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DISSERTATION

Presented in Partial Fulfillment of the Requirements for
The Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
You-Hwa King Lee, B.Ed., M.S.

* * * * * *

The Ohio State University
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Chapter 1

INTRODUCTION AND RESULTS

1.1 Let \((a_n)\) and \((p_n)\) be two sequences. The convolution product of \((a_n)\) and \((p_n)\) is the sequence \(((p\#a)(n))\) defined by

\[
(p\#a)(n) = \sum_{k=1}^{n} p_k a_{n-k}, \quad n = 1, 2, \ldots
\]

One of the earliest results in the study of convolution products can be stated as follows:

Suppose that \((p_n)\) is a sequence of real numbers and that the series \(\sum_{k=1}^{\infty} p_k\) converges absolutely. If \((a_n)\) is a convergent sequence with \(\lim_{n \to \infty} a_n = c\), then the convolution product is a convergent sequence and

\[
\lim_{n \to \infty} (p\#a)(n) = c \sum_{k=1}^{\infty} p_k.
\]

If \(c \neq 0\), we can rewrite this result as follows:

\[
\lim_{n \to \infty} \frac{(p\#a)(n)}{a_n} = \frac{\infty}{\sum_{k=1}^{\infty} p_k}.
\]

A more general result can be obtained if instead of a convergent sequence \((a_n)\) we consider a positive sequence \((a_n)\) which has the property that \(\lim_{n \to \infty} a_{n+1}/a_n\) exists.
In this case the asymptotic behavior of the convolution product \(((p^#a)(n))\) is exactly the same as the asymptotic behavior of the sequence \((a_n)\). More precisely, we have the following result:

**Theorem 1.** Suppose that \((p_n)\) is a sequence of real numbers such that the series \(\sum_{k=1}^{\infty} p_k x^k\) has a positive radius of convergence \(R\). If

\[
\lim_{n \to \infty} a_{n+1}/a_n = \lambda^{-1}
\]

exists, and if \(0 < \lambda < R\), then

\[
\lim_{n \to \infty} \frac{(p^#a)(n)}{a_n} = \sum_{k=1}^{\infty} p_k \lambda^k.
\]

The following example shows that the condition \(0 < \lambda < R\) in Theorem 1 is essential.

**Example.** Let

\[ p_k = k^{-5/4}, \quad k = 1, 2, \ldots \]
\[ a_k = e^{-\sqrt{k}}, \quad k = 0, 1, 2, \ldots \]

It is obvious that the series \(\sum_{k=1}^{\infty} p_k\) converges, \(R = 1\), and

\[
\lim_{n \to \infty} a_{n+1}/a_n = 1.
\]

On the other hand we have
\[
\frac{(p^*a)(n)}{a_n} = \frac{\sum_{k=1}^{n} p_k \frac{a_{n-k}}{a_n}}{a_n} = \frac{n}{\sum_{k=1}^{n} k^{-5/4} e^{\sqrt{n} - \sqrt{n-k}}}
\]

If \( x \geq 0 \), then \( e^x \geq x \). Hence, we have

\[
\frac{(p^*a)(n)}{a_n} \geq \frac{n}{\sum_{k=1}^{n} k^{-5/4}} \frac{k}{\sqrt{n} + \sqrt{n-k}} \geq \frac{1}{2d_n} \sum_{k=1}^{n} k^{-3/4} \geq \frac{1}{2r(n)} \int_{\sqrt{n}}^{n} t^{-\frac{3}{4}} dt \geq \frac{2}{3r(n)} (n^{3/4} - 1),
\]

and so

\[
\lim_{n \to \infty} \frac{(p^*a)(n)}{a_n} = \infty.
\]

We shall refer to Theorem 1 as a direct theorem, since from the existence of \( \lim_{n \to \infty} a_{n+1}/a_n \) we conclude that the limit \( \lim_{n \to \infty} (p^*a)(n)/a_n \) also exists.

If the sequences \( (a_n) \) and \( (p_n) \) are sequences of non-negative numbers satisfying the asymptotic relation

\[(1.2) \quad \lim_{n \to \infty} \frac{(p^*a)(n)/a_n}{a_n} = L \quad (0 < L < \infty)\]
and if we know that \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) exists then the following theorem allows us to express the limit in terms of the numbers \( R \) and \( L \).

**Theorem 2.** Let \( (p_n) \) and \( (a_n) \) be sequences of non-negative numbers such that the series \( \sum_{k=1}^{\infty} p_k x^k \) has positive radius of convergence \( R \). If (1.2) holds and if \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) exists, then

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \vartheta,
\]

where

\[
(1.4) \quad \vartheta = \begin{cases} 
R^{-1}, & \text{if } \sum_{k=1}^{\infty} p_k R^k \leq L \\
\vartheta'^{-1}, & \text{if } \sum_{k=1}^{\infty} p_k R^k > L 
\end{cases}
\]

and \( \vartheta' \) is the unique number in \((0, R)\) such that

\[
\sum_{k=1}^{\infty} p_k \vartheta' R^k = L.
\]

It is easy to see that

\[
\vartheta'^{-1} = \sup \left\{ x : 0 < x \leq R \text{ and } \sum_{k=1}^{\infty} R^k \leq L \right\}
\]

and that the series \( \sum_{k=1}^{\infty} p_k \vartheta' R^k \) is always convergent, but its sum is \( L \) if and only if \( L \leq \sum_{k=1}^{\infty} p_k R^k \).

1.2 The converse problem now can be stated as follows: Suppose that we know that a convolution product \(((p^a)(n))\)
of two sequences \((a_n)\) and \((p_n)\) behaves asymptotically as the sequence \((a_n)\), i.e., suppose that (1.2) holds. Is it true then that \(\lim_{n \to \infty} a_{n+1}/a_n\) exists? It is easy to see that condition (1.2) alone is not sufficient and we shall need additional conditions on the sequences \((a_n)\) and \((p_n)\) in order to show that (1.2) implies the existence of \(\lim_{n \to \infty} a_{n+1}/a_n\). Usually we shall have to assume that both sequences \((a_n)\) and \((p_n)\) are sequences of non-negative numbers.

First results in this direction can be obtained from a result of P. Erdős, W. Feller and H. Pollard [1] in 1949. Their result in its simplest form can be stated as follows:

**Theorem A.** Let \((p_n)\) be a sequence of positive numbers such that \(\sum_{k=1}^{\infty} p_k = 1\) and let

\[ a_n = \frac{n}{\sum_{k=1}^{n} p_k} a_{n-k}, \quad n = 1, 2, \ldots \]

Then

\[ \lim_{n \to \infty} a_n = \frac{1}{\sum_{k=1}^{\infty} k p_k}. \]

If \(\sum_{k=1}^{\infty} k p_k < \infty\), then it follows clearly that

\[ \lim_{n \to \infty} a_{n+1}/a_n = 1. \]

However, if \(\sum_{k=1}^{\infty} k p_k = \infty\), then \(\lim_{n \to \infty} a_n = 0\) and from the preceding result we can obtain no information about the existence of \(\lim_{n \to \infty} a_{n+1}/a_n\).
A more complete discussion of the existence of the limit of the sequence \((a_{n+1}/a_n)\) was first given by N. G. De Bruijn and P. Erdős in the early 1950's. Again, instead of considering a sequence \((a_n)\) satisfying the asymptotic relation (1.2), they have considered a sequence \((a_n)\) which satisfies the relation

\[(1.5) \quad a_0 = 1, \quad a_n = \sum_{k=1}^{n} p_k a_{n-k}, \quad n = 1, 2, \ldots\]

They have proved the following result in 1950 in [2]:

**Theorem E.** Let \(p_k > 0, k = 1, 2, \ldots, \) and \(\lim_{k \to \infty} p_k = 1\) and let the sequence \((a_n)\) satisfies (1.5). If

\[p_{k-1} p_{k+1} > p_k^2, \quad k = 2, 3, \ldots\]

then

\[\lim_{n \to \infty} a_{n+1}/a_n = 1.\]

In 1951, N. G. De Bruijn and P. Erdős have considered the same problem in [3] and [4] from a more general point of view. Assuming that (1.5) holds and that \((p_n)\) is an arbitrary sequence of positive numbers, they have proved first that

\[\lim_{n \to \infty} \frac{n a_n}{\sqrt{(n+1)(n+2)}} = \gamma^{-1}\]

where
\[ \gamma = \sup \left\{ x > 0 : \sum_{k=1}^{\infty} p_k x^k \leq 1 \right\}. \]

In view of the relation
\[ \liminf_{n \to \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \to \infty} \frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_n}} \leq \limsup_{n \to \infty} \frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_n}} \leq \limsup_{n \to \infty} \frac{a_{n+1}}{a_n}, \]
this result gives no information about the existence of \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \). However, if \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) exists, then it is clearly equal to \( \gamma^{-1} \).

The most general result proved by N. G. de Bruijn and P. Erdős in [3] can be stated as follows:

**Theorem C.** Let \((p_n)\) be a sequence of positive numbers such that the series \( \sum_{k=1}^{\infty} p_k x^k \) have a positive radius of convergence \( R \) and let \((a_n)\) be a sequence of positive numbers such that \((1.5)\) holds. The necessary and sufficient condition for the existence of \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) is that
\[
(1.6) \quad \phi(A) \to 0 \quad (A \to \infty),
\]
where
\[ \phi(A) = \limsup_{n \to \infty} \frac{\sum_{k=A}^{n} p_k \frac{a_{n-k+1}}{a_n}}{\sqrt[n]{\sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}}} - G \frac{\sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}}{\sqrt[n]{\sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}}} \]
and \( G \) is defined by \((1.4)\).

Many sufficient conditions for the existence of the
\( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) are known. A typical result of this type, due to N. G. De Bruijn and P. Erdős [3] can be stated as follows:

**Theorem D.** If

\[
\lim_{n \to \infty} \frac{p_{n+1}}{p_n} = \sigma,
\]

where \( \sigma \) is defined by (1.4), then

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \sigma.
\]

Independently, in 1962, A. M. Garsia, S. Orey and E. Rodemich [5] and in 1963, A. M. Garsia [6] investigated the same problem. One of the simplest results that they have proved can be considered as an extension of the preceding result when \( \sum_{k=1}^{\infty} p_k R^k > 1 \).

