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COMPUTATION TECHNIQUES FOR VARIOUS GRAVITY ANOMALY CORRECTION TERMS AND THEIR EFFECT UPON DEFLECTION OF THE VERTICAL COMPUTATIONS FOR MOUNTAINOUS AREAS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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The Ohio State University

1973

Approved by

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Department of Geodetic Science
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ABSTRACT

An introductory background section is concerned with the discussion of terminology, concepts, discoveries and events lending to an understanding of the Classical and Modern Theories of Gravimetric Geodesy.

The various correction terms to the gravity anomaly and some alternative computational procedures are discussed. The recommended computational procedures are an Iterative Solution (Bjerhammer and Moritz) and a Nonlinear Solution (Moritz).

The recent and current investigations by: Arnold, Bjerhammer, Bursa, De Graff-Hunter, Heiskanen-Moritz, Koch, Mather, Molodensky, Moritz, Pellinen, Pick, Pola, Strange and Woollard, Tengstrom and Wrobel are discussed.

A 6 degree by 6 degree anomaly field, which is representative of a mountainous area, is established using available gravity and elevation data and Rapp's prediction program.

The Deflection of the Vertical computation procedures are discussed and a recommended approach is given using a modified Fischer technique.

The results upon which the conclusions are drawn are illustrated with several tables and figures. Also included in an Appendix are brief derivations, illustrations of the data fields, and some suggestions for the computer.
CHAPTER I
INTRODUCTION

The requirement for the application of the Theories of Physical Geodesy is well known to those dedicated scientists and agencies engaged in an effort to achieve the highest possible accuracies in position determination on an earth whose parameters are approximately known or set by international agreement to the best known or most probable values. Indeed, the precise determination of these parameters requires an interaction between all branches of Geodesy, such as the following: Gravimetric Methods, Statistical Methods, Astro-Geodetic Methods, Celestial Methods, Map Projections, Geometric Geodesy and Adjustment Computations, as well as the more recent contributions from Computer Science. In this study I shall deal mainly with the Gravimetric Methods and shall discuss the solutions obtained both manually and with the aid of high speed digital computers (The Ohio State University’s IBM 7094 and the United States Air Force Academy’s Burroughs 5500). The particular subject which will be investigated in this study is the determination of the Deflections of the Vertical in an area of rugged topography, i.e., the Pikes Peak Region of the Rocky Mountains.

One method that is used for determining the Deflections of the Vertical utilizes the results of gravimetry in subsequent mathematical computations. Gravimetry pertains to the measurement of the gravi-
atural force and to the study of its use. The total force, the resultant of the gravitational force and centrifugal force, is called gravity (Heiskanen-Moritz, 1967, p47). The gravity at two or more locations can be measured relative to each other by a precision scientific instrument called a gravimeter. The gravimeter enables the scientist to determine the derivative of the potential of gravity in the direction which is normal to the equipotential surface (surface of constant potential, \( W=\text{const.} \)) passing through the measuring point. The first derivative of this potential normal to the surface (\( W=\text{constant} \)) is the force of gravity. The second derivative of this potential normal to the surface (\( W=\text{constant} \)) is the gravity gradient. For a complete discussion of these terms refer to Heiskanen-Moritz (1967) or Brovar (1964). The unit of the force of gravity is the dyne (\( \text{gm}^{-2}\text{cm}\cdot\text{sec}^{-2} \)). The acceleration called gravity is the actual quantity determined by the gravity measurements with the gravimeter, and its unit is the gal (\( \text{cm}^{-2}\text{sec}^{-2} \)). The unit most commonly used in geodetic applications is the milligal (\( 10^{-3} \) gal). The modern gravimeters can determine differences in the acceleration due to gravity to \( 10^{-5} \) gals and a precision of \( 10^{-6} \) gals is possible with tidal gravimeters.

The traditional way of describing the measured values has been after a mathematical reduction to an equipotential surface called the geoid. The undisturbed surfaces of the oceans, which define a particular equipotential surface (the geoid) were proposed by C.F. Gauss as the mathematical figure of the earth (Heiskanen-Moritz, 1967, p49). Various approximations are sometimes applied to this
surface such as the ellipsoid (2nd approximation) which mathematically most approximately represents the equipotential surface, the sphere (1st approximation) which is mathematically simpler to use but does not represent the equipotential surface as accurately as the ellipsoid, and the plane which is simplest to use mathematically but does not represent the equipotential surface as accurately as the sphere.

Clairaut (A.C. Clairaut, 1713-1765) first established the relation of the acceleration of gravity as a function of latitude and the flattening of the earth's best fitting ellipsoid. After Clairaut, the investigations of Stokes (G. Stokes, 1819-1903) solved the problems of determining the shape of the gravitational force potential's equipotential surface from measurements of the acceleration (gravity). The lack of data often discouraged further investigations, however, important inquiries were made by Helmert (F. Helmert, 1843-1917). In 1928, Vening-Meinesz (F.A. Vening-Meinesz, 1887-1966) obtained formulae which permitted determination of the deflections of a plumb line from the vertical direction in terms of gravimetric data. Only then was it possible to project the results of geodetic measurements onto an ellipsoid which approximated the geoid.

The geoid cannot be regarded everywhere as an external equipotential surface, and consequently the theory devised by Stokes is not universally applicable. Several methods of transferring the masses external to the geoid to within the geoid were advanced and
will be discussed in part later. These methods of transference (usually called reductions) had theoretical and practical drawbacks, which presented the problem: how to represent the earth's surface by an external equipotential surface requiring only highly approximate reduction procedures? By 1945, M.S. Molodensky had formulated a new basis for solving this problem by designing a new theory for determining the figure of the earth's physical surface and the external gravitational field in a direct manner. Considering the accurate instruments available today for the measurement of the acceleration (gravity), Molodensky (Molodensky et al., 1962a, p29) advocated a change of the Classical Theory and its application in order to keep pace with the instrument technology. Some of the practical solutions involving the theory of Molodensky, which will be referred to as the Modern Theory, will be tested and compared to the traditional or Classical Theory in this investigation. The remainder of this chapter will be used to introduce and define some of the terms and concepts which are necessary for some readers to understand in order to fully comprehend the solutions which will follow.

The actual gravity field may be divided into two parts for computational purposes. The normal part is represented by an equipotential surface on which the Normal Gravity Potential \(U\) is constant. The normal gravity field is defined precisely by closed formulae shown in Heiskanen-Moritz (1967, p82). The geoid is also an equipotential surface on which the potential \(W\) is constant. This potential \(W\) is called the geopotential when it refers to the gravity potential of
the actual earth. The deviations from the normal field make up what is called a disturbing field. These disturbances may usually be considered linear since they are so small. The word linear here means that the deviation of the actual value from the approximate value may be represented by an equation containing no higher order terms of the parameter in which the expression is made or derived and which is called a linear equation (Moritz, 1964, p11). The Modern Theory enables us to determine the surface of the earth and the Deflections of the Vertical at the surface of the earth (the Deflections of the Vertical will be discussed later) by utilizing computations involving the normal and disturbing fields.

An ellipsoid which has the same volume and flattening as the real earth is called the Earth Ellipsoid (Mueller-Rockie, 1966, p28). This Earth Ellipsoid very closely approximates the geoid. If a line is extended normal to the surface of the Earth Ellipsoid and also passes through the observation point on the earth's surface (see Figure 1), we obtain the geometric height of the point. This geometric height cannot be determined directly. In Figure 1 and in the text the large case letter P represents the point with astronomic coordinates \( (\phi, \bar{\lambda}) \) in the actual gravity field and the small case letter p represents the point with the geodetic coordinates \( (\Phi, \lambda) \) in the normal gravity field. There is a certain point Q on the plumb line of P where the normal potential \( U \) at Q is equal to the actual potential \( W \) at P (Heiskanen-Moritz, 1967, p170). The normal height \( (H^*) \) of Q or p is nothing but the geometric height of Q or p respectively above the ellipsoid (Ibid.).
For a more detailed discussion of the various systems of heights, the reader is referred to the works of Krawkiwsky and Mueller (1965) and Rapp (1961). An error in the spirit leveling may also be caused by not considering an anomalous mass of unusual density near the leveling route (Kraikiwsky-Mueller, 1965, pp56-59). Since spirit leveling is the main technique used for determining height differences on the earth's surface, observations of gravity are required along the leveling route for use in the calculation of the height differences.

We can determine the normal height (H*) by knowing the latitude and potential difference between the geoid and the point in question and then using an iterative type of formula (Heiskanen-Moritz, 1967, p17) in which (H*) can be determined. Astronomic observations made at the point P of latitude \( \phi_p \) and longitude \( \lambda_p \) would completely define the location of the point if we also knew the geopotential \( W \) at the point. We are also able to determine the normal gravity field \( U \) at a point whose geodetic coordinates are \( \phi_p, \lambda_p \) for latitude and longitude respectively. The geographic coordinates in the normal gravity field \( \phi_p, \lambda_p \) and the normal potential \( U \) are equal to the astronomic coordinates and the actual potential in the actual gravity field \( \phi_p, \lambda_p \) and \( W_p \) (Moritz, 1964, p11). We have thus determined the equipotential surface of point Q \( U_Q = U_p = W_p \), which is on a normal level surface. The term level is used to describe each of the series of equipotential surfaces which 

"... share the intuitive and physical significance of the horizontal; and they share the geodetic importance of the plumb line because they are normal to it." (Heiskanen-Moritz, 1967, p49). The difference between the normal gravity at
(The nonsuperscripted set of points \( (P, p, Q) \)'s refer to the computation point. The superscripted set of points \( (p^n, p^n, q^n) \)'s refer generally to all other points on the surface profile.)

MODERN THEORY
point Q (\( \Delta Q \)) and the surface measurement (\( g_P \)) is called the gravity anomaly (\( \Delta g \)) of the Modern Theory. These gravity anomalies may be used to determine the distance \( \xi = PQ \), which is called the height anomaly (Moritz, 1964, p13), and are used in this study to determine the Deflections of the Vertical at the surface point P. Notice that the loci of the points p and Q do not necessarily lie on the same normal level surface but are represented by the lines \( \Sigma_p \) and \( \Sigma_Q \) respectively in Figure 1, i.e. any other point on the topographic surface would have subsequent representation by \( (p^N) \) on \( \Sigma_p \) and \( (Q^N) \) on \( \Sigma_Q \) which are not on the \( U=U_Q=U_P=W_p \) surface. The surface formed by the \( \Sigma_Q \)'s is called the telluroid (Hirvonen, 1959, p2). Molodensky used a similar surface located \( \xi \) above the reference ellipsoid and called it the quasi-geoid (Moritz, 1964, p38).

The actual direction of a plumb line (normal to the geoid) at a point (P) on the earth's surface may differ from the normal direction with respect to the earth-ellipsoid at that point (P) due to the effect of anomalous masses at varying distances and directions from the point on the earth's surface. If the difference between these directions is "... determined from gravity values evenly distributed over the whole of the earth's surface through the use of, e.g., the Vening-Meinesz formula ..." (Mueller-Rockie, 1966, p22), we calculate the Gravimetric Deflection of the Vertical. The Gravimetric Deflection of the Vertical at the earth's surface that is determined by the generalized Vening-Meinesz formula (which will be discussed later) is the Deflection of the Vertical referred to for the remainder of this study unless a different deflection of the vertical is specifically noted. Classical definitions
of the deflection of the vertical are mentioned here for completeness:

Absolute deflection of the vertical, which is the angle between the normal to the geoid and the normal to the earth-spheroid and which is usually identified with the Earth Ellipsoid (Ibid., p99) at a point on the geoid; Astro-Geodetic deflection of the vertical, which is the angle between the normal to some reference ellipsoid of a selected datum and the normal to the geoid at a point on the geoid; Topographic deflection of the vertical which is due solely to the changes in the mass distribution brought about by the various gravity reductions (also called the indirect effect on the deflections - Ibid., p23). The Deflection of the Vertical referring to the earth's surface will have the D and V in capital letters and the deflection of the vertical referring to the geoid will have the d and v in small case letters.

Moritz, (1964, pp14-17) shows the following development for the Deflections of the Vertical. Usually the deflection is given in terms of east-west (Prime Vertical) and north-south (Meridian) components and these components are represented by the Greek letters \( \eta \) and \( \xi \) respectively. The Deflection of the Vertical values on the actual surface at \( p \) are:

\[
\xi = \Phi_p - \Phi_P \quad \text{and} \quad \eta = (\lambda_p - \lambda_P) \cos \phi.
\]  

(1)

The disturbing potential (deviation of the actual from the normal) is:

\[
T = W_p - U_P.
\]  

(2)

The vector \( pP \) gives the deviation of the computation point (P) on the topographic surface \( (S) \) from the point \( (p) \) on the normal surface \( (\Sigma) \). The horizontal vector \( pQ \) has the components \( R\xi, R\eta \) (where \( R \) is the mean radius of curvature of the normal level surface - Moritz, 1966,
The normal potential at $P$ is $U_P$ which may be determined from:

$$U_P = U_Q + \frac{\partial U}{\partial n} = U_Q - \gamma \zeta \quad (3)$$

(where $\gamma = - \partial U / \partial n$ is the normal gravity at $Q$ and the derivative is taken along the ellipsoidal normal $n$, along which $h$ is measured — see Figure 2).

The disturbing potential at $P$ is:

$$T = W_P - U_Q + \gamma \zeta \quad (4)$$

since $(W_P - U_Q) = 0$, which will be discussed later:

$$T = \gamma \zeta \quad \text{and} \quad \zeta = T / \gamma \quad (5)$$

which relates $\zeta$ with $T$ and (5) is called the Bruns Theorem. The vertical derivative of $T$ is:

$$\left( \frac{\partial T}{\partial h} \right) = \left( \frac{\partial W}{\partial h} \right) - \left( \frac{\partial U}{\partial h} \right) = - \gamma P + \gamma P = - \gamma P + \left[ \gamma Q + \left( \gamma Y / \partial h \right) \zeta \right] = - \gamma P + \gamma Q + \left( \gamma Y / \partial h \right) \zeta \quad (6)$$

and then the gravity anomaly is defined as:

$$\Delta g = g_P - \gamma Q \quad (7)$$

and then from (6) and (7):

$$\left( \frac{\partial T}{\partial h} \right) = - \Delta g + \left( \gamma Y / \partial h \right) (T / \gamma) \quad \text{or} \quad \left( \frac{\partial T}{\partial h} \right) - \left( 1 / \gamma \right) \left( \gamma Y / \partial h \right) \frac{T}{\gamma} + \Delta g = 0 \quad (8)$$

which is known as the Fundamental Equation of Geodesy in that it relates the measured quantity $\Delta g$ to the unknown anomalous potential $T$ (Heiskanen-Moritz, 1967, p86).

The Deflections of the Vertical components on the normal level surface are:

$$\xi = -\partial \zeta / \partial x \quad \text{and} \quad \eta = -\partial \zeta / \partial y \quad (9)$$

where the $x$ and $y$ axes point north and east respectively. The
telluroid does not necessarily coincide with the normal level surface and Moritz (1964, p16) continues to show that:

\[
\xi = - \frac{\partial \phi}{\partial x_{\text{telluroid}}} - \frac{\partial \phi}{\partial y_{\text{telluroid}}},
\]

(10)

(where the angles of inclination of the telluroid with the horizontal are \(\beta_1\) and \(\beta_2\) in the north-south and east-west directions respectively and \(\partial h/\partial x = \tan \beta_1\) and \(\partial h/\partial y = \tan \beta_2\).)

The geodetic coordinates of the surface point determined from a Deflection of the Vertical solution at the surface of the earth may be compared with the geodetic coordinates determined from a deflection of the vertical solution at the geoid (Heiskanen-Moritz, 1967, pp315-317), which technique allows for a conversion from the surface to geoid deflections of the vertical. The Absolute deflection of the vertical is equivalent to the Gravimetric deflection of the vertical plus the Topographic deflection of the vertical and this sum is equivalent to the Astro-Geodetic deflection of the vertical plus a coordinate transformation which transforms the geodetic coordinates referred to a reference ellipsoid to a geocentric datum which has the same volume and flattening and center of mass as the geoid. Thus we are able to convert from one set of deflection values to the other if required.

The Moritz definition of the telluroid (Moritz, 1964, p13) is used for the purposes of this paper. Actually we utilize a different normal level surface of the computation surface (Figure 1) for each
computation point. Mather (1968, p517) states that the differences of potential \((U_Q - U_0)\) and geopotential \((W_P - W_0)\) should be assumed rather than \(W_0 = U_0\) a la Hirvonen (1959, p1) and Heiskanen and Moritz (1967, p83). Hirvonen, Heiskanen and Moritz (Ibid.) have recognized this discrepancy to be dependent upon the preciseness to which the parameters of the reference ellipsoid represent the parameters of the actual earth. Hirvonen discounts the value of discriminating between the geoid and the ellipsoid potentials in his derivation. Heiskanen and Moritz (1967, p114) mention that the computations of the geoid heights will be off a scale factor if the potentials on the geoid and the ellipsoid are not taken into account; however, they also show that the zero order deflections are not affected by this situation. The difference between \(U_0\) and \(W_0\) or \(W_P\) and \(U_Q\) should be used in equation (4) for geoid height determination. Because this paper deals mainly with the correction terms to be applied to the practical Deflection of the Vertical computation this correction recommended by Mather will not be used.

In the Classical Theory (Figure 2), the mass of topography existing above the geoid is reduced to within the geoid. This reduction should be accomplished without changing or disturbing the potential of the computation point relative to all other points. In practice several reduction or regularization techniques are used but not without encountering theoretical and/or practical difficulties. The condition that the disturbing potential satisfies Laplace's equation \((\Delta T=0)\) and is harmonic on the geoid should be enforced by the reduction technique used (Heiskanen-Moritz,
1967, p5, p86). This condition must be met for the solutions of the formulae which determine the geoid-ellipsoid separation (N by Stokes formula) and the deflections of the vertical at the geoid (\( \xi \) and \( \eta \) by the Vening-Meinesz Formula). In order to determine the surface of the earth in the classical sense, it is also necessary to determine the orthometric height (H) which is the distance along the curved plumb line from the ground to the geoid in addition to the geoid height (N). The orthometric height "... depends on the gravity between the geoid and the ground, which must be computed using estimated values for the rock density in the earth's crust" (Moritz, 1964, p61).

Although this density is seldom precisely known, the assumptions made usually have an effect of less than 10 centimeters (Ibid.).

If we compute the angular difference between the normal to the ellipsoid and the curved plumb line at the regularized geoid, we obtain the Deflections of the Vertical for the Classical Theory. The regularized geoid is one of the various surfaces used to approximate the geoid that is obtained from Stokes formula using gravity anomalies in the calculations. A different cogeoid is thus obtained for each reduction system. These cogeoids give various approximations to the geoid of a regularized earth in which all of the masses are transferred to within the geoid to varying degrees of accuracy. The distance between the geoid and a cogeoid is called the indirect effect (Mueller-Rockie, 1966, p18).

The most commonly described reduction techniques are the following: the free air reduction, which condenses the masses
Figure 2

Topographic Surface

Cugeoid Surface

\( U = U \)

\( U = U_0 \)

\( \delta N \)

\( \delta h \)

Referene Ellipsoid Surface

Geoid (\( W = W_0 \))

Classical Theory
above the reference surface into an infinitely thin layer on the surface, and which also reduces the gravity from the terrain to the reference surface (Ibid., p36); the Bouguer reduction, which removes all masses above the reference surface and then reduces the gravity from the terrain to the reference surface (Ibid., p13); and the isostatic reduction, which removes the masses above the geoid for land masses and removes the deficiencies in the ocean areas to the localities inside of the geoid where their compensation is defined to exist in a prescribed system of isostasy; the Rudzki reduction transfers the masses from outside of to inside the geoid so that the potential on its surface is unchanged, thus constraining the cogeoid to equal the geoid; however, a different mass is transferred which changes the total mass of the earth (Ibid., p93). Further details on these reductions may be found in Heiskanen and Vening-Meinesz (1958), and in Heiskanen and Moritz (1967). The importance of these reductions is that they reduce the gravity (g) used in the Classical Theory to a cogeoid. The indirect effect (effect for the geoid and the cogeoid not coinciding) must be determined to fit each reduction in the Classical Theory.

In the Modern Theory, the gravity anomaly used is the difference between the gravity at the earth's surface and the normal gravity computed at the telluroid. The normal gravity at the telluroid is computed by the Normal Gravity Formula and a Free Air reduction is now applied upward (Ibid., p29). An integral equation may be transformed in the Modern Theory to fit each reduction method (Moritz, 1964, p41).
In the solutions used in this study we shall be dealing with
the following specific gravity anomalies:

1. The Simple Bouguer Anomaly excluding any terrain correction,
   which was one product of Rapp's prediction program which will
   be discussed later in Chapter 3 (Rapp, 1964, p29).

2. The Free Air Anomaly excluding any terrain correction,
   which also was an output from Rapp's program;
   \[
   \Delta g_{\text{Free Air Bouguer}} = \Delta g + 0.1119 H \quad (11)
   \]

3. Terrain Corrected Bouguer Anomaly (Moritz, 1969b, p32);
   \[
   \Delta g_{\text{Bouguer Free Air}} = \Delta g - 2\pi kH_p\rho - C \quad (12)
   \]
   where \( C = \int \left( \frac{1}{\mathbf{l}_0} - \frac{1}{\mathbf{l}} \right) d\sigma \)
   in which \( k = \) gravitational constant \( (6.67 \times 10^{-9} \text{cm g}^{-1} \text{sec}^{-2}) \)
   \( \rho = \) density (approximately \( 2.67 \text{ gm/cm}^3 \) )
   \( \mathbf{l}_0 = 2R \sin \frac{\Psi}{2} \) where \( \Psi \) is the spherical distance
   between the computation point and the anomaly
   location and \( R \) is the mean radius of the earth.

4. The Faye Anomaly which is the sum of the Free Air Anomaly
   and the Terrain Correction as in 3 above;
   \[
   \Delta g_{\text{Faye Free Air}} = \Delta g + C \quad (13)
   \]

5. The Complete Topographic Anomaly which consists of the
   Free Air Anomaly and two reductions for the direct effect
   and the indirect effect and also this anomaly is analogous
   to the conventional isostatic anomaly but refers to the
earth's surface and can be represented by the following formula (Moritz, 1969b, p29):

\[ \Delta g^c = \Delta g_{\text{Free Air}} - 2\pi k\rho H_p + C + \frac{3}{2R} V_T \chi \]  

where \( V_T = 4\pi k\rho H_p R + k\rho R^2 \int \ln \frac{\rho + h - h_p}{\rho_0} \, d\sigma \)

which represents the indirect effect and the other symbols are the same as before.

