.</td>
<td>175</td>
</tr>
<tr>
<td>V</td>
<td>188</td>
</tr>
<tr>
<td>WORKS CITED</td>
<td>192</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>2-1</td>
<td>The Function ( m(z) )</td>
</tr>
<tr>
<td>2-2</td>
<td>The Planning Epoch</td>
</tr>
<tr>
<td>2-3</td>
<td>The Production Function, ( g(z) )</td>
</tr>
<tr>
<td>2-4</td>
<td>Graph of the function ( f(u_0,u_1) )</td>
</tr>
<tr>
<td>2-5</td>
<td>Graph of the Function ( u_0 = u_0(u_1) )</td>
</tr>
<tr>
<td>2-6</td>
<td>Graph of the Function ( u_1 = u_1(u_0) )</td>
</tr>
<tr>
<td>2-7</td>
<td>Graph of the Functions ( h(u_1,u_0) )</td>
</tr>
<tr>
<td>2-8</td>
<td>Graph of the Function ( I(u_0) )</td>
</tr>
<tr>
<td>2-9</td>
<td>Graph of the Function ( u_{1\text{II}}(u_0) )</td>
</tr>
<tr>
<td>2-10</td>
<td>Graph of the Function ( u_{1\text{II}}(u_0) ): ( m = \bar{m} = \text{constant} )</td>
</tr>
<tr>
<td>2-11</td>
<td>Graph of the Function ( u_1 = u_{1\text{II}}(u_0) )</td>
</tr>
<tr>
<td>2-12</td>
<td>Graph of the Function ( F(u_0) )</td>
</tr>
<tr>
<td>2-13</td>
<td>Graph of the Function ( u_{1\text{I}}(u_0) : 0 = 0 )</td>
</tr>
<tr>
<td>2-14</td>
<td>Graph of the Function ( u_{1\text{II}}(u_0) : 0 = 0 )</td>
</tr>
<tr>
<td>2-15</td>
<td>Graph of the Functions ( f(u_0,\bar{u}_1) )</td>
</tr>
<tr>
<td>2-16</td>
<td>Graph of the Function ( u_0 = u_0(\bar{u}_1) )</td>
</tr>
<tr>
<td>2-17</td>
<td>Graph of the Function ( \bar{u}<em>1 = \bar{u}</em>{1\text{II}}(u_0) )</td>
</tr>
<tr>
<td>2-18</td>
<td>Graph of the Functions ( h(\bar{u}_1,u_0) )</td>
</tr>
<tr>
<td>2-19</td>
<td>Graph of the Function ( I(u_0) )</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>2-20</td>
<td>Graph of the Function $\tilde{u}_{II}(u_0)$</td>
</tr>
<tr>
<td>2-21</td>
<td>Graph of the Function $K(u_0)$</td>
</tr>
<tr>
<td>2-22</td>
<td>The Firm's Planned Factor Demand Over the Planning Horizon</td>
</tr>
<tr>
<td>2-23</td>
<td>Determination of the Firm's Plan for Factor Demand</td>
</tr>
<tr>
<td>3-1</td>
<td>Solution of the Conventional Factor-Augmenting Model</td>
</tr>
<tr>
<td>3-2</td>
<td>Solution of the Conventional Model with Cobb-Douglas Production Function</td>
</tr>
<tr>
<td>3-3</td>
<td>Effect of an Increase in the Relative Wage on the Determination of Strategy A</td>
</tr>
<tr>
<td>3-4</td>
<td>Effect on the Level of Employment of an Increase in the Relative Wage</td>
</tr>
<tr>
<td>3-5</td>
<td>Effect of Increase in the Relative Wage on the Graph of $u^*_I(u_0)$</td>
</tr>
<tr>
<td>3-6</td>
<td>Effect of Increase in the Relative Wage on the Graph of $u^*_III(u_0)$</td>
</tr>
<tr>
<td>3-7</td>
<td>Determination of Strategy B: $u_{0m} &lt; \hat{u}_{0B}$</td>
</tr>
<tr>
<td>3-8</td>
<td>Strategy B: $u_{0m} = \hat{u}_{0B}$</td>
</tr>
<tr>
<td>3-9</td>
<td>Relationship of the Quantities x and y.</td>
</tr>
<tr>
<td>3-10</td>
<td>Determination of Strategy B: $u_{0m} = \hat{u}<em>{0B}$, $g''(u</em>{0m}) &gt; x$</td>
</tr>
<tr>
<td>3-11</td>
<td>Determination of Strategy B: $u_{0m} = \hat{u}<em>{0B}$, $g''(u</em>{0m}) &lt; x$</td>
</tr>
</tbody>
</table>
List of Figures (Continued)

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-12</td>
<td>Determination of Strategy B: $u_{0m} &gt; \hat{u}_{OB}$</td>
<td>109</td>
</tr>
<tr>
<td>4-1</td>
<td>Graph of $h(u_0; u_1)$</td>
<td>126</td>
</tr>
<tr>
<td>4-2</td>
<td>Graph of the Function $u_0(u_1)$</td>
<td>127</td>
</tr>
<tr>
<td>4-3</td>
<td>Graph of the Function $u_1 = u_1I(u_0)$</td>
<td>129</td>
</tr>
<tr>
<td>4-4</td>
<td>Graph of the Function $u_1 = u_1II(u_0)$</td>
<td>131</td>
</tr>
<tr>
<td>4-5</td>
<td>Effect of Increase in $n$: $u_{0mx} = \bar{w}/(1+m)$</td>
<td>140</td>
</tr>
<tr>
<td>4-6</td>
<td>Effect of Increase in $n$: $u_{0mx} &gt; \bar{w}/(1+m)$</td>
<td>143</td>
</tr>
<tr>
<td>4-7</td>
<td>Effect of Increase in $n$: $u_{0mx} &lt; \bar{w}/(1+m)$</td>
<td>145</td>
</tr>
<tr>
<td>4-8</td>
<td>Graph of the Functions $k(u_0; \bar{u}_1)$</td>
<td>153</td>
</tr>
<tr>
<td>4-9</td>
<td>Graph of the Function $u_0 = u_0(\bar{u}_1)$</td>
<td>154</td>
</tr>
<tr>
<td>4-10</td>
<td>Graph of the Function $\bar{u}_1 = \bar{u}_1(u_0)$</td>
<td>157</td>
</tr>
<tr>
<td>4-11</td>
<td>Graph of the Function $\bar{u}_1 = \bar{u}_1II(u_0)$</td>
<td>159</td>
</tr>
<tr>
<td>4-12</td>
<td>Graph of the Function $J(n)$</td>
<td>164</td>
</tr>
<tr>
<td>4-13</td>
<td>Effect of Increase in $n$: $u_{0mn} = \bar{w}/(1+m)$</td>
<td>168</td>
</tr>
<tr>
<td>4-14</td>
<td>Effect of Increase in $n$: $u_{0mn} &lt; \bar{w}/(1+m)$</td>
<td>170</td>
</tr>
<tr>
<td>4-15</td>
<td>Effect of Increase in $n$: $u_{0mn} &gt; \bar{w}/(1+m)$</td>
<td>173</td>
</tr>
<tr>
<td>4-16</td>
<td>The Critical Values for the C-D Example.</td>
<td>186</td>
</tr>
</tbody>
</table>
### LIST OF PLATES

<table>
<thead>
<tr>
<th>Plate</th>
<th>Superposition of the Graphs of $u_{1I}(u_0)$ and $u_{1II}(u_0)$ $m(x)=m(x)=m=constant$</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Superposition of the Graphs of $u_{1I}(u_0)$ and $u_{1II}(u_0)$ $m(x)=m(x)=m=constant$</td>
<td>35</td>
</tr>
<tr>
<td>II</td>
<td>The Dynamic Factor Demand Associated with Strategy B</td>
<td>37</td>
</tr>
<tr>
<td>III</td>
<td>Determination of B-Strategy by Superposition of Graphs of $u_{1I}(u_0)$ and $u_{1II}(u_0)$</td>
<td>66</td>
</tr>
<tr>
<td>IV</td>
<td>The Dynamic Factor Demand Associated with Strategy B</td>
<td>69</td>
</tr>
<tr>
<td>V</td>
<td>Transformation of the Axes from Variable $u_1$ to $u_1$</td>
<td>72</td>
</tr>
<tr>
<td>VI</td>
<td>Synthesis: $m(·)=\bar{m}(·)=constant$</td>
<td>74</td>
</tr>
<tr>
<td>VII</td>
<td>Dynamic Factor Demand: $m(·)=\bar{m}(·)=constant$</td>
<td>78</td>
</tr>
<tr>
<td>VIII</td>
<td>Depiction of the Results of Proposition VI</td>
<td>112</td>
</tr>
<tr>
<td>IX</td>
<td>Comparison of Sensitivities of Two Models to an Increase in Relative Wage</td>
<td>115</td>
</tr>
<tr>
<td>X</td>
<td>Intersections of the Graphs of $u_{1I}(u_0)$ and $u_{1II}(u_0)$</td>
<td>133</td>
</tr>
<tr>
<td>XI</td>
<td>Intersections of the Graphs of $u_{1I}(u_0)$ and $u_{1II}(u_0)$</td>
<td>161</td>
</tr>
<tr>
<td>XII</td>
<td>Juxtaposition of the A and B Strategies</td>
<td>178</td>
</tr>
<tr>
<td>Proposition</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>-------------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>34</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>43</td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>68</td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>77</td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>80</td>
<td></td>
</tr>
<tr>
<td>VI</td>
<td>110</td>
<td></td>
</tr>
<tr>
<td>VII</td>
<td>138</td>
<td></td>
</tr>
<tr>
<td>VIII</td>
<td>146</td>
<td></td>
</tr>
<tr>
<td>IX</td>
<td>164</td>
<td></td>
</tr>
<tr>
<td>X</td>
<td>174</td>
<td></td>
</tr>
<tr>
<td>XI</td>
<td>181</td>
<td></td>
</tr>
<tr>
<td>XII</td>
<td>187</td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER I
INTRODUCTION

The Horndahl iron works in Sweden had no new investment (and therefore presumably no significant change in its methods of production) for a period of 15 years, yet productivity (output per manhour) rose on the average close to 2% per annum......

I advance the hypothesis here that technical change in general can be ascribed to experience, that it is the very activity of production which gives rise to problems for which favorable responses are selected over time.1 (Emphasis mine)

With these remarks Arrow ushered a concept of learning by doing into the mainstream of economic theory. The model which Arrow analyzed rests upon a fixed proportions aggregate production function and upon the assumption that embodied technical progress depends upon the amount of cumulative gross investment in the economy, as of the vintage date of the capital. Thus, the efficiency of labor which combines with a given vintage of capital is determined and unchanging. As labor becomes more

experienced, the greater experience of the labor, associated with a given vintage of capital, is not reflected in its greater efficiency. Only labor associated with a later vintage of capital may have a different, greater efficiency. In other words, experience is interpreted as the experience of producing capital goods.

Four years later Levhari generalized Arrow's putty-clay aggregate production function to allow substitution of capital and labor. Labor-augmenting technical progress is made to depend again upon the amount of cumulative gross investment in the economy as a proxy for experience. Thus, again, experience is interpreted essentially as experience obtained in the production of capital goods.

What implications does this have in relation to the Horndahl Effect? If there is no new investment, a particular firm does not obtain later vintage equipment and its work force using capital of earlier vintage does not become more efficient as it gains experience. This implication suggests the use of another assumption which implies that for fixed vintage, as

---

a unit of labor cumulates time in contributing to production, it becomes more efficient. Presumably it is not merely the lapsing time, but the pace of productive activity of the factor, which necessarily consumes time, that explains the increased efficiency.

Dudley\(^3\) in a recent empirical paper on the Colombian Metal Products industry recognizes a form of learning which is dependent upon input experience. Though he found this kind of learning insignificant in explaining productivity gains for the particular industry and country that he studies, it does not mean that such learning does not take place at all. Who can deny that, as a novice typist acquires experience, he becomes more efficient, and yet the vintage of the capital with which he works (namely, the typewriter) is unchanged? Similarly, no seasoned lawyer will deny that lawyers, recently admitted to practice, become more efficient with experience, while capital and plant unquestionably do not change. These are two examples of a multitude which the reader may multiply by a minimum of effort.

We propose to focus, at the outset, upon the firm—assuming the quantity of capital and its vintage given—in order to study the firm's demand for the variable

\(^3\)Leonard Dudley, "Learning and Productivity Change in Metal Products," *AER*, vol. 62, p. 662.
factor. Experience in the model is taken to mean the cumulative experience of each particular worker engaged in a particular process of production. As the worker cumulates time on the job, he becomes tempered by the solution of problems arising in his activity and he becomes differentiated from his fellow workers by his increased efficiency. In comparison to a worker who has not participated in the production of the particular firm as long a period of time, the more experienced worker is more efficient. In the previously mentioned models of Arrow and Levhari, in contrast, the greater experience of the worker amounts to naught.

Our approach also differs from the usual model of factor-augmenting technological progress (as it relates to this discussion). In this common form of technological progress, labor is not differentiated by the length of its service in a particular productive activity; as technology progresses, all labor becomes more efficient uniformly despite the possibility that some labor may have more seniority.

We shall, therefore lay the groundwork in the second chapter by analyzing a general model basic to the later chapters. In chapter three we introduce a version of the conventional factor-augmenting model and compare it with a basic learning model. Finally, in the last chapter, we analyze a learning model in which the firm expects a changing relative wage.
CHAPTER II
MODELS OF LEARNING AND THE FIRM'S EMPLOYMENT DECISION

A. Statement of the General Model

We assume at the outset that the level of capital is given. The effective remaining choice variable then is the quantity of variable factor, denoted \( z \). A unit of the variable factor changes depending on the cumulative time it has been employed; indeed this represents the assumption that the variable factor learns from its experience in production. As the working factor accumulates experience in production, it becomes more efficient - the longer a factor has contributed to production, the more efficient it becomes. In order to use two-dimensional geometric arguments, however, some special assumptions are made regarding timing and the form of the functions involved.

The duration of production, taken to be the same as the planning horizon of the firm, is subdivided into two periods. In the first period, a particular unit of factor learns and accumulates experience in production
resulting in greater efficiency in its contribution toward production during the second period. In other words, we assume that the zth unit of factor employed during the first period produces as effectively as \((1 + m(z)) \cdot \Delta z\) units of factor during the second period. Though all units improve in efficiency during the second period of their participation in production, the units vary in the degree to which their efficiency improves. We assume that the workers are ranked according to their potential for increased efficiency such that this index, at the margin, decreases with additions to the labor force. We assume, in order to be definite, that the function \(m\) has the properties

\[
m(0) \text{ is a finite positive constant,} \\
m'(z) \leq 0 \\
m''(z) > 0 \\
m(z) > 0 \text{ for all } z < \infty, \text{ and} \\
im m(z) \text{ exists and is } > 0.
\]

Similarly, the zth unit of factor employed during the second period produces as effectively in the third period as \((1 + \bar{m}(z)) \cdot \Delta z\) units of factor. Again, it is assumed that the degree to which a factor improves in efficiency is inhomogeneous in the sense that the function \(\bar{m}(z)\) enjoys the same properties, (2-1), as the function \(m(z)\).
In order to facilitate the use of two-dimensional geometric arguments, we assume that the second period is not long enough for the additional factor hired during that period to become more efficient by the end of the planning epoch.

With reference to the chronology of the increased efficiency of the senior factor hired in the first period, the entire planning epoch is subdivided into a learning period $[0,T)$, during which the senior factor gains experience and the second period $[T,T]$ in which the senior factor becomes more efficient as a consequence of its earlier experience. It should be noted that if the end of the first period coincides with the end of the second period, $t=T$, then we are in the traditional static case of the theory of the firm.¹

---

The firm, which operates in an environment of competitive output and input markets, is aware of the potential time-dependent improvement of the variable factor and seeks to plan its production in order to maximize its discounted profits over the duration of the planning epoch by choosing the level of input for the first period and again for the second period. The learning by the variable factor takes place, in a sense, without the direct influence of the firm since as the factor acquires experience in production, it almost automatically learns and becomes more efficient in the second period of production. But the firm does influence the learning by the factor in the sense that the firm decides whether or not to give the factor the opportunity to learn from the particular mode of production. In determining the quantity of factor to hire in the first period, the firm takes into account the relation of lapsed time in production and the acquisition of experience by the factor.

The production function of the firm, we assume, has the following properties for $z > 0$:

1. $g(z) > 0$ for $z$ in the interval $(0, \infty)$,
2. $g'(z) > 0$ with $g'(0) = \infty$ and $g'(\infty) = 0$, and
3. $g''(z) < 0$.

![The Production Function, $g(z)$](Figure 2-3)
The conditions under (b) are known as the Inada derivative conditions in the growth theory literature. For brevity in the statement of assumptions and results, hereafter we shall mean by the term "Inada Conditions" all of the conditions enumerated in (2-2). These conditions play crucial roles in establishing the existence and uniqueness of optimal strategies to be discussed below. Condition (b) should be noted and contrasted with the common assumption found in the theory of static optimization of the firm which recognizes an initial phase of increasing marginal productivity before ultimate diminishing marginal productivity sets in.

For convenience, the price of the output is taken as numéraire (a common practice in models of the firm): the "price" of output appears as unity and the relative magnitude of factor cost per unit to output price appears in the place of the usual factor cost per unit.

The assumptions introduced thus far require the firm to consider two objective functions depending upon whether the firm is considering increasing or decreasing the amount of variable factor employed in the second period.

---


3Henderson and Quandt, Microeconomic Theory, p. 46.
To help keep the analytics clear, we shall refer to the options connected with increasing the variable factor as Strategies of type A and those options connected with decreasing the variable factor as Strategies of type B. The optimizing procedure of the firm comprises first choosing an optimal strategy of type A which we shall call its A-Strategy, and secondly choosing an optimal strategy of type B, which we shall naturally call its B-Strategy. Finally, the firm chooses the one of the two previously determined A-Strategy or B-Strategy which yields the greater discounted profits. It is this plan of production which the firm will execute if its expectations regarding price of output, wage, and other impinging circumstances are substantially fulfilled.

To complete the formulation of the model, we turn to an explicit statement of the two objective functions. First, assume that the amount of factor in the second period is to be increased by an amount $c > 0$. Suppose the market price, $k$, per unit of factor is expected to remain constant during the entire planning epoch (the contrary case is considered in Chapter III), then the discounted profits resulting from strategies of type A are given by

$$\Pi_A^* = D_0 \{ g(z) - k z \} + D_1 \{ g(\int_0^{z}(1+m(x))dx + \int_0^{\bar{z}}(1+\bar{m}(y))dy) - k(z+c) \}.$$  

The symbols $D_0$ and $D_1$ represent the discount factors associated with each period. They may be given, for
example, by the formulas

\[ (2-4) \quad D_0 = \int_0^T e^{-rt} dt \quad \text{and} \quad D_1 = \int_T^T e^{-rt} dt, \]

where the continuous discounted rate is the constant \( r \).

The quantity enclosed by the first pair of braces in (2-3) is the net profit at each instant of time during the first period and is straightforward enough.

The quantity enclosed by the second pair of braces is different, however, and contains the major innovations of the model. This quantity is the discounted profits of the second period in the planning epoch. The first integral in the argument of the production function represents in efficiency units the factor employed during the first period. The second integral, with \( t = 1 \), would represent the effectiveness of the additional factor (hired in the second period) in its participation in production in the third period if there were a third period. The parameter \( 0 \leq \theta \leq 1 \) is introduced as a means of representing learning by the firm from the earlier first period experience of the now (in the second period) senior factor. As it were, the accumulation of experience by the factor in the first period becomes part of the folklore of the firm so that any newly hired factor in the second period benefits vicariously from the first period experience. It does not have to contribute toward production for a full period before it attains part or all of its potential efficiency.
If θ=0, no such firm-learning takes place; while if θ=1, the opposite extreme is attained and the newly hired factor attains its potential efficiency without experience before its first period of employment. Thus, the second integral represents in efficiency units the amount of factor newly hired in the second period.

On the other hand, if the amount of factor in the second period is to be decreased by an amount (-c) ≥ 0, the objective function takes on a different form because of the asymmetry, implied by the earlier assumptions, between increasing and decreasing the amount of variable factor employed during the second period. The discounted profits resulting from what we have called strategies of type B are given by

\[
(2-5) \quad \pi^B = D_0 \{g(z) - kz\} + D_1 \{g(\int_0^{z+c} (1+m(x)dx) - k(z+c))
\]

The various symbols, including the discount factors, D_0 and D_1, are defined the same as those in formula (2-4) in the preceding paragraph. The argument of the production function g(•) which is contained in the second pair of braces now comprises a single integral with the upper limit of integration, z+c, where c ≤ 0 and |c| ≤ z, since in decreasing the amount of factor the firm will lose the potentially increased efficiency of the factor resulting from the accumulated experience of the first period. The
quantity must satisfy, besides the requirement of non-
negativity, the inequality, \( z \geq (-c) \), since no more factor
can be dismissed in the second period than was hired in
the first period.

The main result of the chapter, Proposition V, is
that it never pays the firm, when \( \theta = 0 \), to augment its
input during the second period.

B. Choice of Strategy A

By substituting the symbols \( u_0 = z \) and \( u_1 = c \), we
convert the objective function associated with strategies
of type A, formula (2-3), to

\[
(2-6) \Pi_A = D_0 \{ g(u_0) - ku_0 \}
+ D_1 \left\{ \int_0^{u_0} (1+m(x)) dx + \int_0^{u_1} (1+m(y)) dy \right\} - k(u_0 + u_1) .
\]

The goal of the firm is to maximize discounted profits
\( \Pi_A(u_0, u_1) \) by suitably choosing \( u_0, u_1 \geq 0 \).

1. Concavity of Objective Function

It is easily seen that the function \( \Pi_A (u_0, u_1) \) is
strictly concave in the variables \( u_0 \) and \( u_1 \). To establish
strict concavity, it is sufficient to show that the
diagonal elements of the Hessian are negative and that
the determinant of the Hessian is positive.\textsuperscript{4,5}

Let $H_A$ denote the Hessian of the objective function $\Pi_A(u_0, u_1)$:

$$H_A = \begin{pmatrix} \frac{\partial^2 \Pi_A}{\partial u_0^2} & \frac{\partial^2 \Pi_A}{\partial u_0 \partial u_1} \\ \frac{\partial^2 \Pi_A}{\partial u_0 \partial u_1} & \frac{\partial^2 \Pi_A}{\partial u_1^2} \end{pmatrix}$$

where

$$\frac{\partial^2 \Pi_A}{\partial u_0} = D_0 \cdot \left\{ g''(u_0) \right\} + D_1 \cdot \left\{ g'' \left( \int_0^{u_0} (1+m(x))dx + \int_0^{u_1} (1+\Theta M(y))dy \right) \right. \\
\cdot \left( 1+m(u_0) \right)^2 + \left. g' \left( \int_0^{u_0} (1+m(x))dx + \int_0^{u_1} (1+\Theta M(y))dy \right) \right.$$ \\
$$\cdot \frac{dm(u_0)}{du_0} \right\},$$

$$\frac{\partial^2 \Pi_A}{\partial u_0 \partial u_1} = D_1 \cdot \left\{ g'' \left( \int_0^{u_0} (1+m(x))dx + \int_0^{u_1} (1+\Theta M(y))dy \right) \right.$$ \\
$$\cdot \left( 1+m(u_0) \right) \cdot \left( 1+\Theta M(u_1) \right) \right\},$$

$$\frac{\partial^2 \Pi_A}{\partial u_1^2} = D_1 \cdot \left\{ g'' \left( \int_0^{u_0} (1+m(x))dx + \int_0^{u_1} (1+\Theta M(y))dy \right) \right.$$

$$\cdot \left( 1+m(u_0) \right)^2 \right.$$ \\
$$\left. + \Theta \cdot g' \left( \int_0^{u_0} (1+m(x))dx + \int_0^{u_1} (1+\Theta M(y))dy \right) \cdot \frac{dm(u_1)}{du_1} \right\}.$$
Since by assumption $g'(\cdot) > 0$, $g''(\cdot) < 0$, and $m'(\cdot) \leq 0$, it follows that

$$\frac{\partial^2 \Pi_A}{\partial u_0^2} < 0, \quad \frac{\partial^2 \Pi_A}{\partial u_1^2} < 0, \quad \text{and} \quad \frac{\partial^2 \Pi_A}{\partial u_0 \partial u_1} < 0.$$ 

Thus,

$$|H_A| = \frac{\partial^2 \Pi_A}{\partial u_0^2} - \frac{\partial^2 \Pi_B}{\partial u_1^2} - \left( \frac{\partial^2 \Pi_A}{\partial u_0 \partial u_1} \right)^2$$

$$= \text{[positive terms]} +$$

$$D_1g''\left(\int_0^{u_0} (1+m(x))dx + \int_0^{u_1} (1+m(y))dy \right)$$

$$\cdot (1+m(u_0)) \cdot (1+\bar{m}(u_1))^2$$

$$- \left\{ D_1g''\left(\int_0^{u_0} (1+m(x))dx + \int_0^{u_1} (1+\bar{m}(y))dy \right)$$

$$(1+m(u_0)) \cdot (1+\bar{m}(u_1))^2 \right\}^2$$

$$= \text{positive terms} > 0.$$

This establishes the strict concavity of $\Pi_A(u_0,u_1)$.

Since $\Pi_A(u_0,u_1)$ is strictly concave, any local maximum is a global maximum and any global maximum is unique.\(^6\)


\(^7\)Ibid., p. 127, theorem 2.10 (b).
2. The Necessary Conditions

We obtain the necessary conditions by using the apparatus of discrete optimal control theory. Let $L(0)$ denote the first term in the formula $\pi_A(u_0,u_1)$ for discounted profits resulting from a strategy of type A (see (2-3)):

$$L(0) = D_0 \{ g(u_0) - k u_0 \}$$

and let $L(1)$ denote the second term in the same formula:

$$L(1) = D_1 \{ g'(\int_0^u (1+m(x))dx + \int_0^u (1+\Theta m(y))dy)$$

$$- k(u_0 + u_1) \}.$$

The firm's problem is to maximize the discounted profits by choosing $u_0, u_1 \geq 0$:

$$\max_{u_0, u_1 \geq 0} [L(0) + L(1)].$$

Let $x_1 = (df) f(0) = u_0$. Then,

$$\lambda(1) = \frac{\partial L(1)}{\partial x_1} = \frac{\partial D_1}{\partial x_1} g' \left( \int_0^u (1+m(x))dx + \int_0^u (1+\Theta m(y))dy \right)$$

$$- k(x_1 + u_1) \}$$

$$= D_1 \{ g' \left( \int_0^u (1+m(x))dx + \int_0^u (1+m(y))dy \right) \cdot (1+m(x_1)) - k \}.$$

---

Let
\[ H(0) = (df) L(0) + \lambda(1) \cdot f(0) = D_0 \{ g(u_0) - ku_0 \} + \lambda(1) \cdot u_0. \]
then the optimal \( u_0, \hat{u}_0, \) must satisfy the condition
\[ D_0 \cdot \{ g(\hat{u}_0) - ku_0 \} + \lambda(1) \cdot \hat{u}_0 = \max_{u_0} H(0) = \max_{u_0 \geq 0} \left[ D_0 \{ g(u_0) - ku_0 \} + \lambda(1) \cdot u_0 \right], \]
where \( \lambda(1) \) is the optimal value of \( \lambda(1) \) which corresponds to the optimal values \( \hat{u}_0 \) and \( \hat{u}_1 \). If an internal solution exists, the necessary condition takes the form: choose \( u_0 = \hat{u}_0 \) such that
\[ 0 = \frac{\partial H(0)}{\partial u_0} = D_0 \cdot \{ g'(\hat{u}_0) - k \} + \lambda(1) \]
\[ (2-6) = D_0 \cdot \{ g'(\hat{u}_0) - k \} + D_1 \cdot \{ g'(\hat{u}_0) (1+m(x)) dx + \int_0^{\hat{u}_0} (1+\theta \hat{m}(y)) dy \}
\]
\[ \cdot (1+m(\hat{u}_0)) - k \}. \]
We shall refer to this condition as the first necessary condition for the A-Strategy.

We show next that an internal solution \( u_0 \) exists for any \( u_1 \geq 0 \) (we have removed the "hats" from the variables for convenience). Define the function
\[ (2-7) f(u_0;u_1) = D_0 g'(u_0) + D_1 g'(\int_0^{u_0} (1+m(x)) dx + \int_0^{u_1} (1+\theta \hat{m}(y)) dy \)
\[ \cdot (1+m(u_0)). \]
The quantity \( u_1 \) is regarded as a fixed parameter for the discussion. The values of the function \( f(u_0; u_1) \) at the extremes of the domain of definition are

\[
\begin{align*}
\text{at } u_1 = 0: & \quad f(0; u_1) = 0^0 \quad \text{since } g'(0) = 0^0 \text{ and } 0 < g'(\int_0^{u_1} (1 + \theta \bar{m}(y))dy) \leq 0^0 \\
\text{and } u_1 = 0: & \quad f(0; u_1) = 0^0 \quad \text{since } g'(\int_0^{u_1} (1 + \theta \bar{m}(y))dy) = 0^0.
\end{align*}
\]

Moreover, the function \( f(u_0; u_1) \) is monotonic decreasing in the variable \( u_0 \) since

\[
\begin{align*}
\frac{df(u_0; u_1)}{du_0} &= D_0 \frac{g''(u_0)}{u_0} + D_1 \left\{ g'' \left( \int_0^{u_0} (1 + \theta \bar{m}(y))dy \right) \right. \\
&\left. \cdot \left(1 + m(u_0)\right)^2 + g' \left( \int_0^{u_0} (1 + \theta \bar{m}(y))dy \right) \cdot \frac{dm(u_0)}{du_0} \right\}
\end{align*}
\]

is negative \( g''(\cdot) < 0, g'(\cdot) > 0, \) and \( m'(\cdot) \leq 0 \) by assumption. Thus, given \( u_1 \), there exists a unique \( u_0 \) such that

\[
f(u_0; u_1) = (D_0 + D_1) \cdot k.
\]

This is the first necessary condition (2-6). Thus, the first necessary condition has an internal solution \( u_0 \).

The marginal productivity of the \( u_0 \)th unit of factor, hired in the first period, is

\[
(1 + m(u_0)) \cdot g'(\int_0^{u_1} (1 + m(x))dx + \int_0^{u_1} (1 + \theta \bar{m}(y))dy)
\]

in the second period. Thus, the meaning of the first necessary condition is that the discounted marginal pro-
duct of the $u_0$th factor be equated to discounted unit cost.

To derive the second necessary condition, let

$$H(1)=L(1)=D_1 \{ g(\int_0^{u_0}(1+m(x))dx+\int_0^{u_1}(1+\theta \bar{m}(y))dy) -k(u_0+u_1)^2 \}.$$

The necessary condition requires, given the optimum $\hat{u}_0$, to select $u_1=\hat{u}_1$ such that

$$\max_{u_1 \geq 0} H(1)=D_1 \{ g(\int_0^{\hat{u}_0}(1+m(x))dx+\int_0^{\hat{u}_1}(1+\theta \bar{m}(y))dy) -k(\hat{u}_0+\hat{u}_1) \}.$$

If an internal solution exists, it is necessary that

$$0=\frac{\partial H(1)}{\partial u_1}=D_1 \{ g'(\int_0^{\hat{u}_0}(1+m(x))dx+\int_0^{\hat{u}_1}(1+\theta \bar{m}(y))dy) \cdot (1+\theta \bar{m}(\hat{u}_1)) - k \} ; \text{ or,}$$

$$(2-8) \quad g'(\int_0^{\hat{u}_0}(1+m(x))dx+\int_0^{\hat{u}_1}(1+\theta \bar{m}(y))dy) \cdot (1+\theta \bar{m}(\hat{u}_1))=k.$$

If an internal solution does not exist (for some values of $u_1$ we shall see below it does not), because of the simple form of the function $g(\cdot)$, the necessary condition for a maximum requires that, given $\hat{u}_0$, select $\hat{u}_1$ such that

$$(2-9) \quad g'(\int_0^{\hat{u}_0}(1+m(x))dx+\int_0^{\hat{u}_1}(1+\theta \bar{m}(y))dy) \cdot (1+\theta \bar{m}(\hat{u}_1))$$

comes as close as possible to the constant $k$. We shall refer to the condition (2-8) and (2-9) as the second necessary condition for the A-Strategy.

