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STRESS WAVE PROPAGATION IN A CURVED TRANSMISSION LINE

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Nelson Tsaichuang Ma, B.S., M.S.

* * * * *

The Ohio State University
1972

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INTRODUCTION

Developments in ultrasonic and power sonic applications in industry have lead to several problems involving vibrations and wave propagation in curved rings and plates. For power sonic applications, the two most important items of equipment are the transducer and the transmission line. The transducer is a device which converts electrical input energy into vibratory mechanical energy. It consists of piezoelectric ceramic rings sandwiched in a resonant horn assembly, vibrating at frequencies in the range of 10 to 40 Khz. The horn, which is an energy concentrator, is in turn connected to a transmission line. The transmission line is a device which is used to transmit mechanical vibratory energy from the tip of the transducer to a receiver or load. It usually is made of the same material as the transducer, such as steel or aluminum, and is in the form of a straight or curved rod or plate. In order to have the whole system work efficiently, it is necessary to have the resonant frequency of the transducer match the resonant frequency of the transmission line. Under these conditions, the transmission line can transmit a large part of the mechanical energy to the work surface. As a consequence, the rational design and optimum use of transmission lines, be they straight or curved, requires an accurate governing theory for describing their vibration and wave propagation characteristics.
If the transmission lines are straight, either in the form of rods or plates, there is no particular problem in this respect and the theory has been developed by numerous authors. In the exact theory approach, the characteristic equations have been derived for the rod by Pochhammer [1]* and Chree [2] and for the plate by Rayleigh [3] and Lamb [4,5] from the general theory of elasticity. The resulting frequency equations are so complex that only in recent years have the detailed frequency spectrums been obtained [6]. This difficulty has given rise to a number of intermediate approximate theories. Approximate theories have been developed by Timoshenko [7,8], Mindlin and McNiven [9], Onoe, McNiven and Mindlin [10], Mindlin and Herrmann [11] for the rod, and Kane and Mindlin [12] and Mindlin and Medick [13] for the plate. Detailed accounts of these and other developments in the theory of elastic rods and plates have been given by Miklowitz [14] and Meeker and Meitzler [15].

On the other hand, the vibration and wave propagation characteristics of curved transmission lines are not as well established as for straight lines. It is extremely difficult to derive and evaluate the dispersion equation for a curved wave guide by exact theory and in fact only few approximate theories are available such as those of Graff [17] and Morely [16]. The complicating effect of curvature alters the response of such transmission lines from that of the straight counterparts. The physical basis for the more complicated behavior of curved lines lies mainly in the coupling between flexural and longitudinal

*Number in brackets designates reference at end of paper.
motions of the line. In the case of straight lines, there is no tendency for such coupling; in curved ones, it is unavoidable.

While the physical basis for the more complicated behavior of curved lines is quite clear, the mathematical description of their vibration and wave propagation characteristics has been only partially established. It is the object of this study to develop both an exact and an approximate theory for harmonic stress propagation in a curved transmission line. The basic configuration studied was that of a cylindrical shell under conditions of plane strain. Although the vibrations for hollow cylinders have been investigated by Gazis [18] and McNiven, Shah, and Sackman [20], their attention has been mainly focused on the axially symmetric vibrations along the longitudinal axis of the cylinder. In the present case, the curved elastic plate, or shell, is assumed to have infinite extent in the axial direction with the stress wave propagating in the circumferential direction. Under these conditions, both longitudinal and flexural waves may propagate along the circumferential direction. In addition, the SH wave (shear horizontal wave) may propagate along the circumferential direction with a pure shear motion in the z-direction only.

Utilizing exact theory, a frequency equation has been obtained, within the framework of the linear theory of elasticity, for the appropriate free surface boundary conditions. This equation, while rather complicated, was numerically evaluated by the use of the digital computer. In this manner the frequency spectrum has been determined for several values of curvatures and Poisson's ratio.
As we have mentioned before, it is quite difficult to derive and evaluate a characteristic equation for a curved elastic plate by the exact theory. Therefore, a relatively simple approximate theory is desirable for describing the characteristics of sonic transmission lines. The second chapter in this study presents the development of an approximate theory which involves three selected modes of propagation. This theory can be used to predict a relationship between angular frequency and angular wave number.

For the approximate theory, the one-dimensional governing equations are deduced from the equations of elasticity by a procedure based on the series expansion methods of Poisson [26] and Cauchy [27], and the integral method of Kirchhoff [28]. The procedure involves expansion of displacement in a series of orthogonal polynomials of the radial coordinate, followed by truncating the series and retaining only the desired modes. There follows an integration across the thickness which converts the elasticity equations into the approximate one-dimensional equations of motion. The procedure, which is analogous to Poisson's series, serves to uncouple the modes retained from the higher order modes without seriously affecting the behavior of the lower ones. Adjustments are made to improve the match between the approximate frequency spectrum of the curved plate with the exact spectrum. This is accomplished by the introduction of adjustment factors analogous to the shear coefficient used in Timoshenko beam theory [29]. The adjustment factors are determined in such a way that the behavior of the three selected modes are reproduced at long wavelength. The final
results of the investigation shows an excellent agreement in the fre-
quency spectrum of the exact and the approximate theories.
CHAPTER 1

EXACT THEORY

1-1. FORMULATION OF THE PROBLEM

A curved elastic plate in a cylindrical coordinate \((r, \theta, z)\) system with mean radius \(R\) and thickness \(H\), as shown in Figure 1, is used in the present study. The governing differential equations are based on the three-dimensional theory of elasticity. Three types of waves are considered here; the longitudinal and flexural waves are propagating along the circumferential direction (\(\theta\)-direction) of the curved plate and the SH wave (shear horizontal wave) is also propagating the circumferential direction of the curved plate but its particle motion is along the generatrices (\(z\)-direction) of the curved surface. The derivation of the frequency equation for wave propagation in the plate is based on the following assumptions:

a. The material of the curved plate is elastic, isotropic, and homogeneous.

b. The strain-displacement relations are linear; i.e., the linear theory of elasticity is adopted.

c. The variation of stress, strain, and displacement components along the axial direction \(z\) vanishes; i.e., \(\frac{\partial}{\partial z} = 0\).

d. In the normal application of a curved transmission line, the ratio of the mean radius to the plate thickness is
relatively large, therefore, it is assumed that R/H is large (i.e., R/H \gg 0.5). It is also assumed the range of angular interval \( \theta \) is large; i.e., \(-\infty < \theta < \infty\). In other words, a transmission line in the shape of a spiral is allowed.
Figure 1. Reference coordinates and dimensions
1-2. F R E Q U E N C Y  E Q U A T I O N *

From the previous assumption (a), the three dimensional displacement equations of motion (i.e., Navier's equation) governing the small elastic motions of an isotropic elastic medium can be written as

\[ \mu \nabla^2 \vec{u} + (\lambda + 2\mu) \nabla (\nabla \cdot \vec{u}) = \rho \frac{\partial^2 \vec{u}}{\partial t^2} \]

(1)

where \( \vec{u} \) is the displacement vector, \( t \) is the time variable, \( \lambda \) and \( \mu \) are the two Lamé's constants required to specify the elastic properties of the medium, and \( \nabla^2 \) is Laplace's operator. Body forces are assumed to be absent.

The displacement vector \( \vec{u} \) can be represented in terms of Lamé potential \([18]\), that is a dilational scalar potential \( \Phi \) and an equal-voluminal vector potential \( \vec{H} \) defined by

\[ \vec{u} = \nabla \Phi + \nabla \times \vec{H} \]

(2)

with

\[ \nabla \cdot \vec{H} = G(\vec{r}, t) \]

(3)

In (3) \( G \) is a function of the spatial vector \( \vec{r} \) and the time \( t \), which can be chosen arbitrarily due to the Gauge Invariance (also

*The derivation of this section follows closely Gazis \([21]\) and Lamb \([5]\).
called Gradient Invariance) of the field transformation described by (2).
The various governing equations are set in the cylindrical coordinate
system (r, θ, z) as shown in Figure 1.

Substituting the displacement expression (2) into (1), we
obtain the scalar and vector wave equations,

\[ \nabla^2 \Phi = \frac{1}{C_1^2} \dddot{\Phi}, \quad \nabla^2 \mathbf{H} = \frac{1}{C_2^2} \dddot{\mathbf{H}} \tag{4} \]

where the dot notation represents differentiation with respect to
time and where \( C_1, C_2 \) are the dilatational and shear wave velocities
respectively, given by

\[ C_1^2 = \frac{\lambda + 2\mu}{\mu}, \quad C_2^2 = \frac{\mu}{\mu} \tag{5} \]

In cylindrical coordinate form, (4) is given as

\[ \left( \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \dddot{\Phi} = \frac{1}{C_1^2} \dddot{\Phi} \]

\[ \left( \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \dddot{H}_r - \frac{H_r}{r^2} - \frac{2}{r^2} \frac{\partial H_\theta}{\partial \theta} = \frac{1}{C_2^2} \dddot{H}_r \]

\[ \left( \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \dddot{H}_\theta + \frac{2}{r^2} \frac{\partial H_r}{\partial \theta} - \frac{H_\theta}{r^2} = \frac{1}{C_2^2} \dddot{H}_\theta \]

\[ \left( \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \dddot{H}_z = \frac{1}{C_2^2} \dddot{H}_z \]
We now consider the conditions under which free harmonic waves may propagate in the curved plate. For this purpose, it is sufficient to consider solutions of the form

\[ \Phi = f_1(r) \cos(\gamma \theta - \omega t) \]
\[ H_r = f_2(r) \sin(\gamma \theta - \omega t) \]
\[ H_\theta = f_3(r) \cos(\gamma \theta - \omega t) \]
\[ H_z = f_4(r) \sin(\gamma \theta - \omega t) \]

where \( \gamma \) is the real wave number and \( \omega \) is the radial frequency.

Substituting relation (7) into the equations of motion (6), we obtain

\[ r^2 f_1'' + rf_1' + (K_2^2 r^2 - \gamma^2) f_1 = 0 \]
\[ r^2 f_2'' + rf_2' + (K_2^2 r^2 - \gamma^2) f_2 = 2 \gamma f_3 = 0 \]
\[ r^2 f_3'' + rf_3' + (K_2^2 r^2 - \gamma^2) f_3 + 2 \gamma f_2 = 0 \]
\[ r^2 f_4'' + rf_4' + (K_2^2 r^2 - \gamma^2) f_4 = 0 \]

where \( K_2^2 = \omega^2/C_1^2 \) and \( K_t^2 = \omega^2/C_2^2 \). The general solution of (8) is

\[ f_1 = A_1 J_r(K_2 r) + B_1 N_\gamma(K_2 r) \]
\[ 2g_1 = f_2 + f_3 = 2A_4 J_{\gamma-1}(K_2 r) + 2B_4 N_{\gamma-1}(K_2 r) \]
\[ 2g_2 = f_2 - f_3 = 2A_3 J_{\gamma+1}(K_2 r) + 2B_3 N_{\gamma+1}(K_2 r) \]
\[ f_4 = A_2 J_\gamma(K_2 r) + B_2 N_\gamma(K_2 r) \]
where $J(K_{\lambda, \lambda} r)$ is the $r$th-order Bessel Function of first kind, and $N(K_{\lambda, \lambda} r)$ is the $r$th-order Neumann Function, also known as the Bessel Function of the second kind.