Figures 3 and 4 which are based on Heiskanen and Moritz (1967, p292) and Brovar (1964, p256) illustrate respectively represent a general description of the "modern" approach used by Molodensky (Molodensky et al., 1962a) and Hirvonen (1959). In Figures 3 and 4 the orthometric height is \( H^* \), the normal height is \( H \), the separation between the geoid and ellipsoid (geoid undulation) is \( N \) and the height anomaly is \( \xi \). Conceptually, Figures 3 and 4 both represent the same modern statement of the problem. Descriptively, either the telluroid (\( \xi \) distance from the surface) or quasigeoid (\( \xi \) distance from the ellipsoid) may be desired for illustrative purposes.

Hirvonen (1959, p1) claims that the telluroid (measuring the height anomalies from the surface downward) is the most desirable since it does not confuse the real physical meaning of the geoid as could the quasigeoid (measuring the height anomaly up from the reference surface) which is close to the geoid (Hirvonen, 1959, p15). The normal height of Molodensky differs slightly from that of Hirvonen because of the computation technique used (Krakiwsky and Mueller, 1965, p27).

This chapter has broadly described some of the terminology,
equations, and rationale for an understanding of the Modern and Classical Theories. A closer look at various solutions for the modern approach and an insight into some practical applications will occupy the remainder of this study.
Figure 3

Topographic Surface

Telluroid

H

H*

Geoid

Reference Ellipsoid

Hirvonen's Telluroid
Figure 4

Topographic Surface

Geoid

Quasigeoid

Reference Ellipsoid

Molodensky's Quasigeoid

(All symbols depicted here are representative of the same numerical values as Figure 3)
CHAPTER II
BACKGROUND DISCUSSION OF THE DEFLections OF THE
VERTICAL AND THE GRAVITY ANOMALY CORRECTION
TERMS USED IN THIS STUDY

This chapter will discuss the Deflection of the vertical Formula and the Linear, Nonlinear and Iterative solutions as derived by Moritz (1964, 1966c and 1969).

The Vening—Meinesz Formula for the classical theory which was referred to earlier (p8) regarding the computation of the gravimetric deflection of the vertical is (Heiskanen—Moritz, 1967, p114):

\[
\eta = \frac{1}{4G} \int \frac{dS}{d\psi} \left[ \frac{\cos \alpha}{\sin \alpha} \right] d\sigma
\]

where: \( G \) is the mean value of gravity over the earth; \( \alpha \) represents the azimuth of the surface element \( d\sigma \) from the computation point;

\[
dS/d\psi = - \left[ \cos (\Psi/2)/2\sin^2(\Psi/2) + 8\sin \Psi - 6 \cos (\Psi/2) - 3[1-\sin(\Psi/2)/\sin (\Psi)] + 3\sin \Psi \ln \left[ \sin (\Psi/2) + \sin^2(\Psi/2) \right] \right] ;
\]

\[
\psi = \cos^{-1} \left[ \sin \phi \sin \phi' + \cos \phi \cos \phi' \cos (\lambda' - \lambda) \right] ;
\]
ψ is the central angle between the fixed and moving points;

\[ \alpha = \arctan \left[ \cos \psi' \sin (\lambda' - \lambda) / (\cos \psi \sin \phi' - \sin \phi \cos \phi' \cos \lambda' - \lambda) \right]. \]

I will now show how Moritz develops a similar equation for use in the modern theory.

**Linear Solutions**

For the Linear Solutions of the Geodetic Boundary Value Problem, Moritz (1966c, pi) deals with what he calls "Molodensky's Problem," which is the gravimetric determination of the earth's physical surface and external gravity field from free air anomalies. The linear solutions are those in which second and higher powers of the elevation (which must be considered small when compared to the dimensions of the earth) and of the terrain inclination are systematically neglected.

Moritz (Ibid., p43) shows that the integral equation for the unknown surface density \( \phi \) is:

\[
2 \pi \phi \cos \beta - \oint \left[ (\partial / \partial h_p)(1/\lambda) - (1 / \gamma) (\partial \gamma / \partial h) (1/\lambda) \right] \phi \, ds = \Delta g ,
\]

where: \( S \) = any known surface approximating the physical surface of the earth such as the normal surface or telluroid;
\( \Delta g \) = free air gravity anomaly;
\( \phi \) = unknown surface density;
\[ h = \text{elevation of the terrain}; \]
\[ \gamma = \text{normal gravity}; \]
\[ \beta = \text{angle of inclination of the topography}; \]
\[ \ell = \text{chord distance between the computation point and gravity anomaly location.} \]

In order to simplify this integral equation for ease of solving, certain approximations are applied by Moritz (Ibid., pp25-35). The first approximation is the spherical approximation, which may be visualized as plotting the heights, \( h \), above the reference ellipsoid as heights above a mean sphere of radius \( R \). This approximation changes the integral equation (16) to the following form:

\[
2\pi \phi (\cos \beta) - \iint \left[ \left( \frac{3}{2\rho} \ell + \left( r^2 - r_p^2 \right) / \left( 2\rho \ell \right) \right) r^2 \sec \phi \right] d\phi d\theta = \Delta g , \quad (17)
\]

where: \( R = \text{mean radius of the earth}; \)
\( r = R + h ; \)
\( r_p = R + h_p . \)

The next approximation used is called the planar approximation and can be used in expressions where an omission of the elevation term causes an error on the order of \( (h/R) \), which causes a relative error of less than 0.1 percent (Ibid., p31). Moritz also states that \( (h) \) can only be neglected in formulae that do not enter through the horizontal derivatives such as in \( \tan \beta \).
approximation of (16) where \( r = r_p = R \) is:

\[
2\pi \phi \cos \beta - \int \left[ \frac{(3R/2J) + R^2(h-h_p)}{L^3} \right] \sec \beta \phi \, d\sigma = \Delta g .
\]  

(18)

If the expressions for the planar solution are expanded into power series retaining only the linear terms, the result is a linear approximation. This linear approximation takes into account the main part of the topography and should be sufficient, however, the use of this approximation may cause an error of as much as 50% in very rugged terrain. The linear approximation of (16) where \( \phi = \phi_0 \) and \( \cos \beta = \sec \beta \) is:

\[
2\pi \phi - \frac{(3R/2J)}{\int \left[ \phi/\phi_0 \right] \, d\sigma} = \Delta g + R^2 \int \left[ \frac{(h-h_p)}{L^3} \right] \, d\sigma ,
\]  

(19)

or

\[
\phi - \frac{(3R/4\pi)}{\int \left[ \phi/\phi_0 \right] \, d\sigma} = \frac{(1/2\pi)}{\left( \Delta g + G_i \right)} ,
\]  

(20)

where: \( G_i = R^2 \int \left[ \frac{(h-h_p)}{L^3} \right] \, d\phi \, d\sigma \),

(21)

and since \( G_i \) is a small correction term, Moritz approximates \( \phi \) in formula (20) by \( (\Delta g + 3T/2R) \) and also shows that neglecting the \( 3T/2R \) term is consistent with a planar approximation, therefore:

\[
G_i = \frac{R^2}{2\pi} \int \left[ \frac{(h-h_p)}{L^3} \right] \Delta g \, d\sigma .
\]  

(22)
The Deflection of the Vertical formula that results from this solution is (Ibid., p96):

\[
\xi = \frac{1}{(4\pi G)} \int \frac{(\Delta g+G)}{(dS/d\psi)} \left\{ \cos \alpha \right\} \left( -\frac{\Delta g}{G} \right) \tan \beta_1 \frac{\tan \beta_2}{(dS/d\psi)} \right)^2 \right)^{1/2} \] 

(23)

where \( \tan \beta_1 \) is the horizontal derivative of the topography in the \( \xi \) direction and \( \tan \beta_2 \) is the horizontal derivative of the topography in the \( \eta \) direction (Ibid., p29).

We have now demonstrated in general terms how Moritz arrived at the linear correction term. In order to eliminate the inclination of the topography (\( \tan \beta_1 \) term) in the solution, he ingeniously adopted a point level solution (a solution for the specific point \( \Lambda \) on the earth's surface where the Deflection of the Vertical or the geoid height is to be computed). Moritz (Ibid., p62) gives:

\[
a \frac{\Delta g}{\partial h} = \frac{R^2}{2\pi} \int \frac{[(\Delta g-\Delta g_p)/\lambda^3]}{\xi} \ d\xi , \] 

(24)

and therefore the correction term to the free air anomaly for use in equation (23) becomes:

\[
-[(\partial \Delta g/\partial h)(h-h_\Lambda)]G = (h-h_\Lambda)(R^2/2\pi) \int \frac{[(\Delta g-\Delta g_p)/\lambda^3]}{\xi} \ d\xi , \] 

(25)

and this is referred to as the Gradient Solution. Slightly different versions attributed to various
approaches are also given as follows (Ibid., p102):

Molodensky type solution:

$$G_1 = \frac{R}{2\pi} \iiint (h-r)(\Delta g - \Delta g_A) / (\ell^3_0) \, d\sigma,$$

(26)

Pellinen type solution;

$$G_2' = \frac{R}{2\pi} \iiint (h-r)(\Delta g - \Delta g_g) / (\ell^3_0) \, d\sigma,$$

(27)

Terrain Correction type solution (Ibid., p88):

$$G_1' = \frac{1}{2} k \rho R \iiint (h-r)^2 / (\ell^3_0) \, d\sigma.$$

(28)

(In deriving this equation, Moritz (Ibid.) assumed the relationship of formula (11) was valid).

The Gradient Solution is recommended for use in the Linear Approximation case by Moritz (Ibid., p121) since the correction terms of the other solutions, such as Molodensky's are strongly dependent upon the irregularities of the terrain.

Nonlinear Solutions

Quoting the recent report of Uotila (1971, pIUGG 13), ... For higher accuracy, especially in computations of the deflection of the vertical, however, approximations of a higher order are required. In his new reports Moritz uses instead of integral equations, the method of analytical continuation using power series, extending it to approximations of an arbitrary order. In this manner, he has not only recovered the essential results of the second order theory of Arnold, but also has obtained a complete series solution of a new type that he has shown to be equivalent to the Molodensky-Brovar series, but simpler and practically more convenient. This equivalence gives a physical
explanation of the divergence of the Molodensky series. The exclusion of topographic masses to improve convergence is also discussed, and the computation of the formulae for height anomalies and deflections of the vertical are given.

The Deflection of the Vertical formula to which the correction terms shown in this section will refer is as shown by Moritz (1969b, p37):

\[ \xi = \frac{1}{4 \pi G} \int \int \left( \Delta g + g_1 + g_2 \right) \frac{ds}{d\nu} \left\{ \cos \alpha \right\} d\sigma \]  

(29)

In the solutions that follow, I have followed Moritz's example and use the spherical approximation and the planar approximation (Ibid., p5). He shows that the sea-level anomaly \( \Delta g^* \) in terms of the ground level anomaly \( \Delta g \) can be expressed as follows (Ibid.):

\[ \Delta g^* = \Delta g - (\partial \Delta g / \partial H) H - \left( \frac{H^2}{2} \right) (\partial^2 \Delta g / \partial H^2) \]  

(30)

where:

\[ (\partial \Delta g / \partial H) = \left( \frac{R^2}{2 \pi} \right) \int \int (\Delta g - \Delta g_p) / (L_0^3) d\sigma; \]  

(31)

and transposing into Moritz's notation:

\[ \Delta g^* = \Delta g - HL_1(\Delta g) + HL_1 [HL_1(\Delta g)] - \frac{H^2}{2} L_2(\Delta g) \]  

(32)

where: \( H \) is the elevation above sea level;

\[ L_1(\Delta g) = \left( \frac{R^2}{2 \pi} \right) \int \int (\Delta g - \Delta g_p) / (L_0^3) d\sigma; \]  

(33)

\[ HL_1 [HL_1(\Delta g)] = \left( \frac{HR^2}{2 \pi} \right) \int \int \left[ \left( \frac{HR^2}{2 \pi} \right) \int \int (\Delta g - \Delta g_p) / (L_0^3) d\sigma \right] d\sigma; \]  

(34)

\[ L_2(\Delta g) = \frac{1}{2} L_1^2(\Delta g). \]  

(35)
Equations (33) through (35) show the expressions obtained for \( L_1(Ag) \) and \( L_2(Ag) \) in terms of the surface values of the anomalies \( Ag \). The free air anomalies refer to that non-level (not equipotential) surface of the earth (Ibid., p7). \( L_1(Ag) \) has the value \( \frac{\Delta g}{\Delta h} \) only if \( Ag \) refers to a level surface, therefore, \( Ag \) is continued to the level surface through \( A \).

Moritz continues to show that the anomaly may analogously be continued to point level as follows (Ibid., p8):

\[
\Delta g' = \Delta g - zL_1(\Delta g) + zL_1[zL_1(\Delta g)] - z^2L_2(\Delta g) \tag{36}
\]

where:

\[
z = H - H_A \tag{37}
\]

and \( H \) is the elevation of the computation point.

Then Moritz shows that a recursive process may be continued to any higher order desired (Ibid., pp20-22):

\[
\Delta g' = \Delta g + g_1 + g_2 + g_3 + \ldots \tag{38}
\]

where:

\[
g_1 = -zL_1(\Delta g),
\]

\[
g_2 = -z^2L_2(\Delta g) + zL_1[zL_1(\Delta g)],
\]

\[
g_3 = -z^3L_3(\Delta g) + zL_1[z^2L_2(\Delta g)] + z^2L_2[zL_1(\Delta g)]
\]

\[
- zL_1[zL_1[zL_1(\Delta g)]]; \tag{39}
\]

which agrees to the second order with (32) or more
generally to any order as follows (Ibid.):

\[ g_m = D_m (\Delta g), \]

and \( D_m(\Delta g) = -U_m(\Delta g) - \sum_{r=1}^{m-1} D_{m-r} [U_r(\Delta g)]; \) (40)

where:

\[ U_m(\Delta g) = z^m L_m (\Delta g); \]

\[ z = H - H_A \text{ as before;} \]

\[ L_m(\Delta g) = (1/n) L_1 [L_{m-1}(\Delta g)]; \]

and \( L_1 (\Delta g) \) is as in formula (33) before.

If the anomaly used for \( \Delta g \) in Formula (40) is the free air anomaly, then \( g_1 \) thru \( g_m \) are computed using free air anomalies, however, if the anomaly used for \( \Delta g \) in Formula (40) is the Faye anomaly of Formula (13) then \( g_1 \) thru \( g_m \) are computed using the complete topographic anomalies of Formula (14) or in the Planar approximation the terrain corrected Bouguer anomaly of Formula (12) is used (Ibid., p37).

**Iterative Solution**

The planar iterative solution of Bjerhammer is shown by Heiskanen-Moritz (1967, p318) to use the following Iterative Solution:

\[ \Delta g^{(h)} = \Delta g, \] (41)
\[ \Delta g^* = \Delta g_p - \frac{t}{4} \int_0^\infty \frac{1}{D^3} \Delta g_p \; d\sigma, \quad (42) \]

where: \( D = \frac{R}{r} \),
\( r = R + h_p \),
\( t = R/r \),
\( \Delta g^* \) is the sea level anomaly,
\( \Delta g_p = \Delta g' \) is the point level anomaly.

An iterative solution is generally acceptable if the surface in question is continuous, not only in the physical sense but also in the mathematical sense. On the computer this means that after each iteration, when a correction term is added to the previous calculation \( (\Delta g^\omega) \), the anomaly field should be approaching a limiting set of values, or put another way, the corrections should be constantly becoming smaller in their absolute values.

The Bjerhammer technique generates a set of fictitious gravity anomalies \( (\Delta g^\omega) \) on the sphere representing the reference ellipsoid (Ibid., p58). The gravity anomalies \( (\Delta g) \) on the earth's surface as obtained by measurement are then related to \( (\Delta g^\omega) \) by the "upward continuation integral" (e.g., Heiskanen-Moritz, 1967, sec. 6-8):

\[ \Delta g_p = \Delta g' = \frac{t^2 (1-t^2)}{4} \int_0^\infty \frac{\Delta g^*}{D^3} \; d\sigma, \quad (43) \]

where: \( t = R/r \), \( r = R + h_p \).
Then the \((\Delta g_j^\prime)\) values from equation (42) are used in the
following Deflection of the Vertical formula (Ibid., p320):

\[
\frac{\xi}{\eta} = \frac{t}{4\pi \sigma} \int_{\sigma'} \int \Delta g^\prime \frac{dS(\eta, \psi)}{d\psi} \left\{ \frac{\cos \alpha}{\sin \alpha} \right\} d\sigma' \tag{44}
\]

where: \(t = \frac{R}{r}\) as in (42)

\(\xi = \gamma \) (Ibid., p309).

Bjerhammer (1969, pp173-203) illustrates a different but equivalent version of this solution.

In the iterative solution used in this study \((h_p)\)
(in the formula for \(r\) and \(t\)) was replaced with \(z = h - h_p\),
which reduced the anomaly to the spherical surface thru
the computation point. Then the spherical formula (44)
was used for the Deflection of the Vertical at the earth's
surface, however, \(\Delta g_j^\prime\) was replaced with \(\Delta q^\prime\). The specific
equations which were used are those given by Moritz (1966c,
p58) as modified for the solution at the earth's surface:

\[
\Delta g' = \Delta g, \quad (45)
\]

\[
\Delta g^\prime = \Delta g - z(R^3/2\pi) \left[ \frac{\Delta q^\prime - \Delta g_p^\prime}{f_0^3} \right] d\sigma', \quad (46)
\]

(where \(\Delta g\) is the uncorrected point level free air
anomaly and \(z\) is the height above or below the
computation point).

Equation (44) thus becomes:

\[
\frac{\xi}{\eta} = \frac{1}{4\pi G} \int_{\sigma'} \int \Delta g' \frac{dS}{d\psi} \left\{ \frac{\cos \alpha}{\sin \alpha} \right\} d\sigma', \quad (47)
\]

since \(t=1\) and \(R=r\) outside of the geoid (Heiskanen-Moritz, 1967, p235).
Previous Correction Term Investigations

The International Association of Geodesy formed Special Group 16 in 1962 for the purpose of comparing the gravimetric Deflections of the Vertical over a test area in the West Alps. Five methods are being investigated by various groups. The size area \((d\delta)\) used for averaging the values of the gravity anomalies \((\Delta g)\) and the elevations \((H)\) in the following computations consists of a set of geographic trapezoids 5' in longitude and 5' in latitude within a circle with a 200 km diameter. The anomalies used were formed by Rapp's Prediction program (Rapp, 1966) which will be discussed in Chapter III.

Special Group 16 consisted of Pellinen, Bursa, Pick, and Pola as principal investigators (Tengstrom, 1969, pp1-20). Pellinen (1964, p327) suggested the following gravity anomaly correction term (Ibid.):

\[
G_p = \frac{R^2}{2\pi} \int \int d\delta \frac{(H-H_p)(\Delta g-\Delta g_p)}{d\delta} ,
\]

which he used in an expression for the disturbing potential which is shown by Moritz (1966c, p84) to be of the following form:

\[
T = \frac{R}{4\pi} \int \int (\Delta g + C') S(\psi) d\delta + t ,
\]

where \(T\) is the disturbing potential, \(S(\psi)\) is the Stokes function, and \(t\) is a correction term. Moritz (1966c, p88) proves the Pellinen correction to be the same as the Terrain Correction for the assumptions that the Free Air Anomalies can be expressed in a linear relationship:

\[
\Delta g = a + bH ,
\]

where \(a\) is the Bouguer anomaly and \(b\) is the Bouguer gradient. For the specific assumptions that the Free Air anomalies are correlated
with elevation Moritz (1966c, p88) proves Formula (48) to be equivalent to:

$$
G' = \left( - k \int \phi \, d \phi \right) \left( \frac{H - H_p}{\lambda^3} \right) d \phi
$$

where $k$ = gravitational constant and $\phi$ = density as before.

For the assumptions just mentioned, Moritz (1966c, p101) shows that the correction term $t$ vanishes for the Deflection of the Vertical solutions and that $G'_p$ can be used in the same manner as $G'$ for a correction to each gravity anomaly. Pellinen (1969, p102) shows that Arnold's method for determining the gravity anomaly correction terms (which will be discussed later in this chapter) provides the same results as Molodensky's first approximation formula where:

$$
G_i = \left( \frac{R}{2\pi} \right) \left( \frac{1}{\lambda^3} \right) \int \frac{(\phi) (H - H_p)}{d \phi} d \phi
$$

where $2\pi(\phi) = \Delta g + \left( \frac{3\pi}{2R} \right)$ from which $(\phi)$ may be obtained.

These formulae have achieved accuracies up to the order of $\Delta g(H/R)$ in the West Alps. He also shows (Ibid.) solutions at sea level where the topography has been removed before applying the $G_i$ correction. Pellinen then recommends the following type of formula for the gravity anomaly correction term (Pellinen, 1965c, p126 and 1968, p347):

$$
G_i = \frac{R}{2\pi} \left( \frac{1}{\lambda^3} \right) \int \frac{(\Delta g - \Delta g_p)(H - H_p)}{d \phi} d \phi
$$

Pellinen (1965b, p97) considers the 2nd ($G_2$) and third ($G_3$) correction terms of Molodensky and explains that $G_{n=1}$ must be known everywhere for the computations. Pellinen (Ibid.) recommends that rough estimates of $G_2$ and $G_3$ be made once and for all for large areas. Bursa (1964c, p2) mentions that since the distance from the computation point to the
moving point \( l \) is in the denominator, it decreases the value of the double integral rapidly (proportional to \( l^{-3} \)) and that we can proceed from the sphere to the plane and use a formula such as Formula (53).

Bursa, in effect, sets \( \Delta g_p \) of Formula (48) to zero and adds another term to the deflection formula for moving from a zero elevation to the earth's surface. For computational purposes Bursa (Ibid.) divides the computation area into three parts: 0-5 kms, 5-42.6 kms, and beyond 42.6 kms. According to Tengstrom (1969, pI-22), Bursa was the first investigator to discover a gross error in an astronomical latitude (14°67) by the use of Molodensky's method inclusive of the \( G \) term. For the inner area (0-5 kms), he uses a Gauss numerical integration method (Bursa, 1967, pp9-11), which will be mentioned in more detail later. Tengstrom (1969, pI-23) recommends the investigations of Pick and Pola (Pick, 1964 and Pick and Pola, 1968) as the best contributions to the test area studies up to the 1969 time period.