The quantity appearing in (2-9) is the marginal product in the second period of the additional factor hired in that period. The economic meaning of the second necessary condition is that the marginal product of the
incremental factor come as close as possible to its unit cost.

3. Analysis of the First Necessary Condition

In order to determine the A-Strategy, we find the unique intersection of the graphs of two functions, denoted \( u_1 = u_{11}(u_0) \) and \( u_1 = u_{111}(u_0) \), which are derived, respectively, from the first and second necessary conditions for the A-Strategy. In this section, we determine the properties of the first function, \( u_{11}(u_0) \). In the last section it was shown that the function \( f(u_0; u_1) \), defined in (2-7), is monotonic decreasing with \( f(0; u_1) = \infty \) and \( f(\infty; u_1) = 0 \). The graph shifts downward, as indicated in Figure 2-4 since

\[
\frac{2f}{f} = D_1 g'' \left( \int_0^{u_0} (1+m(x)) dx + \int_0^{u_1} (1+m(y)) dy \right) \cdot (1+m(u_0)) \cdot (1+\eta(u_1)) < 0.
\]

![Graph of the function \( f(u_0; u_1) \)](Figure 2-4)
The uppermost curve is associated with the function 
\( f(u_0;0) \); i.e., with parameter value \( u_1 = 0 \). The first 
necessary condition requires (see (2-6) and (2-7)) that 
\[ (2-10) \quad f(u_0;u_1) = (D_0 + D_1) \cdot k. \]

In the preceding section we saw, as is plain from figure 2-4 
that for each \( u_1 \geq 0 \) there is exactly one value \( u_0 \) which 
satisfies this condition: \( u_0 = u_0(u_1) \). The function \( u_0(u_1) \) 
achieves a maximum for \( u_1 = 0 \). Denote this value by \( u_{0\text{max}} \).

Graph of the Function \( u_0 = u_0(u_1) \)

![Graph of the Function](image)

Figure 2-5

Also, as \( u_1 \to \infty \),
\[
\begin{align*}
D_1 g' \left( \int_0^{u_0} (1+m(x)) dx + \int_0^{u_1} (1+\theta \bar{m}(y)) dy \right) & \to \\
D_1 g' \left( \int_0^{u_0} (1+m(x)) dx + \int_0^{\infty} (1+\theta \bar{m}(y)) dy \right) & = D_1 g'(\infty) = 0;
\end{align*}
\]

while
\[
1 \leq (1+m(u_0)) \leq (1+m(0)) = \text{finite number exceeding one.}
\]

Thus,
\[ (2-11) \quad D_1 g' \left( \int_0^{u_0} (1+m(x)) dx + \int_0^{u_1} (1+\theta \bar{m}(y)) dy \right) \cdot (1+m(u_0)) \to 0 
\text{as } u_1 \to \infty. \]
The expression in (2-11) is the second term of the function $f(u_0; u_1)$. For the first necessary condition (2-10) to hold, the first term of $f(u_0; u_1)$ (see (2-7)) which is $D_0 g'(u_0)$ must adjust. Thus, as $u_1 \to \infty$, $u_0$ approaches some quantity, finite and denoted $\hat{u}_0$, such that

$$D_0 g'(u_0) \rightarrow (D_0 + D_1) \cdot k.$$  

Thus,

$$u_0(u_1) \rightarrow \hat{u}_0 \quad \text{as} \quad u_1 \rightarrow \infty.$$  

The function $u_0(u_1)$ is monotonic decreasing as evaluation of the derivative indicates. Take the total derivative with respect to $u_1$ of the first necessary condition (2-6):

$$D_0 g''(u_0) \cdot u_0'(u_1) + D_1 g'\left(\int_0^{u_0} (1 + m(x)) \, dx + \int_0^{u_1} (1 + \theta \bar{m}(y)) \, dy\right)$$

$$\cdot (1 + m(u_0)) \cdot \left[(1 + m(u_0)) \cdot u_0'(u_1) + (1 + \theta \bar{m}(u_1))\right]$$

$$+ D_1 g'\left(\int_0^{u_0} (1 + m(x)) \, dx + \int_0^{u_1} (1 + \theta \bar{m}(y)) \, dy\right) \cdot m'(u_0) \cdot u_0'(u_1) = 0;$$

or,

$$u_0'(u_1) = \frac{\left[D_1 g''\left(\int_0^{u_0} (1 + m(x)) \, dx + \int_0^{u_1} (1 + \theta \bar{m}(y)) \, dy\right) \cdot (1 + m(u_0)) \cdot (1 + \theta \bar{m}(u_1))\right]}{\left[D_0 g''(u_0) + D_1 g'\left(\int_0^{u_0} (1 + m(x)) \, dx + \int_0^{u_1} (1 + \theta \bar{m}(y)) \, dy\right) \cdot (1 + m(u_0))^2 \right.}$$

$$+ g'\left(\int_0^{u_0} (1 + m(x)) \, dx + \int_0^{u_1} (1 + \theta \bar{m}(y)) \, dy\right) \cdot m'(u_0)$$

$$< 0 \quad \text{since} \quad g''(\cdot) < 0, \quad g'(\cdot) > 0, \quad m'(\cdot) < 0.$$
The function \( u_0 = u_0(u_1) \) is graphed in Figure 2-5. Because of its monotonicity the function \( u_0 = u_0(u_1) \) has a unique inverse, which is denoted \( u_1 = u_{1I}(u_0) \), the subscript "I" indicating the origin of the function in the first necessary condition. The graph of the function \( u_{1I}(u_0) \) is shown in Figure 2-6. The domain of definition of the function is the interval \( [\bar{u}_0, u_{0mx}] \). In addition, \( u_{1I}(u_0) \to \infty \) as \( u_0 \to \bar{u}_0 \). The function is monotonic decreasing with derivative

\[
(2-13) \quad u_{1I}'(u_0) = \frac{1}{u_0'(u_1)} = \frac{D_0 g''(u_0) + D_1 \left[ g'' \left( \int_0^{u_0(1+m)} dx + \int_0^{u_1(1+\tilde{m})} dy \right) \cdot (1+m(u_0))^2 
+ g' \left( \int_0^{u_0(1+m)} dx + \int_0^{u_1(1+\tilde{m})} dy \right) m'(u_0) \right] \}
\]

\[ \cdot \frac{1}{\left[ D_1 g'' \left( \int_0^{u_0(1+m)} dx + \int_0^{u_1(1+\tilde{m})} dy \right) \cdot (1+m(u_0)) \cdot (1+\tilde{m}(u_1)) \right]} < 0. \]
4. Analysis of the Second Necessary Condition

In this section we derive a function, denoted \( u_1 = u_{1II}(u_0) \), from the second necessary condition and study the properties of the function implied by its defining condition.

The second necessary condition, (2-8) and (2-9), requires that

\[
g'(\int_0^{u_0}(1+m)dx + \int_0^{u_1}(1+m)dy) \cdot (1+\text{finite positive number}) = 0
\]

the marginal product of the \( u_1 \)-th unit of factor hired in the second period, come as close as possible to the constant \( k \), given \( u_0 \) and the positivity constraint on \( u_1 \).

We shall establish that, for each \( u_0 \), a unique \( u_1 \) exists satisfying the condition.

Let

\[
(2^{-14}) \ h(u_1; u_0) = g'(\int_0^{u_0}(1+m)dx + \int_0^{u_1}(1+m)dy) \cdot (1+\tilde{m}(u_1))
\]

then,

\[
h(0; u_0) = g'(\int_0^{u_0}(1+m)dx + \int_0^{\infty}(1+m)dy) \cdot (1+\tilde{m}(\infty))
\]

\[
= g'(\int_0^{u_0}(1+m)dx + \infty) \cdot (1+\text{finite positive number})
\]

\[
= g'(\infty) \cdot [\text{finite positive number}] = 0, \quad \text{and}
\]

\[
h(\infty; u_0) = g'(\int_0^{u_0}(1+m)dx) \cdot (1+\tilde{m}(0)).
\]

The slope of the \( h(u_1; u_0) \) function is given by

\[
\frac{\partial h(u_1; u_0)}{\partial u_1}
\]

\[
g''(\int_0^{u_0}(1+m)dx + \int_0^{u_1}(1+m)dy) \cdot (1+\tilde{m}(u_1))^2
\]

\[
+ g'(\int_0^{u_0}(1+m)dx + \int_0^{u_1}(1+m)dy) \cdot \frac{\tilde{m}(u_1)}{du_1} < 0
\]
Since by assumption \( g''(\cdot) < 0, \bar{m}'(\cdot) \leq 0, \) and \( g'(\cdot) > 0. \) The graphs of \( h(u_1; u_0) \) shift downward as the parameter \( u_0 \) increases since

\[
\frac{\partial h(u_1; u_0)}{u_0} = g'' \left( \int_0^{u_0} (1+m)dx + \int_0^{u_1} (1+\bar{m})dy \right) (1+\bar{m}(u_0)) (1+\bar{m}(u_1)) < 0
\]

![Graph of the Functions \( h(u_1; u_0) \)](image)

Figure 2-7

The properties of the functions \( h(u_1; u_0) \) are represented by the graphs in figure 2-7. From the figure, it is seen that if the graph of \( h(u_1; u_0) \) starts above or at the line representing the value \( k \), then the marginal product can actually be equated to \( k \).

We shall show that there exists a unique \( \hat{u}_{0A} \) such that for \( u_0 \leq \hat{u}_{0A} \) the graph of the function \( h(u_1; u_0) \) starts
above or at the line representing \( k \) and decreases monotonically toward the \( u_1 \)-axis; whereas, if \( u_0 > \hat{u}_{0A} \), the graph starts below the \( k \)-line.

Define

\[
(2-15) \quad I(u_0) = g'(\int_0^{u_0} (1+m)dx) \cdot (1+\theta \bar{m}(0)).
\]

Then,

\[
I(0) = g'(0) \cdot (1+\theta \bar{m}(0)) = \infty \cdot (1+\theta \bar{m}(0)) = \infty, \quad \text{and}
\]

\[
I(\infty) = g'(\infty) \cdot (1+\theta \bar{m}(0)) = 0 \cdot (1+\theta \bar{m}(0)) = 0.
\]

The function is also monotonic decreasing in the variable \( u_0 \) since

\[
\frac{dI(u_0)}{du_0} =
\]

\[
g''(\int_0^{u_0} (1+m)dx) \cdot (1+m(u_0))' (1+\theta \bar{m}(0)) < 0.
\]

Therefore, there exists a unique \( u_0, \hat{u}_{0A} \), such that

\[
(2-16) \quad k = I(\hat{u}_{0A})
\]

\[
= g''(\int_0^{\hat{u}_{0A}} (1+m)dx) \cdot (1+\theta \bar{m}(0)).
\]
In figure 2-7, we now see that, if \( u_0 < \hat{u}_{0A} \), the graph of \( h(u_1; u_0) \) begins above or at the \( k \)-line and then asymptotically and monotonically approaches the \( u_1 \)-axis.

The formula, resulting from the second condition in light of the preceding discussion, for selecting \( u_1 \), given \( u_0 \), is:

(a) for \( u_0 < \hat{u}_{0A} \), \( u_1 \) is determined uniquely by

\[
(2-17) \quad g^* \left( \int_0^{u_0} (1+m)dx + \int_0^{u_1} (1+\theta\bar{m})dy \right) \cdot (1+\theta\bar{m}(u_1)) = k,
\]

so that equality is actually achieved, and

(b) for \( u_0 > \hat{u}_{0A} \), \( u_1 = 0 \) is the choice which results in the value of

\[
\int_0^{u_0} (1+m)dx + \int_0^{u_1} (1+\theta\bar{m})dy \cdot (1+\theta\bar{m}(u_1))
\]

which is closest (or on, if \( u_0 = \bar{u}_0 \)) the \( k \)-line.

Thus, the function \( u_1 = u_{1II}(u_0) \) is specified as follows.

(a) for \( u_0 < \hat{u}_{0A} \), \( u_1 \) is determined implicitly by

\[
(2-18) \quad \text{the relation (2-16), and}
\]

(b) for \( u_0 > \hat{u}_{0A} \), \( u_1 = 0 \).

---

Graph of the Function \( u_{1II}(u_0) \)

Figure 2-9
For the part of the graph that does not coincide with the \( u_0 \)-axis, the slope is obtained by implicitly differentiating the defining relation (2-16) totally with respect to \( u_0 \):

\[
\frac{g'(\int_0^{u_0} (1+m)dx + \int_0^{u_1} (1+\theta\bar{m})dy) \cdot (1+\theta\bar{m}(u_1))}{[1+m(u_0)] + (1+\theta\bar{m}(u_1))u_{III}'(u_0)} + \theta \cdot g'(\int_0^{u_0} + \int_0^{u_1}) \cdot \bar{m}'(u_1) \cdot u_{III}'(u_0) = 0;
\]

or,

\[
(2-19) \quad u_{III}'(u_0) = \frac{-g''(\int_0^{u_0} + \int_0^{u_1}) \cdot (1+\theta\bar{m}(u_1)) \cdot (1+m(u_0))}{g''(\int_0^{u_0} + \int_0^{u_1}) \cdot (1+\theta\bar{m}(u_1))^2 + \theta g'(\int_0^{u_0} + \int_0^{u_1}) \cdot \bar{m}'(u_1)} < 0
\]

since \( g''(\cdot) < 0, \bar{m}'(\cdot) \leq 0, \) and \( g'(\cdot) > 0. \)

Let \( \bar{u}_1 \) denote the \( u_1 \)-intercept of the function \( u_{III}(u_0) \). We see that \( \bar{u}_1 > \hat{u}_{0A} \) as follows. By the monotonicity of \( g'(\cdot) \) and \( \bar{m}(\cdot) \)

\[
g'(\int_0^{\bar{u}_1} (1+m)dx) = \frac{k}{(1+\theta\bar{m}(\bar{u}_1))} \leq \frac{k}{(1+\theta\bar{m}(0))} = g'(\int_0^{\hat{u}_{0A}} (1+m)dx),
\]

where \( \hat{u}_{0A} \) denotes the \( u_0 \)-intercept of the graph of the function \( u_{III}(u_0) \), implies

\[
\int_0^{\bar{u}_1} (1+m)dx \geq \int_0^{\hat{u}_{0A}} (1+m)dx; \text{ or, } \bar{u}_1 \geq \hat{u}_{0A}.
\]

Thus, the \( u_1 \)-intercept is not less than the \( u_0 \)-intercept.
5. Determination of Strategy A

The next step is to determine the A-Strategy. We do this for two cases: in the first, \( m(x) = \bar{m}(x) = m \), a constant. This assumption means that there is no individual variation in the degree of increased efficiency resulting from the accumulation of experience. In the second, \( \theta = 0 \); that is, there is no firm-learning, as we have called it.

a. Submodel: \( m(x) = \bar{m}(x) = m \)

A preliminary look at the relative magnitude of the slopes of the graphs of the functions \( u_{1I}(u_0) \) and \( u_{1II}(u_0) \) will be helpful in superposing the two graphs on the same figure.

The slope of the graph of \( u_{1I}(u_0) \) is given by the formula (2-15), page 26, when the assumption of the submodel is taken into account:

\[
\frac{d}{du} u_{1I}(u_0) = 
\]

\[
(2-20) \quad \frac{-{D_0g}''(u_0)+D_1g''((1+m)u_0+(1+\theta m)u_1) \cdot (1+m)^2}{D_0g''((1+m)u_0+(1+\theta m)u_1) \cdot (1+m) \cdot (1+\theta m)} 
\]

\[
= \frac{-g''(u_0)}{g''((1+m)u_0+(1+\theta m)u_1) \cdot (1+m) \cdot (1+\theta m)} - \frac{(1+m)}{(1+\theta m)}. 
\]

Thus,

\[
(2-21) \quad u_{1I}^{'}(u_0) < -\frac{(1+m)}{(1+\theta m)} 
\]

at each point \((u_0, u_1)\) of the graph of \( u_{1I}(u_0) \).
The slope of the part of the graph of the function \( u_{1 II}(u_0) \) which does not coincide with the \( u_0 \)-axis is given by the formula (2-19) page 28, when the assumption of the sub-model is taken into account:

\[
(2-22) \quad u'_{1 II}(u_0) = - \frac{g''((1+m)u_0+(1+\theta m)u_1) \cdot (1+\theta m) \cdot (1+m)}{g''((1+m)u_0+(1+\theta m)u_1) \cdot (1+\theta m)^2}
\]

Thus,

\[
(2-23) \quad u'_{1 II}(u_0) = - \frac{(1+m)}{(1+\theta m)}
\]

at each point \((u_0, u_1)\) of this portion of the graph of \( u_{1 II}(u_0) \).

Before superposing the two graphs on the same figure to determine Strategy A, we summarize the relevant properties of each of the two functions. The function \( u_{1 I}(u_0) \)

\[9\] is determined implicitly by the relation, see (2-6), page 13,

\[
(2-24) \quad D_0 g'(u_0) + D_1 g'((1+m)u_0+(1+\theta m)u_1) \cdot (1+m) = (D_0 + D_1) \cdot k.
\]

Graph of the Function \( u_{1 II}(u_0) \), \( m = \tilde{m} = \text{constant} \)

\[\text{Figure 2-10}\]

---

9The special assumptions of the submodel of this section are added to the discussion of section 3, page 20, et seq.
The domain of definition of the function is the interval \([\hat{u}_0, u_{0mx}]\) on the \(u_0\)-axis. The function is monotonic decreasing and finally intercepts the \(u_0\)-axis at \(u_0 = u_{0mx}\). As \(u_0 \to \hat{u}_0\) from the right, \(u_{1\varphi}(u_0) \to \infty\). The curve is always steeper than \((1+m)/(1+\Theta m)\). The graph of the function is given in figure 2-10.

The function \(u_{1\varphi}(u_0)^{10}\) is given by the specification, see (2-18), page 27:

(a) for \(u_0 > \hat{u}_{0A}\), the function \(u_{1\varphi}(u_0)\) is determined implicitly by the relation

\[
(2-25) \quad g'((1+m)u_0 + (1+\Theta m)u_1) \cdot (1+\Theta m) = k, \quad \text{and}
\]

(b) for \(u_0 \leq \hat{u}_{0A}\), \(u_1 = 0\).

The point where the graph "enters" the \(u_0\)-axis is given by

\[
(2-26) \quad g'((1+m) \hat{u}_{0A}) = k/(1+\Theta m),
\]

see (2-16), page 26.

\[u_1 = \frac{(1+m)}{(1+\Theta m)} \hat{u}_{0A}\]

Graph of the Function \(u_1 = u_{1\varphi}(u_0)\)

Figure 2-11

---

\(^{10}\)Again the discussion of Section 4, page 24 et seq. is adapted to the submodel.
We saw from (2-23) that the part of the graph of $u_{1II}(u_0)$ which does not coincide with the $u_0$-axis has constant positive slope; and, therefore, is a straight line segment.

Since the $u_0$-intercept must be $\hat{u}_{0A}$ by definition and since $\hat{u}_{0A}$ also satisfies (2-25), the straight line segment portion of the graph of $u_{1II}(u_0)$ which does not coincide with the $u_0$-axis is given by

\[(2-27) \quad (1+m)u_0 + (1+\theta m) u_1 = (1+m) \hat{u}_{0A}\]

Hence, the $u_1$-intercept of this portion of the graph is

\[\hat{u}_1 = \frac{1+m}{1+\theta m} \cdot \hat{u}_{0A} \geq \hat{u}_{0A}\]

since $0 < \theta \leq 1$. This simply confirms the fact, established in the more general setting, that the $u_1$-intercept is never less than the $u_0$-intercept.

Next, we see that the $u_0$-intercept of the graph of $u_{1I}(u_0)$, $u_{0mx}$, is not constrained by the model in relation to the "$u_0$-intercept" of the graph of $u_{1II}(u_0)$, $\hat{u}_{0A}$. Define the function $F(u_0)$ by

\[(2-28) \quad F(u_0) = D_0 g''(u_0) + D_1 g'((1+m)u_0) - (D_0 + D_1) \cdot k.\]

Since the derivative

\[F'(u_0) = D_0 g''(u_0) + D_1 g''((1+m)u_0) \cdot (1+m) < 0,\]

the function is monotonic decreasing.
The $u_0$-intercept, $u_{0mx}$, of the function $u_{1I}(u_0)$ is determined implicitly by the condition

$$D_0 g'(u_{0mx}) + D_1 g'(u_{0mx}) \cdot (1+m)-(D_0+D_1)k = 0$$

(see (2-24), and discussion following it). In terms of the function $F(u_0)$, the condition (2-29) is

$$F(u_{0mx}) = 0.$$ 

We next evaluate $F(\hat{u}_{0A}) = 0$

$$D_0 g'(\hat{u}_{0A}) + D_1 k \frac{(1+m)}{(1+\theta m)} -(D_0+D_1) \cdot k$$

$$= D_0 \left[ g'(\hat{u}_{0A}) - k \right] + D_1 k \left\{ \frac{(1+m)}{(1+\theta m)} -1 \right\},$$

where condition (2-26) has been substituted for the original second term of $F(u_0)$. Since $0 \leq \theta \leq 1$ and since by the monotonicity of $g'(\cdot)$,

$$g'(\hat{u}_{0A}) \geq g'((1+m)\hat{u}_{0A}) = k \frac{1}{1+\theta m} \leq k,$$

the first term in braces of the expression (2-31) may be positive, negative, or zero; and the second term in braces is never negative. Thus, $F(\hat{u}_{0A}) \geq 0$, which is to say $\hat{u}_{0A}$ may be $\hat{u}_{0A} < u_{0mx}$.

We must therefore, allow for the various relative possibilities for $\hat{u}_{0A}$ and $u_{0mx}$ in superposing the graphs of $u_{1I}(u_0)$ and $u_{1III}(u_0)$. The result is that three types of intersections of the graphs may arise. These are
indicated in Plate I and are discussed in turn. (See page 35 for the Plate.)

**Case (a):** \( \hat{u}_{OA} < u_{0mx} \) The A-Strategy \((\hat{u}_0, \hat{u}_1)\) = \((u_{0mx}, 0)\), and \( \hat{u}_{OA} < u_{0mx} \).

**Case (b):** \( \hat{u}_{OA} = u_{0mx} \) The A-Strategy \((\hat{u}_0, \hat{u}_1)\) = \((u_{0mx}, 0)\). In both case (a) and case (b), the firm requires in the second period the same amount of factor input as it does in the first period.

**Case (c):** \( \hat{u}_{OA} > u_{0mx} \) The A-Strategy \((\hat{u}_0, \hat{u}_1)\) is such that \( \hat{u}_0 < u_{0mx} \) and \( \hat{u}_1 > 0 \). This must be the case because the graph of \( u_{1II}(u_0) \) runs from \(+00\) to 0 and has slope greater, at each point, than the graph of \( u_{1III}(u_0) \). The firm, in this case, "underhires" in the first period and augments the factor pool in the second period.

These results are summarized in the following proposition.

**Proposition I.** In the model stated in section A of the chapter with the additional assumption that \( m(x) = \bar{m}(x) = \text{constant} \), the behavior of a profit-maximizing firm, restricted to choosing from strategies of type A, is described as follows.

There exist unique \( u_{0mx} \) and \( u_{0A} \) satisfying, respectively, the conditions

\[
D_0 g''(u_{0mx}) + D_1 g''((1+m)u_{0mx})(1+m) = (D_0 + D_1)k,
\]

and

\[
g'((1+m)u_{0A}) = k/(1+\theta m).
\]
Superposition of the Graphs of $u_{1I}(u_0)$ and $u_{1II}(u_0)$

$m(x) = m(x) = m = \text{constant}$

Plate I
Then,

(a) if \( \hat{u}_{0A} \leq u_{0mx} \), the firm will demand \( u_{0mx} \) units of variable input in each of the two periods, and

(b) if \( \hat{u}_{0A} > u_{0mx} \), the firm will demand \( u_0 \) units of variable input in the first period and \( u_0 + u_1 \) units in the second period. The quantities \( u_0, u_1 \) satisfy the inequalities
\[
u_0 < u_{0mx} \quad \text{and} \quad 0 < u_1 < (1 + \theta)(\hat{u}_{0A}/(1 + \theta_m)).
\]
In this case the firm increases its input requirement in the second period.

Briefly, when there is firm-learning in addition to factor-learning, the firm considering only strategies of type A, may or may not increase its demand for input in the second period and the exact conditions in which it will increase the demand are given.

The conclusions of the proposition are illustrated in Plate II.

b. Submodel: \( \theta = 0 \)

Case (b) in the above proposition is a logical result of having \( \theta > 0 \), i.e., newly hired workers in the second period also benefit from the learning by the senior workers. As might be expected, if we exclude this possibility, i.e., if \( \theta = 0 \), it turns out that case (b) never arises as we shall now show.

We start out by showing that in contrast to the preceding submodel, the "\( u_0 \)-intercept" of the graph of \( u_{1II}(u_0) \) is never larger than the \( u_0 \)-intercept of the graph of \( u_{1I}(u_0) \): \( \hat{u}_{0A} \leq u_{0mx} \). To this end, let
The Dynamic Factor Demand
Associated with Strategy B

Plate II
Then

\[ F(0) = D_0 g'(0) + D_1 g'(0) \cdot (1 + m(0)) - (D_0 + D_1)k = \infty \]

since \( g'(0) = \infty \), and

\[ F(\infty) = D_0 g'(\infty) + D_1 g'(\infty) \cdot (1 + m(\infty)) - (D_0 + D_1)k = -(D_0 + D_1)k \]

since \( g'(\infty) = 0 \).

In addition, \( F(u_0) \) is monotonic decreasing since

\[
\frac{dF}{du_0} = D_0 g''(u_0) + D_1 g''\left(\int_0^{u_0} (1 + m(x))dx\right) \cdot (1 - m(u_0))^2 \\
+ D_1 g'(\int_0^{u_0} (1 + m(x))dx) \cdot \frac{dm}{du_0} < 0
\]

[recalling that \( g''(\cdot) < 0 \), \( g'(\cdot) > 0 \), and \( m'(\cdot) \leq 0 \)].

The "\( u_0 \)-intercept," \( \hat{u}_{0A} \), for the graph of \( u_{1II}(u_0) \) is determined, see (2-16), page 26, by the relation

\[ g'(\int_0^{\hat{u}_{0A}} (1 + m(x))dx) = k; \]

whereas, the \( u_0 \)-intercept, \( u_{0mx} \), for the graph of \( u_{1I}(u_0) \) is determined by the condition

\[ F(u_{0mx}) = 0. \]

(See discussion following (2-10), page 21.) Since \( F(\cdot) \) is monotonic, \( u_{0mx} \) is unique in satisfying the condition (2-34).

Now evaluate \( F(\cdot) \) at \( \hat{u}_{0A} \):

\[
F(\hat{u}_{0A}) = D_0 g'(u_0) + D_1 g'(\int_0^{\hat{u}_{0A}} (1 + m(x))dx) \cdot (1 + m(\hat{u}_{0A})) \\
-(D_0 + D_1)k \\
=D_0 g'(u_0) + D_1 k(1 + m(\hat{u}_{0A})) - (D_0 + D_1)k
\]
\[ D_0 g'(u_0) + D_1 k m(u_{0A}) - D_0 k D_1 k m(u_{0A}) \geq 0. \]

The next-to-the last inequality follows since by the monotonicity of \( g'(\cdot) \) and condition (2-33),
\[
k = g'\left( \int_0^{u_{0A}} (1 + m(x)) \, dx \right) = g'\left( \hat{u}_{0A} + \int_0^{u_{0A}} m(x) \, dx \right) \leq g'(\hat{u}_{0A});
\]
or,
\[
D_0 \cdot \left[ g'(\hat{u}_{0A}) - k \right] \geq 0.
\]
Therefore, \( \hat{u}_{0A} \leq u_{0mx}. \)

The slopes of the graphs are easily obtained by specializing the formulas developed for the more general model earlier. For the slope of the graph of \( u_{11}(u_0) \), formula (2-13), page 23, upon the making of appropriate changes reflecting the assumption \( \theta = 0 \), gives
\[
(2-35) \quad u_{11}'(u_0) =
- \left[ D_0 g''(u_0) + D_1 \cdot \left( g''\left( \int_0^{u_0} (1 + m) \, dx + u_1 \right) \cdot (1 + m(u_0))^2 \right. \right.
\]
\[
+ g'\left( \int_0^{u_0} (1 + m) \, dx + u_1 \right) \cdot \frac{dm(u_0)}{du_0} \left. \right] \div D_1 g''\left( \int_0^{u_0} (1 + m) \, dx + u_1 \right)
\]
\[
\cdot (1 + m(u_0)) \]

\[ ^{11} \text{Note that, if } m(x) \text{ is a positive constant, then } F(\hat{u}_{0A}) > 0, \text{ and } \hat{u}_{0A} < u_{0mx}. \]
Therefore,

\[(2-36) \quad u_{1III}(u_0) < -(1+m(u_0))\]  for each \((u_0, u_1)\) on the graph.

Similarly, upon making the changes to reflect the assumption \(\theta = 0\) in formula (2-19), page 28, we obtain for the slope of the portion of the graph of \(u_{1III}(u_0)\) which does not coincide with the \(u_0\)-axis

\[u_{1III}'(u_0) = \frac{-g''(\int_0^{u_0}(1+m)dx+u_1)(1+m(u_0))}{g''(\int_0^{u_0}(1+m)dx+u_1)} = -(1+m(u_0)),\]

Thus,

\[(2-37) \quad u_{1III}'(u_0) = -(1+m(u_0))\]

is the slope of the part of the graph of \(u_{1III}(u_0)\) which does not coincide with the \(u_0\)-axis.

Since the slope of the part of the graph of \(u_{1III}(u_0)\) coinciding with the \(u_0\)-axis is obviously zero, it follows from (2-36) and (2-37) that for each \(u_0\) in the intersection of the domains of definition of the functions
u_{1I}(u_0) and u_{1II}(u_0)

(2-38) \quad u_{1I}'(u_0) < u_{1II}'(u_0).

We summarily describe the functions \( u_{1I}(u_0) \) and \( u_{1II}(u_0) \) before taking the last step of superposing their graphs on the same figure.

Graph of the Function \( u_{1I}(u_0) \); \( \theta = 0 \)

Figure 2-13

The function \( u_{1I}(u_0) \)\textsuperscript{12} is determined implicitly by the relation

\[
(2-39) \quad D_0g''(u_0) + D_1g'(\int_0^{u_0}(1+m)dx + u_1) \cdot (1+m) = (D_0 + D_1)k
\]

The domain of definition of the function is the interval \([\tilde{u}_0, u_{0mx}]\), where \( \tilde{u}_0 \) satisfies, uniquely, the condition

\[
(2-40) \quad D_0g'(\tilde{u}_0) = (D_0 + D_1)k
\]

and \( u_{0mx} \) satisfies, uniquely, the condition

\textsuperscript{12}The general discussion of section 3, page 20 et seq. is specialized to the requirements of the submodel of this discussion.
(2-41) \[ D_0 g'(u_{0mx}) + D_1 g'(\int_0^{u_{0mx}} (1+m) dx) \cdot (1+m) = (D_0 + D_1) k. \]

The function is monotonic decreasing with \( u_{1II}(\hat{u}_0) = 0 \) and \( u_{1II}(u_{0mx}) = 0 \). The slope of the graph of the function exceeds \( 1+m(u_0) \).

