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A THEORY FOR THICK ELASTIC PLATES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree of Doctor of Philosophy in the Graduate School of The Ohio State University

By

Ting-Hwa Wang, B.S., M.S.

******

The Ohio State University
1972

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Studies in Applied Mathematics. Professors A.M. Buoncristiani, F.W. Carroll
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CHAPTER I
INTRODUCTION

1.1 Background and Review of the Literature

Although the classical theory of the plates has been well established and widely applied since Lagrange corrected the work of S. Germain early in the 19th century, the literature indicates that recent attempts to improve on this theory are numerous and diverse. The basic assumption (due to Kirchhoff) of classical plate theory is that normals to the middle surface remain straight, normal, and unstrained as the plate deforms. This assumption neglects the effects of transverse shear deformation and the normal stress $\sigma_y$ and, as a result, only two boundary conditions are allowed on an edge. The errors in such a theory naturally increase as the plate thickness increases. A thorough discussion on the effects of shear deformation and rotary inertia is seen in Leissa's monograph (ref. 1) on plate vibrations.

Throughout most modern literature the method proposed by Reissner (refs. 2, 3, and 4) in 1944 has become the standard to which other thick plate theories are compared. Reissner's theory was obtained by use of the principle of virtual work. He assumed a linear distribution of the stresses $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ across the plate thickness, at an edge the normal and tangential displacements $u_n$ and $u_\theta$ vary linearly and the lateral displacement $w$ does not vary through the thickness. His theory permits the satisfaction of three boundary conditions on each edge. Salerno and

1
Goldberg (ref. 5), Koeller and Essenber (ref. 6), Carley and Langhaar (ref. 7) applied this theory to rectangular plates having two opposite sides simply supported and the other two edges with various supporting conditions. The applications to thick plates having elliptic holes, rigid circular inclusions, and on an elastic foundation can be found in the papers by Naghdi (ref. 8), Hirsch (ref. 9), and Frederick (ref. 10). Reissner's theory subsequently was rederived from the general equations of elasticity by Green (refs. 11 and 12) who also indicated a series method for the three-dimensional solution of flexure of plates. An analogous theory to that of Reissner also has been developed by Mindlin (ref. 13) for flexural motion of elastic plates where the influence of rotary inertia and shear deformation has been taken into account.

Donnell and Lee (refs. 14 and 15) proposed a method in which one starts with expressions of classical thin plate theory for the stresses, and improves the solution by successive addition of terms, so as to satisfy both the boundary conditions on lateral surfaces and the differential equation of elasticity in three dimensions more and more accurately. Lee (ref. 16) applied this method to simply supported thick rectangular plates.

Srinivas, Rao, and Rao (refs 17 and 18) investigated flexure of simply supported thick homogeneous and laminated rectangular plates by the three-dimensional elasticity displacement equilibrium equations. The solution is formally exact when infinite terms are taken in the series. However, this method can only be applied to simply supported rectangular plates.
Luré (ref. 19) proposed a symbolic method for the solution of three-dimensional elasticity displacement equilibrium equations. Several stress functions and the method of complex variables are then employed to obtain the final solution.

Goldvenelzer (ref. 20) gave an approximate theory by the method of asymptotic integration of the governing equilibrium differential equations in elasticity theory. However, no numerical results or comparison with other theories were given.

1.2 Statement of Research Problem

The main purpose of this investigation is to develop a general static theory for the small deflection of thick elastic plates. Homogeneous as well as heterogeneous plates will both be considered.

As stated in the previous section the Reissner theory is the most widely-accepted theory in dealing with thick plate problem. However, some shortcomings do exist in regard to his basic assumptions. After careful study of his theory, the following several deficiencies which should be improved:

(1) The stress components $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ are not in fact linear variations across the plate thickness. The distribution of these stresses is actually unknown. Hence, a more general form should be taken to start with.

(2) In regard to the assumption of edge displacements, Goodier (ref. 21) has the following comment,

"The second raises some questions because the essence
of the theory is to take into account deformation due to shear \( (\tau_{xy}, \tau_{yz}) \) and transverse stress \( (\sigma_y) \) as well as bending stress \( (\sigma_x, \sigma_y, \tau_{xy}) \). The shear deformation will cause departure from linearity of \( u_n, u_e \), and the effects of \( \sigma_y \) will cause \( w \) to vary through the thickness.

(3) In the case of simply supported or free edges, the normal displacement \( u_n \) at the mid-plane \((z=0)\) is not zero due to Poisson's effect. Furthermore, the lateral deflection \( w \) at top and bottom surfaces of the plate is not the same as the mid-plane surface if the plate is relatively thick. The Reissner theory does not have these features.

(4) Hooke's law is not satisfied by Reissner's assumptions, i.e., relation between his stress and displacement assumptions.

The above-mentioned shortcomings are not intended as negative criticism of Reissner's theory. Indeed, Reissner's theory is about the best that can be done with a two-dimensional, sixth order plate theory to represent the three-dimensional static behavior reasonably well in most applications. It is intended in this work to develop an improved thick plate theory which eliminates the above shortcomings, but this can only be done by raising the order of the governing system of differential equations. Thus a more general set of displacement-functions is assumed in the form

\[
U(x, y, z) = U_0(x, y) + \frac{z}{2} U_1(x, y) + \frac{z^2}{3!} U_2(x, y) + \frac{z^3}{3!} U_3(x, y)
\]

\[
V(x, y, z) = V_0(x, y) + \frac{z}{2} V_1(x, y) + \frac{z^2}{3!} V_2(x, y) + \frac{z^3}{3!} V_3(x, y)
\]

\[
W(x, y, z) = W_0(x, y) + \frac{z}{2} W_1(x, y) + \frac{z^2}{2!} W_2(x, y)
\]
Although more terms could be taken in the power series expansion of displacements \( u, v, \) and \( w \), the technique and calculations necessary for such extension soon become prohibitive. Another factor in terminating the power series expansion of displacements \( w \) at \( \bar{z}^2 \) is due to the number of edge boundary conditions. This will become clear in the subsequent chapters.

The theory based on the preceding displacement approximation is developed directly from the general equations of three-dimensional elasticity. Comparing with other elaborate thick plate theories, the present theory is more straightforward, yet all shortcomings of the Reissner theory are removed.
CHAPTER II
GENERAL EQUATIONS OF LINEAR HOMOGENEOUS
ISOTROPIC THICK ELASTIC PLATE THEORY

2.1 Notation and Basic Assumption

A notation will be developed which is consistent with that of elasticity; that is, at a point the directions of positive stress will be taken as shown on the element of figure 2.1. Figure 2.2 shows a plate element of thickness $h$ and incremental dimensions $dx$ and $dy$. The $x$- and $y$- axes are chosen to contain the middle surface of the plate. This plane is called the "neutral plane" in the classical thin plate theory. The stress and moment resultants are shown in figures 2.2 and 2.3, respectively.

Let the components of the elastic displacement in the $x$, $y$ and $z$ directions of a rectangular plate of sides $a$ and $b$ be

\[
U(x, y, z) = \Phi(x, y) + \frac{\partial^2}{2} \Psi(x, y) + \frac{3}{2} \frac{\partial^3}{3!} \Phi(x, y) + \frac{3}{3!} \Phi(x, y) \quad (2.1)
\]

\[
V(x, y, z) = \Psi(x, y) + \frac{\partial^2}{2} \Phi(x, y) + \frac{3}{2} \frac{\partial^3}{3!} \Psi(x, y) + \frac{3}{3!} \Psi(x, y) \quad (2.2)
\]

\[
W(x, y, z) = \Phi(x, y) + \frac{\partial^2}{2} \Psi(x, y) + \frac{3}{2} \frac{\partial^3}{3!} \Phi(x, y) \quad (2.3)
\]
Figure 2.1. - Notation and positive directions of stress.

Figure 2.2. - Stress resultants nomenclature.
Figure 2.3: Moment resultants nomenclature.
The above approximation of plate displacement components is consistent with the results of strength of materials that shearing stresses vary parabolically through the thickness. However, instead of the usual linear bending-stress distribution as assumed by most researchers in dealing with the thick plate problem, a more general expression for the bending stresses containing third powers of the thickness co-ordinate $z$ is obtained from this displacement approximation.

As for the accuracy of the present analysis based on the above displacement approximation, it is best described by Reissner (ref. 3) in the following paragraphs.

"Instead of the linear bending-stress distribution take a more general expression for the bending stresses containing third powers of the thickness co-ordinate $z$. Determine the corresponding transverse shear and normal stresses and again apply Castigliano's theorem. If the more accurate results thus obtained are in good agreement with the results based on the linear bending-stress distribution, those may be assumed to be final from a practical point of view."

As will be seen later, the conventional elasticity theory instead of Castigliano's theorem will be used to obtain the solution of the thick plate problem. The results thus obtained will be compared with those based on the Reissner and Donnell-Lee theories.

2.2 Formulation of the Problem

Let the surface tractions of a plate (fig. 2.4) be prescribed as follows:
Figure 2.4. - Surface tractions in a plate.
\[ \tau_{3x} = \tau_{3x}^+(x, y), \quad \tau_{3y} = \tau_{3y}^+(x, y), \quad \sigma_3 = \sigma_3^+(x, y) \text{ for } \frac{\gamma}{2} = \pm \frac{h}{2} \quad (2.4) \]

\[ \tau_{3x} = \tau_{3x}^-(x, y), \quad \tau_{3y} = \tau_{3y}^-(x, y), \quad \sigma_3 = \sigma_3^-(x, y) \text{ for } \frac{\gamma}{2} = -\frac{h}{2} \quad (2.5) \]

where \( \tau_{3x}^+, \ldots \) are given functions of the points of the surface boundaries.

Following a frequently used method of structural mechanics, the case of symmetrical and antisymmetrical loading of the plate will be considered separately. First, this leads to less cumbersome formulae, and secondly, each of these loadings is important in itself, since they correspond to definite types of deformation of the plate.

Using the quantities

\[ \tau_1(x, y) = \left( \tau_{3x}^+ - \tau_{3x}^- \right)/2 \]

\[ \tau_2(x, y) = \left( \tau_{3y}^+ - \tau_{3y}^- \right)/2 \]

\[ \sigma_1(x, y) = \left( \sigma_3^+ + \sigma_3^- \right)/2 \]

and

\[ S_1(x, y) = \left( \tau_{3x}^+ + \tau_{3x}^- \right)/2 \]

\[ S_2(x, y) = \left( \tau_{3y}^+ + \tau_{3y}^- \right)/2 \]

\[ \sigma_2(x, y) = \left( \sigma_3^+ - \sigma_3^- \right)/2 \]

the boundary value problem (2.4) and (2.5) can be split into two problems, namely, that of finding the solution for the face boundary conditions

(A) \[ \tau_{3y} = \pm \tau_1, \quad \tau_{3x} = \pm \tau_2, \quad \sigma_3 = \sigma_1 \text{ for } \frac{\gamma}{2} = \pm \frac{h}{2} \quad (2.8) \]

(B) \[ \tau_{3y} = S_1, \quad \tau_{3x} = S_2, \quad \sigma_3 = \pm \sigma_2 \text{ for } \frac{\gamma}{2} = \pm \frac{h}{2} \quad (2.9) \]

Clearly, the solution of the original problem (2.4) and (2.5) is obtained by addition of the problems (A) and (B).
Thus, in problem (A), the middle surface experiences an extension (or compression) and in problem (B) bending; consequently, the first problem is called the symmetric (or extensional) problem, and the second the antisymmetric (or flexural) problem.

In problem (A), the displacements $u$ and $v$ are even functions and $w$ is an odd function of $z$; hence

$$\frac{\partial u}{\partial z} = 0, \quad \frac{\partial v}{\partial z} = 0, \quad w = 0 \quad \text{for} \quad z = 0 \quad (2.10)$$

On the other hand, in problem (B), the displacements $u$ and $v$ are odd functions and $w$ is an even function of $z$; hence

$$u = 0, \quad v = 0, \quad \frac{\partial w}{\partial z} = 0 \quad \text{for} \quad z = 0 \quad (2.11)$$

For example, let a concentrated normal force $P$ act on the plate at the point $(x, y, h/2)$ of one of the faces. The decomposition of this problem into the problems (A) and (B) is shown in figure 2.5(a), whence it is readily seen by symmetry considerations that in the first case the middle surface will undergo no bending ($w=0$ at $z=0$), while it will be stretched ($u, v \neq 0$ at $z=0$), but in the second case, it will neither be stretched nor compressed ($u=v=0$ at $z=0$), but it will undergo bending ($w \neq 0$ at $z=0$). Figure 2.5(b) shows the decomposition into the cases of extension and flexure for the case of the action of a concentrated shearing force $T$ on a plate.

From the above discussion, we can express the total displacements as,
Figure 2.5. - Decomposition of forces.
Subscripts e and f are added to the displacements for the distinction of displacements due to extension and flexure.

2.3 Equations of Equilibrium

The stress equations of equilibrium of three-dimensional elasticity in the absence of body forces and couples are well-known as

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \quad (2.13a)
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \quad (2.13b)
\]

\[
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \quad (2.13c)
\]

These equations are converted into plate-stress equations of equilibrium by the method of Boussinesq (ref. 22). Integrating equations (2.13) over the plate thickness then we get

\[
\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} + p_x = 0 \quad (2.14a)
\]

\[
\frac{\partial N_{yx}}{\partial x} + \frac{\partial N_y}{\partial y} + p_y = 0 \quad (2.14b)
\]

\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0 \quad (2.14c)
\]
where
\[ (N_x, N_{x,y}, N_y, Q_x, Q_y) = \int_{-h/2}^{h/2} (\sigma_x, \tau_{xy}, \sigma_y, \tau_{x,y}, \tau_{y,y}) \, d\bar{z} \]

\[ p_x = \tau_{x,y} - \tau_{y,y} \]
\[ p_y = \tau_{y,y} - \tau_{y,y} \]
\[ p = \sigma_{11} - \sigma_{22} \]

Multiplying equations (2.13a) and (2.13b) by \( z \) and integrating over the plate thickness we get

\[ \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x + M_x = 0 \quad (2.15a) \]
\[ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y + M_y = 0 \quad (2.15b) \]

where
\[ (M_x, M_{xy}, M_y) = \int_{-h/2}^{h/2} (\sigma_x, \tau_{xy}, \sigma_y) \cdot \bar{z} \, d\bar{z} \]

\[ m_x = \frac{h}{2} (\tau_{x,y} + \tau_{y,x})/2 \]
\[ m_y = \frac{h}{2} (\tau_{y,y} + \tau_{y,y})/2 \]

2.4 Kinematics of Deformation (Strain-Displacement Relations)

From the linear theory of elasticity the strain-displacement relations are

\[ \varepsilon_x = \frac{\partial u}{\partial x} \]
\[ \varepsilon_y = \frac{\partial v}{\partial y} \]
\[ \varepsilon_z = \frac{\partial w}{\partial z} \]
\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]
\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \]

\[ \gamma_{yx} = \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \]

(2.16)

Substituting equations (2.1) - (2.3) into (2.16), then we obtain the strains in terms of displacement components

\[ \varepsilon_x = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{3}{2} \frac{\partial^2 u}{\partial y \partial x} + \frac{3}{2} \frac{\partial^3 u}{\partial y^3} + \frac{3}{6} \frac{\partial u}{\partial x} \]

(2.17a)

\[ \varepsilon_y = \frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial x^2} + \frac{3}{2} \frac{\partial^2 v}{\partial x \partial y} + \frac{3}{2} \frac{\partial^3 v}{\partial x^3} + \frac{3}{6} \frac{\partial v}{\partial y} \]

(2.17b)

\[ \varepsilon_z = w_1 + \frac{3}{2} w_2 \]

(2.17c)

\[ \gamma_{xy} = \left( \frac{\partial u}{\partial y} + \frac{2}{3} \frac{\partial v}{\partial x} \right) + \frac{3}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{3}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{3}{6} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) \]

(2.17d)

\[ \gamma_{yz} = \left( \frac{1}{2} + \frac{3}{2} \frac{\partial w}{\partial x} \right) + \frac{3}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) + \frac{3}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) \]

(2.17e)

\[ \gamma_{xz} = \left( \frac{1}{2} + \frac{3}{2} \frac{\partial w}{\partial y} \right) + \frac{3}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) + \frac{3}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) \]

(2.17f)

### 2.5 Stress-Strain Relations (Constitutive Equations)

For a homogeneous isotropic elastic plate, the stress-strain relations are

\[ \sigma_x = \lambda \varepsilon_x + 2 \mu \frac{\partial u}{\partial x} \]

\[ \sigma_y = \lambda \varepsilon_y + 2 \mu \frac{\partial v}{\partial y} \]

\[ \sigma_z = \lambda \varepsilon_z + 2 \mu \frac{\partial w}{\partial z} \]
where
\[
\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}
\]
\[
G = \frac{E}{2(1 + \nu)}
\]
\[
e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}
\]

and \(\nu\) is the Poisson ratio.

At this stage, the symmetric and antisymmetric problems will be considered separately.

2.5.1 Problem (A): Symmetric Problem (Extension)

As discussed in section 2.2, the displacements for the symmetric problem are taken as follows from equations (2.1) - (2.3)

\[
U_e = U_o + \frac{3}{2} \cdot U_1 \quad (2.19a)
\]
\[
V_e = V_o + \frac{3}{2} \cdot V_1 \quad (2.19b)
\]
\[
W_e = \frac{3}{2} \cdot W_1 \quad (2.19c)
\]

Substituting equations (2.19) into (2.18), we get

\[
\sigma_x = \lambda \left[ \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \partial w \right) + \frac{3}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] + 2G \left( \frac{\partial u}{\partial x} + \frac{3}{2} \cdot \frac{\partial u}{\partial x} \right) \quad (2.20a)
\]
\[
\sigma_y = \lambda \left[ \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \partial w \right) + \frac{3}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] + 2G \left( \frac{\partial v}{\partial y} + \frac{3}{2} \cdot \frac{\partial v}{\partial y} \right) \quad (2.20b)
\]
The stress and moment resultants are

\[
\sigma_{\delta} = \lambda \left[ \left( \frac{2u}{2x} + \frac{2v}{2y} + w_1 \right) + \frac{\delta^3}{3} \left( \frac{2u}{2x} + \frac{2v}{2y} \right) \right] + 2Gw_1 \tag{2.20c}
\]

\[
\tau_{xy} = G \left[ \left( \frac{2u}{2y} + \frac{2v}{2x} \right) + \frac{\delta^2}{2} \left( \frac{2u}{2y} + \frac{2v}{2x} \right) \right] \tag{2.20d}
\]

\[
\tau_{xz} = G \frac{\delta}{2} \left( u_1 + \frac{\partial w_1}{\partial x} \right) \tag{2.20e}
\]

\[
\tau_{yz} = G \frac{\delta}{2} \left( v_1 + \frac{\partial w_1}{\partial y} \right) \tag{2.20f}
\]

\[
N_x = G \left[ \lambda \left( \frac{2u}{2x} + \frac{2v}{2y} + w_1 \right) + 2G_1 \frac{2u}{2x} \right] + \frac{\delta^3}{24} \left[ \lambda \left( \frac{2u}{2x} + \frac{2v}{2y} \right) + 2G_1 \frac{2u}{2x} \right] \tag{2.21}
\]

\[
N_y = G \left[ \lambda \left( \frac{2u}{2x} + \frac{2v}{2y} + w_1 \right) + 2G_1 \frac{2v}{2y} \right] + \frac{\delta^3}{24} \left[ \lambda \left( \frac{2u}{2x} + \frac{2v}{2y} \right) + 2G_1 \frac{2v}{2y} \right] \tag{2.22}
\]

\[
N_{xz} = G \left[ \lambda \left( \frac{2u}{2y} + \frac{2v}{2x} \right) + \frac{\delta^2}{24} \left( \frac{2u}{2y} + \frac{2v}{2x} \right) \right] \tag{2.23}
\]

\[
Q_x = Q_y = M_x = M_y = M_{xy} = 0 \tag{2.24}
\]
2.5.2 Problem (B): Antisymmetric Problem (Flexure)

Similar to problem (A), the displacements are taken as

\[ U_f = \frac{y}{8} U_1 + \frac{3y}{b} U_3 \]  
\[ V_f = \frac{y}{8} V_1 + \frac{3y}{b} V_3 \]  
\[ W_f = W_0 + \frac{3y}{2} W_L \]

Substituting equations (2.25) into (2.18), we get

\[ G_x = \lambda \left[ \frac{y}{8} \left( \frac{2U_1}{2x} + \frac{2V_1}{2y} + W_L \right) + \frac{3y}{b} \left( \frac{2U_3}{2x} + \frac{2V_3}{2y} \right) + 2G_1 \left( \frac{3}{8} \frac{2U_1}{2x} + \frac{3}{b} \frac{2U_3}{2x} \right) \right] \]  
\[ G_y = \lambda \left[ \frac{y}{8} \left( \frac{2U_1}{2x} + \frac{2V_1}{2y} + W_L \right) + \frac{3y}{b} \left( \frac{2U_3}{2x} + \frac{2V_3}{2y} \right) + 2G_1 \left( \frac{3}{8} \frac{2V_1}{2y} + \frac{3}{b} \frac{2V_3}{2y} \right) \right] \]  
\[ G_z = \lambda \left[ \frac{y}{8} \left( \frac{2U_1}{2x} + \frac{2V_1}{2y} + W_L \right) + \frac{3y}{b} \left( \frac{2U_3}{2x} + \frac{2V_3}{2y} \right) + 2G_1 \frac{3}{8} \frac{2W_0}{2y} \right] \]  
\[ \tau_{xy} = G \left[ \frac{y}{8} \left( \frac{2U_1}{2y} + \frac{2V_1}{2x} \right) + \frac{3y}{b} \left( \frac{2U_3}{2y} + \frac{2V_3}{2x} \right) \right] \]  
\[ \tau_{xz} = G \left[ \left( U_1 + \frac{2W_0}{2x} \right) + \frac{3y}{b} \left( U_3 + \frac{2W_0}{2x} \right) \right] \]  
\[ \tau_{yz} = G \left[ \left( V_1 + \frac{2W_0}{2y} \right) + \frac{3y}{b} \left( V_3 + \frac{2W_0}{2y} \right) \right] \]

The corresponding stress and moment resultants are

\[ N_x = N_y = N_{xy} = 0 \]  
\[ (2.27) \]
2.6 Governing Equilibrium Equations in Terms of Displacements

As can be seen from section 2.5, the symmetric and antisymmetric cases are uncoupled. For the symmetric case, we have two equilibrium equations, namely equations (2.14a) and (2.14b), and three equilibrium equations, (2.14c), (2.15a) and (2.15b) for the antisymmetric case.