They used the free air anomalies which were obtained from Bouguer anomalies (since the Bouguer anomaly field is smoother and therefore the interpolation procedures are simplified). Then a correction to the free air anomaly is included for the terrain as in equation (12). The corrected anomalies are then used in Deflection of the Vertical and other gravimetric computations. The value of \( G \) [Formula (53)] represents also the results of Pick and Pola or as reported by Tengstrom (1969, pI-23) to be "... closely the same." In a letter from Pick (1968b) the equations used in his investigation were the same as used by Bursa [Formula (53)] for the first approximation. He also revealed that his value of \( G \) was about \( \pm 60 \) mgals maximum for
1' x 1' squares and that the influence of the first correction term (G₁) was less than 2" when used in calculating the Deflections of the Vertical. He used 5' x 5' squares out to a 50 km radius but then recommends that the density of the gravity observations be increased to at least one point per 5 square kilometers within an 80 km radius from the computation point. Pick also uses a Gauss numerical integration method which is similar to that of Bursa (1967, pp9-11) for the inner 2.73 km radius where he recommends that at least 5 to 10 gravimetric stations be located. Pick uses the second order terms only if the gravity data field is very dense since, as he states, they are meaningless otherwise. He basically uses Formula (23) for his solution.

The second group conducting an investigation in the West Alps, has utilized the techniques of Arnold (1958-1965-1968). Moritz (1967, p22) and Pellinen (1968, p347) use the equations of Arnold as a jumping off point for further discussions. Arnold's earlier formulae (Moritz, 1966c, pp73-74) for the integral equations contained corrections which presupposed knowledge of the Deflection of the Vertical Field is:

\[ T = \left( \frac{R}{4 \pi} \right) \sum_{\sigma} \left[ \Delta g - \gamma \left( \frac{\xi}{\tan \beta} + \eta \tan \beta \right) \right] S(\psi) d \sigma \]
\[ + \left( \frac{R}{2 \pi} \right) \sum_{\sigma} \left[ h \frac{b'}{b} \gamma \left( \frac{\cos \alpha}{\tan \beta} + \eta \sin \alpha \right) \right] d \sigma \]  

(54)

however, his most recent studies use as a correction term to the Vening-Meinesz formula the following (Tengstrom, 1969, p1-23):

\[ \frac{d \xi}{d \eta} = - \left( \frac{R}{2 \pi} \right) \int \frac{K G (\Delta g)}{\sigma} \left\{ \frac{\sin \alpha}{\cos \alpha} \right\} d \sigma d A \]  

(55)

(\text{where } KG(\Delta g) = - (H_Q - H_P)/(2 \pi) \sum_{\sigma} \int \frac{\Delta g_f - (\Delta g)_0}{b^3} dS)  

(56)
and $P$ is the computation point and $Q$ is the moving point
and $\Delta g_f$ is the surface free air anomaly),
which is actually similar to the previously discussed formulae for
the first order approximations.

The third group investigating in the Alps is under the
leadership of Bjerhammer. Bjerhammer (1969, pp173-203) reduces
the surface observations downward to a spherical reference surface
closely approximating the geoid. Bjerhammer discounts that there
can be any analytical continuation of gravity within the earth.
His solution for the gravity anomalies reduced to the spherical
reference surface recognizes that gravity is usually only known
from measurements at discrete points on the surface of the earth;
however, he integrates over finite surface elements in his
solution. For the solution he utilizes an iterative procedure after
having solved for the values of the surface elements by a system of
linear equations (Ibid., pp180-181). Bjerhammer's method, repre-
senting a different type of Modern Theory solution at the topographic
surface is of interest for a comparison with the results of the
other solutions mentioned thus far by Molodensky, Moritz etc. The
computational technique of the Bjerhammer method is analogous to
to the so called Downward Continuation solution (Heiskanen-Moritz,

De Graaff-Hunter's Model Earth Method is also being investigated
as part of the Alps study. Pellinen (1962, p61) suggests the use
of the Graaff-Hunter "Model Earth" anomalies for the gravity obser-
vation stations which are situated far from each other. In essence
the "Model Earth" anomalies contain smoothed out terrain corrections in the classical sense. Strange and Woollard (1964, p64) comment that the theoretical and computational problems of the Classical Method are not solved by this technique. Heiskanen and Moritz (1967, p330) mention that through the application of a smoothing procedure or an isostatic type reduction, the "modern" Soviet scientists are reconciled to using an almost classical technique.

The final solution being investigated by the West Alps Study Group uses Rudzki anomalies, and Tengstrom (1969, p1-24) is the chief investigator. As is stated by Heiskanen and Moritz (1967, p144-145), "Since the indirect effect is zero, the cogeoid of Rudzki coincides with the actual geoid, but the gravity field outside of the earth is changed. In addition, the Rudzki reduction does not correspond to a geophysically meaningful model." Strange and Woollard (1964, p75) report that this anomaly is useful for interpolation purposes over a short distance only and that they do not believe there is sufficient justification for the additional effort involved in the computation.

The results of the five groups should be very interesting for comparison purposes, however, the final computations are awaiting a densification of the gravity measuring station network.

Several additional Soviet authors have been active in recent investigations which utilize the Modern Theory. In 1962 Molodensky (1962, p29) was very critical of other authors for building assumptions into the solution of the integral equation for "... repeating the deducing rather than simplifying the known exact integral equation."
Today we find these assumptions evident in their current literature (Pellinen, 1968, p345). This does not mean to imply that we should remove the higher order approximation terms from our theoretical knowledge but that we should use in practice only those terms which are meaningful to our results.

Investigations also have been conducted by Koch (1967a-1967b-1968) into the theory and into the practical applications with models. He reaches conclusions that as the inclination of the terrain approaches 45 degrees, the solutions for the Deflections of the Vertical become weaker or nonexistent and that instead of sophisticated numerical integration techniques, a reduction of the integration square size should be accomplished.

Writing about various aspects of gravimetric leveling in the German Alps, Wrobel (1967) discusses integration errors and decided upon square sizes of approximately 1.5 and 6' out to 44 kms. He showed that a standard error of 0.45 mgals for the Alps anomaly field resulted in an approximate 0.1 error in the Deflection of the Vertical components. The average anomaly for the German Alps was 120 mgals. He decided that the Bouguer point anomaly values resulted in less error than averaged Bouguer anomaly values. He is considerably concerned with the determination of the density of the topography which as discussed by Heiskanen-Moritz (1967, p319) need not be known. The linearity of the anomaly field was the basic criteria for the size of his innermost area. He determined that at least 8 point gravity anomalies were required to determine the horizontal derivatives.
Mather (1970, p90) presented a method for the solution in the innermost area, which is essentially the same as Moritz's Nonlinear Technique.

An approach has also been recommended by Baussus von Luetzow (1970) which is nothing more than Moritz's solution for the surface anomalies reduced to the level of the computation point.

Summary

This chapter has provided a background and a justification for the spherical Formula (47)'s use as the basic Deflection of the Vertical Formula for the investigation of the Linear, Nonlinear and Iterative Solutions in this study. The value of $\Delta g'$ is determined differently for each of the Solutions: for the Linear Solution $\Delta g' = \Delta g + (G_1 \text{ or } G_\text{p} \text{ or } G' \text{ or } G'\text{ [Formulae (25-28)]})$; for the Nonlinear Solution $\Delta g' = \Delta g + g_1 + g_2 \text{ [Formula (29)]}$; for the Iterative Solution $\Delta g' = \Delta g (i+1)' \text{ [Formulae (45-46)]}$. 
CHAPTER III

ANOMALY FIELD COMPUTATION PROCEDURE

In this experiment we shall test various correction terms discussed in Chapter II and the resulting corrected anomalies will be used for comparative determinations of Deflections of the Vertical. The gravity observations and elevations used were the best available for a 6 degree by 6 degree area centered on Pikes Peak, Colorado. A realistic model of an anomaly field for this rugged area on the Front Range of the Colorado Rockies will be used to compare the results of several Deflection of the Vertical solutions, which were previously discussed. The Aeronautical Chart and Information Center furnished approximately 18,000 gravity observations in the test area as well as the mean elevations for all 5' x 5' squares (the term square actually refers to the geographic area bounded by lines of constant latitude and longitude and more approximates a trapazoid, however, the terms squares and blocks are common in the literature and will be used interchangeably throughout this study) and for about one-half of the 37.5 squares that were used in this study. The remainder of the required mean elevations were calculated using the mean of 64 evenly spaced point values read from USGS 1:24,000 topo sheets as discussed by Smith (1963). The need for mean gravity anomalies is apparent when the computation of the Deflections of the Vertical and the correction
terms is considered. The numerical integration of formulae such as (25) and (26) involves the selection of the appropriate sized small blocks, represented by $d\sigma$ in the formulae, which will give results sufficiently close to those obtained by using an infinite number of infinitesimal blocks. Once the appropriate sized block has been determined, it is imperative to obtain a "mean value anomaly" from those point anomalies available from gravity surveys in each block. The accuracy of these mean values is of course dependent upon the quantity, quality, and geographical distribution (spacing between/density) of the survey point data. The point values used in this study were a composite of those available at the Ohio State University Department of Geodetic Science's Gravity Library and the Aeronautical Chart and Information Center's Gravity Branch Library. Since some of the 37'5 elevations were also available from the study of Beers (1965), it was decided to begin using 5'x 5' and 37'5 x 37'5 block sizes.

If we have available a field of closely spaced gravity anomalies, then we may integrate over the area of this field and obtain mean gravity anomalies by the following formula:

$$\Delta \bar{g} = \frac{1}{A} \int \int A \Delta g \ dA$$

(57)

(where $A$ is the area of the square).

The mean anomalies which we are interested in predicting are the Bouguer anomaly and the free air anomaly which has been shown to be closely correlated with the terrain on a local basis. Because of this unwanted correlation it is desirable to predict values for the mean Bouguer anomalies, which are not as distinctively correlated with the terrain (Uotila, 1960), and then to convert the results
to free air anomalies using the mean elevations of the squares
and an assumed mean density for the topography.

We may determine the mean anomaly for each block (Rapp, 1966a)
from the following formula:

$$\Delta g = \frac{1}{A} \int \int [C'] C^{-1} G \ dA,$$

where: $G$ is a column vector of known anomalies;
$A$ is the area of the block;
$C'$ is a function of the position of the
predicted value within the block $A$;
$C^*$ is a full symmetric matrix composed of
mean values of anomaly products whose
separation is the distance between the
individually known points [the mean
value of the product of the vectors $G$
and $G'$, $C^* = M \ (G G')$].

If we consider a block on the surface of a sphere then:

$$A = R^2 \Delta \phi \Delta \lambda \ \cos \phi \ m,$$

where: $R$ is the radius of the sphere;
$\Delta \phi$ is the latitude extent of the block in radians;
$\Delta \lambda$ is the longitudinal extent of the block in radians;
$\phi_m$ is the mean latitude of the block;
$\delta \phi$ is the spacing along $\Delta \phi$;
$\delta \lambda$ is the spacing along $\Delta \lambda$;
and also:

$$dA = R^2 \ d\phi \ d\lambda \ \cos \phi_m \ m,$$

The difference between $\cos \phi_m$ and $\cos \phi$ is insignificant
and Formula (58) becomes:
\[ \Delta g = \frac{\Delta \phi}{\Delta \psi} (\frac{\Delta \lambda}{\Delta \lambda}) \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} [C'] \mathbf{C}^{-1} \mathbf{G} \]

where the summation limits \( n_x \) and \( n_y \) are:

\[ n_x = (\Delta \phi/\Delta \psi) \quad \text{and} \quad n_y = (\Delta \lambda/\Delta \lambda) \quad . \]  \hspace{1cm} (62)

Next we obtain Free Air Anomalies from the elevation correlated Bouguer anomalies using the following formula:
\[ \Delta \bar{g} = -bH' + \frac{1}{n_x n_y} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} [Z'] \mathbf{Z}^{-1} (G+bH) \]

where: \( b \) is the localized Bouguer coefficient, usually, \(-0.1119\) mgals/meter is used, \( H' \) is the centered elevation of the point at which the prediction is to take place and \( Z \) is the covariance vector as explained by Rapp (1964, p33) of the Bouguer anomalies which depend on the known point location and cannot be precomputed and \( Z^* \) is a symmetric full matrix composed of mean values of anomaly products whose separation is that between the individual known points.

The polynomial representation of the values of the covariance vector for several mountainous and smooth areas have been computed by Rapp (1966a, p10) by determining a polynomial of degree 10 which fits the observed covariances.

Essentially the computer program which was used in this study is the same as in Rapp (Ibid.); however, several interesting changes and/or verifications were made where deemed useful. One reason for these changes was the length of the computer program and the time required for the predictions, of which 1024 different predictions were needed for the mean anomalies of the 37\(1^\circ5 \) blocks and 5184 different predictions were required for the mean anomalies.
of the 5' blocks. Considering the rapid calculation of one 5'
prediction which is approximately 0.2 minutes on the IBM 7094,
one might easily be misled concerning the total job time required
which was approximately 17.3 hours of 7094 computer time. To
conserve time the computer program of Rapp was truncated to exclude
the prediction accuracy computation except in random instances for
a check. Also the value of the sine function of the central angle
between the predicted point and the known point was substituted
for the distance computation which speeded up the sorting process
for the determination of the closest points to be used for the
prediction. After testing various predictions it was found that
the results yielded when using distances computed by ellipsoidal
or spherical formulae were nearly the same as those obtained when
computing the distances on a plane, and therefore, the plane or
linear distances were used in the computations. Closely spaced
points (less than 30" apart) were eliminated from the data since the
sine function value of the central angles would be approximately the
same and which the computer can not discriminate between. Another
interesting test determined that the Bouguer coefficient of 0.1119
mgals/meter which assumes a density of 2.67 gm/cm$^2$ yielded the same
results for the area where the observed points were closest together
(just to the west of Colorado Springs) as when using a geophysically
determined density for the topography, which for this area was 2.88
gm/cm$^2$ (Jackson, 1966) resulting in a Bouguer coefficient of 0.1174
mgals/meter. Several methods were attempted to derive a proper
Bouguer coefficient for each 37\textdegree'5x37\textdegree'5 block, however, a more dense gravity network is required. Therefore, the Bouguer anomalies used were considered uncorrelated with the topography for the specific use in the prediction program. An experiment was also conducted with various covariance polynomial coefficients. The set of coefficients of Rapp (1966, p10) for the rugged terrain was the most compatible to both the hills and the plains of the test area. This area is approximately two-thirds rugged mountains and one-third sloping plains. His rugged terrain polynomial was selected for use since the overall difference between the actual and predicted free air anomaly field (+0.4 mgals) would be less than if the polynomial for the plains was used (+0.5 mgals). Also the individual differences between the values obtained using both the rugged and plain polynomials were less than the accuracy of the prediction program which was ±1.7 mgals for the 5' blocks and ±0.6 mgals for the 37\textdegree'5 blocks. These values assumed that no errors existed in the data used. If the computer time and program are available to the user and a sufficient quantity of data is also available, then the polynomial coefficients should be computed for each prediction area that is to be used.

An excerpt for the central portion of the original gravity observations is shown in Figure 5 where the plots indicate the location of the gravity observations from which Bouguer anomalies were computed for use in the prediction program. These plots give an indication of the spacing of the original data and some of the values. The observations in the immediate vicinity of Pikes Peak
are shown in Figure 6, which includes some data points which were added subsequent to the plot for Figure 5.
Gravity Station Locations
Figure 6

Inner Area for 37'5 Squares and 5' Squares Showing Gravity Station Locations (x).
CHAPTER IV

GRAVITY ANOMALY CORRECTION TERM

COMPUTATIONAL PROCEDURES AND RESULTS

If the correction terms shown in Chapter 4 are added to the original predicted anomalies, then we have established model anomaly fields for 3 degrees around Pikes Peak for use later in Deflection of the Vertical comparison tests. The correction terms to be used for this comparison include: Linear Solutions, Nonlinear Solutions and Iterative Solutions as well as the original uncorrected anomaly field. Even though there may be some error in the predicted anomalies due to the sparseness of the observed gravity and some conceivable errors in the mean elevation data, the consistency of the data used for the calculations of the correction terms makes a comparison valid for the model field which may be slightly different from the actual existing anomaly field. This slight variation is to be the subject of future investigations by this author when field results become available, however, at present we shall use the data we have at hand.

Procedure for Reduction to Point Level

Most of the discussions in the literature concerning this topic have been efforts concentrated on applications to airbourne
gravimetry or the reductions of surface gravity anomalies to the geoid. The application of several of these techniques for reducing surface gravity anomalies to the elevation of the Deflection of the Vertical computation point will be discussed in this chapter later on. The specific techniques that will be investigated are: a) Moritz's Indefinite Integral Procedure (Moritz, 1966a, pp89-95); b) Gottschalk's (1969, p192) Mechanical Quadrature Procedure; c) Groten's (1964, p3) procedure using the 4 corner and center values; d) Iterative Simpson's Rule for the kernal of the integral within each block; e) Double Summation for the kernal of the integral within each block; f) A method similar to Fischer's (1966a, pp 267-275). The procedure for the reduction to point level actually becomes a procedure for computing the correction terms. Of the above mentioned techniques; a), b) and f) will be used to actually calculate a correction term \( G \) using equation (25). Additionally, techniques c, d and e will be used to examine the determination of \( (1/\mathcal{L}_o^3) \) in equation (25).

Integration Requirement for Innermost Areas

The fundamental question to be answered in this section concerns: whether the value for the kernal of the integral \( (1/\mathcal{L}_o^3) \) computed at the center of a square is adequate and also how far out from the center need the various smaller sized squares be used?

A test was completed using Formula (25) with a representative elevation difference \( (z-h-h_A) \) of 1000 meters and an arbitrary anomaly difference value of 1 mgal with only \( \mathcal{L}_o \) and \( d\kappa \) as variables. Then if
This result is multiplied by the average anomaly of an area the approximate effect of the square being investigated is obtained. The square sizes tested were 7°5', 37°5', 5' and 1 degree. Each square was divided into 25 smaller squares of almost equal size and then the arithmetic mean of the sum of the 25 smaller squares was compared with the value for the center of the undivided square. The values were also computed at each of the four corners of the square and the mean of the sum of these four values was also compared with the center value. Table 1 shows the numerical results of this short experiment.

<table>
<thead>
<tr>
<th>Square Size</th>
<th>Distance from Computation Point</th>
<th>Value at Squares Center</th>
<th>( \frac{1}{\sum_{i=1}^{25} \frac{1}{L_{i}^2}} )</th>
<th>( \frac{1}{\sum_{i=1}^{4} \frac{1}{L_{i}^2}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7°5'</td>
<td>.082 kms</td>
<td>1.5x10^-6</td>
<td>1.0x10^-5</td>
<td>4.0x10^-6</td>
</tr>
<tr>
<td></td>
<td>.240</td>
<td>1.4x10^-7</td>
<td>1.6x10^-7</td>
<td>1.5x10^-7</td>
</tr>
<tr>
<td></td>
<td>.410</td>
<td>3.2x10^-7</td>
<td>3.4x10^-7</td>
<td>3.4x10^-7</td>
</tr>
<tr>
<td>37°5'</td>
<td>.410</td>
<td>3.2x10^-7</td>
<td>2.0x10^-6</td>
<td>7.0x10^-7</td>
</tr>
<tr>
<td></td>
<td>1.228</td>
<td>3.0x10^-8</td>
<td>3.0x10^-8</td>
<td>3.0x10^-8</td>
</tr>
<tr>
<td></td>
<td>2.047</td>
<td>7.0x10^-9</td>
<td>7.0x10^-9</td>
<td>7.0x10^-9</td>
</tr>
<tr>
<td>5'</td>
<td>3.276</td>
<td>3.8x10^-8</td>
<td>2.6x10^-7</td>
<td>9.0x10^-8</td>
</tr>
<tr>
<td></td>
<td>9.828</td>
<td>3.0x10^-9</td>
<td>4.0x10^-9</td>
<td>4.0x10^-9</td>
</tr>
<tr>
<td></td>
<td>16.380</td>
<td>8.0x10^-10</td>
<td>8.0x10^-10</td>
<td>8.0x10^-10</td>
</tr>
<tr>
<td>1 degree</td>
<td>117.940</td>
<td>3.0x10^-10</td>
<td>3.0x10^-10</td>
<td>3.0x10^-10</td>
</tr>
</tbody>
</table>

Several Evaluations of Formula (25) for Several Types of Approximations of the Kernel of the Integral in mgals.

The results of Table 1 indicate that there definitely exists an evaluation and integration problem for \( L_c \) in the innermost area
for the correction terms to be computed in this study since \( J_0 \) appears in each of them.

**Numerical Integration Methods Used**

The following sections describe the methods or techniques that were used in a test to determine the most practical manner in which to numerically integrate the correction terms which were previously described.

**Indefinite Integral for Computing Correction Terms**

Moritz (1966a, p80) uses the following indefinite integral for an upward continuation solution and it seemed appropriate to include this type of solution for Formula (25) in this experiment:

\[
F_n(x,y) = \frac{z}{2\pi} \int \Delta g(\text{d}x \text{d}y) / (D^3) = (1/2\pi)\Delta g \arctan(xy/yzD), (64)
\]

where: \( z = (H-H_p) \).

Also Moritz (Ibid. p83 and p91) uses several of the following type indefinite integral with block averages:

\[
F_{II} (x,y) = \frac{z}{2\pi} \int \Delta g(x^2/D^3) \text{d}x \text{d}y = (zy/2\pi)\Delta g \ln(xD) - z^2 F_c(x,y). (65)
\]

These type formulae are then used by Moritz (Ibid. pp89-95) to develop appropriate formulae using block averages where the final surface gravity anomaly is:

\[
\Delta g' = \Delta g_o + \sum d_i \Delta g_i, \quad (66)
\]

where the coefficients \( d_i \) are expressed as in Moritz's diagram 12-1 (Ibid., p93) and shown in Figure 7 where:

\[
L = \frac{z(b/32 a^2)}{2} \ln\left[\frac{\sqrt{a^2+b^2}+a}{\sqrt{a^2+b^2}-a}\right],
\]

\[
M = \frac{z(a/32 b^2)}{2} \ln\left[\frac{\sqrt{a^2+b^2}+b}{\sqrt{a^2+b^2}-b}\right]. \quad (67)
\]

These Formulae (64 - 67) are derived in Appendix A.