**Theorem E.** Let \((p_n)\) be a sequence of positive numbers such that \( \sum_{k=1}^{\infty} p_k = 1 \) and let \((a_n)\) be a sequence of positive numbers such that (1.5) holds. A necessary and sufficient condition for the existence of the limit \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) is that

\[
W(\Delta) \to 0 \quad (\Delta \to \infty),
\]

where

\[
W(\Delta) = \limsup_{n \to \infty} \sup_{k=\Delta}^{n} \frac{a_{n-k}}{a_n}.
\]
They have also proved

**Theorem F.** If \((p_n)\) and \((a_n)\) are sequences as in Theorem E and if

\[
\limsup_{n \to \infty} \frac{p_{n+1}}{p_n} \leq 1,
\]

then

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1.
\]

The necessary and sufficient condition of Theorem E is clearly much simpler than the necessary and sufficient condition of Theorem C. This is due to the fact that the condition

\[
\sum_{k=1}^{\infty} p_k = 1 \implies \sum_{k=1}^{\infty} \frac{p_k}{R^k} > 1.
\]

If \(\sum_{k=1}^{\infty} \frac{p_k}{R^k} < 1\), the condition (1.8) does not hold as we shall see later.

To see how Theorem F is related to de Bruijn and Erdős's result (Theorem D), observe that \(\sum_{k=1}^{\infty} p_k = 1\) implies that \(R \geq 1\). If \(R > 1\), we have \(\sum_{k=1}^{\infty} k p_k < \infty\), and \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1\) by Theorem A. If \(R = 1\), we have, by (1.4), \(q = R^{-1} = 1\). The interesting fact about the result of Garsia, Orey and Rodemich is that in this case, in Theorem F, it is sufficient to assume that

\[
\limsup_{n \to \infty} \frac{p_{n+1}}{p_n} \leq 1
\]

and not
as in Theorem D.

1.3 While De Bruijn and Erdős always assume that \( (p_n) \) is a sequence of positive numbers, the principal aim of the work of Garsia, Orey and Rodemich was to weaken the hypothesis about the positivity of the coefficients \( p_n \). In Theorems 3-5, we shall always assume that \( (p_n) \) is a sequence of non-negative numbers with \( p_1 > 0 \), and that the radius of convergence \( R \) of the series \( \sum_{k=1}^{\infty} p_k x^k \) is a positive finite number. We shall study the necessary and sufficient conditions in order that (1.2) implies the existence of \( \lim_{n \to \infty} a_{n+1}/a_n \). The necessary and sufficient condition for the existence of \( \lim_{n \to \infty} a_{n+1}/a_n \) depends on the relation between the two quantities \( L \) and \( \sum_{k=1}^{\infty} p_k R^k \). First we generalize the result of De Bruijn and Erdős mentioned earlier (Theorem 0) as follows:

**Theorem 3.** Let \( (a_n) \) be a sequence of positive numbers satisfying the asymptotic relation

\[ (1.10) \quad (L + \varepsilon_n) a_n = \sum_{k=1}^{n} p_k a_{n-k}, \quad n = 1, 2, \ldots \]

where the sequence \( (\varepsilon_n) \) converges to zero and \( 0 < L < \infty \).

Then a necessary and sufficient condition for the existence of \( \lim_{n \to \infty} a_{n+1}/a_n \) is that, for every fixed \( A \),

\[ (1.11) \quad \lim_{n \to \infty} c_n(A) = 0 \]
where

\[ (1.12) \quad \phi_n(A) = \sum_{k=A}^{n} p_k \frac{a_{n-k+1}}{a_n} - \gamma \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n} \]

and \( \gamma \) is the number defined by (1.4).

In some cases it can be shown that the condition (1.11) can be replaced by a simpler condition. The following result can be considered as a generalization of Garsia, Orey and Rodemich's Theorem E:

**Theorem 4.** Let \((a_n)\) be a sequence of positive numbers satisfying the asymptotic relation (1.10). If \(\sum_{k=1}^{\infty} p_k R^k > L\), then a necessary and sufficient condition for the existence of \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n}\) is

\[ (1.13) \quad \lim_{A \to \infty} \lim_{n \to \infty} \sup_{n \in \mathbb{N}} w_n(A) = 0 \]

where

\[ (1.14) \quad w_n(A) = \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n} \]

It is easy to see that if \(\sum_{k=1}^{\infty} p_k R^k < L\), condition (1.13) can never be satisfied. If \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n}\) exists, by Theorem 2 it is equal to \(R^{-1}\). Hence, by (1.10), we have

\[ \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n} = L + \varepsilon_n - \frac{A^{-1}}{\sum_{k=A}^{n} p_k} \frac{a_{n-k}}{a_n} \]

and so
\[
\lim \sup_{n \to \infty} W_n(A) = L - \sum_{k=1}^{\infty} p_k R^k.
\]

But then
\[
\lim_{A \to \infty} \lim \sup_{n \to \infty} W_n(A) = L - \sum_{k=1}^{\infty} p_k R^k > 0.
\]

If we know that \( \sum_{k=1}^{\infty} p_k R^k > L \), it is possible to give a still simpler necessary and sufficient condition.

**Theorem 5.** Let \((a_n)\) be a sequence of positive numbers satisfying the asymptotic relation \((1.10)\). If \( \sum_{k=1}^{\infty} p_k R^k > L \), the necessary and sufficient condition for the existence of
\[
\lim_{n \to \infty} a_{n+1}/a_n
\]
is
\[
(1.15) \quad \lim \sup_{n \to \infty} \frac{n a_n}{a_{n+1}} > R^{-1}
\]

If \( \sum_{k=1}^{\infty} p_k R^k \leq L \) then the condition \((1.15)\) cannot be true. It is easy to see that if \( \lim_{n \to \infty} a_{n+1}/a_n \) exists then by \((1.4)\), this limit is equal to \( R^{-1} \). Hence \( \lim \sup_{n \to \infty} n a_n = R^{-1} \).

Finally, we shall use Theorem 4 to derive the following extensions of Theorems D and F:

**Theorem 6.** Let \((a_n)\) and \((p_n)\) be defined as in Theorem 3 and let \((1.10)\) hold. If
\[
(1.16) \quad \lim_{n \to \infty} p_{n+1}/p_n = \sigma,
\]
then
Theorem 7. Let \((a_n)\) and \((p_n)\) be defined as in Theorem 3 and let (1.10) hold. If \(\frac{\sum_{k=1}^{\infty} p_k R^k}{R} > L\) and if
\[
\limsup_{n \to \infty} \frac{p_{n+1}}{p_n} \leq \sigma,
\]
then (1.17) holds.

In both Theorems 6 and 7, the number \(\sigma\) is defined by
(1.4) and \(p_k > 0, k = 1, 2, \ldots\)

1.4 The proofs of the Theorems 1 - 7 will be given in the next three chapters. In Chapter 2 we shall prove the direct Theorems 1 and 2. The three converse results, Theorems 3, 4, and 5, will be proved in Chapter 3. In the final Chapter 4, we shall prove Theorems 6 and 7.

The proofs of Theorems 1 and 2 are straightforward. Theorems 1 and 2 are used in the proof of the necessity parts of Theorems 3 - 5. The method used here for the proofs of the sufficiency parts of Theorems 3 and 4 is essentially an extension of the method which De Bruijn and Erdős have used to prove Theorem C.

In order to prove the sufficiency part of Theorem 3 we first show that the asymptotic relation (1.10) and the condition (1.11) imply that
\[ 0 < \lambda = \liminf_{n \to \infty} \frac{a_{n+1}}{a_n} \leq \limsup_{n \to \infty} \frac{a_{n+1}}{a_n} = \Lambda < \infty. \]

Next, if \( \sigma \) is defined by (1.4) we prove that \( \Lambda \leq \sigma \)
and \( \sigma \leq \lambda \) by showing that the hypothesis \( \sigma > \lambda \) or \( \lambda > \sigma \)
leads to a contradiction.

Assuming that \( \lambda > \sigma \), and using (1.11) and the inequality

\[
\left| d^0_n(A) \right| + I(\Lambda + \epsilon - \frac{a_{n+1}}{a_n}) - \epsilon \frac{a_{n+1}}{a_n} + (\Lambda + \epsilon) c_n \geq
\]

\[
\sum_{k=1}^{A-1} p_k \left( \frac{a_{n-k} - a_{n-k+1}}{a_n} \right) + (\Lambda - \sigma) \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n},
\]

we find that

\[
\frac{a_{n_i+1}}{a_{n_i}} \rightarrow \Lambda \quad (i \rightarrow \infty)
\]

implies

\[
\frac{a_{n_i-j+1}}{a_{n_i-j}} \rightarrow \Lambda \quad (i \rightarrow \infty)
\]

for any positive integer \( j \) such that \( p_j > 0 \), and

\[
\limsup_{k \to \infty} \sum_{k=A}^{n_i} p_k \frac{a_{n_i-k}}{a_{n_i}} \rightarrow 0 \quad (A \rightarrow \infty).
\]

Using these results we next prove that \( L = \sum_{k=1}^{\infty} p_k \Lambda^{-k} \).
which is impossible by the definition of $\gamma$. This shows that $\Lambda \leq \gamma$. Likewise, we get $\gamma \leq \Lambda$.

In the proof of sufficiency part of Theorem 4, using the asymptotic relation (1.10) and the condition (1.13) we again prove that $0 < \Lambda \leq \Lambda < \infty$.

Next, using (1.13) and the inequality
\[
(\Lambda + \varepsilon)^{n+1} - L((\Lambda + \varepsilon) - \frac{a_{n+1}}{a_n}) - \varepsilon_n \frac{a_{n+1}}{a_n} + (\Lambda + \varepsilon)\varepsilon_n \geq \varepsilon_{n+1} \frac{a_{n+1}}{a_n} - \varepsilon_n \frac{a_{n-k+1}}{a_n},
\]
we prove again that
\[
\frac{a_{n+1}}{a_n} \rightarrow \Lambda \quad (n \rightarrow \infty)
\]
implies
\[
\frac{a_{n-j+1}}{a_{n-j}} \rightarrow \Lambda \quad (n \rightarrow \infty)
\]
for any positive integer $j$ such that $p_j > 0$.

Using this result we can prove again that
\[
L = \sum_{k=0}^{N} p_k \Lambda^{-k}.
\]
Since $\sum_{k=1}^{\infty} p_k \Lambda^{-k} \geq L$, we have $\Lambda^{-1} = \xi$, where $\xi$ is the unique
number in \((0, R)\) such that \(\sum_{k=1}^{\infty} p_k \xi^k = L\).

Likewise, \(\alpha^{-1} = \xi\) and the theorem is proved.