2.6.1 Problem (A): Symmetric Problem

Substituting the constitutive equations (2.21) - (2.23) into (2.14a) and (2.14b), we get

\[
Q_x = G \left[ \frac{\partial}{\partial x} \left( u_1 + \frac{3w_1}{2x} \right) + \frac{G_1^3}{2}\left( U_5 + \frac{3w_1}{2x} \right) \right] \tag{2.28}
\]

\[
Q_y = G \left[ \frac{\partial}{\partial y} \left( v_1 + \frac{3w_1}{2y} \right) + \frac{G_1^3}{2}\left( V_3 + \frac{3w_1}{2y} \right) \right] \tag{2.29}
\]

\[
M_{x} = \lambda \left[ \frac{G_1^3}{2} \left( \frac{2u_1}{2x} + \frac{3v_1}{2y} + w_1 \right) + \frac{G_1^5}{48} \left( \frac{2u_1}{2x} + \frac{3v_1}{2y} \right) \right] + 2G \left( \frac{G_1^3}{2} \cdot \frac{2u_1}{2x} + \frac{G_1^5}{48} \cdot \frac{2u_1}{2x} \right) \tag{2.30}
\]

\[
M_{y} = \lambda \left[ \frac{G_1^3}{2} \left( \frac{2u_1}{2x} + \frac{3v_1}{2y} + w_1 \right) + \frac{G_1^5}{48} \left( \frac{2u_1}{2x} + \frac{3v_1}{2y} \right) \right] + 2G \left( \frac{G_1^3}{2} \cdot \frac{2v_1}{2y} + \frac{G_1^5}{48} \cdot \frac{2v_1}{2y} \right) \tag{2.31}
\]

\[
M_{xy} = G \left[ \frac{G_1^3}{2} \left( \frac{2u_1}{2y} + \frac{3v_1}{2x} \right) + \frac{G_1^5}{48} \left( \frac{2u_1}{2y} + \frac{3v_1}{2x} \right) \right] \tag{2.32}
\]
Equations (2.33) and (2.34) constitute a set of two equations with five unknowns, i.e., $u_0, v_0, u_z, v_z$ and $w_1$. The other three equations will be obtained from the result of satisfying plate face boundary conditions at $z = \pm h/2$.

### 2.6.2 Problem (B): Antisymmetric Problem

Substituting the constitutive equations (2.28) - (2.32) into (2.14c), (2.15a) and (2.15b), we get

\[
\frac{h}{2} \left\{ (\lambda + G) \frac{\partial u_z}{\partial x} + \left[ G \frac{\partial^2 v_z}{\partial x^2} + (\lambda + 2G) \frac{\partial^2 v_z}{\partial y^2} \right] v_z + \frac{h}{2} \left( \frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2} \right) + p_y = 0 \tag{2.34}
\]

\[
G \left[ \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} \right] + \nabla^2 w_z + \nabla^2 w_1 \right] + p = 0 \tag{2.35}
\]

\[
\left\{ \frac{h}{12} \left[ (\lambda + 2G) \frac{\partial^2 u_3}{\partial x^2} + G \frac{\partial^2 v_3}{\partial y^2} \right] - G \frac{\partial^2 w_1}{\partial x^2} \right\} u_1 + \frac{h}{12} \left( \lambda + G \right) \frac{\partial^2 v_1}{\partial x^2} + \frac{h}{24} \left( 2\lambda - G \right) \frac{\partial^2 w_1}{\partial y^2} + \frac{h}{480} \left( \lambda + 2G \right) \frac{\partial^2 w_1}{\partial x^2} + \frac{h}{480} \left( 2\lambda - G \right) \frac{\partial^2 w_1}{\partial y^2} + m_y = 0 \tag{2.36}
\]

\[
\frac{h}{12} \left( \lambda + G \right) \frac{\partial^2 u_3}{\partial x^2} + \left\{ \frac{h}{12} \left[ G \frac{\partial^2 v_3}{\partial x^2} + (\lambda + 2G) \frac{\partial^2 v_3}{\partial y^2} \right] - G \frac{\partial^2 w_1}{\partial x^2} \right\} v_1 - G \frac{\partial^2 w_1}{\partial y^2} + \frac{h}{480} \left( \lambda + 2G \right) \frac{\partial^2 w_1}{\partial x^2} + \frac{h}{480} \left( 2\lambda - G \right) \frac{\partial^2 w_1}{\partial y^2} + m_y = 0 \tag{2.37}
\]
where
\[ \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]

Equations (2.35) - (2.37) constitute a set of three simultaneous differential equations with six unknowns \((u_1, u_2, v_1, v_2, w_0\) and \(w_1)\). The other three differential equations will be obtained from the result of satisfying plate face boundary conditions at \(z = \pm h/2\).

2.7 Boundary Conditions

A plate is defined as part of a layer bounded by a cylindrical surface \(C\) with generators perpendicular to the plane faces of the layer. For the sake of brevity, the intersection of the cylindrical surface \(C\) with the middle plane \((z=0)\) will henceforth be called the curve \(c\).

For a system of linear differential equations, appropriate boundary conditions are those quantities, in number and combination which are necessary to insure an unique solution. Thus, in the problem of plate equilibrium, not only the boundary conditions on the plate faces must be satisfied, but also the conditions along the edge surface \(C\). Complete edge boundary conditions in any plate problem involve the specification of either the displacement or the stress at every point on the edge surface \(C\). Such complete solutions are in general extremely difficult to obtain. Therefore one finds oneself forced to limit consideration to weak edge boundary conditions. It will be required that those conditions are only satisfied along the curve \(c\) for the average of these or other quantities over the thickness of the plate. For instance, in many cases it may be sufficient to satisfy conditions for the resultant forces and moments per unit length of edge curve \(c\), thus sacrificing accuracy near
the edges, but retaining relative exactness at other points.

The acceptability of such a solution is justified by Saint-Venant's principle, according to which statically equivalent systems of forces distributed over a small part C of the boundary of an elastic body produce in the body systems of stresses which differ from each other only in the immediate vicinity of the area of loading C. However, at sufficient distances from this section, the stress aroused by one or the other system of forces are practically identical.

2.7.1 Face Boundary Conditions

Assume the plate is loaded on its top surface \((z=+h/2)\) by a normal load \(p(x, y)\) and tangential loads \(T_x(x, y)\) and \(T_y(x, y)\). The bottom surface \((z=-h/2)\) is free from any loading.

According to the discussion in section 2.2, the face boundary conditions are

\[ \tau_{yx} = \frac{1}{2} T_y / 2, \quad \tau_{yy} = \frac{1}{2} T_y / 2, \quad \sigma_z = \frac{p}{2} \text{ at } \gamma = \pm h/2 \]  \hspace{1cm} (2.38)

for the symmetric problem, and

\[ \tau_{yx} = T_y / 2, \quad \tau_{yy} = T_y / 2, \quad \sigma_z = \frac{p}{2} \text{ at } \gamma = \pm h/2 \]  \hspace{1cm} (2.39)

for the antisymmetric problem.

From equations (2.20 c, e, f) we obtain the following three constraining equations for the symmetric problem:

\[ U_z + \frac{2w}{2x} = \frac{T_x}{G\bar{h}} \]  \hspace{1cm} (2.40a)

\[ V_z + \frac{2w}{2y} = \frac{T_y}{G\bar{h}} \]  \hspace{1cm} (2.40b)
Similarly, for the antisymmetric problem we have

\[ U_1 + \frac{2w_k}{\partial x} + \frac{A^2}{8} (u_1 + \frac{2w_k}{\partial x}) = \frac{T_x}{2G} \]  
(2.41a)

\[ V_1 + \frac{2w_k}{\partial y} + \frac{A^2}{8} (v_1 + \frac{2w_k}{\partial y}) = \frac{T_y}{2G} \]  
(2.41b)

\[ \lambda \left[ \frac{2u_t}{\partial x} + \frac{2v_t}{\partial y} + w_t + \frac{A^2}{24} \left( \frac{2u_t}{\partial x} + \frac{2v_t}{\partial y} \right) \right] + 2G \cdot w_t = \frac{P}{A} \]  
(2.41c)

The two equilibrium equations (2.33) and (2.34) plus the three constraining equations (2.40) constitute a system of five differential equations with five unknown functions \((u, v, u_t, v_t, w)\) for the symmetric problem. For the antisymmetric problem the six governing differential equations are those of (2.35), (2.36), (2.37) and (2.41).

In case the tangential loadings \(T_x\) and \(T_y\) are absent equations (2.40) and (2.41) become

\[ U_t = - \frac{2w_0}{\partial x} \]  
(2.42a)

\[ V_t = - \frac{2w_0}{\partial y} \]  
(2.42b)

\[ w_t = \frac{1}{\lambda + 2G} \left[ \frac{A^2}{8} \nabla^2 w_t - \lambda \left( \frac{2u_t}{\partial x} + \frac{2v_t}{\partial y} \right) + \frac{P}{2} \right] \]  
(2.42c)

\[ U_3 = - \frac{2w_t}{\partial x} - \frac{B}{A^2} (u_t + \frac{2w_t}{\partial x}) \]  
(2.43a)

\[ V_3 = - \frac{2w_t}{\partial y} - \frac{B}{A^2} (v_t + \frac{2w_t}{\partial y}) \]  
(2.43b)

\[ w_2 = \frac{1}{\lambda + 2G} \left[ \frac{A^2}{24} \nabla^2 w_2 - \frac{2\lambda}{3} \left( \frac{2u_t}{\partial x} + \frac{2v_t}{\partial y} \right) + \frac{\lambda}{3} \nabla^2 w_0 + \frac{P}{A} \right] \]  
(2.43c)
2.7.2 Edge Boundary Conditions

The independent displacement component functions are reduced to five when the constraining equations are used in the present theory. The order of the resulting set of linear differential equations is twice the number of independent functions. Thus, the present set of differential equations is of the tenth order. From the uncoupled characteristics of extension and flexure, this system of tenth order differential equation is decomposed into two groups of differential equations, namely, a fourth order system for the symmetric problem and a sixth order system for the antisymmetric problem. Because of this, five boundary conditions are required to be prescribed at each edge.

A natural way to derive edge boundary conditions is by the well-known variational principle. Hildebrand, Reissner, and Thomas (ref. 23) applied the principle of minimum potential energy to derive their shell equilibrium equations and edge boundary conditions. The assumption of displacements $u$, $v$ and $w$ in their theory is similar to that of the present theory except the latter involves the third power terms of $z$ in $u$ and $v$. They treated all displacement components, namely $u_0, \ldots, w_2$, as independent functions and satisfaction of face boundary conditions were ignored. As a result of this treatment, nine equilibrium equations along with nine boundary conditions at an edge were obtained. Application of this theory to practical problems is extremely difficult. The reason lies mainly with those higher moments of the stresses which are outside the realm of what we generally understand as plate theory.

In the present theory the generalized force analysis method is adopted to derive edge boundary conditions. The edge conditions may be
simply supported, clamped or free. The stress and moment resultants acting on an edge \( y=\text{constant} \) of a Cartesian coordinate system is shown in figure 2.6. The total work done by the generalized forces at the mid-plane \( (z=0) \) of the plate is

\[
W = \int_{x_i}^{x_f} \left( \mathbf{F}_y \cdot \mathbf{U} + \mathbf{C}_y \cdot \mathbf{\Omega} \right) dx
\]  

(2.44)

where

\[
\mathbf{F}_y = N_x \mathbf{i} + N_y \mathbf{j} + Q_y \mathbf{k} \\
\mathbf{U} = u_x \mathbf{i} + v_x \mathbf{j} + w_x \mathbf{k} \\
\mathbf{C}_y = -M_y \mathbf{i} + M_{xy} \mathbf{j} \\
\mathbf{\Omega} = -\theta_x \mathbf{i} + \theta_y \mathbf{j} \\
\theta_x = \frac{\partial v}{\partial y} = v_i \\
\theta_y = \frac{\partial u}{\partial y} = u_i
\]

and \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) are the unit vectors in the directions of the \( x, \) \( y \) and \( z \)-axes, respectively. Performing the scalar multiplications and setting the total work to zero in equation (2.44), we then have

\[
W = \int_{x_i}^{x_f} \left( N_{xy} u_x + N_y v_x + Q_y w_x + M_y v_i + M_{xy} u_i \right) dx = 0
\]  

(2.45)

From equation (2.45) the appropriate boundary conditions at \( y=\text{constant} \) are

\[
\begin{align*}
N_{xy} &= 0 \quad \text{or} \quad u_x = 0 \\
N_y &= 0 \quad \text{or} \quad v_x = 0 \\
Q_y &= 0 \quad \text{or} \quad w_x = 0 \\
M_y &= 0 \quad \text{or} \quad v_i = 0 \\
M_{xy} &= 0 \quad \text{or} \quad u_i = 0
\end{align*}
\]  

(2.46)
Figure 2.6. - Stress and moment resultants at an edge $y=$constant.
The first two of equations (2.46) are for the symmetric problem and the last three for the antisymmetric problem.

The combination of boundary conditions for simply supported, clamped, and free edges at \( y = \text{constant} \) are as follows:

**Simply supported edge:**
\[
\begin{align*}
    u_s &= 0, 
    N_y &= 0; \quad (2.47a) \\
    w_s &= 0, 
    M_y &= 0, 
    u_s &= 0 \quad (2.47b)
\end{align*}
\]

**Clamped edge:**
\[
\begin{align*}
    u_0 &= 0, 
    v_0 &= 0; \quad (2.48a) \\
    w_0 &= 0, 
    v_0 &= 0, 
    u_0 &= 0 \quad (2.48b)
\end{align*}
\]

**Free edge:**
\[
\begin{align*}
    N_{x_0} &= 0, 
    N_y &= 0; \quad (2.49a) \\
    Q_y &= 0, 
    M_y &= 0, 
    M_{x_0} &= 0 \quad (2.49b)
\end{align*}
\]

### 2.8 Discussion

Two systems of differential equation governing the symmetric and antisymmetric problems have been obtained in previous sections. In the symmetric problem, in addition to the two equilibrium equations (2.33) and (2.34) there are three constraining equations (2.40). We wish to eliminate \( u_z, v_z \) and \( w_i \) in the equilibrium equations through the use of three constraining equations. However, this cannot be done because of equation (2.40c). We encountered the same difficulty in the antisymmetric problem. For the time being, we treat \( w_i \) and \( w_2 \) as independent functions and eliminate \( u_z, v_z, u_i, \) and \( v_3 \) from the equilibrium equations. Thus, in the absence of surface tangential loadings, we have
for the symmetric problem, and

\[
\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} + \nabla^2 w_i = -\frac{3}{2G_4} \rho 
\]  

(2.51a)

\[
\left\{-\frac{4}{3} G_1 + \frac{4}{15} (\lambda + 2G_4) \frac{\partial^2}{\partial x^2} + G_1 \frac{\partial^2}{\partial y^2}\right\} u_i + \frac{4^3}{15} (\lambda + G_1) \frac{\partial^2 v_i}{\partial x \partial y} - \left[\frac{2G_4}{3} \frac{\partial^2}{\partial x^2} + \frac{A_1^3}{60} (\lambda + 2G_4) \frac{\partial^2}{\partial x \partial y} \nabla^2 \right] W_x + \left[\frac{A_1^3}{12} \lambda \frac{\partial}{\partial y} - \frac{A_1^3}{480} (\lambda + 2G_4) \frac{\partial}{\partial y} \nabla^2 \right] W_z = 0
\]  

(2.51b)

\[
\frac{A_1^3}{15} (\lambda + G_1) \frac{\partial^2 u_i}{\partial x^2 y} + \left\{-\frac{2}{3} G_1 + \frac{A_1^3}{15} \left[G_1 \frac{\partial^2}{\partial x^2} + (\lambda + 2G_4) \frac{\partial^2}{\partial y^2}\right]\right\} v_i - \left[\frac{2G_4}{3} \frac{\partial}{\partial x} + \frac{A_1^3}{60} (\lambda + 2G_4) \frac{\partial^2}{\partial y^2} \nabla^2 \right] W_x + \left[\frac{A_1^3}{12} \lambda \frac{\partial}{\partial y} - \frac{A_1^3}{480} (\lambda + 2G_4) \frac{\partial}{\partial y} \nabla^2 \right] W_z = 0
\]  

(2.51c)

\[
\frac{2\lambda}{3} \left[\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} - \frac{1}{2} \nabla^2 w_i\right] + \left[(\lambda + 2G_4) - \frac{A_1^3}{24} \lambda \nabla^2 \right] W_z = \frac{\rho}{\rho_0} 
\]  

(2.51d)

for the antisymmetric problem.

It is noted that equations (2.50c) and (2.51d) are constraining
equations. From the appearance of above equations if we assume $\nabla^2 w_1$ and $\nabla^2 w_2$ equal to constants we will be able to eliminate $w_1$ and $w_2$ entirely from the two sets of equilibrium equations. In some cases, as will be seen in next chapter, this assumption is true. With this assumption we arrive at the following two sets of equilibrium equations in matrix form.

The symmetric problem:

\[
\begin{bmatrix}
\frac{4G_l(\lambda+G_l)}{\lambda+2G_l} \cdot \frac{\partial^2}{\partial x^2} + G_l \frac{\partial}{\partial y} \\
\frac{G_l(3\lambda+2G_l)}{\lambda+2G_l} \cdot \frac{\partial^2}{\partial x \partial y} \\
\frac{G_l(3\lambda+2G_l)}{\lambda+2G_l} \cdot \frac{\partial^2}{\partial x \partial y} + G_l \frac{\partial}{\partial y} \\
\frac{G_l(\lambda+G_l)}{\lambda+2G_l} \cdot \frac{\partial}{\partial y} + \frac{G_l}{\lambda+2G_l} \cdot \frac{\partial^2}{\partial y^2}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial^2}{\partial x^2} + G_l \frac{\partial}{\partial y} \\
\frac{\partial^2}{\partial x \partial y} + \frac{G_l(\lambda+G_l)}{\lambda+2G_l} \cdot \frac{\partial}{\partial y} \\
\frac{\partial}{\partial y} \\
\frac{\partial^2}{\partial y^2}
\end{bmatrix}
\begin{bmatrix}
u_x \\
v_y \\
\psi_x \\
\psi_y
\end{bmatrix} = \begin{bmatrix}
\frac{\partial P}{\partial x} \\
\frac{\partial P}{\partial y}
\end{bmatrix}
\]

(2.52)
The antisymmetric problem:

\[
\begin{pmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \nabla^2 \\
-\frac{2}{3} G \hat{K} + \frac{\hat{A}^3}{q_0} K_1 \frac{\partial^2}{\partial x^2} + \frac{\hat{A}^3}{15} G \frac{\partial^2}{\partial y^2} & \frac{\hat{A}^3}{q_0} K_2 \frac{\partial^2}{\partial x \partial y} & -\frac{2}{3} G \hat{K} \frac{\partial}{\partial x} + \frac{\hat{A}^3}{q_0} K_3 \frac{\partial}{\partial x} \nabla^2 \\
\frac{\hat{A}^3}{q_0} K_1 \frac{\partial^2}{\partial x \partial y} & -\frac{2}{3} G \hat{K} + \frac{\hat{A}^3}{15} G \frac{\partial^2}{\partial x^2} + \frac{\hat{A}^3}{q_0} K_1 \frac{\partial^2}{\partial y^2} & -\frac{2}{3} G \hat{K} \frac{\partial}{\partial y} + \frac{\hat{A}^3}{q_0} K_3 \frac{\partial}{\partial y} \nabla^2 \\
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}_1 \\
\mathbf{v}_1 \\
\mathbf{w}_1 \\
\end{pmatrix}
= \begin{pmatrix}
-\frac{3}{2 G \hat{K}} \mathbf{p} \\
-\frac{3}{12 \lambda + 2 \hat{G}} \frac{2 \mathbf{p}}{\partial x} \\
-\frac{3}{12 \lambda + 2 \hat{G}} \frac{2 \mathbf{p}}{\partial y}
\end{pmatrix}
\]

where

\[
K_1 = \frac{\lambda^2 + 24 \lambda G + 24 G^2}{\lambda + 2G}
\]

\[
K_2 = \frac{\lambda^2 + 18 \lambda G + 12 G^2}{\lambda + 2G}
\]

\[
K_3 = \frac{\lambda^2 - 6 \lambda G - 6 G^2}{\lambda + 2G}
\]
CHAPTER III
A THREE-DIMENSIONAL SOLUTION FOR HOMOGENEOUS ISOTROPIC
THICK ELASTIC PLATES HAVING SIMPLY SUPPORTED EDGES

3.1 Introduction

A linear theory for homogeneous isotropic thick elastic plates was established in Chapter II. In this chapter, the thick plate theory developed in Chapter II is used to investigate rectangular plates having simply supported edges. Two types of loading are considered in the study, namely, the uniformly and sinusoidally distributed loads. In the case of uniform load, a Levy type solution is obtained. Results of displacements, stresses, stress and moment resultants are compared to those from the classical, Reissner, and Donnell-Lee plate theories. Detailed numerical comparisons for various $b/a$ and $h/a$ ratios will be presented in Chapter IV.

Figure 3.1 shows a plate under a uniformly distributed load $P$ and its coordinate system which will be used in the subsequent sections.

3.2 Symmetric Problem Under Uniform Load (Extension)

For the symmetric problem, the displacements are those shown in equations (2.19). Rewrite them once again as follows,

\begin{align}
U_e &= U_0 + \frac{x^2}{2} U_1 \\
V_e &= V_0 + \frac{x^2}{2} V_1
\end{align}

(3.1a) (3.1b)
Figure 3.1. - Simply supported rectangular plate under uniform load.
The equilibrium equations in terms of $u_0$ and $v_0$ for uniform load $P$ are

\[
\begin{align*}
[4G_0(\lambda+2G_0) \frac{\partial^2}{\partial x^2} + G_0 \frac{\partial^2}{\partial y^2}] u_0 + \frac{G_0(3\lambda+2G_0)}{\lambda+2G_0} \frac{\partial^2 v_0}{\partial x \partial y} &= 0 \quad (3.2a) \\
\frac{G_0(3\lambda+2G_0)}{\lambda+2G_0} \frac{\partial^2 u_0}{\partial x \partial y} + \left[ G_0 \frac{\partial^2}{\partial x^2} + \frac{4G_0(\lambda+G_0)}{\lambda+2G_0} \frac{\partial^2 v_0}{\partial y^2} \right] v_0 &= 0 \quad (3.2b)
\end{align*}
\]

As discussed in section 2.8, equations (3.2) were obtained by assuming $\nabla^2 w_1 =$ constant. In the present case we assume $\nabla^2 w_1 = 0$. The proof of this assumption will be presented later in this section. A very interesting thing about equations (3.2) is that if we rewrite them in terms of $\gamma$ (Poisson's ratio) instead of $G$ and $\lambda$, we obtain the following equations.

\[
\begin{align*}
\frac{\partial^2 u_0}{\partial x^2} + \frac{1}{2} (1 - \gamma) \frac{\partial^2 v_0}{\partial y^2} + \frac{1}{2} (1 + \gamma) \frac{\partial^2 v_0}{\partial x \partial y} &= 0 \quad (3.3a) \\
\frac{1}{2} (1 + \gamma) \frac{\partial^2 u_0}{\partial x \partial y} + \frac{1}{2} (1 - \gamma) \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} &= 0 \quad (3.3b)
\end{align*}
\]

Comparing equations (3.3) with Love's equations for extensional vibration of plates (i.e., eq. (97), p. 497), we find that the left-hand sides of the corresponding equations are identical.