The property of the Gauge Invariance can now be utilized in order to eliminate two of the integration constants entering (9). It may be shown that any one of the three potentials can be set equal to zero, without loss of generality of the solution. Physically, this implies that the displacement field corresponding to an equivoluminal potential $g_1$, $g_2$, and $f_4$ of (9) can also be derived by a combination of the other two equivoluminal potentials. This is seen to be true for the potential function (7),[18]. Thus, setting $g_1 = 0$ we obtain

$$f_2 = -f_3 = g_2$$

and (7) can be rewritten as

$$\Phi = (A_1 J_y(K_\lambda r) + B_1 N_y(K_\lambda r)) \cos(y \theta - \omega \tau)$$

$$H_r = (A_3 J_{y+1}(K_\lambda r) + B_3 N_{y+1}(K_\lambda r)) \sin(y \theta - \omega \tau)$$

$$H_\theta = -(A_3 J_{y+1}(K_\lambda r) + B_3 N_{y+1}(K_\lambda r)) \cos(y \theta - \omega \tau)$$

$$H_z = (A_2 J_y(K_\lambda r) + B_2 N_y(K_\lambda r)) \sin(y \theta - \omega \tau)$$

(10)
Consequently, the displacement field is then

\[ U_r = \frac{\partial \Phi}{\partial r} + \frac{1}{r} \frac{\partial H_r}{\partial \theta} = (f_1 + \frac{r}{r} f_4') \cos(\gamma \theta - \omega t) \]

\[ U_\theta = \frac{1}{r} \frac{\partial \Phi}{\partial \theta} - \frac{\partial H_r}{\partial r} = -(\frac{r}{r} f_1 + f_4') \sin(\gamma \theta - \omega t) \]

\[ U_z = \frac{1}{r} \frac{\partial \Phi}{\partial \theta} (r H_\theta) - \frac{1}{r} \frac{\partial H_r}{\partial \theta} \]

\[ = -(f_2' + \frac{1}{r} f_2 + \frac{1}{r} f_2') \cos(\gamma \theta - \omega t) \]  

(11)

where a prime denotes differentiation with respect to \( r \). The strain components are

\[ \varepsilon_r = \frac{\partial U_r}{\partial r} = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial H_r}{\partial \theta} - \frac{r}{r^2} \frac{\partial H_r}{\partial \theta} \]

\[ \varepsilon_\theta = \frac{1}{r} \frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r} = \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial H_r}{\partial \theta} - \frac{1}{r} \frac{\partial^2 H_r}{\partial \theta^2} \]

\[ \varepsilon_z = \frac{\partial U_z}{\partial z} = 0 \]

\[ \varepsilon_r \theta = \frac{1}{2} \left( \frac{1}{r} \frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_r}{r} \right) \]

\[ = \frac{1}{2} \left( \frac{2}{r} \frac{\partial^2 \Phi}{\partial r \partial \theta} - \frac{2}{r} \frac{\partial \Phi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 H_r}{\partial \theta^2} - \frac{\partial^2 H_r}{\partial \theta^2} + \frac{1}{r} \frac{\partial H_r}{\partial \theta} \right) \]  

(12)

\[ \varepsilon_\theta z = \frac{1}{2} \left( \frac{\partial U_\theta}{\partial z} + \frac{\partial U_z}{\partial \theta} \right) = \frac{1}{2} \left( \frac{1}{r} \frac{\partial^2 H_r}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial H_r}{\partial \theta} - \frac{1}{r^2} \frac{\partial H_r}{\partial \theta} \right) \]

\[ \varepsilon_r z = \frac{1}{2} \left( \frac{\partial U_z}{\partial r} + \frac{\partial U_r}{\partial z} \right) = \frac{1}{2} \left( \frac{\partial^2 H_r}{\partial z^2} + \frac{1}{r} \frac{\partial H_r}{\partial r} - \frac{H_r}{r^2} - \frac{1}{r} \frac{\partial^2 H_r}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial H_r}{\partial \theta} \right) \]
The stress components are given by

$$
\sigma_r = (\lambda + 2\mu) \varepsilon_r + \lambda \varepsilon_\theta
$$

$$
= (\lambda + 2\mu) \frac{\partial^2 \phi}{\partial r^2} + \frac{\lambda}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\lambda}{r} \frac{\partial \phi}{\partial \theta} + \frac{2\mu}{r} \frac{\partial^2 \psi}{\partial \theta \partial \varphi} + \frac{2\mu}{r^2} \frac{\partial \psi}{\partial \varphi}
$$

$$
\sigma_{r\theta} = \mu \varepsilon_{r\theta} = \frac{\mu}{2} \left( \frac{2}{r} \frac{\partial^2 \phi}{\partial \theta \partial \varphi} - \frac{2}{r^2} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{3}{r} \frac{\partial \psi}{\partial \varphi} - \frac{1}{r^2} \frac{\partial \psi}{\partial \theta} \right)
$$

$$
\sigma_{r\varphi} = \mu \varepsilon_{r\varphi} = \frac{\mu}{2} \left( \frac{3}{r} \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{\partial \psi}{r} - \frac{1}{r^2} \frac{\partial \psi}{\partial \theta} + \frac{1}{r^2} \frac{\partial \psi}{\partial \varphi} \right)
$$

Substituting (10) into (13) and utilizing (4), the stress components can be expressed as

$$
\sigma_r = \left\{ -\lambda K_2 f_1 + 2\mu \left[ f_1'' + \frac{\chi}{r} \left( f_4' - \frac{1}{r} f_4 \right) \right] \right\} \cos (r \theta - \omega t)
$$

$$
\sigma_{r\theta} = \frac{\mu}{2} \left\{ -\frac{2\chi f_1'}{r} + \frac{2\chi}{r^2} f_1 - \frac{\chi^2}{r^2} f_4 - f_4'' + \frac{1}{r} f_4' \right\} \sin (r \theta - \omega t)
$$

$$
\sigma_{r\varphi} = \frac{\mu}{2} \left\{ f_3'' + \frac{1}{r} f_3' - \frac{1}{r^2} f_3 - \frac{\chi}{r} f_2 + \frac{\chi}{r^2} f_2 \right\} \cos (r \theta - \omega t)
$$

The unknown coefficients $A_1, A_2, A_3, B_1, B_2$, and $B_3$ in (10) are determined by application of the boundary conditions assuming that the cylindrical surface is traction free. These condition are
Substituting (14) in (15) leads to six homogeneous equations in terms of the unknowns $A_1$, $A_2$, $A_3$, $B_1$, $B_2$, and $B_3$. The only nontrivial solutions to these equations are those for which the determinant of coefficients is equal to zero. This determinant is given by

$$|d_{ij}| = 0 \quad (i, j = 1 \text{ to } 6)$$

where $i$ indicates the row and $j$ the column of the determinant and

$$d_{11} = J^+(x_1) - 2\left(\frac{K^2}{K_2^2} - 1\right) J_Y(x_1)$$

$$d_{12} = N^+(x_1) - 2\left(\frac{K^2}{K_2^2} - 1\right) N_Y(x_1)$$

$$d_{13} = J^-(y_1)$$

$$d_{14} = N^-(y_1)$$
\[ d_{21} = J^+(x_2) - 2 \left( \frac{K_2^2}{K_1^2} \right) J_y(x_2) \]
\[ d_{22} = N^+(x_2) - 2 \left( \frac{K_2^2}{K_1^2} \right) N_y(x_2) \]
\[ d_{33} = J^-(y_2) \quad d_{24} = N^-(y_2) \]
\[ d_{31} = J^-(x_1) \quad d_{32} = N^-(x_1) \]
\[ d_{33} = J^+(y_1) \quad d_{34} = N^+(y_1) \]
\[ d_{41} = J^-(x_2) \quad d_{42} = N^-(x_2) \]
\[ d_{43} = J^-(y_2) \quad d_{44} = N^+(y_2) \]
\[ d_{55} = J_{y+1}(y_1) - J_{y-1}(y_1) \]
\[ d_{56} = N_{y+1}(y_1) - N_{y-1}(y_1) \]
\[ d_{66} = N_{y+1}(y_2) - N_{y-1}(y_2) \]
\[ d_{65} = J_{y+1}(y_2) - J_{y-1}(y_2) \]
\[ d_{15} = d_{16} = d_{25} = d_{26} = d_{35} = d_{36} = d_{45} = d_{46} = 0 \]
\[ d_{51} = d_{52} = d_{53} = d_{54} = d_{61} = d_{62} = d_{63} = d_{64} = 0 \]
where

\[ J^+(s) = J_{\gamma-2}(s) + J_{\gamma+2}(s) \]
\[ J^-(s) = J_{\gamma-2}(s) - J_{\gamma+2}(s) \]
\[ N^+(s) = N_{\gamma-2}(s) + N_{\gamma+2}(s) \]
\[ N^-(s) = N_{\gamma-2}(s) - N_{\gamma+2}(s) \]

\( \chi_1 = K_x a, \chi_2 = K_x b, \gamma_1 = K_x a, \gamma_2 = K_x b \)

The equation formed by expanding (16) is the frequency equation. If the elastic properties of the medium are specified in terms of Poisson's ratio, \( \nu^* \), rather than \( \lambda \) and \( \mu \), the frequency equation can be regarded as relating four independent variables. In dimensionless form these are the angular wave number (or propagation constant) \( \gamma \), the geometric parameter of the curved plate \( R/H \), Poisson's ratio \( \nu \) and the angular frequency \( \Omega = \omega / \omega_s \), where \( \omega_s = \pi c_s / H \) is the lowest simple thickness shear frequency of an infinite flat plate with thickness \( H \). The relationship between angular frequency \( \Omega \) and propagation constant \( \gamma \) is evaluated for a given value of \( \nu \) and \( R/H \).

The solutions to the frequency equation generally take the form of a series of continuous curves or branches, each branch representing the relationship of \( \gamma \) and \( \Omega \) for a given mode of propagation. These branches constitute the frequency spectrum of the modes in the curved plate. In terms of these branches, the dimensionless phase velocity is defined as \( \Omega / \gamma \) and the dimensionless group velocity as \( d \Omega / d \gamma \).

*Poisson's ratio expressed in terms of the Lame' constants is \( \nu = \frac{\lambda}{2(\lambda + \mu)} \)
As was indicated by Gazis in his analysis of wave propagation in hollow circular cylinders, the frequency equation (16) can be uncoupled into two subdeterminants as

\[ D_2 \cdot D_3 = 0 \]  

(19)

where

\[
D_2 = \begin{vmatrix}
  d_{11} & d_{12} & d_{13} & d_{14} \\
  d_{21} & d_{22} & d_{23} & d_{24} \\
  d_{31} & d_{32} & d_{33} & d_{34} \\
  d_{41} & d_{42} & d_{43} & d_{44}
\end{vmatrix}
\]  

(20)

\[
D_3 = \begin{vmatrix}
  d_{55} & d_{56} \\
  d_{65} & d_{66}
\end{vmatrix}
\]  

(21)

Obviously, (10) is satisfied if \( D_2 = 0 \), or \( D_3 = 0 \). For \( D_2 = 0 \), circumferential motion under plane-strain condition results. On the other hand, the case \( D_3 = 0 \) represents circumferential propagation of a pure shear wave having only \( u_z \) displacement (the so-called SH wave). The results of \( D_2 = 0 \) and \( D_3 = 0 \) can also be obtained individually by separately formulating the plane-strain problem and the SH wave problem. This uncoupling nature between a plane strain vibration and SH motion is also exhibited in the case of a flat elastic plate.