On the other hand, the function \( u_{1III}(u_0) \) is determined according to the rule, see (2-18), page 27.

(a) for \( u_0 > \hat{u}_{0A} \), the value \( u_1 \) is determined by the relation

\[ g'(\int_0^{\hat{u}_{0A}} (1+m(x)) dx + u_1) = k, \text{ and} \]

(b) for \( u_0 \leq \hat{u}_{0A} \), \( u_1 = 0 \).

The quantity \( \hat{u}_{0A} \) is determined by the relation

\[ g'(\int_0^{\hat{u}_{0A}} (1+m(x)) dx) = k \]

The slope of the part of the graph of \( u_{1III}(u_0) \) which does not coincide with the \( u_0 \)-axis is \( -(1+m(u_0)) \). The \( u_1 \)-intercept, denoted by \( \tilde{u}_1 \), is determined by the relation

\[ g'(\tilde{u}_1) = k, \]

and, as we have seen (pp. 27 and 28) \( \tilde{u}_1 > \hat{u}_{0A} \). Figure 2-14 shows the graph of the function \( u_{1III}(u_0) \).

---

\[ ^{13} \text{Again the general discussion at page 23, et seq, is adapted to the particular assumptions of this section.} \]
The superposition of the two graphs of \( u_{1I}(u_0) \) and \( u_{1II}(u_0) \) is shown, for all practical purposes, in Plate I, page 35. The case \( u_{0mx} < \hat{u}_{0A} \) cannot occur when \( \theta=0 \); that is, case (c) depicted in the plate is superfluous for the sub-model under discussion. The discussion of the two cases (a) and (b) which are relevant for this submodel may be found at page 34.

The results for this submodel are summarized in the proposition.

**Proposition II** In the model stated in Section A of the chapter with the additional assumption that \( \theta=0 \), the behavior of a profit-maximizing firm, restricted to choosing from strategies of type A, is described as follows: The firm will demand \( u_{0mx} \) units of variable input in each of the two periods. The quantity \( u_{0mx} \) is determined by the condition
\[ D_0 g'(u_{0mx}) + D_1 g'(\int_0^{u_{0mx}} (1+m(x)) dx) \cdot (1+m(u_{0mx})) = (D_0 + D_1) k. \]

That is to say, a firm restricted to choices from strategies of type A requires the same amount of input in each of the two periods - when there is only factor-learning and not firm-learning. This is to be contrasted with Proposition I, page 34, for a special firm-learning model.

The result of Proposition II is depicted in Plate II, case (a). (Case (b) is inapplicable.)

C. Choice of Strategy B

For the determination of the B-Strategy, we shall follow a pattern of reasoning similar to that followed in the determination of the A-Strategy. The objective function which the firm is assumed to maximize in evaluating the strategies of type B (those strategies which permit the firm to maintain or decrease its demand for input during the second period) is, it will be recalled,\(^{14}\)

\[(2-45) \quad \Pi_B = D_0 \{ g(z) - k z \} + D_1 \{ g(\int_0^{z+c} (1+m(x)) dx) - k(z+c) \},\]

where \(z \geq 0 \geq c\) and \(0 \leq |c| \leq z.\)

---

\(^{14}\) Vid., page 12, formula (2-5).
As we did in the case of the objective function associated with the strategies of type A, we revert to the customary symbols of control theory by setting $u_0 = z$ and $\bar{u}_1 = -c$ so that the objective function associated with strategies of type B becomes

$$
(2-46) \quad \tau_B = D_0 \left\{ g(u_0) - ku_0 \right\} + D_1 \left\{ g\left( \int_0^{u_0 - \bar{u}_1} (1 + m(x)) \, dx \right) - k(u_0 - \bar{u}_1) \right\}
$$

The firm's problem is to maximize this objective function by determining $u_0$, $\bar{u}_1 \geq 0$ subject to $\bar{u}_1 \leq u_0$.

1. Concavity of Objective Function

The function $\tau_B(u_0, \bar{u}_1)$ of (2-46) is strictly concave in the variables $u_0$ and $\bar{u}_1$. This is established by showing that the diagonal elements of the Hessian of the objective function $\tau_B(u_0, \bar{u}_1)$ are both negative and the determinant of the Hessian is positive.\(^{15, 16}\)

Use of the general formula for differentiation of an integral with respect to a variable appearing as an upper limit of integration

$$
(2-47) \quad \frac{d}{dy} \left\{ \int_0^y (1 + m(x)) \, dx \right\} = (1 + m(f(y))) \cdot f'(y),
$$

\(^{15}\) Hadley and Kemp, *Variational Methods*, p. 99, corollary.

gives

\begin{equation}
(2-48) \quad \frac{\partial \Pi_B}{\partial u_0} = D_0 \cdot \left\{ g'(u_0) - k \right\} + D_1 \cdot \left\{ g' \left( \int_0^{u_0 - \bar{u}_1} (1+m(x)) dx \right) \right. \\
\left. \cdot (1+m(u_0 - \bar{u}_1)) - k \right\}
\end{equation}

and

\begin{equation}
(2-49) \quad \frac{\partial \Pi_B}{\partial \bar{u}_1} = -D_1 \cdot \left\{ g' \left( \int_0^{u_0 - \bar{u}_1} (1+m(x)) dx \right) \cdot (1+m(u_0 - \bar{u}_1)) - k \right\}.
\end{equation}

Hence, again using the general formula (2-47) for differentiation of an integral, the Hessian of the objective function \( \Pi_B(u_0, \bar{u}_1) \), denoted \( H_B \), is obtained as

\begin{equation}
(2-50) \quad H_B = \begin{pmatrix}
\frac{\partial^2 \Pi_B}{\partial u_0^2} & \frac{\partial^2 \Pi_B}{\partial u_0 \partial \bar{u}_1} \\
\frac{\partial^2 \Pi_B}{\partial u_0 \partial \bar{u}_1} & \frac{\partial^2 \Pi_B}{\partial \bar{u}_1^2}
\end{pmatrix}
\end{equation}

where

\begin{equation}
(2-51) \quad \frac{\partial^2 \Pi_B}{\partial u_0^2} = D_0 g''(u_0) + \\
D_1 \left\{ g' \left( \int_0^{u_0 - \bar{u}_1} (1+m(x)) dx \right) \cdot (1+m(u_0 - \bar{u}_1))^2 \right. \\
+ g' \left( \int_0^{u_0 - \bar{u}_1} (1+m(x)) dx \right) \cdot \frac{dm(u_0 - \bar{u}_1)}{du_0} \right\},
\end{equation}
\[
\frac{\partial^2 \Pi_B}{\partial u_0 \partial u_1} = D_1 \left\{ -g'' \left( \int_0^{u_0-\bar{u}_1} (1+m(x)) \, dx \right) (1+m(u_0-\bar{u}_1))^2 \\
+ g' \left( \int_0^{u_0-\bar{u}_1} (1+m(x)) \, dx \right) \frac{dm(u_0-\bar{u}_1)}{du_1} \right\} = \\
-D_1 \left\{ g'' \left( \int_0^{u_0-\bar{u}_1} (1+m(x)) \, dx \right) (1+m(u_0-\bar{u}_1))^2 \\
+ g' \left( \int_0^{u_0-\bar{u}_1} (1+m(x)) \, dx \right) \frac{dm(u_0-\bar{u}_1)}{du_0} \right\},
\]

and

\[
\frac{\partial^2 \Pi_B}{\partial u_1^2} = D_1 \left\{ g' \left( \int_0^{u_0-\bar{u}_1} (1+m(x)) \, dx \right) (1+m(u_0-\bar{u}_1))^2 \\
- g'' \left( \int_0^{u_0-\bar{u}_1} (1+m(x)) \, dx \right) \frac{dm(u_0-\bar{u}_1)}{du_1} \right\} = \\
D_1 \left\{ g'' \left( \int_0^{u_0-\bar{u}_1} (1+m(x)) \, dx \right) (1+m(u_0-\bar{u}_1))^2 \\
+ g' \left( \int_0^{u_0-\bar{u}_1} (1+m(x)) \, dx \right) \frac{dm(u_0-\bar{u}_1)}{du_0} \right\}.
\]

Since \( g'' (\cdot) < 0 \), \( m'(\cdot) \leq 0 \), and \( g'(\cdot) > 0 \), the diagonal elements \( \frac{\partial^2 \Pi_B}{\partial u_0^2} \) and \( \frac{\partial^2 \Pi_B}{\partial u_1^2} \), (2-51) and (2-53), above, respectively, are negative and the off-diagonal elements \( \frac{\partial^2 \Pi_B}{\partial u_0 \partial u_1} \), (2-52), above, are positive.
In addition, the determinant of the Hessian matrix of the objective function is given by

\[ |H_B| = \frac{\partial^2 \pi_B}{\partial u_0^2} \cdot \frac{\partial^2 \pi_B}{\partial u_1^2} - \left[ \frac{\partial^2 \pi_B}{\partial u_0 \partial u_1} \right]^2 \]

\[ = \left[ D_0 g''(u_0) + \frac{\partial^2 \pi_B}{\partial u_1^2} \right] \cdot \frac{\partial^2 \pi_B}{\partial u_1^2} - \left[ \frac{\partial^2 \pi_B}{\partial u_1^2} \right]^2 \]

\[ = D_0 g''(u_0) \cdot \frac{\partial^2 \pi_B}{\partial u_1^2} > 0 \text{ since } g''(\cdot) < 0 \]

and \( \frac{\partial^2 \pi_B}{\partial u_1^2} < 0 \) and since \( \frac{\partial^2 \pi_B}{\partial u_0^2} = D_0 g''(u_0) + \frac{\partial^2 \pi_B}{\partial u_1^2} \)

and \( \frac{\partial^2 \pi_B}{\partial u_0 \partial u_1} = - \frac{\partial^2 \pi_B}{\partial u_1^2} \).

Thus, the objective function \( \pi_B(u_0, u_1) \) is strictly concave.

Since \( \pi_B(u_0, u_1) \) is strictly concave, any local maximum is a global maximum\(^{17}\) and any global maximum is unique.\(^{18}\)

2. The Necessary Conditions

To obtain the necessary conditions,\(^{19}\) let \( L(0) \) denote the first term in the formula \( \pi_B(u_0, u_1) \) for discounted

\(^{17}\)Benavie, Mathematical Techniques, p. 126, theorem 2.9.

\(^{18}\)Ibid., p. 127, theorem 2.10 (b).

\(^{19}\)Ibid., p. 256, theorem 5.1 and Bryson and Ho, Control, pp. 42-47.
profits resulting from a strategy of type B (see (2-46), page 45)

\[ L(0) = D_0 \cdot \{ g(u_0) - ku_0 \} \]

and let \( L(1) \) denote the second term in the same formula:

\[ L(1) = D_1 \cdot \{ g(\int_0^{u_0} - \bar{u}_1 (1 + m(x)) dx) - k(u_0 - \bar{u}_1) \} \cdot \]

The firm seeks to maximize discounted profits \( \Pi_B(u_0, \bar{u}_1) \)
by choosing a suitable strategy of type B; that is, \( u_0, \bar{u}_1 \geq 0 \) and \( \bar{u}_1 \leq u_0 \):

\[
\max_{u_0, \bar{u}_1 \geq 0} \left[ L(0) + L(1) \right] \quad \bar{u} \leq u_0
\]

Let \( x_1 = f(0) = u_0 \). Then,

\[
\lambda(1) = \frac{\partial L(1)}{\partial x_1} = \frac{\partial}{\partial x_1} \left\{ g(\int_0^{x_1} - \bar{u}_1 (1 + m(x)) dx) - k(x_1 - \bar{u}_1) \right\}
\]

\[
= D_1 \left\{ g(\int_0^{x_1} - \bar{u}_1 (1 + m(x)) dx) \cdot (1 + m(x_1 - \bar{u}_1) - k) \right\}
\]

Let

\[
H(0) = (df) \cdot L(0) + \lambda(1) \cdot f(0) = D_0 \cdot \{ g(u_0) - ku_0 \} + \lambda(1) \cdot u_0.
\]

Then the first necessary condition requires a choice of \( u_0, \hat{u}_0 \), which maximizes \( H(0) \), given optimal \( \lambda(1) \):

\[
\max_{u_0 \geq 0} H(0) = \max_{u_0 \geq 0} \left[ D_0 \cdot \{ g(u_0) - ku_0 \} + \hat{\lambda}(1) \cdot u_0 \right]
\]

\[
= D_0 \cdot \{ g(\hat{u}_0) - k\hat{u}_0 \} + \hat{\lambda}(1) \cdot \hat{u}_0.
\]
where \( \hat{\lambda}(1) \) denotes the optimal value of \( \lambda(1) \) corresponding to the optimal values \( \hat{u}_0 \) and \( \hat{u}_1 \). If an internal solution exists, the condition may be re-expressed as requiring a choice \( u_0=\hat{u}_0 \) such that

\[
(2-55) \quad 0 = \frac{\partial H(0)}{\partial u_0} = D_0 \{ g'(u_0) - k \} + \hat{\lambda}(1) \\
= D_0 \{ g'(u_0) - k \} \\
+ D_1 \{ g'' \left( \int_0^{u_0-\hat{u}_1} (1+m(x)) \, dx \right) \cdot (1+m(u_0-\hat{u}_1)) - k \}.
\]

We refer to this condition as the first necessary condition for the B-Strategy.

Next, we show that, given \( \hat{u}_1 > 0 \), there exists a unique \( \hat{u}_0 \) which satisfies the first necessary condition. For simplicity we use the variables without the "hats."

Define the function

\[
(2-56) \quad f(u_0, \hat{u}_1) = D_0 g'(u_0) \\
+ D_1 \cdot g' \left( \int_0^{u_0-\hat{u}_1} (1+m(x)) \, dx \right) \cdot (1+m(u_0-\hat{u}_1)).
\]

Given \( \hat{u}_1 \), at the extremes of the domain of definition \([\hat{u}_1, \infty] \), the function \( f(\cdot, \hat{u}_1) \) takes the values

\[
f(\hat{u}_1, \hat{u}_1) = D_0 g'(\hat{u}_1) + D_1 g'(0) = D_0 g'(\hat{u}_1) + \infty \cdot (1+m(0)) = \infty,
\]
and,

\[
f(\infty, \hat{u}_1) = D_0 g'(\infty) + D_1 g'(\int_0^{\infty} (1+m(x)) \, dx) \cdot (1+m(\infty)) \\
= D_0 g'(\infty) + D_1 g'(0) = 0.
\]
since $g'(0) = 0$ and $g'(\infty) = 0$, by assumption. Also, $f(u_0; \bar{u}_1)$ is monotonic decreasing in the variable $u_0$:

$$\frac{\partial f(u_0; u_1)}{\partial u_0} = D_0 g''(u_0) + D_1 \left\{ \int_{-\infty}^{u_0-\bar{u}_1} (1+m(x)) \, dx \right\} \cdot (1+m(u_0-\bar{u}_1))^2 + g'(\int_{0}^{u_0-\bar{u}_1} (1+m(x)) \, dx) \cdot \frac{dm(u_0-\bar{u}_1)}{du_0} \} < 0$$

since $g''(\cdot) < 0$, $g'(\cdot) > 0$, and $m'(\cdot) \leq 0$. The first necessary condition requires a choice of $u_0$, given $\bar{u}_1$, such that

$$f(u_0; \bar{u}_1) = (D_0 + D_1) k$$

The monotonicity of $f(u_0; \bar{u}_1)$ thus guarantees the existence of a unique $u_0$ satisfying the conditions. This establishes the existence of an internal solution for the first necessary condition.

Transposition of the constant terms to one side of the equality in the first condition (2-55) yields

$$D_0 g'(u_0) + D_1 g'(\int_{0}^{u_0-\bar{u}_1} (1+m(x)) \, dx) \cdot (1+m(u_0-\bar{u}_1)) = (D_0 + D_1) k.$$  

The second term on the right hand side of the equality is the discounted value of the marginal product of the $(u_0-u_1)_{th}$ factor in the second period. With this comment, it becomes evident that the above condition requires the equating of the sum of the discounted marginal product of the $u_0$th unit of factor in the first period.
and the discounted marginal product of the \((u_0 - \bar{u}_1)\)th unit of factor in the second period to the discounted unit cost, \((D_0 + D_1)k\).

We turn next to the derivation of the second necessary condition. Let

\[
H(1) = L(1) = D_1 \{ g(\int_0^{u_0 - \bar{u}_1} (1 + m(x)) dx) - k(u_0 - \bar{u}_1) \}.
\]

The necessary condition for the B-Strategy requires, given the optimal \(\hat{u}_0\), the selection of \(\bar{u}_1 = \hat{u}_1\) such that

\[
\max_{\bar{u}_1 > 0, \bar{u}_1 \leq \hat{u}_0} H(1) = D_1 \{ g(\int_0^{\hat{u}_0 - \bar{u}_1} (1 + m(x)) dx) - k(\hat{u}_0 - \bar{u}_1) \}.
\]

First, if an internal solution exists, the condition becomes

\[
(2-57) \quad 0 = \frac{\partial H(1)}{\partial \bar{u}_1} = -D_1 \{ g \left( \int_0^{u_0 - \bar{u}_1} (1 + m(x)) dx \right) \cdot (1 + m(u_0 - \bar{u}_1) - k) \}
\]

(the "hats" have been removed from the variables for convenience)

Second, if an internal solution does not exist (in the analysis below we see that this may be the case), because of the simple form of the function \(g(\cdot)\), the necessary condition requires us, given \(u_0\), to determine \(\bar{u}_1\) such that

\[
(2-58) \quad g \left( \int_0^{u_0 - \bar{u}_1} (1 + m(x)) dx \right) \cdot (1 + m(u_0 - \bar{u}_1))
\]
come as close as possible to $k$.

In economic terms, the second necessary condition requires that the marginal product of (2-58) of the $(u_0-u_1)$th unit of factor be made to come as close as possible to the unit cost $k$.

We shall refer to the condition (2-57) and (2-58) as the second necessary condition for the B-Strategy.

3. **Analysis of the First Necessary Condition**

In this section, we study the properties of the function, denoted by $\bar{u}_{11}(u_0)$, which is determined by the first necessary condition for the B-Strategy, (2-55), page 50. In the last section (2-56), page 50, it was shown that the function

$$f(u_0; u_1) = D_0 g'(u_0) + D_1 g'(\int_0^{u_0} \bar{u}_1 (1+m(x)) dx) \cdot (1+m(u_0-\bar{u}_1))$$

is a monotonic decreasing function of $u_0$, given $\bar{u}_1$, defined on the interval $[\bar{u}_1, \infty]$ such that $f(\bar{u}_1; \bar{u}_1) = 0$ and $f(\infty; \bar{u}_1) = 0$. The graph of the function $f(u_0; u_1)$ is shown in figure 2-15. As $\bar{u}_1$ increases, the graph shifts outward since

$$\frac{\partial f(u_0; u_1)}{\partial u_1} =$$

$$D_1 \left\{ -g''(\int_0^{u_0} \bar{u}_1 (1+m(x)) dx) \cdot (1+m(u_0-\bar{u}_1))^2 \right\}$$
\begin{align*}
+ g' & \left( \int_{0}^{u_0-\bar{u}_1} (1+m(x)) \, dx \right) \cdot \frac{dm(u_0-\bar{u}_1)}{du_1} \\
= & -D_1 \left\{ g'' \left( \int_{0}^{u_0-\bar{u}_1} (1+m) \, dx \right) \cdot (1+m(u_0-\bar{u}_1))^2 \\
+ g' \left( \int_{0}^{u_0-\bar{u}_1} (1+m) \, dx \right) \cdot \frac{dm(u_0-\bar{u}_1)}{du_0} \right\} > 0
\end{align*}

(recalling the assumptions $g''(\cdot) < 0$, $g'(\cdot) > 0$, and $m'(\cdot) \leq 0$). The lowest graph on the figure belongs to the parameter value $\bar{u}_1 = 0$, that is, represents the function $f(u_0;0)$.

Graph of the Functions $f(u_0;\bar{u}_1)$

Figure 2-15
The first necessary condition requires that, (2-55), page 50,
\[ f(u_0, \bar{u}_1) = (D_0 + D_1) k. \]

In the preceding section, we established the existence of a function, which we denote here by \( u_0 = u_0(\bar{u}_1) \) and which is implicitly defined by the first necessary condition.

It is clear from figure 2-15 that the function \( u_0 = u_0(\bar{u}_1) \) has a minimal functional value, denoted \( u_{0mn} \), which corresponds to the value \( \bar{u}_1 = 0 \). Moreover, the constraint \( u_0 > \bar{u}_1 \) requires that, as \( \bar{u}_1 \to 0^+ \), \( u_0 \to 0^+ \) in such a way that the equality in the condition is maintained. The slope of the graph is obtained by implicitly taking the total derivative of the first condition, (2-55), page 50, with respect to the variable \( \bar{u}_1 \):

\[
D_0 \ g''(u_0) \cdot \frac{du_0}{d\bar{u}_1} +
\]

\[
D_1 \cdot \left\{ g' \left( \int_0^{u_0 - \bar{u}_1} (1+m(x)) \, dx \right) \cdot (1+m(u_0 - \bar{u}_1))^2 \right. \\
\left. \cdot \left( \frac{du_0}{d\bar{u}_1} - 1 \right) + g' \left( \int_0^{u_0 - \bar{u}_1} (1+m(x)) \, dx \right) \cdot m' \left( u_0 - \bar{u}_1 \right) \right. \\
\left. \cdot \left( \frac{du_0}{d\bar{u}_1} - 1 \right) \right\} = 0.
\]
Thus,

\[
\frac{d\mu_0}{d\mu_1} = \frac{D_1 \left( g' \left( \int_0^{u_0} u_1 (1+m) dx \right) m' (u_0 - \tilde{u}_1) + g'' \left( \int_0^{u_0} u_1 (1+m) dx \right) \right) (1+m(u_0 - \tilde{u}_1))^2}{D_0 g''(u_0) + D_1 g' \left( \int_0^{u_0} u_1 (1+m) dx \right) m' (u_0 - \tilde{u}_1) + g'' \left( \int_0^{u_0} u_1 (1+m) dx \right) (1+m(u_0 - \tilde{u}_1))^2} > 0.
\]

Since in absolute value the denominator is greater than the numerator, \( \frac{d\mu_0}{d\mu_1} < 1 \).

Graph of the Function \( u_0 = u_0(\tilde{u}_1) \)

Figure 2-16
Since the function \( u_0 = u_0(\bar{u}_1) \) is monotonic, it has an inverse which we denote \( \bar{u}_1 = \bar{u}_{11}(u_0) \) and depict in figure 2-17. The domain of definition of the function is \([u_{0mn}, \infty)\) and the function is monotonic increasing with slope, in light of (2-59), page 56,

\[
\frac{\mathrm{d}u_{11}}{\mathrm{d}u_0} = \frac{1}{\frac{\mathrm{d}u_0}{\mathrm{d}u_1}} = D_0 g''(u_0)
\]

\[
\begin{align*}
\frac{\partial}{\partial u_0} & \{ g'(\int u_0 - \bar{u}_1 (1+m(x)) \, dx) \cdot m'(u_0 - u_1) \\
& \quad + g''(\int u_0 - \bar{u}_1 (1+m(x)) \, dx) \cdot (1+m(u_0 - \bar{u}_1))^2 \} + 1 > 1.
\end{align*}
\]

Graph of the Function \( \bar{u}_1 = \bar{u}_{11}(u_0) \)

Figure 2-17
4. **Analysis of the Second Necessary Condition**

In this section, we study the properties of the function denoted by \( \tilde{u}_1 = \tilde{u}_{1II}(u_0) \) and implied by the second necessary condition, (2-58), page 52.

Let

\[
h(\tilde{u}_1; u_0) = (df) \int_{u_0}^{u_0-\tilde{u}_1} (1+m(x)) dx \cdot (1+m(u_0-\tilde{u}_1)).
\]

The domain of this function is clearly \([0, u_0]\) on the \( \tilde{u}_1 \)-axis and evidently increases in length with \( u_0 \). It will be helpful to consult figure 2-18 for the following discussion. At the extremes of the domain of...
The function is given by

$$h(u_0; u_0) = g'(0) \cdot (1 + m(0)) = 0 \quad \text{finite}$$

The slope of the function is given by

$$\frac{\partial h(\bar{u}_1; u_0)}{\partial \bar{u}_1} = -g'' \left( \int_{0}^{u_0-\bar{u}_1} (1 + m(x)) \, dx \right) \cdot (1 + m(u_0 - \bar{u}_1))^2$$

$$+ g' \left( \int_{0}^{u_0-\bar{u}_1} (1 + m(x)) \, dx \right) \cdot \frac{dm(u_0 - \bar{u}_1)}{du_0}$$

since $g''(\cdot) < 0$, $g'(\cdot) > 0$, and $m'(\cdot) \leq 0$. When the parameter $u_0$ increases, the graph of $h(\bar{u}_1; u_0)$ shifts downward:

$$\frac{\partial h}{\partial u_0} = g'' \left( \int_{0}^{u_0-\bar{u}_1} (1 + m(x)) \, dx \right) \cdot (1 + m(u_0 - \bar{u}_1))^2$$

$$+ g' \left( \int_{0}^{u_0-\bar{u}_1} (1 + m(x)) \, dx \right) \cdot \frac{dm(u_0 - \bar{u}_1)}{du_0} < 0$$

and the domain of definition $[0, u_0]$ on the $\bar{u}_1$-axis of the function $h(\bar{u}_1; u_0)$ becomes larger.

Now let $I(u_0)$ denote the ordinate-intercept of the $h(\bar{u}_1; u_0)$-graph.
\[ I(u_0) = h(0; u_0) = g' \left( \int_0^{u_0} (1 + m(x)) \, dx \right) \cdot (1 + m(u_0)). \]

The domain of definition of the function \( I(\cdot) \) is the entire \( u_1 \)-axis (see figure 2-19).

Graph of the Function \( I(u_0) \)

Figure 2-19

Again, clearly,

\[ I(0) = g'(0) \cdot (1 + m(0)) = \infty, \text{ and} \]
\[ I(\infty) = g'(\infty) \cdot (1 + m(\infty)) = 0 \cdot \text{positive (finite \#)} = 0. \]

Moreover, \( I(\cdot) \) is monotonic decreasing

\[ \frac{\partial I}{\partial u_0} = \frac{\partial h(0; u_0)}{\partial u_0} < 0 \]

from (2-63) with \( \bar{u}_1 = 0 \). Thus, as is so clearly depicted in figure 2-19, there exists a unique \( u_0, \bar{u}_0 \), such that
\[ I(\bar{u}_0) = h(0; \bar{u}_0) = g' \left( \int_0^{\bar{u}_0} (1+m(x)) \, dx \right) \cdot (1+m(\bar{u}_0)) = k \]

If \( u_0 < \bar{u}_0 \), then \( I(u_0^-, k) \); while if \( u_0^+ > \bar{u}_0 \), \( I(u_0^+) < k \).

This property of the ordinate-intercept of the graphs of the functions \( h(\bar{u}_1; u_0) \) is represented in figure 2-18, where the graph of the function \( h(\bar{u}_1; \bar{u}_0) \), associated with parameter \( \bar{u}_0 \), is the only one which passes through the intersection of the horizontal line, representing the constant \( k \), and the ordinate axis. The graph of a function \( h(\bar{u}_1; u_0^-) \) with parameter \( u_0^- < \bar{u}_0 \) lies entirely above the \( k \)-line and the graph of a function with parameter \( u_0^+ > \bar{u}_0 \) has an ordinate-intercept below the \( k \)-line and intersects the \( k \)-line for some \( \bar{u}_1 \) such that \( 0 < \bar{u}_1 < u_0^+ \) and passes above the \( k \)-line.

Thus, for \( u_0 < \bar{u}_0 \), the second condition gives rise to a corner solution such that \( \bar{u}_1 = 0 \) since

\[ h(\bar{u}_1; u_0) = g' \left( \int_0^{\bar{u}_0} (1+m(x)) \, dx \right) \cdot (1+m(\bar{u}_0 - \bar{u}_1)) \]

is closest to the constant \( k \) for such \( \bar{u}_1 \) in accordance with the second necessary condition (2-58), page 52. On the other hand, for \( u_0 > \bar{u}_0 \), the second necessary condition may be satisfied exactly, i.e., has an internal solution, which satisfies

(2-64)

\[ h(\bar{u}_1; u_0) = g' \left( \int_0^{u_0} (1+m(x)) \, dx \right) \cdot (1+m(u_0 - \bar{u}_1)) = k. \]
In summary, the function $\tilde{u}_{1II}(u_0)$ is defined as follows:

(i) for $u_0 > \tilde{u}_0$, $\tilde{u}_1$ is implicitly determined by (2-65) the condition (2-64), and

(ii) for $u_0 \leq u_0$, $\tilde{u}_1 = 0$.

(See Figure 2-20.)

\[ g'' \left( \int_0^{u_0-\tilde{u}_1} (1+m(x)) \, dx \right) \cdot (1+m(u_0-\tilde{u}_1))^2 \cdot \left\{ 1 - \frac{d\tilde{u}_1}{du_0} \right\} + \]
+g'(\int_0^{u_0-\bar{u}_1} (1+m(x)) dx) \cdot m'(u_0-\bar{u}_1) \cdot \left\{1- \frac{d\bar{u}_1}{du_0}\right\} = 0; \text{ or,}

\frac{d\bar{u}_1}{du_0} = 1

(2-66)

5. Determination of Strategy B

To study the B-Strategy determined by the profit-maximizing behavior of the firm, the two graphs of the functions \( \bar{u}_{1II}(u_0) \) and \( \bar{u}_{1III}(u_0) \) are superposed on the same figure. In preparation for this, recall that the \( u_0 \)-intercept of the graph of \( \bar{u}_{1II}(u_0) \) is uniquely determined by the condition

\[ D_0 \cdot g'(u_{0mn}) + D_1 \cdot g'(\int_0^{u_{0mn}} (1+m(x)) dx) \cdot (1+m(u_{0mn})) = (D_0+D_1)k \]

(2-67)

(see page 54) and that the point from which the graph of \( \bar{u}_{1III}(u_0) \) leaves the \( u_0 \)-axis is uniquely determined by the condition

\[ g'(\int_0^{u_0} (1+m(x)) dx) \cdot (1+m(\bar{u}_0)) = k; \text{ or,}
\]

\[ g'(\int_0^{u_0} (1+m(x)) dx) = \frac{k}{(1+m(\bar{u}_0))} \]

(2-68)

(See (2-64), page 61). Since \( g'(\bar{u}_0) > g'(\int_0^{u_0} (1+m(x)) dx) \) by the assumption \( g''(\cdot) < 0 \) and since \( k > k/(1+m(u_0)) \) by
the assumption that \( m(\cdot) \geq 0 \), we have the three possible cases:

\[ (2-69) \]

(a) \( k > g'(u_0) > g'(\int_0^{u_0}(1+m(x))dx) = k/(1+m(u_0)) \), or

(b) \( g'(u_0) > k > k/(1+m(u_0)) \), or

(c) \( g'(u_0) = k > k/(1+m(u_0)) \).

Now define the function

\[ K(u_0) = D_0 g'(u_0) + D_1 g'(\int_0^{u_0}(1+m(x))dx) \cdot (1+m(u_0)) \]

\[ -(D_0 + D_1) \cdot k. \]

(2-70)

The condition which determines the \( u_0 \)-intercept of the graph of the function \( u_{11}(u_0) \) may be written by (2-67), page 63, as

\[ (2-71) \quad K(u_{0mn}) = 0 \]

The function \( K(u_0) \) is monotonic decreasing since

\[ K'(u_0) = D_0 g''(u_0) + D_1 g''(\int_0^{u_0}(1+m(x))dx) \cdot (1+m(u_0))^2 \]

\[ + D_1 g'(\int_0^{u_0}(1+m(x))dx) m'(u_0) \]

(recalling that \( g''(\cdot) < 0, g'(\cdot) > 0 \), and \( m'(\cdot) \leq 0 \) by assumption).