It is the purpose of this section to find $u_0$, $u_2$, $v_0$, $v_2$, and $w_1$ which will not only satisfy the equilibrium equations but also the appropriate edge boundary conditions. The coordinate system adopted is shown in figure 3.1.

The edge boundary conditions for simply supported plates at $x=0$ and $x=a$ are $v_0 = 0$ and $N_x = 0$. 
Let
\[ u_0 = P \left[ \sum_{m=1}^{\infty} Y_1(y) \cos \alpha_m x - \frac{\lambda}{\beta G (\lambda + \beta)} \left( x - \frac{a}{2} \right) \right] \quad (3.4) \]
\[ v_0 = P \sum_{m=1}^{\infty} Y_2(y) \sin \alpha_m x \quad (3.5) \]

where \( Y_1 \) and \( Y_2 \) are functions of \( y \) only and \( \alpha_m = \frac{m\pi}{a} \) \((m=1, 3, 5, \ldots)\) and throughout this chapter only odd integers will be used for \( m \). The boundary condition \( v_0 = 0 \) at \( x=0, a \) is obviously satisfied. From equation (2.21) we have the following expression for \( N_x \),
\[ N_x = \lambda \left[ \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + w_1 \right) + 2G \frac{\partial u}{\partial x} \right] + \frac{P}{2} \left[ \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + 2G \frac{\partial u}{\partial x} \right) \right] \quad (3.6) \]

The constraining equations corresponding to \( \nabla^2 w_1 = 0 \) are (from equations (2.42))
\[ u_2 = -\frac{\partial w_1}{\partial x} \quad (3.7a) \]
\[ v_2 = -\frac{\partial w_1}{\partial y} \quad (3.7b) \]
\[ w_1 = \frac{P}{\lambda + 2G} \left[ \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{P}{2} \right] \quad (3.7c) \]

Substituting equations (3.7) into (3.6) we have \( N_x \) expressed in terms of \( u_0 \) and \( v_0 \) as follows,
\[ N_x = \frac{\lambda}{\lambda + 2G} \left[ 4G \left( \lambda + \beta \right) \frac{\partial u}{\partial x} + \frac{\lambda^3}{12} \lambda G \frac{\partial u}{\partial x}^2 + 2\lambda G \frac{\partial v}{\partial y} + \frac{\lambda^3}{12} \lambda G \frac{\partial v}{\partial x}^2 + \frac{\lambda}{2} \right] \quad (3.8) \]

A direct substitution of equations (3.4) and (3.5) into (3.8) yields
\[ N_x = \frac{G}{\lambda + 2G} \frac{\lambda}{\lambda + \beta} \sum_{m=1}^{\infty} \left\{ \left[ \frac{\lambda^3}{12} \lambda \alpha_m^3 - 4(\lambda + \beta) \alpha_m \right] Y_1 + \left[ 2\lambda - \frac{\lambda^3}{12} \lambda \alpha_m^2 \right] Y_2 \right\} \sin \alpha_m x \quad (3.9) \]

where \( Y_2 \) indicates differentiation with respect to \( y \). Clearly, the boundary condition \( N_x = 0 \) is satisfied at \( x=0, a \).
Next, we seek the solution of the differential equations (3.2). Substituting equations (3.4) and (3.5) into (3.2), we get

\[-4(\lambda + G_1)\alpha_m^2 Y_1 + (\lambda + 2G_1) Y_1'' + (3\lambda + 2G_1)\alpha_m Y_2' = 0 \quad (3.10)\]
\[-(3\lambda + 2G_1)\alpha_m Y_1' - (\lambda + 2G_1)\alpha_m^2 Y_1 + 4(\lambda + G_1) Y_2'' = 0 \quad (3.11)\]

Let \(Y_1 = \alpha e^{\delta y}\) and \(Y_2 = \beta e^{\delta y}\) where \(\alpha\), \(\beta\), and \(\delta\) are constants. Equations (3.10) and (3.11) become

\[
\begin{bmatrix}
(\lambda + 2G_1)\delta^2 - 4(\lambda + G_1)\alpha_m^2 \\
-(3\lambda + 2G_1)\alpha_m\delta \\
4(\lambda + G_1)\delta^2 - (\lambda + 2G_1)\alpha_m^2
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta \\
0
\end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}
\]

(3.12)

For a non-trivial solution, the determinant of the coefficients of \(\alpha\) and \(\beta\) must vanish. Hence we obtain the following characteristic equation

\[\delta^4 - 2\alpha_m^2 \delta^2 + \alpha_m^4 = 0 \quad (3.13)\]

The four roots of equation (3.13) are

\[\delta_1 = \delta_2 = \alpha_m \quad , \quad \delta_3 = \delta_4 = -\alpha_m\]

The general solution thus becomes

\[Y_1 = (C_1 e^{\alpha_m y} + C_2 e^{-\alpha_m y}) + y(C_3 e^{\alpha_m y} + C_4 e^{-\alpha_m y}) + \frac{5\lambda + 2G_1}{3\lambda + 2G_1} \frac{1}{\alpha_m} (C_1 e^{\alpha_m y} - C_4 e^{-\alpha_m y}) \quad (3.14a)\]
\[Y_2 = (C_1 e^{\alpha_m y} - C_3 e^{-\alpha_m y}) + y(C_3 e^{\alpha_m y} - C_4 e^{-\alpha_m y}) \quad (3.14b)\]

where \(C_1, ..., C_4\) are constants. Converting the exponentials in equations (3.14) into hyperbolic functions, we get
\[
Y_1 = (A_1 \cosh \alpha_m y + A_3 \sinh \alpha_m y) + \frac{\pi}{\lambda + 2q} (A_2 \sinh \alpha_m y + A_4 \cosh \alpha_m y)
\]
\[
Y_2 = (A_1 \sinh \alpha_m y + A_3 \cosh \alpha_m y) + \frac{\pi}{\lambda + 2q} (A_2 \sinh \alpha_m y + A_4 \cosh \alpha_m y)
\]
(3.15a)

where \(A_1, \ldots, A_4\) are again constants.

Now, we shall prove the assumption \(\nabla^2 w = 0\) which we made at the beginning of this section. Substituting equations (3.4), (3.5) and (3.15) into (3.7c), then we get

\[
w_i = \frac{P}{\lambda + 2q} \left[ \frac{(\lambda + 2q)^2}{8q(\lambda + q)} + \frac{\pi}{3(\lambda + 2q)} \sum_{m=1}^{\infty} (A_2 \sinh \alpha_m y + A_4 \cosh \alpha_m y) \sin \alpha_m x \right]
\]
(3.16)

Clearly, equation (3.16) is a harmonic function. Hence \(\nabla^2 w_i = 0\) is proved.

As can be seen from equations (3.15), there are four constants \(A_1, \ldots, A_4\) to be determined. These four constants can be determined completely from the four boundary conditions at \(y = \pm b/2\). In the case of simply supported edges at \(y = \pm b/2\), the boundary conditions to be satisfied are \(u_0 = 0\) and \(N_y = 0\). Using equations (2.22), (3.4), (3.5), (3.7) and (3.15) we obtain the following four algebraic equations as a consequence of satisfying boundary conditions \(u_0 = 0\) and \(N_y = 0\) at \(y = \pm b/2\).

\[
cosh \frac{\alpha_m b}{2} A_1 + \left( \frac{b}{2} \cosh \frac{\alpha_m b}{2} + \frac{1}{\alpha_m} \frac{5\lambda + 6q}{3(\lambda + q)} \sinh \frac{\alpha_m b}{2} \right) A_2 + \sinh \frac{\alpha_m b}{2} A_3
\]
\[
+ \left( \frac{b}{2} \sinh \frac{\alpha_m b}{2} + \frac{1}{\alpha_m} \frac{5\lambda + 6q}{3(\lambda + q)} \cosh \frac{\alpha_m b}{2} \right) A_4 = -\frac{1}{2a} \frac{\lambda}{G(\lambda + q)} \frac{1}{\alpha_m^2}
\]
(3.17a)
Solving the simultaneous equations (3.17), we obtain the following results,

\[ A_1 = - \frac{1}{\cosh \frac{d_{mb}}{2}} \left[ \frac{1}{2a} \cdot \frac{\lambda}{G(\lambda+\delta)} \cdot \frac{1}{d_m} \right] \]

\[ A_2 = A_3 = A_4 = 0 \]
Hence

\[ U_e = -\frac{\lambda P}{8 G_1(\lambda + G_1)} \left[ (x - \frac{a}{2}) + \frac{4}{a} \sum_{m=1,3}^{\infty} \frac{\cosh \alpha_m^2 y}{\alpha_m^2 \cosh \frac{\alpha_m^2 b}{2}} \cos \alpha_m^2 x \right] \]  \hspace{1cm} (3.18a)

\[ V_e = -\frac{\lambda P}{G_1(\lambda + G_1)} \cdot \frac{1}{2a} \sum_{m=1,3}^{\infty} \frac{\sinh \alpha_m^2 y}{\alpha_m^2 \cosh \frac{\alpha_m^2 b}{2}} \sin \alpha_m^2 x \]  \hspace{1cm} (3.18b)

\[ w_1 = \frac{P}{8} \cdot \frac{\lambda + 2G_1}{G_1(\lambda + G_1)} \]  \hspace{1cm} (3.18c)

\[ U_2 = 0 \]  \hspace{1cm} (3.18d)

\[ V_2 = 0 \]  \hspace{1cm} (3.18e)

Equation (3.18a) can be put into another form as follows,

\[ U_e = -\frac{\lambda P}{G_1(\lambda + G_1)} \cdot \frac{1}{2a} \sum_{m=1,3}^{\infty} \frac{1}{\alpha_m^2} \left( 1 - \frac{\cosh \alpha_m^2 y}{\cosh \frac{\alpha_m^2 b}{2}} \right) \cos \alpha_m^2 x \]  \hspace{1cm} (3.19)

Note the Fourier series expansion for \( x = a/2 \) in the interval \( 0 \leq x \leq a \)

\[ x - \frac{a}{2} = -\frac{4}{a} \sum_{m=1,3}^{\infty} \frac{\cos \alpha_m^2 x}{\alpha_m^2} \]

has been used in converting equation (3.18a) into (3.19).

The displacement components \( u_e, u_2, v_e, v_2, \) and \( w_1 \) are thus completely determined. The stresses and stress resultants can be obtained simply by substituting equations (3.18) into (2.20) - (2.23).
3.3 Antisymmetric Problem Under Uniform Load (Flexure)

The antisymmetric problem of a plate is more complex than the symmetric problem since on the one hand we have three equilibrium equations while on the other hand $w_2$ is not a harmonic function. First, let us examine the displacement expressions

$$U_f = \frac{y}{3} u_1 + \frac{y^3}{6} u_3 \quad (3.20a)$$

$$V_f = \frac{y}{3} V_1 + \frac{y^3}{6} V_3 \quad (3.20b)$$

$$W_f = W_o + \frac{y^3}{2} W_2 \quad (3.20c)$$

If we assume $\nabla^2 w_2 =$constant, then the equilibrium equations (2.51a, b, and c) take the following forms for the case of a uniform load $P$.

$$\frac{\partial U_i}{\partial x} + \frac{\partial V_i}{\partial y} + \nabla^2 W_o = - \frac{3}{2g_h} P \quad (3.21a)$$

$$\left(- \frac{2}{3} g_h^3 + \frac{h^3}{q_0} K_1 \frac{\partial^2}{\partial x^2} + \frac{h^3}{15} G \frac{\partial^2}{\partial y^2}\right) U_i + \frac{h^3}{q_o} K_2 \frac{\partial^2}{\partial x^2} V_1 + \left(- \frac{2}{3} g_h^2 \frac{\partial}{\partial x} + \frac{h^3}{q_0} K_3 \frac{\partial^2}{\partial y^2} \nabla^2\right) W_o = 0 \quad (3.21b)$$

$$\frac{h^3}{q_0} K_2 \frac{\partial^3}{\partial x^2} + \left(- \frac{2}{3} g_h^3 + \frac{h^3}{15} G \frac{\partial^3}{\partial x^2} + \frac{h^3}{q_0} K_3 \frac{\partial^2}{\partial y^2}\right) V_1 + \left(- \frac{2}{3} g_h^3 \frac{\partial^2}{\partial y^2} + \frac{h^3}{q_0} K_3 \frac{\partial^2}{\partial y^2} \nabla^2\right) W_o = 0 \quad (3.21c)$$

where $K_1$, $K_2$, and $K_3$ are the same as in equation (2.53). The proof of $\nabla^2 w_2 =$constant will be presented later in this section. The constraining equations are those of equations (2.43).

Let

$$U_i = P \left[ (a - 2x) S_i + \sum_{m=1,3}^{\infty} Y_3(cy) \cos \alpha_mx \right] \quad (3.22)$$
\[ V_i = \mathcal{P} \sum_{n=1,3}^{\infty} Y_n(y) \sin \alpha_n x \] (3.23)

\[ W_0 = \mathcal{P} \left[ \left( \frac{3}{4g\eta} - S_1 \right) (ax - x^2) + \sum_{n=1,3}^{\infty} Y_n(y) \sin \alpha_n x \right] \] (3.24)

\[ \nabla^2 W_2 = \frac{P}{i^2} (S_1 + S_2) \] (3.25)

where \( S_1 \) and \( S_2 \) are constants. There are particular reasons for using two constants in the expression of \( \nabla^2 W_2 \). The same reasons also apply to the expressions of equations (3.22) and (3.24). As the calculations proceed, we will be able to see those reasons.

Equations (3.23) and (3.24) identically satisfy the following boundary conditions at \( x = 0, a \),

\[ w_0 = v_i = 0 \]

As for the third boundary condition, \( M'_x = 0 \) at \( x = 0, a \), its satisfaction is not readily seen. However, later in this section we can find a relationship between the constants \( S_1 \) and \( S_2 \) such that this boundary condition will be satisfied.

In anticipation of subsequent use we note that the Fourier series expansions for \( (ax - x^2) \) and \( (a - 2x) \) in the interval \( 0 \leq x \leq a \) are

\[ ax - x^2 = \frac{a}{\alpha} \sum_{n=1,3}^{\infty} \frac{\sin \alpha_n x}{\alpha_n^3} \] (3.26)

\[ a - 2x = \frac{a}{\alpha} \sum_{n=1,3}^{\infty} \frac{\cos \alpha_n x}{\alpha_n^2} \] (3.27)

Substituting equations (3.22) - (3.24) into (3.21) and making use of (3.26) and (3.27) leads to the following set of nonhomogeneous simultaneous linear differential equations; namely,
The particular solution of the above set of simultaneous differential equations can be easily obtained. Let \( Y_{3p} \), \( Y_{4p} \), and \( Y_{5p} \) be the particular solutions and set them all equal to constants. From equation (3.28c) we get \( Y_{4p} = 0 \) and from (3.28a) and (3.28b) we can solve for \( Y_{3p} \) and \( Y_{5p} \), thus

\[
Y_{3p} = -\frac{360}{a^4 \delta^5} \cdot \frac{1}{K_1 - K_3} \cdot \frac{1}{\alpha_m^4}
\]  
(3.29a)

\[
Y_{4p} = 0
\]  
(3.29b)

\[
Y_{5p} = \frac{360}{a^4 \delta^5} \cdot \frac{1}{K_1 - K_3} \cdot \frac{1}{\alpha_m^5}
\]  
(3.29c)

The related homogeneous solution can be obtained from the set of differential equations below,

\[-\alpha_m Y_{3p} + Y_{3p}' - \alpha_m^2 Y_{3p} + Y_{3p}'' = 0 \]  
(3.30a)
Let \( S \) be equal to the constants. Substituting them into equations (3.30) then we get

\[
\begin{pmatrix}
-\alpha_m & \xi & \zeta - \alpha_m^2 \\
(\frac{2}{3} G T + \alpha_m \frac{43}{15} G - \frac{43}{14 K_1\delta}) & \alpha_m \frac{43}{15} G & -\alpha_m (\frac{2}{3} G T + \alpha_m \frac{43}{15} G - \frac{43}{14 K_1\delta}) \\
-\alpha_m \frac{43}{15} G & (\frac{2}{3} G T + \alpha_m \frac{43}{15} G - \frac{43}{14 K_1\delta}) & -\alpha_m (\frac{2}{3} G T + \alpha_m \frac{43}{15} G - \frac{43}{14 K_1\delta})
\end{pmatrix}
\begin{pmatrix}
\eta \\
\xi \\
\zeta
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

(3.31)

For a non-trivial solution, the determinant of the coefficients of \( \eta, \xi, \) and \( \zeta \) must vanish in the above matrix expression. Hence we obtain the following characteristic equation

\[
\delta^6 - (3\alpha_m^4 + \frac{10}{\eta^4}) \delta^4 + (3\alpha_m^4 + \frac{20}{\eta^4} \alpha_m^2) \delta^2 - (3\alpha_m^4 + \frac{10}{\eta^4} \alpha_m^2) = 0
\]

(3.32)
The six roots of equation (3.32) are

\[ \delta_1 = \sqrt{\lambda_m^2 + \frac{10}{n^2}} , \]
\[ \delta_2 = -\sqrt{\lambda_m^2 + \frac{10}{n^2}} , \]
\[ \delta_3 = \delta_4 = \alpha_m , \quad \delta_5 = \delta_6 = -\alpha_m . \]

The related homogeneous solution corresponding to the above six roots are,

\[ Y_{3h} = l_m (c_1 e^{\lambda_m y} + c_2 e^{-\lambda_m y}) + (c_3 e^{\alpha_m y} + c_5 e^{-\alpha_m y}) + \frac{y}{\alpha_m} (c_4 e^{\alpha_m y} + c_6 e^{-\alpha_m y}) \]  
(3.33a)

\[ Y_{4h} = \alpha_m (c_1 e^{\lambda_m y} + c_2 e^{-\lambda_m y}) + (c_3 e^{\alpha_m y} - c_5 e^{-\alpha_m y}) + \frac{1}{\alpha_m} (c_4 e^{\alpha_m y} + c_6 e^{-\alpha_m y}) \]
\[ + \frac{y}{\alpha_m} (c_4 e^{\alpha_m y} - c_6 e^{-\alpha_m y}) \]  
(3.33b)

\[ Y_{5h} = -\frac{1}{\alpha_m} (c_5 e^{\alpha_m y} + c_6 e^{-\alpha_m y}) + \frac{\lambda + \alpha_m}{\lambda + 2\alpha_m} \alpha_m^2 (c_4 e^{\alpha_m y} - c_6 e^{-\alpha_m y}) - \frac{y}{\alpha_m} (c_4 e^{\alpha_m y} + c_6 e^{-\alpha_m y}) \]  
(3.33c)

where \( l_m = \sqrt{\lambda_m^2 + \frac{10}{n^2}} \) and \( c_1, \ldots, c_6 \) are constants. Converting the exponentials in equations (3.33) into hyperbolic functions, then we get

\[ Y_{3h} = l_m (B_1 \sinh \lambda_m y + B_2 \cosh \lambda_m y) + (B_3 \cosh \alpha_m y + B_5 \sinh \alpha_m y) \]
\[ + \frac{y}{\alpha_m} (B_4 \cosh \alpha_m y + B_6 \sinh \alpha_m y) \]  
(3.34a)

\[ Y_{4h} = \alpha_m (B_1 \cosh \lambda_m y + B_2 \sinh \lambda_m y) + (B_3 \sinh \lambda_m y + B_5 \cosh \lambda_m y) \]
\[ + \frac{1}{\alpha_m} (B_4 \cosh \lambda_m y + B_6 \sinh \lambda_m y) + \frac{y}{\alpha_m} (B_4 \sinh \alpha_m y + B_6 \cosh \alpha_m y) \]  
(3.34b)
where $B_1, \ldots, B_6$ are constants.