When the wave length is infinite (propagation constant \( \gamma \) is zero), the dilatational and equivoluminal modes become uncoupled. As the propagation constant approaches zero, these frequencies are called the cutoff frequencies of the appropriate extensional (or radial) or shear modes. The uncoupled determinants are
\[ D_4 \cdot D_5 = 0 \]  

where

\[
D_4 = \begin{vmatrix}
\left( \frac{K_4^2}{K_4} - 1 \right) J_0(x_4) - J_2(x_4) & \left( \frac{K_4^2}{K_4} - 1 \right) N_0(x_4) - N_2(x_4) \\
\left( \frac{K_4^2}{K_4} - 1 \right) J_0(x_2) - J_2(x_2) & \left( \frac{K_4^2}{K_4} - 1 \right) N_0(x_2) - N_2(x_2)
\end{vmatrix}
\]  

and

\[
D_5 = \begin{vmatrix}
J_{-2}(y_1) + J_2(y_1) & N_{-2}(y_1) + N_2(y_1) \\
J_{-2}(y_2) + J_2(y_2) & N_{-2}(y_2) + N_2(y_2)
\end{vmatrix}
\]

Equation (22) is satisfied if \( D_4 = 0 \), or \( D_5 = 0 \). It can easily be shown [19] that \( D_4 = 0 \) corresponds to extensional (or radial) modes. Similarly, \( D_5 = 0 \) corresponds to shear modes. In addition, it is seen that, at the cutoff frequency, shear modes are independent of changes in Poisson's ratio, but that extensional modes are not independent of such changes.
1-3 NUMERICAL RESULTS

§ Frequency Spectrum for a Flat Plate:

Because of the many similarities between wave propagation in curved plates and wave propagation in flat plates, especially whenever the R/H ratio becomes large for the curved plate case, it is desirable to discuss results for a flat plate to facilitate a comparison between these two cases.

Figure 2 is the frequency spectrum for SH waves in a flat elastic plate and Figure 3 is the spectrum for longitudinal and flexural waves of a flat plate with Poisson's ratio equal to 0.31. Figure 3 is based on Mindlin's work \[6, 23\], where he has used a method of determining the roots of the Rayleigh-Lamb equation which allows an approximate, but quite detailed, mode spectrum to be sketched rather easily without extensive numerical calculations. The symmetric modes (symmetric motion with respect to the middle surface of the flat plate), often simply referred to as the longitudinal modes, are designated by the symbols \( S(m) \), where \( m \) is an integer representing the order of the branches. All of the antisymmetric modes (antisymmetric motion with respect to the middle surface of the flat plate) are designated by the symbol \( AS(n) \), where \( n \), as \( m \) for the symmetric modes, represents the order of the branches. The AS modes are often simply called the flexural modes.

Several important observations of the SH wave spectrum and of the longitudinal and flexural wave spectrums were made by Meeker and Meitzler \[15\]. These results are summarized as follows:
For the SH wave spectrum:

1. The group velocity and phase velocity of the lowest symmetric mode SH(1) are independent of frequency and equal to $C_2$. This mode of propagation is therefore the only nondispersive SH mode.

2. The SH modes depend only upon one elastic constant, the shear modulus.

3. Figure 2 shows that at a given frequency there is only a finite number of solutions with real $\gamma H/2$ for SH wave motion, where $\gamma$ is the wave number for a flat plate with thickness $H$. Consequently, there is only a finite number of freely propagating SH modes.

4. For all the modes with the exception of the lowest symmetric mode SH(1), the curve is a hyperbola starting at the cutoff frequency and the slope of the curve is zero at the cutoff frequency. In Figure 2 the SH(1) mode is represented by a simple straight line passing through the origin and having a slope equal to $2/\pi$. 
Figure 2. Frequency spectrum of SH waves in an infinite flat plate

Figure 3. Frequency spectrum of longitudinal and flexural waves in an infinite flat plate having a Poisson's ratio of 0.31
For the longitudinal and flexural wave spectrum:

1. The Rayleigh velocity $V_R$ is the high-frequency limiting velocity for both the first longitudinal and first flexural modes in a plate.

2. For nonzero wave number $\gamma$, both the longitudinal and flexural modes couple together in the sense that two displacement components are nonzero and the total wave motion involves a combination of shear and dilatational wave motions.

3. At the cutoff frequency ($\gamma = 0$), the longitudinal and flexural modes of propagation reduce to either thickness shear motions or thickness extensional motions.

4. The longitudinal and flexural modes depend on two elastic constants.

5. Similar to the SH wave case, for a given frequency, there are a finite number of modes propagating in the plate with real $\gamma$.

§ Numerical Results for a Curved Plate:

The numerical evaluation of the roots of (16) is a problem of some difficulty unless a high speed digital computer is used. With the aid of an electronic computer and good Bessel function and Neumann function subroutines, the roots of $\Omega$ may be computed for fixed $\gamma$ or vice versa, for a given material and geometric parameters of the curved plate. The methods which are used to develop the subroutines for the Bessel and Neumann functions are based on the technique given by Goldstein and Thaler [24, 25], and the subroutines are a modification of SHARE, a program developed at New York University.
As was mentioned in the previous section, for given elastic constants and geometric dimensions of the curved plate, the frequency equation (16) constitutes an implicit transcendental function of angular wave number \( \gamma \) and nondimensional angular frequency \( \Omega \). The results of a numerical evaluation of the complete frequency spectrum was obtained for various sets of the physical parameters by means of an IBM 370 computer. The frequency spectrum \( \Omega \) versus \( \gamma \) was computed by an "interval halving" iteration technique and the range of parameters covered was as follows:

Poisson's ratio: \( \nu = 0.25 \) and \( \nu = 0.3 \)

Ratio of mean radius \( R \) to thickness \( H \), \( R/H = 10.0, 25.0, \) and 50.0

In Figures 4 and 5, normalized displacement components \( u_r \) and \( u_\theta \) are plotted as functions of the radial coordinate for specified value of \( R/H \) and \( \gamma \). Mode shapes are plotted for \( \gamma = 0, \gamma = 10, \) and \( R/H = 10.0 \), and Poisson's ratio = 0.3. It is seen that the displacement profiles for the shell resemble those of the flat plate. However, these displacements are not purely symmetric or anti-symmetric with respect to the middle plane of the curved plate.

A convenient means of labeling the various modes is defined as follows. All modes are designated by the symbol \( M(p) \), where \( p \) is an integer representing the order of the branches. Thus the first (or lowest) branch of the mode is designated by the symbol \( M(1) \). For SH waves, all modes are designated by the symbol \( SH(q) \), these q's, as for the longitudinal and flexural waves, represent the order of the branches.
Figure 4, Mode shapes for \( R/H = 10.0, \ \nu = 0.30, \ \text{and} \ \gamma = 0.0 \) (i.e., variation of normalized displacements along the thickness)
\( \Omega = 0.13260 \quad 0.53436 \quad 1.15624 \quad 1.77245 \quad 2.17968 \quad 2.99804 \quad 3.7955 \)

<table>
<thead>
<tr>
<th>Distance along thickness ( r = \frac{r_0}{H} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
</tr>
<tr>
<td>0.20</td>
</tr>
<tr>
<td>0.30</td>
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<tr>
<td>0.40</td>
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<tr>
<td>0.50</td>
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<td>0.80</td>
</tr>
<tr>
<td>0.90</td>
</tr>
<tr>
<td>1.00</td>
</tr>
</tbody>
</table>

Circumferential Component of Displacement \( \hat{U}_\theta \)

Radial Component of Displacement \( \hat{U}_r \)

Figure 5, Mode shapes for \( R/H = 10.0 \), \( \nu = 0.30 \), and \( \gamma = 10.0 \) (i.e., variation of normalized displacements along the thickness)
The results of the numerical calculations of the spectrum are shown in Figures 6 through 11. After comparing these figures with Figures 2 and 3, it is evident that there exists a great similarity between the wave characteristics of a curved plate and those of the flat plate. Some of the comparisons between the curved plate with the previous observations of the flat plate are as follows:

1. The SH(1) mode is the only nondispersive SH mode. The only differences between the flat plate case and the curved plate case are the values of the phase and group velocities. For example, the phase and group velocities for a flat plate are $2/\pi = 0.6366$, while for curved plate with $R/H = 50.0$ are 0.6329.

2. It can be seen in (21) that the SH wave for a curved plate also involves only one elastic constant, the shear modulus $\mu$, similar to the flat plate. Similarly, the longitudinal and flexural modes for the curved plate are seen from (22) to be dependent on two elastic constants, $\lambda$ and $\mu$, a behavior analogous to the flat plate.

3. For a given frequency, there are a finite number of propagating modes for SH waves and longitudinal-flexural waves in the curved plate.

4. For the SH wave, all the curves with the exception of the curve corresponding to the SH(1) mode are hyperbolas with horizontal tangents at $\gamma = 0$.

5. The pattern of the curves for the longitudinal-flexural waves for the curved plate are similar to the corresponding curves for the flat plate. Thus, comparing the cutoff frequencies of the two plates, it is
observed that not only do the orders of the vibration modes match each other but also the values of the cutoff frequencies agree excellently as the R/H ratio of the curved plate becomes large.

6. Furthermore, at the cutoff frequency $\gamma = 0$, the longitudinal and flexural modes are uncoupled into either extensional modes (thickness extensional motions) or shear modes (thickness shear motions), similar to the flat plate case. Therefore, it is very easy to match the corresponding modes between two plates.

7. As we have mentioned in the previous section, the sensitivity of the cutoff frequency for extensional modes $M(2)$, $M(4)$, and $M(7)$ to the change of Poisson's ratio of the curved plate is also observed here by comparing Figure 6 to Figure 9, Figure 7 to Figure 10, and Figure 8 to Figure 11. This phenomenon also exists for the flat plate case.

8. In addition, it is seen that for a given Poisson's ratio and propagation constant (or angular wave number) $\gamma$, the frequency for any specified mode increases substantially as the ratio of mean radius to plate thickness (R/H) decreases. In other words, for large R/H value (plates with large curvature or small thickness) the wavelength $\lambda$ is smaller and the propagation constant $\gamma$ is larger than the plate with smaller R/H value.

With the previous observations, it may be concluded that when the R/H ratio becomes larger, the behavior of the curved plate for wave propagation is closer to the flat plate. On the other hand, when the R/H ratio is not large, the pattern of the frequency spectrum for the curved plate is similar to that of the flat plate, but the frequency for any mode and any angular wave number is shifted.
Figure 6. Frequency spectrum of the SH modes, and the longitudinal flexural modes for a curved plate having a Poisson's ratio of 0.3 and $R/H = 10.0$
Figure 7. Frequency spectrum of the SH modes, and the longitudinal flexural modes for a curved plate having a Poisson's ratio of 0.3 and \( R/H = 25.0 \)
Figure 8. Frequency spectrum of the SH modes, and the longitudinal flexural modes for a curved plate having a Poisson's ratio of 0.3 and R/H = 50.0
Figure 9. Frequency spectrum of the SH modes, and the longitudinal flexural modes for a curved plate having a Poisson's ratio of 0.25 and $R/H = 10.0$.
Figure 10. Frequency spectrum of the SH modes, and the longitudinal flexural modes for a curved plate having a Poisson's ratio of 0.25 and R/H = 25.0.
Figure 11. Frequency spectrum of the SH modes, and the longitudinal flexural modes for a curved plate having a Poisson's ratio of 0.25 and R/H = 50.0
2-1. EXPANSION IN INFINITE SERIES

An approximate analysis will now be developed that will predict a relationship between frequency and angular wave number of three propagation modes. The following development is the beginning in the generation of a theory capable of accounting for $n$ modes. A curved elastic plate in a cylindrical coordinate $(r, \theta, z)$ system with mean radius $R$ and thickness $H$, as shown in Figure 1, is used in the present investigation. All assumptions which were indicated in the previous exact analysis are still applied to this approximate analysis*. For the plane strain case, the axial component of displacement $u_z$ is taken as zero and the radial, circumferential components of displacement are expressed in the infinite series expansion form as

\[
\begin{align*}
U_r &= \sum_{n=0}^{\infty} P_n(r) U_n(\theta, z) \\
U_\theta &= \sum_{n=0}^{\infty} q_n(r) U_n(\theta, z)
\end{align*}
\]  

*(25)

*See page 6.
where the polynomials $p_n(r)$ and $q_n(r)$ are functions of the radial coordinate $r$, and $u_n$ and $v_n$ are functions of the coordinate $\theta$ and the time $t$. The amplitudes of polynomial distributions of displacements across the thickness of the plate are $u_n$ and $v_n$.