By evaluating \( K(u_0) \) in light of (2-68), page 63, we obtain

\[ K(u_0) = D_0 \cdot \{ g'(u_0) - k \}. \]
By the decreasing monotonicity of $K(\cdot)$, we have corresponding to the three cases enumerated in (2-69), page 64, (see figure 2-21)

(a) $K(\bar{u}_0) < 0$, or $\bar{u}_0 < u_{0mn}$

(b) $K(\bar{u}_0) > 0$, or $\bar{u}_0 > u_{0mn}$

(c) $K(\bar{u}_0) = 0$, or $\bar{u}_0 = u_{0mn}$.

Graph of the Function $K(u_0)$

Figure 2-21

Taking consideration of these possibilities in the superposition of the graphs of the functions $u_{1i}(u_0)$ and $u_{1ii}(u_0)$ gives rise to three cases depicted in Plate III.
Determination of B-Strategy by Superposition of Graphs of $u_{1I}(u_0)$ and $u_{1II}(u_0)$

Plate III
We recall that at each point of the graph of the function \( u_{1I}(u_0) \) the slope exceeds unity; while the slope of the part of the graph of the function \( u_{1II}(u_0) \) which does not coincide with the \( u_0 \)-axis has slope exactly one.

**case (a):** \( \hat{u}_0 < u_{0mn} \). The graph of the function \( \bar{u}_{1I}(u_0) \) intersects the graph of the function \( \bar{u}_{1II}(u_0) \) in the part where the latter does not coincide with the \( u_0 \)-axis. The B-Strategy is \( (\hat{u}_0, \hat{u}_1) \), where \( \hat{u}_0 \) is such that \( \hat{u}_0 > u_{0mn} > \hat{u}_0 \) and \( \hat{u}_1 > 0 \). The firm has a demand for \( \hat{u}_0 > 0 \) units of input during the first period and a demand \( (\hat{u}_0 - \hat{u}_1) \) units during the second period.

**case (b):** \( \hat{u}_0 > u_{0mn} \). The graph of the function \( u_{1I}(u_0) \) intersects the graph of \( u_{1II}(u_0) \) where the latter coincides with the \( u_0 \)-axis. The B-Strategy is \( (\hat{u}_0, \hat{u}_1) \) where \( \hat{u}_0 \) is such that \( 0 < \hat{u}_0 < u_{0mn} < \hat{u}_{0B} \) and \( \hat{u}_1 = 0 \).

**case (c):** \( \hat{u}_0 = u_{0mn} \). The graph of the function \( u_{1I}(u_0) \) intersects the graph of \( u_{1II}(u_0) \) precisely where the latter leaves the \( u_0 \)-axis. Thus, the B-Strategy is \( (\hat{u}_0, \hat{u}_1) \), where \( \hat{u}_0 \) is such that \( \hat{u}_0 = \hat{u}_{0B} = u_{0mn} \) and \( \hat{u}_1 = 0 \). In both cases (b) and (c) the firm requires \( \hat{u}_0 = u_{0mn} \) units of input during each period of production.

We summarize the findings concerning the B-Strategy in the following proposition.
Proposition III. In the model stated in section A of the chapter, the behavior of a profit-maximizing firm, restricted to choosing from strategies of type B, is described as follows:

There exist unique \( u_{0mn} \) and \( \hat{u}_{0B} \) satisfying, respectively, the conditions

\[
D_0 \cdot g'(u_{0mn}) + D_1 \cdot g'\left( \int_0^{u_{0mn}} (1+m(x))dx \right) \cdot (1+m(u_{0mn})) = (D_0+D_1)k
\]

and

\[
g'(\int_0^{\hat{u}_{0B}} (1+m(x))dx) \cdot (1+m(\hat{u}_{0B})) = k.
\]

then,

(a) if \( \hat{u}_{0B} > u_{0mn} \), the firm will demand \( u_0 \) units of variable input in the first period and \( (u_0-u_1) \) units in the second period. The quantities \( u_0 \) and \( u_1 \) satisfy the inequalities \( u_0 > u_{0mn} \) and \( u_1 > 0 \).

(b) If \( \hat{u}_{0B} \leq u_{0mn} \), the firm will demand \( u_{0mn} \) units of variable input in each of the two periods.

The conclusions of the proposition are illustrated in Plate IV.
The Dynamic Factor Demand Associated with Strategy B.

Plate IV
D. Synthesis

Some preliminaries are needed before the final choice, as the more profitable, of the A-Strategy or the B-Strategy.

1. Preliminaries

First, we show that the value $u_{0mx}$, related to the A-Strategy (see propositions I and II, pages 34 and 43, respectively), and the value $u_{0mn}$, related to the B-Strategy (see proposition III, page 68), are equal. Recall the function $K(\cdot)$, (2-70), page 64,

$$K(u_0) = D_0 g'(u_0) + D_1 g'(\int_0^{u_0}(1+m(x))dx) \cdot (1+m(u_0))$$

$$-(D_0+D_1)k$$

is a monotonic decreasing function (see page 64). Both values $u_{0mx}$ and $u_{0mn}$ satisfy the condition

$$K(u_0) = 0.$$  

Thus, $u_{0mx} = u_{0mn}$. Hereafter, denote the common value by $u_{0m}$.

Second, extend the definition of the non-negative variable $u_1$ of the type A strategies to include negative values by relating it to the non-negative variable $\bar{u}_1$ of the type B strategies by

$$(2-73) \quad u_1 = -\bar{u}_1$$
Thus, the functions $\tilde{u}_{1I}(u_0)$ and $\tilde{u}_{1II}(u_0)$ derived for the B-Strategy are transformed to

\begin{equation}
(2-74) \quad u_1 = \tilde{u}_{1I}(u_0) = -\tilde{u}_{1II}(u_0) \quad \text{and} \\
\quad u_1 = \tilde{u}_{1III}(u_0) = -\tilde{u}_{1II}(u_0).
\end{equation}

The graphs of these functions are obtained by reflecting the graphs of the functions $u_{1I}(u_0)$ and $u_{1II}(u_0)$ in the $u_0$-axis. In this way, the relevant graphs of Plate III are transformed to the graphs depicted in Plate V.

For the determination of the better of the two Strategies A and B, we consider two cases corresponding to the two cases studied in relation to the A-Strategy:

(i) $m(\cdot) = \bar{m}(\cdot) = m$, a constant, and

(ii) $\theta = 0$.

Briefly, recall that the first case means that all the units of senior factor increase in efficiency uniformly and the second case means that only the senior factor learns and the "firm does not learn."
Transformation of the Axes from Variable $\bar{\mathbf{u}}_1$ to $\mathbf{u}_1$

Plate V
2. **The Submodel:** \( m(\cdot)=\bar{m}(\cdot) = \text{constant} \)

Recall that the quantity \( \hat{u}_{0A} \), the \( u_0 \)-coordinate of the point where the graph of the function \( u_{1\text{III}}(u_0) \) emerges from the \( u_0 \)-axis, is given by

\[
(2-74) \quad g'((1+m)\hat{u}_{0A}) = k/(1+\theta m), \quad 0 < \theta \leq 1, \quad (\text{proposition I, page } 34)
\]

and the quantity \( \hat{u}_{0B} \), the \( u_0 \)-coordinate of the point where the graph of the function \( u_{1\text{III}}^*(u_0) \) emerges from the \( u_0 \)-axis, is given by

\[
(2-75) \quad g'((1+m)\hat{u}_{0B}) = \frac{k}{1+m} \geq \frac{k}{1+\theta m} = g'((1+m)\hat{u}_{0A}),
\]

(proposition III, page 68).

By the decreasing monotonicity of \( g'(\cdot) \), this implies that \((1+m)\hat{u}_{0A} \leq (1+m)\hat{u}_{0B} \); or, \( \hat{u}_{0A} \leq \hat{u}_{0B} \). Thus, the "\( u_0 \)-intercept" of the graph of \( u_{1\text{III}}(u_0) \) never lies to the right of the "\( u_0 \)-intercept" of the graph of \( u_{1\text{III}}^*(u_0) \).

Finally, to obtain the sought-after synthesis, using propositions I and III and the facts just developed, simply set the graphs of the functions \( u_{1\text{I}}(u_0) \) and \( u_{1\text{III}}(u_0) \) on "top" of the graphs of the functions \( u_{1\text{I}}^*(u_0) \) and \( u_{1\text{III}}^*(u_0) \), as indicated in Plate VI, page 74.
Synthesis: \( m(\cdot) = \overline{m}(\cdot) = \text{constant} \)

Plate VI
We discuss the three cases in turn to determine the better strategy.

case (a): \( \hat{u}_{OA} \leq \hat{u}_{OB} < u_{0m} \) The A-Strategy \((u_{0A}, u_{1A}) = (u_{0m}, 0)\); while the B-Strategy \((u_{0B}, u_{1B})\) is such that \( u_{0B} > u_{0m} \) and \( u_{1} < 0 \). In this case, it is the B-Strategy which gives the greater profits.

To see this, recall the definitions of the discounted profits-functions \( \Pi_{A}(u_{0}, u_{1}) \), (2-6), page 13, and \( \Pi_{B}(u_{0}, \bar{u}_{1}) \), (2-46), page 45, and write

\[
\Pi_{A}(u_{0A}, u_{1A}) = \Pi_{A}(u_{0m}, 0) = D_{0} \cdot g(u_{om}) + D_{1} g\left(\int_{0}^{u_{om}} (1 + m(x)) dx\right)
\]

\[-(D_{0} + D_{1}) k u_{om} = \Pi_{B}(u_{0m}, 0) = \Pi_{B}(u_{0A}, u_{1A}).\]

We have already shown that the discounted profits function \( \Pi_{B}(u_{0}, \bar{u}_{1}) \) possesses a unique maximum, which we have called the B-Strategy, \((u_{0B}, u_{1B})\). (See proposition I, page 34.) That is \( \Pi_{B}(u_{0B}, u_{1B}) > \Pi_{B}(u_{0A}, u_{1A}) \); or, \( \Pi_{B}(u_{0B}, u_{1B}) > \Pi_{A}(u_{0A}, u_{1A}) \); and the B-Strategy yields greater profits.

In this case, it pays the firm to "overhire" in the first period and to dismiss in the second period a portion of the amount of factor originally hired in the first period.
case (b): \( \hat{u}_{0A} \leq u_{0m} \leq \hat{u}_{0B} \) The A-Strategy coincides with the B-Strategy \((u_{0A}, u_{1A}) = (u_{0B}, u_{1B}) = (u_{0m}, 0)\). The firm hires a certain quantity of factor during the first period and is content with exactly the same quantity during the second period. This case gives the condition under which the dynamic model is equivalent to the static case.

\( \hat{u}_{0A} \leq u_{0m} \leq \hat{u}_{0B} \) The A-Strategy \((u_{0A}, u_{1A})\) is such that \(u_{0A} < u_{0m}\) and \(u_{1A} > 0\), while the B-Strategy \((u_{0B}, u_{1B}) = (u_{0m}, 0)\). In this case, it is the A-Strategy which is preferred to the B-Strategy: the opposite of case (a).

To show this:
\[
\Pi_A(u_{0B}, u_{1B}) = \Pi_B(u_{0m}, 0) = D_0 g(u_{0m}) + D_1 \int_0^{u_{0m}} (1 + m(x)) dx
\]
\[
-(D_0 + D_1) ku_{0m} = \Pi_A(u_{0m}, 0) = \Pi_A(u_{0B}, u_{1B})
\]

The unique maximum of the discounted profits function \( \Pi_A(u_0, u_1) \) is the A-Strategy \((u_{0A}, u_{1A})\) (see proposition I, page 34). Thus, \( \Pi_A(u_{0A}, u_{1A}) > \Pi_A(u_{0B}, u_{1B}) = \Pi_B(u_{0B}, u_{1B}) \) and we conclude that the profits resulting from the A-Strategy are greater than the profits from the B-Strategy. In this case, the firm "underhires" in the first period and augments the factor-pool in the second period.

These findings are summarized in the next proposition.
Proposition IV. In the model stated in section A of the chapter, the behavior of a profit-maximizing firm is described as follows when \( m(\cdot)=\bar{m}(\cdot)=m \), a constant.

There exist unique \( u_{0m}, \hat{u}_{0A}, \) and \( \hat{u}_{0B} \) satisfying, respectively, the conditions

\[
D_0 g'(u_{0m}) + D_1 g'((1+m)u_{0m}) \cdot (1+m) = (D_0 + D_1)k,
\]

\[
g'((1+m)\hat{u}_{0A}) \cdot (1+m) = k,
\]

and \( \hat{u}_{0A} \leq \hat{u}_{0B} \). Then,

(a) If \( \hat{u}_{0A} < \hat{u}_{0B} < u_{0m} \), the firm will demand \( u_0 \) units of factor in the first period and \( (u_0+u_1) \) units of factor in the second period, where \( u_0 > u_{0m} \) and \( u_1 < 0 \). The firm "overhires" in the first period and dismisses surplus factor in the second period.

(b) If \( \hat{u}_{0A} < u_{0m} \leq \hat{u}_{0B} \), the firm will demand \( u_0 = u_{0m} \) units of factor in each of the two periods. Under these conditions, the firm behaves as it would in the usual static analysis.

(c) If \( u_{0m} < \hat{u}_{0A} \leq \hat{u}_{0B} \), the firm will demand \( u_0 \) units of factor in the first period and \( (u_0+u_1) \) units of factor in the second period, where \( u_0 < u_{0m} \) and \( u_1 > 0 \). The firm "underhires" in the first period and augments its factor-pool in the second period.

The conclusions of the proposition are illustrated in Plate VII.
Dynamic Factor Demand: \( m(\cdot) = \bar{m}(\cdot) = \text{constant} \)

Plate VII
3. **The Submodel: \( \theta=0 \)**

In order to ultimately select between the A Strategy and the B Strategy we consider this second submodel corresponding to the second submodel analyzed for the strategies of type A (see page 36).

Recall that the quantity \( \hat{u}_{0A} \), the \( u_0 \)-coordinate of the point where the graph of the function \( u_{III}(u_0) \) emerges from the \( u_0 \)-axis is given by

\[
(2-76) \quad g'(\int_0^{\hat{u}_{0A}} (1+m(x))dx) = k, \quad \text{(proposition II, page 43)},
\]

and the quantity \( \hat{u}_{0B} \), the \( u_0 \)-coordinate of the point where the graph of the function \( u_{III}(u_0) \) leaves the \( u_0 \)-axis, is given by

\[
(2-77) \quad g'(\int_0^{\hat{u}_{0B}} (1+m(x))dx) \cdot (1+m(\hat{u}_{0B})) = k, \quad \text{(proposition III, page 68)}.
\]

Combining (2-76) and (2-77), we have

\[
(2-78) \quad g'(\int_0^{\hat{u}_{0A}} (1+m(x))dx) = k \geq \frac{k}{1+m(\hat{u}_{0B})} = g'(\int_0^{\hat{u}_{0B}} (1+m(x))dx).
\]

Since \( g'(\cdot) \) is monotonic decreasing, the inequalities (2-78) imply that

\[
(2-79) \quad \int_0^{\hat{u}_{0A}} (1+m(x))dx \leq \int_0^{\hat{u}_{0B}} (1+m(x))dx; \quad \text{or,} \quad \hat{u}_{0A} \leq \hat{u}_{0B}.
\]

That is, the "\( u_0 \)-intercept" of the graph of \( u_{III}(u_0) \) never lies to the right of the "\( u_0 \)-intercept" of the graph of \( u_{III}(u_0) \).
Finally, in light of proposition II and III, and the additional facts just established, we bring together the graphs of the functions $u_{1I}(u_0)$ and $u_{1II}(u_0)$ and the graphs of the functions $u_{1I}^*(u_0)$ and $u_{1II}^*(u_0)$. The results are the same as those presented in Plate VI except that the last case, case (c), is ruled out.

The discussion of each of the cases proceeds in the manner of that preceding proposition IV beginning at page 77, except that the discussion of case (c) is ignored.

In conclusion of the discussion, we summarize the results in the next proposition.

**Proposition V.** In the model stated in section A at the beginning of the chapter, when $\theta=0$, i.e., when the firm does not learn, the behavior of a profit-maximizing firm is described as follows:

There exist unique quantities $u_{0m}$, $\hat{u}_{0A}$, and $\hat{u}_{0B}$ satisfying, respectively, the conditions

\[ D_0 \ g'(u_{0m}) + D_1 \ g'\left(\int_0^{u_{0m}} (1+m(x))dx\right) \cdot (1+m(u_{0m})) = (D_0+D_1)k, \]

\[ g''\left(\int_0^{\hat{u}_{0A}} (1+m(x))dx\right) = k, \text{ and} \]

\[ g'\left(\int_0^{\hat{u}_{0B}} (1+m(x))dx\right) \cdot (1+m(\hat{u}_{0B})) = k. \]
These quantities also satisfy the inequalities
\[ u_{0m} > \hat{u}_{0A} \quad \text{and} \quad \hat{u}_{0A} \leq \hat{u}_{0B}. \]

Then,
(a) If \( \hat{u}_{0A} \leq \hat{u}_{0B} < u_{0m} \), the firm will demand \( u_0 \) units of factor in the first period and \( (u_0 + u_1) \) units of factor in the second period, where \( u_0 > u_{0m} \) and \( u_1 < 0 \). The firm "overhires" in the first period and dismisses surplus factor in the second period.

(b) If \( \hat{u}_{0A} \leq u_{0m} < \hat{u}_{0B} \), the firm will demand \( u_0 = u_{0m} \) units of factor in each of the two periods. Under these conditions, the firm behaves as it would in the usual static analysis.

In other words, when only the senior factor learns, the firm's needs of factor never increase during the second period. The model in which \( \theta = 0 \) differs from the model in which \( m(\cdot) = \bar{m}(\cdot) = a \) constant, in that proposition V, which relates to the former model, militates against the conclusion, labeled (c), in proposition IV, which relates to the latter model.

In section B of the next chapter, at page 93, we present an example which uses a Cobb-Douglas function and in which \( \theta = 1 \) (a conventional factor-augmenting model). This example shows that proposition V which rules out one of the cases of proposition IV (namely, the case of a positive increment in the second period
demand for input over the first period demand) actually adds knowledge to that of proposition IV.

E. Example

As an application and illustration of the preceding theory, let the production function be the Cobb-Douglas function (hereafter, C-D).

\[(2-80) \quad g(z) = z^\alpha, \]

where \(0 < \alpha < 1\). This function clearly satisfies the Inada conditions ((2-2, page 8). For,

a) \(g'(z) = \alpha z^{\alpha-1} = \alpha (1/z)^{1-\alpha}\)

with \(g'(0) = \alpha (1/0)^{1-\alpha} = \infty\) and

\[(2-81) \quad g'(\infty) = \alpha (1/\infty)^{1-\alpha} = 0; \quad \text{and} \]

b) \(g''(z) = \alpha (\alpha-1) z^{\alpha-2} = -\alpha (1-\alpha) (1/z)^{2-\alpha} < 0.\)

Recall that capital is assumed to be of the putty-clay variety and that the amount and vintage is taken as a given for our analysis.

In addition, we assume that \(\theta = 0\) (no "firm-learning") and \(m(z) = m\), a constant, (that is, the increase in efficiency in the second period affects the senior factor uniformly). Thus, proposition V, page 80, applies. To determine the dynamic demand of the firm for the variable factor, according to the
proposition, we need to deduce the relationship between the quantities \( u_{0m} \) and \( \hat{u}_{0B} \).

The quantity \( u_{0m} \) is determined by the condition

\[
D_0 \cdot g'(u_{0m}) + D_1 \cdot (1+m) \cdot g'((1+m)u_{0m}) = (D_0 + D_1)k,
\]

which upon substitution of the C-D function for \( g(\cdot) \) gives

\[
(2-82) \quad D_0 \cdot \alpha \cdot u_{0m}^{\alpha-1} + D_1 \cdot \alpha \cdot (1+m) \cdot [(1+m)u_{0m}]^{\alpha-1} = (D_0 + D_1)k.
\]

Simplification yields

\[
\alpha \cdot u_{0m}^{\alpha-1} \left\{ \frac{D_0 + D_1 \cdot (1+m)^\alpha}{D_0 + D_1} \right\} = (D_0 + D_1) \cdot k; \quad \text{or}
\]

\[
(2-83) \quad u_{0m} = \left[ \frac{D_0 + D_1 \cdot (1+m)^\alpha}{D_0 + D_1} \cdot \frac{\alpha}{k} \right]^{1/(1-\alpha)}
\]

The quantity \( \hat{u}_{0B} \) is determined, according to the proposition, by the condition

\[
(1+m)g'((1+m)\hat{u}_{0B}) = k,
\]

which upon substitution of the C-D function for \( g(\cdot) \) becomes

\[
(1+m) \cdot \alpha \cdot \left[ (1+m) \hat{u}_{0B} \right]^{\alpha-1} = k; \quad \text{or}
\]

\[
(2-84) \quad \hat{u}_{0B} = \left[ (1+m)^\alpha \cdot \frac{\alpha}{k} \right]^{1/(1-\alpha)}.
\]
Thus,

\[(2-85) \quad \hat{u}_{OB} > \left( \frac{\alpha}{k} \right)^{1/(1-\alpha)} \]

Clearly, if \( m > 0 \),

\[(D_0+D_1) \cdot (1+m)^\alpha > D_0+D_1 \cdot (1+m)^\alpha \]

and

\[\frac{(D_0+D_1)}{D_0+D_1 (1+m)^\alpha} > \frac{1}{(1+m)^\alpha}\]

since all symbols denote positive numbers. Therefore,

\[(2-86) \left\{ \frac{D_0+D_1 \cdot (1+m)^\alpha \cdot \alpha}{k} \right\}^{1/(1-\alpha)} < \left[ (1+m)^\alpha \cdot \frac{\alpha}{k} \right]^{1/(1-\alpha)}\]

The left side of the inequality is \( u_{om} \), while the right side is \( \hat{u}_{OB} \). Thus, \( u_{om} < \hat{u}_{OB} \).

In this case, the proposition predicts that the firm will demand the same amount of labor in each period, namely, by (2-83),

\[(2-87) \quad u = u_{om} = \left\{ \frac{D_0+D_1 \cdot (1+m)^\alpha \cdot \alpha}{k} \right\}^{1/(1-\alpha)}\]

Since we are now assuming a production function having a particular form, we verify directly what was done more abstractly in the theoretical discussion.

First, proposition V predicts that \( \hat{u}_{0A} \leq \hat{u}_{0B} \). We already have \( \hat{u}_{0B} \) in (2-84), page 83. The formula for
\( \hat{u}_{0A} \), as given by proposition \( V \), is

\[
g'((1+m)\hat{u}_{0A}) = k
\]

which yields upon substitution of the C-D function for \( g(\cdot) \)

\[
\alpha \cdot \left[ (1+m)\hat{u}_{0A} \right]^{\alpha-1} = k; \quad \text{or}
\]

\[
(2-88) \quad \hat{u}_{0A} = \left[ \frac{1}{(1+m)(1-\alpha)} \right]^{1/(1-\alpha)}
\]

Hence \( u_{0A} < (\alpha/k)^{1/(1-\alpha)} \). In light of (2-85), we can conclude that

\[
(2-89) \quad \hat{u}_{0A} < \hat{u}_{0B}.
\]

Second, we verify that both the A-Strategy and the B-Strategy are corner solutions. As to the A-Strategy, the two necessary and sufficient conditions are

\[
(2-90-I) \quad D_0 \cdot g'(u_0) + D_1 \cdot (1+m) \cdot g'((1+m)u_0+u_1) - (D_0+D_1)k = 0
\]

(see (2-6), page 13), and

\[
(2-90-II) \quad g'((1+m)u_0+u_1) \text{ as close as possible to } k
\]

(see (2-8), page 19), such that \( u_0, u_1 \geq 0 \). If an internal solution exists, (2-90-II) is an equality. Substituting (2-90-II), in the equality form, into (2-90-I)
yields

\[ D_0 \cdot g'(u_0) + D_1 \cdot (1+m) \cdot k - (D_0 + D_1) k = 0; \]

or,

(2-91) \[ D_0 \cdot \{g'(u_0) - k\} + D_1 mk = 0. \]

This implies that \{g'(u_0) - k\} < 0; or

(2-92) \[ g'(u_0) < k. \]

Since \( g'(\cdot) \) is monotonic decreasing, (2-90-II), in equality form, is inconsistent with (2-92). Thus, the conditions (2-90) cannot have an internal solution. The method used here differs from that used in the proof of propositions II and V (see page 43).

As to the B-Strategy, our argument depends on the particular assumed form of the production function. The two necessary and sufficient conditions for an internal solution for the B-Strategy are

(2-93-I) \[ D_0 \cdot g'(u_0) + D_1 \cdot (1+m) \cdot g'((1+m)(u_0 - \bar{u}_1)) - (D_0 + D_1) k = 0, \]

(see (2-55), page 50), and

(2-93-II) \[ (1+m) \cdot g'((1+m)(u_0 - \bar{u}_1)) = k \]

(see (2-57), page 52), such that \( u_0, \bar{u}_1 \geq 0 \) and \( \bar{u}_1 \leq u_0 \). Substitution of (2-93-II) into (2-93-I) yields
This means, in terms of the C-D function,
\[ \alpha u_0^{\alpha-1} = k; \text{ or } g'(u_0) = k. \]

Thus, condition (2-93-III) as it applies to our example becomes
\[ (1+m) \cdot \alpha \cdot [(1+m) \cdot (u_0 - \bar{u}_1)]^{\alpha-1} = k. \]

Hence,
\[ u_0 - \bar{u}_1 = \left( \frac{1}{1+m} \right)^{\alpha/(\alpha-1)} \cdot \left( \frac{k}{\alpha} \right)^{1/(\alpha-1)} = (1+m)^{\alpha/(1-\alpha)} \left( \frac{\alpha}{k} \right)^{1/(1-\alpha)}. \]

Using (2-94) in (2-95) gives
\[ \bar{u}_1 = u_0 - (1+m)^{\alpha/(1-\alpha)} \left( \frac{\alpha}{k} \right)^{1/(1-\alpha)} = \left( \frac{\alpha}{k} \right)^{1/(1-\alpha)} [1 - (1+m)^{\alpha/(1-\alpha)}]. \]

Since \( \frac{\alpha}{1-\alpha} > 0 \), \( (1+m)^{\alpha/(1-\alpha)} > 1 \). Hence, the factor in square brackets in (2-96) is negative. This means that \( (\alpha-1) \) must be such that extracting the \( (\alpha-1) \)st root yields a negative root. In addition, the negativity of \( \bar{u}_1 \) in (2-96) is implied by the assumed existence of an internal solution. The non-negativity constraint on \( \bar{u}_1 \), however, militates against such an internal solution. It follows, as it did in the case of the A-Strategy, that the B-Strategy must be a corner
solution. The plan of the firm is, if you will, a "double" corner solution - a corner solution both for Strategy A and for Strategy B. This result is depicted in the two following figures.

The Firm's Planned Determination of the Factor Demand Over the Planning Horizon

Figure 2-22

Determination of the Firm's Plan for Factor Demand

Figure 2-23
CHAPTER III
COMPARISON OF A SIMPLE LEARNING MODEL
WITH A CONVENTIONAL FACTOR-AUGMENTING MODEL

In this chapter, we introduce a conventional factor-augmenting model and compare it, vis-à-vis the firm's employment plans and the sensitivity thereof to the relative wage, with a simple model without firm-learning. In addition, we develop an example of the conventional model with the Cobb-Douglas production function and compare it to the example, discussed at the end of the preceding chapter, which also uses the Cobb-Douglas production function in the simple model without firm-learning.

A. Conventional Factor-Augmenting Model

In the general framework of section A of the preceding chapter, we set \( \theta = 1 \) and \( m(\cdot) = \bar{m}(\cdot) = m \), a constant, to obtain the factor-augmenting model which we now discuss.

The reason for the distinction of the last chapter between strategies of type A and strategies of type B disappears and the pair of necessary conditions for each type of strategy becomes a single pair
of necessary conditions. The first necessary condition for the A-Strategy (2-6), page 17, becomes

\[ 0 = D_0 \cdot \left\{ g'(u_0) - k \right\} + D_1 \cdot \left\{ g'(\int_0^{u_0} (1+m(x)) \, dx + \int_0^{u_1} (1+\theta \tilde{m}(x)) \, dx) \right\} \cdot (1+m(u_0)) - k \]

(3-1)

\[ = D_0 \cdot \left\{ g'(u_0) - k \right\} + D_1 \cdot \left\{ g'((1+m)(u_0+u_1)) \cdot (1+m) - k \right\} , \]

where \( u_0, u_1 \geq 0 \); and the first necessary condition for the B-Strategy (2-8), page 21, becomes

\[ 0 = D_0 \cdot \left\{ g'(u_0) - k \right\} + D_1 \cdot g'\left(\int_0^{u_0} - \tilde{u}_1 (1+m(x)) \, dx \right) \cdot (1+m(u_0)) - k \]

(3-2)

\[ = D_0 \cdot \left\{ g'(u_0) - k \right\} + D_1 \cdot \left\{ g'((1+m)(u_0-\tilde{u}_1)) \cdot (1+m) - k \right\} , \]

where \( u_0, \bar{u}_1 \geq 0 \) and \( \bar{u}_1 \leq u_0 \). Both of these conditions may be expressed as

\[ 0 = D_0 \cdot \left\{ g'(u_0) - k \right\} + D_1 \cdot \left\{ g'((1+m)(u_0+u_1)) - k \right\} , \]

(3-3)

where \( u_0 \geq 0 \) and, if \( u_1 < 0 \), then \( |u_1| \leq u_0 \).

By similar reasoning, the two second necessary conditions for the A-Strategy and for the B-Strategy may be expressed as one. The second necessary condition for the A-Strategy (2-55), page 50, becomes
\[ k = g' \left( \int_{0}^{u_0} (1+m(x)) \, dx + \int_{u_0}^{u_1} (1+\theta m(x)) \, dx \right) \cdot (1+\theta m(u_1)) \]

\[ = g'((1+m)(u_0+u_1)) \cdot (1+m), \]

(3-4)

where \( u_0, u_1 \geq 0 \); and the second necessary condition for the B-Strategy (2-58), page 52, becomes

\[ k = g' \left( \int_{0}^{u_0-\bar{u}_1} (1+m(x)) \, dx \right) \cdot (1+m(\bar{u}_1)) \]

\[ = g'((1+m)(u_0-\bar{u}_1)) \cdot (1+m), \]

(3-5)

where \( u_0, \bar{u}_1 \geq 0 \) and \( |\bar{u}_1| \leq u_0 \). These two conditions collapse into the single condition

\[ k = g'((1+m)(u_0+u_1)) \cdot (1+m), \]

(3-6)

where \( u_0 \geq 0 \) and, if \( u_1 < 0 \), then \( |u_1| \leq u_0 \).

Suppose that condition (3-6) holds. Then substitution in (3-3) implies that

\[ g'(u_0) = k \]

(3-7)

This completely determines \( u_0 \). In the \((u_0, u_1)\)-plane, the graph of (3-7) is represented as a vertical line.

On the other hand, in relation to condition (3-6), since \( g'(\cdot) \) is monotonic decreasing, there exists a \( w \) such that

\[ g'(w) = k/(1+m). \]

(3-8)
Thus, $(1+m)(u_0+u_1)=w > 0$ and $(u_0+u_1) > 0$; or $u_1 \geq -u_0$.

If $u_1$ should be negative, then the preceding inequality implies $|u_1| \leq u_0$. Hence, any solution $u_1$ satisfying (3-6) automatically satisfies the accompanying constraint that $|u_1| \leq u_0$.