The general solution for the set of differential equations (3.28) is

$$Y_3 = \alpha_m \left( B_1 \sinh \alpha_m \gamma + B_2 \cosh \alpha_m \gamma \right) + \left( B_3 \cosh \alpha_m \gamma + B_5 \sinh \alpha_m \gamma \right)$$

$$Y_4 = \alpha_m \left( B_1 \cosh \alpha_m \gamma + B_2 \sinh \alpha_m \gamma \right) + \left( B_3 \sinh \alpha_m \gamma + B_5 \cosh \alpha_m \gamma \right)$$

$$Y_5 = -\frac{1}{\alpha_m} \left( B_3 \cosh \alpha_m \gamma + B_5 \sinh \alpha_m \gamma \right) + \frac{\lambda + \sigma}{\lambda + 2\sigma} \alpha_m^2 \left( B_4 \sinh \alpha_m \gamma + B_6 \cosh \alpha_m \gamma \right)$$

It is a suitable time at this stage to prove the assumption we made at the beginning of this section, namely that $\nabla^2 W_2 = \text{constant}$. Substituting equations (3.22) - (3.25) and (3.35) into (2.43c), i.e.,

$$W_2 = \frac{1}{\lambda + 2\sigma} \left[ \frac{\lambda + \sigma}{24} \nabla^2 W_2 - \frac{2\lambda}{3} \left( \frac{\partial W_1}{\partial x} + \frac{\partial W_1}{\partial y} \right) + \frac{\lambda}{3} \nabla^2 W_0 + \frac{P}{A} \right]$$
we then get

\[ W_z = \frac{\lambda P}{\lambda + 2\varrho} \left[ \frac{1}{24} (4q S_1 + S_2) + \frac{1}{2} \frac{2\varrho - \lambda}{\lambda \varrho} - \frac{4\varrho}{4\varrho} \cdot \frac{1}{8} \frac{1}{k_1 - k_3} (ax - x^2) \right. \]

\[ \left. - 2 \sum_{m=1}^{\infty} \left( B_m \sinh \delta_m y + B_m \cosh \delta_m y \right) \sin \delta_m x \right] \tag{3.36} \]

Applying the Laplace operator to the two sides of equation (3.36) and noting \( \sinh \delta_y \sin \delta_x \) and \( \cosh \delta_y \sin \delta_x \) are both harmonic functions, we obtain

\[ \nabla^2 W_z = \frac{\lambda P}{\lambda + 2\varrho} \cdot \frac{q_0}{4\varrho(K_1 - K_3)} \tag{3.37} \]

Clearly, the right-hand side of equation (3.37) is a constant. Furthermore, in view of equations (3.25) and (3.37) we get the following relationship between \( S_1 \) and \( S_2 \)

\[ S_1 + S_2 = \frac{\lambda}{\lambda + 2\varrho} \cdot \frac{q_0}{4\varrho(K_1 - K_3)} \tag{3.38} \]

It remains to find the values of \( S_1 \) and \( S_2 \). One more equation relating \( S_1 \) and \( S_2 \) has to be found. This equation can be obtained by satisfying the boundary condition \( M_x = 0 \) at \( x = 0 \), a. Equation (2.30) can be transformed into terms of \( u_1 \), \( v_1 \), \( w_o \), and \( w_2 \) by using the constraining equations (2.43a) and (2.43b).

\[ M_x = \frac{4}{15} \left[ (\lambda + 2\varrho) \frac{\partial u_1}{\partial x} + \lambda \frac{\partial v_1}{\partial y} \right] - \frac{4}{60} \left[ (\lambda + 2\varrho) \frac{\partial^2 w_o}{\partial x^2} + \lambda \frac{\partial^2 w_o}{\partial y^2} \right] \]

\[ + \frac{4}{12} \lambda w_2 - \frac{4}{480} \left( \lambda \nabla^2 w_2 + 2\varrho \frac{\partial^2 w_o}{\partial x^2} \right) \tag{3.39} \]

Substituting equations (3.22) - (3.25), (3.35), and (3.36) into (3.39), we have \( M_x \) in the following form
As a result of satisfying the boundary condition \( M_x = 0 \) at \( x = 0, a \), we get

\[
\frac{a}{3} \left( \frac{\lambda^2 - 483 \lambda G - 480 G^2}{\lambda + 2G} S_1 + \frac{\lambda^2 - 3\lambda G}{\lambda + 2G} S_2 \right) + \frac{4\lambda^2 + 13\lambda G + 62 \lambda G^2 + 24 G^3}{G (\lambda + G)(\lambda + 2G)} = 0
\]

(3.41)

Equations (3.38) and (3.41) are two algebraic equations with two unknowns \( S_1 \) an \( S_2 \). The solution for \( S_1 \) and \( S_2 \) is

\[
S_1 = \frac{1}{160 \pi} \cdot \frac{3\lambda^3 - 34 \lambda G - 62 \lambda G^2 - 24 G^3}{G^2 (\lambda + G)^2}
\]

(3.42)

\[
S_2 = \frac{1}{160 \pi} \cdot \frac{3\lambda^3 + 446 \lambda G + 41 G^2 + 24 G^3}{G (\lambda + G)^2}
\]

(3.43)

The boundary conditions at \( x = 0, a \) are thus all satisfied. It remains to apply the boundary conditions at \( y = \pm b/2 \) to determine the six constants \( B_1 \ldots B_6 \). For simply supported edges at \( y = \pm b/2 \), we have the following boundary conditions
\[ W_0 = M_y = u_1 = 0 \]

The deflection at the mid-plane \( w_0 \) is apparently symmetrical about \( x \)-axis due to the plate geometry and the uniformly distributed load, hence from equations (3.24) and (3.35c) we find

\[ B_4 = B_5 = 0 \]

Application of boundary condition \( u_1 = 0 \) at \( y = \pm b/2 \) also yields

\[ B_1 = 0 \]

From equation (2.31) we have \( M_y \) in terms of \( u, \nu, w_0, \) and \( w_2 \) as follows,

\[ M_y = \frac{d^3}{60} \sum_{m=1}^{\infty} \left\{ (8 G \alpha_m \lambda_m \cosh \alpha_m y) B_2 + (10 G \alpha_m \cosh \alpha_m y) B_3 
+ \left[ \frac{4 G \alpha_m (\lambda + \phi)}{\lambda + 2G} - \frac{\lambda^2 \alpha_m}{\lambda + G} \right] \cosh \alpha_m y + 10 G \alpha_m \frac{\sinh \alpha_m y}{\lambda + \phi} \right\} B_6 
+ \frac{120}{\alpha_m^3 \alpha_m^3} \frac{\lambda}{\lambda + \phi} + \frac{1}{\alpha_m^3 \alpha_m^3} \frac{-3 \lambda^3 + 6 \lambda^2 G + 2G \Lambda G^2}{2G (\lambda + \phi)^2} \right\} \sin \alpha_m x \]

(3.45)

Applying

\[ 8 G \alpha_m \lambda_m \cosh \alpha_m y \] \[ 5 b G \alpha_m \sinh \alpha_m y \]

\[ B_2 = -\frac{120}{\alpha_m^3 \alpha_m^3} \frac{\lambda}{\lambda + G} - \frac{1}{\alpha_m^3 \alpha_m^3} \frac{-3 \lambda^3 + 6 \lambda^2 G + 2G \Lambda G^2}{2G (\lambda + G)^2} \]

(3.46)
Application of $w_0 = 0$ and $u_1 = 0$ at $y = \pm b/2$ yields equations (3.47) and (3.48), respectively.

\[
- \frac{\cosh \frac{\alpha_m b}{2}}{\alpha_m} B_3 + \left( \frac{\lambda + 2\alpha_1}{\lambda + 2\alpha_2} \right) \frac{\alpha_m b}{2} \sinh \frac{\alpha_m b}{2} = -\frac{b}{2 \alpha_m} \sinh \frac{\alpha_m b}{2} B_6
\]

\[
= - \frac{360}{\alpha_m^5 \alpha_1^3} \frac{1}{k_1 - k_3} - \frac{8}{\alpha_m^2 a_1^2 \alpha_1} \left( \frac{3}{4 G_1} - \alpha_1 S_1 \right) \tag{3.47}
\]

\[
I_m \cosh \frac{\alpha_m b}{2} B_2 + \cosh \frac{\alpha_m b}{2} B_3 + \frac{b}{2} \sinh \frac{\alpha_m b}{2} B_6 = \frac{360}{\alpha_m^5 \alpha_1^3} \frac{1}{k_1 - k_3} - \frac{8}{\alpha_m^2 a_1^2 \alpha_1} S_1 \tag{3.48}
\]

Equations (3.46) - (3.48) are three algebraic equations with three unknowns $B_2$, $B_3$, and $B_6$. The solution is

\[
B_2 = 0 \tag{3.49a}
\]

\[
B_3 = \frac{1}{\cosh \frac{\alpha_m b}{2}} \left[ \frac{12}{\alpha_m^4 \alpha_1^3} \frac{\lambda + 2\alpha_1}{4 \alpha_1} + \frac{3 b}{\alpha_m^3 \alpha_1^3} \frac{\lambda + 2\alpha_1}{4 \alpha_1} \tanh \frac{\alpha_m b}{2} + \frac{1}{\alpha_m^3 \alpha_1^3} \frac{3 \lambda^2 - 2 \lambda \alpha_1^2 - \alpha_1^2 - 2 \lambda^3}{20 \alpha_1^2 (\lambda + \gamma_1)^2} \right] \tag{3.49b}
\]

\[
B_6 = -\frac{1}{\cosh \frac{\alpha_m b}{2}} \frac{6}{\alpha_m a_1^2 \alpha_1^3} \frac{\lambda + 2\alpha_1}{\alpha_1^2 (\lambda + \gamma_1)} \tag{3.49c}
\]

Knowing $B_1 \ldots B_6$, the displacements $u_1$, $v_1$, and $w_0$ are completely determined. The displacement $w_2$ can be found from equation (3.36). Once $w_2$ is known, $w_3$ and $v_3$ can be also found from the constraining equations (2.43a) and (2.43b), respectively. The stress and moment resultants
can be calculated accordingly.

The deflection at the mid-plane \( w_o \) can be rearranged as shown in equation (3.50) after substituting all known values into (3.24).

\[
\begin{align*}
  w_o = \frac{4P}{\pi D} \sum_{n=1,3} \left\{ \frac{1}{\alpha_n} \left[ 1 - \left(1 + \frac{3\lambda^2 + 86\lambda^4 + 178\lambda^4 + 96\lambda^6}{240 \alpha_1 (\lambda + \alpha_1) (\lambda + 2\alpha_1)} \alpha_n^{-2} + \frac{b}{4} \alpha_n \coth \frac{\alpha_n b}{2} \right) \frac{\cosh \alpha_n b}{\cosh \frac{\alpha_n b}{2}} 
  
  + \frac{\alpha_n^2}{2 \cosh \frac{\alpha_n b}{2}} \sinh \alpha_n b \right] + \frac{3\lambda^2 + 86\lambda^4 + 178\lambda^4 + 96\lambda^6}{240 \alpha_1 (\lambda + \alpha_1) (\lambda + 2\alpha_1)} \frac{\alpha_n^2}{\cosh \frac{\alpha_n b}{2}} \right\} \sin \alpha_n x 
  
  \right. 
\end{align*}
\]

where \( D = \frac{E t^3}{12(1-\nu^2)} \)

3.4 Stresses, Stress and Moment Resultants Under Uniform Load

Stresses, stress and moment resultants can be calculated from equations (2.20) - (2.24) and (2.26) - (2.32). As mentioned in Chapter II, the final results of displacements \( u, v, w \), stresses, stress and moment resultants are the superposition of symmetric and antisymmetric problems. In order to compare the results of the present theory with classical, Reissner, and Donnell-Lee plate theories, some representative quantities, namely, \( (Q_x)_{x=0}, (M_x)_{\text{max}}, (w_0)_{\text{max}}, \) and \( (\sigma_x)_{\text{max}} \) will be given in this section. These quantities will be used in next chapter for detailed numerical comparisons.

\[
(Q_x)_{x=0} = \frac{4P\alpha}{\pi^3} \sum_{n=1,3} \frac{1}{m^2} \left( 1 - \frac{\cosh \alpha_n b}{\cosh \frac{\alpha_n b}{2}} \right) 
\]
\[
(M_x)_{\text{max}} = \frac{P \alpha^2}{\delta} \left\{ 1 - \frac{8}{\pi^3} \sum_{n=1,3}^{\infty} (-1)^{(m-1)/2} \left[ \frac{4 + \frac{\lambda + 2G}{2(\lambda + G)} m \pi \frac{b}{a} \tanh \frac{\delta_m b}{2}}{m^3 \cosh \frac{\delta_m b}{2}} \right] \right\} \\
+ \frac{P \alpha^2}{120 \pi} \sum_{m=1,3}^{\infty} (-1)^{(m-1)/2} \frac{1}{m \cosh \frac{\delta_m b}{2}} \left( \frac{\delta}{\alpha} \right)^2 \tag{3.52}
\]

\[
(W_0)_{\text{max}} = \frac{4P \alpha^2}{a \delta} \sum_{n=1,3}^{\infty} (-1)^{(m-1)/2} \left\{ \frac{1}{\lambda_m^2} \left[ 1 - \left( 1 + \frac{3\lambda^3 + 8G\lambda^2G + 178\lambda G + 96 G^3}{240 G_1 (\lambda + G_1) (\lambda + 2G)} \right) \lambda_m \right] \right\} \\
+ \frac{b}{4} \delta_m \tanh \frac{\delta_m b}{2} \left( \frac{1}{\cosh \frac{\delta_m b}{2}} \right) \\
+ \frac{3\lambda^3 + 8G\lambda^2G + 178\lambda G + 96 G^3}{240 G_1 (\lambda + G_1) (\lambda + 2G)} \left( \frac{\delta}{\delta_m^3} \right) \frac{\delta_m^2}{\delta_m^3} \frac{\delta_m b}{\Delta} \left( \frac{\delta}{\Delta} \right)^2 \tag{3.53}
\]

\[
(\sigma_x)_{\text{max}} = \frac{P \alpha^2}{\pi^2} \left\{ \frac{3}{4} \left[ 1 - \frac{8}{\pi^3} \sum_{m=1,3}^{\infty} (-1)^{(m-1)/2} \left[ 4 + \frac{\lambda + 2G}{2(\lambda + G)} m \pi \frac{b}{a} \tanh \frac{\delta_m b}{2} \right] \frac{1}{m \cosh \frac{\delta_m b}{2}} \right] \right\} \\
+ \frac{1}{\pi} \sum_{n=1,3}^{\infty} (-1)^{(m-1)/2} \frac{1}{m} \left[ \frac{(\lambda + 2G)(\lambda + G)}{10 G_1 (\lambda + G_1)} + \frac{3\lambda + 2G\lambda G + 12\lambda G^2 - 16 G^3}{20 G_1 (\lambda + G_1)^2} \right] \left( \frac{\delta}{\delta_m b} \right) \left( \frac{\delta}{\Delta} \right)^2 \tag{3.54}
\]

It should be pointed out that in the above equations, \((M_x)_{\text{max}}\) occurs at \(x=a/2, y=0\) and \((\sigma_x)_{\text{max}}\) at \(x=a/2, y=0\) and \(z=h/2\).
3.5 Simply Supported Rectangular Plate Under Sinusoidal Load

Assume that the loading condition is given by

$$\mathcal{P}(x,y) = p_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{b}$$

(3.55)

where \(p_0\) is a constant and \(a, b\) are the lengths of the sides (fig. 3.2). The edge boundary conditions to be satisfied by the present theory are

\[
x = \pm \frac{a}{2} : \quad v_0 = N_x = w_0 = M_x = v_1 = 0; \\
y = \pm \frac{b}{2} : \quad u_0 = N_y = w_0 = M_y = u_1 = 0. \tag{3.56}
\]

The technique of solving this problem based on the present theory is different from that of under uniform load. It is clear that the presumptions used in the case of uniform load, namely, \(\nabla^2 w = 0\) and \(\nabla^2 w = 0\) constant, are no longer true in the case of sinusoidal load. In addition to \(u, v, u_1, v_1\) and \(w, w_1\), displacement components \(w_1\) and \(w_2\) will be treated as independent functions for the symmetric and antisymmetric problems. Thus, equations (2.50) are the set of governing simultaneous differential equations for the symmetric problem and (2.51) for the antisymmetric problem.

Let

\[
U_0 = C_1 \sin \frac{\pi x}{a} \cos \frac{\pi y}{b} \tag{3.57a}
\]

\[
V_0 = C_2 \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} \tag{3.57b}
\]

\[
W_1 = C_3 \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \tag{3.57c}
\]
Figure 3.2. - Simply supported rectangular plate under sinusoidal load.
\[ U_i = C_4 \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) \] (3.58a)

\[ V_i = C_5 \cos \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \] (3.58b)

\[ W_i = C_6 \cos \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) \] (3.58c)

\[ W_2 = C_7 \cos \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) \] (3.58d)

where \( C_1, \ldots, C_7 \) are constants.

Clearly, equations (3.57) and (3.58) satisfy boundary conditions (3.56). Substituting equations (3.57) and (3.58) into (2.50) and (2.51), when reduced to a square plate, we obtain the following two sets of simultaneous algebraic equations

\[
(\lambda + 3G_1)\left( \frac{\pi}{a} \right) C_1 + (\lambda + G_1)\left( \frac{\pi}{a} \right) C_2 + \left[ \lambda + \frac{h^2}{12} (\lambda + 2G_1)\left( \frac{\pi}{a} \right)^2 \right] C_3 = 0
\] (3.59a)

\[
(\lambda + G_1)\left( \frac{\pi}{a} \right) C_1 + (\lambda + 3G_1)\left( \frac{\pi}{a} \right) C_2 + \left[ \lambda + \frac{h^2}{12} (\lambda + 2G_1)\left( \frac{\pi}{a} \right)^2 \right] C_3 = 0
\] (3.59b)

\[
\lambda \left( \frac{\pi}{a} \right) C_1 + \lambda \left( \frac{\pi}{a} \right) C_2 + \left[ (\lambda + 2G_1) + \frac{h^2}{4} \lambda \left( \frac{\pi}{a} \right)^2 \right] C_3 = \frac{P_0}{2}
\] (3.59c)

\[ C_4 + C_5 - 2 \left( \frac{\pi}{a} \right) C_6 = - \frac{3}{2G_1h} \left( \frac{a}{h} \right) P_0
\] (3.60a)

\[
\left[ \frac{2h}{3} G_1 + \frac{h^3}{15} (\lambda + 3G_1)\left( \frac{\pi}{a} \right)^2 \right] C_4 + \frac{h^3}{15} (\lambda + G_1)\left( \frac{\pi}{a} \right)^2 C_5 - \left[ \frac{h^3}{3} G_1 \left( \frac{\pi}{a} \right) - \frac{h^3}{30} (\lambda + 2G_1)\left( \frac{\pi}{a} \right)^3 \right] C_6
\]

\[ + \left[ \frac{h^3}{12} \lambda \left( \frac{\pi}{a} \right) + \frac{h^3}{240} (\lambda + 2G_1)\left( \frac{\pi}{a} \right)^2 \right] C_7 = 0
\] (3.60b)
The solution for $C_4$, $C_5$, $C_6$ is

$$C_4 = C_5 = -\frac{12\lambda + \frac{h^3}{3}(\lambda + 2G)(\pi/a)^2}{192G(\lambda + G)(\pi/a) + 8\frac{h^3}{3}\lambda(\lambda + 2G)(\pi/a)^3} \cdot P_0 \quad (3.61a)$$

$$C_3 = \frac{3(\lambda + 2G)}{24G(\lambda + G) + \frac{h^3}{3}\lambda(\lambda + 2G)(\pi/a)^2} \cdot P_0 \quad (3.61b)$$

$$C_4 = C_5 = \frac{120G(\lambda + 2G)(a/\pi)^2 + 2\frac{h^3}{3}(2\lambda^2 - 17\lambda G - 12G^2) - \frac{h^3}{3}G(\lambda + 2G)(\pi/a)^2}{160 \frac{h^3}{3}G(\lambda + G)(\pi/a) + (4/3) \frac{h^3}{3}\lambda G(\lambda + 2G)(\pi/a)^3} \cdot P_0 \quad (3.61c)$$

$$C_6 = \frac{120 G(\lambda + 2G)(a/\pi)^2 + 2\frac{h^3}{3}(2\lambda^2 + 43\lambda G + 48G^2)(a/\pi) + \frac{h^3}{3}(\lambda + 2G)(\lambda - G)(\pi/a)}{160 \frac{h^3}{3}G(\lambda + G)(\pi/a) + (4/3) \frac{h^3}{3}\lambda G(\lambda + 2G)(\pi/a)^3} \cdot P_0 \quad (3.61d)$$

$$C_7 = -\frac{360G(\lambda + 2G)(a/\pi)^2 + 6\frac{h^3}{3}(2\lambda^3 + 3\lambda^2 G - 32\lambda G^2 - 40G^3) + \frac{h^3}{3}\lambda(\lambda + 2G)(\lambda - 2G)(\pi/a)^2}{[160 \frac{h^3}{3}G(\lambda + G)(\pi/a) + (4/3) \frac{h^3}{3}\lambda G(\lambda + 2G)(\pi/a)^3][9/2](\lambda + 2G)(a/\pi) + (1/8) \frac{h^3}{3}\lambda(\pi/a)} \cdot P_0 \quad (3.61e)$$

The displacements, stresses, stress and moment resultants are completely determined by substituting equations (3.57) and (3.58) into appropriate expressions. A direct substitution gives the following expressions for the present theory:
\begin{align}
(Q_x)_{x=0/2} &= -\frac{P_0 a}{2 \pi} \cos \frac{\pi y}{a} \tag{3.62a} \\
(M_x)_{\text{max}} &= \frac{P_0 a^2}{8 \pi^2} \cdot \frac{3 \lambda + 2 \mu}{\lambda + \mu} \left[ 1 + \frac{\pi^2}{120} \cdot \frac{\lambda (-3 \lambda^2 + 16 \lambda \mu + 2 \mu^2)}{G(\lambda + \mu)(3 \lambda + 2 \mu)} \cdot \left( \frac{a}{h} \right)^2 \\
&\quad - \frac{\pi^4}{14400} \cdot \frac{\lambda^2 (\lambda + 2 \mu)(-3 \lambda^2 + 16 \lambda \mu + 2 \mu^2)}{G^2 (\lambda + \mu)^2 (3 \lambda + 2 \mu)} \left( \frac{a}{h} \right)^4 + \cdots \right] \tag{3.62b} \\
(W_x)_{\text{max}} &= \frac{3 P_0 a^4}{4 \pi^4} \cdot \frac{\lambda + 2 \mu}{\lambda^2 G(\lambda + \mu)} \left[ 1 + \frac{\pi^2}{120} \cdot \frac{3 \lambda^2 + 8 \lambda \mu + 17 \lambda \mu^2 + 9 \mu \mu^3}{G(\lambda + \mu)(\lambda + 2 \mu)} \cdot \left( \frac{a}{h} \right)^2 \\
&\quad - \frac{\pi^4}{14400} \cdot \frac{3 \lambda^4 - 3 \lambda^2 \mu^2 + 5 \lambda^2 \mu^2 + 2 \lambda \mu^3 + 12 \mu^4 G^3}{G^4 (\lambda + \mu)^2} \left( \frac{a}{h} \right)^4 + \cdots \right] \tag{3.62c} \\
(U_x)_{\text{max}} &= \frac{3}{4 \pi^2} \cdot \frac{3 \lambda + 2 \mu}{\lambda + \mu} \cdot \frac{P_0 a^3}{h^3} \left[ 1 + \frac{\pi^2}{120} \cdot \frac{9 \lambda^3 + 76 \lambda^2 \mu + 3 \lambda \mu^2 + 16 \mu^3}{G(\lambda + \mu) (3 \lambda + 2 \mu)} \cdot \left( \frac{a}{h} \right)^2 \\
&\quad - \frac{\pi^4}{14400} \cdot \frac{9 \lambda^4 + 76 \lambda^3 \mu + 12 \lambda^2 \mu^2 + 2 \lambda \mu^3 + 12 \mu^4 G^3}{G^4 (\lambda + \mu)(\lambda + 2 \mu)^2 (3 \lambda + 2 \mu)} \left( \frac{a}{h} \right)^4 + \cdots \right] \tag{3.62d}
\end{align}

where the first term of each series, except \(Q_x\), represents the classical plate theory and the rest represent the correction terms due to shear deformations.
3.6 Discussion

Two different types of loading have been studied in previous sections in solving a simply supported rectangular thick plate. As can be seen clearly, the methods employed for solution were also different. In the case of an uniformly distributed load, if the load is expanded into a double Fourier series the method similar to that for the sinusoidal load can be used to solve the problem. However, the convergence of the resulting double Fourier series is much slower than the Levy type solution.