The choice of polynomials $p_n(r)$ and $q_n(r)$ was determined by several important factors, including

1. Each term in the polynomial should be similar to a deformation mode which can exist during wave propagation in the curved plate. This requirement is necessary because the truncation of the expansion during the process will reduce the capability of the series to represent the exact displacement. With this requirement, the difficulty for matching the frequency spectrum between the approximate theory and the exact theory in the late section will be minimized.

2. The polynomials $p_n(r)$ and $q_n(r)$ must satisfy the following orthogonality conditions between inner radius $a$ and outer radius $b$ so that some uncoupling of the equations of motion will be possible.

\[
\int_{\text{vol}} p_m(r) p_n(r) \, dB = \int_{\alpha}^{b} \int_{\alpha}^{b} p_m(r) p_n(r) r \, dr \, d\theta \, dz = 0 \quad m \neq n
\]

\[
\int_{\text{vol}} q_m(r) q_n(r) \, dB = \int_{\alpha}^{b} \int_{\alpha}^{b} q_m(r) q_n(r) r \, dr \, d\theta \, dz = 0 \quad m \neq n
\]

where $dB = r \, dr \, d\theta \, dz$ represents a unit volume of the body.

In order to be able to find a set of polynomials $p_n(r)$ and $q_n(r)$ that satisfy these two requirements, a large number of polynomials were considered. For example, the Legendre and Jacobi polynomials
satisfy the first requirement, but they do not satisfy the second requirement of orthogonality between inner radius $a$ and outer radius $b$. After much exploration, two polynomials $p_n(r)$ and $q_n(r)$ were selected as follows

$$p_0(r) = 1$$

$$p_1(r) = 1 - A_{11} r^2$$

$$p_n(r) = 1 + \sum_{k=1}^{n} (-1)^k A_{nk} r^{2k}$$

$$q_0(r) = \frac{r}{a}$$

$$q_1(r) = \frac{r}{a} - \frac{B_{11}}{a} r^3$$

$$q_2(r) = \frac{r}{a} - \frac{B_{21}}{a} r^3 + \frac{B_{22}}{a} r^5$$

$$q_{nk}(r) = \frac{r}{a} + \sum_{k=1}^{n} (-1)^k \frac{B_{nk}}{a} r^{2k+1}$$

The coefficients $B_{nk}$ and $A_{nk}$ are given as

$$A_{11} = \frac{2}{b^2 + a^2}$$

$$B_{11} = \frac{3(b^2 + a^2)}{2(b^2 + b^2a^2 + a^4)}$$

$$B_{21} = \frac{24(b^4 - a^4)(b^{10} - a^{10}) - 20(b^6 - a^6)(b^8 - a^8)}{16(b^6 - a^6)(b^{10} - a^{10}) - 15(b^8 - a^8)^2}$$

$$B_{22} = \frac{10[9(b^4 - a^4)(b^8 - a^8) - 8(b^6 - a^6)^2]}{3[16(b^4 - a^4)(b^{10} - a^{10}) - 15(b^8 - a^8)^2]}$$
In this approximate analysis, we are interested in the three modes, $M(2)$, $M(4)$, and $M(5)$, which are essentially associated with the transmission of longitudinal mechanical energy in a curve plate. Considering each term in the power series expansion and retaining only the terms which correspond to $M(2)$, $M(4)$, and $M(5)$ modes gives displacement relations of the form:

$$U_r = \varphi_1(r) \ U_1(\theta, \kappa) = (1 - A_{11} r^3) \ U_1(\theta, \kappa)$$

$$U_\theta = q_0(r) \ U_0(\theta, \kappa) + q_2(r) \ U_2(\theta, \kappa)$$

$$= \frac{r}{a} \ U_0(\theta, \kappa) + \left( \frac{r}{a} - \frac{B_{31}}{a} r^3 + \frac{B_{22}}{a} r^5 \right) \ U_2(\theta, \kappa)$$

Figure 12 shows the mode shape of $p_1(r)$, $q_0(r)$, and $q_2(r)$ in the expansion.

Figure 12. Displacement distributions for $p_1(r)$, $q_0(r)$ and $q_2(r)$.
§ STRESS EQUATION OF MOTION

To derive the governing differential equations, we apply Hamilton's Principle. In the absence of body forces and surface tractions, the principle appears in variational form as

$$\delta \int_{t_1}^{t_2} (T - V) \, dt = \int_{t_1}^{t_2} (\delta T - \delta V) \, dt = 0$$

where $t_1, t_2$ are end points of time $t$, $T$ is the kinetic energy, $V$ is the internal strain energy.

The internal strain energy is expressed as

$$V = \int_{\text{vol}} (\sigma_r \varepsilon_r + \sigma_\theta \varepsilon_\theta + 2 \sigma_{r\theta} \varepsilon_{r\theta}) \, r \, dr \, d\theta \, dz$$

where $\sigma_r$, $\sigma_\theta$, and $\sigma_{r\theta}$ are components of stress derivable from a strain-energy function $U$ and $\varepsilon_r$, $\varepsilon_\theta$, and $\varepsilon_{r\theta}$ are components of strain.
The kinetic energy of the curved plate is expressed as

\[ T = \frac{g}{2} \int_{\text{Vol}} \left[ \left( \frac{\partial u_r}{\partial r} \right)^2 + \left( \frac{\partial u_\theta}{\partial r} \right)^2 \right] r \, dr \, d\theta \, dz \]  \hspace{1cm} (32)

where \( g \) is the density of the curved plate and \( t \) is the time variable.

After using the strain-displacement relations, \( \delta T \) and \( \delta V \) can be expressed as

\[ \delta T = -g \int_{\text{Vol}} \left[ \frac{\partial^2 u_r}{\partial r^2} \delta u_r + \frac{\partial^2 u_\theta}{\partial r^2} \delta u_\theta \right] r \, dr \, d\theta \, dz \]  \hspace{1cm} (33)

\[ \delta V = -\int_{\text{Vol}} \left[ \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} \right] \delta u_r \, r \, dr \, d\theta \, dz 
\] 
\[ -\int_{\text{Vol}} \left[ \frac{\partial \sigma_\theta}{\partial r} + \frac{1}{r} \frac{\partial \sigma_r}{\partial \theta} + 2 \frac{\sigma_r - \sigma_\theta}{r} \right] \delta u_\theta \, r \, dr \, d\theta \, dz \] 
\[ + \int_{S} \sigma_r \, r \, \delta u_r |_{\theta=0} \, d\theta \, dz + \int_{S} \sigma_\theta \, r \, \delta u_\theta |_{\theta=0} \, d\theta \, dz \] 
\[ + \int_{S} \sigma_\theta \, \delta u_\theta |_{\theta=1} \, d\theta \, dz + \int_{S} \sigma_r \, \delta u_r |_{\theta=1} \, d\theta \, dz \]  \hspace{1cm} (34)
The last two terms in (34) represent the boundary condition for a finite curved plate. However, in the present study we have already assumed that there are no waves reflected from the edge of the boundary, it is therefore justified to assume that the last two terms vanish. By substituting (29) into (33) and (34), performing the integrations with respect to \( r \) and in turn substituting into (30), we obtain the following relations.

\[
\int_{\lambda_1}^{\lambda_2} \int_{S} \left\{ \left[ r(1 - A_{11} r^2) \sigma_r|_{\alpha}^b + \frac{\partial E_{11}}{\partial \theta} + E_{12} - \frac{\partial^3 U_1}{\partial \theta^3} \right] \delta U_1 
+ \left[ \frac{r^2}{a} \sigma_{r0}|_{\alpha}^b + \frac{1}{a} \frac{\partial E_{21}}{\partial \theta} - \frac{\sigma}{a} C_2 \frac{\partial^3 U_0}{\partial \theta^3} \right] \delta U_0 
+ \left[ \frac{1}{a} (1 - B_{11} r^2 + B_{22} r^4) r^2 \sigma_{r0}|_{\alpha}^b + \frac{1}{a} \frac{\partial E_{31}}{\partial \theta} + \frac{E_{32}}{a} \right. 
- \frac{\sigma}{a} C_3 \frac{\partial^3 U_1}{\partial \theta^3} \right\} \delta U_2 
- \left[ \frac{r^2}{a} \sigma_{r0}|_{\alpha}^b \right] \delta U_0 
- \left[ \frac{1}{a} (1 - B_{21} r^2 + B_{32} r^4) r^2 \sigma_{r0}|_{\alpha}^b \right] \delta U_2 
\}
\, d\theta \, dz \, d\lambda = 0
\]
where the components of the stress resultants are

\[ E_{11} = \int_{a}^{b} (1 - A_{11} r^2) \sigma_{r\theta} \, dr \]

\[ E_{12} = \int_{a}^{b} A_{11} (2 \sigma_r + \sigma_{\theta}) r^3 \, dr - \int_{a}^{b} \sigma_{\theta} \, dr \]

\[ E_{21} = \int_{a}^{b} r \sigma_{\theta} \, dr \]

\[ E_{31} = \int_{a}^{b} (1 - B_{21} r^3 + B_{22} r^4) r \sigma_{\theta} \, dr \]

\[ E_{32} = \int_{a}^{b} 2 (B_{21} - 2 B_{22} r^2) \sigma_{r\theta} r^3 \, dr \]

and \( c_1, c_2, c_3 \) are given by

\[ c_1 = \frac{1}{2} (b^2 - a^2) - \frac{1}{2} A_{11} (b^4 - a^4) + \frac{1}{6} A_{11} (b^6 - a^6) \]

\[ c_2 = \frac{1}{4} a (b^4 - a^4) \]

\[ c_3 = \frac{1}{a} \left[ \frac{1}{4} a (b^4 - a^4) - \frac{1}{3} B_{21} (b^6 - a^6) + \frac{1}{6} (2 B_{22} + B_{21}^2) (b^8 - a^8) \right. \]

\[ - \left. \frac{1}{5} B_{21} B_{22} (b^{10} - a^{10}) + \frac{1}{12} B_{22}^2 (b^{12} - a^{12}) \right] \]

It is easily seen that the terms associated with the boundary conditions for the inner radius \( a \) and outer radius \( b \) are cancelled. After setting the coefficients of \( \delta u_1, \delta v_0, \) and \( \delta v_2 \) equal to zero,
we obtain three stress equations of motion involving the stress-resultant forces.

\[ \frac{\partial E_{11}}{\partial \theta} + E_{12} = 3 \zeta c_1 \frac{\partial^3 u_1}{\partial \theta^2 \partial r^2} \]

\[ \frac{\partial E_{31}}{\partial \theta} = 3 \zeta c_2 \frac{\partial^3 u_0}{\partial \theta^2 \partial r^2} \] (38)

\[ \frac{\partial E_{31}}{\partial \theta} + E_{32} = 3 \zeta c_3 \frac{\partial^3 u_2}{\partial \theta^2 \partial r^2} \]

§ COMPONENTS OF STRAINS

By substituting the displacement expression (29) into the usual strain-displacement relations, we obtain

\[ \varepsilon_r = \frac{\partial u_r}{\partial r} = -2A_{11} \frac{\partial u_1}{\partial r} \]

\[ \varepsilon_\theta = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{i}{a} \left[ \frac{u_0'}{(1 - B_{21} r^2 + B_{22} r^4)} + \frac{1}{r} \left( 1 - A_{11} r^2 \right) u_1 \right] \] (39)

\[ \varepsilon_{r\theta} = \frac{i}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \]

\[ = \frac{i}{2} \left[ \frac{1}{r} \left( 1 - A_{11} r^2 \right) u_1' - \frac{2}{a} \left( B_{21} r^2 - 2B_{22} r^4 \right) u_2 \right] \]

where prime indicates differentiation with respect to \( \theta \).
§ ENERGY DENSITY