The graph of condition (3-8) is simply the straight line-segment with the algebraic representation

$$ (3-9) \quad (1+m)(u_0+u_1) = w, $$

where $w$ is defined by (3-8) and $u_0 \geq 0$. Its graph has a slope of negative unity and $u_0$-intercept, denoted $u_{00}$.

$$ (3-10) \quad u_{00} = \frac{w}{1+m}. $$

Solution of the Conventional Factor-Augmenting Model

Figure 3-1

As figure 3-1 suggests, we differentiate three cases depending on whether $u_{00}$, the $u_0$-intercept of the relation (3-9), is to the left, coincident with, or
to the right of $u_0$. The relative positions of $u_0$ and $u_{00}$ naturally depend upon the properties of the particular production function one uses as the example presented in the next subsection illustrates.

B. Example of the Conventional Model Using a Cobb-Douglas Production Function

The example about to be discussed shows that proposition V contains information additional to that contained in proposition IV. In the example, the firm's planned demand for input in the second period increases over that for input in the first period - a phenomenon which proposition V says is excluded when $\theta=0$ (i.e., when no firm-learning takes place).

Again, we let

\[(3-11) \quad g(z) = z^\alpha,\]

the Cobb-Douglas (hereafter, C-D) production function. We have noted at (2-81), page 82, that this production function satisfies the Inada Conditions of Chapter II.

We determine the firm's plan for inputs by using conditions (3-7), (3-8), and (3-9). Condition (3-7) becomes

$$\alpha u_0^{\alpha-1} = k; \text{ or,}$$

\[(3-12) \quad u_0 = \left(\frac{\alpha}{k}\right)^{1/(1-\alpha)}.\]
Conditions (3-8) and (3-9) become

\[
\frac{k}{1+m} = \alpha \left[ (1+m)(\hat{u}_0 + \hat{u}_1) \right]^{\alpha-1}, \quad \text{or,}
\]

\[
\hat{u}_0 + \hat{u}_1 = \left[ \frac{\alpha}{k} (1+m)^\alpha \right]^{1/(1-\alpha)}.
\]

(3-13)

Thus, solving for \( \hat{u}_1 \) in (3-13) and substituting for \( \hat{u}_0 \) from (3-12), we obtain

\[
\hat{u}_1 = \left( \frac{\alpha}{k} \right)^{1/(1-\alpha)} \left[ (1+m)^{\alpha/(1-\alpha)-1} \right] > 0,
\]

where the term in the brackets exceeds one if \( m, \alpha > 0 \).

We see that the \( u_0 \)-intercept, \( u_{00} \), of relation (3-9) is given by (3-13) as

\[
u_{00} = \left[ \frac{\alpha}{k} (1+m)^\alpha \right]^{1/(1-\alpha)}
\]

and \( \hat{u}_1 = u_{00} - \hat{u}_0 > 0 \).

Solution of the Conventional Model with Cobb-Douglas Production Function

Figure 3-2
Thence, we see that with a Cobb-Douglas production function, the learning model with \( \theta=1 \) and \( m(\cdot)=\bar{m}(\cdot) \) a constant (which we have called the conventional factor-augmenting model) predicts an increase in the firm's planned input requirement for the second period over that for the first period. This is to be contrasted with the result obtained for the learning model with \( \theta=0, \) \( m=\)constant, in section E of the preceding chapter at page 82, et seq., where it was shown that the firm never increases its demand in the second period.

C. The Price-Sensitivity of the Firm's Employment Plans in the Conventional Model

We start with the relations (3-7), (3-8), and (3-9), which determine the firm's demand for inputs in each period. Differentiation of the relation (3-7) with respect to the relative wage \( k \) yields

\[
(3-16) \quad \frac{d u_0}{d k} = \frac{1}{g''(u_0)} < 0.
\]

If the wage increases, the demand for first period input decreases.

Differentiating relations (3-8) and (3-9) with respect to \( k \) results in

\[
(3-17) \quad \frac{d(u_0+u_1)}{d k} = \frac{du_0}{d k} + \frac{du_1}{d k} = \frac{1}{(1+m)2g''((1+m)(u_0+u_1))} < 0.
\]

The demand for input in the second period, likewise, decreases in the second period.
By solving for \( \frac{du_1}{dk} \) in (3-13) and substituting for \( \frac{du_0}{dk} \) from (3-12), we obtain

\[
(3-18) \quad \frac{du_1}{dk} = \frac{1}{(1+m)^2} \cdot \frac{1}{g''((1+m)(u_0+u_1))} - \frac{1}{g'(u_0)}
\]

Though the input requirement in the second period decreases, the difference \( u_1 \) between the first period and the second period demand may increase:

\[
\frac{du_1}{dk} \leq 0 \text{ if, and only if } (3-19) \quad (1+m)^2 \cdot g''((1+m)(u_0+u_1)) \geq g''(u_0).
\]

In the example with the Cobb-Douglas production function in the preceding section, this difference \( u_1 \) was an increment. In this special case we show that the increment in input requirement for the second period decreases with an increase in the relative wage.

The change in first period demand for input given by (3-16) becomes, in light of (3-12):

\[
(3-20) \quad \frac{du_0}{dk} = \frac{1}{g''(u_0)} = \frac{1}{\alpha(\alpha-1)u^{\alpha-2}} = \frac{u^{\alpha-2}}{\alpha(\alpha-1)} = \frac{\left(\frac{u}{k}\right)^{\frac{2-\alpha}{1-\alpha}}}{\alpha(1-\alpha)} < 0
\]

since \( \alpha < 1 \). The change in second period demand for input follows from (3-17) by virtue of (3-13):
\[
\frac{d(u_0 + u_1)}{dk} = \frac{1}{(1+m)^2 \cdot g''((1+m)(u_0+u_1))} = \\
\frac{(1+m)^2 \cdot \alpha \cdot (1 - \alpha) \cdot \left[\frac{-(u_0 + u_1)^{1-\alpha}}{\alpha \cdot (1-\alpha) \cdot (1+m)^{1-\alpha}} \right]^2}{[(1+m)(u_0+u_1)]^{-1}} \\
\tag{3-21}
\]

Thus,

\[
\frac{du_1}{dk} = \frac{d(u_0+u_1)}{dk} - \frac{du_0}{dk} = -\left(\frac{\alpha}{1-\alpha}\right)^{2-\alpha} \cdot \left(\frac{k}{1-\alpha}\right)^{2-\alpha} \cdot \left[(1+m)^{1-\alpha} - 1\right] < 0
\]

since \( \alpha < 1 \) and since the term in the brackets is positive when \( m > 0 \). Thus, the second period increment in input demand decreases with an increase in the relative wage when the production function is Cobb-Douglas.

D. Price Sensitivity of the Firm's Employment Plans for the Case \( \theta = 0, m=\text{constant} \)

A discussion of the sensitivity of the model with \( \theta = 0 \) and \( m=\text{constant} \) (that is, no firm-learning takes place and efficiency increases in the second period are uniformly distributed among the senior factor) to the price of output or to the wage requires the consideration of several different cases because of the presence of discontinuities in the conditions governing the choice of the optimal employment plan. We consider
the different cases permissible for the A-Strategy, then naturally move to those cases permissible for the B-Strategy, and finally gather the results in a proposition at the end of the section.

According to proposition V, page 80, the three quantities \( u_{0m}, \hat{u}_{0A}, \) and \( \hat{u}_{0B} \) are determined uniquely by the following conditions, which are obtained by specializing the conditions, listed in the proposition, to our submodel:

\[
(D_0 + D_1)k = D_0 \cdot g'(u_{0m}) + D_1 \cdot (1 + m(u_{0m}))
\]

\[
\cdot g'(\int_0 u_{0m}(1 + m(x))dx)
\]

(3-23)

\[
=D_0 \cdot g'(u_{0m}) + D_1 \cdot (1 + m) \cdot g'((1 + m)u_{0m}).
\]

(3-24) \( k = g'(\int_0 ^A (1 + m(x))dx) = g'((1 + m)\hat{u}_{0A}). \)

(3-25) \( k = g'(\int_0 ^B (1 + m(x))dx) \cdot (1 + m(\hat{u}_{0B})) = g'((1 + m)\hat{u}_{0B}). \)

For the A-Strategy, the conclusion of proposition V states that only one case is permissible: \( u_0 > 0, \ u_1 = 0, \) where \( u_{0m} > \hat{u}_{0A}. \) The first necessary condition for the A-Strategy is obtained by setting \( \theta = 0 \) and \( m = \) constant in (2-6), page 17,

\[
(D_0 + D_1)k = D_0 \cdot g'(u_0) + D_1 \cdot (1 + m)^2 \cdot g'((1 + m)u_0 + u_1),
\]

(3-26)
where $u_0$, $u_1 > 0$. With the variable $u_1$ held constant and the variable $u_0$ permitted to adjust, implicit differentiation of (3-26) with respect to $k$ gives

$$D_0 \cdot g''(u_0) \frac{du_0}{dk} + D_1 \cdot (1+m)^2 \cdot g''((1+m)u_0 + u_1) \frac{du_0}{dk} = (D_0 + D_1),$$

which upon solution for $\frac{du_0}{dk}$ becomes

$$(3-27) \quad \frac{du_0}{dk} = \frac{D_0 + D_1}{D_0 \cdot g''(u_0) + D_1 \cdot (1+m)^2 \cdot g''((1+m)u_0 + u_1)} < 0$$

since $g''(\cdot) < 0$. Thus, the graph of $u_1(u_0)$ shifts to the left; as does the $u_0$-intercept, a result obtained from (3-23) by a similar procedure, or by setting $u_1 = 0$ in (3-27):

$$(3-28) \quad \frac{du_0}{dk} = \frac{D_0 + D_1}{D_0 \cdot g''(u_0) + D_1 \cdot (1+m)^2 \cdot g''((1+m)u_0 + u_1)} < 0.$$  

The graph of the second necessary condition for the $A$-Strategy is essentially two straight line-segments, one coinciding with the $u_0$-axis and the other having a negative slope. The portion of the graph not coinciding with the $u_0$-axis has slope given by the following formula, obtained by specializing (2-19), page 28, to the present model:
\[ u'_{1III}(u_0) = \frac{\left[ g''(\int_0^u (1+m(x))dx+\int_0^{u_1(1+m(x))dx}\cdot(1+\theta m(u_1))(1+m(u_0)) \right]}{\left[ g''(\int_0^u (1+m(x))dx+\int_0^{u_1(1+m(x))dx}\cdot(1+\theta m(u_1))^2 \right]} + \theta \cdot g'\left(\int_0^u (1+m(x))dx+\int_0^{u_1(1+m(x))dx}\cdot m'(u_1)\right) \]

\[ = \frac{-g''((1+m)(u_0+u_1))(1+m)}{g''((1+m)(u_0+u_1))} = -(1+m), \]

a constant.

Implicit differentiation of (3-24) with respect to \( k \) yields

\[ (3-29) \quad \frac{d\hat{u}_0}{dk} = \frac{1}{(1+m)} - \frac{1}{g''((1+m)\hat{u}_0)} < 0. \]

Thus, the graph of the function \( u'_{1III}(u_0) \) shifts to the left just as that of the function \( u'_{1I}(u_0) \).

Effect of an Increase in the Relative Wage on the Determination of Strategy A

![Figure 3-3](image_url)
The only possible case that arises for the A-Strategy is depicted in figure 3-3. The shifts of the respective graphs resulting from an increase in $k$ are represented by the dotted graphs: both graphs shifting to the left. Since $u_{0m} > \hat{u}_{0a}$ and the optimal plan requires the firm to hire $u_{0m}$ units of factor in each period, the wage/price sensitivity in this case is given by (3-28):

$$\frac{d\hat{u}_0}{dk} = \frac{du_{0m}}{dk} = \frac{D_0 + D_1}{D_0 \cdot g''(u_{0m}) + D_1 \cdot (1+m) \cdot g''((1+m)u_{0m})} < 0$$

(3-30)

and $$\frac{d\hat{u}_1}{dk} = 0.$$

Hence, the level of employment is decreased by the same amount in each period when the wage increases or the price of output decreases. This result is shown in Figure 3-4.

Effect on the Level of Employment of an Increase in the Relative Wage

Figure 3-4
In contrast to the single case for the A-Strategy, three cases are allowable for the B-Strategy. The first necessary condition for the B-Strategy (2-55), upon substitution of $\theta=0$ and $m=\text{constant}$, becomes

\[(D_0+D_1)\cdot k = D_0\cdot g'(u_0) + D_1\cdot (1+m(u_0+u_1))\]

\[(3-29)\]

\[g'(\int_0^{u_0+u_1} (1+m(x))\,dx)\]

\[= D_0\cdot g'(u_0) + D_1\cdot (1+m)\cdot g'((1+m)(u_0+u_1))\]

where $u_0 \geq 0$, $u_1 \leq 0$, and $|u_1| \leq u_0$. Implicit differentiation of (3-29) with respect to the relative wage $k$, allowing $u_0$ to adjust while holding $u_1$ fixed, gives

\[D_0+D_1 = D_0\cdot g''(u_0)\frac{du_0}{dk} + D_1\cdot (1+m)^2\cdot g''((1+m)(u_0+u_1))\frac{du_0}{dk}\]

Finally, solution for $\frac{du_0}{dk}$ yields the expression

\[(3-30)\]

\[\frac{du_0}{dk} = \frac{D_0+D_1}{D_0\cdot g''(u_0)+D_1\cdot (1+m)^2\cdot g''((1+m)(u_0+u_1))} < 0\]

which indicates that the graph of the function $u_{11}^*(u_0)$ shifts to the left. Upon setting $u_1=0$, we obtain the formula (3-28), above, which describes the sensitivity of the $u_0$-intercept of the graph of $u_{11}^*(u_0)$ to a wage increase or a decrease in the price of the output.

Figure 3-5 depicts the graph of the function $u_{11}^*(u_0)$.
after an increase in the relative wage, as a dotted graph.

![Graph of the effect of increase in the relative wage](image)

**Effect of Increase in the Relative Wage on the Graph of** $u_{iI}^*(u_0)$

**Figure 3-5**

The graph of the second necessary condition for the B-Strategy is composed of two straight line-segments, one coinciding with the $u_0$-axis from the origin to the point $u_0 = \hat{u}_0B$, the other emerging from the point $u_0 = \hat{u}_0B$ with the constant slope, (2-66), page 63,

\[(3-31) \quad \frac{du_1}{du_0} = -1.\]

This graph of the function $u_{iIII}^*(u_0)$ also shifts to the left as may be easily seen by implicitly differentiating (3-25) with respect to the relative wage $k$

\[(3-32) \quad \frac{\hat{u}_{0B}}{dk} = \frac{1}{(1+m)^2 \cdot g''((1+m)\hat{u}_{0B})} < 0.\]
that is, the point from which the second infinite segment of the graph of the function \( u^*_{III}(u_0) \) emerges from the \( u_0 \)-axis moves closer to the origin as \( k \) increases. The shifting of the graph of \( u^*_{III}(u_0) \) as \( k \) increases is depicted in figure 3-6.

![Graph illustration](image)

**Effect of Increase in the Relative Wage on the Graph of \( u^*_{III}(u_0) \)**

Figure 3-6

Strategy B may arise from three sets of circumstances which we now discuss.

**case (a):** \( u_{0m} < \hat{u}_{0B} \). The case is depicted in figure 3-7. Since the intersection of the graphs of the functions \( u^*_{II}(u_0) \) and \( u^*_{III}(u_0) \) occurs in the interior of the portion of the graph of \( u^*_{III}(u_0) \) which coincides with the \( u_0 \)-axis, the sensitivity of the B-Strategy is determined by the sensitivity of the \( u_0 \)-intercept \( u_{0m} \) of the graph of \( u^*_{II}(u_0) \), (3-28), page 99. Thus, the firm decreases its demand for input equally in each period. This is illustrated in
case (b): $u_{0m} = \hat{u}_{OB}$

This case, depicted in figure 3-8, is more delicate since it requires a comparison of the two graphs of the functions $u_{II}^{*}(u_0)$ and $u_{III}^{*}(u_0)$ to determine which shifts greater to the left. In order to accomplish this end, we compare the expression (3-28), page 99, and (3-32), page 103. Define

$$x = \frac{1}{\frac{d\hat{u}_{OB}}{dk}}$$

(3-33)

\[
(1+m)^2 \cdot g''((1+m)\hat{u}_{OB}),
\]

and

$$y = \frac{1}{\frac{du_{0m}}{dk}} = \frac{D_0 \cdot g'(u_{0m}) + D_1}{D_0 + D_1} \cdot x$$

Strategy B: $u_{0m} = \hat{u}_{OB}$

Figure 3-7
We are interested in the relationship between $x$ and $y$. Since $y$ is a weighted average of $g''(u_{0m})$ and $x$, with weights $D_0/(D_0+D_1)$ and $D_1/(D_0+D_1)$, which are obviously positive and sum to unity, the quantity $y$ always lies between $g''(u_0)$ and $x$. Figure 3-9 shows the case $g''(u_{0m}) > x$; but the inequality may run in the opposite direction because we have placed no restrictions on $g''(\cdot)$ other than that it be negative. In this case, the points marked $x$ and $g''(u_{0m})$ would evidently be switched around.

\[
\begin{array}{ccc}
x & y & g''(u_{0m}) \\
\end{array}
\]

Relationship of the Quantities $x$ and $y$

Figure 3-9

If $g''(u_{0m}) > x$, then $y < x$ and $\frac{du_{0m}}{dk} > \frac{du_{0B}}{dk}$; or $\left|\frac{du_{0m}}{dk}\right| < \left|\frac{du_{0B}}{dk}\right|$. The graph of the function $u_{II}^*(u_0)$ shifts to the left less than the graph of the function $u_{II}^*(u_0)$. Hence, as figure 3-10 suggests, when the relative wage $k$ increases, the character of the original Strategy B is changed; and this subcase may be treated like an "internal" intersection; that is, one which is internal to the graph of the function $u_{II}^*(u_0)$ and internal to the portion of the graph of the func-
tion $u_{III}^*(u_0)$ which does not coincide with the $u_0$-axis. This circumstance is treated as case (c), below.

If, instead, $g''(u_{0m}) < x$, then $y > x$ and

$$\frac{du_{0m}}{dk} < \frac{\hat{u}_{OB}}{dk}, \text{ or } \left| \frac{du_{0m}}{dk} \right| > \left| \frac{\hat{u}_{OB}}{dk} \right|. $$

The graph of the function $u_{II}^*(u_0)$ shifts more to the left than does that of $u_{III}^*(u_0)$ so that the former graph comes to intersect the latter graph in the interior of the portion coinciding with the $u_0$-axis. The Strategy B is completely determined; the firm requiring $u_{0m}$ units of factor in each period. The sensitivity of this employment plan of the firm to an increase in the relative wage is given by (3-28), page 99:

$$\frac{du_0}{dk} = \frac{D_0 + D_1}{D_0 \cdot g''(u_{0m}) + D_1 \cdot (1+m)^2 \cdot g''((1+m)u_{0m})}$$ and
This subcase is illustrated in figure 3-11.

\[
\frac{du_1}{dk} = 0.
\]

Determination of Strategy B:

\[
u_{0m} = \hat{u}_{OB}, \quad g''(u_{0m}) < x
\]

Figure 3-11

\textbf{case (c): } u_{0m} > \hat{u}_{OB}. \text{ The first necessary condition for the B-Strategy was obtained as (3-29), page 102. The second necessary condition for the B-Strategy for our submodel is obtained from (2-57), page 52, by substituting } \theta=0 \text{ and } m=\text{constant.}

\begin{align*}
0 &= g' \left( \int_0^{u_0+u_1} (1+m(x)) \, dx \right) \cdot (1+m(u_0+u_1)) - k \\
&= g'((1+m)(u_0+u_1)) \cdot (1+m) - k,
\end{align*}

where \( u_0 > 0, \quad u_1 \leq 0 \) \text{ and } |u_1| \leq u_0. \text{ Conditions (3-29) and}
(3-34) are exactly the conditions (3-3) and (3-6), pages 90 and 91, which determine the firm's employment plans in what we called the conventional model. Thus, the price sensitivity in this case of the B-Strategy is given by formulas (3-16), (3-17), and (3-18), pages 95 and 96. It will be recalled that with an increase in the relative wage both the first period input and the second period input demands decrease, in general, by different amounts; while the difference between the two may decrease, increase, or remain the same.

In the preceding case, we had to leave unfinished the analysis of the subcase, corresponding to \( g''(u_{0m}) < (1+m)^2 \cdot g'(1+m)u_{0m} \), since it had to be treated as an "internal" solution. We see that both \( u_0 \), the first period input demand, decreases and \( u_0 + u_1 \), the second period input demand also decreases. The effect on the
difference $u_1$ can be unambiguously predicted in this subcase:

\[ \frac{du_1}{dk} = \frac{1}{(1+m)^2 g''((1+m)u_0m)} - \frac{1}{g''(u_0m)} < 0. \]

Finally, we summarize the results just obtained in the following proposition.

**Proposition VI.** The sensitivity to a change in the relative wage of the optimal employment plans of a firm, hypothesized in section A of Chapter II, which in addition satisfies the assumptions that there is no firm-learning ($\Theta=0$) and that the senior factor becomes uniformly more efficient in the second period ($m(\cdot)=\text{constant}$), is described as follows:

There exist quantities $u_{0m}$, $\hat{u}_{0A}$, and $\hat{u}_{0B}$ which uniquely satisfy the relations

\[ D_0 \cdot g'(u_{0m}) + D_1 \cdot (1+m) \cdot g'((1+m)u_{0m}) = (D_0 + D_1) \cdot k, \]

\[ g'((1+m)\hat{u}_{0A}) = k, \text{ and} \]

\[ g'((1+m)\hat{u}_{0B}) \cdot (1+m) = k. \]

(a) If $u_{0m} < \hat{u}_{0B}$, then at the optimal solution

\[ (\hat{u}_0, \hat{u}_1) = (u_{0m}, 0), \]
\[
\frac{d\hat{u}_0}{dk} = \frac{D_0 + D_1}{D_0 \cdot g''(u_{0m}) + D_1 \cdot (1+m)2g''((1+m)u_{0m})} < 0
\]

and

\[
\frac{d\hat{u}_1}{dk} = 0.
\]

(b) If \( u_{0m} > \hat{u}_{OB} \), then at the optimal solution \((\hat{u}_0, \hat{u}_1)\)

\[
\frac{d\hat{u}_0}{dk} = \frac{1}{g''(\hat{u}_0)} < 0,
\]

\[
\frac{d(\hat{u}_0 + \hat{u}_1)}{dk} = \frac{1}{(1+m)^2 \cdot g''((1+m)(\hat{u}_0 + \hat{u}_1))} < 0, \text{ and}
\]

\[
\frac{d\hat{u}_1}{dk} = \frac{1}{(1+m)^2 \cdot g''((1+m)(\hat{u}_0 + \hat{u}_1))} - \frac{1}{g''(\hat{u}_0)}.
\]

The sign of the last derivative is in general indeterminate.

(c) If \( u_{0m} = \hat{u}_{OB} \), and, in addition,

(i) \( g''(u_{0m}) < (1+m)^2 \cdot g''((1+m)u_{0m}) \), then the conclusion of case (a) holds; or

(ii) \( g''(u_{0m}) > (1+m)^2 \cdot g''((1+m)u_{0m}) \), then the conclusion of case (b) holds and the ambiguity of the sign of \( \frac{d\hat{u}_1}{dk} \) is resolved:

\[
\frac{d\hat{u}_1}{dk} < 0.
\]

The results of the proposition are displayed in Plate VIII.
Depiction of the Results of Proposition VI

Plate VIII
By comparison with section C, we see that the conventional factor-augmenting model may coincide with the model under discussion, insofar as price sensitivity is concerned, only in cases (b) and (c)(ii).

Next, we apply proposition VI to the example of section E of Chapter II, page 82. The example, it will be recalled dealt with a learning model in which there is no firm-learning (θ=0) and in which the increase in efficiency in the second period affects the senior factor uniformly (m(·)=constant). Moreover, the production function was assumed to be Cobb-Douglas. As figure 2-23, page 88, clearly shows, the example satisfies the inequality, \( u_{0m} < \hat{u}_B \), and hence falls under case (a) of the proposition. Substitution of \( g''(x) = -\alpha(1-\alpha)x^{\alpha-2} \) and of (2-87), page 84, for \( u_{0m} \) into the formulas of the proposition gives

\[
\frac{d\hat{u}_0}{dk} = \frac{D_0+D_1}{D_0 \cdot g''(u_0) + D_1 \cdot (1+m)^2 \cdot g''((1+m)u_0)}
\]

\[
= \frac{D_0+D_1}{D_0 \cdot \alpha \cdot (\alpha-1) \cdot u_{0m}^{\alpha-2} + D_1 \cdot \alpha \cdot (\alpha-1) \cdot (1+m) \cdot u_{0m}^{\alpha-2}}
\]

(3-36)

\[
= \frac{D_0+D_1}{\alpha \cdot (\alpha-1) \cdot [D_0+D_1 \cdot (1+m)^\alpha]} \cdot u_{0m}^{2-\alpha}
\]

\[
= \frac{D_0+D_1}{\alpha \cdot (\alpha-1) \cdot [D_0+D_1 \cdot (1+m)^\alpha]} \left[ \frac{D_0+D_1 \cdot (1+m)^\alpha}{D_0+D_1} \cdot \frac{\alpha}{k} \right]^{2-\alpha} \left( \frac{\alpha}{k} \right)^{1-\alpha} < 0;
\]
Thus, the demand for input in the first period decreases with an increase in the relative wage, and so does, by an equal amount, the demand for input in the second period.

Comparison of the price sensitivity (3-36) of the differentiated learning model with Cobb-Douglas production function, the example just discussed, with the conventional factor-augmenting model which also uses the Cobb-Douglas production function, (3-20), page 96, indicates that the first period demand for input of the learning model is more sensitive by a factor of

\[
\frac{D_0 + D_1}{D_0 + D_1} \cdot (1+m)^\alpha
\]

than that for the conventional model and that the second period increment for the conventional model is decreased, (3-22), page 97; whereas the level of second period demand for input in the learning model is, as it was without the relative wage increase, unchanged by (3-37) above. In short, the two examples of the conventional and the learning models using the Cobb-Douglas function differ in the sensitivity of their optimal employment plans to an increase in the relative wage. Plate IX illustrates these differences.
Comparison of Sensitivities of Two Models to an Increase in Relative Wage

Plate IX
CHAPTER IV
EXPECTED CHANGING PRICES AND WAGES IN A LEARNING MODEL

In this chapter, we concentrate on a modification of a model deduced as a special case of the general theory developed in chapter two. We apply the techniques of that chapter directly to a modified version of the submodel, which corresponds to uniformity in the increased efficiency of the experienced factor ($m(\cdot)=\text{constant}$) and to an absence of firm-learning ($\theta=0$), in order to extend the earlier results to include a firm's expectation of changing prices and wages. The outcome of this chapter provides insight into the behavior of the firm when it expects prices and wages to remain constant over the planning period - this was the case in chapter two. We call attention to the example following proposition XI, page 183, as a means of integrating information scattered throughout the paper. Moreover, naturally flowing from our techniques are facts relating to the sensitivity of the firm's employment plan to changes in the firm's expectation of a changing relative wage.
A. Statement of the Model

We start by briefly recalling some assumptions of the submodel of chapter two (Section D, paragraph 3, at page 79) which with the modification to be introduced below will comprise the basis of search in this chapter. The firm's production function $g(\cdot)$ is again assumed to satisfy the Inada conditions, (2-2), page 8. The quantity of factor whose services are required during the first period is denoted by $z$ and the quantity of factor required during the second period is given by $z + c$, where $c$ is the change in the factor demand over the two periods. The quantity $c$ may be positive, corresponding in an increase in the demand for factor services during the second period, or negative, corresponding to a decrease in second period demand for factor services (or, of course, zero corresponding to no change). The $z$ units of factor hired during the first period have the efficiency of $(1+m)z$ units in the second period because of the increased knowledge and skill resulting from on-the-job experience during the first period; and the amount of factor newly hired in the second period $c (>0)$, not having gained any experience, has the same efficiency as the older factor when it was just hired in the first period.
The firm faces competitive markets both in the selling of output and in the hiring of input services. The price of the output is taken as numéraire (as unity). In this chapter, however, it is assumed that the wage of the first period, $k$, changes to $(1+n)\cdot k$ in the second period, where $n$ satisfies the inequality $-1 \leq n < \infty$. If $n > 0$, the firm expects that during the second period either the wage it pays increases; or, equivalently, the price it obtains for its product decreases. Similarly, if $n \leq 0$, the firm expects that the second period wage decreases; or, equivalently, that the second period price of the product increases. In order to carry over the analytic and geometric techniques of the earlier chapters, it is assumed that the timing of the expected price changes coincides with that of the increase in efficiency of the senior factor.

For analytic purposes, we again divide into two categories all of the options open to the firm. We call strategies of type A those strategies which do not entail a decrease in the demand for factor services during the second period and strategies of type B those which do not entail an increase in the demand for second period factor services. The program for maximization followed by the firm requires selection
of the optimal strategy of type A and selection of the optimal strategy of type B, and finally a choice between the optimal A strategy and the optimal B strategy.

The discounted profits of the firm which follows a strategy of type A are given by

\[
\Pi_A = D_0 \cdot \{g(z) - k z\} + D_1 \cdot \{g((1+m)z+c) - (1+n) \cdot k \cdot (z+c)\},
\]

where again the discount weights may be chosen, for example, as

\[
D_0 = \int_0^T e^{-rt} \, dt \quad \text{and} \quad D_1 = \int_T^\infty e^{-rt} \, dt.
\]

The quantity \( c > 0 \) is the increment of the second period demand for factor services over the level of the first period demand for factor services, and \( n \) is the percentage change in the relative wage (relative to the price of the output).

The discounted profits of the firm which follows a strategy of type B are given by

\[
\Pi_B = D_0 \cdot \{g(z) - k z\} + D_1 \cdot \{g((1+m)(z+c)) - (1+n) \cdot k \cdot (z+c)\},
\]

where \( c \) is the decrement for the second period in the demand for factor services from the first period. Thus, \( c < 0 \) and \( |c| < z \). Notice that in the argument of the function \( g(\cdot) \) in the second term we have \( (1+m) \cdot (z+c) \),
with \( c < 0 \), whereas the analogous argument for type A strategies is \((1+m) \cdot z + c\).

B. Choice of Strategy A

For the first step in the program to find the maximizing strategy the firm must decide upon the strategy of type A which is the optimal one. This strategy we shall again call the A Strategy or Strategy A.

1. Concavity of Objective Function

We show in the manner of chapter two (see Section B, paragraph one, page 13, of chapter two) that the function \( \Pi_A \) is strictly concave in the variables \( z \) and \( c \), and claim the resulting benefits.

Denote by \( H_A \) the Hessian matrix of the discounted profits function \( \Pi_A(z, c) \):

\[
H_A = \begin{pmatrix}
\frac{\partial^2 \Pi_A}{\partial z^2} & \frac{\partial^2 \Pi_A}{\partial z \partial c} \\
\frac{\partial^2 \Pi_A}{\partial z \partial c} & \frac{\partial^2 \Pi_A}{\partial c^2}
\end{pmatrix}
\]

The entries of \( H_A \) are as follows:

\[
\frac{\partial^2 \Pi_A}{\partial z^2} = D_0 \cdot g''(z) + D_1 \cdot (1+m) \cdot g''(1+m)z + c < 0
\]
\[ \frac{\partial^2 \Pi_A}{\partial c^2} = D_1 \cdot g''((1+m)z+c) < 0, \text{ and} \]
\[ \frac{\partial^2 \Pi_A}{\partial z \partial c} = D_1 \cdot (1+m) \cdot g''((1+m)z+c) < 0. \]

Note that the diagonal entries, \( \frac{\partial^2 \Pi_A}{\partial z^2} \) and \( \frac{\partial^2 \Pi_A}{\partial c^2} \), are negative. It also follows quickly that the determinant \( |H_A| \) is positive:
\[
|H_A| = \frac{\partial^2 \Pi_A}{\partial z^2} \cdot \frac{\partial^2 \Pi_A}{\partial c^2} - \left( \frac{\partial^2 \Pi_A}{\partial z \partial c} \right)^2
\]
\[
= \left[ D_0 \cdot g''(z) + D_1 \cdot (1+m)^2 \cdot g''((1+m)z+c) \right] \cdot \left[ D_1 \cdot g''((1+m)z+c) \right]
\]
\[
- D_1 \cdot (1+m)^2 \cdot \left[ g''((1+m)z+c) \right]^2
\]
\[
= D_0 \cdot D_1 \cdot g''(z) \cdot g''((1+m)z+c) > 0.
\]

Thus, the function \( \Pi_A(z,c) \) is strictly concave.