Moritz also gives an expression for the second derivative (Ibid.,
which may be extended to the third derivative for a test on the extent of its effect for which the Taylor series from Moritz's Formula (12-1), (Ibid., p89), becomes:

\[
\Delta g' = \Delta g - (\Delta g/\Delta h)z + (1/2!)(\Delta^2 g/\Delta h^2)z^2 - \\
- (1/3!)(\Delta^3 g/\Delta h^3)z^3 + \ldots 
\]

(68)

\[\begin{array}{ccc}
-L & -18L+2M & -L-M \\
-18L+2M & \Delta g_2 & -L-M \\
2L-18M & 38L+38M & 2L-18M \\
-(\Delta g_4) & (\Delta g_6) & (\Delta g_3) \\
-L-M & -18L+2M & -L-M \\
-L & -L-M
\end{array}\]

Coefficients for \(d_1\)

and Formula (12-2), (Ibid.), becomes:

\[
\partial^3 g/\partial h^3 = \xi , (1/2)(\partial^2 g/\partial h^2) = \chi , \\
\text{and } (1/6)(\partial^3 g/\partial h^3) = \mu , 
\]

(69)

and Formula (12-3), (Ibid.), becomes:

\[
\Delta g' = \Delta g - z\xi + z^2\chi - z^3\mu 
\]

(70)

and continuing as does Moritz for the first and second derivative and then the third derivative becomes:

\[
\partial^3 g/\partial z^3 = -[(\partial^2 f/\partial x^2) + (\partial^2 f/\partial y^2)](\partial g/\partial z) = -2[(1/2a^2)(S_z + S_1 - 2S_{c}) + (1/2b^2)(S_z + S_1 - 2S_{c})] = [(1/2a^2) + (1/b^2)]S_z - \\
\quad \quad - [(1/2a^2)(S_z + S_1)] - [(1/2b^2)(S_z + S_1)], 
\]

(71)
and then for \( a=b \):

\[
\mu z^3 = (2z^3/a^2)[\Sigma_o - (\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4)/4],
\]

(72)

where the subscripts of the \( \Sigma \)'s refer to the same relative square locations as the subscripted \( \Delta g \)'s.

Moritz (Ibid., pp94-95) furnishes formulae for approximating the results of the 1st and 2nd derivatives and we can estimate the 3rd derivatives by Formula (72). Formulae (71) and (72) are left in terms of gravity gradients which were sequentially obtained from gravity anomalies. Thus, for the three derivatives \[ 1.3(z/a)\Delta g_o \]

is the first derivative, \[ 2(z^2/a^2)[\Delta g_o - (\Delta g_1 + \Delta g_2 + \Delta g_3 + \Delta g_4)/4] \]

is the second derivative and Formula (72) is the third derivative] and using values which approximated the actual elevation difference in the study area of \( z=100, 300, \) and \( 500 \) meters and \( a=b \) of \( 7\,\text{"}5, 37\,\text{"}5 \) and \( 5' \) respectively, these formulae for the 1st, 2nd and 3rd derivatives were broken down in Tables 2, 3 and 4 respectively for an illustration of the magnitude of each term.

<table>
<thead>
<tr>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derivative</td>
</tr>
<tr>
<td>------------</td>
</tr>
<tr>
<td>Size Block</td>
</tr>
<tr>
<td>7&quot;5</td>
</tr>
<tr>
<td>37&quot;5</td>
</tr>
<tr>
<td>5'</td>
</tr>
</tbody>
</table>

Numerical Approximation of 1st Terms of 1st, 2nd and 3rd Derivatives
Then we obtain the 2nd term in each formula from the $\Delta g'$s and $\mathcal{S}'$s as follows in Table 3.

Table 3

<table>
<thead>
<tr>
<th>Derivative 2nd Term</th>
<th>$\Delta g_0$</th>
<th>$\Delta g_0 - \left(\frac{\Delta g_1 + \Delta g_2 + \Delta g_3 + \Delta g_4}{4}\right)$</th>
<th>$\frac{\mathcal{S}_0 - \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4}{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size Block</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7\text{&quot;}5</td>
<td>110.0</td>
<td>97.0</td>
<td>0.65</td>
</tr>
<tr>
<td>37\text{&quot;}5</td>
<td>100.0</td>
<td>86.0</td>
<td>0.39</td>
</tr>
<tr>
<td>5\text{'}</td>
<td>80.0</td>
<td>78.0</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Numerical Approximation of 2nd Terms of 1st, 2nd and 3rd Derivatives (subscripted $\Delta g'$s and $\mathcal{S}'$s locations are shown in Figure 7).

The results of multiplying the 1st and 2nd terms above are shown in Table 4.

Table 4

<table>
<thead>
<tr>
<th>Derivative</th>
<th>1st(mgals)</th>
<th>2nd(mgals)</th>
<th>3rd(mgals)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block Size</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7\text{&quot;}5</td>
<td>71.5</td>
<td>48.5</td>
<td>34.0</td>
</tr>
<tr>
<td>37\text{&quot;}5</td>
<td>39.0</td>
<td>15.5</td>
<td>21.0</td>
</tr>
<tr>
<td>5\text{'}</td>
<td>4.0</td>
<td>0.5</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Numerical Approximation of 1st, 2nd and 3rd Derivative

From the size of the results it appears that the derivatives of an order higher than the 3rd is necessary for the 7\text{"}5 and 37\text{"}5 square's results to converge. The approximate results from Table 4 agree conservatively with the correction terms shown in Appendix C.
Gottschalk's Mechanical Quadrature

The coordinates \((0,0,z)\) are assigned to the field point in a rectangular coordinate system as shown in Figure 8.

![Figure 8](image)

Gottschalk's Coordinate System

Then:

\[
\Delta g' = \frac{z}{2\pi} \sum_{R} \Delta g \left/ (x^2 + y^2 + z^2)^{3/2} \right. \text{ dydx }, \tag{73}
\]

and after dividing the integration plane into finite area elements \(E(i,k)\), within which the anomalies \(\Delta g\) are regarded as constant:

\[
\Delta g' = g(z) = \left( \frac{z}{2\pi} \sum_{R} \Delta g \right) \left/ \frac{1}{z} \left[ \arctan(x y) / z(x^2 + y^2 + z^2)^{3/2} \right] \arctan(x y) / \left[ k+1 \right], \right. \tag{74}
\]

The derivation of the indefinite integral which is the basis for Formula (74) and a sample computation can be found in Appendix A.

A Fischer Type Solution

The following formula was derived using the assumptions and grid of Fischer (1966b) for Formula (25) instead of Formula (15).
Using the same coordinate system as in the Gottschalk presentation of Figure 8, the Formula (74) can be presented as follows:

$$\Delta g' = \sum_{ki} \sum_{i} \Delta g \left( \frac{y}{zx} \right) (\ln \left[ \frac{y}{z} (y + x)^{\frac{1}{2}} \right] - k_{i+1} E(i,k) i k+1 i k+1$$

$$- \ln \left[ \frac{y}{z} (y + x)^{\frac{1}{2}} \right] + (x/zy) (\ln [x / i+1 i+1 k+1 k i+1 i+1 k+1] z(y + x) \right] \right) .$$

The derivation of Formula (75) and a sample numerical computation can be found in Appendix B.

**Groten's Procedure**

In this procedure (Groten, 1964, p3) the kernel of the integral \((F)\) was evaluated at the centerpoint \((P_o)\) and the four corners of the square \((P_1, P_2, P_3, P_4)\) which are shown in Figure 9.

The function is evaluated by this method as follows:

$$F(\text{mean}) = \frac{1}{s} \int F \, d\xi = \frac{1}{4} \sum_{i=1}^{4} \left( \frac{1}{S_i} \right) \left( \text{function} \right) \frac{1}{1} .$$

$$= \frac{1}{4} \sum_{i=1}^{4} \left( \frac{1}{2} \right) \left[ \left( \text{function} \right) + \left( \text{function} \right) \right] .$$

(Figure 9)

**Illustration For Groten's Procedure**

**Mechanical Cubature-Iterative Simpson's Rule**

Numerical double integration is sometimes called Mechanical Cubature and is the process of calculating the value of a definite
integral of a function of two variables (Nielson, p.130). The following rule explains the application: "The value of the double integral may be found by applying to each horizontal row (or to each vertical column) any quadrature formula employing equidistant ordinates. Then, to the results thus obtained for the row (or columns), again apply a similar formula." The vertical and horizontal columns which were quoted refer to the two dimensional matrix where \( i = 1 \) thru \( n \) and \( j = 1 \) thru \( m \) for the usual case. At each \( i,j \) coordinate the value of \( 1/\int_{x_i}^{x_j} \) is required, computed and subsequently integrated by Simpson's Rule. The value of the double integral can thus be found by repeated application of Simpson's Rule, Weddle's Rule, or any other quadrature formula (Ibid., p131).

Simpson's Rule is a quadrature formula for an odd number of points which are evenly spaced. This is theoretically acceptable for our computations if the innermost area is omitted when the function being integrated is not continuous at the computation point.

The formula for Simpson's Rule is:

\[
\int_{x_0}^{x_n} y \, dx = \left(\frac{h}{3}\right) [y_0 + 4y_1 + 2y_2 + \ldots + 4y_{n-1} + y_n] = \\
= \left(\frac{h}{3}\right) \sum_{i=0}^{n} c_i y_i \tag{77}
\]

where:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>\ldots</th>
<th>( n-2 )</th>
<th>( n-1 )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_i )</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>\ldots</td>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

and the integral \( (x_n - x_0) = nh \).

Formula (77) applies when there is an even number of points and for an even number of intervals (an odd number of points) the following form may be used (Conte, p135):
\[
\int_{x_i}^{x_{2N}} y \, dx = \left( h/3 \right) \left( y_0 + 4 y_1 + 2 y_2 + \ldots + 4 y_{2N-1} + y_{2N} \right),
\]
where: \( 2Nh = x_{2N} - x_0 \).

The above process is called an Iterative Simpson's Rule since the process for evaluating \( (1/\int_0^3) \) continues by increasing \( (n) \) until the improvement resulting from each iteration meets an established criteria. The iterative procedure is to determine the number of intervals \( (n) \) needed for any particular kernel of an integral over any size square for which evaluation is required.

**Polynomial Quadrature**

Legendre and Chebyshev Polynomials (Conte, 1965, pp138-142) may also be used in the formula for Gauss and Chebyshev respectively. The polynomials would be fitted to the data points of the two dimensional matrix as described previously, however, a fewer number of evaluations would be required. The advantage of the Gauss Quadrature Formula over the Simpson's Rule Formula is that for the same accuracy about one-half of the computational work is required (Ibid., p142), i.e., the Simpson's Rule will require about 20 subdivisions for the same accuracy as a Gaussian Formula for 4 subdivisions. The Chebyshev Formula offers the advantage of making all of the weight coefficients equal, which in effect spreads out evenly the computational errors of the individual terms. This type of an approach may be desired if the computer system to be used or the using agency or individual has the programs available in their library.
Double Summation for the Kernel

This is a very simple procedure which involves substitution of the summation symbols ($\sum^{}$) for the integration symbols ($\int^{}$) in any of the integration formulae. The summations $\sum^{i=1}_{i=n}$ then proceed by incrementing for each step by $i=1$, $i=2$, ..., $i=n$ for all of the steps by $j$ (i thru m). The indexes $i$ and $j$ actually identify the horizontal and vertical grid squares and are therefore related to the increments of longitude and latitude used in the study.

The summations $\sum^{i=1}_{i=n}\sum^{j=1}_{j=m}$ were made evaluating the kernel of the integral at: a) the center point only [(\(\lambda_1\), +\(\lambda_2\))/2; (\(\lambda_1\), +\(\lambda_2\))/2]; b) Groten's mean value of the function as shown previously in Formula (76); c) the center of 25 smaller squares which were formed by subdividing the larger squares (5', 37.5' and 7.5') into 25 approximately equal smaller squares.

Results

The results shown below in Table 5 were calculated using an anomaly difference (\(\Delta g - \Delta g_p\)) of 1 mgal and the actual mean elevation difference (z) from the sample data field. The results shown are for the mathematical average of three blocks including and continuing west from Pikes Peak (\(\sum^{3}_{i=1} \Delta g'_i /3\)) for the three different size blocks used (7.5', 37.5', and 5'). The Deflection of the Vertical computation point is at 38 50'N, 105 W with an elevation of 3044 meters.

When the sample data field's mean anomalies are utilized in the computation, the results are as shown in Table 6.

The numbers in Table 6 may appear to be too large, however,
it must be remembered that they are not additive, i.e., the Area II correction includes area I and area III includes area I and II and also only one innermost area is used for any one computation.

Table 5

<table>
<thead>
<tr>
<th>Method</th>
<th>Integrating $\frac{z(A_g-A_g)d\sigma}{\lambda_c^3}$</th>
<th>Integrating $\frac{d\sigma}{\lambda_c^3}$ only</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moritz</td>
<td>Gottschalk (Fischer)</td>
<td>Groten</td>
</tr>
<tr>
<td>Area</td>
<td></td>
<td>Simpson's Rule</td>
</tr>
<tr>
<td>I (7°5)</td>
<td>0.67</td>
<td>0.61</td>
</tr>
<tr>
<td></td>
<td>0.59</td>
<td>0.49</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.46</td>
</tr>
<tr>
<td>II (37°5)</td>
<td>1.32</td>
<td>1.47</td>
</tr>
<tr>
<td></td>
<td>1.35</td>
<td>1.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.27</td>
</tr>
<tr>
<td>III (5°)</td>
<td>0.24</td>
<td>0.29</td>
</tr>
<tr>
<td></td>
<td>0.26</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.16</td>
</tr>
</tbody>
</table>

Pike's Peak Inner Area Correction Term Computation Results [Formula (25) for 1 mgal Anomaly Difference (results are in mgals)]

Table 6

<table>
<thead>
<tr>
<th>Method</th>
<th>Integrating $\frac{z(A_g-A_g)d\sigma}{\lambda_c^3}$</th>
<th>Integrating $\frac{d\sigma}{\lambda_c^3}$ only</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moritz</td>
<td>Gottschalk (Fischer)</td>
<td>Groten</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Simpson's Rule</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Double Summation</td>
</tr>
<tr>
<td>I (7°5)</td>
<td>46.91</td>
<td>40.92</td>
</tr>
<tr>
<td></td>
<td>49.45</td>
<td>32.81</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30.82</td>
</tr>
<tr>
<td>II (37°5)</td>
<td>46.1</td>
<td>51.45</td>
</tr>
<tr>
<td></td>
<td>47.23</td>
<td>43.71</td>
</tr>
<tr>
<td></td>
<td></td>
<td>44.45</td>
</tr>
<tr>
<td>III (5°)</td>
<td>3.61</td>
<td>4.35</td>
</tr>
<tr>
<td></td>
<td>3.91</td>
<td>2.41</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.41</td>
</tr>
</tbody>
</table>

Pike's Peak Inner Area Correction Term Computation Results [Formula (25) for Actual Data Anomaly Field (in mgals)]

The Iterative Simpson's Rule technique gives the preferred results since it is iterative and converges. The other techniques are more than 25% in error.
The Gottschalk and Fischer methods for the integration rendered
similar results (+1.7%) and therefore were averaged and given as one
entry in Tables 5 and 6.

The Moritz and Gottschalk columns of Tables 5 and 6 were
determined using the Fischer Formula (75) and the Moritz Technique
expressed in Formula (66). The results in the Groten, Simpson's Rule
and Double-Summation columns of the Tables 5 and 6 were determined
using the Double Summation Technique for integrating Formula (25),
however, the value of \(1/\lambda^3\) was determined separately by each
method (Groten, Simpson's Rule and Double Summation).

**Methods for Integrating the Outer Area Using Discrete**
**Mean Block Values**

The two methods investigated here were the Moritz Analytical
Continuation Technique (1969, sec.8), which was introduced briefly
in Chapter 4 and included equations (36)-(40) and an Iterative
Solution (Moritz, 1966c, sec.3) which is the one explained by
equations (45)-(47) previously. In our computations, both the
Free Air Anomaly (11) and the Complete Topographic Anomaly (14) are
used. The Iterative Method is most highly recommended since it is
the only method that does not assume that the data is given on a
horizontal plane (Ibid., p70). Not only can the process be iter-
atively handled by a computer subroutine, but also the results of
each term may be computed separately. The following is an example
where:
\[ \Delta g^{(1)} = \Delta g^{(1)} \]
\[ \Delta g^{(i)} = \Delta g^{(i-1)} - \frac{z}{2\pi} \int \left( \frac{\Delta g^{(i-1)} - \Delta g^{(i-1)}}{D} \right) dx dy. \]  
(79)

and each \( \Delta g^{(i)} \) may be printed out or plotted while in the iteration loop. The \( \Delta g^{(i)} \) and \( \Delta g^{(i-1)} \) should be compared during each iteration and the process should continue until the difference between the \( i \)th and the \( (i-1) \)th term is less than the precision desired.

A major problem encountered in the iterative solution was to determine how to handle the data in that area necessary for the correction but in which area no data had been computed. For a complete solution the correction should be made worldwide for each iteration. To compute the \((i+1)\)th correction using Formula (79) for each square within a 3 degree square block, the \( i \)th corrections are required for the area within and immediately adjacent to this 3 degree square block. We are therefore forced to extrapolate those corrections in the bordering area which are necessary for the calculation of the \((i+1)\)th corrections or to use the method illustrated by Madkour (1968, pp105-107) or Gottschalk (1969, pp193). The method which was most conveniently adapted to my iterative computer program was Gottschalk's. In Figure 10, the uncorrected anomalies are located in Area I, however, they are used to correct the anomalies in II and III. Only the corrected anomalies from II are used in the computation of the corrections for III. Gottschalk (Ibid.) states that, "... the influence of the unreduced, i.e. erroneous values of zone I may be regarded as vanishingly small." Gottschalk (Ibid.) also finds
that five iterations were sufficient in his study. He also found that it is sufficient to integrate over a square with a side length of 20 $z$, which in our case would be different for each correction. For an elevation difference ($z$) of 2000 meters, $20z$ is 40 kilometers, therefore, the width of region II in this study of 3 degrees should be more than adequate.

Figure 10

Iterative Solution Data Processing Illustration

He also finds that the integration error which originates from the neglect of the region outside the integration area may be as much as 10% of the value of $\Delta B'$. This is a correction to
the inner area computations representing the omitted outer area.

Gottschalk reduces this error by a correction term to $\Delta g'$:

$$v\Delta g = (1 - \sum_k A(i,k) \Delta g'),$$

where: $A(i,k) = (z/2\pi)\int\int [(dydx)/ (x^2+y^2+z^2)^{3/2}].$ (81)

This correction method of Gottschalk was incorporated into

the iterative solutions used in this study.

Alternative Approach

Instead of using very small subdivisions as we have done

previously, another possibility is to utilize the prediction

program for Terrain Corrected Bouguer point anomalies within

each square (at some interval predetermined by the numerical

integration process selected for use) and then to store the

free air anomalies computed from the terrain corrected point

anomalies on a magnetic tape or disk. Then utilize a numerical

integration technique throughout the square for the kernal of the

integral, which is stored on a tape or disk for each prediction

point. Next the integrated value of the kernals are used to

calculate the desired integral (Vening-Meinesz, Stokes or Correction

Term) by the Fischer or Uotila technique. Another alternative is

to numerically integrate in the square only the functions dependent

upon the distance from the computation point, such as $(1/\lambda_o^3)$ and

the Vening-Meinesz or Stokes functions. These integral functions

would then be multiplied by the respective predicted mean values

for each square respectively for use as the kernal of the integral

to be evaluated.
The values for the kernels obtained by the two methods last mentioned are the same unless there exists a great number (at least more than 4 in my data area) of measured gravity values in any particular square. If an evenly spaced set of gravity anomaly points or discrete mean values are available for the 7.5 or smaller squares, then the methods of Moritz (1966a, pp89-95) should be tried for the integration. If this data is not available, then the smallest square for which the data is available or necessary should be used and the kernel should be integrated throughout each square by one of the more precise methods mentioned previously. I would recommend an iterative Simpson's Rule which cuts off when the desired difference between iterations is reached, which would have to be determined as a function of the computer capability and money availability at the users facility.

Test to Determine the Use of Linear, Nonlinear or Iterative Solutions

The following set of steps and the flow chart (Figure 11) illustrate the approach taken in this study to determine the appropriate computer procedure to follow.

1. Obtain Elevations and Anomalies for 5' squares around the computation point.

2. Test this sample to determine the number of iterations required to meet the desired correction term accuracy.
   a) If the iteration required is one, then the Linear solutions are recommended.
      i) If the Linear Solution are sufficient for the
5' then there is no need to use 37.5 squares and the 4 innermost 5' squares is the innermost area.
b) If the Iterations required equals 2 for the 5' squares then the Non-linear Solutions are recommended, however, the test procedure should be extended to a smaller size square such as but not necessarily the 37.5 squares used here.

i) If the iterations are one for the 37.5 squares then the Linear Solutions are recommended as before and there is no need to use smaller squares such as the 7.5 squares used in this study.

ii) If the iterations are 2 for the 37.5 squares then the Non-linear Solution should be used and the test procedure should be repeated for the 7.5 squares for solution determination and to see if any solution is feasible.*

iii) If Iterations required are greater than 2 for the 37.5 square then the Iterative Solution is required and the test procedure should be repeated for the 7.5 square for solution determination and to see if a 7.5 square solution is feasible.*

c) If the Iterations required are greater than two for the 5' squares then the Iterative procedure is recommended if feasible* and then the 37.5 squares and the 7.5 squares should be treated as in b) above.

* In these instances it is very conceivable that the solution will
not converge within the economic limits of computer usage and therefore the solution should be abandoned or possibly more data smoothing should be investigated.

Graphical examples of the numerical values of most of the various correction terms may be found in Appendix C as well as the numerical values of the uncorrected sample data anomaly field.