In three-dimensional elasticity theory, the strain energy density is

$$ U = \frac{1}{2}(\sigma_r \varepsilon_r + \sigma_\theta \varepsilon_\theta + 2 \sigma_\phi \varepsilon_\phi) \quad (40) $$

We define an energy density per unit length

$$ \bar{U} = \int_a^b U_r \, dr \quad (41) $$

Substituting the strain expression (39) into (40) and in turn substituting into (41) we obtain, after carrying out the integrations,

$$ 2 \bar{U} = S_{12} E_{12} + S_{11} E_{11} + S_{21} E_{21} + S_{31} E_{31} + S_{32} E_{32} \quad (42) $$
where the modified strains are defined by

\[
\begin{align*}
S_{12} &= -u_1, \\
S_{11} &= u_1', \\
S_{21} &= v_{/a}' , \\
S_{32} &= -v_{/a}' , \\
S_{31} &= v_{/a}' 
\end{align*}
\]  \( (43) \)

From (39), the strain components, expressed in terms of the modified strain, are

\[
\begin{align*}
\epsilon_r &= 2A_{11} r S_{12} \\
\epsilon_\theta &= S_{21} + (1 - B_{21} r^2 + B_{22} r^4) S_{31} - \frac{1}{r} (1 - A_{11} r^2) S_{12} \quad (44) \\
\epsilon_{r\theta} &= \frac{1}{2} \left[ \frac{1}{r} (1 - A_{11} r^2) S_{11} + 2 (B_{21} r^2 - 2 B_{22} r^4) S_{32} \right]
\end{align*}
\]

Similarly, in ordinary three-dimensional theory, the kinetic energy density is

\[
K = \frac{\rho}{2} \left[ \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_0}{\partial x_1} \right)^2 \right] \quad (45)
\]
We define a kinetic energy density per unit length

\[ \bar{K} = \int_a^b K r \, dr = \frac{\rho}{2} \left[ c_1 \left( \frac{\partial u}{\partial x} \right)^2 + c_2 \left( \frac{\partial v}{\partial x} \right)^2 + c_3 \left( \frac{\partial w}{\partial x} \right)^2 \right] \]  \hspace{2cm} (46)

where \( c_1, c_2, \text{ and } c_3 \) are defined in (37)

§ STRESS-STRAIN RELATIONS

Substituting the strain expression from (44) into the stress strain relations

\[ \sigma_r = \frac{\partial U}{\partial \varepsilon_r} = (\lambda + 2\mu) \varepsilon_r + \lambda \varepsilon_\theta \]  \hspace{2cm} (47)

\[ \sigma_\theta = \frac{\partial U}{\partial \varepsilon_\theta} = (\lambda + 2\mu) \varepsilon_\theta + \lambda \varepsilon_r \]

\[ \sigma_{r\theta} = 2\mu \varepsilon_{r\theta} \]

where \( \lambda \) and \( \mu \) are Lame' constants, we obtain the following relations between the stresses and strains.
\[ \sigma_r = \left( -\frac{\lambda}{r^2} + 3\lambda A_{11} r + 4\mu A_{11} r \right) S_{12} + \lambda S_{21} \]

\[ + \lambda (1 - B_{21} r^2 + B_{22} r^4) S_{31} \]

\[ \sigma_\theta = \left[ (3\lambda + 2\mu) A_{11} r - \frac{1}{r} (\lambda + 2\mu) \right] S_{12} + (\lambda + 2\mu) S_{21} \]

\[ + (\lambda + 2\mu) (1 - B_{21} r^2 + B_{22} r^4) S_{31} \]

\[ \sigma_\phi = \mu \left[ \frac{1}{r} (1 - A_{11} r^2) S_{11} + 2 (B_{21} r^2 - 2B_{22} r^4) S_{32} \right] \]

Substituting the stress expression in (48) into the previously defined components of the stress resultants, we obtain the following relations between the components of the stress resultants and the modified strains

\[ E_{11} = T_1 S_{11} + T_2 S_{32} \]

\[ E_{12} = T_3 S_{12} + T_4 S_{21} + T_5 S_{31} \]

\[ E_{21} = T_4 S_{12} + T_6 S_{21} + T_7 S_{31} \]

\[ E_{31} = T_5 S_{12} + T_7 S_{21} + T_8 S_{31} \]

\[ E_{32} = T_2 S_{11} + T_9 S_{32} \]
Where

\[ T_1 = \mu \left[ \log \frac{b}{a} - A_{11}(b^3 - a^3) + \frac{1}{4} A_{11}^2 (b^4 - a^4) \right] \]

\[ T_2 = 2\mu \left[ \frac{1}{3} B_{21} (b^3 - a^3) - \frac{1}{3} (A_{11} B_{21} + 2 B_{23}) (b^5 - a^5) + \frac{3}{7} A_{11} B_{22} \right] \cdot (b^2 - a^2) \]

\[ T_3 = (\lambda + 2\mu) \log \frac{b}{a} - (3\lambda + 2\mu)(b^3 - a^3) A_{11} + \frac{1}{4\pi} (9\lambda + 10\mu) \cdot A_{11}^2 (b^4 - a^4) \]

\[ T_4 = \frac{1}{3} (3\lambda + 2\mu) A_{11} (b^3 - a^3) - (\lambda + 3\mu) (b - a) \]

\[ T_5 = -(\lambda + 2\mu) (b - a) + \frac{1}{3} \left[ (3\lambda + 2\mu) A_{11} + (\lambda + 3\mu) B_{21} \right] (b^5 - a^5) \]

\[ - \frac{1}{5} \left[ (\lambda + 2\mu) B_{22} + B_{21} A_{11} (3\lambda + 2\mu) \right] (b^5 - a^5) + \frac{1}{7} (3\lambda + 2\mu) \cdot A_{11} B_{22} (b^2 - a^2) \]

\[ T_6 = \frac{1}{2} (\lambda + 2\mu) (b^4 - a^4) \]

\[ T_7 = (\lambda + 2\mu) \left[ \frac{1}{2} (b^3 - a^3) - \frac{1}{4} B_{21} (b^4 - a^4) + \frac{1}{6} B_{23} (b^6 - a^6) \right] \]

\[ T_8 = (\lambda + 2\mu) \left[ \frac{1}{2} (b^3 - a^3) - \frac{1}{2} B_{21} (b^4 - a^4) + \frac{1}{6} (2B_{22} + B_{21}) (b^6 - a^6) \right. \]

\[ - \frac{1}{4} B_{21} B_{22} (b^8 - a^8) + \frac{1}{60} B_{22}^3 (b^{10} - a^{10}) \]

\[ T_9 = 2\mu \left[ \frac{1}{3} B_{21}^3 (b^6 - a^6) - B_{21} B_{23} (b^8 - a^8) + \frac{4}{5} B_{23}^2 (b^{10} - a^{10}) \right] \]
2-2. **INTRODUCTION OF ADJUSTMENT FACTORS**

The quality of the approximate analysis is judged by a comparison between the frequency spectrums of the curved elastic plate as given by approximate analysis and by the exact elasticity analysis. The deviation of the frequency spectrum of the curved elastic plate for the approximate theory from the three-dimensional exact analysis is expected due to the following: (1) The present approximate analysis only retains the three lowest symmetric modes selected from the series expansion of the displacement expression. Therefore, the truncation of the series expansion will affect, to some degree, the phase and group velocities. (2) By choosing a particular set of displacement patterns that satisfy a radial orthogonality condition, the displacement patterns in the approximate theory are not exactly the same as those in the exact theory.

However, this deviation can be reduced substantially by introducing adjustment factors $k_i$ ($i = 1$ to $4$). The additional adjustments are made to improve the match between the frequency spectrum of a curved plate as obtained from the approximate and the exact spectrum. The introduction of adjustment factors is analogous to the shear coefficient in the Timoshenko beam equations [29] and the analogous equations for plates [31].

We replace $S_{12}$ by $k_1 S_{12}$ and $S_{32}$ by $k_2 S_{32}$ in the strain energy density and $u_1$ by $k_3 u_1$ and $v_2$ by $k_4 v_2$ in the kinetic energy density, where $k_i$ are constants for a given curved plate. Then, we can obtain the following adjusted relations:
Adjusted strain energy density

\[ 2 \bar{U} = T_3 \kappa_3 S_{12}^2 + T_1 S_{11}^2 + T_6 S_{21}^2 + T_8 S_{31}^2 + 2 T_4 \kappa_1 S_{12} S_{21} + 2 T_5 \kappa_1 S_{12} S_{31} + 2 T_7 S_{21} S_{31} + T_9 \kappa_3 S_{33}^2 \]

(51)

Adjusted kinetic energy density

\[ 2 \bar{K} = \mathfrak{g} (c_1 \kappa_3 \dot{u}_1^2 + c_2 \dot{v}_0^2 + c_3 \kappa_4 \dot{u}_1^2) \]

(52)

§ Stress-Strain Relations

The adjusted stress-resultant and strain-displacement relations, derived from the strain-energy density function (51) are

\[ E_{12} = \frac{\partial \bar{U}}{\partial S_{12}} = -T_3 \kappa_3 \dot{u}_1 + \frac{1}{\alpha} T_4 \kappa_1 \dot{v}_0 + \frac{1}{\alpha} T_5 \kappa_1 \dot{u}_2' \]

\[ E_{11} = \frac{\partial \bar{U}}{\partial S_{11}} = T_1 \dot{u}_1 - \frac{1}{\alpha} T_3 \kappa_2 \dot{v}_2 \]

\[ E_{21} = \frac{\partial \bar{U}}{\partial S_{21}} = -T_4 \kappa_1 \dot{u}_1 + \frac{1}{\alpha} T_6 \dot{v}_0' + \frac{1}{\alpha} T_7 \dot{u}_2' \]

\[ E_{32} = \frac{\partial \bar{U}}{\partial S_{32}} = T_2 \kappa_2 \dot{u}_1 + \frac{1}{\alpha} T_9 \kappa_3 \dot{v}_2 \]

\[ E_{31} = \frac{\partial \bar{U}}{\partial S_{31}} = -T_5 \kappa_1 \dot{u}_1 + \frac{1}{\alpha} T_7 \dot{v}_0' + \frac{1}{\alpha} T_8 \dot{u}_2' \]

(53)
§ DISPLACEMENT EQUATIONS OF MOTION

By substituting the stress-resultant expression (53) into the stress equation of motion in (38), we obtain the following displacement equations of motion.

\[
T_1 u''_1 - T_3 k_1 u_1 + \frac{1}{\alpha} T_4 k_1 v_0' + \frac{1}{\alpha} (T_5 k_1 - T_2 k_2) v_2' = \gamma_1 c_1 k_3 u_1
\]

\[-T_4 k_1 u_1' + \frac{1}{\alpha} T_6 v_0'' + \frac{1}{\alpha} T_7 v_2'' = \gamma_2 c_2 v_0
\]

\[(T_2 k_2 - T_5 k_1) u_1' + \frac{1}{\alpha} T_7 v_0'' + \frac{1}{\alpha} T_9 k_2 v_2'' - \frac{1}{\alpha} T_9 k_2 v_2 = \gamma_3 c_3 k_2\]