For plates with boundary conditions other than simply supported along all edges or having two opposite edges simply supported, solution methods of the classical thin plate theory can be adopted as well.
CHAPTER IV

NUMERICAL RESULTS AND DISCUSSION FOR A HOMOGENEOUS ISOTROPIC
RECTANGULAR THICK ELASTIC PLATE HAVING SIMPLY SUPPORTED EDGES

4.1 Introduction

In this chapter, detailed numerical results for displacements, stresses, stress and moment resultants based on the present theory are presented for a homogeneous isotropic rectangular thick plate subjected to uniformly and sinusoidally distributed loads. Results are compared to those based on classical thin plate, Reissner and Donnell-Lee thick plate theories. In general, a good agreement is observed among all thick plate theories.

4.2 Comparison of Displacement w Under Uniform Load

The maximum lateral deflection of mid-surface \((z=0)\) occurs at the point \(x=a/2, y=0\). Define the non-dimensional deflection parameter \(\alpha\) as

\[
\alpha = \left(\frac{w}{W_0}\right)_{\text{max}} \frac{E h^3}{P a^4}
\]  

(4.1)

The parameter \(\alpha\) for various theories are as follows,

Classical theory (ref. 25):

\[
\alpha_C = \frac{48(1-\nu^2)}{\pi^4} \sum_{m=1,3}^{\infty} (-1)^{m-1/2} \frac{1}{m^4} \left( 1 - \frac{n \pi}{a} \frac{b}{a} \frac{\tanh \mu}{\cosh \mu} \right)
\]

(4.2a)
Present theory:

\[
\alpha_p = \alpha_c + \left[ \frac{1-\nu^2}{\pi^2} \cdot \frac{3\lambda^3 + 86\lambda^2\gamma + 178\lambda\gamma^2 + 96\gamma^3}{5\mu(\lambda + \epsilon)(\lambda + 2\gamma)} \sum_{m=1,3}^{\infty} (-1)^{(m-1)/2} \frac{1}{m^3} \left(1 - \text{sech} \mu \right) \right] \left( \frac{h}{a} \right)^2
\] (4.2b)

Reissner theory (ref. 5):

\[
\alpha_R = \alpha_c + \left[ \frac{24(1+\nu)(2-\nu)}{5\pi^3} \sum_{m=1,3}^{\infty} (-1)^{(m-1)/2} \frac{1}{m^3} \left(1 - \text{sech} \mu \right) \right] \left( \frac{h}{a} \right)^2
\] (4.2c)

Donnell-Lee theory (ref. 16):

\[
\alpha_D = \alpha_c + \left[ \frac{6(1+\nu)(2-3\nu)}{5\pi^3} \sum_{m=1,3}^{\infty} (-1)^{(m-1)/2} \frac{1}{m^3} \left(1 - \text{sech} \mu \right) \right] \left( \frac{h}{a} \right)^2
\]

\[- \left\{ \frac{(1+\nu)(227-15\nu^2)}{1400\pi} \sum_{m=1,3}^{\infty} (-1)^{(m-1)/2} \frac{1}{m} \left[ 1 - \frac{1 + \frac{40(1-\nu)h^3 + 16n h^2 \mu}{40 + (8+\nu)\mu^2 + (h/a)^2}}{\text{cosh} \mu} \right] \right\} \left( \frac{h}{a} \right)^4
\] (4.2d)

where \( \mu = \frac{m\pi b}{2a} \); \( m = 1, 3, 5, \ldots \)

Table 4.1 gives the non-dimensional deflection parameter \( \alpha \) for various values of \( b/a \) and \( h/a \) with \( \gamma = 0.3 \).

It is noted that the maximum deflection due to shear deformation increases with increase of \( b/a \) and \( h/a \). For the ranges chosen in the foregoing, the maximum percentage of deviation from classical, Reissner and Donnell-Lee theories are by about 19.57, 1.66 and 0.71, respectively, for the case \( h/a = 0.200 \) and \( b/a = 1 \) (square plate).
Table 4.1 - Non-Dimensional Deflection Parameter $\alpha$ for Various Values of $b/a$ and $h/a$ with $\gamma = 0.3$

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>$h/a$</th>
<th>Classical Theory</th>
<th>Present Theory</th>
<th>Reissner Theory</th>
<th>Donnell-Lee Theory</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.005</td>
<td>0.044361</td>
<td>0.044366</td>
<td>0.044366</td>
<td>0.044366</td>
</tr>
<tr>
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<td>0.044381</td>
<td>0.044381</td>
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<tr>
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<td>0.044849</td>
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<td>0.046315</td>
<td>0.046410</td>
</tr>
<tr>
<td>1.0</td>
<td>0.200</td>
<td>0.0444361</td>
<td>0.053043</td>
<td>0.052175</td>
<td>0.052670</td>
</tr>
<tr>
<td>1.5</td>
<td>0.005</td>
<td>0.084346</td>
<td>0.084354</td>
<td>0.084353</td>
<td>0.084353</td>
</tr>
<tr>
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<td>0.010</td>
<td>0.084346</td>
<td>0.084376</td>
<td>0.084373</td>
<td>0.084374</td>
</tr>
<tr>
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<td>0.084346</td>
<td>0.085089</td>
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<tr>
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<td>0.087140</td>
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<td>0.096222</td>
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<td>0.095557</td>
</tr>
<tr>
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<td>0.110605</td>
<td>0.110613</td>
<td>0.110613</td>
<td>0.110613</td>
</tr>
<tr>
<td>2.0</td>
<td>0.010</td>
<td>0.110605</td>
<td>0.110639</td>
<td>0.110635</td>
<td>0.110637</td>
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<tr>
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<td>0.110605</td>
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<tr>
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<tr>
<td>5.0</td>
<td>0.005</td>
<td>0.141642</td>
<td>0.141651</td>
<td>0.141650</td>
<td>0.141650</td>
</tr>
<tr>
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<tr>
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<td>0.142562</td>
<td>0.142470</td>
<td>0.142506</td>
</tr>
<tr>
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<td>0.145322</td>
<td>0.144954</td>
<td>0.145096</td>
</tr>
<tr>
<td>5.0</td>
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<td>0.141642</td>
<td>0.156361</td>
<td>0.154890</td>
<td>0.155411</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.005</td>
<td>0.142188</td>
<td>0.142197</td>
<td>0.142196</td>
<td>0.142196</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.010</td>
<td>0.142188</td>
<td>0.142225</td>
<td>0.142221</td>
<td>0.142222</td>
</tr>
<tr>
<td>$\infty$</td>
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<td>0.143108</td>
<td>0.143017</td>
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</tr>
<tr>
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<td>0.142188</td>
<td>0.145871</td>
<td>0.145503</td>
<td>0.145645</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.200</td>
<td>0.142188</td>
<td>0.156919</td>
<td>0.155447</td>
<td>0.155967</td>
</tr>
</tbody>
</table>
Table 4.2 gives the ratios of lateral deflection \( w(a/2, 0, z)/w(a/2, 0, 0) \) across the plate thickness for a square plate with \( V = 0.3 \). The general equation for \( w(a/2, 0, z) \) based on the present theory is

\[
W(a/2, 0, z) = (W_*)_{\text{max}} + \frac{P\alpha^4}{Eh^3}(1 - \nu^2) \left\{ \frac{3}{4} \frac{\lambda}{\lambda + 2\mu} \left[ -1 + \sum_{m=1}^{\infty} \left( \frac{m\pi}{a} \right)^2 \frac{1}{m^2 \cosh \mu} \right] \left( \frac{a}{h} \right)^2 \left( \frac{z}{h} \right)^2 \right\} + \frac{1}{2} \left[ \left( \frac{3}{a} \right)^2 + \frac{-3\lambda^2 + 16\lambda G + 2G^2}{20G(\lambda + G)} \right] \left( \frac{3}{h} \right)^2 \left( \frac{a}{h} \right)^4 \right\} \quad (4.3)
\]

where

\[(W_*)_{\text{max}} = w(a/2, 0, 0) = \kappa_p \frac{P\alpha^4}{Eh^3}\]

As mentioned in Chapter 1 the lateral deflection \( w \) does not vary through the plate thickness in the Reissner theory. This is almost true when the plate is very thin as can be seen from Table 4.2. However, as the thickness increases the Poisson effect causes \( w(a/2, 0, z) \) to become less than \( w(a/2, 0, 0) \). The maximum percentage deviation from the mid-surface deflection is by about -3.68 at \( z = -h/2 \) and \( b/a = 0.200 \). The Donnell-Lee theory also has this feature. However, the general form of \( w(a/2, 0, z) \) is not obtainable hence no comparison is made.

### 4.3 Comparison of Displacement u Under Uniform Load

Reissner (ref. 3) assumed the normal and tangential displacements at an edge vary linearly through the plate thickness. This was
Table 4.2 - Ratios of Lateral Deflection $w(a/2, 0, z)/w(a/2, 0, 0)$ Across the Plate Thickness (for $b/a=1$ and $\tau=0.3$)

<table>
<thead>
<tr>
<th>$\frac{h}{a}$</th>
<th>$z=-h/2$</th>
<th>$z=-3h/8$</th>
<th>$z=-h/4$</th>
<th>$z=-h/8$</th>
<th>$z=0$</th>
<th>$z=h/8$</th>
<th>$z=h/4$</th>
<th>$z=3h/8$</th>
<th>$z=h/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>0.999976</td>
<td>0.999986</td>
<td>0.999994</td>
<td>0.999998</td>
<td>1.000000</td>
<td>0.999998</td>
<td>0.999994</td>
<td>0.999998</td>
<td>0.999976</td>
</tr>
<tr>
<td>0.010</td>
<td>0.999903</td>
<td>0.999945</td>
<td>0.999976</td>
<td>0.999994</td>
<td>1.000000</td>
<td>0.999994</td>
<td>0.999976</td>
<td>0.999945</td>
<td>0.999903</td>
</tr>
<tr>
<td>0.050</td>
<td>0.997581</td>
<td>0.998633</td>
<td>0.999387</td>
<td>0.999843</td>
<td>1.000000</td>
<td>0.999859</td>
<td>0.999419</td>
<td>0.998681</td>
<td>0.997644</td>
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<tr>
<td>0.100</td>
<td>0.990431</td>
<td>0.994526</td>
<td>0.997485</td>
<td>0.999310</td>
<td>1.000000</td>
<td>0.999555</td>
<td>0.997974</td>
<td>0.995259</td>
<td>0.991409</td>
</tr>
<tr>
<td>0.200</td>
<td>0.963195</td>
<td>0.978010</td>
<td>0.989083</td>
<td>0.996413</td>
<td>1.000000</td>
<td>0.999844</td>
<td>0.995945</td>
<td>0.988304</td>
<td>0.976919</td>
</tr>
</tbody>
</table>
one of the principal assumptions in deriving his theory. However, as commented by Goodier (ref. 21), "the shear deformation will cause departure from linearity of $u_n, u_b$". The present theory provides a way of finding the degree of deviation from linearity in displacement $u$.

The expressions of displacement $u$ for the present and Reissner theories at $x=a$ are

Present theory:

$$u(a, y, z) = \frac{P a^4}{E h^3} (1 - \nu^2) \left\{ - \frac{\lambda}{4(\lambda + 2G)} \left( 1 - \frac{96}{\pi^2} \sum_{m=1,3} \frac{\cosh \alpha_m y}{m^3 \cosh \mu} \right) \left( \frac{\mu}{\alpha} \right)^3 \right. + \left. \left[ \frac{1}{2} \left( 1 - \frac{96}{\pi^2} \sum_{m=1,3} \frac{\cosh \alpha_m y}{m^3 \cosh \mu} \right) + \frac{12}{\pi^3} \sum_{m=1,3} \frac{\cosh \alpha_m y}{m^3 \cosh \mu} \left( 2 \frac{\mu}{\alpha} \tanh \alpha_m y - \frac{b}{a} \tanh \mu \right) \right] \left( \frac{\mu}{\alpha} \right)^3 \right\}$$

$$+ \frac{3\lambda^3 - 34\lambda^2 G - 62\lambda G^2 - 24G^3}{40 G (\lambda + G)} \left( 1 - \frac{8}{\pi^2} \sum_{m=1,3} \frac{\cosh \alpha_m y}{m^3 \cosh \mu} \right) \cdot \left( \frac{\mu}{\alpha} \right)^3 \left( \frac{\mu}{\alpha} \right)^3$$

$$+ \frac{3\lambda + 4G}{\lambda + 2G} \left( 1 - \frac{8}{\pi^2} \sum_{m=1,3} \frac{\cosh \alpha_m y}{m^3 \cosh \mu} \right) \cdot \left( \frac{\mu}{\alpha} \right)^3 \left( \frac{\mu}{\alpha} \right)^3 \right\} \quad (4.4a)$$

Reissner theory (ref. 5):

$$u(a, y, z) = \frac{P a^4}{E h^3} (1 - \nu^2) \left\{ \left[ \frac{1}{2} \left( 1 - \frac{96}{\pi^2} \sum_{m=1,3} \frac{\cosh \alpha_m y}{m^3 \cosh \mu} \right) + \frac{12}{\pi^3} \sum_{m=1,3} \frac{\cosh \alpha_m y}{m^3 \cosh \mu} \left( 2 \frac{\mu}{\alpha} \tanh \alpha_m y \right. \right. \left( \frac{\mu}{\alpha} \right)^3 \right\}$$

$$- \frac{b}{a} \tanh \mu \right) \cdot \left( \frac{\mu}{\alpha} \right)^3 \left( \frac{\mu}{\alpha} \right)^3 \right\} - \frac{3\nu^2}{5(1-\nu)} \left( 1 - \frac{8}{\pi^2} \sum_{m=1,3} \frac{\cosh \alpha_m y}{m^3 \cosh \mu} \right) \cdot \left( \frac{\mu}{\alpha} \right)^3 \left( \frac{\mu}{\alpha} \right)^3 \right\} \right\} \quad (4.4b)$$
The values of $u$ for several values of $h/a$ with $b/a=1$ and Poisson's ratio equal to 0.3 are tabulated in Tables 4.3 and 4.4 for the present and Reissner theories, respectively. The result from Donnell-Lee theory is not available.

Figure 4.1 shows variation of $u$ at the top, middle and bottom surfaces of a square plate along the $y$-axis when $h/a=0.2$. The largest deviation of $u$ from that of Reissner theory is about 9.68 per cent.

Figure 4.2 shows variation of $u$ across the plate thickness at $x=a$, $y=0$. Although the present theory assumes nonlinear displacement $u$ across the plate thickness, the result is very close to Reissner's linear assumption.

### 4.4 Comparisons of Moment Resultant $M_x$ and Stress $\sigma_x$ Under Uniform Load

In this section, comparisons of maximum moment resultant $M_x$ which occurs at $x=a/2$, $y=0$ and maximum stress $\sigma_x$ at $x=a/2$, $y=0$, $z=h/2$ are investigated.

Define the non-dimensional moment and stress parameters as follows:

\[ \beta = \frac{(M_x)_{\text{max}}}{P a^2} \tag{4.5} \]

\[ \gamma = \frac{(\sigma_x)_{\text{max}} a^2}{P a^2} \tag{4.6} \]

The parameters $\beta$ and $\gamma$ for various theories are

**Classical theory (ref. 25):**

\[ \beta_c = \frac{1}{8} \left[ 1 - \frac{\nu}{4 \pi^2} \sum_{m=1,3}^{\infty} \frac{(-1)^{m+1/2} 4 + (1-\nu)m \pi (b/a) \tanh \mu}{m^3 \cosh \mu} \right] \tag{4.7a} \]
Table 4.3. - Displacement $u$ Based on the Present Theory at $x=a$ (for $b/a=1$ and $\gamma=0.3$).

<table>
<thead>
<tr>
<th>$\frac{z}{h}$</th>
<th>$\frac{h}{a}$</th>
<th>$y=0$</th>
<th>$y=\frac{a}{10}$</th>
<th>$y=\frac{a}{4}$</th>
<th>$y=\frac{3}{8}a$</th>
<th>$y=\frac{1}{2}$</th>
</tr>
</thead>
<tbody>
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<td>0.005</td>
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<td>0.342090x10^{-3}</td>
<td>0.268777x10^{-3}</td>
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<td>0.143998x10^{-3}</td>
<td>0.134071x10^{-3}</td>
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<td>-0.300816x10^-3</td>
<td>0</td>
</tr>
<tr>
<td>-0.500</td>
<td>0.050</td>
<td>-0.367063x10^3</td>
<td>-0.341966x10^3</td>
<td>-0.267958x10^3</td>
<td>-0.149858x10^-3</td>
<td>0</td>
</tr>
<tr>
<td>-0.500</td>
<td>0.100</td>
<td>-0.728201x10^3</td>
<td>-0.678240x10^3</td>
<td>-0.530979x10^3</td>
<td>-0.296277x10^-3</td>
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</tr>
<tr>
<td>-0.500</td>
<td>0.200</td>
<td>-0.140899x10^3</td>
<td>-0.131094x10^3</td>
<td>-0.102245x10^3</td>
<td>-0.565047x10^-3</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 4.1 - Normal displacement $u$ at $x=a$. 

\[
\frac{b}{a} = 1, \quad \frac{d}{a} = 0.2
\]

--- REISSNER THEORY

--- PRESENT THEORY
Figure 4.2 - Normal displacement $u$ across the plate thickness at $x=a$, $y=0$. 

\[ \frac{b}{a} = 1, \quad \frac{h}{a} = 0.2, \quad y = 0 \]

--- REISSNER THEORY

--- PRESENT THEORY
\[
\gamma_c = \frac{3}{4} \left[ 1 - \frac{8}{\pi} \sum_{m=1,3}^{\infty} \left( \frac{-1}{m^{11/2}} \frac{4 + (1-\nu)m \pi (b/a) \tanh \mu}{m^3 \cosh \mu} \right) \right] (4.7b)
\]

**Present theory:**

\[
\beta_p = \beta_c + \left[ \frac{1}{12 \pi} \sum_{m=1,3}^{\infty} \left( \frac{-3 \lambda^2 + 16 \lambda \delta + 20 \lambda G^2}{G_1 (\lambda + G_1)^2} \right) \frac{1}{m} \frac{1}{m \cosh \mu} \right] \left( \frac{\delta}{\alpha} \right)^2 \quad (4.8a)
\]

\[
\gamma_p = \gamma_c + \left[ \frac{1}{12 \pi} \sum_{m=1,3}^{\infty} \left( \frac{m+2G_1/3}{G_1 (\lambda + G_1)} \right) \frac{1}{m} \frac{1}{m \cosh \mu} \right] \left( \frac{\delta}{\alpha} \right)^2 \quad (4.8b)
\]

**Reissner theory (ref. 5):**

\[
\beta_R = \beta_c + \left[ \frac{2 \pi}{5 \pi} \sum_{m=1,3}^{\infty} \left( \frac{-1}{m^{11/2}} \frac{1}{m \cosh \mu} \right) \right] \left( \frac{\delta}{\alpha} \right)^2 \quad (4.9a)
\]

\[
\gamma_R = \gamma_c + \left[ \frac{12 \pi}{5 \pi} \sum_{m=1,3}^{\infty} \left( \frac{-1}{m^{11/2}} \frac{1}{m \cosh \mu} \right) \right] \left( \frac{\delta}{\alpha} \right)^2 \quad (4.9b)
\]

**Donnell-Lee theory (ref. 16):**

\[
\beta_D = \beta_c + \left[ \frac{2 \pi}{5 \pi} \sum_{m=1,3}^{\infty} \left( \frac{-1}{m^{11/2}} \frac{1}{m \cosh \mu} \right) \right] \left( \frac{\delta}{\alpha} \right)^2
\]

\[
+ \left[ \frac{\pi (227-1577\nu)}{16 \pi \alpha} \sum_{m=1,3}^{\infty} \left( \frac{-1}{m^{11/2}} \frac{1 + 40 \pi m \pi (1-\nu) m^3 \cosh \mu}{4 \alpha + (8+\nu)m^2 (\delta/\alpha)^2 \cosh \mu} \right) \right] \left( \frac{\delta}{\alpha} \right)^4 \quad (4.10a)
\]
It is noted that in the classical and Reissner theories $\sigma_x$ is obtained from the simple equation

$$\sigma_x = \frac{M_x}{A^{1/6}} \cdot \frac{3}{h/2}$$

Tables 4.5 and 4.6 give the parameters $\beta$ and $\gamma$ for various values of $b/a$ and $h/a$ with $\nu = 0.3$.

Table 4.5 shows that $(M_x)_{\text{max}}$ is practically identical for the three thick plate theories. The largest deviation of the maximum bending moment from that of classical theory is about 0.78 per cent, the moments containing the thickness effect being larger than those of classical theory.