Figure 11

Method Selection Flow Chart
CHAPTER V

DEFLECTION OF THE VERTICAL COMPUTATION PROCEDURES

The gravity anomalies and their correction terms are used in the computation of height anomalies (Moritz, 1964, p13) and the Deflection of the Vertical (see Formulae 29 and 47). The Deflection of the Vertical values are affected to a greater degree by the corrections to the gravity anomalies than is the height anomaly (Heiskanen-Moritz, 1967, p329). A practical comparison of the Deflection of the Vertical results using the various corrections to the gravity anomalies was therefore decided upon for the data field area.

The first topic discussed in this chapter is the selection of the square size to be used in the computation of the Deflection of the Vertical. Next, several methods for computing the Deflection of the Vertical which are discussed in the literature are described and/or modified for use with the specific square sizes available. The concluding section discusses a method for interpolating the Deflection of the Vertical values between closely spaced points (less than 5 km apart) if the gravity anomaly field can be represented by a linear function. This technique is included to determine the Deflections of the Vertical at any location. Most methods only allow the determination of deflections at the common corner of the innermost 4 squares or at a square's center.
Square Size Selection for Integration

The inner area as used in this study is the set of squares surrounding the computation point: for the 5' squares (1296), this would be the 4 innermost 5' squares or a large 10' square; for the 37.5 squares (256), this would be the 4 innermost 37.5 squares or a large 1.25 square into which fit 100 of the 7.5 squares etc. The 7.5 - 37.5 - 5' squares are the ones used in this study since the data was either available or it was reasonable to independently determine mean anomalies and mean elevations for them. The problem is to determine if these various square sizes are in fact the proper ones to use. Perhaps some other square size sequence such as 7.5 - 15.0 - 30.0 - 1.0 etc. would be more appropriate for a production type operation.

The effect on the Deflection of the Vertical by one square of various sizes was determined by the formula:

\[
\frac{\delta F}{\delta \eta} = \left\{ \Delta g \left( \frac{dS}{d\Psi} \right) \right\} \left\{ \cos \alpha \right\} \left\{ \sin \alpha \right\} \left\{ 1/(4\pi G) \right\}.
\]  

(82)

(where \( \alpha \) is 0° and \( \Delta g = 1 \) mgals and \( dS/d\Psi \) is computed at the center of the square).

In order to obtain the maximum value, the meridian deflection is evaluated for an azimuth of zero degrees and the prime vertical deflection is evaluated at ninety degrees.

When the effect of a single square approached 0.001 as an arbitrary limit, the maximum square size had been reached for the criteria used in this study. This limit is arbitrary depending on anomaly size and the number of anomalies used in the integration,
either of which could add to an error accumulation. The \(0.001\) is the effect of one compartment in Rice's Rings procedure and was selected here as a "first cut" criteria. Since \(\pm 0.3\) corresponds to about \(\pm 10\) meters in position location, it is important to keep any errors as small as possible. If the effect of \(0.001\) is multiplied by 100 representing an anomaly of 100 mgals then the actual effect becomes \(0.1\) or approximately 3 meters, which is generally used as a criteria for accuracy in practice. Also care must be taken in programming to select the combination of arithmetic operations that will lead to the least cumulative round off and truncation errors (Wilkenson, 1963). Additionally each square was subdivided in to 25 smaller squares and the mean effect of these 25 smaller squares was calculated and compared to the effect computed at the center point of the square. This subdivision was used here to investigate the effect on the Deflection of the Vertical of changes in the Vening-Meinesz Function \((dS/d\psi)\) which varies from point to point within the square as \(d_0^3\) previously.

The errors of Representation and Integration as discussed by Heiskanen-Moritz (1967, pp264-270) and Wrobel (1967, pp48-67) respectively are not a part of the problem being studied here and are not included in the investigation. For an actual practical computation these errors should be considered. For the present we are accepting this limited anomaly field to be correct and we are not considering the effect of the anomalies outside of the study area.
The results of the test were (where the distance is to the center of the square) are shown in Table 7. These results show the recommended distance limits for integration from the computation point for several square size selections. The selection of the limits was based on criteria that limit the variation of the effect of a given square on the Deflection of the Vertical for the case where the Vening-Meinesz Function is computed for the center of the square versus the case where the mean effect is calculated from the subdivided squares where the Vening-Meinesz Function is computed at the center of each subdivision.

In order to meet the criteria that the effect of the subdivided square equals the effect of the non-subdivided square and that the effect of any square size be 0.001 or less, for a mean anomaly of one milligal, the minimum distance limits for use of the non-subdivided squares would appear from Table 7 to be the following: 0.819 kms for 7.5 squares; 4.095 kms for 37.5 squares; 8.19 kms for 1.25 squares; 16.38 kms for 2.5 squares; 32.76 kms for 5 squares; 65.52 kms for 10 squares; 131.04 kms for 20 squares; 185.33 kms for 40 squares; 166.80 kms for 1 degree squares. The criteria decided upon by another investigator may determine that different limits are necessary. The limits determined in this study will be at least as effective as those used in the Rice's Rings method (Rice, 1952) for computing the Deflections of the Vertical.

To meet the criteria established for this study, the 7.5 squares and the innermost 16 of the 37.5 squares were subdivided.
<table>
<thead>
<tr>
<th>Distance from Computation Size</th>
<th>Square Size (Kms)</th>
<th>Effect of Given Square at Given Distance Evaluated</th>
<th>Square Size Used For Actual Computation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point to Center of Square</td>
<td>(Arc Units /Kms)</td>
<td>At Center Of Square By Subdividing The Given Square</td>
<td>Subdivided 7°5 Squares</td>
</tr>
<tr>
<td>0.164</td>
<td>7°5/.232</td>
<td>0.085 0.014</td>
<td>Subdivided</td>
</tr>
<tr>
<td>0.366</td>
<td>&quot;</td>
<td>0.009 0.011</td>
<td>&quot;</td>
</tr>
<tr>
<td>0.591</td>
<td>&quot;</td>
<td>0.004 0.004</td>
<td>&quot;</td>
</tr>
<tr>
<td>0.819</td>
<td>&quot;</td>
<td>0.001 0.001</td>
<td>&quot;</td>
</tr>
<tr>
<td>0.819</td>
<td>37°5/1.16</td>
<td>0.09 0.014</td>
<td>Subdivided 37°5 Squares</td>
</tr>
<tr>
<td>1.831</td>
<td>&quot;</td>
<td>0.009 0.011</td>
<td>&quot;</td>
</tr>
<tr>
<td>2.953</td>
<td>&quot;</td>
<td>0.004 0.004</td>
<td>&quot;</td>
</tr>
<tr>
<td>4.095</td>
<td>&quot;</td>
<td>0.001 0.001</td>
<td>&quot;</td>
</tr>
<tr>
<td>1.638</td>
<td>1°25/2.31</td>
<td>0.09 0.014</td>
<td>Non-Sub-divided 37°5 Squares</td>
</tr>
<tr>
<td>3.66</td>
<td>&quot;</td>
<td>0.009 0.011</td>
<td>&quot;</td>
</tr>
<tr>
<td>5.91</td>
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</tr>
<tr>
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<tr>
<td>7.32</td>
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<td>11.82</td>
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<td>0.004 0.004</td>
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</tr>
<tr>
<td>16.38</td>
<td>&quot;</td>
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<td>&quot;</td>
</tr>
<tr>
<td>6.552</td>
<td>5°10/9.27</td>
<td>0.09 0.014</td>
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<tr>
<td>14.64</td>
<td>&quot;</td>
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</tr>
<tr>
<td>23.64</td>
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<td>32.76</td>
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<tr>
<td>13.11</td>
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<td>47.28</td>
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</tr>
<tr>
<td>65.52</td>
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<td>&quot;</td>
</tr>
<tr>
<td>26.21</td>
<td>20°/37.1</td>
<td>0.09 0.014</td>
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<td>58.56</td>
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<td>94.56</td>
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</tr>
<tr>
<td>131.04</td>
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<td>&quot;</td>
</tr>
<tr>
<td>185.326</td>
<td>40°74.1</td>
<td>0.001 0.001</td>
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</tr>
<tr>
<td>166.80</td>
<td>1 degree</td>
<td>0.001 0.001</td>
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</tr>
</tbody>
</table>

Effect of Integration Square Size in Meridian Direction (arc-sec)
Numerical Integration For The Vening-Meinesz Function

The numerical value of the Vening-Meinesz Function (Formula 15) varies rapidly with distance from the computation point. It is especially difficult to estimate the value of the function to use as the mean value throughout an area. Table 8 shows the value of the function and estimates of its mean value computed along a meridian out from the computation point using the following integration methods along the meridian: a) corner values at the extremities of the square divided by two; b) Simpson's Rule between the corners of the square along the meridian; c) computed at 10 evenly spaced points between the corners of the square along the meridian and then summed and divided by 10; d) computed at 4 evenly spaced points between the corners of the square and along the meridian and then summed and divided by 4. Also computed and shown are (e) which were the results obtained for the complete Vening-Meinesz Function at the center of the square and (f) which was an approximation of the function using \((1/\psi^2)\) where \((\psi)\) is the arc distance from the computation point to the moving point located at the center of the square.

From the results shown in Table 8, the problem of accurately obtaining a mean value for the Vening-Meinesz Function becomes discernible. The subdivision approach of 4 or more divisions gives the best non-iterative solution. The Iterative Simpson's Rule as explained before does converge to the same results as obtained by subdividing into 10 increments. The Iterative Simpson's Rule approach is preferred since it has already been programmed for the correction term procedure as previously mentioned. The mathemati-
cal average difference between the 10 increment and the Simpson's Rule approaches was +0.003% which is insignificantly small and has no apparent effect on the Deflection of the Vertical results.

Deflection Solutions

There are several types of Deflection of the Vertical programs that were tried here: (1) programming of Vening-Meinesz Formula in a straight forward manner, (2) programming of Vening-Meinesz a la Fischer (1966b, p4909), (3) programming of Vening-Meinesz a la Campbell (1963, p281), (4) Rice's Rings Template Solutions (Rice, 1952).

The straight-forward method of programming the Generalized Vening-Meinesz Formula for the Deflection of the Vertical is shown in Heiskanen-Moritz (1967, p315) for an equivalent Molodensky 1st order solution with a terrain type correction which utilizes Formula (47) where $\Delta g' = \Delta g + \overline{G}_1$ and $\overline{G}_1$ is determined by Formula (26).

The Fischer Method (1966b, p4909) appeared interesting and applicable to the type of data available. Since no published derivation of the formula showing the integration by parts could be located, a short derivation is shown in Appendix D for the Fischer Formulae. The Fischer method, (Ibid.) described is for an area of one and one-half degrees around the computation point, which precisely fits our model. However, it could be easily adapted to other size areas. The Deflection station is surrounded by belts of equal width in latitude (a) and in longitude (b) as shown in Figure 12. Sections I, II and III are each (axb) in area and as we continue outward in Figure 12 sections IV, V, and VI are (2a x 2b) in area. Continuing
### Table 8

<table>
<thead>
<tr>
<th>Distance (Kms)</th>
<th>Solution a</th>
<th>Solution b</th>
<th>Solution c</th>
<th>Solution d</th>
<th>Solution e</th>
<th>Solution f</th>
<th>Exponent</th>
<th>Square Size</th>
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<td>4.72724090</td>
<td>3.81680932</td>
<td>3.77658472</td>
<td>3.77629998</td>
<td>3.36186857</td>
<td>3.36159353</td>
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<td>7'5</td>
</tr>
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<td>0.579</td>
<td>1.36564743</td>
<td>1.26199829</td>
<td>1.26021107</td>
<td>1.21033875</td>
<td>1.21017372</td>
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<tr>
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<td>6.36561256</td>
<td>6.30477461</td>
<td>6.30282611</td>
<td>6.17553452</td>
<td>6.17435564</td>
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<td>1.042</td>
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<td>2.52173519</td>
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<td>2.50035892</td>
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<tr>
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</tr>
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<td>3.571</td>
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<tr>
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<td>0.954202</td>
<td>0.945506</td>
<td>0.841776</td>
<td>0.840398</td>
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<td></td>
</tr>
<tr>
<td>16.216</td>
<td>1.64140</td>
<td>1.57619</td>
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<td>1.54359</td>
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<td>23.166</td>
<td>8.5353</td>
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<td>7.9173</td>
<td>7.6051</td>
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<td>$5'1$</td>
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<tr>
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<td>3.9405</td>
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<td>2.3866</td>
<td>2.3576</td>
<td>2.3344</td>
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</tr>
<tr>
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<td>1.1256</td>
<td>1.1416</td>
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<td>1.1189</td>
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<tr>
<td>83.396</td>
<td>6.057</td>
<td>5.909</td>
<td>6.026</td>
<td>6.024</td>
<td>5.953</td>
<td>5.836</td>
<td>$10^3$</td>
<td></td>
</tr>
</tbody>
</table>

(The Solutions a-f are explained on page 71 and the results are in radians and the numbers in the Solution Column are to be multiplied by the Exponent.)

Characteristics of the Vening-Meinesz Function During Integration
outward sections VII, VIII and IX will be (4ax4b) and the lengths of the sides continue to progress outward at the fixed ratio 1:2. The inner area may also be broken into subbelts in a ratio of 2:1 progressing inward. The final formulae shown by Fischer (Ibid.) are:

\[
\xi = \frac{\csc \theta}{2\pi G} \ln \left[ \sqrt{1+(b/a)^2}+(b/a) \right] \left[ \sqrt{1+(b/2a)^2}-(b/2a) \right], (83)
\]

\[
\eta = \frac{\csc \theta}{2\pi G} \ln 2 \left[ \sqrt{1+(2a/b)^2}-(2a/b) \right] \left[ \sqrt{1+(a/b)^2}+(a/b) \right], (84)
\]

\[
\xi = \frac{\csc \theta}{2\pi G} \ln \left[ \sqrt{1+(b/2a)^2}+(b/2a) \right] \left[ \sqrt{1+(2b/a)^2}+(2b/a) \right]
\]

\[
\xi = \frac{\csc \theta}{2\pi G} \ln \left[ \sqrt{1+(b/a)^2} + (b/a) \right] - (b/a), (85)
\]

\[
\eta = \frac{\csc \theta}{2\pi G} \ln \left[ \sqrt{1+(a/2b)^2} + (a/2b) \right] \left[ \sqrt{1+(a/b)^2} + (a/b) \right]
\]

\[
\eta = \frac{\csc \theta}{2\pi G} \ln \left[ \sqrt{1+(b/a)^2} + (b/a) \right] \left[ \sqrt{1+(2a/b)^2} - (2a/b) \right], (86)
\]

\[
\zeta = \frac{\csc \theta}{2\pi G} \ln \left[ \sqrt{1+(a/b)^2} + (a/b) \right] \left[ \sqrt{1+(a/2b)^2} + (a/2b) \right]
\]

\[
\zeta = \frac{\csc \theta}{2\pi G} \ln \left[ \sqrt{1+(a/b)^2} + (a/b) \right] \left[ \sqrt{1+(a/2b)^2} - (a/2b) \right]. (87)
\]

The final computation of Formulae (83-88) was accomplished by integrating separately the 5' and 37'5 squares outside of and inside of the center 10' block respectively. The squares bordering along the east-west or north-south axis were used with the formula for I and III and the remainder with the formula of type II (see Figure 14). A simple substitution enables a further subdivision of each of Fischer's subbelts into smaller squares. The subdivisions are formed by substituting nx(a/5) for A1 and nx(b/5) for B1 where n is an integer varying from 1 to 5. If we substitute A for a, B for b, (A1+A) for 2a and (B1+B) for 2b in Fischer's Formula, the following
Fischer's Belts
Figure 13

Fischer's Subbelts
**Figure 14**

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>II</th>
<th>II</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>A1</td>
<td>II</td>
<td>II</td>
<td>II</td>
<td>II</td>
<td>II</td>
</tr>
</tbody>
</table>

Equal Division of Subbelt Area
computational formulae are achieved:

$$\Phi = \left( \frac{\csc \theta}{2W} \right) \ln \left( \sqrt{1+\left[ \frac{B}{(A1+A)} \right]} + \frac{B}{(B1+B)} \right) \left( \sqrt{1+\left[ \frac{(B1+B)/A}{B1+B} \right]} \right)$$

$$\eta = \left( \frac{\csc \theta}{2W} \right) \ln \left( \sqrt{1+\left[ \frac{A}{(B1+B)} \right]} + \frac{A}{(B1+B)} \right) \left( \sqrt{1+\left[ \frac{(A1+A)/B}{A1+A} \right]} \right)$$

and similarly a substitution may be made for I and III which are used only along the axes as described previously.

Mrs. Fischer (1966b, p269) warns that for the rectangles of the second subbelt and all of the rectangles contained therein "... it is preferable to integrate over the area because of the increasing discrepancy between the geometrical midpoint and the center of gravity in relation to the station P, due to the rapid change of the Vening-Meinesz function when approaching P."

For this reason each square in the innermost area was divided into 25 still smaller squares for each of which the Vening-Meinesz function was computed, summed, and averaged using Formulae (89 and 90). This is the same approach that was used for the straightforward solution which evaluated Formula (47) at the center of each of the 25 smaller squares. The squares were tested, one at a time, using the subdivided formula versus the unsubdivided formula and when the results agreed (approximately) the subdividing process was discontinued. This technique was also used for the Uotila and Bursa type integration techniques as discussed earlier. The Fischer technique was altered to be compatible with the rapid elevation and anomaly changes in the mountainous areas. In the computations for the Fischer technique the
actual mean elevation of the subsquares was used, however, due to the lack of gravity data, the value of $\Delta g'$ of the larger square was used or the next larger square where the $\Delta g'$ value was known.

A thorough explanation of the formulae used in the inner area (the first subbelt) is given by Fischer (Ibid., p273) in which the integration is performed as demonstrated for Formulae (83) and (84). The formulae for the inner area are as follows (Ibid.):

$$\xi_{inner} = \left(\csc 1''/4\Pi G\right)(B/A)\left[(4.)\ln \sqrt{(A/B)^2 + 1} + (B/A)(-\Delta g_1 + \Delta g_2 + \Delta g_3 - \Delta g_4)\right]$$

$$(91)$$

$$\eta_{inner} = \left(\csc 1''/4\Pi G\right)(A/B)\left[(4.)\ln \sqrt{(B/A)^2 + 1} + (B/A)(\Delta g_1 + \Delta g_2 + \Delta g_3 + \Delta g_4)\right]$$

$$(92)$$

(91) (92)

(92)

(where the $\Delta g$'s location are shown in Figure 15. The algebraic sign of the deflection components depend upon the orientation of the axes. The positive direction for this study is to the south for the meridian deflection and to the east for the prime vertical deflection).

The Vening-Meinesz Formulae (where the Vening-Meinesz Function was evaluated at the center of each square) for Deflections of the Vertical were also programmed using Formula (47) and evaluated by substituting a double summation for the double integration and then using Formulae (91) and (92) for the innermost area.

The other type computer solution was accomplished by Andrew Campbell at the Navy Oceanographic Office using his well tested and publicized method (Campbell, 1962, 1963 and 1967). Campbell uses zones of 72 mean values in an 81 square anomaly block omitting
Figure 15

\[ \Delta g_4 \quad \Delta g_1 \]
\[ \Delta g_3 \quad \Delta g_2 \]

Inner Area
the center 9 squares and dividing the remainder into 81 square blocks, etc. He also determines the center of gravity for each square which approximates a quadrilateral on the earth's surface. Since my study is only of a comparative nature, the precise center of gravity of each square was not computed, but the mean latitude and mean longitude were used. His program averaged or lumped together my mean values to fit his 81 block criteria, whereas, I had: 1276-5' blocks; 1025-37"5 blocks; and 400-7"5 blocks. The actual computation of Campbell is different from that of Fischer's in that the logarithms are not used in the Campbell computation as in Formulae (83-92) but rather the explicit computational form of Formula (47) is used. Also the Vening-Meinesz Function is evaluated as \( (1/\psi) \) where \( \psi \) is the distance from the center of the square to the computation point (this approximation results in relative errors of 1/10,000 for the 7"5 squares and 1/3,000 for the 37"5 squares and 1/500 for the 5' squares). In this system all 72 squares, for all subzones, approach the central point in precisely the same mathematical manner. The side of each square is increased by a ratio of 3 to 2 as the distance from the computation point increases, i.e., if the size of the inner belt of squares is 7"5x7"5, then the size of the next belt would be \((3/2)\times(7''5\times7''5) = 11''25\times11''25\), etc.

By knowing only the first and last terms of the series, the summation for the iteration is as follows:

\[
\sum_{i=1}^{p} \Delta g = \Delta g_1 + \Delta g_2 + \ldots + \Delta g_p
\]

(93) (where successive terms of \( \Delta g \) are determined by the
previous and last term as follows):

$$\Delta g = \frac{2}{3} \left( \Delta g_p - \Delta g_{n-1} \right) + \Delta g_{n-1}$$  \hspace{1cm} (94)

Also a manual computation was accomplished using the method of Rice's Rings (Emrick, 1961). This involved making an anomaly map and two types were used: the 5 mgal contours of the first map were interpolated for the 37°5 mean free air anomalies obtained from the prediction program; the second map's 5 mgal contours were interpolated directly from the observed free air anomalies. The elevations for each compartment of Rice's Rings were obtained from various scale maps and charts: 1:12,500 (where available) for the zones 1-20; 1:24,000/25,000 for most of the zones 1-20 (1.304-3.641 Kms); 1:50,000 for the zones 21-24 (3.641-7.216 Kms); 1:100,000 for the zones 24-28 (7.216-14.29 Kms); 1:250,000 for the zones 28-32 (14.29-28.25 Kms); 1:500,000 for the zones 32-36 (28.25-55.66 Kms), and 1:1,000,000 for the remaining zones 36-43 (55.66-180 Kms). Sixty-four evenly spaced values were read in each compartment, and the arithmetic mean was used for the compartment elevation. Since the area was very mountainous and in some parts the control used was very old, the reliability could most certainly be questioned. The average elevations of the compartments was comparable with the mean elevations used in the computer for the 37°5 and 5' squares for the random instances where the compartments and squares were almost coincident.

**Fischer's Curvature Technique**

Fischer's Curvature Technique is a method of determining the
Deflections of the Vertical at a point where they are unknown from the values at a nearby point where they are known. The application to this study is to transfer deflection values from the intersection point of the four innermost area squares, where they have been computed, to the actual observation points where they might be compared to realistic values such as the astronomical observations at Pikes Peak.