§ FREQUENCY EQUATION

Since we are interested in the study of continuous wave propagation, it is justifiable to consider harmonic waves propagating in the plate, limiting ourselves to work with the real angular wave number \( \gamma \). Thus, let

\[
u_1 = A_1 \cos (\gamma \theta - \omega t)
\]

\[
u_0 = A_2 \sin (\gamma \theta - \omega t)
\]

\[
u_2 = A_3 \sin (\gamma \theta - \omega t)
\]
Upon substituting (55) into the displacement equations of motion (54), the characteristic equation in determinant form can be easily obtained as

\[
\begin{vmatrix}
  f_{11} & f_{12} & f_{13} \\
  f_{12} & f_{22} & f_{23} \\
  f_{13} & f_{23} & f_{33}
\end{vmatrix} = 0
\]

where

\[
\begin{align*}
  f_{11} &= -(T_{11} \gamma^2 + T_{12} \kappa^2 + T_{13} \kappa_3 \eta \delta^2 \Omega^2) \\
  f_{22} &= -(3^2 - 1) (T_{21} \gamma^2 + T_{22} \eta \delta^2 \Omega^2) \\
  f_{33} &= -(3^2 - 1) (T_{31} \gamma^2 + T_{32} \kappa_3 + T_{33} \kappa_4 \eta \delta^2 \Omega^2) \\
  f_{12} &= (3 - 1) T_{41} \kappa_1 \gamma \\
  f_{13} &= (3 - 1) (T_{51} \kappa_1 + T_{52}) \gamma \\
  f_{23} &= (3^2 - 1) T_{61} \gamma^2
\end{align*}
\]
\[ z = \frac{b}{a}, \quad \theta^2 = \frac{\lambda}{\mu} = \frac{2\nu}{1-2\nu}, \quad \delta = \frac{a}{H}, \]

\[ \Omega^2 = \frac{\omega^2}{\omega_0^2}, \quad \omega_5^2 = \frac{\pi^2 C^2}{H}, \quad \alpha_{11} = \frac{2}{\frac{\theta^2}{\Omega^2} + 1} \quad (58) \]

\[ b_{21} = \frac{24(\frac{\theta^2}{\Omega^2} - 1)(\frac{\theta_0^2}{\Omega_0^2} - 1) - 20(\frac{\theta_0^2}{\Omega_0^2} - 1) }{16(\frac{\theta_0^2}{\Omega_0^2} - 1)(\frac{\theta_0^2}{\Omega_0^2} - 1) - 15(\frac{\theta_0^2}{\Omega_0^2} - 1)^2} \]

\[ b_{22} = \frac{16(\frac{\theta^2}{\Omega^2} - 1)(\frac{\theta_0^2}{\Omega_0^2} - 1) - 8(\frac{\theta_0^2}{\Omega_0^2} - 1)^2 }{3(16(\frac{\theta_0^2}{\Omega_0^2} - 1)(\frac{\theta_0^2}{\Omega_0^2} - 1) - 15(\frac{\theta_0^2}{\Omega_0^2} - 1)^2)} \]

and

\[ T_{11} = \log z - a_{11}(\frac{\theta^2}{\Omega^2} - 1) + \frac{1}{4} a_{11}^2(\frac{\theta^2}{\Omega^2} - 1) \]

\[ T_{12} = (\frac{\theta^2}{\Omega^2} + 2) \log z - (3\frac{\theta^2}{\Omega^2} + 2) a_{11}(\frac{\theta^2}{\Omega^2} - 1) + \frac{1}{4} (9\frac{\theta^2}{\Omega^2} + 10) a_{11}^2(\frac{\theta^2}{\Omega^2} - 1) \]

\[ T_{13} = -\frac{1}{2} \left( (\frac{\theta^2}{\Omega^2} - 1) - a_{11}(\frac{\theta^2}{\Omega^2} - 1) + \frac{1}{3} a_{11}^2(\frac{\theta^2}{\Omega^2} - 1) \right) \quad (59) \]

\[ T_{21} = \frac{1}{2} (\frac{\theta^2}{\Omega^2} + 2) \]

\[ T_{22} = -\frac{1}{4} (\frac{\theta^2}{\Omega^2} + 1) \]

\[ T_{31} = \frac{1}{2} (\frac{\theta^2}{\Omega^2} + 2) \left[ 1 - b_{21}(\frac{\theta^2}{\Omega^2} + 1) + \frac{1}{3} (2 b_{22} + b_{21}^2)(\frac{\theta^2}{\Omega^2} + 1) - \frac{1}{3} b_{21} b_{22}(\frac{\theta^2}{\Omega^2} + 1)(\frac{\theta^2}{\Omega^2} + 1) + \frac{1}{8} b_{21}^2(\frac{\theta^2}{\Omega^2} + 3\frac{\theta^2}{\Omega^2} + 1) \right] \]

\[ T_{32} = 2 \left[ \frac{1}{3} b_{21}^2(\frac{\theta^2}{\Omega^2} + 3\frac{\theta^2}{\Omega^2} + 1) - b_{21} b_{22} (\frac{\theta^2}{\Omega^2} + 1)(\frac{\theta^2}{\Omega^2} + 1) + \frac{4}{8} b_{22}^2 - \right. \\
\left. (\frac{\theta^2}{\Omega^2} + 3\frac{\theta^2}{\Omega^2} + 3\frac{\theta^2}{\Omega^2} + 1) \right] \]
\[ T_{33} = - \left[ \frac{1}{4} (3^3 + 1) - \frac{1}{3} b_{21} (3^4 + 3^3 + 1) + \frac{1}{8} (2 b_{22} + b_{22}^2) (3^4 + 1) \right] \]

\[ - \left[ (3^3 + 1) - \frac{1}{5} b_{21} b_{22} (3^6 + 3^4 + 3^2 + 1) + \frac{1}{12} b_{22}^3 \right] \]

\[ (3^3 + 1) (3^4 + 3^2 + 1) \]

\[ T_{41} = \frac{1}{3} (3 g^2 + 2) a_{11} (3^3 + 3 + 1) - (g^2 + 2) \]

\[ T_{51} = -(g^2 + 2) + \frac{1}{3} \left[ (3 g^2 + 2) a_{11} + (g^2 + 2) b_{21} \right] (3^3 + 3 + 1) \]

\[- \frac{1}{6} \left[ (g^2 + 2) b_{22} + (3 g^2 + 2) b_{21} a_{11} \right] (3^4 + 3^3 + 3^2 + 3 + 1) \]

\[ + \frac{1}{7} (3 g^2 + 2) a_{11} b_{22} (3^6 + 3^5 + 3^4 + 3^3 + 3^2 + 3 + 1) \]

\[ T_{52} = - 2 \left[ \frac{1}{3} b_{21} (3^2 + 3 + 1) - \frac{1}{5} (a_{11} b_{21} + 2 b_{22}) (3^4 + 3^3 + 3^2 + 3 + 1) \right] \]

\[ + \frac{2}{7} a_{11} b_{22} (3^6 + 3^5 + 3^4 + 3^3 + 3^2 + 3 + 1) \]

\[ T_{61} = - \frac{1}{2} (g^2 + 2) \left[ 1 - \frac{1}{2} b_{21} (3^2 + 1) + \frac{1}{3} b_{22} (3^4 + 3^2 + 1) \right] \]

In order to evaluate the adjustment factors \( k_1 \), and to plot the approximate frequency spectrum, it is desirable to write the frequency equation in polynomial form as

\[ l_1 (\pi \delta \omega)^6 + l_2 (\pi \delta \omega)^4 + l_3 (\pi \delta \omega)^2 + l_4 = 0 \]  (60)
where

\[ \lambda_1 = -\left(3^2 - 1\right)^3 T_{13} T_{23} T_{33} \kappa_3 \kappa_4 \]

\[ \lambda_2 = -\left(3^2 - 1\right)^2 \left( (T_{11} \gamma^2 + T_{12} \kappa_1^2) \kappa_4 \gamma^2 T_{22} T_{33} + (T_{11} \gamma^2 + T_{12} \kappa_1^2) \kappa_3 \gamma^2 T_{13} T_{23} \right) \]

\[ \lambda_3 = -\left(3^2 - 1\right)^2 \left( (T_{31} \gamma^2 + T_{32} \kappa_2^2) \gamma^2 T_{22} + (T_{31} \gamma^2 + T_{32} \kappa_2^2) \kappa_3 \gamma^2 T_{13} T_{23} \right) + (\gamma - 1)^2 \]

\[ \lambda_4 = -\left(3^2 - 1\right)^3 \left( (T_{11} \gamma^2 + T_{12} \kappa_1^2) (T_{31} \gamma^2 + T_{32} \kappa_2^2) T_{22} \gamma^2 + 2 (\gamma - 1)^2 (3^2 - 1) \right) \]

\[ \left( T_{61} \kappa_1 + T_{62} \right) T_{61} T_{62} \kappa_1 \gamma^4 + (\gamma - 1)^2 (3^2 - 1) \left( T_{51} \kappa_1 + T_{52} \right)^2 \gamma^2 T_{22} \]

\[ + (\gamma - 1)^2 (3^2 - 1) (T_{31} \gamma^2 + T_{32} \kappa_2^2) T_{41} \kappa_1 \gamma^2 + (\gamma - 1)^2 (T_{11} \gamma^2 + T_{12} \kappa_1^2) T_{61} \gamma^4. \]

5. EVALUATION OF THE ADJUSTMENT FACTORS \( k_i \)

The frequency equation (60) which relates the square of the frequency \( \Omega^2 \) to the square of the angular wave number \( \gamma^2 \), with the radii, Poisson's ratios, and the adjustment factor \( k_i \) as the parameters. In general the frequency \( \Omega \) must be real, but the wave number \( \gamma \) may be real, imaginary, or complex. As mentioned, in this investigation, we have already restricted ourselves to real wave numbers.

The relation between \( \Omega \) and \( \gamma \) from the approximate analysis should match, as closely as possible, the corresponding relation
obtained from the three-dimensional exact analysis. A desirable match between these two analyses may be achieved by choosing appropriate values for the adjustment factors $k_i$.

In our previous displacement expressions (29), the higher order terms which corresponded to the high frequency modes were dropped. Therefore, it is reasonable, as well as important, to match the approximate analysis with the exact analysis for long wave length, such as at the cutoff frequencies. In other words, the applicability of the approximate equations is limited to frequencies below the lowest frequency of the lowest neglected mode and the correspondingly small wave numbers.

In order to be able to determine the four adjustment factors, it would be desirable to match four quantities at the cutoff frequencies, namely the two intercepts and the two curvatures of the second and third branches. This is the procedure in the case of the solid rod and the flat plate [13, 9]. For the curved plate theory, the match at the intercepts can easily be achieved. However, because of the complexity of the frequency equation, it is not practical to match the intercept curvatures. As an alternative approach, paralleling the arguments used in the hollow rod theory of McNiven, Shah, and Sackman [20], the branches from the approximate analysis are made to pass through two points on the branches of the exact spectrum offset from the $\omega$ axis; that is, at points for which the wave length is finite.
For given radii and Poisson's ratio of the curved plate, the cutoff frequency from the exact theory for the M(2), M(4), and M(5) modes are

\[ \Omega_1, \quad \Omega_2, \quad \Omega_3 \] (62)

From the approximate analysis, we set \( \gamma = 0 \), and solve for the three cutoff frequencies from (60), obtaining

\[ \Omega, - \frac{T_{12} \cdot k_3^2}{T_{13} \cdot k_3^2} \cdot \frac{1}{\pi^2 \delta^2}, \quad - \frac{T_{32} \cdot k_2^2}{T_{33} \cdot k_4^2} \cdot \frac{1}{\pi^2 \delta^2} \] (63)

Setting comparable intercepts equal gives the three equations

\[ \Omega_1 = 0 \]

\[ \Omega_2 = - \frac{T_{12} \cdot k_3^2}{T_{13} \cdot k_3^2} \cdot \frac{1}{\pi^2 \delta^2} \]

\[ \Omega_3 = - \frac{T_{32} \cdot k_2^2}{T_{33} \cdot k_4^2} \cdot \frac{1}{\pi^2 \delta^2} \] (64)
It is seen that the first cutoff frequency $\Omega_1 = 0$ matches perfectly with the exact theory.

As we have stated before, it remains to establish two more matching points leading to two more equations. It was decided to match the second and third branches at $\gamma = 10.0$ with its corresponding frequencies $\Omega_4$ and $\Omega_5$ from the exact theory. Matching at these two points maintained a match for longer wave-lengths and gave real, positive adjustment factors, which are necessary conditions to maintain real strains and positive definite energy densities.