The maximum stress $\sigma_x$ of the present theory is generally larger than all other theories. This is mainly due to the inclusion of the effect of plate extension. The maximum percentage of deviation from the classical, Reissner, and Donnell-Lee theories are by about 4.65, 3.35 and 2.79, respectively, at $h/a = 0.200$ and $b/a = 1$ (square plate).
Table 4.5 - Non-Dimensional Moment Parameter $\beta$ for Various Values of $b/a$ and $h/a$ with $\nu=0.3$

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>$h/a$</th>
<th>Classical Theory</th>
<th>Present Theory</th>
<th>Reissner Theory</th>
<th>Donnell-Lee Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.005</td>
<td>0.047886</td>
<td>0.047886</td>
<td>0.047887</td>
<td>0.047887</td>
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<td>0.047888</td>
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Table 4.6. - Non-Dimensional Stress Parameter $\tau$ for Various Values of
$b/a$ and $h/a$ with $\nu=0.3$

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>$h/a$</th>
<th>Classical Theory</th>
<th>Present Theory</th>
<th>Reissner Theory</th>
<th>Donnell-Lee Theory</th>
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<tbody>
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</tr>
<tr>
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<tr>
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</tr>
</tbody>
</table>
4.5 Comparisons of Displacement, Stress, Stress and Moment Resultants Under a Sinusoidal Load

For a square plate under a sinusoidal loading condition

\[ p(x, y) = p_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} \]

at the top surface, the following expressions are derived from the present theory

\[ (Q_x)_{x=a/2} = -\frac{p_0 a}{2 \pi} \cos \frac{\pi y}{a} \]  

(4.11a)

\[ (M_x)_{\text{max}} = \frac{1 + \nu}{4 \pi^2} \frac{p_0 a^3}{D} \left[ 1 + \frac{\pi^2}{20} \frac{\lambda (-3 \lambda^2 + 16 \lambda G + 20 G^2)}{G_1 (\lambda + G)(3 \lambda + 2G)} \left(\frac{A}{a}\right)^2 \right. \]

\[ - \left. \frac{\pi^4}{14400} \frac{\lambda^2 (\lambda + 2G) (-3 \lambda^2 + 16 \lambda G + 20 G^2)}{G_1^2 (\lambda + G)^2 (3 \lambda + 2G)} \left(\frac{A}{a}\right)^4 + \ldots \right] \]  

(4.11b)

\[ (W_o)_{\text{max}} = \frac{p_0 a^3}{4 \pi^4 D} \left[ 1 + \frac{\pi^2}{20} \frac{\lambda^2 + 8 \lambda^2 G + 178 \lambda G^2 + 96 G^3}{G_1 (\lambda + G)(\lambda + 2G)} \left(\frac{A}{a}\right)^2 \right. \]

\[ - \left. \frac{\pi^4}{14400} \frac{3 \lambda^4 - 34 \lambda^2 G + 58 \lambda G^2 + 216 \lambda G^2 + 120 G^4}{G_1^2 (\lambda + G)^2} \left(\frac{A}{a}\right)^4 + \ldots \right] \]  

(4.11c)

\[ (G_x)_{\text{max}} = \frac{3(1+\nu)}{2 \pi^2} \frac{p_0 a^2}{D} \left[ 1 + \frac{\pi^2}{120} \frac{9 \lambda^3 + 96 \lambda^3 G + 8 \lambda G^2 + 16 G^3}{G_1 (\lambda + G)(3 \lambda + 2G)} \left(\frac{A}{a}\right)^2 \right. \]

\[ - \left. \frac{\pi^4}{14400} \frac{-9 \lambda^2 + 270 \lambda^2 G^2 + 3112 \lambda G^2 + 128 G^4 + 1272 \lambda G^2 + 1272 \lambda G^2 + 1272 \lambda G^2 + 17152 \lambda G^2 + 3840 G^2}{G_1^2 (\lambda + G)^2 (3 \lambda + 2G)^2} \left(\frac{A}{a}\right)^4 + \ldots \right] \]  

(4.11d)
Equations (4.11) are the same as those of (3.62). The only difference is the terms outside the bracket which have been converted to \( V \) for the purpose of to be consistent with the results of other theories.

The corresponding solution from classical plate theory (ref. 25) is well known, they are listed below for the comparison purpose.

\[
(Q)_{x=a/2} = -\frac{3-\nu}{4\pi} P_o a \cos \frac{\pi y}{a} \quad (4.12a)
\]

\[
(M_x)_{\max} = \frac{1+\nu}{4\pi^3} P_o a^2 \quad (4.12b)
\]

\[
(W_o)_{\max} = \frac{P_o a^4}{4\pi^3 D} \quad (4.12c)
\]

\[
(\sigma_x)_{\max} = \frac{3(1+\nu)}{2\pi^2} \cdot \frac{P_o a^2}{h^2} \quad (4.12d)
\]

It should be noted that there is a concentrated force at the corner which is

\[
R = \frac{1-\nu}{2\pi^2} P_o a^2 \quad (4.13)
\]

The results from Reissner's thick plate theory (ref. 16) are

\[
(Q_x)_{x=a/2} = -\frac{P_o a}{2\pi} \cos \frac{\pi y}{a} \cos \frac{\pi y}{a} \quad (4.14a)
\]

\[
(M_x)_{\max} = \frac{1+\nu}{4\pi^3} P_o a^2 \left[ 1 + \frac{\pi^4}{5} \cdot \frac{\nu}{1+\nu} \cdot \left( \frac{h}{a} \right)^2 \right] \quad (4.14b)
\]
The results from the Donnell-Lee thick plate theory (ref. 16) are

\[ (Q_x)_{x=a/2} = - \frac{P_o a}{2 \pi} \cos \frac{\pi y}{a} \] (4.15a)

\[ (M_x)_{\text{max}} = \frac{1+\nu}{4 \pi^3} P_o a^3 \left[ 1 + \frac{\pi^2}{5} \frac{\nu}{1+\nu} \left( \frac{h}{a} \right)^2 - \frac{\pi^2}{1050} \frac{\nu}{1+\nu} \left( \frac{h}{a} \right)^4 + \cdots \right] \] (4.15b)

\[ (W_x)_{\text{max}} = \frac{P_o a^4}{4 \pi^3 D} \left[ 1 + \frac{\pi^2}{20} \frac{9-3\nu}{1-\nu} \left( \frac{h}{a} \right)^2 - \frac{\pi^2}{16800} \frac{227-157\nu}{1-\nu} \left( \frac{h}{a} \right)^4 + \cdots \right] \] (4.15c)

\[ (G_x)_{\text{max}} = \frac{3(1+\nu)}{2 \pi^3} \frac{P_o a^3}{h^2} \left[ 1 + \frac{\pi^2}{15} \left( \frac{h}{a} \right)^2 - \frac{71\pi^4}{3150} \left( \frac{h}{a} \right)^4 + \cdots \right] \] (4.15d)

It is interesting to compare the results among the present theory, equations (4.11), Reissner plate theory, equation (4.14), and Donnell-Lee plate theory, equations (4.15). The stress resultant \((Q_x)_{x=a/2}\) is identical in all three theories, and is independent of the plate thickness. All other quantities calculated depend on the thickness ratio \(h/a\). Since the series in equations (4.11) converge very rapidly, the third term is small comparing to the second term.
Define the non-dimensional parameters in the same way as before, namely,

\[ \alpha = (w_0)_{\text{max}} E h^2/\rho_0 a^4 \]
\[ \beta = (M_x)_{\text{max}}/\rho_0 a^2 \]
\[ \gamma = (\sigma_x)_{\text{max}} h^2/\rho_0 a^4 \]

we have the following numerical comparison of parameters \( \alpha, \beta, \) and \( \gamma \) for various plate theories with \( v=0.3 \).

**Classical theory:**
\[ \alpha_c = 0.028026 \]
\[ \beta_c = 0.032929 \]
\[ \gamma_c = 0.197575 \]

**Present theory:**
\[ \alpha_p = \alpha_c + 0.143269 \cdot \left( \frac{h}{a} \right)^2 - 0.084407 \cdot \left( \frac{h}{a} \right)^4 + \cdots \]
\[ \beta_p = \beta_c + 0.016095 \cdot \left( \frac{h}{a} \right)^2 - 0.002987 \cdot \left( \frac{h}{a} \right)^4 + \cdots \]
\[ \gamma_p = \gamma_c + 0.134028 \cdot \left( \frac{h}{a} \right)^2 - 0.498324 \cdot \left( \frac{h}{a} \right)^4 + \cdots \]

**Reissner theory:**
\[ \alpha_R = \alpha_c + 0.134351 \cdot \left( \frac{h}{a} \right)^2 \]
\[ \beta_R = \beta_c + 0.015000 \cdot \left( \frac{h}{a} \right)^2 \]
\[ \gamma_R = \gamma_c + 0.090000 \cdot \left( \frac{h}{a} \right)^2 \]
Donnell-Lee theory:

\[ \alpha_d = \alpha_c + 0.140279 \cdot \left( \frac{t}{a} \right)^2 - 0.041763 \cdot \left( \frac{t}{a} \right)^4 + \cdots \]

\[ \beta_d = \beta_c + 0.01500 \cdot \left( \frac{t}{a} \right)^2 - 0.000705 \cdot \left( \frac{t}{a} \right)^4 + \cdots \]

\[ \gamma_d = \gamma_c + 0.13000 \cdot \left( \frac{t}{a} \right)^2 - 0.43375 \cdot \left( \frac{t}{a} \right)^4 + \cdots \]

Values of \( \alpha, \beta, \) and \( \gamma \) for the present, Reissner, Donnell-Lee, and classical plate theories are shown in Table 4.7 and figures 4.3, 4.4, and 4.5. These values are functions of the thickness ratio \( h/a \), and \( \gamma = 0.3 \) has been used in the calculation. It is noted that values of \( \alpha_c, \beta_c, \) and \( \gamma_c \) for the classical plate theory has been normalized in order to have a better comparison.

4.6 Discussion

It has been shown in detailed numerical calculations that displacements, stresses, stress and moment resultants based on the present theory are in good agreement with the results of the Reissner and Donnell-Lee plate theories. Three boundary conditions are satisfied at each edge in the Reissner theory whereas four boundary conditions are required in the Donnell-Lee theory. The present theory requires five boundary conditions at each edge.

Reissner indicated that concentrated reactions will not occur at the corners of the plate. We take the case of sinusoidal loading condition to demonstrate this fact. From equation (4.11a) for the edge \( x = a/2 \) we have
Table 4.7. - Non-Dimensional Parameters \( \alpha, \beta, \) and \( \gamma \) for Various Plate Theories Under a Sinusoidal Load

<table>
<thead>
<tr>
<th></th>
<th>( a/h=5 )</th>
<th>( a/h=10 )</th>
<th>( a/h=15 )</th>
<th>( a/h=20 )</th>
<th>( a/h=25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_P )</td>
<td>1.199660</td>
<td>1.050818</td>
<td>1.022660</td>
<td>1.012760</td>
<td>1.008171</td>
</tr>
<tr>
<td>( \alpha_R )</td>
<td>1.191751</td>
<td>1.047937</td>
<td>1.021305</td>
<td>1.011984</td>
<td>1.007669</td>
</tr>
<tr>
<td>( \alpha_D )</td>
<td>1.197827</td>
<td>1.049904</td>
<td>1.02216</td>
<td>1.012504</td>
<td>1.008004</td>
</tr>
<tr>
<td>( \beta_P )</td>
<td>1.019405</td>
<td>1.004878</td>
<td>1.002170</td>
<td>1.001221</td>
<td>1.000781</td>
</tr>
<tr>
<td>( \beta_R )</td>
<td>1.018221</td>
<td>1.004555</td>
<td>1.002024</td>
<td>1.001139</td>
<td>1.000729</td>
</tr>
<tr>
<td>( \beta_D )</td>
<td>1.018187</td>
<td>1.004553</td>
<td>1.002024</td>
<td>1.001138</td>
<td>1.000729</td>
</tr>
<tr>
<td>( \gamma_P )</td>
<td>1.023098</td>
<td>1.006531</td>
<td>1.002964</td>
<td>1.001679</td>
<td>1.001078</td>
</tr>
<tr>
<td>( \gamma_R )</td>
<td>1.018220</td>
<td>1.004555</td>
<td>1.002024</td>
<td>1.001138</td>
<td>1.000728</td>
</tr>
<tr>
<td>( \gamma_D )</td>
<td>1.022805</td>
<td>1.006359</td>
<td>1.002880</td>
<td>1.001631</td>
<td>1.001046</td>
</tr>
</tbody>
</table>
Figure 4.3 - Parameter $\alpha$ of maximum lateral deflection $w_0$ under a sinusoidal load.
Figure 4.4 - Parameter $\beta$ of maximum moment $M_x$ under a sinusoidal load.
Figure 4.5 - Parameter $\gamma$ of maximum stress $\sigma_x$ under a sinusoidal load.
\[(Q_x)_{x=a/2} = -\frac{P_a}{2\pi} \cos \frac{\pi y}{a}\] (4.16)

In the same manner, for the edge \(y=a/2\)

\[(Q_y)_{y=a/2} = -\frac{P_a}{2\pi} \cos \frac{\pi x}{a}\] (4.17)

The minus sign indicates that the reactions on the plate act downward.

From symmetry it may be concluded that equations (4.16) and (4.17) also represent stress resultant distributions along the sides \(x=-a/2\) and \(y=-a/2\), respectively. The resultant of \(Q_x\) and \(Q_y\) is

\[2\left[\frac{P_a}{2\pi} \left(\int_{-a/2}^{a/2} \cos \frac{\pi y}{a} dy + \int_{-a/2}^{a/2} \cos \frac{\pi x}{a} dx\right)\right] = \frac{4P_a}{\pi}\]

The resultant of the sinusoidal load \(p(x,y)\) is

\[\int_{-a/2}^{a/2} \int_{-a/2}^{a/2} P \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} dxdy = \frac{4P_a}{\pi}\]

Hence the corners of the plate are free from concentrated reactions.

Another interesting thing worthy of discussion is the transverse normal stress \(\sigma_z\). In the Reissner theory, \(\sigma_z\) has the following expression for a rectangular plate under uniformly distributed loading condition

\[\sigma_z = \frac{3}{4} P \left[ \frac{2}{3} + \frac{2}{3} \left(\frac{a}{2}\right)^2 - \frac{1}{3} \left(\frac{a}{2}\right)^3 \right]\]

which is obviously independent of material properties. The expression of \(\sigma_z\) based on the present theory is
Clearly, \( \sigma_z \) of the present theory is also a function of material constants \( \lambda \) and \( G \).

\[
\sigma_z^{'} = \frac{3}{4} P \left[ \frac{2}{3} + \frac{3\lambda^2 + 4 \lambda G + 8G^2}{12G(\lambda + G)} \left( \frac{3}{4} \right)^2 \right] + \frac{\lambda(3\lambda + 4G)}{12G(\lambda + G)} \left( \frac{3}{4} \right)^3
\]
CHAPTER V
SOLUTION FOR HOMOGENEOUS ISOTROPIC THICK ELASTIC CIRCULAR PLATES

5.1 Introduction

A theory for the homogeneous isotropic thick elastic plates in the Cartesian coordinates has been established in the previous chapters. Detailed numerical results for uniformly and sinusoidally distributed loads indicated a close agreement with other thick plate theories.

It is well known in the classical thin plate theory that the moment resultant approaches infinity at the point of application of a concentrated load or moment. It is rather difficult to observe this nature of singularities in the Cartesian coordinates. Besides, circular plates and plates with holes often arise in the practical engineering structures. Therefore it is necessary to extend the present theory into polar coordinates.

5.2 Governing Equilibrium Equations in Terms of Displacements

The coordinate system and loading condition of a circular plate are shown in figure 5.1. Assume the following surface tractions

\[
\begin{align*}
\sigma_z(r, \theta, \pm h/2) &= P(r, \theta) \\
\sigma_r(r, \theta, -h/2) &= T_{rz}(r, \theta, \pm h/2) = T_{\theta z}(r, \theta, \pm h/2) = 0
\end{align*}
\]

Following the same procedure used in section 2.3, the plate equa-
Figure 5.1. - Coordinate system and loading of a circular plate.
tions of equilibrium are obtained from the stress equilibrium equations in the cylindrical coordinates.

\[
\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{1}{r} \frac{\partial Q_\theta}{\partial \theta} = -p \tag{5.2a}
\]

\[
\frac{\partial M_r}{\partial r} + \frac{1}{r} \frac{\partial M_\theta}{\partial \theta} + \frac{1}{r} (M_r - M_\theta) = Q_r \tag{5.2b}
\]

\[
\frac{\partial M_\theta}{\partial r} + \frac{1}{r} \frac{\partial M_\theta}{\partial \theta} + \frac{2}{r} M_\theta = Q_\theta \tag{5.2c}
\]

\[
\frac{\partial N_r}{\partial r} + \frac{1}{r} \frac{\partial N_\theta}{\partial \theta} + \frac{1}{r} (N_r - N_\theta) = 0 \tag{5.3a}
\]

\[
\frac{\partial N_\theta}{\partial r} + \frac{1}{r} \frac{\partial N_\theta}{\partial \theta} + \frac{2}{r} N_\theta = 0 \tag{5.3b}
\]

where

\[
(Q_r, Q_\theta, N_r, N_\theta, N_\theta) = \int_{-\pi/2}^{\pi/2} (T_{rz}, T_{\theta z}, \sigma_r, \sigma_\theta, \tau_{r\theta}) d\gamma \tag{5.4}
\]

\[
(M_r, M_\theta, M_\theta) = \int_{-\pi/2}^{\pi/2} (\sigma_r, \sigma_\theta, \tau_{r\theta}) d\gamma \tag{5.5}
\]

The displacements \(u, v,\) and \(w\) are assumed in the same way as equations (2.1) - (2.3) except \(u_0, v_0, \ldots w_2\) are functions of \(r\) and \(\theta\). The strain-displacement and stress-strain relations in the cylindrical coordinates can be found in reference 19. As before, the cases of symmetric
(extensional) problem and antisymmetrical (flexural) problem will be investigated separately due to the uncoupled characteristics. The stress components thus obtained and the stress and moment resultants by the definition of equations (5.4) and (5.5) are listed below for future use.

Symmetrical problem:

\[ \sigma_r = \lambda e + 2G \left( \frac{\partial u_r}{\partial r} + \frac{3}{2} \frac{\partial v_t}{\partial r} \right) \]  \hspace{1cm} (5.6a)

\[ \sigma_\theta = \lambda e + \frac{2G}{r} \left[ (u_0 + \frac{\partial v_\theta}{\partial \theta}) + \frac{3}{2} (u_z + \frac{\partial v_z}{\partial \theta}) \right] \]  \hspace{1cm} (5.6b)

\[ \sigma_z = \lambda e + 2Gw_1 \]  \hspace{1cm} (5.6c)

\[ \tau_{r\theta} = G \left[ \left( \frac{\partial v_r}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \frac{3}{2} \left( \frac{\partial v_z}{\partial r} - \frac{v_z}{r} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) \right] \]  \hspace{1cm} (5.6d)

\[ \tau_{r\theta} = G \left( u_2 + \frac{\partial w_1}{\partial r} \right) \]  \hspace{1cm} (5.6e)

\[ \tau_{\theta \theta} = G \left( v_2 + \frac{1}{r} \frac{\partial w_1}{\partial \theta} \right) \]  \hspace{1cm} (5.6f)

\[ N_r = \mathring{H} \left\{ \lambda \left[ \left( \frac{u_0}{r} + \frac{2u_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \right) + w_1 \right] + \frac{\mathring{H}}{24} \left( \frac{u_r}{r} + \frac{2u_z}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \right) \right\} \]  \hspace{1cm} (5.7a)

\[ N_\theta = \mathring{H} \left\{ \lambda \left[ \left( \frac{u_0}{r} + \frac{2u_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \right) + w_1 \right] + \frac{\mathring{H}}{24} \left( \frac{u_r}{r} + \frac{2u_z}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \right) \right\} \]  \hspace{1cm} (5.7b)
\[ N_{r\theta} = G_{r} \left[ \left( \frac{\partial v_{r}}{\partial r} - \frac{v_{r}}{r} + \frac{1}{r} \frac{\partial u_{r}}{\partial \theta} \right) + \frac{v_{r}^{2}}{2 r} \left( \frac{\partial v_{r}}{\partial r} - \frac{v_{r}}{r} + \frac{1}{r} \frac{\partial u_{r}}{\partial \theta} \right) \right] \] (5.7c)

\[ M_{r} = M_{\theta} = M_{r\theta} = Q_{r} = Q_{\theta} = 0 \] (5.7d)

where
\[ e = \left( \frac{u_{r}}{r} + \frac{\partial u_{r}}{\partial r} + \frac{1}{r} \frac{\partial u_{r}}{\partial \theta} \right) + \frac{v_{r}^{2}}{2} \left( \frac{u_{r}}{r} + \frac{\partial u_{r}}{\partial r} + \frac{1}{r} \frac{\partial u_{r}}{\partial \theta} \right) \] (5.8)