The general formulae for the curvature components are derived from the Vening-Meinesz Formula which is differentiated along the meridian and prime-vertical directions, and we obtain (Fischer, 1966c, p4911):

\[
\frac{d^2 \xi}{Rd \psi} = -\csc \psi \int \Delta g \left( \frac{d^2 s}{d \psi^2} - \frac{ds}{d \psi} \cot \psi \right) \cos \alpha + \frac{ds \cot \psi}{d \psi} \, d\alpha, \quad (95)
\]

\[
\frac{d^2 \eta}{Rd \psi} = -\csc \psi \int \Delta g \left( \frac{d^2 s}{d \psi^2} - \frac{ds}{d \psi} \cot \psi \right) \sin \alpha \cos \alpha \, d\alpha, \quad (96)
\]

\[
\frac{d^2 \xi}{Rd \psi \cos \psi} = -\csc \psi \int \Delta g \left( \frac{d^2 s}{d \psi^2} - \frac{ds}{d \psi} \cot \psi \right) \sin \alpha \cos \alpha \, d\alpha - \frac{\eta'' \tan \psi}{R}, \quad (97)
\]

\[
\frac{d^2 \eta}{Rd \psi \cos \psi} = -\csc \psi \int \Delta g \left( \frac{d^2 s}{d \psi^2} - \frac{ds}{d \psi} \cot \psi \right) \sin \alpha + \frac{ds \cot \psi}{d \psi} \, d\alpha + \frac{\eta'' \tan \psi}{R}. \quad (98)
\]

In the following formulae, (a) is the width of a square in the north-south direction, (b) is the width of a square in the east-west direction, and \((\Delta g_\psi)\) is the anomaly at point 0 (see Figure 16). The anomaly \(\Delta g_\psi\) is formed by averaging the mean anomalies of the surrounding four squares or \(\Delta g_\psi = \frac{1}{4} [\Delta g(1,-1)+\Delta g(1,1)+\Delta g(-1,-1)+\Delta g(-1,1)].\)

The center of Figure 16 (0) represents the point where a Deflection of the Vertical has been determined. The formulae are then used to interpolate the Deflections of the Vertical at other stations.
Figure 16

Second Derivative Anomaly Guide
in the near vicinity of station (0). The procedure for simplifying and obtaining a computable form is similar to the previous Fischer Deflection Techniques, and she obtains (Fischer, 1966c, p4914):

\[
\frac{d^2 \phi''}{Rd \phi} = \left( \frac{\csc \alpha''/2 \pi G}{Rd \lambda \cos \phi} \right) \left[ \Delta g \left( \frac{(4b)/(a \sqrt{a^2 + b^2})}{\frac{2 \sqrt{a^2 + b^2}}{a}} \right) + \frac{f}{\chi} \left[ (2ab) / \sqrt{a^2 + b^2} + \right.ight.

\[
\left. + 4b \ln \left( \frac{\sqrt{a^2 + b^2} - a}{b} \right) - \frac{2 \sqrt{a^2 + b^2}}{a} \right] \right]
\]

\[
\frac{d \eta''}{Rd \phi} = \frac{d^2 \phi''}{Rd \lambda \cos \phi} = \left( -\csc \alpha''/2 \pi G \right) \left( 4ab / \sqrt{a^2 + b^2} \right)
\]

The second derivative may be read from the anomaly chart (Figure 16) as follows:

\[
f_{x^2} = \left( 1/4a^2 \right) \left[ \Delta g(2,-1) + \Delta g(2,1) - \Delta g(1,-1) - \Delta g(1,1) \right]
\]

\[
- \Delta g(-2,1) + \Delta g(-2,-1) + \Delta g(-1,1) \right] , \quad (102)
\]

\[
f_{y^2} = \left( 1/4b^2 \right) \left[ \Delta g(1,2) + \Delta g(-1,2) - \Delta g(1,-1) - \Delta g(-1,1) \right]
\]

\[
- \Delta g(1,-1) - \Delta g(-1,1) + \Delta g(1,2) + \Delta g(-1,-2) \right] , \quad (103)
\]

\[
f_{xy} = \left( 1/ab \right) \left[ \Delta g(1,1) - \Delta g(-1,1) - \Delta g(1,-1) + \Delta g(-1,-1) \right] . \quad (104)
\]

When the horizontal gradient of the Deflections of the Vertical from Formulae (99), (100), and (101) are multiplied by the distances \( d \phi \), and \( d \lambda \) in kilometers, the Deflection of the Vertical change is that direction is obtained (Ibid., p4915).
CHAPTER VI

RESULTS

Integration Technique Comparison

One of the double integration techniques utilized for the correction terms of Formulae (25-28), (36), (45), and (46), as well as the straight forward solution of the Vening-Meinesz Formula (47), was the procedure recommended by Uotila (1960, pp52-53). This method simply replaces the double integration by a double summation. There is also an iteration loop in the computer program for the Vening-Meinesz Function which was utilized in my investigation by an iterative use of Simpson's Rule as described in Chapter IV.

\[ \frac{\xi_j}{\eta_i} = \sum \sum \left\{ c_i^j \right\} \left( \phi_i \right) \Delta g_m \left( \phi_i \right) \left( \lambda_j \right) \]

(105)

(Where \( \left\{ c_i^j \right\} \left( \phi_i \right) \left( \lambda_j \right) \) = \( \frac{1}{2\pi G} \frac{dS}{d\psi} \left\{ \cos \alpha \right\} dq \),

and dq is the area of the square and the \( c'(\phi_i, \lambda_j) \)

can be computed independently and stored for each block).

Another method which was also tested was reported by Bursa (1967, pp13-49) and involves the Gauss Mechanical Quadrature Technique for the area within 5 Kms of the computation point. This method was found to be greatly dependent upon the position of the gravity measurements. For optimum results in mountainous terrain Bursa (Ibid, p32 and p38) recommends using 16 equal sectors around
the point with 6 or 7 gravity observations located within each sector. The horizontal gradient of gravity is also assumed to be constant within this radius out from the computation point. The following formula was utilized in a computer program:

$$
\frac{A_x}{\Delta x} = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\bar{W}_k \Delta g(r_{ij})}{2 \pi G} \left\{ \frac{\cos A}{r} \right\} dAdr,
$$

(106)

where

$$
\bar{W} = \frac{-\csc \theta''}{4 \pi G} \frac{Q(m) - Q(\phi)}{Q(m) + Q(\phi)} \left\{ \frac{\sin A}{\sin A} \right\} \left\{ \frac{\cos A}{\cos A} \right\}
$$

and \( \csc \theta'' \) is the conversion from radians to seconds of arc; \( Q(m) \) and \( Q(\phi) \) are the partial limits of distance determined by the nodes of an orthogonal polynomial where \( r(k) = (1/2) \left[ Q(m) + Q(\phi) \right] \). This indicates that the anomaly \( \Delta g(r(k)) \) lies at the node of the polynomial \( r(k) \), which is halfway between \( Q(m) \) and \( Q(\phi) \). The nodes of a polynomial are located at the intersection of the polynomial with the reference surface, which is the earth's surface in this instance. The degree of the polynomial and the spacing of the nodes may be determined in this method, Bursa (Ibid.) recommends a polynomial of a degree of 6 or 7 in the high mountains. The Fischer Formulae (89) and (90) were also used for this investigation for the innermost squares.

Two terrain situations were used for the determination of the anomaly field to be used for a comparative study of the integration methods mentioned above. For a test of the computer program the first situation used consisted of a hypothetical terrain model which was
composed of four quadrants of flat terrain which differed from each other by 1000 feet in elevation, i.e., the NW quadrant was 10,000 feet in elevation, the NE quadrant was 11,000 feet in elevation, the SE quadrant was 8,000 feet in elevation and the SW quadrant was 9,000 feet in elevation. The second terrain situation consisted of the actual mean terrain of the 7'5 squares (see the crosshatched area in Figure 6) taken from 1:24,000 topographic sheets by the technique of Smith which was mentioned previously. Since there were few gravity measurements within the 5 Km circle, the Bouguer anomalies were interpolated from a U.S. Coast and Geodetic Survey Bouguer Anomaly Contour Map (Rice, 1967). The free air anomalies were determined for this area of a 5 km radius by the familiar correction to the Bouguer Anomaly of (0.0341 h) where h is in feet. The 37'5 and 5' predicted means, which have been previously described, were also used in this comparison of techniques for use in an inner area. The computation point for this comparison was taken as 38°50'N and 105°00'W, which is the center of the data area for the predicted 5' mean anomalies. The results of this test are shown in Table 9.

It can be noted that for the flat model all three methods were in close agreement, but that they diverged only slightly as the real terrain was used. It was concluded from this test that a higher degree polynomial than the seven degree polynomial used here is not needed to use the Gauss type formula even in rough terrain.

**Terrain Corrections**

The relation between the conventional Terrain Correction and the correction in terms of the Modern Theory has been derived and
### TABLE 9

<table>
<thead>
<tr>
<th></th>
<th>FISCHER</th>
<th>UOTILA</th>
<th>BURSA</th>
</tr>
</thead>
</table>
| 7\textsuperscript{5} Model  
(flat hypothetical-integrated from point out for a radius of 37\textsuperscript{5} at a 37\textsuperscript{5}/25 interval) | 10.01   | 10.1   | 10.1  |
|                    | -21.69  | -21.8  | -21.9 |
| 7\textsuperscript{5} Terrain Model 
(from computation point out to 1\textdegree25)* | -1.47   | -1.45  | -1.50 |
|                    | -2.02   | -2.00  | -2.03 |
| 37\textsuperscript{5} Terrain Model 
(from 1\textdegree25 out to 5\textdegree)* | -1.23   | -1.21  | -1.25 |
|                    | -4.17   | -4.15  | -4.18 |
| 5\textdegree Terrain Model 
(from 10\textdegree out to 1 1/2 degrees)* | -1.92   | -1.93  | -1.91 |
|                    | -9.78   | -9.81  | -9.80 |

Total of 7\textsuperscript{5}, 37\textsuperscript{5} and 5\textdegree Terrain Models from the computation point out to 1 1/2 degrees. 
-4.62  -4.59  -4.66
-15.97  -15.96  -16.01

Integration Technique Comparisons of Deflections of the Vertical

* The Fischer Technique was used in all cases for at least the inner 15\textdegree radius.
investigated by Moritz (1968). The comparison of results of these two methods was considered by me to be applicable to this investigation. The mean terrain corrections which were added to the mean anomaly for each 37'-5" and 5' block were determined by two methods. One method was manual determination of the Terrain Correction using the well-known Hayford zones (Hayford and Bowie, 1937) at two geographic locations one of which was located in rugged terrain near Pikes Peak and the other in a relatively flat area. The second method was the computer program of Rapp and Snowden (Rapp, 1967) using Formula (28). The results of these two techniques are compared in Table 10 with the corrections as determined by the Linear Correction Terms Formulae (25), (26), and (27) for the two points. These results were all determined for the center of a square and were not integrated throughout the square.

The two Hayford Terrain correction computations seemed to validate the Rapp-Snowden computer program method. The Pellinen Linear Corrections are shown in Table 10 to be very similar to the Terrain Correction which was shown analytically by Moritz (1968).

**Linear and Nonlinear Gravity Anomaly Correction Terms**

Table 10 shows a comparison of various Linear Type 1st Order Correction Terms to the gravity anomaly.

As previously mentioned, in order to compute the correction terms properly, the gravity anomaly and its Linear Correction terms are needed for the surrounding area. This problem was solved by using the technique described by Gottschalk (1969) and discussed
Table 10

<table>
<thead>
<tr>
<th>Location</th>
<th>Snowden/Rapp Formula (28)</th>
<th>Hayford</th>
<th>Pellinen Formula (27)</th>
<th>Molodensky Formula (26)</th>
<th>Gradient Formula (25)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>38°50'00&quot;N 105°00'00&quot;W</td>
<td>38°49'14&quot;N 105°02'12&quot;W</td>
<td>38°49'14&quot;N 105°01'34&quot;W</td>
<td>38°49'14&quot;N 105°00'56&quot;W</td>
<td>38°49'14&quot;N 105°59'41&quot;W</td>
</tr>
<tr>
<td>38°50'00&quot;N 105°00'00&quot;W</td>
<td>51.13</td>
<td>7.9</td>
<td>18.2</td>
<td>15.3</td>
<td>9.9</td>
</tr>
<tr>
<td>38°49'14&quot;N 105°02'12&quot;W</td>
<td>50.41</td>
<td>7.5</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>38°49'14&quot;N 105°01'34&quot;W</td>
<td>42.1</td>
<td>6.0</td>
<td>18.8</td>
<td>14.5</td>
<td>12.2</td>
</tr>
<tr>
<td>38°49'14&quot;N 105°00'56&quot;W</td>
<td>94.2</td>
<td>-0.3</td>
<td>28.0</td>
<td>7.5</td>
<td>-2.4</td>
</tr>
<tr>
<td>38°49'14&quot;N 105°59'41&quot;W</td>
<td>97.2</td>
<td>-1.2</td>
<td>28.0</td>
<td>7.0</td>
<td>-3.5</td>
</tr>
</tbody>
</table>

Linear Correction Comparisons for Various Locations (in mgals)
### Table 11

<table>
<thead>
<tr>
<th>Term</th>
<th>Maximum Value (mgals)</th>
<th>Mean Value (mgals)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta g )</td>
<td>+205.0</td>
<td>+91.6</td>
</tr>
<tr>
<td>( \bar{g}_2 )</td>
<td>[37'5\ squares]</td>
<td>[5' squares]</td>
</tr>
<tr>
<td>( \bar{g}_3 )</td>
<td>86.0</td>
<td>10.0</td>
</tr>
</tbody>
</table>

Comparison of \( \Delta g \), \( \bar{g}_2 \), and \( \bar{g}_3 \) Using the Nonlinear Solution Formulae (38) - (40)

### Table 12

<table>
<thead>
<tr>
<th>Term</th>
<th>Maximum Value (mgals)</th>
<th>Mean Value (mgals)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Iteration</td>
<td>+94.2</td>
<td>+12.1</td>
</tr>
<tr>
<td>2nd Iteration</td>
<td>-15.9</td>
<td>-2.2</td>
</tr>
<tr>
<td>3rd Iteration</td>
<td>+8.2</td>
<td>+1.1</td>
</tr>
<tr>
<td>4th Iteration</td>
<td>-3.1</td>
<td>-0.8</td>
</tr>
<tr>
<td>5th Iteration</td>
<td>+1.7</td>
<td>+0.4</td>
</tr>
<tr>
<td>6th Iteration</td>
<td>-0.3</td>
<td>-0.1</td>
</tr>
</tbody>
</table>

Changes During the First Thru Sixth Iterations Using Formulae (45)-(46)
maximum results of the first, thru the sixth iterations using Formulae (45) and (46) are shown in Table 12. The evaluation of all terms utilized the mean values for the free air anomalies and elevations. Perhaps, the point values would have produced different maximum results, but the average results obtained here are assumed to be approximately representative -- at least for a comparison with the other results utilizing mean values. The Nonlinear terms were computed in the same manner as the Linear Correction Terms. The results of Tables 11 and 12 show the need for the Nonlinear corrections including at least $g_3$ and an Iterative Solution up to at least the fifth iteration for this particular terrain model and the equations used.

Comparison of Deflection of the Vertical Results

Utilizing the Various Correction Terms

The Deflection of the Vertical results shown in Table 13 were computed basically for 2 points based on data extending out to a 3 degree radius from the Pikes Peak area. The 7°5 squares were used for the mean elevation and mean anomaly data out to a radius of 1°25 from the computation points. The 37°5 squares were used for the mean elevation and mean anomaly data out to a radius of 10' from the computation points. The 5' squares were used for the mean elevation and mean anomaly data out to 1.5 degrees from the computation points. The data between 1.5 degrees and 3 degrees was used to compute the correction terms only and was not included
<table>
<thead>
<tr>
<th>Interval</th>
<th>(1) Uncorrected Using Δg</th>
<th>(2) Δg + Pellinen Correction Formula (27)</th>
<th>(3) Δg + Molodensky Correction Formula (26)</th>
<th>(4) Δg + Gradient Correction Formula (25)</th>
<th>(5) Δg + Terrain Correction Formula (28)</th>
<th>(6) Δg + Nonlinear Correction Formula (36)</th>
<th>(7) Iterative Solution Formulae (45-47)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0° to 7°5°</td>
<td>-0.20</td>
<td>-0.27</td>
<td>-0.29</td>
<td>-0.29</td>
<td>-0.27</td>
<td>-0.29</td>
<td>-0.29</td>
</tr>
<tr>
<td>using 7°5°</td>
<td>(0.03)</td>
<td>(0.11)</td>
<td>(0.13)</td>
<td>(0.13)</td>
<td>(0.11)</td>
<td>(0.13)</td>
<td>(0.13)</td>
</tr>
<tr>
<td>squares</td>
<td>-0.11</td>
<td>-0.21</td>
<td>-0.22</td>
<td>-0.22</td>
<td>-0.21</td>
<td>-0.22</td>
<td>-0.22</td>
</tr>
<tr>
<td></td>
<td>(-0.02)</td>
<td>(-0.16)</td>
<td>(-0.18)</td>
<td>(-0.18)</td>
<td>(-0.16)</td>
<td>(-0.18)</td>
<td>(-0.18)</td>
</tr>
<tr>
<td>7°5 to 1°25°</td>
<td>-1.27</td>
<td>-1.31</td>
<td>-1.33</td>
<td>-1.32</td>
<td>-1.31</td>
<td>-1.35</td>
<td>-1.37</td>
</tr>
<tr>
<td>using 7°5°</td>
<td>(0.32)</td>
<td>(0.40)</td>
<td>(0.42)</td>
<td>(0.43)</td>
<td>(0.40)</td>
<td>(0.37)</td>
<td>(0.39)</td>
</tr>
<tr>
<td>squares</td>
<td>-1.91</td>
<td>-2.01</td>
<td>-2.06</td>
<td>-2.07</td>
<td>-2.01</td>
<td>-2.12</td>
<td>-2.15</td>
</tr>
<tr>
<td></td>
<td>(-1.42)</td>
<td>(-1.75)</td>
<td>(-1.78)</td>
<td>(-1.77)</td>
<td>(-1.75)</td>
<td>(-1.87)</td>
<td>(-1.93)</td>
</tr>
<tr>
<td>1°25 to 10°</td>
<td>-1.23</td>
<td>-1.70</td>
<td>-1.90</td>
<td>-1.91</td>
<td>-1.75</td>
<td>-1.92</td>
<td>-1.95</td>
</tr>
<tr>
<td>using 37°5°</td>
<td>(-0.62)</td>
<td>(-0.83)</td>
<td>(-0.96)</td>
<td>(-0.97)</td>
<td>(-0.87)</td>
<td>(-0.86)</td>
<td>(-0.89)</td>
</tr>
<tr>
<td>squares</td>
<td>-4.17</td>
<td>-4.31</td>
<td>-4.35</td>
<td>-4.34</td>
<td>-4.30</td>
<td>-4.47</td>
<td>-4.55</td>
</tr>
<tr>
<td></td>
<td>(-3.87)</td>
<td>(-4.09)</td>
<td>(-4.25)</td>
<td>(-4.24)</td>
<td>(-4.07)</td>
<td>(-4.28)</td>
<td>(-4.34)</td>
</tr>
<tr>
<td>10° to 1°5°</td>
<td>-1.92</td>
<td>-1.88</td>
<td>-1.89</td>
<td>-1.88</td>
<td>-1.88</td>
<td>-1.92</td>
<td>-1.96</td>
</tr>
<tr>
<td>using 5°</td>
<td>(-1.84)</td>
<td>(-1.82)</td>
<td>(-1.83)</td>
<td>(-1.83)</td>
<td>(-1.81)</td>
<td>(-1.89)</td>
<td>(-1.92)</td>
</tr>
<tr>
<td></td>
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<td>(-10.28)</td>
<td>(-10.27)</td>
<td>(-10.30)</td>
<td>(-10.76)</td>
<td>(-10.79)</td>
</tr>
<tr>
<td>Total 0°</td>
<td>-4.62</td>
<td>-5.16</td>
<td>-5.41</td>
<td>-5.40</td>
<td>-5.21</td>
<td>-5.48</td>
<td>-5.57</td>
</tr>
<tr>
<td>out to 1°5°</td>
<td>(-2.17)</td>
<td>(-2.14)</td>
<td>(-2.24)</td>
<td>(-2.24)</td>
<td>(-2.17)</td>
<td>(-2.25)</td>
<td>(-2.29)</td>
</tr>
</tbody>
</table>

Deflection of the Vertical [Formula (47) where $Δg' = Δg$ for (1) and $Δg' = Δg + $correction for (2-7)]

Results at 38°50'N and 105°00'W in Seconds of Arc (Results in parentheses are for 38°50'37.5'N and 105°02'37.5'W) Using the Uotila Technique Except For The Innermost Squares Where The Subdivided Fischer Technique Was Used.
in the Deflection of the Vertical computations.

The computation of the Deflection of the Vertical for the corner of the 7°5 square containing Pikes Peak (38°50'37"5N, 105°02'37"5W) in Table 13 appears in parentheses.

In addition to the computation with Free Air Anomalies of Formula (11), the mean Terrain Corrected Bouguer Anomalies of Formula (12) and the FAYE Anomalies of Formula (13) were utilized for the Deflection of the Vertical Computation for the 37°5 squares. The results are shown in Table 14 as well as a repetition of the Uncorrected and Terrain Corrected results from Table 13 for the purpose of comparison.

Table 14

<table>
<thead>
<tr>
<th></th>
<th>( \xi )</th>
<th>( \eta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uncorrected Free Air Anomalies</td>
<td>(-1,\prime,23)</td>
<td>(-4,\prime,17)</td>
</tr>
<tr>
<td>Free Air Anomalies Computed From Mean Terrain Corrected Bouguer Anomalies</td>
<td>(-1,\prime,54)</td>
<td>(-4,\prime,25)</td>
</tr>
<tr>
<td>FAYE Anomalies</td>
<td>(-1,\prime,73)</td>
<td>(-4,\prime,28)</td>
</tr>
</tbody>
</table>

Comparison of the Deflection of the Vertical Results Using Various Types of Anomalies for the 37°5 Squares (38°50' N and 105°00' W)

The free air anomalies for the 7°5 squares were computed from Formula (11) using a Bouguer anomaly map (Woollard, 1964) to determine the value of \( a \) and \( b \) was \(-0.1119\) and \( h \) was the mean elevation of the 7°5 square in meters. Since the Bouguer anomalies
did not vary greatly throughout the region as shown on the map, the free air anomalies are almost directly correlated with the elevation for the 7\"5 squares. Thus, the criteria described by Moritz (1966c, p88) for the Terrain Correction of Formula (28) to be equivalent to the Pellinen Correction of Formula (27) are apparently met for the 7\"5 squares, which the results in Table 13 verify.