For the proof of the sufficiency part of Theorem 5, assuming that

\[ \alpha^{-1} = \limsup_{n \to \infty} \frac{n}{a_n} > R^{-1}, \]

instead of (1.10) we consider the asymptotic relation

(1.19) \((L + \gamma_n(\xi))A_n(\xi) = \frac{n}{\sum_{k=1}^{\infty} p_k \xi^k} A_{n-k}(\xi)\)

where

\[ A_n(\xi) = \sum_{k=0}^{n} a_k \xi^k, \quad \xi \in (\xi_1, R) \]

and

\[ \lim_{n \to \infty} \gamma_n(\xi) = 0. \]

If

\[ \alpha'_\xi = \liminf_{n \to \infty} \frac{a_n \xi^n}{A_n(\xi)} \leq \limsup_{n \to \infty} \frac{a_n \xi^n}{A_n(\xi)} = \beta'_\xi. \]

We first show that there exists \(r \in (\xi_1, R)\) such that

\[ 0 < \beta'_r < 1. \]

Next, using the asymptotic relation (1.19) we show that
\[
\frac{a_{n_1} r^{n_1}}{A_{n_1}(r)} \to \beta_r \quad (i \to \infty)
\]

implies

\[
\frac{a_{n_1-j} r^{n_1-j}}{A_{n_1-j}(r)} \to \beta_r \quad (i \to \infty)
\]

for any positive integer \( j \) such that \( p_j > 0 \), and likewise, that

\[
\frac{a_{m_1} r^{m_1}}{A_{m_1}(r)} \to \alpha_r \quad (i \to \infty)
\]

implies

\[
\frac{a_{m_1-j} r^{m_1-j}}{A_{m_1-j}(r)} \to \alpha_r \quad (i \to \infty)
\]

for any positive integer \( j \) such that \( p_j > 0 \).

From these results follows immediately that

\[
\sum_{k=1}^{\infty} p_k r^k (1 - \alpha_r)^k = L = \sum_{k=1}^{\infty} p_k r^k (1 - \beta_r)^k,
\]

and we conclude that \( \alpha_r = \beta_r = c_r \).

Finally, from
\[ \frac{a_n}{a_{n+1}} = r \frac{r^n a_n}{A_n(r)} \frac{A_n(r)}{A_{n+1}(r)} \frac{A_{n+1}(r)}{a_{n+1} r^{n+1}}, \]

and \( r \in (0, 1) \), we conclude that

\[ \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = r \frac{1}{c_r} (1 - c_r) c_r = r (1 - c_r). \]

It should be mentioned here that more general results, for the inversion of convolution products of integrable functions, which correspond to Theorem 5, have been obtained by A. Edrei and W. H. J. Fuchs [7] and D. Drasin [8].

The proofs of Theorems 6 and 7 are similar to the proofs of Theorems 3 and 4.

We first show that the hypothesis

\[ \limsup_{n \to \infty} \frac{p_{n+1}}{p_n} \leq \sigma \]

implies that \( \lambda \leq \sigma \) and, likewise, that

\[ \liminf_{n \to \infty} \frac{p_{n+1}}{p_n} \geq \sigma \]

implies that \( \sigma \leq \lambda \). Hence, if

\[ \lim_{n \to \infty} \frac{p_{n+1}}{p_n} = \sigma, \]

we have \( \lambda = \sigma = \lambda \), and Theorem 6 follows.
To prove Theorem 7, we observe that if $\Lambda \leq \mathcal{C}$, then the condition (1.13) holds. Thus if we assume in addition that

$$L \leq \sum_{k=1}^{\infty} p_k \ r_k$$

then the proof of Theorem 7 follows from Theorem 4.
2.1 Proof of Theorem 1. Let

\[ S_n = \sum_{k=1}^{n} p_k \frac{a_{n-k}}{a_n} - \sum_{k=1}^{\infty} p_k \lambda^k. \]

To prove the theorem it is clearly sufficient to show that

\[ \lim_{n \to \infty} S_n = 0. \]

We first note that if

\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lambda^{-1}, \]

then

\[ \lim_{n \to \infty} \frac{\sqrt[n]{a_n}}{\lambda} > R^{-1} = \lim_{n \to \infty} \sup \frac{\sqrt[n]{|p_n|}}{\sqrt[n]{|p_n|}}. \]

Next, choose \( \gamma \in (0, \frac{R-\lambda}{R+\lambda}) \). We have then \( 0 < \gamma < 1 \) and

\[ \frac{1-\gamma}{\lambda} > \frac{1+\gamma}{R}. \]

From (2.1), (2.2) and (2.3) we find that the following inequalities hold for all \( n > N_\gamma \):

\[ \frac{a_{n+1}}{a_n} > \frac{1-\gamma}{\lambda}. \]
(2.5) \[ a_n > \left( \frac{1+\gamma}{R} \right)^n \]

(2.6) \[ |p_n| < \left( \frac{1+\gamma/2}{R} \right)^n. \]

Hence, for any positive integer \( A \) and \( n > A + N_\gamma \), we have

\[ |s_n| \leq \left( \sum_{k=1}^{A-1} \frac{n}{k} + \sum_{k=A}^{n-N_\gamma} \frac{n}{k} + \sum_{k=n-N_\gamma}^{n} \right) |p_n| \left| \frac{a_n-k}{a_n} - \lambda \right| \]

\[ = I_n + J_n + K_n. \]

We first consider \( I_n \):

\[ I_n \leq \max_{1 \leq k \leq A} \left| \frac{a_n-k}{a_n} - \lambda \right| \sum_{k=1}^{A-1} \frac{n}{k} |p_k|. \]

Next, consider \( J_n \). By (2.4) we find that

\[ J_n \leq \sum_{k=A}^{n-N_\gamma-1} \frac{n}{k} |p_k| \left( \frac{\lambda}{1-\gamma} \right)^k + \sum_{k=n-N_\gamma}^{n} |p_k| \left( \frac{\lambda}{1-\gamma} \right)^k \]

\[ \leq 2 \sum_{k=A}^{n-N_\gamma-1} |p_k| \left( \frac{\lambda}{1-\gamma} \right)^k. \]

Finally, by (2.5) and (2.6), we have

\[ K_n \leq \left( \max_{0 \leq k \leq N_\gamma} a_k \right) \sum_{k=n-N_\gamma}^{n} \frac{n}{k} |p_k| / a_n + \sum_{k=n-N_\gamma}^{n} |p_k| \left( \frac{\lambda}{1-\gamma} \right)^k \]

\[ \leq \left( \max_{0 \leq k \leq N_\gamma} a_k \right) \sum_{k=n-N_\gamma}^{n} \left( \frac{1+\gamma/2}{R} \right)^k \left( \frac{R}{1+\gamma} \right)^n + \sum_{k=n-N_\gamma}^{n} |p_k| \left( \frac{\lambda}{1-\gamma} \right)^k. \]
\[ \leq \left( \max_{0 \leq k \leq N} a_k \right) \left( \frac{1 + \sqrt{2}}{1 + \sqrt{1}} \right)^n \sum_{k=0}^\infty \frac{\lambda^k}{k!} k + 2 \sum_{k=0}^N |p_k| \left( \frac{\lambda}{1-\sqrt{1}} \right)^k. \]

Hence,

\[ |S_n| \leq \max_{1 \leq k \leq N} \left| \frac{a_{n-k}}{a_n} - \lambda^k \right| \left( \sum_{k=1}^{\infty} |p_k| \right) + 2 \sum_{k=0}^N |p_k| \left( \frac{\lambda}{1-\sqrt{1}} \right)^k + \left( \max_{0 \leq k \leq N} a_k \right) \left( \frac{1 + \sqrt{2}}{1 + \sqrt{1}} \right)^n \sum_{k=0}^\infty \frac{\lambda^k}{k!} k. \]

Since \( \frac{1 + \sqrt{2}}{1 + \sqrt{1}} < 1 \) and \( \frac{a_{n-k}}{a_n} \to \lambda^k, \ k = 1, 2, \ldots, \) we have by (2.3),

\[ \limsup_{n \to \infty} |S_n| \leq 2 \sum_{k=0}^\infty |p_k| \left( \frac{\lambda}{1-\sqrt{1}} \right)^k \leq 2 \sum_{k=0}^\infty |p_k| \left( \frac{\lambda}{1+\sqrt{1}} \right)^k \]

and the theorem follows by choosing \( A \) sufficiently large.

2.2 Proof of Theorem 2. Suppose that

(2.7) \[ \lim_{n \to \infty} a_{n+1}/a_n = \alpha \]

exists. We shall prove that \( \alpha = \sigma \), where \( \sigma \) is defined by (1.4). Rewrite (1.2) as follows

(2.8) \[ (L + \varepsilon_n)a_n = \sum_{k=1}^n p_k a_{n-k}, \quad n = 1, 2, \ldots \]

where \( \varepsilon_n \to 0 \) (\( n \to \infty \)). Then

\[ (L + \varepsilon_n)a_n > a_0 p_n \]
and so

\[(2.9) \quad \limsup_{n \to \infty} \frac{a_n}{a_{n+1}} \geq R^{-1} = \limsup_{n \to \infty} \frac{\sqrt[p]{p_n}}{p_{n+1}}.\]

If (2.7) holds, it follows from (2.9) that we always have

\[\alpha \geq R^{-1}.\]

Suppose first that

\[\sum_{k=1}^{\infty} p_k R^k \leq L.\]

We have to show that \(\alpha = R^{-1}\). If we had \(\alpha > R^{-1}\), we would have, by Theorem 1,

\[L = \lim_{h \to \infty} \frac{n}{a_n} \frac{p_k a_{n-k}}{a_n} = \sum_{k=1}^{\infty} p_k \alpha^{-k} < \sum_{k=1}^{\infty} p_k R^k,\]

which is impossible. Hence, we have

\[\alpha = R^{-1} = \kappa.\]

Next, suppose that

\[\sum_{k=1}^{\infty} p_k R^k > L.\]

From (2.8), we have

\[L + e_n \geq \sum_{k=1}^{A} p_k \frac{a_{n-k}}{a_n},\]

for any \(n > A\) and it follows immediately that

\[L \geq \sum_{k=1}^{A} p_k \alpha^{-k}.\]
Since $A$ can be chosen arbitrarily large, it follows that

$$L \geq \sum_{k=1}^{\infty} p_k \alpha^{-k}.$$  

If $\alpha = R^{-1}$, then $L \geq \sum_{k=1}^{\infty} p_k R^k$ which is impossible. Hence we must have $\alpha > R^{-1}$ and by Theorem 1

$$L = \lim_{n \to \infty} \sum_{k=1}^{n} p_k \frac{a_{n-k}}{a_n} = \sum_{k=1}^{\infty} p_k \alpha^{-k}.$$  

Hence, by (1.4), we have $\alpha^{-1} = \gamma = \sigma^{-1}$, or $\alpha = \sigma$. This completes the proof of Theorem 2.
Chapter 3

PROOFS OF THE CONVERSE RESULTS

3.1 Proof of Theorem 3. First we shall prove the necessity of the condition (1.11). If \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) exists, we have by Theorem 2

\[
(3.1) \quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \sigma.
\]

Then, by (1.12), if \( n > A \), we have

\[
a_n \phi_n(A) = \frac{n+1}{k=A} p_k a_{n-k+1} - \sigma \frac{n}{k=A} p_k a_{n-k} = (I+\varepsilon_{n+1}) a_{n+1} - \sigma (I+\varepsilon_n) a_n + \frac{A}{k=A} p_k (v a_{n-k} - a_{n-k+1})
\]