Antisymmetric problem:

\[ \sigma_{r} = \lambda e + 2 G_{r} \left( \frac{u_{r}}{r} + \frac{\partial u_{r}}{\partial r} + \frac{v_{r}^{2}}{2} \right) \] (5.9a)

\[ \sigma_{\theta} = \lambda e + \frac{2 G_{r}}{r} \left[ \frac{1}{r} (u_{r} + \frac{\partial u_{r}}{\partial \theta}) + \frac{v_{r}^{2}}{2} (u_{r} + \frac{\partial u_{r}}{\partial \theta}) \right] \] (5.9b)

\[ \sigma_{\phi} = \lambda e + 2 G_{r} \frac{\partial w_{z}}{\partial \phi} \] (5.9c)

\[ \tau_{r\theta} = G_{r} \left[ \frac{1}{r} (u_{r} + \frac{\partial u_{r}}{\partial \theta}) + \frac{v_{r}^{2}}{2} (u_{r} + \frac{\partial u_{r}}{\partial \theta}) \right] \] (5.9d)

\[ \tau_{r\phi} = G_{r} \left[ (u_{r} + \frac{1}{r} \frac{\partial w_{z}}{\partial \phi}) + \frac{v_{r}^{2}}{2} (v_{r} + \frac{\partial w_{z}}{\partial \phi}) \right] \] (5.9e)

\[ \tau_{\theta\phi} = G_{r} \left[ (v_{r} + \frac{1}{r} \frac{\partial w_{z}}{\partial \phi}) + \frac{v_{r}^{2}}{2} (v_{r} + \frac{\partial w_{z}}{\partial \phi}) \right] \] (5.9f)

\[ M_{r\theta} = G_{r} \left[ \frac{v_{r}^{3}}{12} (\frac{1}{r} \frac{\partial u_{r}}{\partial \theta} - \frac{v_{r}}{r} + \frac{\partial v_{r}}{r} \frac{\partial u_{r}}{\partial \theta}) + \frac{v_{r}^{5}}{480} (\frac{1}{r} \frac{\partial u_{r}}{\partial \theta} - \frac{v_{r}}{r} + \frac{\partial v_{r}}{r} \frac{\partial u_{r}}{\partial \theta}) \right] \] (5.10a)
The equilibrium equations in terms of displacements are obtained by direct substitution of equations (5.7) and (5.10) into (5.3) and (5.2), respectively. The results are; for the symmetric problem,

$$
\begin{align*}
M_r &= \lambda \left[ \frac{R^3}{12} \left( \frac{u_1}{r} + \frac{2u_1}{r^2} + \frac{1}{r} \cdot \frac{2u_1}{r^2} + \frac{4}{480} \left( \frac{u_3}{r} + \frac{2u_3}{r^2} + \frac{1}{r} \cdot \frac{2V_3}{r^2} \right) \right) \\
+ 2G \left( \frac{R^3}{12} \cdot \frac{2u_1}{r^2} + \frac{R^5}{480} \cdot \frac{8u_3}{r^2} \right) \right] \\
M_\theta &= \lambda \left[ \frac{R^3}{12} \left( \frac{u_1}{r} + \frac{2u_1}{r^2} + \frac{1}{r} \cdot \frac{2u_1}{r^2} + \frac{4}{480} \left( \frac{u_3}{r} + \frac{2u_3}{r^2} + \frac{1}{r} \cdot \frac{2V_3}{r^2} \right) \right) \\
+ \frac{2G}{r} \left[ \frac{R^3}{12} \left( \frac{u_1}{r} + \frac{2u_1}{r^2} \right) + \frac{R^5}{480} \left( \frac{u_3}{r} + \frac{2u_3}{r^2} \right) \right] \\
Q_r &= G \left[ \left( \frac{u_1}{r} + \frac{2w_3}{r^2} \right) + \frac{R^2}{24} \left( \frac{u_3}{r} + \frac{2w_3}{r^2} \right) \right] \\
Q_\theta &= G \left[ \left( \frac{2w_3}{r^2} \right) + \frac{R^2}{24} \left( \frac{2w_3}{r^2} \right) \right]
\end{align*}
$$

\begin{align*}
N_r &= N_\theta = N_{r\theta} = 0
\end{align*}

where

$$
E = \frac{1}{2} \left( \frac{u_1}{r} + \frac{2u_1}{r^2} + \frac{1}{r} \cdot \frac{2u_1}{r^2} + \frac{4}{480} \left( \frac{u_3}{r} + \frac{2u_3}{r^2} + \frac{1}{r} \cdot \frac{2V_3}{r^2} \right) \right)
$$

$$
\begin{align*}
\left[ -\left( \lambda + 2G \right) \cdot \frac{1}{r} + \left( \lambda + 2G \right) \cdot \frac{1}{r} \cdot \frac{2}{r^2} + \left( \lambda + 2G \right) \cdot \frac{2}{r^2} + G \left( \frac{1}{r} \cdot \frac{2}{r^2} \right) \right] u_r + \left[ \left( \lambda + 2G \right) \cdot \frac{1}{r} \cdot \frac{2}{r^2} - \left( \lambda + 2G \right) \cdot \frac{1}{r} \cdot \frac{2}{r^2} \right] V_r \\
+ \frac{R^2}{24} \left[ -\left( \lambda + 2G \right) \cdot \frac{1}{r} + \left( \lambda + 2G \right) \cdot \frac{1}{r} \cdot \frac{2}{r^2} + \left( \lambda + 2G \right) \cdot \frac{2}{r^2} + G \left( \frac{1}{r} \cdot \frac{2}{r^2} \right) \right] u_z \\
+ \frac{R^2}{24} \left[ \left( \lambda + 2G \right) \cdot \frac{1}{r} \cdot \frac{2}{r^2} - \left( \lambda + 2G \right) \cdot \frac{1}{r} \cdot \frac{2}{r^2} \right] V_z + \lambda \frac{2w_3}{r^2} = 0
\end{align*}
$$
\[
\begin{align*}
&\left[ (\lambda + 3\alpha) - \frac{2}{r^2} + (\lambda + 6\beta) - \frac{12}{r^2} \right] u_0 + \left[ -G_1 \frac{1}{r} + G_1 \frac{1}{r^2} + G_1 \frac{2}{r^2} + (\lambda + 2\delta) \frac{1}{r^2} \right] V_0 \\
&+ \frac{K_0}{24} \left[ (\lambda + 3\alpha) - \frac{2}{r^2} + (\lambda + 6\beta) - \frac{12}{r^2} \right] u_2 \\
&+ \frac{K_0}{24} \left[ -G_1 \frac{1}{r} + G_1 \frac{1}{r^2} + G_1 \frac{2}{r^2} + (\lambda + 2\delta) \frac{1}{r^2} \right] \frac{\partial w_1}{\partial \theta} = 0 \quad (5.12b)
\end{align*}
\]

and for the antisymmetric problem,

\[
\begin{align*}
&\{- \left( G_1 \frac{K_0}{12} + \frac{K_0}{12} \frac{\lambda + 3\alpha}{r^2} \right) + \frac{K_0}{12} \left[ \frac{\lambda + 2\beta}{r} \frac{\partial}{\partial r} + (\lambda + 2\delta) \frac{\partial}{\partial r^2} \right] \left( \frac{\partial w_0}{\partial \theta} + \frac{K_0}{24} \left( 2\lambda - 6\beta \right) \frac{\partial w_2}{\partial \theta} \right) \} \frac{\partial w_0}{\partial \theta} \\
&+ \frac{K_0}{24} \left[ \frac{\lambda + 6\beta}{r} \frac{\partial}{\partial r} + (\lambda + 2\delta) \frac{\partial}{\partial r^2} \right] \frac{\partial w_1}{\partial \theta} = 0 \quad (5.13a)
\end{align*}
\]

\[
\begin{align*}
&\frac{K_0}{12} \left( \frac{\lambda + 3\alpha}{r} \frac{\partial}{\partial r^2} + \frac{\lambda + 2\beta}{r^2} \right) u_1 + \left[ -G_1 \frac{1}{r} + G_1 \frac{1}{r^2} + G_1 \frac{2}{r^2} + (\lambda + 2\delta) \frac{1}{r^2} \right] V_1 \\
&- G_1 \frac{1}{r} \frac{\partial w_0}{\partial \theta} + \frac{K_0}{24} \left( 2\lambda - 6\beta \right) \frac{\partial w_2}{\partial \theta} + \frac{K_0}{480} \left( \frac{\lambda + 6\beta}{r} \frac{\partial}{\partial r} + (\lambda + 2\delta) \frac{\partial}{\partial r^2} \right) \frac{\partial w_1}{\partial \theta} \\
&\left[ -G_1 \frac{1}{r} \frac{\partial w_0}{\partial \theta} + \frac{K_0}{24} \left( 2\lambda - 6\beta \right) \frac{\partial w_2}{\partial \theta} + \frac{K_0}{480} \left( \frac{\lambda + 6\beta}{r} \frac{\partial}{\partial r} + (\lambda + 2\delta) \frac{\partial}{\partial r^2} \right) \frac{\partial w_1}{\partial \theta} \right] V_3 = 0 \quad (5.13b)
\end{align*}
\]
\[
\left( \frac{1}{r} + \frac{2}{r^2} \right) u_1 + \frac{1}{r} \frac{2u_t}{r} + \nabla^2 w_0 + \frac{r^2}{r} \left[ \left( \frac{1}{r} + \frac{2}{r^2} \right) u_0 + \frac{1}{r} \frac{2u_t}{r} + \nabla^2 w_0 \right] = -\frac{P}{q_h}
\]
(5.13c)

where \( \nabla^2 \equiv \frac{1}{r} \frac{2}{r} + \frac{2}{r^2} + \frac{1}{r} \frac{2}{r} \frac{2}{r} \)

### 5.3 Boundary Conditions

#### 5.3.1 Face Boundary Conditions

In the symmetric problem, the face boundary conditions are

\[
\begin{align*}
\sigma_0^\gamma(r, \theta, \pm \phi/2) &= \frac{P}{2} \\
\tau_0^\gamma(r, \theta, \mp \phi/2) &= \tau_0^\gamma(r, \theta, \mp \phi/2) = 0
\end{align*}
\]
(5.14)

A direct substitution of equation (5.14) into (5.16c, e, and f) yields the following three equations

\[
\lambda \left( \frac{u_0}{r} + \frac{2u_t}{r} + \frac{1}{r} \frac{2u_t}{r} + w_1 \right) + \frac{r^2}{B} \left( \frac{u_0}{r} + \frac{2u_t}{r} + \frac{1}{r} \frac{2u_t}{r} \right) + 2 \gamma w_1 = \frac{P}{2}
\]
(5.15a)

\[
u_2 = -\frac{2w_1}{\partial r}
\]
(5.15b)

\[
u_2 = -\frac{1}{\partial \theta}
\]
(5.15c)

Substituting equations (5.15b and c) into (5.15a) we then have

\[
w_1 = -\frac{1}{\lambda + 2 \gamma} \left[ \frac{P}{2} + \frac{r^2}{B} \lambda \nabla^2 w_1 - \lambda \left( \frac{u_0}{r} + \frac{2u_t}{r} + \frac{1}{r} \frac{2u_t}{r} \right) \right]
\]
(5.16)

Similarly, the face boundary conditions in the antisymmetric problem are
\[
\begin{align*}
\sigma_z^-(r, \theta, +\frac{r}{2}) &= \frac{P}{2} \\
\sigma_z^-(r, \theta, -\frac{r}{2}) &= -\frac{P}{2} \\
\tau_{rz}(r, \theta, \pm\frac{r}{2}) &= \tau_{\theta z}(r, \theta, \pm\frac{r}{2}) = 0
\end{align*}
\]

(5.17)

From equations (5.9c, e, and f) and (5.17) we have the following three equations:

\[
\lambda \left[ \frac{K}{2} \left( \frac{u_1}{r} + \frac{2u_1}{\partial r} + \frac{1}{r} \cdot \frac{2v_1}{\partial \theta} + w_2 \right) + \frac{K^3}{48} \left( \frac{u_3}{r} + \frac{2u_3}{\partial r} + \frac{1}{r} \cdot \frac{2v_3}{\partial \theta} \right) \right] + \epsilon \gamma_i w_i = \frac{P}{2}
\]

(5.18a)

\[
u_3 = -\frac{1}{\mu} \left( \frac{u_3}{r} + \frac{2w_3}{\partial \theta} \right) - \frac{1}{\mu} \frac{2w_3}{\partial \theta}
\]

(5.18b)

\[
u_3 = -\frac{1}{\mu} \left( v_3 + \frac{1}{r} \cdot \frac{2w_3}{\partial \theta} \right) - \frac{1}{\mu} \frac{2w_3}{\partial \theta}
\]

(5.18c)

Equation (5.18a) can be simplified by using (5.18b and c) as follow

\[
w_2 = \frac{1}{\lambda + 2\mu} \left[ \frac{P}{A} + \frac{K^2}{24} \nabla^2 w_2 + \frac{K^2}{3} \nabla^2 w_2 - \frac{2K}{3} \left( \frac{u_3}{r} + \frac{2u_3}{\partial r} + \frac{1}{r} \cdot \frac{2v_3}{\partial \theta} \right) \right]
\]

(5.19)

5.3.2 Edge Boundary Conditions

The edge boundary conditions at the plate mid-plane \( z = 0 \) for simply supported, clamped, and free edges are given below.

Simply supported edge:

\[
u_0 = N_r = w_0 = M_r = v_1 = 0
\]

(5.20)

Clamped edge:

\[
u_0 = \nu_0 = w_0 = u_1 = v_1 = 0
\]

(5.21)
Free edge:

\[ N_{r\theta} = N_r = Q_r = M_r = M_{r\theta} = 0 \quad (5.22) \]

It should be pointed out that in equations (5.20) - (5.22) the first two boundary conditions are for the symmetric problem and the last three for the antisymmetric problem.

5.4 Uniformly Loaded Circular Plates

Assume a circular plate of radius \( a \) carrying a load of intensity \( P \) uniformly distributed over the entire top surface \( (z = h/2) \) of the plate. Solutions for the symmetric and the antisymmetric problems will be investigated separately.

5.4.1 Symmetric Problem

First we consider the symmetric problem. Because of the axisymmetric property of the problem, the displacement components \( u_0, v_0 \).

\( v_0 \) and \( w_0 \) are functions of \( r \) only. As in the problem of rectangular plates, we assume \( \nabla^2 w_0 = 0 \). The proof of this assumption will be presented later in this section. Equations (5.15b and c) and (5.16) then take the following forms

\[ u_z = -\frac{d w_0}{d r} \quad (5.23a) \]

\[ v_z = 0 \quad (5.23b) \]

\[ w_i = \frac{1}{\lambda + 2\sigma} \left[ \frac{P}{2} - \lambda \left( \frac{u_z}{r} + \frac{du_z}{dr} \right) \right] \quad (5.23c) \]

Substituting equations (5.23) into (5.12), the following two equilibrium
equations are obtained

\[
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) V_0 = 0 \tag{5.24}
\]

\[
\left( \frac{d^2}{dt^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right)(u_0 - \frac{k^2}{24} \frac{d}{dr} \frac{d}{dt}) + \frac{\lambda}{\lambda + 2G} \frac{d}{dr} \frac{d}{dt} = 0 \tag{5.25}
\]

The general solution of equation (5.24) is

\[ V_0 = C_1 r + \frac{C_2}{r} \tag{5.26} \]

where \( C_1 \) and \( C_2 \) are the integration constants. Using equation (5.23c) in (5.25) and noting the assumption \( \nabla^2 w_1 = 0 \), we have

\[
\left[ \frac{4G(\lambda + 2G)}{\lambda (\lambda + 2G)} - \frac{k^2}{24} \frac{1}{r^2} \right] \frac{d}{dr} \frac{d}{dt} = 0
\]

For arbitrary \( r \) we must have \( \frac{d}{dr} \frac{d}{dt} = 0 \), hence

\[ w_1 = C_3 \tag{5.27} \]

A direct substitution of equation (5.27) into (5.23c) yields

\[
\frac{d}{dr} u_0 + \frac{u_0}{r} = \frac{1}{\lambda} \left( \frac{P}{2} - (\lambda + 2G) C_3 \right)
\]

The general solution of \( u_0 \) is

\[ u_0 = \frac{r}{\lambda} \left( \frac{P}{4} - \frac{\lambda + 2G}{2} C_3 \right) + \frac{C_4}{r} \tag{5.28} \]

From equation (5.7a) the stress resultant \( N_r \) has the expression

\[ N_r = \frac{1}{\lambda} \left[ \lambda \left( \frac{u_0}{r} + \frac{du_0}{dr} + w_1 \right) + 2G \frac{du_0}{dr} \right] \tag{5.29} \]

or

\[ N_r = \frac{1}{\lambda} \left[ \frac{\lambda + 2G}{2} P - G (3 \lambda + 2G) C_3 \right] - \frac{2G}{r^2} C_4 \tag{5.30} \]
If the plate contains no hole and is simply supported along its edge, we have the boundary conditions at $r = a$

$$v_o = N_r = 0$$

and at $r = 0$ the displacement components $u_o$ and $v_o$ must be finite values. From these boundary conditions the constants $C_1 \ldots C_4$ are obtained as follows

$$C_1 = C_2 = C_4 = 0$$
$$C_3 = \frac{P}{2} \frac{\lambda + G}{G_1(3\lambda + 2\mu)}$$

(5.31)

The displacement components are thus completely determined.

$$u_o = -\frac{P}{4} \frac{\lambda}{G(3\lambda + 2\mu)} \cdot r$$

(5.32a)

$$w_1 = \frac{P}{2} \frac{\lambda + G}{G_1(3\lambda + 2\mu)}$$

(5.32b)

$$u_2 = v_0 = v_2 = 0$$

(5.32c)

From the expression of $w_1$ in equation (5.32b) it is clearly seen that $\nabla^2 w_1 = 0$ is identically satisfied. Changing the parameters $\lambda$ and $G$ into $\gamma$ and $E$ we have the following expressions of $u_o$ and $w_1$ which will be used later for comparison purpose.

$$u_o = -\frac{P}{E} \frac{\gamma}{2} \cdot r$$

(5.33a)

$$w_1 = \frac{P}{E} \frac{1}{2}$$

(5.33b)

5.4.2 Antisymmetric Problem

Assuming $\nabla^2 w_2 = S$ ($S$ is a constant), equations (5.18b and c) and
(5.19) then take the following forms

\[ u_3 = - \frac{\varrho}{\lambda t^2} (u_1 + \frac{dw_3}{dt}) - \frac{dW_3}{dr} \]  (5.34a)

\[ V_3 = - \frac{\varrho}{\lambda t^2} V_1 \]  (5.34b)

\[ W_1 = \frac{1}{\lambda + 2\delta} \left[ \frac{P}{\lambda} + \frac{\lambda t^4}{24} S + \frac{\lambda}{3} \nabla^2 W_0 - \frac{2\lambda}{3} \left( \frac{u_1}{r} + \frac{du_1}{dr} \right) \right] \]  (5.34c)

Applying the above expressions in the equilibrium equations (5.13) then we get

\[ \frac{u_1}{r} + \frac{du_1}{dr} + \nabla^2 W_0 = - \frac{3P}{2\delta^2} \]  (5.35)

\[ -\frac{2\delta^2}{3} u_1 + \frac{d^3}{dr^3} \left( \frac{\lambda + 2\delta}{\lambda^2} \right) \frac{d}{dr} \left( \frac{u_1}{r} + \frac{du_1}{dr} \right) + \frac{d}{dr} \left( \frac{2\delta^2}{3} W_0 + \frac{\lambda^3}{\delta} \frac{\lambda + 2\delta}{\lambda + 2\delta} \nabla^2 W_0 \right) = 0 \]  (5.36)

\[ \frac{d^2 V_1}{dr^2} + \frac{1}{r} \frac{dV_1}{dr} - \left[ \left( \frac{\lambda}{\delta} \right)^2 + \left( \frac{1}{r} \right)^2 \right] V_1 = 0 \]  (5.37)

From equations (5.35) and (5.36) we obtain the following two equations

\[ \frac{\lambda^3}{3} \frac{\lambda + 2\delta}{\lambda + 2\delta} \nabla^2 W_0 = P \]  (5.38)

\[ u_1 = - \frac{d}{dr} \left( W_0 + \frac{\lambda t^4}{2} \frac{\lambda + 2\delta}{\lambda + 2\delta} \nabla^2 W_0 \right) \]  (5.39)

where \( \nabla^4 = \nabla^2 \nabla^2 \). The general solution of equation (5.38) is

\[ W_0 = \frac{P}{D} \frac{t^4}{64} + A_1 + A_2 r + A_3 r^2 + A_4 r^3 \ln r \]  (5.40)

where \( A_1 \ldots A_4 \) are constants and
\[ D = \frac{R^3}{3} \frac{G(\lambda + 6\alpha)}{\lambda + 2\alpha} = \frac{E R^3}{12(1 - \nu^2)} \]  

Equation (5.37) is the modified Bessel equation and its general solution is

\[ V_i = A_1 I_1(\frac{\sqrt{10}}{R} r) + A_6 K_1(\frac{\sqrt{10}}{R} r) \]  

where \( I_1 \) and \( K_1 \) are the modified Bessel functions of first order of the first and second kind, respectively, and \( A_5 \) and \( A_6 \) are constants.

The expression of the moment resultant \( M_r \) can be obtained by direct substitution of the related displacement components into equation (5.10b).

\[ M_r = -\left[ \frac{7\lambda + 6\alpha}{3^2(\lambda + 6\alpha)} r^2 + \frac{R^2}{48} \frac{6(\lambda + 12\lambda + 6\lambda + 206\lambda + 96\lambda)}{\lambda(\lambda + 6\alpha)(\lambda + 2\alpha)} \right] \rho \]

\[ + \frac{R^3}{6} \frac{G_1}{R} A_2 - \left[ \frac{R^3}{5} \frac{G_1(3\lambda + 2\alpha)}{\lambda + 2\alpha} \right] A_3 \]

\[ - \frac{R^3}{6} \frac{G_1(7\lambda + 6\alpha)}{\lambda + 2\alpha} + \frac{26(3\lambda + 2\alpha)}{\lambda + 2\alpha} A_4 r - \frac{G_1(7\lambda + 16\alpha)}{10(\lambda + 2\alpha)} (\frac{R}{r})^2 \]  

Considering a circular plate without hole and having simply supported edge, the boundary conditions at \( r = a \) are

\[ w_o = M_r = v_i = 0 \]

In addition to the boundary conditions at \( r = a \), the displacement components \( v_i \) and \( w_o \) and the moment resultant \( M_r \) should be finite values at \( r = 0 \). From these considerations the constants \( A_1 \ldots A_6 \) are com-
pletely determined.

\[ A_1 = \frac{P}{D} \left[ \frac{a^4}{64} \frac{11\lambda + 10G}{3\lambda + 2G} + \frac{a^2 \lambda^2}{480} \frac{6\lambda^3 + 122\lambda^2 G + 206\lambda G^2 + 96G^3}{G(\lambda + 2G)(3\lambda + 2G)} \right] \]  

\[ A_3 = -\frac{P}{D} \left[ \frac{a^2}{32} \frac{7\lambda + 6G}{3\lambda + 2G} + \frac{a^4 \lambda^2}{480} \frac{6\lambda^3 + 122\lambda^2 G + 206\lambda G^2 + 96G^3}{G(\lambda + 2G)(3\lambda + 2G)} \right] \]  

\[ A_2 = A_4 = A_5 = A_6 = 0 \]  

The displacement components thus have the following forms

\[ W_0 = \frac{1}{8} \frac{P}{D} (a^2 - r^2) \left[ \frac{1}{8} \frac{11\lambda + 10G}{3\lambda + 2G} r^2 + \frac{a^2 \lambda^2}{60} \frac{6\lambda^3 + 122\lambda^2 G + 206\lambda G^2 + 96G^3}{G(\lambda + 2G)(3\lambda + 2G)} \right] \]  

\[ W_2 = -\frac{1}{4} \frac{P}{D} \left[ \frac{\lambda(7\lambda + 6G)}{2(\lambda + 2G)(3\lambda + 2G)} a^2 - \frac{\lambda}{\lambda + 2G} r^2 + \frac{a^2 \lambda^2}{150} \frac{3\lambda^3 - 13\lambda^2 G - 36\lambda G^2 - 20G^3}{G(\lambda + 2G)(3\lambda + 2G)} \right] \]  

\[ u_1 = \frac{1}{8} \frac{P}{D} r \left[ \frac{1}{2} \frac{7\lambda + 6G}{9\lambda + 2G} a^2 - r^2 \right] + \frac{a^2 \lambda^2}{60} \frac{6\lambda^3 - 69\lambda^2 G - 94\lambda G^2 - 24G^3}{G(\lambda + 2G)(3\lambda + 2G)} \]  

\[ u_3 = \frac{P}{D} r \frac{3\lambda + 4G}{2(\lambda + 2G)} \]  

\[ v_1 = v_3 = 0 \]
Applying the Laplace operator $\nabla^2$ to equation (5.45b), then we get

$$\nabla^2 W_2 = \frac{l}{2} \frac{P}{D} \frac{\lambda}{\lambda + 2G}$$

(5.46)

Hence $\nabla^2 W_2 = \text{constant}$ is proved.