After substituting $\Omega_4$ and $\Omega_5$ successively into (60) and then eliminating $\Omega_5$ between these two equations, we obtain the following equation in $k_3$.

\[
M_1 k_3^4 + M_2 k_3^3 + M_3 k_3^2 + M_4 k_3 + M_5 = 0
\]
where

\[ M_1 = E_{12} E_3 - E_{11} E_{32} \]

\[ M_2 = E_{12} E_{41} - E_{11} E_{42} \]

\[ M_3 = E_{12} E_{51} + E_{22} E_{31} - E_{11} E_{52} - E_{21} E_{32} \]

\[ M_4 = E_{22} E_{41} - E_{21} E_{42} \]

\[ M_5 = E_{22} E_{51} - E_{21} E_{52} \]

and for \( i = 1 \) and \( 2 \)

\[ E_{1\lambda} = (\frac{3}{2} - 1) (T_{33} \delta^2 + T_{33} \Omega_{2}^2) \left[ (\frac{3}{2} - 1)^2 \alpha^2 - \frac{3}{2} \right] \]

\[ E_{2\lambda} = - (\frac{3}{2} - 1)^2 T_{11} \frac{\gamma^2}{\lambda^2} \left( T_{21} \frac{\gamma^2}{\lambda^2} + T_{22} \Omega_{2}^2 \Omega_{2}^2 \right) \left( T_{33} \delta^2 + T_{33} \Omega_{2}^2 \Omega_{2}^2 \right) \]

\[ E_{3\lambda} = (\frac{3}{2} - 1)^2 \left( T_{12} \delta^2 + T_{12} \Omega_{2}^2 \Omega_{2}^2 \right) \left[ (\frac{3}{2} - 1)^2 \frac{\gamma^2}{\lambda^2} - (T_{21} \frac{\gamma^2}{\lambda^2} + T_{22} \Omega_{2}^2 \Omega_{2}^2) \right] \]

\[ E_{4\lambda} = 3 (\frac{3}{2} - 1)^2 T_{51} \frac{\gamma^2}{\lambda^2} \Omega_{2}^2 \left[ T_{61} \frac{\gamma^2}{\lambda^2} + T_{51} \left( T_{21} \frac{\gamma^2}{\lambda^2} + T_{22} \Omega_{2}^2 \Omega_{2}^2 \right) \right] \]

\[ E_{5\lambda} = (\frac{3}{2} - 1)^2 \left[ T_{11} \frac{\gamma^2}{\lambda^2} \right] \left[ T_{61} \frac{\gamma^2}{\lambda^2} - (T_{21} \frac{\gamma^2}{\lambda^2} + T_{22} \Omega_{2}^2 \Omega_{2}^2) \right] \]

\[ + (\frac{3}{2} - 1)^2 \frac{\gamma^2}{\lambda^2} T_{51}^2 \left( T_{21} \frac{\gamma^2}{\lambda^2} + T_{22} \Omega_{2}^2 \Omega_{2}^2 \right) \]
Equation (66) has four roots, but only one is positive and real and hence acceptable on physical grounds. The factor $k_4$ is obtained from

$$
k_4 = \frac{-(E_{3i}k_3^2 + E_{4i}k_4 + E_{5i})}{E_{1i}k_3^2 + E_{2i}}
$$

(69)

Finally $k_1$ and $k_2$ are obtained from (65).

Table 1 gives the values of $k_1$ for some typical values of the parameters $\nu$ and $R/H$.

§ NUMERICAL RESULTS

Having the value of the $k_1$, the frequency spectrum for given radii and Poisson's ratio of the curved plate can be obtained by the "interval halving" iteration technique that was used in the exact analysis. The comparison of the frequency spectrum between two theories are shown in Figure 13 through Figure 16. The dotted lines in the frequency spectra represent the approximate analysis, while the solid lines represent the exact analysis. It is clearly seen that the two analyses agree quite well in the region of long wave lengths. For small wave lengths, the deviation is expected because the higher order terms in the series expansion expression are neglected.
It is also seen that the agreement between two theories becomes worse for small wave lengths as the R/H ratio becomes larger (i.e., approaches to the flat plate case). However, intuition would suggest that agreement would be improved as the flat plate case is approached. This poor agreement may be caused by the kind of displacement profile which we selected in the beginning of this study as indicated in (29) and Figure 12. The nonsymmetrical nature of the displacement profile with respect to the middle surface of the curved plate may have led to some difficulty in representing the symmetric displacement of a flat plate. When this effect is coupled with the influence of neglecting the higher order terms in the series expansions, the deviation between two theories becomes substantial at small wave length.
Table 1. Values of $k_i^2$

<table>
<thead>
<tr>
<th>Poisson's Ratio</th>
<th>R/H</th>
<th>$k_1^2$</th>
<th>$k_2^2$</th>
<th>$k_3^2$</th>
<th>$k_4^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>10.0</td>
<td>0.3798</td>
<td>1.4500</td>
<td>0.4630</td>
<td>1.8841</td>
</tr>
<tr>
<td>0.25</td>
<td>10.0</td>
<td>1.4200</td>
<td>0.5185</td>
<td>1.7311</td>
<td>0.6738</td>
</tr>
<tr>
<td>0.30</td>
<td>10.0</td>
<td>0.8085</td>
<td>0.7070</td>
<td>0.9859</td>
<td>0.9187</td>
</tr>
<tr>
<td>0.25</td>
<td>30.0</td>
<td>0.0342</td>
<td>0.1505</td>
<td>0.0420</td>
<td>0.5678</td>
</tr>
<tr>
<td>0.3</td>
<td>30.0</td>
<td>0.0354</td>
<td>0.1832</td>
<td>0.0434</td>
<td>0.6910</td>
</tr>
<tr>
<td>0.25</td>
<td>50.0</td>
<td>0.0108</td>
<td>0.2666</td>
<td>0.0125</td>
<td>0.0627</td>
</tr>
<tr>
<td>0.30</td>
<td>50.0</td>
<td>0.0212</td>
<td>0.6346</td>
<td>0.0246</td>
<td>0.1494</td>
</tr>
</tbody>
</table>
Figure 13. Frequency spectrum showing comparison between the exact and approximate theories for $\nu = 0.20$, $R/H = 10.0$
Figure 14. Frequency spectrum showing comparison between the exact and approximate theories for $\nu = 0.30$, $R/H = 10.0$
Figure 15. Frequency spectrum showing comparison between the exact and approximate theories for $\nu = 0.25$, $R/H = 30.0$
Figure 16. Frequency spectrum showing comparison between the exact and approximate theories for $\nu = 0.3$, $R/H = 50.0$
SUMMARY

1. In the exact theory, a frequency equation derived from three-dimensional linear theory of elasticity and appropriate boundary conditions was developed. The displacement field was obtained from a dilational potential \( f_1 \) and two equivoluminal ones \( g_1 \) and \( f_4 \). The potentials \( f_1 \) and \( g_1 \) are coupled through the boundary conditions and generate the longitudinal flexural modes. Similarly to the case of the propagation of waves in hollow circular cylinders, when the wave number \( \gamma \) propagating along the circumference is equal to zero, the two potentials, \( f_1 \) and \( g_1 \), generate two uncoupled mode families. They are extensional modes and shear modes.

2. The results of a numerical analysis of the frequency equation describing wave propagation in an elastic curved plate has been presented. As was seen in the case of flat plates, there exists an infinite number of modes in the curved plate.

3. An approximate theory was developed that governs the relationship between frequencies and their wave numbers for stress waves propagating in a curved plate. Comparison was made for \( M(2) \), \( M(4) \), and \( M(5) \) modes, with real propagation constants, for this theory and the comparable modes of the exact theory. The final results indicated that the approximate analysis agrees with the exact analysis very well for long wave lengths. The deviations between these
two analyses at small wave lengths are due to the truncation of the series expansion. Some uncertainty exists regarding the cause of increasing deviation for longer R/H ratios.

4. With the technique developed in the approximate theory, a relationship between frequency and angular wave number for as many modes as desired can be predicted. Also, this technique can be used for other selected modes.

5. The equations of motion and stress for the plane stress problem are practically identical with those for the plane strain problem, with the only difference being that \( \lambda = \frac{2\nu}{1-\nu^2} \) appears in the former and \( \lambda \) in the later [32]. In fact, the only difference is in the elastic constants. Therefore, we can easily obtain the solution for a narrow rectangular bar without resolving the plane stress theory. Modes appearing in both theories are qualitatively the same, but their speeds may differ.
APPENDIX A

FUNCTION BESSJ(X,FNUP)

SINGLE-VALUED BESSEL FUNCTION. (BESSEL FUNCTIONS OF THE
FIRST KIND).
FORTRAN IV CODED.
PURPOSE
FOR GIVEN REAL X AND FNUP, TO COMPUTE A BESSEL FUNCTION OF
ARGUMENT X AND ORDER FNUP.
DESCRIPTION OF PARAMETERS:
X THE REAL ARGUMENT OF THE BESSEL FUNCTION DESIRED.
FNUP THE REAL ORDER (POSITIVE OR NEGATIVE, INTEGER OR NON-INTEGER) OF THE BESSEL FUNCTION DESIRED.
REMARK
COMMON BLOCK HAS BEEN USED. --COMMON BJ(1350), BY(1350)
METHOD
GOLDSTEIN, M. AND THALER, R., 'RECURRENCE TECHNIQUES FOR
THE CALCULATION OF BESSEL FUNCTIONS', MTAC, VOL. XIII, NO.66,
APRIL 1966. FOR X GREATER OR EQUAL TO 10.0 ASYMPTOTIC VALUES
ARE COMPUTED USING THE SO CALLED PHASE AMPLITUDE METHOD. SEE
GOLDSTEIN, M., AND THALER, R., 'BESSEL FUNCTIONS FOR LARGE
ARGUMENTS,' MTAC, VOL XII, NO. 61, JANUARY 1958.
SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED:
IABS(N)--- ABSOLUTE VALUE FOR INTEGER NUMBER.
MAXO(X,Y)- SELECT MAXIMUM VALUE BETWEEN X AND Y.
SQRT(X)--- SQUARE ROOT OF X.
SIN(X)---- SINE OF X
GAMMA(X)-- GAMMA FUNCTION OF X (FROM IBM 360 SCIENTIFIC
SUBROUTINE PACKAGE).