Hence,

\[
|\phi_n(A)| \leq L \left| \frac{a_{n+1}}{a_n} - \sigma \right| + \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n} \left| \frac{a_{n-k+1}}{a_{n-k}} - \sigma \right| + |\varepsilon_{n+1}| \left| \frac{a_{n+1}}{a_n} \right| + |\varepsilon_n|.
\]

By (3.1), it follows that

\[
\lim_{n \to \infty} \phi_n(A) = 0.
\]

Before we proceed to the proof of the sufficiency we shall establish the following lemmas.
Lemma 3.1 If the condition (1.11) holds, we have

\[ 0 < \lambda = \lim \inf_{n \to \infty} \frac{a_{n+1}}{a_n} \leq \lim \sup_{n \to \infty} \frac{a_{n+1}}{a_n} = \lambda < \infty. \]

Proof: By (1.10), we can find \( N_L \) such that

\[ |\varepsilon_n| < \frac{1}{2} L \quad \text{for } n \geq N_L. \]

We have then

\[ (3/2) L a_{n+1} \geq (L + \varepsilon_{n+1}) a_{n+1} \]

\[ = \sum_{k=1}^{n+1} p_k a_{n+1-k} > p_1 a_n \]

which implies that

\[ \lambda = \lim \inf_{n \to \infty} \frac{a_{n+1}}{a_n} \geq \frac{2p_1}{3L} > 0. \]

Next, we prove that \( \lambda < \infty \). By (1.10) and (1.12), we have

\[ (L+\varepsilon_{n+1}) \frac{a_{n+1}}{a_n} - (L+\varepsilon_{n}) = \sum_{k=1}^{n+1} p_k \frac{a_{n+1-k}}{a_n} - \varepsilon \sum_{k=1}^{n} p_k \frac{a_{n-k}}{a_n} \]

\[ = \sum_{k=1}^{n+1} p_k \left( \frac{a_{n+1-k}}{a_n} - \varepsilon \frac{a_{n-k}}{a_n} \right) + \phi_n(a) \]

\[ \leq \sum_{k=1}^{n+1} p_k \frac{a_{n+1-k}}{a_n} + \phi_n(a). \]

Using (3.3), for \( n \geq k > N_L \), we find that
Hence, by (3.2) and (3.4), if $n > A + N_L$, we have
\[
\frac{a_{n+1}}{a_n} \leq \frac{3L}{2p_1}.
\]

Since $\lim_{n \to \infty} \phi_n(A) = 0$, we conclude that
\[
\Lambda = \limsup_{n \to \infty} \frac{a_{n+1}}{a_n} \leq 2\epsilon + \frac{2}{L} \sum_{k=1}^{\infty} p_k \left( \frac{3L}{2p_1} \right)^{k-1} < \infty.
\]

In the next two lemmas, let $\sigma$ be defined by (1.4) and suppose that (1.11) holds. Since both lemmas can be proved in the same manner, we shall only prove the first one.

**Lemma 3.2** If $\sigma < \Lambda < \infty$ and if $(n_1)$ is a sequence such that
\[
(3.5) \quad \lim_{i \to \infty} \frac{a_{n_1+1}}{a_{n_1}} = \Lambda,
\]

then for each fixed positive integer $j$ such that $p_j > 0$, we have
\[
(3.6) \quad \lim_{i \to \infty} \frac{a_{n_1+1-j}}{a_{n_1-j}} = \Lambda,
\]

and for any positive integer $A$, we have
Lemma 3.3 If $0 < \lambda < \sigma$ and if $(m_1)$ is a sequence such that

$$\lim_{i \to \infty} \sup_{n \geq N} \frac{a_{n_1-k}}{a_{n_1}} = 0.$$  

then for each fixed positive integer $j$ such that $p_j > 0$, we have

$$\lim_{i \to \infty} \frac{a_{m_1+1-j}}{a_{m_1-j}} = \lambda.$$  

Proof of Lemma 3.2. By (1.10) and (1.12), we have

$$\varphi_n(\lambda) = \frac{a_{n+1}}{a_n} (I + \epsilon + 1) - (\lambda + \epsilon)(I + \epsilon) +$$

$$+ \sum_{k=1}^{n-1} \frac{(\lambda + \epsilon)a_{n-k}}{a_n} - \frac{a_{n+1-k}}{a_n} + (\lambda - \sigma + \epsilon) \sum_{k=1}^{n} \frac{a_{n-k}}{a_n}.$$  

Let $(n_1)$ be a sequence such that (3.5) holds. Replace $n$ by $n_1$ in (3.10) and use the fact that if $\epsilon > 0$ and $k > N_\epsilon$ we have

$$\frac{a_{k+1}}{a_k} < \lambda + \epsilon.$$  

Hence, (3.10) becomes
\[ |\phi_n(A)| \geq \frac{a_{n+1}}{a_n^k} (1 + \varepsilon) a_{n+1} - (\Lambda + \varepsilon) (1 - a_n a_{n+1}) + (\Lambda - \sigma + \varepsilon) \sum_{k=1}^{\infty} p_k \frac{a_{n-1}}{a_n} \]

and so
\[ (\Lambda - \sigma + \varepsilon) \sum_{k=1}^{\infty} p_k \frac{a_{n-1}}{a_n} \leq (\Lambda + \varepsilon) - \frac{a_{n+1}}{a_n} \Lambda + |\phi_n(A)|. \]

Let \( i \to \infty \). Since \( \Lambda > \sigma \) we then have
\[ \lim_{i \to \infty} \sup \frac{w_i}{k=1} p_k \frac{a_{n-1}}{a_n} \leq (\varepsilon \Lambda + \lim_{i \to \infty} \sup |\phi_n(A)|) / (\Lambda - \sigma + \varepsilon) \]
\[ \leq \varepsilon \Lambda / (\Lambda - \sigma), \]

and (3.7) follows, since \( \varepsilon \) can be chosen arbitrarily small.

Next, since \( \sigma < \Lambda \) and \((\Lambda + \varepsilon) a_{n-k} - a_{n+1-k} \geq 0\), we have
\[ |\phi_n(A)| \geq \frac{a_{n+1}}{a_n} - (\Lambda + \varepsilon) \Lambda + p_j \frac{(\Lambda + \varepsilon) a_{n-1} - a_{n+1-j}}{a_n} \]
\[ \geq \frac{a_{n+1}}{a_n} - (\Lambda + \varepsilon) \Lambda + p_j \frac{a_{n-1}}{a_n} ((\Lambda + \varepsilon) - \frac{a_{n+1-j}}{a_{n-1}}). \]

Hence
\[ -\varepsilon \leq \frac{a_{n+1-j}}{a_{n-1}} \leq (\Lambda + \varepsilon) - \frac{a_{n+1-j}}{a_{n-1}} \leq \]
\[ \leq \frac{1}{p_j a_{n-1}} ((\Lambda + \varepsilon) - \frac{a_{n+1}}{a_n}) \Lambda + \phi_n(A). \]
Since $\frac{a_{n_k}}{a_{n_1-1}} < (\lambda + \epsilon)^j$ and $\limsup_{i \to \infty} \varphi_{n_1}(\lambda) = 0$, it follows that

$$-\epsilon \leq \liminf_{i \to \infty} (\lambda - \frac{a_{n_k+1-j}}{a_{n_1-j}}) \leq \limsup_{i \to \infty} (\lambda - \frac{a_{n_k+1-j}}{a_{n_1-j}}) \leq \frac{(\lambda + \epsilon)^j \lambda \epsilon}{p_j}.$$  

Since $\epsilon$ can be chosen arbitrarily small, we conclude that

$$\lim_{i \to \infty} \frac{a_{n_k+1-j}}{a_{n_1-j}} = \lambda$$

for each $j$ such that $p_j > 0$. This completes the proof of the lemma.

We can now prove the sufficiency of the condition $(1.11)$. It is clearly sufficient to show that $\lambda \leq \varphi$ and $\varphi \leq \lambda$. By Lemma 3.1 there exists a sequence $(n_1)$ such that $(3.5)$ holds with $0 < \lambda < \infty$. Suppose that $\lambda > \varphi$. Then, by Lemma 3.2, we have also for all $k$ such that $p_k > 0$

$$\frac{a_{n_1-k}}{a_{n_1}} \to \lambda^{-k} \quad (1 \to \infty).$$

Using $(1.10)$ we find that

$$\left| L - \sum_{k=1}^{\lambda} p_k \frac{a_{n_1-k}}{a_{n_1}} \right| \leq \sum_{k=1}^{\lambda} p_k \frac{a_{n_1-k}}{a_{n_1}} + \left| \epsilon_{n_1} \right|.$$
Hence

\[ \left| L - \sum_{k=1}^{n-1} p_k \Lambda^{-k} \right| \leq \limsup_{t \to \infty} \sum_{k=1}^{n-1} p_k \frac{a_{n-1-k}}{a_{n-1}} . \]

Since \( A \) can be chosen arbitrarily large, by (3.7), we get

\[ L = \sum_{k=1}^{\infty} p_k \Lambda^{-k} . \]

We can now easily show that our hypothesis \( \Lambda > \sigma \) leads to a contradiction.

Suppose that first \( L \geq \sum_{k=1}^{\infty} p_k R_k \). We have then by (1.4) \( \sigma = R^{-1} \). Since \( \Lambda > \sigma = R^{-1} \), we have

\[ L = \sum_{k=1}^{\infty} p_k \Lambda^{-k} < \sum_{k=1}^{\infty} p_k R_k \leq L \]

which is impossible. Hence \( \Lambda \leq \sigma \).

Next, suppose that \( L < \sum_{k=1}^{\infty} p_k R_k \). Then we have by (1.4) \( \sigma = \gamma^{-1} \), where \( \sum_{k=1}^{\infty} p_k \gamma^k = L \). Since \( \Lambda > \sigma = \gamma^{-1} \), we have

\[ L = \sum_{k=1}^{\infty} p_k \Lambda^{-k} < \sum_{k=1}^{\infty} p_k \gamma^k = L \]

which is again impossible. Hence \( \Lambda \leq \sigma \).

To complete the proof of the theorem we have to show that \( \sigma \leq \lambda \).

Suppose that \( \sigma > \lambda \). We have, by (1.10) and (1.12),
Let \((m_1)\) be a sequence so that \((3.8)\) holds and let \(A\) be such that \(p_A > 0\). If \(e > 0\) and \(k > N\), then \(a_{k+1} - (\lambda - e)a_k \geq 0\).

Hence, for \(m_1 > A + N\), we have

\[
L\left(\frac{a_{m_1+1}}{a_{m_1}} - (\lambda - e)\right) \geq e_{m_1} - e_{m_1+1} - \frac{a_{m_1+1}}{a_{m_1}} + \phi_{m_1}(A) +
\]

\[+ (\sigma - \lambda + e) \sum_{k=A}^{m_1} p_k \frac{a_{m_1-k}}{e_{m_1}}.
\]

Let \(i \to \infty\). Since \(\sigma > \lambda\), we have, by Lemma 3.3,

\[e \in L \geq (\sigma - \lambda) p_A \lambda^{-A},\]

a contradiction, since \(\varepsilon\) can be chosen arbitrarily small.