I believe that the following interpretation of the results could have some interesting implications which will be discussed in the final chapter. The 7\"5 and the 37\"5 results indicate that utilizing the Terrain Correction for the Deflection of the Vertical computation is indeed almost equivalent to using the Pellinen Linear Correction Term as shown by Moritz (1968). The omission of the Linear Correction can cause an error of approximately 1\"0 and the omission of the Non-linear can cause an error of approximately 0\"5. The Iterative Solution results indicate an error of approximately 0\"5 may still exist because of the neglect of the higher order terms. The anomaly formula used had no apparent effect on the Deflection of the Vertical computation for the 37\"5 squares. The Gradient and Molodensky Linear Solutions rendered about the same results but differed from the Pellinen Solution by a maximum of about 0\"25. Any of the Linear Correction Term Formulae used for the Iterative Deflection Solution converge to essentially the same answer (+0\"005) after six iterations.

Assuming the Iterative Solution to give the best results for the data sample used, the results of Table 13 indicate that a maximum error of 1\"08 (16\"64 - 15\"56) seconds of arc may occur by neglecting all correction terms or of 0\"53 (16\"91 - 16\"38) by using only
Linear Correction Terms or of 0".45 (17°09' - 16°64') by using the Nonlinear Corrections.

The results for the 38°50'N and 105°W point from this study may be compared to other results reached independently by other agencies. The Aeronautical Chart and Information Center obtained $\xi = -4''78$ and $\eta = -15''95$ for the same coordinates using basically the same uncorrected anomalies and elevations. A manual Rice's Rings method using the uncorrected anomalies gave results of $\xi = -4''49$ and $\eta = -16''13$ and the results for the inner 45' x 45' around the computation point agreed well with the ACIC values: $\xi = -0''80$ and $\eta = -10''28$ versus $\xi = -0''88$ and $\eta = -10''13$ respectively. Another solution by Campbell (1967) of the Naval Oceanographic Office, using my data for the point 38°47'5N and 104°52'5W (he could only compute points at 2.5 intervals) was: for the uncorrected set of data, $\xi = -1''47$ and $\eta = -14''90$; and for the set of data which had Pellinen's Linear Correction added to the uncorrected anomaly, $\xi = -1''25$ and $\eta = -15''23$. These results are compared in Table 15. The ACIC and Rice's Rings solutions were used to check my computer solution for gross errors. The Campbell values were for interest, however, a comparison of the two Campbell solutions were also used to obtain an idea of the effect that the Linear Corrections have on the Deflections of the Vertical which gave an indication that further research was warranted.

Figure 17 represents a study for several points in the close proximity of Pike's Peak. The results in Figure 17 are for the 7'5 squares used out to approximately 1 kilometer or 37'5, and as can be
### Table 15

<table>
<thead>
<tr>
<th>Method</th>
<th>Area and Correction Term Used For Computation</th>
<th>Deflection of the Vertical Components</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Coordinates of Computation Point)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rice's Rings (38°50'N, 105°W)</td>
<td>[Total Using Uncorrected Anomalies]</td>
<td>-4&quot;49</td>
</tr>
<tr>
<td></td>
<td>[Inner 45' Area]</td>
<td>-16&quot;13</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0&quot;80</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-10&quot;28</td>
</tr>
<tr>
<td>ACIC (38°50'N, 105°W)</td>
<td>[Total Using Uncorrected Anomalies]</td>
<td>-4&quot;78</td>
</tr>
<tr>
<td></td>
<td>[Inner 45' Area]</td>
<td>-15&quot;95</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0&quot;88</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-10&quot;13</td>
</tr>
<tr>
<td>Campbell (38°47'5N, 104°52'5W)</td>
<td>[Total Using Uncorrected Anomalies]</td>
<td>-1&quot;47</td>
</tr>
<tr>
<td></td>
<td>[Total Using Pellinen Linear Correction Term (Formula (27))]</td>
<td>-15&quot;90</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1&quot;25</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-15&quot;23</td>
</tr>
</tbody>
</table>

Comparison of Results of Various Methods for the Computation of the Deflection of the Vertical (Using Formula (47) where \( \Delta g' = \Delta g \) or \( \Delta g' = \Delta g + G_p \))

seen in the Figure the contribution to the Deflection of the Vertical value reaches about 2"0 maximum and for the closest 7"5 or approximately 200 meters the Deflection of the Vertical reaches approximately 0"4 maximum for this terrain. In observing the values at the grid intersections, it is obvious that the change in the Deflection of the Vertical values is not linear for this region. The interpolation method of Fischer was also used (Formulae 99-101), and the Deflection of the Vertical differences from the 38°50'30"N, 105°02'37"5W grid intersection to the 38°50'22"5N, 105°02'37"5W grid intersection were: \( df = +0"189 \) and \( d\varphi = +0"521 \) versus the values previously determined by comparing the results of the Fischer's Deflection of the Vertical
\[ \begin{array}{cccc}
105^\circ 02'45"W & 105^\circ 02'37.5"W & 105^\circ 02'30"W \\
\hbar = 13880 & \hbar = 13750 & \hbar = 13610 & \hbar = 1370
\\
\xi = 0.06 (0.03) & \xi = -0.24 (0.405) & & \\
\eta = 1.05 (0.189) & \eta = -0.70 (-0.101) & & \\
\hbar = 13920 & \hbar = 14010 & \hbar = 13990 & \hbar = 13850
\\
\xi = -0.52 (-0.171) & \xi = -0.45 (-0.196) & \xi = -1.80 (-0.057) & \\
\eta = 0.65 (0.082) & \eta = -0.47 (0.019) & \eta = -0.27 (-0.07) & \\
\hbar = 13810 & \hbar = 13850 & \hbar = 13840 & \hbar = 13890
\\
\xi = -0.77 (-0.18) & & & \\
\eta = -0.19 (0.07) & & & \\
\hbar = 13560 & \hbar = 13690 & \hbar = 13760 & \hbar = 13550
\\
\end{array} \]

\( \overline{h} \) indicates the mean elevation in feet for each square.

Deflection of the Vertical Results Using 7.5 Square Values Out To 37.5 Near Pike's Peak For Selected Grid Intersections

(In parentheses are the results only out to 7.5)
Formulae such as shown by Formula (89) and (90) which were $\delta = +0^\circ.21$ and $\phi = +0^\circ.23$. This interpolation method was extended then to the point $(38^\circ 50'15''N, 105^\circ 02'30''W)$ and the results there were $\delta = +0^\circ.008$ and $\phi = +0^\circ.827$ versus the values again determined by the Fischer Deflection of the Vertical Formula which were $\delta = +0^\circ.32$ and $\phi = -0^\circ.28$. The difference in results indicate that this interpolation method versus direct computation may result in large errors for mountainous terrain.

The Iterative Solution was also attempted for a zero elevation using the 37.5 squares rather than a point level solution, however, the results were not convergent. An improved solution was attempted by using the Gottschalk technique for the outer area. After three gravity anomaly correction term iterations the results were $\xi = -0^\circ.78$ and $\eta = -4^\circ.72$ and were converging, therefore, it appears as though the techniques that were used here for point level deflection solutions are also applicable for the sea level deflection solutions [Formulae (41, 42 and 44)]. For each computation point, there exists an area for which the reduction to point level and terrain corrections were not computed. This is the outer area which is referred to in the text and which is compensated for by the Gottschalk Technique. The inner area data set is corrected by this technique for the effect of the outer area.

An analysis of the individual correction terms as shown in Tables 11 and 12 indicated that the 3rd order correction terms should be considered for the 37.5 and the 5' squares and the analysis
of the cumulative effect of the correction terms upon the De-
flection of the Vertical values as shown in Table 13 indicated that
the Iterative Solution or at least the \( g_3 \) Nonlinear term should be
included in the computation for the 37'5 squares and that the 5' Non-
linear terms thru \( g_3 \) are adequate. A logical recommendation is the
use of the Iterative Solution which will stop when the desired
convergence is reached.

**Error Analysis**

The primary purpose of this research is to give the representa-
tive and comparative solutions of the various correction procedures
(Linear, Nonlinear and Iterative) using a fairly realistic terrain
and gravity anomaly model. An estimate of errors which could be
encountered in practical applications is deemed beyond the scope
of this study.

Some errors have been shown by other authors including Pick
(1968, p140) who published a maximum error of 1"4 for mountainous
areas resulting from insufficient gravity data out to 160 kms from
the computation point. Yeremeyev (1965, p137) also shows a reduction
of error in a theoretical model from 2"2 to 0"1 by incorporating
1st and 2nd order corrections. Biro (1967, p86) observed an interpo-
lation error of 1"2 for a Deflection of the Vertical in a mountainous
area with observations out to 100 km from his computation points.
Szabo (1962, p234) shows the expected errors for the Deflection compo-
nents to be on the order of ±0"21 for more level areas.
The possible accumulation of errors resulting from round off in the numerical processing should be avoided by use of adequate computer systems which carry at least 12 significant digits as shown by Wilkinson (1963, pp7-10). The round off errors are dependent upon the number of arithmetic operations involved in the entire procedure. This type of error could possibly occur when the large mean anomalies and mean elevations of the 7'5 squares are used in a double summation procedure for the numerical evaluation of a double integral. If more iterations or summations are to be used then this type of error should again be investigated.
Conclusions

Assuming that the computed free air anomalies are realistic and valid, the major conclusions are as follows:

1. The procedures shown in Figure 18 are those recommended for use in hilly and mountainous areas and are based upon the results of this and other studies, which have been quoted in this text.

2. The Pellinen Linear Solution of Formula (27) is preferred for use at lower elevations and in less rugged terrain since it is more easily mapped and is equivalent to the Terrain Correction Solution using Formula (28). This correction does not depend on the location of the computation point and thus may be stored on maps, data cards, etc.

3. In extremely rugged areas the gravimetric Deflections of the Vertical require a more densified gravity field than was available for this study. Since the results of Table 7 show that 7.5 squares should be used out to 1 kilometer, there should be at least one observation in each square and preferably one at each grid intersection and another at the square's center. The gravity observations
1- Obtain Gravity and Elevation Observation Records
2- Compute Bouguer Anomalies for each Gravity Observation Point
3- Compute Terrain Corrected Bouguer Anomalies [Formula(12)]
4- Predict Terrain Corrected Bouguer Anomalies by Rapp's Prediction Program (Rapp, 1964, p29) for square size to be utilized. (The C term of Formula (14) should be included within 50 kms of the computation point and (3V_/2R) of Formula (14) should be included beyond 50 kms from the computation point)
5- Compute the FAYE Anomalies from the anomalies obtained in 4
6- Compute the Gravity Anomaly Correction Terms
   a) Use the Gottschalk (p56) or Fischer (p57) Numerical Integration Procedure
      i) Use an Iterative Simpson's Rule (pp57-59) for the Kernal of the Integral
7- Compute the Deflections of the Vertical
   a) Use Fischer's Subdivided Solution for the Innermost Area (pp 75-82)
   b) Use Fischer's Solution from the innermost area at least out to 1' and Uotila's Solution to the antipode
      i) Use an Iterative Simpson's Rule for the Kernal as in 6-a) i) above

Recommended Steps for Obtaining the Deflections of the Vertical
should be made similarly for the other sized squares.

4. The Iterative Solution [Formulae (45-47)] for six iterations resulted in a Deflection of the Vertical solution which was significantly different than the other solutions. This type of solution is recommended for use in mountainous areas if a computer with a large storage capacity is available since the input and results of each iteration are needed for the subsequent iteration or iterations.

5. The change to the Deflection of the Vertical resulting from the Nonlinear correction was approximately 1".53 (Table 13) for the data used in this study. The higher order corrections are recommended to be used near the computation point for the anomaly determination in rugged terrain.

6. The change to the Deflection of the Vertical resulting from the Iterative Solution for six iterations was 1".08 (Table 13) for the data used in this study. This value was 0".45 less than the result from the Nonlinear solution (Table 13). Since 0".45 is approximately 15 meters on the earth's surface, it appears that the Iterative Solution or higher order than $g_2$ for the Nonlinear Solution is required for the recommended accuracy of 0".1 or 3 meters.

7. The recommended procedure for the innermost area is to use smaller and smaller squares until the change of gravity can be considered linear throughout the area. Also
the use of an iterative Simpson's Rule or some other iterative Numerical Integration technique is recommended for the computation of the Kernel of the Integrals.

8. The Gottschalk (Chapter V) method for dealing with the influence of the anomalies which lie in the area beyond the actual area of investigation should be used.

9. The Method Selection Flow Chart (Figure 11) should be used to best determine the method and technique to be used outside of the innermost area for the correction terms.

Observations

The following comments and opinions are based on the author's perception of several existing problem areas:

1. Fischer's Deflection of the Vertical solution (Formulae 83-88) agreed with the Uotila type solution (Formula 105) when modified to use equal sub-divisions and then integrated in small increments. This adjustment was probably necessary due to the great elevation and gravity field changes. The original Fischer method which is recommended further out from the computation point gave similar results to Campbell's solution and would probably be best adopted for use in more level terrain.

2. Errors are probably made when using mean anomalies rather than point anomalies in the prediction program, which are recommended in Figure 18; however, consistency
has been maintained, and therefore the comparisons are considered to be valid. This assumes that each type of mean corrected anomaly behaves in the same manner as would the mean of the corrected point anomalies.

3. The Linear Correction Term solutions required less than one minute of computer time for the 37.5 block and two minutes for the 5' block. No time estimates were available for the inner area since the corrections were made in the same program as the Deflection of the Vertical solutions for the 7.5 squares; however, the complete job time was less than one minute on the Burroughs 5500.

4. The Iterative Solution [Formulae (45)-(47)] required 2.8 minutes of Burroughs 5500 time for six iterations using tape input for the 5' squares and 1.7 minutes for the 37.5 squares.

5. An approach for the use of the Iterative Solution has been suggested for mountainous regions. The correction that is made to the anomalies in the outermost area was the one recommended by Gottschalk (1969). The correction term decreases as the inverse of the distance cubed \((1/ \ell^3)\), therefore, the Iterative Solutions probably need not extend farther than 5 degrees to find the square size where the free air anomaly (without further correction) is sufficient for computing the Deflections of the Vertical.

6. Molodensky et al., (1962b, pp35-38) recommends the Linear Correction term be calculated out to at least
4 degrees and the Nonlinear (or Iterative) Solution out to 40'. The results shown in Table 13 indicate that the Linear Term is influential to at least 3 degrees for the Deflection of the Vertical computation and Molodensky's comment on the Nonlinear Solution is valid out beyond 10'.

7. The Fischer Curvature Technique for the interpolation between points (Gravimetric Leveling) 7.5" apart resulted in an error of almost 1" for the points tested. The size of the area tested and the number of Deflections of the Vertical computed rendered these results inconclusive.

8. The Iterative Simpson's Rule technique for the integration of \((1/\lambda^2)\) and \((dS/d\psi)\) is an excellent method, which is available at most computer center libraries and has been shown to be convergent for the 7.5", 37.5" and 5' square sizes.

9. A recommended listing of the distance limits for using the various size squares is shown in Table 7. The list must be considered only in the context of the criteria used for its development.

10. A standard error of 1.90 may be incurred by:
    1) insufficient gravity data information; 2) computation and interpolation and prediction; and 3) correction term omission.

11. Using the Mean Terrain Corrected Bouguer Anomalies of Formula (28) is recommended when possible.
The use of the FAYE anomaly of Formula (13) yielded a result that differed 0".19 in the Prime Vertical direction and only 0".03 in the Meridian direction for the Deflections of the Vertical using the data and techniques recommended in this study.

12. When using only the Linear Corrections, the Gradient and Molodensky Correction Formulae (25) and (26) render about the same results but differ from the Pellinen and Terrain Correction solutions by as much as 0".25 in the Meridian.

13. Considerably more effort is recommended for:
1) the analysis of geological/geophysical observations for the determination of the mean density for any particular small area; 2) the general availability of gravity observations throughout the mountainous areas must be improved with observations taken everywhere and not only along highway routes.

14. If a computer facility and an automatic photogrammetric plotter is available, then terms such as \((X^2 + Y^2 + Z^2)^{1/2}\) could be handled by reading \(X\), \(Y\) and \(Z\) at the appropriate interval on the photography. The integration techniques recommended in this paper could then be used in the computer program. This is provided \(\sqrt{L_0} = (X^2 + Y^2 + Z^2)^{1/2}\) and \(Z = h - h_p\) as per this discussion.
The purpose of this study has been to give to the reader an insight into the theory and practicality of the utilization of gravimetric methods in mountainous areas. Perhaps the next few years will produce a more efficient method of determining the gravity anomalies in the mountains and elsewhere; then the studies such as this one and the study in the Alps will be more meaningful and productive.
APPENDIXES

Appendix A

Brief Derivation of Moritz' Indefinite Integral

Let: \( D = (x^2 + y^2 + H^2)^{\frac{1}{2}} \);

\( \partial D / \partial x = x/D \);

\( \partial D / \partial y = y/D \);

If: \( F(x,y) = (1/2\pi) \arctan(xy/HD) \);

Then: \( F(x,y) / \partial x = (1/2\pi) \cdot 1/[(x^2y^2/H^2D^2)+1] \)

\[ \frac{yHD-xyH(x/D)}{(H^2D^2)} \]

\[ = (1/2\pi)(yH/D)[H^2D^2/(x^2y^2+H^2y^2)] \]

\[ [(D^2-x^2)/H^2D] \]

\[ = (1/2\pi)(yH/D)[(y^2H^2)/(x^2+y^2+H^2)(x^2+H^2)] \]

\[ = (1/2\pi)(yH/D)[(y^2+H^2)/(y^2+H^2)(x^2+H^2)] \]

\[ = Hy/[2\pi (x^2+H^2)D] \]

And: \( F(x,y) / \partial x \partial y = (H/2\pi) \cdot [(x^2+H^2)D-(y^2/D)(x^2+H^2)]/[(x^2+H^2)D^3] \)

\[ = (H/2\pi) \cdot [(x^2+H^2)D^2-y^2(x^2+H^2)]/[(x^2+H^2)D^3] \]

\[ = (H/2\pi) \cdot D^2 - y^2/[(x^2+H^2)D^3] \]

\[ = (H/2\pi) \cdot [x^2+H^2-y^2]/[(x^2+H^2)D^3] \]

\[ = (H/2\pi) \cdot [(x^2+H^2)] [(x^2+H^2)D^3] \]

\[ = H/2\pi \cdot \frac{1}{D^3} \]

Similarly for the other two Indefinite Integrals:

Where: \( F(x,y) = (Hy/2\pi) \ln(x+D)-(H^2/2\pi) \arctan(xy/HD); \)

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Then: \[ F(x,y)/\partial x = (Hy/2\pi) \frac{d(x+D)}{dx} - \frac{Hy}{2\pi(x^2+H^2)D} + 0; \]

\[ = (Hy/2\pi) \left[ 1+(x/D) \right] \left[ x+D \right] - (Hy/2\pi) \left[ \frac{H^2}{(x^2+H^2)D} \right] \]

\[ = (Hy/2\pi) \left[ \left( \frac{1}{D} \right) - \frac{H^2}{(x^2+H^2)D} \right] \]

\[ = (Hy/2\pi) \left[ \frac{(x^2+y^2-H^2)}{D (x^2+H^2)} \right] \]

\[ = (H/2\pi) \left[ \frac{x^2y}{D(x^2+H^2)} \right]; \]

And: \[ \frac{F(x,y)}{\partial x \partial y} \]

\[ = (H/2\pi) \left[ \frac{x^2(x+yH^2)D - x^2y^2(x^2+H^2)}{D} \right] / \left[ \left( x^2+H^2 \right) D^3 \right] \]

\[ = (H/2\pi) \left[ \frac{x^2(x+yH^2)D - x^2y^2(x^2+H^2)}{D^3} \right] / \left[ \left( x^2+H^2 \right) D^3 \right] \]

\[ = (x^2H/2\pi) \left[ \frac{(x^2+y^2+H^2-y^2)}{(x^2+H^2)D^3} \right] \]

\[ = \frac{H}{2\pi} \frac{x^2}{D^3}. \]

The following numerical example uses Gottschalk's Quadrature Formula for a 37°5 square and the Quadrature Formula is based on the above derivation. (Also see last paragraph of Appendix B)

\[ \Delta \phi = \frac{z}{2\pi} \left( \arctan [(2.4)(1.638)/\sqrt{1.638 + 2.4}] - \arctan [(1.638)(1.2)/\sqrt{1.638 + 1.2}] - \arctan [(0.819)(2.4)/\sqrt{0.919 + 2.4}] + \arctan [(0.819)(1.2)/\sqrt{0.819 + 1.2}] \right) \]

\[ = \frac{z}{2\pi} \left[ \arctan (1.352931) - \arctan (0.968023) - \arctan (0.77511) + \arctan (0.676465) \right] \]

\[ = \frac{z}{2\pi} \left[ 59°32' - 44°4' - 37°46' + 34°4' \right] \]

\[ = \frac{z}{2\pi} \left[ 11°46' \right] / \text{(Radian Conversion Unit)} = \frac{z}{2\pi} [0.0205076] \]

\[ = z \left( 0.032127 \right) \text{ mgals}. \]
Fischer Type Correction Term Integration Method Derivation

Using:

\[
\frac{(Z/2\pi)}{[1/\sqrt{(x^2+y^2)}]^{3/2}} \int dx dy = (Z/2\pi) \int [s/s^3] ds d\alpha
\]

\[
\alpha = 2\pi \quad \text{and} \quad \alpha = 0
\]

Where: 
\( s \leftarrow (y/x) \tan \alpha \) from 45° to 90°;
And: 
\( s \leftarrow (x/y) \cot \alpha \) from 0° to 45°;