2. The Necessary Conditions

To obtain the A Strategy, we shift to the notation of the simple control problem. Let \( u_0 = z \), the input required during the first period, and \( u_1 = c \), the increment in the first period input requirement. Let \( L(0) = D_0 \cdot \{ g(u_0) - ku_0 \} \), the discounted net profits resulting from first period operation, and \( L(1) = D_1 \{ g((1+m)u_0 + u_1) - (1+n) \cdot k \cdot (u_0 + u_1) \} \), the discounted net profits resulting from second period operation. The
problem is to choose \( u_0 \geq 0 \) and \( u_1 \geq 0 \) to maximize the discounted profits

\[
\max_{u_0, u_1 \geq 0} [L(0)+L(1)].
\]

In order to calculate \( \lambda(1) \), for the moment define \( x_1 = f(0) = u_0 \). Then,

\[
\lambda(1) = \frac{\partial L(1)}{\partial x_1} = \frac{\partial}{\partial x_1} \left[ D_1 \{ g((1+m)x_1+u_1)-(1+n) \cdot k \cdot (x_1+u_1) \} \right]
\]

\[
= D_1 \cdot \{ (1+m) \cdot g'((1+m)x_1+u_1)-(1+n) \cdot k \}.
\]

For the first necessary condition, let

\[
H(0) = L(0) + \lambda(1) \cdot f(0), \quad \text{or}
\]

\[
H(0) = D_0 \cdot \{ g(u_0) - ku_0 \} + \lambda(1) \cdot u_0.
\]

The first necessary condition requires, given the value of optimal \( \lambda(1) \), to choose \( u_0 \geq 0 \) to maximize \( H(0) \):

\[
\max_{u_0 \geq 0} H(0) = \max_{u_0 \geq 0} \left[ D_0 \cdot \{ g(u_0) - ku_0 \} + \lambda(1) \cdot u_0 \right].
\]

If an internal maximum exists, the necessary condition is equivalent to

\[
0 = \frac{\partial H(0)}{\partial u_0} = \frac{\partial L(0)}{\partial u_0} + \lambda(1) = D_0 \cdot \{ g'(u_0) - k \} + \lambda(1); \quad \text{or,}
\]
upon substitution for $\lambda(1)$,

$$(4-4) \quad 0=D_0 \left\{ g'(u_0) - k \right\} + D_1 \cdot \left\{ (1+m) \cdot g'((1+m)u_0 + u_1) - (1+n) \cdot k \right\}$$

Transposing the price terms to one side of the equation gives

$$D_0 \cdot g'(u_0) + D_1 \cdot (1+m) \cdot g'((1+m)u_0 + u_1) = D_0 \cdot k + D_1 (1+n) \cdot k.$$ 

Since the second term on the left side of the equation is the discounted value of the marginal product of the senior factor in the production of the second period, the first necessary condition requires that the sum of the discounted value of the senior factor's marginal product in each period be equated to the discounted marginal cost. This expression differs from the corresponding expression for the determination of Strategy A in the chapter two submodel with $\theta=0$ and $m(\cdot)\text{constant}$ in that the second term of the right side of the equality contains the factor $(1+n)$, as one would expect.

To obtain the second necessary condition, let $H(1)=L(1)$. The second condition, then, requires the selection of $u_1$, given optimal $u_0$, to maximize $H(1)$:

$$\max_{u_1 \geq 0} H(1) = \max_{u_1 \geq 0} \left[ D_1 \cdot \left\{ g((1+m)u_0 + u_1) - (1+n) \cdot k \cdot (u_0 + u_1) \right\} \right].$$
If an internal maximum exists, the second necessary condition becomes: choose $u_1 > 0$ such that

$$0 = \frac{\partial H(1)}{\partial u_1} = D_1 \cdot \left\{ g'((1+m)u_0 + u_1) - (1+n) \cdot k \right\} ; \text{ or,}$$

$$(4-5) \quad g'((1+m)u_0 + u_1) = (1+n) \cdot k.$$ 

The condition requires that the marginal product of the increment of factor services, newly hired in the second period, be equated to their second period marginal cost. This too differs from the corresponding second necessary condition for the determination of Strategy A in the chapter two submodel with $\theta = 0$, $m(\cdot) = \text{constant}$, in the right side of the equation by the factor $(1+n)$.

When an internal maximum does not exist, then because of the assumptions on the behavior of the function $g(\cdot)$, it is necessary for a maximum to choose $u_1$ such that the derivative $g'(\cdot)$ comes as close to equality with $(1+n) \cdot k$ as the constraints will allow.

3. **Analysis of the First Necessary Condition**

From the first condition we deduce a functional relation denoted by $u_1 = u_1(u_0)$ and study along with its properties the sensitivity of the function to changes in $(1+n)$.
Let
\[ \text{(4-6)} \quad H(u_0, u_1) = D_0 \cdot \{g'(u_0) - k\} + D_1 \cdot \{(1+m) \cdot g'((1+m)u_0 + u_1) - (1+n) \cdot k\}. \]

The first condition requires, given optimal \( u_1 \), to select \( u_0 \) such that
\[ H(u_0, u_1) = 0. \]

Since
\[ \text{(4-7)} \quad H_1(u_0, u_1) = D_0 g''(u_0) + D_1 \cdot (1+m) \cdot g''((1+m)u_0 + u_1) < 0 \]
for \( u_0, u_1 > 0 \), the implicit function theorem guarantees the existence of a function \( u_0 = u_0(u_1) \) such that \( H(u_0(u_1), u_1) = 0 \).

Next, we intend to gain a better understanding of this function \( u_0 = u_0(u_1) \) and ultimately \( u_1 = u_1^{-1}(u_0) \), its inverse, by using geometric and analytic methods.

Let
\[ h(u_0, u_1) = D_0 \cdot g'(u_0) + D_1 \cdot (1+m) \cdot g'((1+m)u_0 + u_1). \]

Regard this as a family of curves with family parameter \( u_1 \) and curve parameter \( u_0 \).

It is easily seen that the curves are negatively sloped; that, for given \( u_1 \), as \( u_0 \to 0 \), \( h(u_0; u_1) \to \infty \); that, as \( u_0 \to \infty \), \( h(u_0; u_1) \to 0 \); and that as \( u_1 \to \infty \), the curves shift toward the \( h \)-axis. These facts are seen
The first necessary condition requires that
\[ h(u_0; u_1) = D_0 \cdot k + D_1 \cdot (1+n) \cdot k. \]
The pairs \((u_0, u_1)\) which satisfy the condition and constitute the function \(u_0 = u_0(u_1)\) may be noted from the graph as the intersection of the horizontal line representing the value \(h = D_0 \cdot k + D_1 \cdot (1+n) \cdot k\) and the family of curves \(h(u_0; u_1)\) (the graph of the function \(u_0 = u_0(u_1)\) is depicted in figure 4-2). The function has negative slope; takes on a maximum value, denoted \(u_{0mx}\), when \(u_1 = 0\); and, as \(u_1 \to \infty\), approaches some \(u_0 > 0\) which satisfies the equation
\[ (4-8) \quad g'(\hat{u}_0) = D_0 \cdot k + D_1 \cdot (1+n) \cdot k. \]
If the percentage change \( n \) in the relative wage increases to \( n' \), the horizontal bar in figure 4-1 shifts upward. The \( u_{0mx} \) value moves to \( u'_{0mx} \), closer to the origin, and similarly the typical value \( u_0 \) moves to \( u'_0 \) closer to the origin. In figure 4-2, the effect of the increase in \( n \) shows up as a downward shift indicated as a dotted curve.

The same results may be obtained analytically. That the \( u_0 \)-intercept, \( u_{0mx} \), shifts downward follows from a differentiation of (4-4), page 123, where \( u_0 = u_{0mx} \) and \( u_1 = 0 \), with respect to \( n \) to obtain

\[
D_0 g''(u_{0mx}) \frac{du_{0mx}}{dn} + D_1 \left\{ (1+m)^2 g''((1+m)u_{0mx}) \frac{du_{0mx}}{dn} - k \right\} = 0;
\]
or,

\[
(4-9) \quad \frac{du_{0mx}}{dn} = \frac{D_1 \cdot k}{D_0 \cdot g''(u_{0mx}) + D_1 \cdot (1+m)^2 \cdot g''((1+m)u_{0mx})} < 0.
\]

That the graph of \( u_0(u_1) \) shifts to the left toward the \( u_0 \)-axis follows from a partial differentiation, holding \( u_0 \) constant, of (4-4), page 123, with respect to \( n \) to obtain:

\[
D_1 \cdot [(1+m) \cdot g''((1+m)u_0+u_1) \cdot \frac{du_1}{dn} - k] = 0; \quad \text{or,}
\]

\[
(4-10) \quad \frac{du_1}{dn} = \frac{k}{(1+m) \cdot g''((1+m)u_0+u_1)} < 0.
\]

The slope of the graph of \( u_0(u_1) \) is obtained by implicitly differentiating the condition (4-4), page 123, taking into account the relation \( u_0 = u_0(u_1) \) to obtain:

\[
D_0 \cdot g''(u_0) \frac{du_0}{du_1} + D_1 \cdot (1+m) \cdot g''((1+m)u_0+u_1) \left[ (1+m) \frac{du_0}{du_1} + 1 \right] = 0; \quad \text{or,}
\]

\[
(4-11) \quad \frac{du_0}{du_1} = \frac{-D_1 \cdot (1+m) \cdot g''((1+m)u_0+u_1)}{D_0 \cdot g''(u_0) + D_1 \cdot (1+m)^2 \cdot g''((1+m)u_0+u_1)} < 0.
\]

Thus, the function \( u_0 = u_0(u_1) \) is a monotonic decreasing function. The inverse of the function, denoted by \( u_1 = u_{11}(u_0) \), therefore, exists. The graph of the inverse function, obtained by flipping the axes of the graph
of the original function \( u_0 = u_0(u_1) \), is shown in figure 4-3. The domain of definition of the inverse function is the closed interval from the value \( u_0 \) to the value \( u_{0mx} \) which is also the \( u_0 \)-intercept of the graph. The graph of the function approaches asymptotically from the right the vertical line \( u_0 = \tilde{u}_0 \), where the quantity \( \tilde{u}_0 \) satisfies the relation (4-8), page 126. As \( n \) increases, the graph of \( u_1 = u_{1I}(u_0) \) shifts toward the \( u_1 \)-axis. The slope of the graph is negative and is given by the formula

\[
\frac{du_{1I}}{du_0} = \frac{1}{\frac{du_1}{du_0}} = \frac{D_0 \cdot g''(u_0) + D_1 \cdot (1+m)^2 \cdot g'((1+m)u_0 + u_1)}{D_1 \cdot (1+m) \cdot g''((1+m)u_0 + u_1)}
\]

(4-12)

\[
= \frac{D_0}{D_1} \cdot \frac{g''(u_0)}{(1+m) \cdot g''((1+m)u_0 + u_1)} + (1+m) > (1+m),
\]

since the first term is positive.

![Graph of the Function \( u_1 = u_{1I}(u_0) \)](image)

**Figure 4-3**
4. **Analysis of the Second Necessary Condition**

The second necessary condition gives rise to a function, denoted by $u_1 = u_{II}(u_0)$, the properties of which are the object of study in this section. Denote by $\bar{w}$ the value of the argument of the function $g'(\cdot)$ such that

$$g'(\bar{w}) = (1+n)k.$$  

The value $\bar{w}$ exists and is unique since $g''(\cdot) < 0$, and $g'(0) = \infty$ and $g'(\infty) = 0$. The second necessary condition requires that $u_1$ be chosen, given optimal $u_0$, such that $g'((1+m)u_0 + u_1)$ come as close to $(1+n)k$ as the constraints on $u_1$ will allow. Two cases are relevant for discussion.

(a) Suppose $(1+m)u_0 \geq \bar{w}$. Then $g'((1+m)u_0) \leq g'(\bar{w}) = (1+n)k$ since $g''(\cdot) < 0$. If there were no non-negativity constraint on $u_1$, the value of $u_1$ which would yield the desired equality would be $\leq 0$. Thus, $u_1$ must be chosen zero in order to keep the divergence of the value $g'((1+m)u_0 + u_1)$ from $(1+n)k$ as small as possible.

(b) Suppose $(1+m)u_0 < \bar{w}$. Letting $u_1 = \bar{w} - (1+m)u_0 > 0$, we see that $g'((1+m)u_0 + u_1) = g'(\bar{w}) - (1+n)k$. 
These two cases give rise to a function, denoted by \( u_{1|1}(u_0) \), defined in the following manner:

\[
\begin{align*}
  u_1 &= 0 \quad \text{if } u_0 \geq \frac{\bar{w}}{1+m}, \\
  u_1 &= \bar{w} - (1+m) \cdot u_0 \quad \text{if } u_0 < \frac{\bar{w}}{1+m}.
\end{align*}
\] (4-14)

Graph of the Function \( u_1 = u_{1|1}(u_0) \)

Figure 4-4

The graph of the function \( u_1 = u_{1|1}(u_0) \), depicted in Figure 4-4, consists of two line segments: the first segment joins the points \((0, \bar{w})\) and \(\left(\frac{\bar{w}}{1+m}, 0\right)\) and the second segment coincides with the \(u_0\)-axis from \( u_0 = \frac{\bar{w}}{1+m} \) out to (positive) infinity.

When the expected change \( n \) in the relative wage increases, the graph shifts to the left, depicted as the dotted graph. This result follows from differentiation of the defining equality for \( \bar{w} \), (4-13), with
regard to \( w = w(n) \) as a function of \( n \):

\[
(4-15) \quad \frac{dw}{dn} = \frac{k}{g''(w)} < 0.
\]

In fact, as \( n \to \infty \), the first segment of the graph of \( u_{1\Pi}(u_0) \) approaches the origin since \( g'(w) = (1+n) \cdot k \to \infty \)
and since \( g'(0) = \infty \).

5. **Simultaneous Solution of the Necessary Conditions**

We next inquire into the variety of ways in which the intersection, determining Strategy A, may occur. We single out three cases, Plate X.

**case (a):** \( u_{0mx} = \overline{w}/(1+m) \)  
First assume \( u_{0mx} = \overline{w}/(1+m) \)
Recall that, for the interval from the origin to the point \( \overline{w}/(1+m) \), \( \frac{du_{1\Pi}}{du_0} > (1+m) = \frac{du_{1\Pi}}{du_0} \) (see (4-12) and (4-14)). Thus, after the initial common starting point, the two graphs diverge as \( u_0 \) approaches the origin since the graph of \( u_{1\Pi}(u_0) \) has a constant slope of \( (1+m) \), while the graph of \( u_{1\Pi}(u_0) \) has a slope always exceeding \( (1+m) \). Since the domain of \( u_{1\Pi}(u_0) \) is the segment between the origin and \( u_{0mx} \), the common starting point is the unique point of intersection.
\[ \frac{w}{(1+m)} = u_{0mx} \]

**case (a)**

\[ \frac{w}{(1+m)} < u_{0mx} \]

**case (b)**

\[ \frac{w}{(1+m)} > u_{0mx} \]

**case (c)**

Intersections of the Graphs of \( u_{1I}(u_0) \) and \( u_{1II}(u_0) \)

Plate X
case (b): \( u_{0_{\text{mx}}} > \frac{w}{1+m} \). Second, we assume \( u_{0_{\text{mx}}} > \frac{w}{1+m} \). Initially the slope of the graph of \( u_{1_{\text{II}}}(u_0) \) is zero and, moving toward the origin, at the point \( \frac{w}{1+m} \), the slope increases to \((1+m)\). But the slope of the graph \( u_{1_{\text{II}}}(u_0) \) is always greater than \((1+m)\). Thus, after the initial common point at \( u_{0_{\text{mx}}} \), the two graphs diverge as \( u_0 \) approaches the origin.

In this, and the preceding case, \( u_1 = 0 \).

case (c): \( u_{0_{\text{mx}}} < \frac{w}{1+m} \). Finally, assume \( u_{0_{\text{mx}}} < \frac{w}{1+m} \). In this case, the graph of \( u_{1_{\text{II}}}(u_0) \) starts at \( (u_{0_{\text{mx}}}, 0) \) and as \( u_0 \) approaches the origin, the graph goes out to infinity. Because of continuity, a point of intersection with the positively sloped portion of the graph of \( u_{1_{\text{II}}}(u_0) \) must exist; for, the values of \( u_{1_{\text{II}}}(u_0) \), when \( u_0 \) satisfies the inequality, \( 0 < u_0 < u_{0_{\text{mx}}} \) (i.e., lies in the domain of definition of \( u_{1_{\text{II}}}(u_0) \)), satisfy the inequality \( \left[ \frac{w}{1+m} \cdot u_{0_{\text{mx}}} \right] < u_{1_{\text{II}}} < \frac{w}{1+m} \).

Let the intersection be denoted by \( P=(u_{0_P}, u_{1_P}) \). If \( (u_0, u_1) \) leaves \( P \) following the graph of \( u_{1_{\text{II}}}(u_0) \) and moving to the right, it follows a curve having a slope exceeding \((1+m)\). If \( (u_0, u_1) \) leaves \( P \) following the graph \( u_{1_{\text{II}}}(u_0) \) and moving to the right, it follows a curve having a slope, at first, of exactly \((1+m)\) and
further to the right beyond \( u_0 = \frac{\bar{w}}{1+m} \) having a slope of zero. Therefore, to the right of the point \( P \), the curve \( u_{11}^I(u_0) \) always lies below the curve \( u_{111}(u_0) \), and there is thus no other intersection of the two curves to the right of the point \( P \).

If, on the other hand, \((u_0, u_1)\) leaves the point \( P \) following the graph of \( u_{11I}(u_0) \) and moving to the left, it follows a curve having a slope exceeding \((1+m)\). If \((u_0, u_1)\) leaves the point \( P \) following the graph of \( u_{11I}(u_0) \) and moving to the left, it follows a curve with a slope of exactly \((1+m)\). Thus, to the left of the point \( P \), the curve \( u_{1I}(u_0) \) always lies above the curve \( u_{11I}(u_0) \) and there is no other intersection of the two curves to the left of the point \( P \).

Thus, for each of the three cases we have studied the intersection determining Strategy A which is unique because of the strict concavity of the objective function.

Next, we inquire into the relationship between \( u_{0mx}(n) \) and \( \bar{w}(n)/(1+m) \) and the dependence of the relationship on \( n \). Two cases are considered.

(a) Suppose \( n > 0 \). It will be shown that \( u_{0mx} > \bar{w}/(1+m) \). Let \( u_0 = \bar{w} \) and \( u_1 = 0 \) in the function \( H(u_0; u_1) \) defined by (4-6), page 125. Noting that because of monotonicity, \( g'(\frac{\bar{w}}{1+m}) > g'(\bar{w}) = (1+n) \cdot k \geq k \), we see
that

\[ H(\bar{w}/(1+m), 0) = \]

\[ D_0 \cdot \{ g'(\bar{w}/(1+m)) - k \} + D_1 \cdot \{(1+m) \cdot g'(\bar{w}) - (1+n) \cdot k \} = \]

\[ D_0 \cdot \{ g'(\bar{w}/(1+m)) - k \} + D_1 \cdot \{(1+m) \cdot (1+n) \cdot k - (1+n) \cdot k \} = \]

\[ D_0 \cdot \{ g'(\bar{w}/(1+m)) - k \} + D_1 \cdot m \cdot (1+n) \cdot k > 0. \]

Since \( H_1 < 0 \) from (4-9), page 125, it follows that \( u_{0mx} > \frac{\bar{w}}{(1+m)} \) for \( n \geq 0 \).

(b) Assume \(-1 < n < 0\). It will be shown that there exists a unique \( n_0 \) such that \(-1 < n_0 < 0\) and such that, if \(-1 < n < n_0\), then \( u_{0mx}(n) < \bar{w}(n)/(1+m) \), and, if \( n_0 < n < 0\), then \( u_{0mx}(n) > \bar{w}(n)/(1+m) \). Here we recognize the dependence of \( \bar{w} \) and \( u_{0mx} \) upon \( n \) with the notation \( \bar{w}(n) \) and \( u_{0mx}(n) \). Then, \( \bar{w}(-1) = \infty \) since \( g'(\bar{w}(-1)) = (1+n) \cdot k = 0 \); and

\[ H(\infty, 0) = D_0 \cdot \{ g'(\infty) - k \} + D_1 \cdot \{ g'(\infty) - (1+n) \cdot k \} = -D_0 \cdot k < 0. \]

Thus, \( u_{0mx}(-1) < \bar{w}(-1)/(1+m) \) since \( H_1 < 0 \). Since for \( n = 0 \), \( u_{0mx}(0) > \bar{w}(0)/(1+m) \), as we have seen in case (a); and since for \( n = -1 \), \( u_{0mx}(-1) < \bar{w}(-1)/(1+m) \), as we have just seen, by continuity there must be at least one value \( n_0 \) such that \(-1 < n_0 < 0\) and such that \( u_{0mx}(n_0) = \bar{w}(n_0)/(1+m) \).
Next, we show the uniqueness of $n_0$. Differentiate the condition (4-13), page 130, which defines $\bar{w}$, with respect to $n$ to obtain:

$$\frac{d\bar{w}}{dn} = \frac{k}{g''(\bar{w}(n))} < 0.$$  

Thus, $\bar{w}(n)$ is a monotonic decreasing function of $n$. Now define the function $J(n)$ by

$$(4-17) \quad J(n)=H(\bar{w}(n)/(1+m),0)=D_0 \cdot \left\{g''(\bar{w}(n)/(1+m)) \cdot \frac{1}{1+m} \cdot \frac{d\bar{w}(n)}{dn} + D_1 \cdot m \cdot k \right\}$$

Take the derivative of $J(n)$ with respect to $n$:

$$\begin{align*}
(4-18) \quad J'(n) &= D_0 \cdot g''(\bar{w}(n)/(1+m)) \cdot \frac{1}{1+m} \cdot \frac{d\bar{w}(n)}{dn} + D_1 \cdot m \cdot k > 0
\end{align*}$$

since both terms are positive; and, therefore, $H(\bar{w}(n)/(1+m),0)$ is a monotonic increasing function of $n$. Thus, $n=n_0$ is the unique value of $n$ for which $H(\bar{w}(n)/(1+m),0)=0$; or, equivalently, $u_{0mx}(n)=\bar{w}(n)/(1+m)$. The value $n_0$ corresponds to the graph labeled case (a) in Plate X on page 133.

To complete the proof of the assertion that $n_0$ is a critical value we rely on the fact that $J(n)$ is a monotonic increasing function of $n$. If $n<n_0$, then $J(n)<0$; or, $H(\bar{w}(n)/(1+m),0)<0$. Since $H(u_0,0)$ is monotonic decreasing in $u_0$, $u_{0mx}(n)$, which by defini-
tion satisfies $H(u_{0mx}(n), 0) = 0$, must be such that $u_{0mx}(n) < \bar{w}(n)/(1+m)$. Thus, the inequality $n < n_0$ corresponds to the graph labeled case (c) in Plate X.

Similarly if $n > n_0$, then $J(n) > 0$; or, $H(\bar{w}(n)/(1+m), 0) > 0$. This implies that $u_{0mx}(n) > \bar{w}(n)/(1+m)$. Therefore, the inequality $n > n_0$ corresponds to the graph labeled case (b) in Plate X.

The results obtained above are summarized in the following proposition.

**Proposition VII.** Suppose the firm maximizes discounted profits over its planning horizon of $T$ time units given by the formula

$$
D_0 \cdot \left\{ g(u_0) - ku_0 \right\} + D_1 \cdot \left\{ g((1+m)u_0 + u_1) - (1+n) \cdot k \cdot (u_0 + u_1) \right\},
$$

where $D_0$ and $D_1$ are discount weights, $n$ is the expected percentage change in the relative wage of the factor in the second period, and where $u_0$ and $u_1$ are restricted to be non-negative. In addition, suppose the production function $g(*)$ satisfies the Inada Conditions.

Then there exists a unique quantity $n_0$ such that $-1 < n_0 < 0$ which is determined (jointly with the quantity $\bar{w}$) by the two equalities

$$
g'(\bar{w}) = (1+n_0) \cdot k, \text{ and }$$

$$
D_0 \cdot g'(\bar{w}/(1+m)) = D_0 \cdot k - D_1 \cdot m \cdot (1+n_0) \cdot k.
$$
(a) If \( n > n_0 \), then the firm hires \( u_{0mx} \) units of factor during the first period and neither augments nor decreases the original size of the labor force during the second period. The quantity \( u_{0mx} \) is determined by the condition

\[
D_0 \cdot g'(u_{0mx}) + D_1 \cdot (1+m) \cdot g'(1+m)u_{0mx} = D_0 \cdot k + D_1 \cdot (1+n) \cdot k.
\]

(b) If \( n < n_0 \), then the firm hires \( u_0 + u_1 \) units of factor during the first period and an additional \( u_1 \) units during the second period. The values \( (u_0, u_1) \) are obtained by solving the two equations:

\[
g'((1+m)u_0 + u_1) = (1+n) \cdot k, \quad \text{and}
\]

\[
D_0 \cdot g'(u_0) + D_1 \cdot (1+m) \cdot g'(1+m)u_0 + u_1 = D_0 \cdot k + D_1 \cdot (1+m) \cdot k.
\]

The theorem asserts the existence of a certain critical level of deflation of the relative wage. If deflation greater than the critical level \( (n < n_0) \) is expected by the firm, then the firm will increase its labor force in the second period. If deflation less than the critical level \( (0 > n > n_0) \) or if any inflation of the relative wage \( (n > 0) \) is expected by the firm, then the firm which is constrained from decreasing its demand for labor during the second period (because we are considering only strategies of type A) hires a
particular labor force in the first period which it keeps on in the second period without augmentation or diminution.

Finally, we study how the firm's plan of input requirements responds to an increase in \( n \). The discussion distinguishes three cases.

**case (a):** \( u_{0mx}(n_0) = \frac{\bar{w}(n_0)}{(1+m)} \). It will be recalled that both the graphs of \( u_{1I}(u_0) \) and \( u_{1II}(u_0) \) shift toward the \( u_1 \)-axis as \( n \) increases ((4-9), page 128, and (4-15), page 132). It is, therefore, necessary to compare the magnitudes of the shift in order to determine the effect of an increase in \( n \) on the intersection of the two graphs. See figure 4-5.

![Figure 4-5](image-url)
Let \( \tilde{u}_0(\hat{n}) \) denote the corner point, on the \( u_0 \)-axis, where the two line segments comprising the graph of \( u_{1II}(u_0) \) meet. This value satisfies the condition (4-13), page 130:

\[
g'(1+m)\tilde{u}_0(\hat{n}) = (1+n) \cdot k.
\]

Take the derivative of the relation with respect to \( n \):

\[
\frac{d\tilde{u}_0}{dn} = \frac{k}{(1+m)g''((1+m)\tilde{u}_0(\hat{n}))} < 0.
\]

Recall (see (4-9), page 128) that

\[
\frac{du_{0mx}}{dn} = \frac{D_1 \cdot k}{D_0 \cdot g''(u_{0mx}(\hat{n}))+D_1 \cdot (1+m)^2 \cdot g''((1+m)u_{0mx}(\hat{n}))}
\]

Take the reciprocal of both of these derivatives and substitute the first in the second as indicated:

\[
\frac{dn}{du_{0mx}} = \frac{D_0 \cdot g''(u_{0mx}(\hat{n}))}{D_1 \cdot k} + \frac{D_1 \cdot k}{D_1 \cdot (1+m)^2 \cdot g''((1+m)u_{0mx}(\hat{n}))}
\]

The last inequality follows since the two summands are both negative. Taking reciprocals, again, yields:

\[
\frac{du_{0m}(\hat{n})}{dn} > \frac{d\tilde{u}_0(\hat{n})}{dn}.
\]
If \( \frac{d[u_{0mx}(\hat{n})]}{dn} \) is negative, then $\frac{d[u_0(n)]}{dn} < \frac{d[u_0(\hat{n})]}{dn}$, and the graph of $u_{1II}(u_0)$ shifts to the left more than does the graph of $u_{1I}(u_0)$. This means that the point of intersection may be regarded as determined by the $u_0$-intercept, $u_{0mx}(\hat{n})$, of the graph of $u_{1I}(u_0)$. The analysis and result of this case is the same as that of the next case from this point on.

**Case (b):** $u_{0mx}(n) > \bar{w}(n)/(1+m)$. This case, represented in figure 4-6, corresponds to $n > n_0$. Proposition VII, page 138. Let $(\hat{u}_0(\hat{n}), \hat{u}_1(\hat{n})) = (u_{0mx}(n), 0)$ denote the intersection of the two graphs which is in the $u_0$-axis. Because the intersection is in the interior of the segment of the graph of $u_{1II}(u_0)$ which coincides with the $u_0$-axis, $(1+m) \hat{u}_0(\hat{n}) > \bar{w}(\hat{n})$ (see (4-14), page 131 for the definition of the function $u_{1II}(u_0)$). This implies that $\hat{u}_1(\hat{n}) = 0$. Since $\hat{u}_0(n)$ and $\bar{w}(n)$ are continuous functions, for $n$ infinitesimally larger than $\hat{n}$, we still have $(1+m) \hat{u}_0(n) > \bar{w}(n)/(1+m)$. This last inequality implies that $\hat{u}_1(n) = 0$ for such $n$. Thus, an infinitesimal increase in $n$ from $\hat{n}$ does not have any effect on $\hat{u}_1 = 0$. 
To see how $\hat{u}_0$ behaves in a neighborhood of $\hat{n}$, we note that $\hat{u}_0(n)=u_{0mx}(n)$ in such a neighborhood by virtue of the constancy of $\hat{u}_1=0$ for infinitesimal increases in $n$. The quantity $u_{0mx}(n)$ is the $u_0$-intercept of the graph of $u_{1I}(u_0)$ and is determined by the condition

$$D_0 \cdot \{g'(u_{0mx}(n))-k\} + D_1 \cdot \{(1+m) \cdot g'((1+m)u_{0mx}(n))-(1+n) \cdot k\}$$

Taking the derivative with respect to $n$ yields:

$$D_0 \cdot g''(u_{0mx}(n)) \frac{du_{0mx}}{dn} + D_1 \cdot \{(1+m)^2 \cdot g''((1+m)u_{0mx}(n)) \frac{du_{0mx}}{dn} - k\} = 0;$$

or,

$$\frac{d\hat{u}_0}{dn} = \frac{du_{0mx}}{dn} = D_1 k \cdot \left[ D_0 \cdot g''(u_{0mx}(n)) + D_1 \cdot (1+m)^2 \cdot g''((1+m)u_{0mx}(n)) \right] < 0.$$

Thus, when $n$ increases from $\hat{n}$, $u_0$ decreases.
Economically, this and the preceding case mean that since the firm is bound by the constraint not to dismiss any factor during the second period, when the expected change in the relative wage increases, the burden is carried by the adjustment of the amount of factor employed during the first period (which is neither increased nor decreased in the second period). Thus, the firm simply decreases its demand for factor when \( n \) increases from \( \hat{n} \).