Thus we must have \(\sigma \leq \lambda\), and the theorem is proved.

3.2 Proof of Theorem 4. We shall first prove the
necessity of the condition \((1.13)\). If \(\lim_{n \to \infty} a_{n+1}/a_n\) exists,
then by Theorem 2 it is equal to \(\sigma\) where \(\sigma\) is defined by
\((1.4)\). We shall first consider the case \(\sum_{k=1}^{\infty} p_k R^k = L\). Then
\(\sigma = R^{-1}\). By \((1.10)\), we have
Since
\[
\lim_{n \to \infty} \frac{a_{n-k}}{a_n} = R^k, \quad k = 1, 2, \ldots
\]
it follows that
\[
\limsup_{n \to \infty} \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n} = L - \sum_{k=A}^{n} p_k R^k.
\]
Hence, we have
\[
\lim_{A \to \infty} \left( \limsup_{n \to \infty} \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n} \right) = 0.
\]
Next, suppose that \( \sum_{k=1}^{\infty} p_k R^k > L \). Then by Theorem 2, we have \( \sigma = \gamma^{-1} \) and \( \sum_{k=1}^{\infty} p_k \gamma^k = L \). Hence
\[
\sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n} = L + e_n - \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}.
\]
Since
\[
\lim_{n \to \infty} \frac{a_{n-k}}{a_n} = \sigma^{-k} = \gamma^k, \quad k = 1, 2, \ldots
\]
we find that
\[
\limsup_{n \to \infty} \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n} = L - \sum_{k=A}^{n} p_k \gamma^k.
\]
Let \( A \to \infty \), and (1.13) follows.
The proof of the sufficiency is based on the following lemmas:

**Lemma 3.4** If the condition (1.13) holds, we have

\[ 0 < \lambda = \lim \inf_{n \to \infty} \frac{a_{n+1}}{a_n} \leq \lim \sup_{n \to \infty} \frac{a_{n+1}}{a_n} = \lambda < \infty. \]

**Proof:** By (1.13), for any \( \varepsilon > 0 \), we can choose first \( A_\varepsilon \) such that

\[ \lim \sup_{n \to \infty} \sum_{k=1}^{A_\varepsilon} \frac{a_{n-k}}{a_n} \leq \frac{1}{2} (L + \varepsilon), \]

then, we can find \( N_\varepsilon > A_\varepsilon \) such that for \( n > N_\varepsilon \) we have

\[ \sum_{k=A_\varepsilon}^{n} \frac{a_{n-k}}{a_n} \leq \frac{1}{2} (L + \varepsilon). \]

If \( n > A_\varepsilon + N_\varepsilon \) we have

\[ (L + \varepsilon) \frac{a_{n+1}}{a_n} = \sum_{k=1}^{A_\varepsilon-1} \frac{a_{n-k}}{a_n} + \sum_{k=1}^{N_\varepsilon} \frac{a_{n-k}}{a_n} + \frac{a_{n+1}}{a_n} \sum_{k=A_\varepsilon}^{n} \frac{a_{n-k}}{a_n} \leq \frac{1}{2} (L + \varepsilon) + \frac{A_{\varepsilon-1}}{A_\varepsilon} \sum_{k=1}^{N_\varepsilon} \frac{a_{n-k}}{a_n} \sum_{k=A_\varepsilon}^{n} \frac{a_{n-k}}{a_n} \frac{a_{n-k}}{a_n} \cdot \]

Hence, using (3.4), we find that

\[ (L + \varepsilon) \frac{a_{n+1}}{a_n} \leq \frac{1}{2} (L + \varepsilon) + \frac{A_{\varepsilon-1}}{2} \sum_{k=A_\varepsilon}^{n} \frac{a_{n-k}}{a_n} (3L/2P_1)^{k-1}. \]

Since \( \varepsilon \) can chosen arbitrarily small, it follows that
Lemma 3.5 Suppose that the condition (1.13) holds. If the sequence \((n_1)\) is such that

\[
\lim_{i \to \infty} \frac{a_{n_1+i}}{a_{n_1}} = \Lambda < \infty,
\]

then for each positive integer \(j\) such that \(p_j > 0\), we have

\[
\lim_{i \to \infty} \frac{a_{n_1+i-j}}{a_{n_1-j}} = \Lambda.
\]

Likewise, if

\[
\lim_{i \to \infty} \frac{a_{m_1+i}}{a_{m_1}} = \Lambda > 0,
\]

then

\[
\lim_{i \to \infty} \frac{a_{m_1+i-j}}{a_{m_1-j}} = \Lambda,
\]

for each fixed \(j\) such that \(p_j > 0\).

Proof: If \(\varepsilon > 0\) and \(k > N_{\varepsilon}\) we have

\[
\frac{a_{k+1}}{a_k} < \Lambda + \varepsilon.
\]
Using (1.10), for \( n_1 > A + N_\varepsilon \), we find that
\[
L(\langle A+\varepsilon \rangle a_{n_1} - a_{n_1+1}) = (\langle A+\varepsilon \rangle \sum_{k=1}^{n_1} p_k a_{n_1-k} - (\langle A+\varepsilon \rangle \varepsilon_{n_1} a_{n_1} - \\
\quad - \sum_{k=1}^{n_1+1} p_k a_{n_1+1-k} + \varepsilon_{n_1+1} a_{n_1+1})
\]

By (1.14) and (3.16), we have
\[
L((\langle A+\varepsilon \rangle - \frac{a_{n_1+1}}{a_{n_1}}) \geq p_j \frac{a_{n_1-j}}{a_{n_1}} (\langle A+\varepsilon \rangle - \frac{a_{n_1+1-j}}{a_{n_1-j}}) - \frac{a_{n_1+1}}{a_{n_1}} W_{n_1+1}(A) \\
- (\langle A+\varepsilon \rangle \varepsilon_{n_1} + \varepsilon_{n_1+1} \frac{a_{n_1+1}}{a_{n_1}})
\]
\[
\geq p_j (\langle A+\varepsilon \rangle - \frac{a_{n_1+1-j}}{a_{n_1-j}}) - (\langle A+\varepsilon \rangle \varepsilon_{n_1} \\
- \frac{a_{n_1+1}}{a_{n_1}} W_{n_1+1}(A) + \varepsilon_{n_1+1} \frac{a_{n_1+1}}{a_{n_1}})
\]

Hence
\[
- \varepsilon \leq A - \frac{a_{n_1+1-j}}{a_{n_1-j}} \leq \varepsilon + \frac{a_{n_1+1-j}}{a_{n_1-j}} \leq \\
\leq \frac{(\langle A+\varepsilon \rangle)^j}{p_j} \left\{ L((\langle A+\varepsilon \rangle - \frac{a_{n_1+1}}{a_{n_1}}) + \frac{a_{n_1+1}}{a_{n_1}} W_{n_1+1}(A) + \\
(\langle A+\varepsilon \rangle \varepsilon_{n_1} + \varepsilon_{n_1+1} \frac{a_{n_1+1}}{a_{n_1}}) \right\}.
\]
Hence

\[ -\varepsilon \leq \lim \inf_{i \to \infty} \left( \Lambda - \frac{a_{n+1-j}}{a_{n_j}} \right) \leq \lim \sup_{i \to \infty} \left( \Lambda - \frac{a_{n+1-j}}{a_{n_j}} \right) \leq \frac{(\Lambda + \varepsilon)^j}{p_j} \left( \varepsilon L + \Lambda \lim \sup_{i \to \infty} W_{n+1}(A) \right) \]

and so

\[ \lim_{i \to \infty} \frac{a_{n+1-j}}{a_{n_j}} = \Lambda, \]

since \( \varepsilon \) can be chosen arbitrarily small and \( \Lambda \) can be chosen arbitrarily large. The proof of (3.15) can be established by same argument.

Now we can prove the sufficiency of the condition (1.13). Let \( (n_1) \) be a sequence so that (3.13) holds. By (1.10), we have

\[ L + \varepsilon n_1 = \sum_{k=1}^{\infty} p_k \frac{a_{n_1-k}}{a_{n_1}} \]

\[ = \frac{\Lambda - 1}{k} p_k \frac{a_{n_1-k}}{a_{n_1}} + \sum_{k=1}^{\infty} p_k \frac{a_{n_1-k}}{a_{n_1}}. \]

Hence

\[ \left| L - \sum_{k=1}^{\infty} p_k \frac{a_{n_1-k}}{a_{n_1}} \right| \leq \sum_{k=1}^{\infty} p_k \frac{a_{n_1-k}}{a_{n_1}}. \]
Using Lemma 3.5, we find that

\[ \left| L - \sum_{k=1}^{\infty} \frac{a_k^{-1}}{p_k} \lambda^{-k} \right| \leq \lim_{\varepsilon \to 0} \sup_{\varepsilon > 0} \sum_{k=1}^{\infty} \frac{p_k}{p_k^{a_k}} \frac{a_{n_k-k}}{a_{n_k}}. \]

Finally, using condition (1.13), we find that

\[ L = \sum_{k=1}^{\infty} \frac{a_k^{n_k}}{p_k} \lambda^{-k}. \]

Similarly, by (3.15), we have

\[ L = \sum_{k=1}^{\infty} \frac{a_k^{n_k}}{p_k} \lambda^{-k}. \]

Since \( \sum_{k=1}^{\infty} p_k \lambda^k \geq L \), we have \( \lambda = \xi^{-1} = \lambda \), where \( \xi \) is the unique number in \((0, R]\) such that \( \sum_{k=1}^{\infty} p_k \xi^k = L \). Hence Theorem 4 is proved.

3.3 **Proof of Theorem 5.** (Necessity) If \( \lim_{n \to \infty} a_{n+1}/a_n \) exists by Theorem 2 it is equal to \( \sigma \), where \( \sigma \) is defined by (1.4). Since \( \sum_{k=1}^{\infty} p_k \lambda^k > L \), we have \( \sigma = \gamma^{-1} \), where \( \sum_{k=1}^{\infty} p_k \gamma^k = L \). Hence

\[ \lim_{n \to \infty} \frac{n}{a_n} = \sigma = \gamma^{-1} \geq R^{-1}. \]

If we had \( \sigma = R^{-1} \), it would follow that

\[ L = \sum_{k=1}^{\infty} p_k \gamma^k = \sum_{k=1}^{\infty} p_k \sigma^{-k} = \sum_{k=1}^{\infty} p_k \lambda^k > L \]

which is impossible. Hence \( \sigma > R^{-1} \) and the condition (1.15) is necessary.
In order to prove the sufficiency of the condition (1.15) we define a sequence \((A_n(\xi))\) by the following relation

\[
A_n(\xi) = \sum_{k=1}^{\infty} \xi^k \alpha_k \quad n = 1, 2, \ldots
\]

Let \(\xi\) be the radius of convergence of the series \(\sum_{k=1}^{\infty} \alpha_k x^k\). Then, by (1.15), \(0 \leq \xi < R\) and we have, for every \(\xi > \xi\),

\[
A_n(\xi) \to \infty \quad (n \to \infty).
\]

We shall first use (1.10) to show that \((A_n(\xi))\) satisfies an asymptotic relation similar to (1.10).