Then: 
\[
\frac{(Z/2\pi)}{2/[(x(k+1)\tan \alpha/y(i)]} d\alpha +
\frac{2/[(y(i+1)\tan \alpha/x(k)]} d\alpha
\]

\[
\arctan[y(i)/x(k+1)] - \arctan[y(i+1)/x(k+1)]
\]

\[
= (Z/\pi) \left[ \frac{-y(i)/x(k+1)}{\cot \alpha} d\alpha + \frac{y(i+1)/x(k)}{\tan \alpha} d\alpha \right]
\]

\[
\arctan[y(i+1)/x(k+1)] - \arctan[y(i+1)/x(k)]
\]

\[
= (Z/\pi) \left[ \frac{y(i)/x(k+1)}{\ln |\sin \alpha|} + \frac{y(i+1)/x(k+1)}{\ln |\cos \alpha|} \right]
\]

\[
\arctan[y(i)/x(k+1)] - \arctan[y(i+1)/x(k+1)]
\]

\[
= (Z/\pi) \left[ \frac{-y(i)/x(k+1)}{\ln |\sin (\arctan y(i)/x(k+1))] - \ln |\sin (\arctan y(i+1)/x(k+1))]} +
\right.

\[
+ \frac{x(k)/y(i+1)}{\ln |\cos (\arctan y(i+1)/x(k))] - \ln |\cos (\arctan y(i+1)/x(k+1))]}
\]

\[
\frac{-y(i+1)/x(k+1)}{\sqrt{(y(i+1)/x(i+1) + x(k+1)]} -
\frac{-y(i+1)/x(k+1)}{\sqrt{(y(i+1)/x(i+1) + x(k+1)]}} +
\frac{x(k)/y(i+1)}{\sqrt{(y(i+1)/x(i+1) + x(k+1)]} -
\frac{x(k)/y(i+1)}{\sqrt{(y(i+1)/x(i+1) + x(k+1)]}}
\]

As in Appendix A the above type formula will be numerically evaluated for a 37°5 square as follows.
\[ \Delta g = \frac{Z}{\eta} \left\{ \frac{(1.2/1.638)[\ln(1.2\sqrt{1.638}+1.2)}{-\ln (2.4/\sqrt{1.638} + 2.4 \right) + (0.819/2.4)} + \frac{\ln (2.4/0.819 + 2.4)}{-\ln (2.4/\sqrt{1.638} + 2.4 \right) \} \}

\]

\[ \Delta g = \frac{Z}{\eta} \left\{ (0.7326 \left[ \ln(1.2/2.03052) - \ln(2.4/2.90569) \right] + (0.34125[\ln(2.4/2.535894)-\ln(2.4/2.90569)]) \} \}

\[ \Delta g = \frac{Z}{\eta} \left\{ (0.7326 \left[ \ln 0.590978 - \ln 0.825965 \right) + (0.43125(\ln 0.946411 - \ln 0.825965 )) \} \}

\[ \Delta g = \frac{Z}{\eta} \left\{ [-0.7326(0.5539-0.4379) + 0.34125(0.3881-0.4379)] \right\}

\[ \Delta g = \frac{Z}{\eta} \left\{ [-0.084981 - 0.016994] \right\}

\[ \Delta g = (Z/\pi) \left\{ 0.101975 \right\}

\[ \Delta g = Z \left[ 0.032459 \right] \text{ mgals.} \]

The numerical results shown in Appendix A and Appendix B show the equivalency of these two types of approaches which lead to the same results. In the numerical examples I set \(Z=1\) and \(\Delta g =1\) and approximated \(x(k), y(i), x(k+1)\) and \(y(i+1)\) with \(0.819, 1.200, 1.638\) and \(2.40\) respectively.
Appendix C

Graphical Illustrations

The following illustrations demonstrate the geographical distribution and values of the anomalies and many of the correction terms computed in this study. The geographical boundaries, the square size for which the represented values were computed, and a reference to a Formula or page of the text are noted on each illustration.
37.5 Squares Uncorrected Free Air Anomaly Values (Δg) in mgals. According to Rapp's Prediction Program (pages 40-48). For the Geographic Area Depicted at the bottom of page.
37°5 Squares Terrain Correction Mean Values (G') in mgals
According to Formula (28)
For the Geographic Area Depicted at bottom of page

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104°55'W
38°55'N
38°45'N
105°05'W
### 37°5 Squares Molodensky Type Linear Correction

Term Mean Values ($\bar{G}$) According to Formula (26) in mgals

For the Geographic Area Depicted at bottom of page

| $\Delta$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|----------|---|-----|---|-----|---|-----|---|---|---|---|---|---|---|----|----|----|----|----|----|
| $\Delta$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

![Geographic Area Diagram](image)
37°5 Squares Mean Free Air Anomalies + Mean Molodensky Type Linear Correction Term Values \((\Delta g + G)\) in mgals
For the Geographic Area Depicted at bottom of page

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| 111.5 | 89.7 | 58.1 | 51.3 | 68.9 | 44.6 | 40.6 | 29.7 | 60.2 | 45.6 |
| 76.0 | 58.4 | 34.3 | 31.2 | 35.3 | 21.3 | 20.1 | 14.5 | 27.3 | 20.9 |
| 102.2 | 58.3 | 58.2 | 57.2 | 84.7 | 78.3 | 72.9 | 63.0 | 70.9 | 65.9 |
| 117.9 | 107.4 | 111.9 | 102.9 | 83.3 | 69.8 | 64.8 | 84.7 | 88.4 | 91.0 |
| 03.1 | 101.6 | 126.2 | 126.0 | 109.3 | 105.2 | 87.6 | 75.5 | 60.8 | 86.4 |
| 119.2 | 101.7 | 127.0 | 109.0 | 102.1 | 88.0 | 88.6 | 64.6 | 64.6 | 68.5 |
| 124.3 | 119.7 | 120.9 | 173.1 | 134.6 | 134.6 | 134.6 | 134.6 | 134.6 | 134.6 |
| 124.2 | 119.7 | 120.9 | 173.1 | 134.6 | 134.6 | 134.6 | 134.6 | 134.6 | 134.6 |
| 144.5 | 118.7 | 122.2 | 148.5 | 133.3 | 133.3 | 133.3 | 133.3 | 133.3 | 133.3 |
| 104.1 | 139.2 | 136.2 | 122.1 | 106.1 | 103.3 | 94.5 | 117.1 | 32.2 | 86.1 |
| 98.8 | 107.9 | 159.9 | 153.2 | 118.3 | 113.4 | 113.4 | 113.4 | 113.4 | 113.4 |
| 82.0 | 74.8 | 74.8 | 92.7 | 39.7 | 112.6 | 110.6 | 112.7 | 113.8 | 114.2 |
| 115.8 | 131.1 | 63.8 | 63.8 | 63.8 | 63.8 | 63.8 | 63.8 | 63.8 | 63.8 |
| 112.3 | 112.3 | 112.3 | 112.3 | 112.3 | 112.3 | 112.3 | 112.3 | 112.3 | 112.3 |
| 112.9 | 112.9 | 112.9 | 112.9 | 112.9 | 112.9 | 112.9 | 112.9 | 112.9 | 112.9 |
37.5 Squares Mean Free Air Anomalies + Mean Bjerhammer Type First Iteration \( \Delta q_{\Delta}^2 \)

According to Formulae (45) and (46) for Sea Level in mgals

For the Geographic Area Depicted at bottom of page

| 111.5 | 111.7 | 122.2 | 167.7 | 37.8 | 48.3 | 62.0 | 65.0 | 21.5 | 35.0 | 58.7 | 66.9 | 74.7 | 80.0 | 65.8 |
| 113.4 | 113.7 | 61.9 | 38.4 | 111.3 | 102.4 | 88.6 | 82.3 | 33.9 | 75.5 | 58.9 | 47.7 | 58.5 | 57.9 | 55.3 |
| 72.0 | 72.9 | 82.2 | 82.8 | 72.4 | 100.4 | 87.3 | 85.2 | 78.6 | 88.2 | 67.5 | 57.7 | 52.5 | 41.3 | 34.1 |
| 43.6 | 117.3 | 114.5 | 82.9 | 83.9 | 85.6 | 78.7 | 77.6 | 52.1 | 57.9 | 22.6 | 34.7 | 15.6 |
| 135.4 | 132.2 | 13.4 | 114.2 | 158.4 | 88.6 | 85.7 | 65.8 | 78.0 | 76.4 | 82.2 | 29.7 | 15.6 |
| 112.0 | 151.1 | 288.9 | 278.8 | 107.9 | 101.9 | 82.6 | 65.9 | 78.3 | 88.9 | 72.4 | 77.5 | 59.9 | 35.4 |
| 112.7 | 178.0 | 84.6 | 251.1 | 311.6 | 102.8 | 153.8 | 73.1 | 62.4 | 57.6 | 49.0 | 58.3 | 63.4 | 72.6 | 71.6 |
| 112.1 | 132.1 | 187.2 | 214.8 | 284.6 | 158.4 | 51.1 | 61.0 | 50.7 | 55.0 | 51.1 | 70.1 | 60.0 | 63.1 |
| 112.6 | 113.7 | 117.7 | 136.8 | 121.1 | 219.1 | 150.4 | 37.6 | 70.6 | 68.6 | 77.1 | 51.8 | 71.4 | 81.4 | 86.6 |
| 208.9 | 218.9 | 180.7 | 120.3 | 132.0 | 31.3 | 70.4 | 73.8 | 73.3 | 71.4 | 71.6 | 84.7 | 83.8 |
| 22.9 | 22.6 | 128.6 | 108.6 | 116.7 | 94.8 | 111.3 | 123.3 | 146.4 | 82.6 | 59.5 | 40.5 | 64.6 | 60.0 |
| 82.3 | 82.6 | 178.0 | 108.6 | 116.7 | 94.8 | 111.3 | 123.3 | 146.4 | 82.6 | 59.5 | 40.5 | 64.6 | 60.0 |
| 143.8 | 158.7 | 123.1 | 107.2 | 93.8 | 82.6 | 102.8 | 145.2 | 85.1 | 70.4 | 51.9 | 54.4 | 112.1 | 85.3 |
| 142.6 | 132.6 | 121.5 | 108.7 | 83.7 | 85.2 | 116.6 | 122.6 | 111.2 | 84.3 | 142.7 | 121.4 | 112.2 | 83.5 |

104.55°W

38°55'N

38°45'N

105°05'W
37°5 Squares Mean Free Air Anomalies + Mean Gradient Linear Correction Term Values ($\Delta g + G$)
According to Formula (25) in mgals
For the Geographic Area Depicted at bottom of page

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$104°55'W$

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$38°55'N$

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$38°45'N$

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$105°05'W$
37°5 Squares Mean Elevations in meters
For the Geographic Area Depicted at bottom of page
37°5 Squares: Mean Values of $L_1(Ag)$ Term of
Formula (33) in mgals
For the Geographic Area Depicted at bottom of page

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Diagram:

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104°57'5W
38°52'5N

38°47'5N
105°02'5W
```
37"S Squares Mean Values of $L_1 \{L_1(\Delta g)\}$ Term of Formula (38) in mgals
For the Geographic Area Depicted at bottom of page

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104°57.5W
38°52.5N

38°47.5N
105°02.5W
37°5 Squares Mean Values of $g_2$ as in Formula (38) in mgals
For the Geographic Area Depicted at bottom of page

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104°57'5W
38°52'5N

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38°47'5N
105°02'5W
5' Squares Uncorrected Predicted Free-Air Anomalies (Ag) According to Rapp's Prediction Procedure (pages 40-48) in mgals.
For the Geographic Area Depicted at bottom of page

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[Map of geographic area with coordinates 106°30'W, 40°20'N, 103°30'W, and 37°20'N shaded]
5' Squares Molodensky Type Linear Correction
Term Mean Values ($G_2$)
According to Formula (26) in mgals
For the Geographic Area Depicted at bottom of page

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Diagram:
- 103°30'W
- 40°20'N
- 37°20'N
- 106°30'W
5' Squares Mean Free Air Anomalies \( + \) Mean Molodensky Type Linear Correction Term \( (\Delta g + \bar{G}) \) in mgals
For the Geographic Area Depicted at bottom of page
5' Squares Pellinen Type Linear Correction
Term Mean Values \((G')\) According to Formula (27) in mgals
For the Geographic Area Depicted at bottom of page

| 13 | 9 | 12 | 17 | 17 | 13 | 17 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
|---|---|---|----|----|----|----|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
| 0.5 | 0.5 | 1.0 | 1.6 | 1.7 | 1.7 | 1.5 | 0.9 | 0.6 | 1.6 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 | 0.3 | 0.2 |
5' Squares Mean Free Air Anomalies + Mean Pellinen Type Linear Correction Term Values \( (\Delta g + \Gamma_j) \) in mgals For the Geographic Area Depicted at bottom of page

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![Image of geographic area]
5' Squares Uncorrected Bouguer Anomalies
According to Rapp's Prediction Procedure (pages 40-48) in
For the Geographic Area Depicted at bottom of page  mgals

-3\theta 20'N 103\degree 30'W

37\degree 20'N 106\degree 30'W

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31.9 & 32.1 & 32.3 & 32.5 & 32.7 & 32.9 & 33.1 & 33.3 \\
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32.2 & 32.4 & 32.6 & 32.8 & 33.0 & 33.2 & 33.4 & 33.6 \\
32.3 & 32.5 & 32.7 & 32.9 & 33.1 & 33.3 & 33.5 & 33.7 \\
\end{array}
5' Squares Mean Elevations in meters
For the Geographic Area Depicted at bottom of page
.5' Mean Values for $L_1(\Delta g)$ According to Formula (33)
in mgals/meter

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5' Mean Values for $L_{1}[L_{1}(\Delta g)]$ According to Formula (39)

in mgals/meter

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104°50'W 39°N

38°40'N 4.7 4.5

105°10'W
5' Mean Values for $g_2$ According to Formula (39) mgals

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<td>5.0</td>
<td>7.0</td>
<td>104°50'W</td>
</tr>
</tbody>
</table>

$105°10'W$
Appendix D

Derivation of Fischer's Formula

The deflection computation in terms of spherical polar coordinates $\psi$ and $\alpha$ as shown in Heiskanen-Moriz (1967, p114) is:

$$
\xi = \frac{1}{4\pi G} \int_{\alpha=0}^{\pi} \int_{\psi=0}^{\pi} \Delta g(\psi, \alpha) \begin{cases} \cos \alpha \\ \sin \alpha \end{cases} \frac{dS}{d\psi} \sin \psi \, d\psi \, d\alpha ; (D1)
$$

[where the variables and constants are as before in Formula (15)]

Integrating within the limits of 5 to 20 kms Fischer does not use the exact value of the Vening-Meinesz Function but uses the approximation as explained by Brovar (1961, p301); $\cos (\psi/2) = 1$ and $\sin (\psi/2) = 0$, which mean:

$$
\sin \psi \left(\frac{dS}{d\psi}\right) = - [\csc (\psi/2) + 3]. \quad (D2)
$$

We then arrive at the following formulae found in Fischer (1966b, p4911) by substituting in Formula (D2) $\sin (\psi/2) = s/2R$ and multiplying by the approximation $dS/d\psi = -(2R/s)$, which holds for small values of $\psi$. (Fischer (Ibid.) uses it out to 20 kms and discussed the possibility of a 150 km radius maximum) Next replace $(dS/d\psi)\sin \psi$ in Formula (D1), which after appropriately changing integration limits, results in the following:

$$
\xi = \frac{\cos c}{2\pi G} \int_{\alpha=0}^{\pi} \int_{s=1}^{s(2)} \Delta g[(1/s) + (3/2R) + \ldots] \frac{dS d\alpha \cos \alpha}{\sin \alpha}; (D3)
$$

The deflection solution for section III is formed from the deflection solution of section I by interchanging $a$ and $b$ and $\xi$ and $\eta$. Fischer also states that the second term in the Vening-
Meinesz Function [Formula (15)] may be neglected. Thus for $\Delta g = 1$ mgal:

$$\frac{\xi}{\Omega} = \frac{\csc l''}{2\pi G} \left\{ \int \frac{ds}{s} d\alpha \left\{ \frac{\cos \alpha}{\sin \alpha} \right\} + \int \frac{d\alpha}{\ln s} \right\}.$$  \hspace{1cm} (D4)

For section I the azimuth varies from 0 to $\arctan b/2a$ and then from $\arctan b/a$ to $\arctan b/2a$. For any $\alpha$, the lower limit of $s$ is $a / \cos \alpha$. The upper limit is $2a / \cos \alpha$; if $\alpha = \arctan b/2a$, or $b / \sin \alpha$, if $\alpha = \arctan b/2a$ (see Figure 13 and Fischer, 1966b, p. 270). By integrating once and substituting we may obtain (Ibid.):

$$\arctan(b/2a) \quad \arctan(b/a)$$

$$\frac{\xi}{\Omega} = \frac{\csc l''}{2\pi G} \left\{ \int \frac{\cos \alpha \ln 2}{\cos \alpha} \cos \alpha \ln \left( \frac{b}{a} \cot \alpha \right) d\alpha \right\}. \hspace{1cm} (D5)$$

Now in order to obtain the completely integrated form let:

$$dv = \cos \alpha d\alpha, \quad v = \sin \alpha, \quad u = \ln \left( \frac{b}{a} \cot \alpha \right), \quad du = - \cos \alpha \sin \alpha d\alpha.$$  \hspace{1cm} (D6)

$$\arctan(b/2a) \quad \arctan(b/a) \quad \arctan(b/2a) \quad \arctan(b/2a)$$

Now in order to obtain the completely integrated form let:

$$dv = \cos \alpha d\alpha, \quad v = \sin \alpha, \quad u = \ln \left( \frac{b}{a} \cot \alpha \right), \quad du = - \cos \alpha \sin \alpha d\alpha.$$  \hspace{1cm} (D6)

$$\arctan(b/2a) \quad \arctan(b/a) \quad \arctan(b/a) \quad \arctan(b/2a)$$

where:

$$\arctan(b/2a) \quad \arctan(b/a)$$

$$\int \frac{d\alpha}{\cos \alpha} = \ln \left( \sec \alpha + \tan \alpha \right). \hspace{1cm} (D7)$$

Since the sum of the first two terms of Formula (D6) approaches zero, we are left to evaluate only Formula (D7) where:

$$\tan \alpha = \frac{b}{2a}, \quad \sec \alpha = \sqrt{1 + \left( \frac{b}{2a} \right)^2}; \hspace{1cm} (D8)$$

$$\tan \alpha = \frac{b}{a}, \quad \sec \alpha = \sqrt{1 + \left( \frac{b}{a} \right)^2}; \hspace{1cm} (D9)$$

hence:

$$\frac{\xi}{\Omega} = \frac{\csc l''}{2\pi G} \ln \left[ \sqrt{1 + \left( \frac{b}{a} \right)^2} + \left( \frac{b}{a} \right) \right] / \left[ \sqrt{1 + \left( \frac{b}{2a} \right)^2} - \left( \frac{b}{2a} \right) \right].$$

Then by multiplying both numerator and denominator by $\left[ \sqrt{1 + \left( \frac{b}{2a} \right)^2} - \left( \frac{b}{2a} \right) \right]$ , we obtain Formula (83) and the others follow in a similar manner.
Appendix E

Suggested Computer Techniques

In order to reduce the job time required on the computer, tests were run to establish the preciseness to which the distances need be calculated. The distances required in the correction term computations are: the chord distance \( l_c \); the distance from the computation point to the moving point \( l_m \); and the distance from the center of mass of the earth to the computation point and moving points \( R_0 \) or \( r \). The elevation differences \( h_1 - h_p \) may usually be excluded when computing the distances between the squares greater than 37°5. The elevation differences must usually be included when computing the distances using the 37°5 or smaller squares or in the unusual case where the relief change is drastic. Omitting the elevation difference when computing the distance (using \( l_0 \) instead of \( l_m \)) changed the correction term results less than 0.1 mgals for the 5' and larger blocks and resulted in a significant saving of computer time on the order of 1.04 seconds per correction. The total job time for each set of corrections was 59 minutes for the 37°5 blocks and 24.2 minutes for the 5' blocks. The formula used in computing the distance for the 7°5 and 37°5 blocks is the equation for the planar distance:

\[
l = \sqrt{l_0^2 + (h_1 - h_p)^2}.
\]

A major computational problem involved in the use of the second and third order correction terms as well as the Iterative Solution is the requirement to store four sets of data: \( \Delta g \)'s, \( g \)'s, and \( g' \)'s and the mean elevations for all of the various square sizes for the Nonlinear Solution as well as \( \Delta g \), and the \( i \)th and \( (i-1) \)th results and elevations for the Iterative Solution for all square sizes. This
storage was accomplished by the use of external storage (tapes and a disc) upon which all of the variable values for each square were recorded in a certain sequence. Then the program referred to the appropriate storage locations during the computation for retrieval purposes. Initially the anomaly, elevations and coordinates of each square were stored with files reserved for the correction terms. The correction terms were stored in core until the next iteration at which time they were transferred to the disc. At the end of a run the results were transferred to tape storage from the disc. The computer program itself was stored on standard data processing cards.
Bibliography


Brovarev, V. V. *The Theory of the Figure of the Earth*. Translated by Foreign Technology Division, Wright-Patterson AFB, Ohio. FTD-TT 64-930, November 27, 1964.


Cassinis, Dore, and Bullarin. Tavole Fondamentali per la Riduzione Dei Valori Osservati Della Gravita. R. Politechnico Milano, Pubblicazioni dell' Instututuo di Topografia e Geodesia, Pavia, 1937.


. Successive Approximation of Solutions of Molodensky's Basic Integral Equation, Reports of the Department of Geodetic Science, Ohio State University, August 1967b.


________. Linear Solution of Molodensky's Problem. Report to the International Association of Geodesy, Vienna, March 1967.


Pellinen, L. "Investigation of the Plumb Line Deflection and Deviation of the Quasigeoidal Figure in the Caucasus," Transactions, Central Scientific Research Institute of Geodesy, Aerial Survey, and Cartography, No. 86, Moscow, 1951.


Pick, M. "About Some Results in the Czechoslovak Test Area," Proceedings, International Symposium for the Figure of the Earth and Refraction, Vienna, March 1967.

_________. "The Figure of the Earth in the West Alps," Studia Geophysica et Geodaetica, Vol. 12, Prague, 1968a.

_________. Personal written communication dated May 12, 1968b.


Personal written communication dated January 3, 1968 from USC&GS.