FUNCTION BESSJ(X,FNUP)
COMMON /BESSL/BJ(1350),BY(1350)
IDIM=1350
R=FNUP
N=0
FNU=0.0

C SPLITTING FNUP INTO TWO PARTS, INTEGER 'N' AND NON-INTEGER 'FNU'
IN(R) 3, 17, 9
3 N=-1
5 R=R+1.0
FNU=R
IF (R) 7, 13, 13
7 N=N-1
GO TO 5
9 FNU=R
R=R-1.0
IF (R) 13, 11, 11
11 N=N+1
GO TO 9
19 IF(FNU) 15, 15, 17
15 N=IABS(N)
FNU=ABS(FNU)
17 NN=IABS(N)
21 CONST=2.0/X

C
23 X10=X+25.0
K10=X10
N10=NN+10
25 M=MAX0(K.0,N10)
31 M=M/2
33 K=2*M+1
IF(K+2-IDIM) 30, 30, 26
30 BJ(K+1)=1.0E-37
BJ(K+2)=0.0

C
DO 35 L=1,K
J=K+1-L
FLI=I
35 BJ(I)=CONST*(FLI+FNU)*BJ(I+1)-BJ(I+2)

C
FIND ARPHA EITHER FROM EQUATION 12, WHEN X IS LESS THAN 10.0
OR FROM BESSEL FUNC. FOR LARGE ARGUMENTS WITH X EQUAL OR GREATER
THAN 10.0
IF(X-10.0) 37, 58, 58
37 PHI=FNU+2.0
ALF=PHI*BJ(3)+BJ(1)
MO=3
DO 39 I=2, M
MO=MO+2
FM2=2*I
FM1=I-1
FI=I
TEMP=((FNU+FM2)*(FNU+FM1))/(FI*(FNU+FM2-2.0))*PHI
PHI=TEMP
39 ALF=PHI*BJ(MO)+ALF
ALF=CONST**FNU*GAMMA(FNU+1.0)*ALF

C
FIND J(N) EQUATION 7, WHEN X IS LESS THAN 10.0 OR J(2),J(3),...
J(N)=F(2)/ALPHA.....F(N)/ALPHA WHERE ALPHA=F(1)/J(1) AND
J(1)=A*COS(PHI) FROM PATH 59.
I1=1
41 DO 43 I=I1,K
43 BJ(I)=BJ(I)/ALF
BESSL = BJ(NN+1)
IF(N) 47, 52, 52
47 BJ(2) = CONST*FNU*BJ(1) - BJ(2)
BESSL = BJ(2)
IF(NN-1) 52, 52, 49
49 FRAC = FNU
N1 = NN+1
DO 51 L = 3, N1
FRAC = FRAC - 1.0
51 BJ(L) = CONST*FRAC*BJ(L-1) - BJ(L-2)
BESSL = BJ(NN+1)
52 GO TO 53
53 BESSJ = BESSL
57 RETURN

C
58 IF(NN+1-IDIM) 59, 59, 26
59 KOUNT = 1
GNU = FNU
C0 = 0.25
C1 = 0.15625
C2 = -0.375
C3 = 0.1171875
C4 = -1.15625
C5 = 1.875
C6 = .952148438E-01
C7 = -2.38671875
C8 = 14.2265625
C9 = -19.6875
C10 = -.809326172E-01
C11 = -.410058593E+01
C12 = .582246094E+02
C13 = -277.875
C14 = 354.375
C15 = .416666667E-01
C16 = -.25
C17 = .125E-1
C18 = -.35
C19 = .558035718E-03
C20 = -.424107143
C21 = 3.60267857
C22 = -5.625
C23 = .30381944E-02
C24 = -.486111111
C25 = .102864583E+03
C26 = -58.0
C27 = 78.75
61 AL1 = (GNU*GNU) - 0.25
A2 =C0*AL1
A4 =C1*AL1
A4 = (A4+C2)*AL1
A6 =C3*AL1
A6 = (A6+C4)*AL1
A6 = (A6+C5)*AL1
A8 =C6*AL1
A8 = (A8+C7)*AL1
A8 = (A8+C8)*AL1
A8 = (A8+C9)*AL1
A10= C10*AL1
A10 = (A10+C11)*AL1
A10 = (A10+C12)*AL1
A10 = (A10+C13)*AL1
A10 = (A10+C14)*AL1
PI=3.14159265
TS=1.0/X
T2=TS*TS
B =A10*T2+A8
B =B*T2+A6
B =B*T2+A4
B =B*T2+A2
BNU=B*T2+1.0
ANLI=BNU/SQRT(.5*PI*X)
A2 =.5*AL1
A4 =C15*AL1
A4 = (A4+C16)*AL1
A6 =C17*AL1
A6 = (A6+C18)*AL1
A6 = (A6+.75)*AL1
A8 =C19*AL1
A8 = (A8+C20)*AL1
A8 = (A8+C21)*AL1
A8 = (A8+C22)*AL1
A10=C23*AL1
A10 = (A10+C24)*AL1
A10 = (A10+C25)*AL1
A10 = (A10+C26)*AL1
A10 = (A10+C27)*AL1
B=A10*T2+A8
B=B*T2+A6
B=B*T2+A4
B=B*T2+A2
TPHI=B*T2+1.0
PHI = TPHI * X - (GNU + 0.5) * (PI / 2.0)
COP = COS(PHI)
SIP = SIN(PHI)
F1 = ANU * COP
Y1 = ANU * SIP
IF(KOUNT - 1) 63, 63, 65
63 FSAVE = F1
   YSAVE = Y1
   GNU = FNU + 1.0
   KOUNT = 2
   GO TO 61
65 F2 = F1
   BY(2) = Y1
   F1 = FSAVE
   BY(1) = YSAVE
67 IF(ABS(F1) - ABS(F2)) 77, 77, 79
77 ALF = BJ(2) / F2
   GO TO 81
79 ALF = BJ(1) / F1
81 BJ(1) = F1
   BJ(2) = F2
   I1 = 3
   GO TO 41
C
ERROR DUE TO THE ARGUMENT IS TOO LARGE, LOSS ITS ACCURACY
26 BESSL = 0.0
   DO 27 I = 1, IDIM
27 BJ(I) = 0.0
   BY(3) = 989898.0
WRITE(6, 28)
28 FORMAT('O', 'ERROR DUE TO THE ARGUMENT IS TOO LARGE --- BESSJ')
WRITE(6, 29) X, FNU
29 FORMAT('O', 'X=', F10.5, 10X, 'FNU=', F10.5)
   GO TO 52
100 BESSJ = 1.0
RETURN
END
FUNCTION BESSY(X,FNUP)
SINGLE-VALUED NEUMANN'S FUNCTION (BESSEL FUNCTION OF THE
SECOND KIND).
FORTRAN IV CORDED
PURPOSE
FOR GIVE N REAL X AND FNUP, TO COMPUTE A NEUMANN'S FUNCTION
OF ARGUMENT X AND ORDER FNUP.
DESCRIPTION OF PARAMETERS:
X THE REAL ARGUMENT OF THE NEUMANN'S FUNCTION.
FNUP THE REAL ORDER (POSITIVE OR NEGATIVE, INTEGER OR NON-
INTEGER) OF THE NEUMANN'S FUNCTION.
REMARK
COMMON BLOCK HAS BEEN USED. --COMMON BJ(1350), BY(1350)
METHOD
GOLDSTEIN, M. AND THALER, R., 'RECUR RENCE TECHNIQUES FOR THE
CALCULATION OF BESSEL FUNCTIONS'. MTAC, VOL. XIII, NO.66
APRIL 1966. FOR X GREATER OR EQUAL TO 10.0 ASYMPTOTIC VALUES
ARE COMPUTER USING THE SO CALLED PHASE AMPLITUDE METHOD. SEE
GOLDSTEIN, M., AND THALER, R., 'BESSEL FUNCTIONS FOR LARGE
ARGUMENTS.' MTAC, VOL. XIII, NO.61, JANUARY 1958.
SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
BESSJ(X,FNUP) --BESSEL FUNCTION SUBROUTINE.
IABS(X) -------ABSOLUTE VALUE OF X (AN INTEGER)
ABS(X) -------ABSOLUTE VALUE OF X (A REAL NUMBER)
MAXO(X,Y) ------MAXIMUM VALUE OF X AND Y.
SIN(X) -------SINE OF X
COS(X) -------COSINE OF X
ALOG(X) -------LOG OF X, IT IS A REAL NUMBER.
GAMMA(X) -------GAMMA FUNCTION SUBROUTINE.

FUNCTION BESSY(X,FNUP)
COMMON /BESSL/BJ(1350),BY(1350)
IDIM=1350
SAVENU=FNUP
SAVEX=X
GETJ=BESSJ(X,FNUP)
YNU=BY(1)
X=SAVES
FNUP=SAVENU
IF(BY(3)-9899898.0) 1, 50, 1
1 R=FNUP
N=0
FNU=0.0
IF(R) 4, 35,7
4 N=-1
5 R=R+1.0
FNU=R
IF(R) 6, 9, 9
6 N=N-1
GO TO 5
7 FNU=R
R=R-1.0
IF(R) 9, 8, 8
8 N=N+1
GO TO 7
9 IF(FNU) 34, 34, 35
34 N=IABS(N)
FNU=ABS(FNU)
35 NN=IABS(N)
N1=NN-1
CONST=2.0/X
PI=3.14159265
IF(X-10.0) 10, 23, 23
10 X10=X+25.0
K10=X10
N10=NN+10
M+MAX0(L10,N10)
12 M=M/2
K=2*M+1
ARG=FNU*PI
GA=Gamma(FNU+1.0)
GARG=GA*GA

C COMPUTE GAMMA ZERO EQUATION 15 IF NU=0 USE EQUATION 16
14 IF(FNU) 15, 16, 15
15 TERM=(1.0/PI)*CONST**((2.0*FNU)
GAM1=COS(ARG)/SIN(ARG)-TERM*(GARG/FNU)
GAM2=2.0*TERM8GARG*(FNU+2.0)/(1.0-FNU)
GO TO 17
C COMPUTE GAMMA FOR NU=0
16 TLOG=ALOG(X/2.0)
A=0.577215665
PIH=2.0/pi
GAM1=PIH*(A+TLOG)
GAM2=4.0/PI
17 BY(2)=0.0
BY(1)=0.0
GAM3=0.0
E1=-(1.0/PI)*CONST**((1.0+2.0*FNU)*GARG
BY(2)=E1*BY(1)+BY(2)
E1=GAM1-GAM2/2.0
BY(2)=E1*BY(2)+BY(2)
YNU=GAM1*BY(1)
TXNU=3.0*FNU/X
AB=ABS(BJ(1))-0.000005
MP1=M+1
KSTOP=1
I2=1
DO 21 I=2,MP1
I2=I2+2
KSTOP=KSTOP+2
FI=I
FIM=I-1
FI2=2*I
DENOM=FI*(FI-FNU)*(FNU+FI2-2.0)
GAM3=(FNU+FI2)*(2.0*FNU+FIM)*(FNU+FIM)/DENOM
GAM3=-GAM3*GAM2
YNU=GAM2*BJ(I2)+YNU
IF(AB) 18, 18, 20
18 E1=TXNU*GAM2
BY(2)=E1*BJ(I2)+BY(2)
IF(KSTOP-K) 19, 38, 38
19 E1=(GAM2-GAM3)/2.0
BY(2)=E1*BJ(I2+1)+BY(2)
20 GAM1=GAM2
21 GAM2=GAM3
38 BY(1)=YNU
IF(AB) 23, 23, 22
22 BY(2)=(YNU*BJ(2)-2.0/(PI*X))/BJ(1)
23 IF(N) 31, 24, 25
24 BESSL=BY(1)
GO TO 29
25 IF(N-1) 24,26, 27
26 BESSL=BY(2)
GO TO 29
27 DO 28 I=1, N1
FI1=I
28 BY(I+2)=CONST*(FI1+FNU)*BY(I+1)-BY(I)
BESSL=BY(NN+1)
29 BESSY=BESSL
30 RETURN
31 IF(FNU-0.5) 43, 39, 43
43 BY(2)=CONST*FNU*BY 1)-BY(2)
BESSL=BY(2)
IF(NN-1) 29, 29, 32
32 FRAC=FNU
DO 33 I=1,N1
FRAC=FRAC-1.0
33 BY(I+2)=CONST*FRAC*BY(I+1)-BY(I)
BESSL=BY(NN+1)
GO TO 29
39 BY(2)=BJ(1)
   BESSL=BY(2)
   IF(NN-1) 29, 29, 40
40 ARG=-1.0
   I2=1
   DO 41 I=1, N1
   I2=I2+1
   BY(I+2)=ARG*BJ(I2)
41 ARG=-ARG
   BESSL=BY(NN+1)
   GO TO 29
50 BESSL=0.0
   DO 45 I=1, IDIM
45 BY(I)= .0
   WRITE(6,46)
46 FORMAT ('0','ERROR DUE TO THE ARGUMENT IS TOO LARGE ----BESSY')
   WRITE(6,47) X, FNUP
47 FORMAT ('0','X=' ,F10.5,10X,'FNUP=' ,F10.5)
   GO TO 29
100 BESSY=1.0
   RETURN
END
BIBLIOGRAPHY