**Lemma 3.6** For every \(\xi \in (\xi, R)\) we have

\[
(L + \gamma_n(\xi))A_n(\xi) = \sum_{k=1}^{\infty} \xi^k \sum_{n=1}^{\infty} \alpha_n \quad \gamma_n(\xi) \to 0 \quad (n \to \infty).
\]

**Proof:** In order to prove this relation observe that, by (1.10), we have

\[
(L + \varepsilon_n)a_n = \sum_{k=1}^{\infty} \varepsilon_k \alpha_n \quad a_n.
\]

Multiplying this equation by \(\xi^n\) and summing with respect to \(n\) from 1 to \(N\) we find that

\[
L \sum_{n=1}^{N} \varepsilon_n \xi^n + \sum_{n=1}^{N} \xi^n \alpha_n \xi^n = \sum_{k=1}^{\infty} \xi^k \sum_{n=k}^{\infty} \sum_{n=1}^{\infty} \alpha_n \xi^{n-k} \alpha_{n-k}
\]

\[
= \sum_{k=1}^{\infty} \xi^k \sum_{n=k}^{\infty} \alpha_n \xi^{n-k} \alpha_{n-k}.
\]
Hence
\[ L A_n(\varphi) + \sum_{k=1}^{N} \varepsilon_k a_k S_k^k = \sum_{k=1}^{N} p_k S_k^k A_{N-k}(\varphi) \]
or, replacing \( N \) by \( n \), we have
\[ (L + \mathcal{V}_n(\varphi))A_n(\varphi) = \sum_{k=1}^{n} p_k S_k^k A_{n-k}(\varphi) \quad n = 1, 2, \ldots \]
where
\[ \mathcal{V}_n(\varphi) = \frac{1}{A_n(\varphi)} \sum_{k=1}^{n} \varepsilon_k a_k S_k^k. \]

It is easy to see that \( \mathcal{V}_n(\varphi) \to 0 \quad (n \to \infty) \) for every \( \varphi \in (S, R) \). For \( k > N_\varepsilon \) we have \( |\varepsilon_k| \leq \varepsilon \). Hence for \( n > N_\varepsilon \), we have
\[ \mathcal{V}_n(\varphi) \leq \frac{1}{A_n(\varphi)} \sum_{k=1}^{N_{\varepsilon}-1} \varepsilon_k a_k S_k^k + \varepsilon. \]

Since \( \varphi \in (S, R) \), we have \( A_n(\varphi) \to \infty \quad (n \to \infty) \). Hence
\[ \limsup_{n \to \infty} \mathcal{V}_n(\varphi) \leq \varepsilon. \]

This completes the proof of Lemma 3.6.

To prove that the limit of the sequence \( \left( a_{n+1}/a_n \right) \) exists, we shall prove first that there exists \( c_r \in (0, 1) \), for some \( r \in (S, R) \), such that
\[ (3.18) \quad \lim_{n \to \infty} \frac{r^n a_n}{A_n(r)} = c_r. \]
We have then
\[ \frac{a_n}{a_{n+1}} = r \frac{r^n a_n}{A_n(r)} \frac{A_n(r)}{A_{n+1}(r)} \frac{A_{n+1}(r)}{r^{n+1} a_{n+1}}. \]

Since
\[ \frac{A_n(r)}{A_{n+1}(r)} = 1 - \frac{r^n a_n}{A_{n+1}(r)} \]

we have, by (3.18),
\[ \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = r c_r (1 - c_r) \frac{1}{c_r} = r(1 - c_r). \]

Thus, it remains only to prove that (3.18) holds. Let
\[ \alpha_\gamma = \liminf_{n \to \infty} \frac{r^n a_n}{A_n(\gamma)} \leq \limsup_{n \to \infty} \frac{r^n a_n}{A_n(\gamma)} = \beta_\gamma. \]

For every \( \gamma \in (\delta, R) \), we have
\[ 0 \leq \alpha_\gamma \leq \beta_\gamma \leq 1. \]

In order to prove (3.18), it is sufficient to prove that
\[ (3.19) \quad \alpha_r = \beta_r \text{ for some } r \in (\delta, R). \]

To establish (3.19) we shall need some lemmas.

**Lemma 3.7** There exists \( r \in (\delta, R) \) such that
\[ (3.20) \quad 0 < \beta_r < 1. \]
Proof: We first prove that $\beta_r < 1$. Suppose that $\beta_r = 1$, for every $r \in (S, R)$. Then there exists a sequence $(m_1)$ such that

$$\lim_{i \to \infty} \frac{r^{m_1} a_{m_1}}{A_{m_1}(r)} = 1,$$

or

$$\lim_{i \to \infty} \frac{A_{m_1-1}(r)}{A_{m_1}(r)} = 0.$$

Hence, given $\varepsilon > 0$ there exists $I_\varepsilon$ such that

$$(3.21) \quad \frac{A_{m_1-1}(r)}{A_{m_1}(r)} < \varepsilon \quad \text{for all } i > I_\varepsilon.$$

By (3.17), we have

$$(L + \gamma_{m_1}(r)) A_{m_1}(r) = \sum_{k=1}^{\infty} p_k r^k A_{m_1-k}(r).$$

Since the sequence $(A_{n}(r))$ is monotone increasing, we have

$$(L + \gamma_{m_1}(r)) A_{m_1}(r) \leq A_{m_1-1}(r) \sum_{k=1}^{\infty} p_k r^k.$$

Divide both sides of this inequality by $A_{m_1}(r)$, we then have

$$L + \gamma_{m_1}(r) \leq \frac{A_{m_1-1}(r)}{A_{m_1}(r)} \sum_{k=1}^{\infty} p_k r^k.$$

It follows from (3.21) that
Since $r < R$ and $\varepsilon$ can be chosen arbitrarily small, we get
\[ L \leq 0, \]
a contradiction.

Next, we prove that
\[ \beta_r > 0 \text{ for some } r \in (\delta, R). \]

Suppose that this were not true. Then for any $r \in (\delta, R)$, we would have $\beta_r = 0$. This would imply that
\[ \lim_{n \to \infty} \frac{r^n a_n}{A_n(r)} = 0 \quad \text{for every } r \in (\delta, R). \]

Choose $\varepsilon \in (0, 1)$ and a number $N$, which depends on both $\varepsilon$ and $r$, so that
\[ (3.22) \quad \frac{r^n a_n}{A_n(r)} < \varepsilon, \]
and so
\[ (3.23) \quad \frac{A_{n-1}(r)}{A_n(r)} > 1 - \varepsilon. \]

Using (3.17) and (3.23), we find that
\[ (L + \gamma_n(r)) A_n(r) \geq \sum_{k=N}^\infty p_k r^k A_{n-k}(r) \]
Let \( n \to \infty \). We then have

\[
L \geq \sum_{k=1}^{\infty} p_k (r(1-\varepsilon))^k .
\]

Hence we must have

\[
L \geq \sum_{k=1}^{\infty} p_k r^k \quad \text{for every } r \in (S, R)
\]

and it would follow that

\[
L \geq \sum_{k=1}^{\infty} p_k R^k.
\]

But this is impossible in view of the condition

\[
L < \sum_{k=1}^{\infty} p_k R^k.
\]

This completes the proof of Lemma 3.7.

From now on we shall fix the number \( r \in (S, R) \) which is determined by Lemma 3.7. The result of Lemma 3.7 enables us to prove the following lemma.

**Lemma 3.8** If \((m_1)\) is a sequence of natural numbers so that

\[
(3.24) \quad \lim_{t \to \infty} \frac{r^m_1 a_{m_1}}{A_{m_1}(r)} = \frac{\beta}{|r|},
\]

then, for each \( j \) such that \( p_j > 0 \), we have
Likewise, if \((n_1)\) is a sequence of natural numbers so that

\[
(3.26) \quad \lim_{i \to \infty} \frac{r^{n_1}a_{n_1}}{A_{n_1}(r)} = \alpha_r,
\]

then, for each \(j\) such that \(p_j > 0\), we have

\[
(3.27) \quad \lim_{i \to \infty} \frac{r^{n_1}a_{n_1}}{A_{n_1}(r)} = \alpha_r.
\]

Proof: We shall prove (3.25) only, the proof of (3.27) can be established by the same argument.

Choose \(\varepsilon > 0\) such that \(1 - \beta_r - \varepsilon > 0\). Next, choose \(N_\varepsilon\) such that for \(k > N_\varepsilon\), we have

\[
(3.28) \quad \frac{r^k a_k}{A_k(r)} < \beta_r + \varepsilon,
\]

or

\[
(3.29) \quad \frac{A_{k-1}(r)}{A_k(r)} > 1 - \beta_r - \varepsilon.
\]

Let \(A\) be a positive integer and let \(m_1 > A + N_\varepsilon\).