**case (c):** \( u_{0mx}(n) < \bar{w}(n)/(1+m) \). This case corresponds to \( n > n_0 \), proposition VII, page 138, and is represented in figure 4-7. That this is an intersection in the interior of the first segment of the graph of \( u_{1II}(u_0) \) means that the coordinates of the intersection, denoted by \( (\hat{u}_0(\hat{n}), \hat{u}_1(\hat{n})) \), satisfy the second condition in equality form ((4-5), page 124):

\[
g'( (1+m) \cdot \hat{u}_0(\hat{n}) + \hat{u}_1(\hat{n}) ) = (1+\hat{n}) \cdot k,
\]

and \( \hat{u}_1(\hat{n}) > 0 \). Therefore, \( (1+m) \cdot \hat{u}_0(\hat{n}) = \bar{w}(\hat{n})/(1+m) - \hat{u}_1(\hat{n}) > 0 \). Because of the continuity of \( \hat{u}_0(n) \) and \( \hat{u}_1(n) \), for an infinitesimal increase of \( n \) from \( \hat{n} \), we have \( \hat{u}_1(n) > 0 \) and \( g'( (1+m) \cdot \hat{u}_0(n) + \hat{u}_1(n) ) = (1+n) \cdot k \). Taking the derivative with respect to \( n \) yields:
\[ g''((1+m)\hat{u}_0(\hat{n})+\hat{u}_1(\hat{n})) \cdot (1+m) \cdot \frac{d\hat{u}_0}{dn} + \frac{d\hat{u}_1}{dn} = k; \text{ or}, \]

\[ (4-21) \quad \frac{d\hat{u}_1}{dn} = \frac{k}{g''((1+m)\hat{u}_0(\hat{n})+\hat{u}_1(\hat{n}))} - (1+m) \cdot \frac{d\hat{u}_0}{dn}. \]

**Effect of Increase in \( n \): \( u_{0 \text{max}} < \tilde{w}/(1+m) \)**

*Figure 4-7*

On the other hand, substituting the second condition (4-5), page 124, into the first condition (4-4), page 123, gives

\[
0 = D_0 \cdot \{g'((1+n)\hat{u}_0(\hat{n}))-k\} + D_1 \cdot \{(1+m) \cdot g'((1+m)\hat{u}_0(\hat{n})+\hat{u}_1(\hat{n})) \cdot \}
\]

\[ -(1+n) \cdot k\}
\]

\[ = D_0 \cdot \{g'((1+n)\hat{u}_0(\hat{n}))-k\} + D_1 \cdot \{(1+m) \cdot (1+n) \cdot k-(1+n) \cdot k\}
\]

\[ = D_0 \cdot \{g'((1+n)\hat{u}_0(\hat{n}))-k\} + D_1 \cdot m \cdot (1+n) \cdot k. \]
Taking the derivative with respect to $n$ yields

$$
\frac{du_0}{dn} = - \frac{D_1 m k}{D_0 \cdot g''(u_0(\hat{n}))} > 0.
$$

Thus, an increase in $n$ causes $u_0$ to increase.

Return to the expression (4-21) for $\frac{du_1}{dn}$; since the right side of the equality is negative (both of the summands are), we have $\frac{du_1}{dn} < 0$. Thus, in this case $\hat{u}_0(n)$ increases and $\hat{u}_1(n)$ decreases when $n$ increases from $\hat{n}$.

Economically, this means that, if the expected change $n$ of the relative wage is less than the critical value $n_0$, an increase in the expected second period change of the relative wage induces the firm to hire more factor in the first period and to hire less additional factor in the second period.

These results are stated in the next proposition.

**Proposition VIII.** Let $n_0$ and $\bar{\bar{w}}$ be defined as in the preceding proposition. Let the intersection which corresponds to the optimum A Strategy be denoted by $(\hat{u}_0(\hat{n}), \hat{u}_1(\hat{n}))$.

If $\hat{n} = n_0$, then $u_{0\text{mx}}(\hat{n}) = \bar{\bar{w}}(\hat{n})$, and

$$
\frac{d\hat{u}_0}{dn}(\hat{n}) < 0 \quad \text{and} \quad \frac{d\hat{u}_1}{dn}(\hat{n}) = 0.
$$
If \( \hat{n} > n_0 \), then \( u_{0m^2}(\hat{n}) > \bar{w}(\hat{n})/(1+m) \), and

\[
\frac{d\hat{u}_0(\hat{n})}{dn} < 0 \quad \text{and} \quad \frac{d\hat{u}_1(\hat{n})}{dn} = 0.
\]

If \( \hat{n} < n_0 \), then \( u_{0m^2}(\hat{n}) = \bar{w}(\hat{n})/(1+m) \), and

\[
\frac{d\hat{u}_0(\hat{n})}{dn} > 0 \quad \text{and} \quad \frac{d\hat{u}_1(\hat{n})}{dn} < 0.
\]

C. Choice of Strategy B

Having selected an A Strategy, the firm next selects a B Strategy which is the optimum of all strategies which preclude the firm from hiring any factor in the second period (but which permit the firm in the second period to dismiss earlier hired factor). The expression giving the discounted profits of the firm adopting such a strategy, it will be recalled (4-3), page 119, is

\[
\overline{W}_B = D_0 \cdot [g(z) - kz] + D_1 \cdot [g((1+m)(z+c)) - (1+n) \cdot k \cdot (z+c)],
\]

where \( z \geq 0 \) is the first period demand for factor services and \( z + c \geq 0 \) is the second period demand for factor services. The constraint on the options, which the firm must follow, is that the second period demand may not exceed the first period demand.
1. Concavity of the Objective Function

With $H_B$ denoting the Hessian matrix of the discounted profits function $\Pi_B(z, c)$, we have

$$H_B = \begin{pmatrix} \frac{\partial^2 \Pi_B}{\partial z^2} & \frac{\partial^2 \Pi_B}{\partial z \partial c} \\ \frac{\partial^2 \Pi_B}{\partial z \partial c} & \frac{\partial^2 \Pi_B}{\partial c^2} \end{pmatrix}$$

the entries of which are

$$\frac{\partial^2 \Pi_B}{\partial z^2} = D_0 \cdot g''(z) + D_1 \cdot (1+m)^2 \cdot g''((1+m)(z+c)) < 0$$

$$\frac{\partial^2 \Pi_B}{\partial c \partial z} = D_1 \cdot (1+m)^2 \cdot g''((1+m)(z+c)) < 0,$$

$$\frac{\partial^2 \Pi_B}{\partial c^2} = D_1 \cdot (1+m)^2 \cdot g''((1+m)(z+c)) < 0.$$ 

The diagonal entries, $\frac{\partial^2 \Pi_B}{\partial z^2}$ and $\frac{\partial^2 \Pi_B}{\partial c^2}$, are clearly negative. The determinant $|H_B|$ is seen to be positive as follows:

$$|H_B| = \frac{\partial^2 \Pi_B}{\partial z^2} \cdot \frac{\partial^2 \Pi_B}{\partial c^2} - \left[ \frac{\partial^2 \Pi_B}{\partial z \partial c} \right]^2$$

$$= \left[ D_0 \cdot g''(z) + D_1 \cdot (1+m)^2 \cdot g''((1+m)(z+c)) \right] \cdot$$

$$\left[ D_1 \cdot (1+m)^2 \cdot g''((1+m)(z+c)) \right] - D_1^2 \cdot (1+m)^4 \cdot \left[ g''((1+m)(z+c)) \right]^2$$
Thus, \( \pi_B(z,c) \) is strictly concave.

2. The Necessary Conditions

To obtain the B Strategy, we again convert to the notation for the simple control problem. Let \( u_0 = z \), the input required during the first period, and \( \bar{u}_1 = -c \), the negative of the decrement in the first period requirement necessary to obtain the second period level of demand. Let \( L(0) = D_0 \cdot \left\{ g(u_0) - ku_0 \right\} \), the discounted net profits from the first period, and \( L(1) = D_1 \cdot \left\{ g((1+m)(u_0 - \bar{u}_1)) - (1+n) \cdot k \cdot (u_0 - \bar{u}_1) \right\} \), the discounted net profits from the second period. The firm must choose \( u_0, \bar{u}_1 \geq 0 \), with \( \bar{u}_1 \leq u_0 \), in order to maximize the total discounted profits:

\[
\max_{u_0, \bar{u}_1 \leq 0 \text{ and } \bar{u}_1 \leq u_0} [L(0) + L(1)].
\]

Define \( x_1 = f(0) = u_0 \). Then \( \lambda(1) \) is given by

\[
\lambda(1) = \frac{2 L(1)}{\partial x_1} = \frac{2 D_1 \cdot g((1+m)(x_1 - \bar{u}_1)) - (1+n) \cdot k}{\partial x_1}
\]

Let

\[
H(0) = L(0) + \lambda(1) \cdot f(0); \quad \text{or,}
\]

\[
H(0) = D_0 \cdot \left\{ g(u_0) - ku_0 \right\} + \lambda(1) \cdot u_0.
\]
The first necessary condition requires, given the optimal value of $\lambda(1)$, to choose $u_0 \geq 0$ to maximize $H(0)$:

$$\max_{u_0 \geq 0} H(0) = \max_{u_0 \geq 0} \left[ D_0 \cdot \{ g(u_0) - ku_0 \} + \lambda(1) \cdot u_0 \right].$$

If an internal maximum exists, the necessary condition is equivalent to

$$0 = \frac{\partial H(0)}{\partial u_0} = \frac{\partial L(0)}{\partial u_0} + \lambda(1) = D_0 \cdot \{ g'(u_0) - k \} + \lambda(1),$$
or,

upon substitution for $\lambda(1)$,

$$(4-23) \quad 0 = D_0 \cdot \{ g'(u_0) - k \} + D_1 \cdot \{(1+m) \cdot g'(u_0 - \bar{u}_1) \} - (1+n) \cdot k.$$ 

When the price terms are transposed to one side of the equation, the condition becomes

$$D_0 \cdot g'(u_0) + D_1 \cdot (1+m) \cdot g'(u_0 - \bar{u}_1) = D_0 \cdot k + D_1 \cdot (1+n) \cdot k.$$ 

The second term on the left side of the equality is the discounted value of the marginal product of the portion of originally hired factor which remains in the second period. Thus, the first necessary condition requires that the total discounted value of the marginal product of the input be equated to the total discounted marginal cost of the input. The condition
differs from the corresponding expression for the
determination of Strategy B in the chapter two sub-
model with $\theta = 0$, $m(\cdot) =$ constant, in that the second
term on the right side of the equality contains the
factor $(1+n)$.

To obtain the second necessary condition, let
$H(1) = L(1)$. This condition requires the firm, given
the optimal value of $u_0$, to select $\bar{u}_1$ to maximize $H(1)$:

$$
\max_{\bar{u}_1 \geq 0} \max_{\bar{u}_1 \geq 0} \left[ D_1 \cdot \left\{ g\left( (1+m)(u_0-\bar{u}_1) \right) - (1+n) \cdot k \cdot (u_0-\bar{u}_1) \right\} \right].
$$

When an internal maximum exists, the condition
for a maximum may be written: choose $u_1 \geq 0$ such that

$$
0 = \frac{\partial H(1)}{\partial u_1} = -D_1 \cdot \left\{ (1+m) \cdot g'\left( (1+m)(u_0-\bar{u}_1) \right) - (1+n) \cdot k \right\}; \text{ or}
$$

(4-24) \quad g'\left( (1+m)(u_0-\bar{u}_1) \right) = \frac{1+n}{1+m} \cdot k.

This condition requires that the marginal product of
the input in the second period, $(u_0-\bar{u}_1)$, be equated to
the marginal cost $(1+n) \cdot k$. This necessary condition
differs from the corresponding expression for the
determination of Strategy B in the chapter two sub-
model in that the right side of the equality contains the
factor $(1+n)$.

Because of the assumed behavior of the function
when an internal maximum cannot be obtained, the decision variable $\tilde{u}_1$ must be chosen, given $u_0$, so that the derivative $g'((1+m)(u_0-\tilde{u}_1))$ is as close to $(1+n)\cdot k/(1+m)$ as the constraints allow.

3. Analysis of the First Necessary Condition

The first condition implies a functional relation denoted by $\tilde{u}_1=\tilde{u}_{1\, I}(u_0)$. We shall establish its existence and study its properties and its dependence upon $(1+n)$.

To establish existence, let

\[
(4-25) \quad H(u_0;\tilde{u}_1) = D_0 \cdot \{g'(u_0) - k\} \\
+ D_1 \cdot \{(1+m) \cdot g'((1+m)(u_0-\tilde{u}_1)) - (1+n) \cdot k\}.
\]

The first necessary condition requires, given $\tilde{u}_1$, the choice of $u_0$ such that $H(u_0;\tilde{u}_1)=0$. Since

\[
(4-26) \quad H_1(u_0;\tilde{u}_1) = D_0 \cdot g''(u_0) + D_1 \cdot (1+m) \cdot g''((1+m)(u_0-\tilde{u}_1)) < 0
\]

for $u_0,\tilde{u}_1 > 0$, the implicit function theorem guarantees the existence of a function $u_0=u_0(\tilde{u}_1)$ such that $H(u_0(\tilde{u}_1),\tilde{u}_1)=0$.

To obtain some properties of $u_0=u_0(\tilde{u}_1)$ and $\tilde{u}_{1\, I}(u_0)$, the inverse of $u_0(\tilde{u}_1)$, and to study the dependence of the functions on $(1+n)$, let

\[
h(u_0;\tilde{u}_1) = D_0 \cdot g'(u_0) + D_1 \cdot (1+m) \cdot g'((1+m)(u_0-\tilde{u}_1)).
\]
As before, we regard this as a family of curves with family parameter \( \bar{u}_1 \) and curve parameter \( u_0 \). This family of curves has the following properties which are depicted in figure 4-8: Each of the curves in the family are negatively sloped. As \( u_0 \to \bar{u}_1 \) from the right, the graph of the curve corresponding to the family parameter \( \bar{u}_1 \) approaches infinity and approaches asymptotically the vertical line \( u_0 = \bar{u}_1 \). Also, for each curve, as \( u_0 \to \infty \), the value of the function approaches 0 and the curve asymptotically approaches the \( u_0 \)-axis. As the family parameter \( \bar{u}_1 \) increases the h-curves shift outward.

Graph of the Functions \( k(u_0; \bar{u}_1) \)

Figure 4-8
The first necessary condition requires that

\[ h(u_0; u_1) = D_0 \cdot k + D_1 \cdot (1+n) \cdot k. \]

The pairs \((u_0, u_1)\) which satisfy the condition and constitute the graph of the function \(u_0 = u_0(\bar{u}_1)\) are the points of the intersections of the horizontal line, representing the value \(h = D_0 \cdot k + D_1 \cdot (1+n) \cdot k\) with the various \(h\)-curves associated with the values of \(\bar{u}_1\).

The function \(u_0 = u_0(\bar{u}_1)\), the graph of which is found in figure 4-9, has positive slope and takes on a minimum positive value, denoted \(u_{0mn}\), when \(\bar{u}_1 = 0\).

![Graph of the Function \(u_0 = u_0(\bar{u}_1)\)]

Figure 4-9
If the percentage change \( n \) in the relative wage increases to \( n' \), the horizontal bar in figure 4-8 shifts upward. The \( u_{0mn} \) value moves to \( u_{0mn}' \), closer to the origin, while \( w_1 \) is given and held constant. The effect of the increase in \( n \) is a downward shift in the graph of \( u_0(\bar{u}_1) \), indicated by the dotted curve.

Analytic methods confirm the geometric arguments. That the \( u_0 \)-intercept, \( u_{0mn}' \), shifts downward follows from a differentiation of the definition of \( u_{0mn} \) with respect to \( n \),

\[
D_0 \cdot \{g'(u_{0mn})-k\} + D_1 \cdot \{(1+m) \cdot g'((1+m)u_{0mn})-(1+n) \cdot k\} = 0
\]
to obtain

\[
D_0 \cdot g''(u_{0mn}) \cdot \frac{du_{0mn}}{dn} + D_1 \cdot \{(1+m)^2 \cdot g''((1+m)u_{0mn}) \cdot \frac{du_{0mn}}{dn} - k\};
\]
or

\[
(4-27) \quad \frac{du_{0mn}}{dn} = \frac{D_1 \cdot k}{D_0 \cdot g''(u_{0mn}) + D_1 \cdot (1+m)^2 \cdot g''((1+m)u_{0mn})} < 0.
\]

That the graph of \( u_0(\bar{u}_1) \) shifts to the right, away from the \( u_0 \)-axis follows from a differentiation of the relation defining the function \( u_0 = u_0(\bar{u}_1) \) (see (4-23), page 150) with respect to \( n \) holding \( u_0 \) constant:

\[
-D_1 \cdot \{(1+m)^2 \cdot g''((1+m)(u_0-\bar{u}_1)) \cdot \frac{d\bar{u}_1}{dn} + k\} = 0.
\]
The slope of the graph of $u_0(\tilde{u}_1)$ is obtained by implicitly differentiating the defining condition (4-23), page 150, with $u_0=u_0(\tilde{u}_1)$:

$$D_0 \cdot g''(u_0) \cdot \frac{du_0}{du_1} + D_1 \cdot (1+m)^2 \cdot g''((1+m)(u_0-\tilde{u}_1)) \left\{ \frac{du_0}{du_1} - 1 \right\} = 0;$$

or,

$$\frac{du_0}{du_1} = \frac{D_1 \cdot (1+m)^2 \cdot g''((1+m)(u_0-\tilde{u}_1))}{D_0 \cdot g''(u_0) + D_1 \cdot (1+m)^2 \cdot g''((1+m)(u_0-\tilde{u}_1))} > 0.$$

Since the function $u_0=u_0(\tilde{u}_1)$ is a monotonic increasing function, it has an inverse, denoted by $\tilde{u}_1 = u_1 I(u_0)$. The graph, obtained by flipping the axes of the graph of the function $u_0=u_0(\tilde{u}_1)$, is depicted in figure 4-10. The domain of definition of the inverse function is the portion of the $u_0$-axis beginning at the point $u_0_{\text{mn}}$, the $u_0$-intercept of the graph, and going out to (positive) infinity. As $n$ increases, the graph of $u_1=\tilde{u}_1(u_0)$ shifts toward the $u_1$-axis. The slope is positive and is given by
\[
\frac{du_1}{du_0} = \frac{1}{\frac{du_0}{du_1}} = \frac{D_0 \cdot g''(u_0) + D_1 \cdot (1+m)^2 \cdot g''((1+m)(u_0-u_1))}{D_1 \cdot (1+m)^2 \cdot g''((1+m)(u_0-u_1))}
\]

(4-30)

\[= \frac{D_0 \cdot g''(u_0)}{(1+m)^2 \cdot g''((1+m)(u_0-u_1))} + 1 > 1 \]

since the first term is positive.

Graph of the Function \( \bar{u}_1 = \bar{u}_1(u_0) \)

Figure 4-10

4. Analysis of the Second Necessary Conditions

From the second necessary condition we deduce the existence of a function denoted by \( \bar{u}_1 = \bar{u}_{1\text{III}}(u_0) \). To obtain the explicit definition of the function, denote by \( \tilde{w} \) the value of the argument of the function \( g'(\cdot) \) such that

(4-31) \[ g'(\tilde{w}) = (1+n) \cdot k/(1+m). \]

The value \( \tilde{w} \) exists and is unique since by assumption \( g'(\cdot) \) is monotonic decreasing with "initial" value
$g'(0)=\infty$ and "terminal" value $g'(\infty)=0$. The second condition requires that $\bar{u}_1$ be chosen, given $u_0$, such that $g'((1+m)(u_0-\bar{u}_1))$ comes as close to $(1+n)k/(1+m)$ as the constraints on $\bar{u}_1$ will allow. It is helpful to discuss two cases:

(a) Suppose $(1+m)u_0 \leq \tilde{w}$. Suppose, in addition, that $\bar{u}_1$ is chosen such that $\bar{u}_1 > 0$ (and $\bar{u}_1 \leq u_0$). Since $g'()$ is monotonic decreasing, we have $\bar{w} > (1+m)$ implies that

$$(1+n)k/(1+m) = g'(\tilde{w}) \leq g'((1+m)u_0) < g'((1+m)(u_0-\bar{u}_1)).$$

Thus, a choice of $\bar{u}_1 > 0$ merely increases the divergence of $g'((1+m)u_0)$ from the constant $(1+n)k/(1+m)$. Thus, $\bar{u}_1 = 0$.

(b) Suppose $(1+m)u_0 > \bar{w}$. Define $\bar{u}_1 = u_0 - \bar{w}/(1+m)$. Thus, $g'((1+m)(u_0-\bar{u}_1)) = g'(\tilde{w}) = (1+n)k/(1+m)$.

In this way, we have obtained the function $\bar{u}_{1\text{II}}(u_0)$ defined as follows:

$$\bar{u}_1 = 0 \text{ if } u_0 \leq \frac{\bar{w}}{(1+m)}, \text{ and}$$

$$\bar{u}_1 = u_0 - \frac{\bar{w}}{(1+m)} \text{ if } u_0 > \frac{\bar{w}}{(1+m)}.$$

The graph of the function $\bar{u}_{1\text{II}}(u_0)$, depicted in figure 4-11, consists of two line segments. The first segment follows the $u_0$-axis from the origin to the point $u_0 = \frac{\bar{w}}{(1+m)}$. The second segment is the line segment beginning at the point $(u_0, u_1) = (\frac{\bar{w}}{(1+m)}, 0)$ with slope $= 1$. 
When the expected change $n$ in the relative wage increases, the graph shifts to the left, indicated on the figure by the dotted graph. This follows from differentiating the equality (4-31), page 157, which defines $\tilde{w}$, with respect to $n$:

$$\frac{d\tilde{w}}{dn} = \frac{k}{(1+m) \cdot g''(\tilde{w})} < 0.$$  

As $n \to \infty$, the first segment of the graph $u_1 = u_{1II}(u_0)$ shrinks to a point (the origin) since $g'(\tilde{w}) = (1+n)k \tilde{w}/(1+m) \to \infty$ and $g'(0) = \infty$.

5. Simultaneous Solution of the Necessary Conditions

First, we shall study the properties of the intersection of the graphs of $u_{1I}(u_0)$ and $u_{1II}(u_0)$. 

Figure 4-11

Graph of the Function $u_1 = u_{1II}(u_0)$
case (a): \( u_{0mn} = \frac{\tilde{w}}{(1+m)} \). In this case, recall that \( \frac{du_{1I}}{du_0} > 1 \) (see (4-30, page 157), where the derivative is evaluated at every \( u_0 \) in the domain of \( \tilde{u}_{1I}(u_0) \). Both graphs go through the point \((u_0, \tilde{u}_1) = (\frac{\tilde{w}}{(1+m)}, 0)\). Since the graph of \( \tilde{u}_{1I}(u_0) \) extends only to the right of this point and since the graph of \( \tilde{u}_{1II}(u_0) \) has constant slope of unity, the graph of \( \tilde{u}_{1II}(u_0) \) lies above the graph of \( \tilde{u}_{1III}(u_0) \) to the right of \((u_{0mn}, 0)\). Thus, the intersection of the two graphs occurs in the point \((u_{0mn}, 0)\).

Case (b): \( u_{0mn} < \frac{\tilde{w}}{(1+m)} \). The graph of \( \tilde{u}_{1I}(u_0) \) begins at the point \((u_0, u_1) = (u_{0mn}, 0)\) and extends to the right. The graph of \( \tilde{u}_{1II}(u_0) \) goes through the point \((u_{0mn}, 0)\) and coincides with the \( u_0 \)-axis to the right of \((u_{0mn}, 0)\) up to the point \((u_0, u_1) = (\frac{\tilde{w}}{(1+m)}, 0)\) where the slope changes to 1. Thus, the graph of \( \tilde{u}_{1I}(u_0) \) lies above the graph of \( \tilde{u}_{1II}(u_0) \) to the right of \((u_{0mn}, 0)\); and the intersection of the two graphs occurs at \((u_{0mn}, 0)\).

Case (c): \( u_{0mn} > \frac{\tilde{w}}{(1+m)} \). The graph of \( \tilde{u}_{1I}(u_0) \) begins at the point \((u_{0mn}, 0)\) and extends to the right. Since the segment of the graph of \( \tilde{u}_{1III}(u_0) \) with unit slope begins at the point \((\frac{\tilde{w}}{(1+m)}, 0)\) which lies to
Intersections of the Graphs of $\tilde{u}_{1I}(u_0)$ and $\tilde{u}_{1II}(u_0)$

Plate XI
the left of \((u_{0mn}, 0)\), the slope of \(u_{1II}(u_0)\) is unity in the domain of definition of the function \(u_{1II}(u_0)\). But the slope of \(u_{1II}(u_0)\) exceeds 1 at each point \(u_0\) in its domain of definition. Thus, the intersection \((u_0, \tilde{u}_1)\) is such that \(u_0 > u_{0mn}\) and \(\tilde{u}_1 > 0\). This is the only case where the \(\tilde{u}_1\)-coordinate is greater than 0.

Next, we study what the relationship between \(u_{0mn}\) and \(\tilde{w}/(1+m)\) is, and how it depends on \(n\), the expected percentage change of the relative wage. We shall discuss three cases corresponding to the cases just discussed.

**case (a):** \(u_{0mn} = \tilde{w}/(1+m)\). Recall the function \(H(\cdot, \cdot)\), (4-25), page 152.

\[
H(u_0, 0) = \tilde{w}/(1+m), 0 = D_0 \{ g'(\tilde{w}/(1+m)) - k \} + D_1 \{ (1+m) \cdot g'(\tilde{w}/(1+m)) - (1+n) \cdot k \} = D_0 \{ g'(\tilde{w}/(1+m)) - k \} + D_1 \{ (1+n) \cdot k - (1+n) \cdot k \} = D_0 \{ g'(\tilde{w}/(1+m)) - k \},
\]

where the substitution \(g'(\tilde{w}) = (1+n) \cdot k/(1+m)\) (see (4-31), page 157) has been made.

Let \(\tilde{w}_0\) be such that

\[
(4-34) \quad g'(\tilde{w}_0/(1+m)) = k.
\]

Because of the monotonicity of \(g'(\cdot)\), this \(\tilde{w}_0\) exists and is unique. Next, let \(\tilde{n}_0\) be defined by
\[ \tilde{n}_0 = (1+m)g'(\tilde{w}_0) - 1; \text{ or,} \]
\[ g'(\tilde{w}_0) = (1+\tilde{n}_0) \cdot k/(1+m). \]

Because of the last equality we may legitimately represent \( \tilde{w}_0 \) as \( \tilde{w}(\tilde{n}_0) \). Since \( \tilde{w}(\tilde{n}_0) \to \infty \), then \( \tilde{n}_0 > -1 \).

Notice that \( \tilde{n}_0 \) exists uniquely. With this definition of \( \tilde{n}_0 \) and \( \tilde{w}(\tilde{n}_0) \),

\[ H(\tilde{w}(\tilde{n}_0)/(1+m),0) = D_0 \cdot \left\{ g'(\tilde{w}_0(\tilde{n}_0)/(1+m)) - k \right\} = 0. \]
Thus, \( \tilde{w}(\tilde{n}_0)/(1+m) = u_{0mn} \)

**case (b):** \( u_{0mn} > \tilde{w}/(1+m) \). Taking the derivative of the relation (4-31), page 157, which defines the quantity \( \tilde{w}(n) \), with respect to \( n \) yields:

\[ \frac{d\tilde{w}}{dn} = \frac{k}{(1+m) \cdot g''(\tilde{w}/n)} < 0. \]
That is, \( \tilde{w}(n) \) is a monotonic decreasing function of \( n \).

Define the function

(4-35) \[ J(n) = H(\tilde{w}/(1+m),0) = D_0 \cdot \left\{ g'(\tilde{w}/(1+m)) - k \right\}. \]
Take the derivative of \( J(n) \) with respect to \( n \) to obtain

(4-36) \[ J'(n) = \frac{dH(\tilde{w}(n)/(1+m),0)}{dn} = D_0 \cdot g''(\tilde{w}/(1+m)) \cdot \frac{1}{1+m} \cdot \frac{d\tilde{w}}{dn} > 0. \]
That is, \( J(n) \) is a monotonic increasing function of \( n \), for all \( n \) such that \(-1 \leq n < \infty \). Thus, for \( n > \tilde{n}_0 \),
J > 0; or, \( H(w(n)/(1+m),0) > 0 \). Recall that \( H_1(u_0,0) < 0 \) ((4-26), page 152), so that \( H(u_0,0) \) is a monotonic decreasing function of \( u_0 \). Thus, \( u_{0mn}(n) > \tilde{w}(n)/(1+m) \) for all \( n \) such that \(-1 < n_0 < n\). (It will be recalled that \( u_{0mn} \) is defined by \( H(u_{0mn})=0 \).) This is case (b) in Plate XI on page 161.

Graph of the Function \( J(n) \)

![Graph of the Function J(n)](image)

**Figure 4-12**

\( u_{0mn} < \tilde{w}/(1+m) \). On the other hand, for \( n < n_0 \), \( J(0) \); or, \( H(\tilde{w}(n)/(1+m),0) < 0 \). Since \( H(u_0,0) \) is monotonic decreasing in \( u_0 \), \( u_{0mn}(n) < \tilde{w}(n)/(1+m) \). This is case (c) in Plate XI on page 161.

We summarize the results in a proposition.

**Proposition IX.** Suppose the firm maximizes the discounted profits function

\[
D(0) \left\{ g(u_0) - ku_0 \right\} + D(1) \left\{ g((1+m)(u_0 - \bar{u}_1)) - (1+n)k(u_0 - \bar{u}_1) \right\},
\]
where \( u_0 \) and \( \tilde{u}_1 \) are restricted to be non-negative and \( \tilde{u}_1 \leq u_0 \), and the production function \( g(*) \) satisfies the Inada Conditions.

Then there exist unique solutions \( \tilde{n}_0 > -1 \) and \( \tilde{w}_0 \) of the two equations

\[
g'(\tilde{w}_0/(1+m)) = k \quad \text{and} \quad g'(\tilde{w}_0) = (1+\tilde{n}_0)k/(1+m).
\]

If \( n \leq \tilde{n}_0 \), then the firm's first and second period demand for input is \( u_{0mn} \), a quantity which is obtained by solving the condition

\[
D_0 \{ g'(u_{0mn}) - k \} + D_1 \cdot \{(1+m)g'((1+m)u_{0mn})-(1+n)k\}.
\]

If \( n > \tilde{n}_0 \), then the firm in the first period hires \( u_0 (>u_{0mn}) \) units of factor and in the second period dismisses \( \tilde{u}_1 (<u_0) \) units of the factor hired during the first period. The input requirements are obtained by solving the equations

\[
g'((1+m)(u_0-\tilde{u}_1)) = (1+n)k/(1+m); \quad \text{and},
\]

\[
D_0 \cdot g'(u_0) + D_1 \cdot (1+m) \cdot g'((1+m)(u_0-\tilde{u}_1)) = D_0 \cdot k + D_1 \cdot (1+n)k.
\]

The theorem states that, if the expected change in the relative wage is greater than a critical level, then the firm will "overhire" in the first period before the increase and dismiss some of the factor in
the second period when the increase takes place. In this case, the firm hires more factor in the first period than it would have if it expected no change in the factor costs. On the other hand, if the firm's expected change in the relative wage does not exceed the critical level, then the firm hires a certain amount of factor at the outset without augmentation and diminution during the rest of the production period.

The next proposition will describe what happens to \( u_0 \) and \( \bar{u}_1 \) as \( n \) increases. Recall, to begin, that the graphs of \( \bar{u}_{I I}(u_0) \) and \( \bar{u}_{I I I}(u_0) \) both shift to the left when \( n \) increases (see (4-28), page 156 and (4-33), page 159). Again, the discussion will fall into the three cases followed by the above discussions in this section.