Multiplying (3.17) by \(\beta_r + \varepsilon\) and (1.10) by \(r^{m_1}\) and subtracting, we have
\[(3.30) \quad L((\beta_r + \epsilon)A_{m_1}(r) - r^{m_1}a_{m_1}) + (\beta_r + \epsilon)\gamma_{m_1}(r)A_{m_1}(r) - e_{m_1}r^{m_1}a_{m_1} = \left\{ \frac{A_{n-k}}{n} + \frac{2}{k=A} \right\} p_\kappa r^k \left((\beta_r + \epsilon)A_{m_1-k}(r) - r^{m_1-k}a_{m_1-k}\right). \]

Let
\[\Psi_n(A) = \frac{\kappa}{\kappa=A} p_\kappa r^k((\beta_r + \epsilon)\frac{A_{n-k}(r)}{A_n(r)} - \frac{r^{n-k}a_{n-k}}{A_n(r)}).\]

Since \(r^{n-k}a_{n-k} \leq A_{n-k}(r)\) and \((A_n(r))\) is an increasing sequence, we have
\[|\Psi_n(A)| \leq \frac{\kappa}{\kappa=A} p_\kappa r^k (\beta_r + \epsilon + 1) \leq 2 \frac{\kappa}{\kappa=A} p_\kappa r^k.\]

Since \(r \in (\delta, R)\), we have
\[(3.31) \quad \lim_{n \to \infty} \sup_{A \leq A} \Psi_n(A) \to 0 \quad (A \to \infty).\]

Dividing (3.30) by \(A_{m_1}(r)\) and using (3.28), for \(j \leq A\) with \(p_j > 0\), we have
\[(3.32) \quad L((\beta_r + \epsilon) - \frac{r^{m_1}a_{m_1}}{A_{m_1}(r)}) + (\beta_r + \epsilon)\gamma_{m_1}(r) - \frac{e_{m_1}r^{m_1}a_{m_1}}{A_{m_1}(r)} - \Psi_{m_1}(A) \geq \]
\[\geq p_j r^j \frac{A_{m_1-j}(r)}{A_{m_1}(r)}((\beta_r + \epsilon) - \frac{r^{m_1-j}a_{m_1-j}}{A_{m_1-j}(r)})\]
\[\geq p_j r^j (1 - \beta_r - \epsilon)j((\beta_r + \epsilon) - \frac{r^{m_1-j}a_{m_1-j}}{A_{m_1-j}(r)}).\]
In order to simplify the notation we shall denote the left hand side of (3.32) by $T_{m_1}(A)$. By (3.31) and (3.24), we have

$$\lim_{t \to \infty} \sup_{A} |T_{m_1}(A)| \to eL \quad (A \to \infty).$$

Hence, by (3.32), we have

$$-\varepsilon \leq \beta_r \frac{r^{m_1-j} \alpha_{m_1-j}}{A_{m_1-j}(r)} \leq (\beta_r + \varepsilon) - \frac{r^{m_1-j} \alpha_{m_1-j}}{A_{m_1-j}(r)} \leq \frac{|T_{m_1}(A)|}{p_j r^j (1 - \beta_r - \varepsilon)^j}.$$ 

Since $A$ can be chosen arbitrarily large, we have

$$-\varepsilon \leq \lim_{t \to \infty} \inf_{A} (\beta_r \frac{r^{m_1-j} \alpha_{m_1-j}}{A_{m_1-j}(r)}) \leq \lim_{t \to \infty} \sup_{A} (\beta_r \frac{r^{m_1-j} \alpha_{m_1-j}}{A_{m_1-j}(r)}) \leq \frac{eL}{p_j r^j (1 - \beta_r - \varepsilon)^j}.$$ 

If $\varepsilon \to 0$, it follows that

$$\lim_{t \to \infty} \frac{r^{m_1-j} \alpha_{m_1-j}}{A_{m_1-j}(r)} = \beta_r.$$ 

Using this result, we can prove now that $\beta_r = \alpha_r$. By Lemma 3.8 there is a sequence $(m_1)$ such that (3.24) holds. Consequently, we have, for all $j$ such that $p_j > 0$,

$$(3.33) \quad \lim_{t \to \infty} \frac{A_{m_1-1-j}(r)}{A_{m_1-j}(r)} = 1 - \beta_r.$$
By (3.17), for \( m_1 > A \), we have

\[
(L + \gamma_{m_1}(r)) = \left( \sum_{k=1}^{A-1} + \sum_{k=A}^{m_1-1} \right) p_k r^k \frac{A_{m_1-k}(r)}{A_{m_1}(r)}. 
\]

By (3.33), since \((A_n(r))\) is an increasing sequence, we find that

\[
L \leq \sum_{k=1}^{A-1} p_k r^k (1 - \beta_r)^k + \sum_{k=A}^{\infty} p_k r^k. 
\]

Since \( A \) can be chosen arbitrarily large and since \( r \in (0, R) \), we then have

\[
(3.34) \quad L \leq \sum_{k=1}^{\infty} p_k r^k (1 - \beta_r)^k. 
\]

On the other hand using (3.17), we have

\[
L + \gamma_n(r) = \sum_{k=1}^{n} p_k r^k \frac{A_{n-k}(r)}{A_n(r)} \geq \sum_{k=1}^{A} p_k r^k \frac{A_{n-k}(r)}{A_n(r)}. 
\]

Replacing \( n \) by \( m_1 \) and using (3.33), we find that

\[
L \geq \sum_{k=1}^{A} p_k r^k (1 - \beta_r)^k. 
\]

Hence, as \( A \to \infty \), we have

\[
(3.35) \quad L \geq \sum_{k=1}^{\infty} p_k r^k (1 - \beta_r)^k. 
\]

The equality
(3.36) \[ I = \sum_{k=1}^{\infty} p_k r^k (1- \beta_r)^k \]

follows by combining (3.34) and (3.35).

Similarly, we have

(3.37) \[ I = \sum_{k=1}^{\infty} p_k r^k (1- \alpha_r)^k. \]

From (3.36) and (3.37) follows that

(3.38) \[ \sum_{k=1}^{\infty} p_k r^k ((1- \alpha_r)^k - (1- \beta_r)^k) = 0 \]

Since \( p_k \geq 0 \), and \( 0 \leq \alpha_r \leq \beta_r \leq 1 \), we have

\[ \sum_{k=1}^{\infty} p_k r^k ((1- \alpha_r)^k - (1- \beta_r)^k) \geq 0. \]

Hence, from (3.38), we conclude that

\[ p_k r^k ((1- \alpha_r)^k - (1- \beta_r)^k) = 0, \] \( k = 1,2, \ldots \)

Finally, since \( p_1 > 0 \), we have \( \alpha_r = \beta_r \), and the theorem is proved.
Chapter 4

APPLICATIONS

In this chapter we shall discuss some applications of the converse results proved in Chapter 3. Let \((a_n)\) and \((p_n)\) be sequences of positive numbers and let the asymptotic relation (1.10) hold. The number \(\sigma\) is defined by (1.4). We shall prove Theorems 6 and 7 which were originally proved by De Bruijn and Erdős [3] and Garsia, Orey and Rodemich [5] respectively in case of sequences \((a_n)\) and \((p_n)\) satisfying the relation

\[
a_0 = 1, \quad a_n = \sum_{k=1}^{n} p_k a_{n-k}, \quad n = 1, 2, \ldots
\]

The proofs of Theorems 6 and 7 are based on the following lemmas.

Lemma 4.1 If \(\limsup_{n \to \infty} \frac{p_{n+1}}{p_n} \leq \sigma\), then

\[
\Lambda = \limsup_{n \to \infty} \frac{a_{n+1}}{a_n} \leq \sigma.
\]

Proof: First we shall prove that \(\Lambda\) is finite. Since

\[
\limsup_{n \to \infty} \frac{p_{n+1}}{p_n} \leq \sigma,
\]

there is a number \(M\) such that
By (1.10), we have

\[
\frac{p_{k+1}}{p_k} \leq M, \quad k = 1, 2, \ldots
\]

\[
(L + \varepsilon_{n+1}) \frac{a_{n+1}}{a_n} = \sum_{k=1}^{n+1} p_k \frac{a_{n+1-k}}{a_n}
\]

\[
= p_1 + \sum_{k=1}^{n} \frac{p_{k+1}}{p_k} \frac{p_k a_{n-k}}{a_n}
\]

\[
\leq p_1 + M (L + \varepsilon_n).
\]

Hence,

\[
\Lambda = \lim \sup_{n \to \infty} \frac{a_{n+1}}{a_n} \leq \frac{p_1}{L} + M < \infty.
\]

Next, suppose \( \Lambda > 0 \). If \( \varepsilon > 0 \) and \( k > N_{\varepsilon} \) then

\[
(\Lambda + \varepsilon)a_k - a_{k+1} \geq 0 \quad \text{and} \quad (r + \varepsilon)p_k - p_{k+1} \geq 0.
\]

Using (1.10) again, if \( n > A + N_{\varepsilon} \), we have

\[
(\Lambda + \varepsilon)(L + \varepsilon_n)a_n - (L + \varepsilon_{n+1})a_{n+1} =
\]

\[
= \sum_{k=1}^{n-1} p_k ((\Lambda + \varepsilon)a_{n-k} - a_{n+1-k}) + (\Lambda + \varepsilon) \sum_{k=1}^{n} p_k a_{n-k} +
\]

\[
+ \sum_{k=A}^{n} ((r + \varepsilon)p_k - p_{k+1})a_{n-k} - p_A a_{n-A+1}.
\]

The first 3 terms of the right hand side of this equation are non-negative. Hence, we have
(4.1) \[ L((\Lambda + \epsilon)a_n - a_{n+1}) + (\Lambda + \epsilon)e_n a_n - \epsilon_{n+1} a_{n+1} \geq \]
\[ \geq (\Lambda - \sigma) \sum_{k=1}^{A} p_k a_{n-k} + \sum_{k=1}^{A} p_k ((\Lambda + \epsilon)a_{n-k} - a_{n+1-k}) - \]
\[ - p_{\Lambda} a_{n+1-A}. \]

Let \( K \leq A \leq 2K \). Then
\[ p_{\Lambda} a_{n+1-A} \geq (\Lambda - \sigma) \sum_{k=1}^{\infty} p_k a_{n-k} + \sum_{k=1}^{K} p_k ((\Lambda + \epsilon)a_{n-k} - a_{n+1-k}) - \]
\[ - L((\Lambda + \epsilon)a_n - a_{n+1}) + (\Lambda + \epsilon)e_n a_n - \epsilon_{n+1} a_{n+1}. \]

Hence,
\[ (4.2) \min_{K \leq A \leq 2K} p_{\Lambda} a_{n+1-A} \geq (\Lambda - \sigma) \sum_{k=1}^{\infty} p_k a_{n-k} + \sum_{k=1}^{K} p_k ((\Lambda + \epsilon)a_{n-k} - a_{n+1-k}) - \]
\[ - L((\Lambda + \epsilon)a_n - a_{n+1}) + (\Lambda + \epsilon)e_n a_n - \epsilon_{n+1} a_{n+1}. \]

But, if \( n > 2K + N_{\epsilon} \), we have
\[ (4.3) \min_{K \leq A \leq 2K} p_{\Lambda} a_{n+1-A} \leq \frac{1}{K+1} \sum_{k=1}^{2K} p_k a_{n-k} \]
\[ \leq \frac{1}{K} \sum_{k=1}^{2K} \frac{p_{k+1}}{p_k} p_k a_{n-k} \]
\[ \leq \frac{q + \epsilon}{K} \sum_{k=K}^{2K} p_k a_{n-k} \]
\[ \leq \frac{q + \epsilon}{K} (L + \epsilon_n) a_n. \]

Combining (4.2) and (4.3), we find that (4.1) becomes
Dividing both sides of (4.4) by $a_n$, if $j \leq K$, we have first

$$\frac{q + \varepsilon}{K} (I + \varepsilon_n) a_n + L((\lambda + \varepsilon) a_n - a_{n+1}) + (\lambda + \varepsilon) \varepsilon_n a_n - \varepsilon_{n+1} a_{n+1}$$

$$\geq \sum_{k=1}^{K} p_k ((\lambda + \varepsilon) a_{n-k} - a_{n+1-k}) + (\lambda - \varepsilon) \sum_{k=0}^{n} p_k a_{n-k}.$$

(4.5) Dividing both sides of (4.4) by $a_n$, if $j \leq K$, we have first

$$\frac{q + \varepsilon}{K} (I + \varepsilon_n) + L((\lambda + \varepsilon) \frac{a_{n+1}}{a_n}) + (\lambda + \varepsilon) \varepsilon_n \frac{a_{n+1}}{a_n}$$

$$\geq p_j (\lambda + \varepsilon)^{-j} \left((\lambda + \varepsilon) \frac{a_{n+1-1}}{a_{n-j}}\right).$$

Let $(n_1)$ be a sequence such that

$$\lim_{i \to \infty} \frac{a_{n_1}+1}{a_{n_1}} = \lambda.$$ 

Replacing $n$ by $n_1$ in (4.5), we find that

$$-\varepsilon \leq \lambda - \frac{a_{n_1}+1-j}{a_{n_1}-j} \leq \lambda + \varepsilon - \frac{a_{n_1}+1-j}{a_{n_1}-j} \leq$$

$$\leq \frac{(\lambda + \varepsilon)^j}{p_j} \left\{ \frac{q + \varepsilon}{K} (I + \varepsilon_{n_1}) + L((\lambda + \varepsilon) \frac{a_{n_1}+1}{a_{n_1}}) + \right.$$

$$+ (\lambda + \varepsilon) \varepsilon_{n_1} + \varepsilon_{n_1+1} \frac{a_{n_1}+1}{a_{n_1}} \left\}.$$

Hence,

$$-\varepsilon \leq \lim_{i \to \infty} \inf \left(\lambda - \frac{a_{n_1}+1-j}{a_{n_1}-j}\right) \leq \lim_{i \to \infty} \sup \left(\lambda - \frac{a_{n_1}+1-j}{a_{n_1}-j}\right) \leq$$
\[ \leq \frac{(\Lambda + \varepsilon)^j}{p_j} \left( \frac{\sigma + \varepsilon}{K} L + \varepsilon L \right). \]

Since \( K \) can be chosen arbitrarily large and \( \varepsilon \) can be chosen arbitrarily small, we find that

\[ (4.7) \quad \lim_{i \to \infty} \frac{a_{n_i+1-j}}{a_{n_i-j}} = \Lambda, \quad j = 1, 2, \ldots \]

On the other hand by \((4.4)\), we have

\[ \frac{Q + \varepsilon}{K} (1 + \varepsilon) a_n + L (\Lambda + \varepsilon) a_n - a_{n+1} + (\Lambda + \varepsilon) a_n - \varepsilon a_{n+1} a_{n+1} \geq \]

\[ \geq (\Lambda - \sigma) \sum_{k=n}^{n_i} p_k a_{n-k}. \]

If \((n_i)\) is as in \((4.6)\), we have

\[ \frac{Q + \varepsilon}{K} + L \geq (\Lambda - \sigma) \lim_{i \to \infty} \sup_{k=n}^{n_i} \sum_{k=n}^{n_i} p_k \frac{a_{n_i-k}}{a_{n_i}}. \]

As \( \varepsilon \to 0 \), we find that

\[ (4.8) \quad \lim_{i \to \infty} \sup_{k=n}^{n_i} \sum_{k=n}^{n_i} p_k \frac{a_{n_i-k}}{a_{n_i}} \leq \frac{\sigma}{K(\Lambda - \sigma)}. \]

Now we shall use \((4.7)\) and \((4.8)\) to prove that

\[ L = \sum_{k=1}^{\infty} p_k \Lambda \cdot k. \]

Let \((n_i)\) be the sequence so that \((4.6)\) holds. By \((1.10)\),
we have

\[ L + \varepsilon_{n_1} \geq \sum_{k=1}^{n_1} p_k \frac{a_{n_1-k}}{a_{n_1}}, \]

for any positive integer \( n_1 \). Using (4.7) we find that

\[ L \geq \sum_{k=1}^{n_1} p_k \wedge^{-k}. \]

Hence,

\[ L \geq \sum_{k=1}^{n_1} p_k \wedge^{-k}. \]

Next, from

\[ L + \varepsilon_{n_1} = \sum_{k=1}^{n_1} p_k \frac{a_{n_1-k}}{a_{n_1}} + \sum_{u>n_1} p_k \frac{a_{n_1-k}}{a_{n_1}}, \]

and the asymptotic relations (4.7) and (4.8) we find that

\[ L \leq \sum_{k=1}^{\infty} p_k \wedge^{-k} + \frac{\sigma}{K(\wedge-\sigma)}. \]

Since \( K \) can be chosen arbitrarily large, it follows that

\[ L \leq \sum_{k=1}^{\infty} p_k \wedge^{-k}. \]

Hence,

\[ L = \sum_{k=1}^{\infty} p_k \wedge^{-k}. \]

Now we shall show that the hypothesis \( \wedge > \sigma \) leads to a contradiction. Suppose first that \( \sum_{k=1}^{\infty} p_k R_k \leq L \). We have
then, by (1.4), $\sigma = R^{-1}$. Hence

$$L = \sum_{k=1}^{\infty} p_k \land_k < \sum_{k=1}^{\infty} p_k \sigma_{-k} = \sum_{k=1}^{\infty} p_k R^k \leq L$$

which is impossible. Next, suppose that $\sum_{k=1}^{\infty} p_k R^k > L$. Then, by (1.4), $Q = \gamma^{-1}$ and $\sum_{k=1}^{\infty} p_k \gamma^k = L$. Hence

$$L = \sum_{k=1}^{\infty} p_k \land_k < \sum_{k=1}^{\infty} p_k \sigma_{-k} = \sum_{k=1}^{\infty} p_k \gamma^k = L$$

which is again impossible. Hence $\land \leq \sigma$, and the lemma is proved.

**Lemma 4.2** If $\liminf_{n \to \infty} \frac{p_{n+1}}{p_n} \geq \sigma$, we have

$$\lambda = \liminf_{n \to \infty} \frac{a_{n+1}}{a_n} \geq \sigma.$$  

**Proof:** By (1.10), we always have $\lambda > 0$. Suppose that $\lambda < \sigma$. If $\epsilon > 0$ and $k > N_\epsilon$, then

$$a_{k+1} - (\lambda - \epsilon)a_k \geq 0 \quad \text{and} \quad p_{k+1} - (\gamma - \epsilon)p_k \geq 0.$$  

Let $n > N_\epsilon + A$. We have

$$(4.9) \quad (L + \epsilon_{n+1})a_{n+1} - (\lambda - \epsilon)(L + \epsilon_n)a_n =$$

$$= \sum_{k=1}^{A-1} p_k (a_{n+1-k} - (\lambda - \epsilon)a_{n-k}) + p_A a_{n+1-A} +$$

$$+ \sum_{k=A}^{n} (p_{k+1} - (\gamma - \epsilon)p_k) a_{n-k} + (\lambda - \epsilon) \sum_{k=A}^{n} p_k a_{n-k}$$

$$\geq \sum_{k=1}^{A-1} p_k (a_{n+1-k} - (\lambda - \epsilon)a_{n-k}).$$
Using this inequality and the same arguments as in the proof of Lemma 4.1, we find that

\[(4.10) \lim_{i \to \infty} \frac{a_{m_{1}+1-j}}{a_{m_{1}-j}} = \lambda, \quad j = 1, 2, \ldots\]

whenever

\[\lim_{i \to \infty} \frac{a_{m_{1}+1}}{a_{m_{1}}} = \lambda.\]

Since each term on the right hand side of (4.9) is non-negative, we have, for \(m_{1} > N_{\epsilon} + \Lambda,

\[(I+\epsilon_{m_{1}+1})a_{m_{1}+1} - (\lambda-\epsilon)(I+\epsilon_{m_{1}})a_{m_{1}} \geq p_{A}a_{m_{1}+1-\Lambda}\]

or

\[\frac{a_{m_{1}+1}}{a_{m_{1}}} - (\lambda-\epsilon)(I+\epsilon_{m_{1}}) \geq p_{A}a_{m_{1}}^{-\Lambda}.\]

Let \(i \to \infty.\) By (4.10), we then have

\[p_{A} \lambda^{-\Lambda+1} \leq \epsilon L\]

which is impossible since \(\epsilon\) can be chosen arbitrarily small. Hence \(\lambda \geq \sigma.\)

Now we are able to prove the theorems.
Proof of Theorem 6. Suppose that

$$\lim_{n \to \infty} p_{n+1}/p_n = \sigma.$$  

Then by Lemma 4.1, we have \( \lambda \leq \sigma \) and by Lemma 4.2, we have \( \sigma \leq \lambda \). Hence \( \lambda = \sigma = \lambda \).

Proof of Theorem 7. Lemma 4.1 shows that \( \lambda \leq \sigma \). We also have the condition

$$L \leq \frac{\infty}{\kappa+1} \cdot p_k \cdot R^k.$$  

If we can show that

$$\lim_{A \to \infty} \left( \lim_{n \to \infty} \sup_{k=A}^{\infty} \frac{p_k}{a_n} \cdot \frac{a_{n-k}}{a_n} \right) = 0,$$

then from Theorem 4 it will follow that \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) exists.

If \( \varepsilon > 0 \), and if \( n \geq k \geq N_\varepsilon \), then

$$\frac{a_k}{a_n} \geq (\sigma + \varepsilon)^{-n+k}.$$  

Let \( n > A + N_\varepsilon \) we have, by (1.10),

$$\frac{\sum_{k=A}^{\infty} p_k \frac{a_{n-k}}{a_n}}{\sum_{k=A}^{\infty} p_k} = L + \varepsilon_n - \frac{\sum_{k=A}^{\infty} p_k \frac{a_{n-k}}{a_n}}{\sum_{k=A}^{\infty} p_k} \cdot (\sigma + \varepsilon)^{-k} \leq L + \varepsilon_n - \frac{\sum_{k=A}^{\infty} p_k \cdot (\sigma + \varepsilon)^{-k}}{\sum_{k=A}^{\infty} p_k}.$$

Hence,
\[
\limsup_{n \to \infty} \frac{\sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}}{n} \leq L - \frac{A^{-1}}{\sum_{k=1}^{\infty} p_k (r+\varepsilon)^{-k}}.
\]

If \( \varepsilon \to 0 \), we find that
\[
\limsup_{n \to \infty} \frac{\sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}}{n} \leq L - \frac{A^{-1}}{\sum_{k=1}^{\infty} p_k \sigma^{-k}}.
\]

But, if \( L \leq \sum_{k=1}^{\infty} p_k r^k \), and \( \sigma \) is defined by (1.4), it is easy to see that
\[
\sum_{k=1}^{\infty} p_k \sigma^{-k} = L.
\]

Hence
\[
\lim_{A \to \infty} (\limsup_{n \to \infty} \frac{\sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}}{n}) = 0.
\]
BIBLIOGRAPHY