**case (a):** \( u_{0mn} = \tilde{w}/(1+m) \). This case corresponds to \( n=n_0 \) (page 160, case (a)). Recall that ((4-27), page 155)

\[
\frac{du_{0mn}}{dn} = \frac{D_1 k}{D_0 g''(u_{0mn}) + D_1 (1+m)^2 g''((1+m)u_{0mn})} < 0.
\]

Let \( \tilde{u}_0 \) be the point of departure from the \( u_0 \)-axis of the graph of \( \bar{u}_{I I I}(u_0) \). Then it satisfies the equation,
obtained from the second condition by setting \( u_1 = 0 \),
\[ g'((1+m)\tilde{u}_0(n)) = (1+n)k/(1+m). \]
Taking the derivative with respect to \( n \) yields
\[
(4-37) \quad \frac{d\tilde{u}_0}{dn} = \frac{k}{(1+m)^2g''((1+m)\tilde{u}_0(n))}
\]
\[
= \frac{k}{(1+m)^2g''((1+m)\tilde{w}/(1+m))} < 0
\]
Take the reciprocals of the derivatives \( \frac{du_{0mn}}{dn} \) and \( \frac{du_0}{dn} \)
(each are \( > 0 \)) and make the substitution of the one into the other as indicated:
\[
\frac{dn}{du_{0mn}} = \frac{D_0g''(u_{0mn}(n))}{D_1k} + \frac{D_1(1+m)^2g''((1+m)u_{0mn}(n))}{D_1k}
\]
\[
= \frac{D_0g''(\tilde{w}(n)/(1+m))}{D_1k} + \frac{(1+m)^2g''(\tilde{w})}{k}
\]
\[
= \frac{D_0g''(\tilde{w}(n)/(1+m))}{D_1k} + \frac{dn}{d\tilde{u}_0} < \frac{dn}{d\tilde{u}_0}
\]
since the first term in the sum is negative. Thus
\[
\frac{dn}{d\tilde{u}_0} \cdot \frac{d\tilde{u}_0}{dn}.
\]
Taking reciprocals again yields \( \frac{du_{0mn}}{dn} > \frac{d\tilde{u}_0}{dn} \).
Since both derivatives are negative, the inequality implies
\[
\left| \frac{du_{0mn}}{dn} \right| < \left| \frac{d\tilde{u}_0}{dn} \right|.
\]
Thus, as indicated in figure 4-13, the initial position marked (1) moves to
the position marked (2). The new position is an intersection in the interior of the second segment of the
\( \tilde{u}_{1II}(u_0) \) curve. This means that the second necessary condition takes the form of an equality \( g'(((1+m)(u_0 - \tilde{u}_1)) = (1+n)k/(1+m) \) at the two points: at the original position \( u_0 = u_{0mn}, u_1 = 0 \); and at the new position after \( n \) is increased.

\[
\frac{\partial}{\partial m}(1+m)g''((1+m)(u_0 - \tilde{u}_1)) = \frac{k}{(1+m)}.
\]

Thus,

\[
(4-38) \quad \frac{du_0}{dn} - \frac{d\tilde{u}_1}{dn} = \frac{k}{(1+m)2g''((1+m)(u_0 - \tilde{u}_1))}.
\]

The first condition is (see (4-23), page 150)

\[
D_0 \{ g'(u_0) - k \} + D_1 \{ (1+m)g'((1+m)(u_0 - \tilde{u}_1)) - (1+n)k \} = 0
\]

both at the initial position \( (u_0, \tilde{u}_1) = (u_{0mn}, 0) \) and at
the new position after the increase in n. Take the derivative of this condition with respect to n and substitute the relation (4-38), page 168, obtained at the end of the preceding paragraph:

\[ 0 = D_0 g''(u_0(n)) \frac{du_0}{dn} + D_1 \{(1+m)^2 g''((1+m)(u_0(n)-\bar{u}_1(n))) \left[ \frac{du_0(n)}{dn} - \frac{du_1(n)}{dn} \right]^k \} \]

\[ = D_0 g''(u_0(n)) \frac{du_0}{dn}, \quad \text{or} \]

\[ (4-39) \quad \frac{du_0}{dn} = 0. \]

On the other hand, from (4-38) and the result (4-39) just obtained, it follows that

\[ (4-40) \quad \frac{du_1}{dn} = \frac{-k}{(1+m)^2 g''((1+m)(u_0-\bar{u}_1))} > 0. \]

Thus, an increase in n from \( n = \bar{n}_0 \) leaves \( u_0 \) unchanged and increases \( \bar{u}_1 \). This means, in this case, that when n increases, the profit maximizing firm adjusts, not by changing the amount of factor hired during the first period, but by increasing the portion of the factor (hired during the first period) dismissed during the second period.
case (b): \[ u_{0m}^{\text{mn}}(n) < \frac{\tilde{w}(n)}{(1+m)} \] Let the coordinates of the intersection be \((\tilde{u}_0(n), \tilde{u}_1(n))\). Since the functions \(u_0(n)\) and \(\tilde{w}(n)\) are continuous, and since we start with the inequality \(u_{0m}^{\text{mn}}(n) < \frac{\tilde{w}(n)}{(1+m)}\), an infinitesimal increase in \(n\) from \(n_0\) will not destroy the validity of the inequality: \(u_{0m}^{\text{mn}}(n) < \frac{\tilde{w}(n)}{(1+m)}\). Thus, for such increases in \(n\), \(\tilde{u}_1(n)\), which is initially 0, remains 0, that is, \(\frac{d\tilde{u}_1}{dn} = 0\).

Effect of Increase in \(n\):
\[ u_{0m}^{\text{mn}} < \frac{\tilde{w}}{(1+m)} \]

Figure 4-14
In order to determine the direction of change in \( \hat{u}_0 = u_{0mn} \), the \( u_0 \)-coordinate of the intersection, differentiate the defining relation ((4-23), page 150),

\[
0 = D_0 \left\{ g'(u_{0mn}(n)) - k \right\} + D_1 \left\{ (1+m)g'((1+m)u_{0mn}(n)) - (1+n)k \right\},
\]

with respect to \( n \) and obtain

\[
0 = D_0 \frac{d}{dn} g''(u_{0mn}(n)) \frac{du_{0mn}}{dn} + D_1 \left\{ (1+m)^2 g''((1+m)u_{0mn}(n)) \frac{du_{0mn}}{dn} - k \right\}; \quad \text{or,}
\]

\[(4-41) \]

\[
\frac{du_0}{dn} = \frac{du_{0mn}}{dn} = \frac{D_1 k}{D_0 g''(u_{0mn}(n)) \frac{du_{0mn}}{dn} + D_1 (1+m)^2 g''((1+m)u_{0mn}(n))} < 0.
\]

Thus, \( u_0(n) \) decreases and \( \bar{u}_1 = 0 \) remains unchanged as \( n \) increases infinitesimally from \( \hat{n} \), the \( n \)-value corresponding to original intersection.

Economically, this means that the original strategy which requires the employment of equal amounts of factor in the first and second periods is changed, when \( n \) increases, to a new nearby strategy which requires again the employment of equal amount of factor in the two periods, but the new amounts are less than those of the original strategy. The decre-
ment \( \bar{u}_1 \) cannot adjust because the constraint that 
\( \bar{u}_1 > 0 \) is binding. Thus, all of the adjustment must 
take place in \( u_0 \), the first period demand for factor 
(which the firm neither increases nor decreases during 
the second period).

\[ \text{case (c): } u_{0mn}(n) > \bar{w}(n)/(1+m). \] 
Let the intersection be denoted by \((\hat{u}_0(n), \hat{u}_1(n))\). This is equiva-
 lent to \( n > n_0 \) (see page 160, case (b)). That the inter-
section is internal to the second segment of the graph 
\( \bar{u}_{1\bar{I}}(u_0) \) implies that \((1+m)\hat{u}_0(n) + \hat{u}_1(n) = \bar{w} \), where
\( \hat{u}_1(n) \neq 0 \) (see (4-32), page 158, the definition of the 
function \( \bar{u}_{1\bar{I}}(u_0) \)). Also, the second necessary con-
dition in the form

\[ (4-42) \quad g'((1+m)(\hat{u}_0(n) - \hat{u}_1(n)) = (1+n)k/(1+m) \]

holds for \( n \) infinitesimally greater than \( n \). Thus, 
the first necessary condition ((4-23), page 150)
becomes, upon substitution of (4-42),

\[ 0 = D_0 \left[ g'(\hat{u}_0(n)) - k \right] + D_1 \left[ (1+m)g'(1+m)(\hat{u}_0(n) - \hat{u}_1(n)) \right. \]
\[ - (1+n)k \]
\[ = D_0 \left[ g'(\hat{u}_0(n)) - k \right] + D_1 \left[ (1+m)(1+n)k/(1+m) - (1+n)k \right] \]
\[ = D_0 \left[ g'(u_0(n)) - k \right]. \]
Taking the derivative with respect to \( n \) yields

\[
(4-43) \quad D_0 g''(\hat{u}_0(n)) \frac{d\hat{u}_0}{dn} = 0; \quad \text{or} \quad \frac{d\hat{u}_0}{dn} = 0.
\]

Taking the derivative of the second necessary condition, \((4-42)\), page 172, which will be satisfied in a neighborhood of \( \hat{u}_1(n) \), since \( \hat{u}_1(n) > 0 \), gives

\[
(1+m)g''((1+m)(\hat{u}_0(n)-\hat{u}_1(n)))[\frac{d\hat{u}_0}{dn} - \frac{d\hat{u}_1}{dn}] = k/(1+m); \quad \text{or},
\]

\[
(4-44) \quad \frac{d\hat{u}_1}{dn} = \frac{-k}{(1+m)^2} \cdot \frac{1}{g''((1+m)(\hat{u}_0(n)-\hat{u}_1(n)))} > 0,
\]

since \( \frac{d\hat{u}_0}{dn} = 0 \). Thus, when \( n \) increases from \( \hat{n} \), \( \hat{u}_0 \) is unchanged and \( \hat{u}_1 \) increases.
Economically, this means that when the expected change in the relative wage increases infinitesimally from \( \hat{n} \), the expected change of the relative wage corresponding to the strategy \( (\hat{u}_0(\hat{n}), \hat{u}_1(\hat{n})) \), the first period demand for input does not change; whereas the decrement from the first period level for the second period, which initially was different from zero, increases. In figure 4-15 which illustrates this case, the old strategy is marked (1) and the new nearby strategy is marked (2). The initial strategy prescribes the hiring in the first period of a certain amount of factor and a dismissal in the second period of a certain portion of the factor hired during the first period; when the expected change of the relative wage increases, the new strategy prescribes the same level of factor employment during the first period as the old strategy, but an increase in the portion of this factor which is dismissed during the second period.

In summary of the above results, we state the following proposition.

**Proposition X.** Let \( n'_0 \) be as in the preceding proposition. Let the intersection of the graphs of \( \bar{u}_{1I}(u_0) \) and \( \bar{u}_{1II}(u_0) \) which comprises the B Strategy be denoted by \( (\hat{u}_0(\hat{n}), \hat{u}_1(\hat{n})) \), where \( \hat{n} \) is the expected change in the relative wage
If $\hat{n}=\bar{n}_0$, then $u_{0mn}=\bar{w}/(1+m)$, and $\frac{du_0}{dn}=0$ and $\frac{du_1}{dn}>0$.

If $\hat{n}<\bar{n}_0$, then $u_{0mn}<\bar{w}/(1+m)$, and $\frac{du_0}{dn}=0$ and $\frac{du_1}{dn}=0$.

If $\hat{n}>\bar{n}_0$, then $u_{0mn}>\bar{w}/(1+m)$, and $\frac{du_0}{dn}=0$ and $\frac{du_1}{dn}>0$.

D. Synthesis

Having determined an A Strategy and a B Strategy, the firm finally compares the two, choosing the more profitable. Before enumerating the various situations, it will be helpful to make some preliminary remarks.

Recall that $\bar{w}(n)/(1+m)$ and $\bar{w}(n)/(1+m)$ are the $u_0$-coordinates of the points on the $u_0$-axis from which the graphs of $u_{1II}(u_0)$ and $\bar{u}_{1II}(u_0)$, respectively, leave the $u_0$-axis. They are determined by the conditions

$$g'(\bar{w}(n))=(1+n)k$$

and

$$g'(\tilde{w}(n))=(1+n)k/(1+m)$$

(see (4-13), page 130 and (4-31), page 157). Thus,

$$g''(\bar{w}(n))>g''(\tilde{w}(n)).$$

Since $g'$ is monotonic decreasing, $\bar{w}(n)<\tilde{w}(n)$; and $\bar{w}(n)/(1+m)\leq\tilde{w}(n)/(1+m)$ for every $n$ such that $-1<n<\infty$.

Next, we show that $\bar{n}_0>n_0$, where $n_0$ and $\bar{n}_0$ are the critical levels of expected changes in the relative wage for Strategy A and Strategy B, respectively (see propositions VII and IX). On the one hand, the
value \( n_0 \) is determined (along with \( \bar{w} \)) by the joint solution of the two conditions:

\[
g'(\bar{w}) = (1+n_0)k \quad \text{and} \quad g'(\bar{w}/(1+m)) = k - D(1) \frac{D(1)}{D(0)} m(1+n_0) \quad k < k.
\]

The inequality holds since \( 0 > n_0 > -1 \) (see proposition VII). On the other hand, the value \( \bar{n}_0 \) is determined (along with \( \bar{w} \)) by the joint solution of the two conditions (see proposition IX):

\[
g'(\bar{w}) = (1+\bar{n}_0)k/(1+m) \quad \text{and} \quad g'(\bar{w}/(1+m)) = k.
\]

Thus,

\[
g'(\bar{w}(n_0)/(1+m)) > g'(\bar{w}(n_0)/(1+m)).
\]

\[
\bar{w}(n_0)/(1+m) < \bar{w}(n_0)/(1+m); \quad \text{or,} \quad \bar{w}(n_0) < \bar{w}(n_0).
\]

\[
g'(\bar{w}(\bar{n}_0)) > g'(\bar{w}(n_0)).
\]

\[
(1+\bar{n}_0)/(1+m) = g'(\bar{w}(\bar{n}_0)) > g'(\bar{w}(n_0)) = (1+n_0)k.
\]

\[
(1+\bar{n}_0)/(1+m) > (1+n_0); \quad \text{or} \quad (1+\bar{n}_0) > (1+n_0)(1+m)/(1+n_0),
\]

where the last inequality is implied by the relation \((1+m) > 1\). Thus, \( \bar{n}_0 > n_0 \).

Finally, we show that \( u_{0mx} = u_{0mn} \). The \( u_0 \)-intercept, \( u_{0mx} \), of the \( u_{1I}(u_0) \) curve is determined by setting \( u_1 = 0 \) in the equation, \((4-4)\), page 123.

\[
D_0 \left\{ g'(u_0) - k \right\} + D_1 \left\{ (1+m)g'((1+m)u_0 + u_1) - (1+n)k \right\} = 0.
\]

The \( u_0 \)-intercept, \( u_{0mn} \), of the \( \bar{u}_{1I}(u_0) \) curve is like-
wise determined by setting \( u_1 = 0 \) in the equation ((4-23), page 150):

\[
D_0 \left\{ g'(u_0) - k \right\} + D_1 \left\{ (1+m)g'((1+m)(u_0 - \bar{u}_1) - (1+n)k \right\} = 0.
\]

When \( u_1 \) is set=0 in the first equation and \( \bar{u}_1 \) is set=0 in the second equation, we see that the two quantities satisfy the same condition

\[
(4-45) \quad D_0 \left\{ g'(u_0) - k \right\} + D_1 \left\{ (1+m)g'((1+m)u_0) - (1+n)k \right\} = 0.
\]

The solution of this condition was seen to be unique. Thus \( u_{0mx} = u_{0mn} \). Let \( u_{0m} \) denote the common value.

In order to show the A Strategy and the B Strategy on one graph, we extend the definition of the variable \( u_1 \) to take on negative values by setting \( u_1 = -\bar{u}_1 \). This definition flips the graphs of the B Strategies about the \( u_0 \)-axis.

The above preliminaries yield three possible cases. We shall consider each of the three possible cases, in turn, to determine which strategy, A or B, is chosen in each case.

\[
\text{Case (a):} \quad n < n_0 < \bar{n}_0; \quad u_{0m} < \bar{w}/(1+m) < \bar{w}/(1+m).
\]

Strategy A, marked "A" on the graph, requires the firm to hire \( u_{0A} = u_0(<u_{0m}) \) units of factor for the first
\[ \frac{v}{l+m} n < n_0 \leq n \]

\[ \text{case (a)} \]

\[ \frac{w}{l+m} n_0 < n \leq n \]

\[ \text{case (b)} \]

\[ \frac{w}{l+m} n_0 < n \leq n \]

\[ \text{case (c)} \]

Juxtaposition of the A and B Strategies

Plate XII
period and to hire another $u_{1A} = u_1 > 0$ additional units of factor for the second period. Strategy B, marked "B" on the graph, requires the firm to hire $u_{0m}$ units of factor in the first period and to keep this factor without additional hiring or dismissal during the second period.

The discounted profits of the firm which result from Strategy A are given by

$$D_0 \{g(u_{0A}) - ku_{0A} \} + D_1 \{g((1+m)u_{0A}+u_{1A}) -(1+n)k(u_{0A}+u_{1A}) \}.$$  

The discounted profits of the firm which result from Strategy B are given by

$$D_0 \{g(u_{0m}) - ku_{0m} \} + D_1 \{g((1+m)u_{0m}) -(1+n)ku_{0m}) \}.$$  

Thus, the two strategies may be regarded as solutions of the problem: to determine $u_0, u_1 > 0$ in order to maximize the discounted profits

$$(4-46) = D_0 \{g(u_0) - ku_0 \}$$

$$+ D_1 \{g((1+m)u_0+u_1) -(1+n)k(u_0+u_1) \}.$$  

This is merely a sub-problem of selecting the most profitable of strategies of type A, the unique solution of which was found to be Strategy A. Thus, in this case the firm chooses Strategy A.
\textbf{case (b):} \quad n_0 \leq n < \tilde{n}_0; \quad \tilde{w}/(1+m) \leq u_{0m} < \tilde{w}/(1+m).

Strategy A and Strategy B coincide in this case. They require the firm to use $u_{0m}$ units of factor services in each period.

\textbf{case (c):} \quad n_0 < \tilde{n}_0 \leq n; \quad \tilde{w}/(1+m) < \tilde{w}/(1+m) \leq u_{0m}.

Strategy A, marked "A" in the graph, requires the firm to hire $u_{0m}$ units of factor services in each period. Strategy B, marked "B" in the graph, requires the firm to hire $u_{0B} = u_0 (> u_{0m})$ units of factor service during the first period and to decrease this initial level by $u_{1B} = \tilde{u}_1 = -u_1$ units in the second period.

The discounted profits of the firm which result from Strategy A are given by

$$D_0 \left[ g(u_{0m}) - ku_{0m} \right] + D_1 \left[ g((1+m)u_{0m}) - (1+n)ku_{0m} \right].$$

The discounted profits of the firm which result from Strategy B are given by

$$D_0 \left[ g(u_{0B}) - ku_{0B} \right] + D_1 \left[ g((1+m)(u_{0B} - \tilde{u}_1B)) -(1+n)k(u_{0B} - \tilde{u}_1B) \right].$$

Thus, the two strategies may be regarded as solutions of the problem: to determine $u_0, u_1 \geq 0$ in order to maximize the discounted profits:
This is a sub-problem of the problem of selecting the most profitable of strategies of type B, the unique solution of which was found to be Strategy B. Thus, in this case, the firm chooses Strategy B.

Proposition XI. Suppose the firm seeks to maximize each of the following two discounted profits functions and to choose the one which results in the greater value:

(1) \[ D_0 \{ g(u_0) - ku_0 \} + D_1 \{ g((1+m)(u_0 - \bar{u}_1)) - (1+n)k(u_0 - \bar{u}_1) \} \]

where \( u_0 \) and \( u_1 \) must be chosen non-negative, and

(2) \[ D_0 \{ g(u_0) - ku_0 \} + D_1 \{ g((1+m)(u_0 - \bar{u}_1)) - (1+n)k(u_0 - \bar{u}_1) \} \]

where \( u_0 \) and \( \bar{u}_1 \) must be chosen non-negative, with \( u_0 \geq \bar{u}_1 \), where \( n \geq -1 \), and where the function \( g(\cdot) \) satisfies the Inada Conditions.

Then there exists a unique solution \( n_0 \), such that \( -1 < n_0 < 0 \), of the two equations:

\[ g'(\bar{w}) = (1+n_0)k; \quad \text{and} \]
$D_0 g'(\tilde{w}/(1+m)) = D_0 k - D_1 m(1+n_0)k$. ($\tilde{w}$ is an unknown which is determined jointly with $n_0$.)

Furthermore, there exists a unique solution $\tilde{n}_0$ of the two equations:

$g'(\tilde{w}/(1+m)) = k$ and $g'(\tilde{w}) = (1+\tilde{n}_0)k/(1+m)$.

($\tilde{w}$ is an unknown which is jointly determined with $\tilde{n}_0$.)

Moreover, $n_0 < \tilde{n}_0$.

(a) If $n < n_0 < \tilde{n}_0$, then the firm hires $u_0$ units of factor during the first period and hires $u_1$ units of factor, in addition to the $u_0$, in the second period. The pair $(u_0, u_1)$ is obtained by solving the two equations:

$g'((1+m)u_0 + u_1) = (1+n)k$; and,

$D_0 [g'(u_0) - k] + D_1 [(1+m)g'((1+m)u_0 + u_1) - (1+n)k] = 0$.

(b) If $n_0 \leq n < \tilde{n}_0$, then the firm hires $u_{0m}$ units of factor in each of the two periods. The quantity $u_{0m}$ is determined by

$D_0 [g'(u_{0m}) - k] + D_1 [(1+m)g'((1+m)u_{0m}) - (1+n)k] = 0$.

(c) If $n_0 < \tilde{n}_0 \leq n$, then the firm hires $u_0$ units of factor in the first period and dismisses during the second period $\bar{u}_1$ units of the factor, hired during the first period. The pair $(u_0, \bar{u}_1)$ is obtained by solving
the two equations

\[ g'((1+m)(u_0 - \bar{u}_1)) = (1+n)k/(1+m), \]

and

\[ D_0 \left\{ g'(u_0) - k \right\} + D_1 \left\{ (1+m)g'((1+m)(u_0 - \bar{u}_1)) - (1+n)k \right\} = 0. \]

The conclusions of the proposition are depicted, case for case, in Plate XII on page 178.

It is interesting to verify the existence of the two critical values, \( n_0 \) and \( \bar{n}_0 \), in the type of learning model, we are now discussing, which uses the Cobb-Douglas (C-D) production function. The example, with \( n=0 \), was discussed at the end of chapter two (Section E, page 81, et seq). The two objective functions on which the firm bases its profit calculations now include an expected percentage change in the relative wage, in the second period, which is in general different from zero.

For an increase in input in the second period the objective function with a C-D production function is (see (4-1), page 119):

\[ \pi_A(u_0, u_1) = D_0 \cdot \int u_0 ^ \alpha - k \right\} + D_1 \cdot \left\{ [(1+m)u_0 + u_1] ^ \alpha - (1+n) \cdot k \cdot (u_0 + u_1) \right\}, \]

where \( u_0, u_1 \geq 0 \).

For a decrease in input in the second period, the objective function with a C-D production function is (see (4-3), page 119):
\[ \Pi_B(u_0, \tilde{u}_1) = D_0 \cdot \left\{ u_0^\alpha - k \right\} + D_1 \cdot \left\{ \left[ (1+m)(u_0 - \tilde{u}_1) \right]^\alpha - (1+n) \cdot k \cdot (u_0 - \tilde{u}_1) \right\}, \]

where \( u_0, \tilde{u}_1 \geq 0 \), and \( |\tilde{u}_1| \leq u_0 \).

Proposition V, page 80, of chapter two informs us that the objective function \( \Pi_A(u_0, u_1) \) should be ignored, always, since the firm in its regard for profits will never augment its demand for input in the second period. In fact, we saw that the firm's demand for input in each period will be \( u_{0m} \) determined, according to (2-87), page 84, solely as the \( u_0 \)-intercept, by the first necessary condition which relates the variables \( u_0 \) and \( u_1 \).

Proposition IX, just proved, concludes that there always exists an expected rate of deflation of the relative wage which will reverse the behavior of the firm just described - the firm will unambiguously increase its demand for input in the second period. The critical level \( n_0 \) of relative wage deflation is given by the following conditions obtained from Proposition XI by specialization to the C-D production function:

1. \( (4-48) \quad \alpha \cdot \left[ \frac{\tilde{w}(n_0)}{w(n_0)} \right]^{\alpha-1} = (1+n_0) \cdot k; \)

and

2. \( (4-49) \quad D_0 \cdot \alpha \left[ \frac{\tilde{w}(n_0)}{1+m} \right]^{\alpha-1} = D_0 \cdot k - D_1 \cdot m \cdot (1+n_0) \cdot k. \)
Substitution of (4-48) into (4-49) gives
\[ D_0 \cdot (1+n_0) \cdot k \cdot (1+m)^{1-\alpha} = \left[ D_0 - D_1 \cdot m \cdot (1+n_0) \right] \cdot k; \]
or,
\[ (4-50) \quad 1+n_0 = \frac{D_0}{D_0 \cdot (1+m)^{1-\alpha} + D_1 \cdot m}. \]

Since \( m > 0 \), the right side is clearly between 0 and 1, thus, \(-1 < n_0 < 0\), as predicted by proposition XI.

The critical value \( \bar{n}_0 \) is determined by the two conditions again obtained from the proposition by specialization:

\[ (4-51) \quad \alpha \cdot \left[ \frac{\bar{w}(\bar{n}_0)}{1+m} \right]^{\alpha-1} = k; \]

and,
\[ (4-52) \quad \alpha \cdot \left[ \bar{w}(\bar{n}_0) \right]^{\alpha-1} = (1+\bar{n}_0) \cdot k/(1+m). \]

Substitution from (4-51) into (4-52) yields:
\[ \frac{k}{(1+m)^{1-\alpha}} = \frac{(1+\bar{n}_0) \cdot k}{(1+m)}; \text{ or,} \]
\[ (4-53) \quad (1+\bar{n}_0) = \frac{(1+m)}{(1+m)^{1-\alpha}} = (1+m)^{\alpha}. \]

The right side of the last equality is clearly greater than 1. Thus, \( \bar{n}_0 > 0 \).
It will be recalled that in this example, when the firm expected prices to remain unchanged in the second period, the firm demanded the same amount of input in each of the two periods. Now we see "why" this is the case. An expectation of unchanging prices corresponds to \( n = 0 \), which lies between \( n_0 \) and \( \bar{n}_0 \), as shown in figure 4-16.

\[1\quad n_0\quad 0\quad \bar{n}_0\quad n\]

The Critical Values for the C-D Example
Figure 4-16

We next ask what happens in each of the three cases described in Proposition XI if \( n \) is increased?

In case (a), proposition VIII on page 146 applies. Thus, \( \frac{dU_0}{dn} > 0 \) and \( \frac{dU_1}{dn} < 0 \).

In case (b), proposition VIII for the case \( n_0 < n \) and proposition X on page 175 for the case \( n < \bar{n}_0 \) both apply and give, naturally, the same result: \( \frac{dU_0}{dn} < 0 \) and \( \frac{dU_1}{dn} = 0 \).
In case (c), proposition X on page 175 for the case $\bar{n}_0 \leq n$ applies. Thus, $\frac{du_0}{dn} = 0$ and $\frac{du_1}{dn} = -\frac{du_1}{dn} = 0$.

We have established the following proposition.

**Proposition XII.** Let the critical values $n_0$ and $\bar{n}_0$ be defined as in the preceding proposition.

(a) If $n < n_0 < \bar{n}_0$, then $\frac{du_0}{dn} > 0$ and $\frac{du_1}{dn} < 0$.

(b) If $n_0 \leq n < \bar{n}_0$, then $\frac{du_0}{dn} < 0$ and $\frac{du_1}{dn} = 0$.

(c) If $n_0 < \bar{n}_0 \leq n$, then $\frac{du_0}{dn} = 0$ and $\frac{du_1}{dn} < 0$. 
CHAPTER V
OVERVIEW AND EXTENSIONS

With the benefit of hindsight, I shall briefly review the results of the preceding pages in regard to the assumptions and to possible natural extensions.

The central assumptions which permit us to deduce unique (non-multiple) employment plans for the firm are, first, the assumption of a production function with diminishing marginal returns, and, second, the assumption that the variable factor may be ranked according to its potential for later increased efficiency. (The functions \( m(\cdot) \) and \( m'(\cdot) \), page 6.) We implicitly assumed that this ranking feature outweighs any advantage that may be gained from increasing the amount of factor employed (economies of scale in learning).

Then we specialized the general model to two cases, in the first, assuming that there occurred no individual variation in the degree of increased efficiency resulting from the accumulation of experience (\( m(\cdot) \)=constant, page 29), and, in the second, that there occurred no firm-learning (\( \theta=0 \), page 36). We deduced that these two sub-models yielded different conclusions; in particular, the submodel without firm-learning logically excludes the
possibility of an increase in demand for factor by a firm maximizing discounted profits (proposition V, page 80). This result was verified in Chapter III by example when the production function was specialized to the Cobb-Douglas production function (end of section B, page 95).

The remainder of Chapter III was devoted to a static sensitivity analysis -- a comparison of the responses to changes in factor and output prices of the conventional factor-augmenting model of technological progress with our submodel in which no firm learning takes place (proposition VI, page 110). Once again, the Cobb-Douglas production function provided a convenient example illustrating the results of this inquiry (Plate IX, page 115).

In Chapter IV we introduced the possibility of expected factor or output price inflation and deflation and derived the employment plans of the firm (proposition XI, page 181). Again, we turned to the stand-by Cobb-Douglas production function as an instance verifying the conclusion of the proposition. Finally, we obtained a bonus from our methods in proposition XII, page 187, which presents the results of the sensitivity analysis of the firm's employment plans as changes in the firm's expected rate of inflation or deflation occur.
A natural step toward extending these results is to increase the number of periods. Several alternatives suggest themselves; among them, that the factor continues to become more efficient at a diminishing rate; or, that the factor's efficiency does not change at all after a few initial increases in efficiency; or, indeed, that after some time the factor may even lose efficiency because of obsolescence or depreciation. These extensions may be implemented by appending two subscripts to the function $m(\cdot)$ to indicate the time of hiring and the seniority of the factor. For the first suggestion (diminishing increases in efficiency) as the seniority-index increases, the positive jumps in the function become successively less and less. For the second suggested extension (eventual absence of increases in efficiency), the successive jumps disappear and the function after a finite seniority-index is reached becomes the same for each succeeding seniority-index thereafter; while for the third suggested extension (eventual decreases in efficiency due to obsolescence or depreciation) the positive jumps corresponding to increases in the seniority-index eventually give way to negative jumps.
When the number of time-periods in the planning horizon is increased, alternative assumptions, regarding the differential behavior of wages become appealing. Thus, one may seek to derive the implications of assuming that the more efficient factor may be paid a higher wage with or without a lag in recognition or a lead in anticipation. This wage behavior may be treated in very much the same way as the suggested extensions of the preceding paragraph regarding the factor's responses in efficiency, only here the changes affect the cost part of the profit functions instead of the output part.

Another interesting extension of our model is to assume that, by incurring additional cost, the firm may affect either the attainable levels of the factor's efficiency or the time-rate at which the factor becomes efficient. In handling, for example, the extension relating to level of efficiency one would parameterize the efficiency-function with the additional cost, assume this cost affects efficiency positively at least at the outset, and introduce the cost in the profit function. The modification for this extension thus would appear in both the output and cost parts of the profit functions.
WORKS CITED