The parameters $\lambda$ and $G$ of equations (5.45) can be changed to $\nu$ and $E$ by using the relations in section 2.5. The results are given below for future use.

$$W_0 = -\frac{1}{8} \frac{P}{D} (a^2 - r^2) \left[ \frac{1}{9} \left( \frac{5+\nu}{1+\nu} a^2 - r^2 \right) + \frac{a^4}{60} \cdot \frac{24-4(1-3\nu^2) + 10\nu^3}{(1-2\nu)(1-\nu^2)} \right]$$

(5.47a)

$$W_2 = -\frac{P}{E} \left\{ \frac{1}{4} \frac{3\nu}{2} \left[ (\nu+1) a^2 - 2(1+\nu) r^2 \right] - \frac{1}{5} \frac{1}{E} \cdot \frac{5-12\nu + \nu^2}{1-2\nu} \right\}$$

(5.47b)

$$u_1 = \frac{P}{E} r \left\{ \frac{1}{4} \frac{3(1-\nu)}{4} \left[ (\nu+1) a^2 - (1+\nu) r^2 \right] - \frac{1}{20} \frac{b+11\nu - 57\nu^2 + 10\nu^3}{1-2\nu} \right\}$$

(5.47c)

$$u_3 = \frac{P}{E} r \frac{b}{4} (2 + \nu - \nu^2)$$

(5.47d)

The antisymmetric problem is thus completely solved. The displacements $u$, $v$, and $w$ can be obtained by adding equations (5.32) to (5.45) (or (5.33) to (5.47)) according to equations (2.1) - (2.3).

$$U = \frac{P r}{E} \left\{ \frac{-\nu}{2} + \frac{3(\nu - 1)}{4} \frac{a^3}{4} \left[ (\nu+1) a^2 - (1+\nu) r^2 \right] - \frac{1}{20} \frac{b+11\nu - 57\nu^2 + 10\nu^3}{1-2\nu} \right\}$$

$$+ \frac{3^3}{4} \frac{a^3}{4} (2 + \nu - \nu^2)$$

(5.48a)
The exact solution of this problem is given by Love (p. 481 of ref. 24). After changing his coordinate system and plate thickness to conform with the present analysis the results take the following forms

\[ U = \frac{P r}{E} \left\{ -\frac{\nu}{2} - \frac{3(1-\nu)}{4} \frac{3}{4} \left[ (3+\nu)(a^2 - (1+\nu)r^2) - \frac{3}{20} \frac{3}{4} (5+2\nu^2) \right] + \frac{\nu}{2} \frac{3}{4} \left( 2 + \nu - \nu^2 \right) \right\} \]  \hspace{1cm} (5.49a)

\[ V = 0 \]  \hspace{1cm} (5.49b)

\[ W = W_o + \frac{P t}{E} \left\{ \frac{1}{2} - \frac{3\nu}{4} \frac{3}{4} \left[ (3+\nu)(a^2 - 2(1+\nu)r^2) + \frac{3}{20} \frac{3}{4} (5+2\nu^2) \right] - \frac{1}{2} \frac{3}{4} \left( 1 + \nu^2 \right) \right\} \]  \hspace{1cm} (5.49c)

where

\[ W_o = \frac{1}{3} \cdot \frac{P}{D} (a^2 - r^2) \left[ \frac{1}{8} \left( \frac{5+\nu}{1+\nu} - \frac{4}{3} \right) + \frac{1}{20} \frac{3}{4} \left( 5+2\nu^2 \right) \right] \]  \hspace{1cm} (5.50)

Comparing equations (5.48) and (5.49) it can be seen that the results based on the present theory agree closely with that of Love's exact solution.

Next, we shall compare the maximum lateral displacement \( w_o \) and moment resultant \( M_r \), both occur at the center of the plate, and the
maximum stress $\sigma_r$ which occurs at the top surface of the plate center with the results from Love's exact solution and the classical thin plate theory (ref. 25). As before, the three nondimensional parameters $\lambda$, $\beta$, and $\eta$ are defined as follows

$$\lambda = (W_0)_{\text{max}} E h^3 / \rho a^4$$

$$\beta = (M_r)_{\text{max}} / \rho a^2$$

$$\eta = (\sigma_r)_{\text{max}} h^2 / \rho a^2$$

The corresponding expressions of the parameters from the classical, the present, and the exact theories are given below

$$\lambda_c = \frac{3}{16} (1 - \nu) (5 + \nu)$$

$$\lambda_p = \lambda_c + \frac{1}{40} \left( \frac{4 \nu^2 - 11 + 10 \nu^2 - 10 \nu^3}{1 - 2 \nu} \right) \left( \frac{a}{h} \right)^2$$

$$\lambda_L = \lambda_c + \frac{3}{40} (8 + \nu + \nu^2) \left( \frac{a}{h} \right)^2$$

$$\beta_c = \beta_p = \beta_L = \frac{1}{16} (3 + \nu)$$

$$\eta_c = \frac{3}{8} (3 + \nu)$$
\[ T_p = T_c + \frac{1}{20} \frac{(1-\nu)(2+\nu)(2-\nu^2)}{(1+\nu)(1-2\nu^2)} \left( \frac{h}{a} \right)^2 \]

\[ T_L = T_c + \frac{1}{20} (2+\nu) \left( \frac{h}{a} \right)^2 \]

where the subscripts C, P, and L denote the results from the classical, the present, and Love's exact theories, respectively. It is interesting to note that all three theories yield the same maximum moment resultant \( M_r \).

Table 5.1 gives the nondimensional parameters \( \chi \) and \( \Upsilon \) for various values of thickness ratio \( h/a \) when \( \nu = 0.3 \). Again, \( \chi \) and \( \Upsilon \) increase as the plate thickness increases. For the ranges chosen in the foregoing, the maximum percentages of deviation of \( \chi \) from the classical and Love's exact theories are by about 3.86 and 0.38, respectively for the case \( h/a = 0.2 \). The corresponding values for \( \Upsilon \) are 1.14 and 0.77.

It should be pointed out here that solution of this problem based on the Reissner and Donnell-Lee thick plate theories is not available. Hence no attempt to compare the present results with their theories was made.

### 5.5 Circular Plate Loaded at the Center

The solution for a concentrated load acting at the center of the plate can be deduced from a plate in which the load \( P \) is uniformly dis-
Table 5.1 - Non-Dimensional Displacement Parameter \( \alpha \) and Stress Parameter \( \tau \) for Various Values of \( h/a \) with \( \nu = 0.3 \)

<table>
<thead>
<tr>
<th>( \frac{h}{a} )</th>
<th>Classical Theory</th>
<th>Present Theory</th>
<th>Exact Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \alpha_c )</td>
<td>( \tau_c )</td>
<td>( \alpha_p )</td>
</tr>
<tr>
<td>0.005</td>
<td>0.695625</td>
<td>1.237500</td>
<td>0.695642</td>
</tr>
<tr>
<td>0.010</td>
<td>0.695625</td>
<td>1.237500</td>
<td>0.695694</td>
</tr>
<tr>
<td>0.050</td>
<td>0.695625</td>
<td>1.237500</td>
<td>0.697360</td>
</tr>
<tr>
<td>0.100</td>
<td>0.695625</td>
<td>1.237500</td>
<td>0.702600</td>
</tr>
<tr>
<td>0.200</td>
<td>0.695625</td>
<td>1.237500</td>
<td>0.723525</td>
</tr>
</tbody>
</table>
tributed along a circle of radius \( c \) (see fig. 5.2). This is done by assuming that the radius \( c \) becomes infinitely small, whereas the total load \( q \) remains finite. For the purpose of investigating the nature of singularities under the concentrated load, we limit ourselves to the antisymmetric (flexural) problem only, namely, assume a plate subjected to concentrated load \( q/2 \) at its top and bottom surfaces.

Dividing the simply supported plate into two parts as shown in figure 5.2b and c, it may be seen that the inner and outer portions of the plate can be solved separately provided that the conditions of continuity are satisfied along the circle \( r = c \).

We begin with the inner portion. The general solution of \( w_0 \) is given as equation (5.40). Because the plate contains no hole hence \( A_2 = A_4 = 0 \), thus

\[
W_0 = \frac{P}{D} \cdot \frac{r^4}{b_4} + A_1 + A_3 r^2 \tag{5.51}
\]

\[
U_1 = -\frac{P}{D} \left( \frac{r^3}{16} + \frac{r_0^2}{4} \cdot \frac{\lambda + 61}{\lambda + z_0} \right) - 2t A_3 \tag{5.52}
\]

The corresponding stress and moment resultants are

\[
Q_r = -\frac{P}{2} r \tag{5.53}
\]

\[
M_r = -P \left[ \frac{r^3}{32} \frac{7\lambda + 61}{\lambda + 61} + \frac{r_0^2}{48} \frac{6\lambda^3 + 12\lambda^2 + 20\lambda + 6}{61(\lambda + 61)(\lambda + z_0)} \right] - \frac{P}{3} \frac{6(3\lambda + 26)}{\lambda + z_0} A_3 \tag{5.54}
\]

At \( r = c \), we have

\[
(W_0)_{r=c} = \frac{P}{D} \cdot \frac{c^4}{b_4} + A_1 + A_3 c^2 \tag{5.55}
\]
Figure 5.2. - Antisymmetric problem of a circular plate concentrically loaded.
\[ (u_1)_{r=c} = -\frac{P}{D} \left( \frac{c^3}{16} + \frac{c^2 \lambda + \eta}{4 \lambda + 2 \eta} \right) - 2c A_3 \quad (5.56) \]

\[ Q_c = -\frac{P}{2} c \quad (5.57) \]

\[ M_c = -P \left[ \frac{c^3}{32} \frac{7\lambda + 6\eta}{\lambda + \eta} + \frac{\eta^2}{480} \frac{6(\lambda + 2\eta)^3 + 9\lambda \eta^2}{G(\lambda + \eta)(\lambda + 2\eta)} \right] - \frac{A_3}{2} \frac{G(3\lambda + 2\eta)}{\lambda + 2\eta} A_5 \quad (5.58) \]

where \( A_1 \) and \( A_3 \) are constants.

For the outer portion \( P=0 \), we have the following general solution for \( w_0^* \). A star superscript is added to all quantities related to the outer portion for the purpose of distinguishing from those of the inner portion.

\[ w_0^* = A_1^* + A_2^* \ln r + A_3^* r + \frac{A_4^*}{r} \ln r \quad (5.59) \]

\[ u_1^* = -\frac{1}{r} A_2^* - 2r A_3^* - r A_3^* \left[ r + 2r \ln r + \frac{A_4^*}{r} \frac{2(\lambda + 6\eta)}{\lambda + 2\eta} \right] A_4^* \quad (5.60) \]

The corresponding stress and moment resultants are

\[ Q_r^* = -\frac{A_3^*}{r} \frac{G(\lambda + 6\eta)}{\lambda + 2\eta} \quad (5.61) \]

\[ M_r^* = \frac{A^*}{6} \frac{G(\lambda + 6\eta)}{\lambda + 2\eta} \left[ \frac{G(\lambda + 2\eta)}{\lambda + 2\eta} A_3^* - \frac{A_3^*}{3} \frac{G(7\lambda + 16\eta)}{\lambda + 2\eta} \right] A_4^* \quad (5.62) \]

At \( r = c \), we have

\[ (w_0^*)_{r=c} = A_1^* + A_2^* \ln c + A_3^* c + \frac{A_4^*}{c} c \ln c \quad (5.63) \]
\[(u_i)_r = -\frac{1}{c} A_1 + 2c A_3 - \left[ C + 2c \ln c + \frac{2}{c} \frac{2(\lambda + \varrho)}{\lambda + 2\varrho} \right] A_4^* \quad (5.64)\]

\[-\frac{p}{2} c = -\frac{4}{3} \frac{A^3}{c} \frac{G(\lambda + \varrho)}{\lambda + 2\varrho} A_4^* \quad (5.65)\]

\[-\frac{\rho}{c^2} \left[ \frac{C^2}{32} \frac{7\lambda + 2\varrho}{\lambda + \varrho} + \frac{A^3}{480} \frac{6\lambda + 2(\lambda + \varrho) + 26\lambda^2 + 96\lambda^3}{G(\lambda + \varrho)} \right] - \frac{\rho}{3} \frac{G(3\lambda + 2\varrho)}{\lambda + 2\varrho} A_3^* \]

\[= \frac{A^3}{6} \frac{G}{c^2} A_1^* - \frac{A^3}{3} \frac{G(\lambda + \varrho)}{\lambda + 2\varrho} A_3^* - \frac{A^3}{6} \left[ \frac{G(7\lambda + 2\varrho)}{\lambda + 2\varrho} + \frac{2G(3\lambda + 2\varrho)}{\lambda + 2\varrho} \ln c \right] A_4^* \]

\[- \frac{1}{10} \left( \frac{A^3}{c} \frac{G(17\lambda + 16\varrho)}{\lambda + 2\varrho} \right) A_4^* \quad (5.66)\]

The boundary conditions at \( r = a \) are \( w_0^* = 0 \) and \( H_r^* = 0 \), hence

\[A_1^* + A_2 \ln a + A_3^* a^3 + A_4^* a^2 \ln a = 0 \quad (5.67)\]

\[\frac{G}{a^2} A_1^* - \frac{2G(3\lambda + 2\varrho)}{\lambda + 2\varrho} A_3^* \left[ \frac{G(7\lambda + 2\varrho)}{\lambda + 2\varrho} + \frac{2G(3\lambda + 2\varrho)}{\lambda + 2\varrho} \ln a - \frac{1}{10} \left( \frac{A^3}{c} \frac{G(17\lambda + 16\varrho)}{\lambda + 2\varrho} \right) \right] A_4^* = 0 \quad (5.68)\]

The conditions of continuity at \( r = c \) are \( w_0 = w_0^* \) and \( u_1 = u_1^* \), thus

\[\frac{P}{D} \frac{c^4}{64} + A_1 + A_3 c^2 = A_1^* + A_2 \ln c + A_3^* c^2 + A_4^* c^2 \ln c \quad (5.69)\]

\[-\frac{P}{D} \left( \frac{c^3}{16} + \frac{c^3}{4} \frac{\lambda + \varrho}{\lambda + 2\varrho} \right) - 2c A_3 = -\frac{1}{c} A_1^* - 2c A_3^* \left[ C + 2c \ln c + \frac{2}{c} \frac{2(\lambda + \varrho)}{\lambda + 2\varrho} \right] A_4^* \quad (5.70)\]

Equations (5.65) - (5.70) are six algebraic equations with six unknown constants \( A_1, A_3, A_1^*, \ldots, A_4^* \). The solution for these constants are
\[ A_1 = \frac{P}{D} \left\{ \frac{c^4}{16} \left[ \frac{c^4}{30} \frac{\lambda^3 + 6 \lambda^2 G + 178 \lambda G^2 + 96 G^3}{G (\lambda + G)(\lambda + 2G)} \right] \ln \frac{\alpha}{c} + \frac{c^4}{16} \left( \frac{\lambda^2 - 7 \lambda + 6 G}{3 \lambda + 2G} \right) - \frac{c^4}{4} \frac{\lambda^3 + 16 \lambda^2 G}{3 \lambda + 2G} \right\} \] (5.71a)

\[ A_3 = \frac{P}{D} \left\{ \frac{c^4}{8} \ln \frac{\alpha}{c} + \frac{c^4}{16} \left[ \frac{\lambda^2}{2} \frac{\lambda}{\lambda + 2G} - \frac{4}{3} \frac{(\lambda + G)}{3 \lambda + 2G} \right] \right\} \] (5.71b)

\[ A_1^* = \frac{P}{D} \left\{ \frac{c^4}{16} \left[ \frac{c^4}{480} \frac{\lambda^3 + 8 \lambda^2 G + 178 \lambda G^2 + 96 G^3}{G (\lambda + G)(\lambda + 2G)} \right] \ln \frac{\alpha}{c} \right. \]
\[ + \frac{c^4}{32} \left[ \frac{c^4}{\lambda + 2G} \frac{\lambda}{G (\lambda + G)(\lambda + 2G)} - \frac{4}{30} \frac{\lambda}{6} \frac{(\lambda^2 - 6 \lambda G - 20 G^2)}{G (\lambda + G)} \right] \] (5.71c)

\[ A_2^* = \frac{P}{D} \left[ \frac{c^4}{16} - \frac{c^4}{480} \frac{\lambda^3 + 8 \lambda^2 G + 178 \lambda G^2 + 96 G^3}{G (\lambda + G)(\lambda + 2G)} \right] \] (5.71d)

\[ A_3^* = \frac{P}{D} \left\{ - \frac{c^4}{8} \ln \alpha + \frac{c^4}{32} \left[ \frac{c^4}{\lambda + 2G} - 2(\lambda + 2G) \right] \right. \]
\[ - \frac{c^4}{960} \frac{c^4}{\lambda + 2G} \frac{\lambda}{G (\lambda + G)(\lambda + 2G)} \] (5.71e)

\[ A_4^* = \frac{P}{D} \left( \frac{c^4}{8} \right) \] (5.71f)
The displacement components of the inner and outer portions of the plate are thus completely determined. To investigate the nature of singularities under the concentrated loads let us examine \( w_0 \) and \( M_r \) at \( r = 0 \). From equations (5.51) and (5.54) we have the following expressions after replacing \( P \) by \( q/\pi c^2 \) where \( q \) is the total load.

\[
(W_0)_{r=0} = \frac{q}{\pi D} \cdot \frac{1}{16} \left\{ \left( a^2 - \frac{2\lambda + 6G}{3\lambda + 2G} \right) + \frac{a^2}{3\lambda + 2G} \cdot \frac{6\lambda^3 + 2\lambda G^2 + 20\lambda G^2 + 126G^3}{G(\lambda + 6)(3\lambda + 2G)} \right\} \ln \frac{a}{c} \tag{5.72}
\]

\[
(M_r)_{r=0} = \frac{q}{4\pi} \left[ (1 - \frac{\lambda + 2G}{8(\lambda + G)}) \cdot \frac{c^2}{\alpha^2} + \frac{a^2}{24}\cdot \frac{\lambda(3\lambda^2-16\lambda G-20G^2)}{G(\lambda + G)^2} \left( \frac{1}{\alpha^2} - \frac{1}{c^2} \right) \right.
\]

\[
+ \frac{3\lambda + 2G}{2(\lambda + G)} \ln \frac{a}{c} \right\} \tag{5.73}
\]

When \( c \) approaches zero, equations (5.72) and (5.73) both approach infinity and hence the solution is not valid at the point of application of the concentrated load. Koehler and Essenburg (ref. 6) also obtained the same conclusion from the application of Reissner’s plate theory in a simply supported rectangular plate when subjected to a concentrated load.

In the classical thin plate theory the corresponding \( w_0 \) and \( M_r \) at \( r = 0 \) are obtained from reference 25.

\[
(W_0)_{r=0} = \frac{q}{\pi D} \cdot \frac{1}{16} \left[ a^2 \cdot \frac{3+\nu}{1+\nu} - \frac{c^2}{4} \cdot \frac{7+3\nu}{1+\nu} - c^2\ln \frac{a}{c} \right] \tag{5.74}
\]
The deflection \( w_0 \) has a finite value as \( c \) approaches zero in equation (5.74) whereas \( M_r \) approaches infinity. Hence, the nature of singularity of \( w_0 \) at \( r = 0 \) for the present theory is due to the inclusion of shear deformation. It is clear that if a uniform load is distributed over a small concentrical area, in the vicinity of the loaded region the improvement of the present theory prediction for the deflection \( w_0 \) (over the classical thin plate theory prediction) will be substantial.

\[
(M_r)_{r=a} = \frac{q}{4\pi} \left[ (1 - \frac{1-v^2}{4} \cdot \frac{c^2}{a^2}) + (1+v)ln\frac{a}{c} \right]
\] (5.75)
CHAPTER VI

GENERALIZATION FOR HETEROGENEOUS THICK ELASTIC PLATES

6.1 Introduction

In recent years some interest arose towards the development of elasticity theories for plates whose elastic properties vary from point to point. The variation of elastic properties of the medium may occur continuously or abruptly, whereby the continuous variation might be in the form of discrete steps. The later is the case in synthetic laminated materials; within each layer there is a continuous variation of elastic properties which terminates in an abrupt jump on the boundaries of the layers.

Laminated plate theory based on the classical Kirchhoff hypotheses has been well established. The elastostatic bending and stretching theory for plates that are non-homogeneous in the thickness direction was established by Reissner and Stavsky (ref.26). It was shown that the plate heterogeneity introduced a coupling phenomenon between bending and stretching. Whitney and Leissa (ref.27) have recently solved a number of problems relating to the bending, vibration, and stability of coupled laminates.

The introduction of shear deformation into heterogeneous plate theory was evidently first accomplished by Stavsky (ref.28), for
Isotropic layers having identical Poisson ratios. Yang, Norris, and Stavsky (ref. 29) extended Mindlin's linear theory for homogeneous plates (ref. 13) to laminates consisting of an arbitrary number of bonded anisotropic layers. Agrawal (ref. 30) developed a general dynamic theory for the large deflection of thin, aeolotropic, heterogeneous elastic plates. Thermal stresses and transverse shear deformations are also considered. Whitney and Pagano (refs. 31 and 32) have recently extended Ambartsumyan's approach (ref. 33) to solve certain specific boundary-value problems of heterogeneous anisotropic plates.

A common feature of the references cited above is the following assumption in displacement field,

\[
\begin{align*}
U(x, y, z, t) &= U_0(x, y, t) + \psi_x(x, y, t) \\
V(x, y, z, t) &= V_0(x, y, t) + \psi_y(x, y, t) \\
W(x, y, z, t) &= W_0(x, y, t)
\end{align*}
\]  

(6.1)

where \( t \) indicates time. This displacement assumption makes solution to some heterogeneous anisotropic plates possible. However, as in the homogeneous isotropic plate theory, contradiction to elasticity theory still exists. It is the purpose of this chapter to extend the displacement assumption used in the previous chapters to heterogeneous plates.

In this chapter, the non-homogeneity of the plate is assumed only in the thickness direction and it may be of two types: (1) the elastic moduli vary continuously in the \( z \) direction of the so-called "heterogeneous plates", (2) thin homogeneous layers of different
elastic properties are composed to form a "laminated plate" in which the moduli are step functions of \( z \). Furthermore, the plates are assumed to be heterogeneous orthotropic material. For general heterogeneous anisotropic plate problems, solutions to the set of complicate and high order differential equations are often impossible. This can be seen from Agrawal's work (ref. 30) in which he derived a tenth order system of differential equations for the most general case of heterogeneous anisotropic plates based on the displacement assumption as shown in equation (6.1). Observing the set of differential equations, he stated; "Obviously they are too difficult (rather impossible) to solve for the general case." It is for this reason we restrict our plate problems to heterogeneous orthotropy. In fact, many practical engineering materials possess this property.

6.2 Governing Equilibrium Equations of Heterogeneous Orthotropic Plate

Consider a thick heterogeneous elastic plate of thickness \( h \), referred to an \( x, y, z \) system of Cartesian coordinates. The origin of the coordinate system is located within the central plane \( (x-y) \) with the \( z \)-axis being normal to this plane. Assume the plate surfaces at \( z = \pm h/2 \) are subjected to surface tractions defined by

\[
\begin{align*}
\sigma_y(x, y, \pm h/2) &= \tau(x, y) \\
\sigma_z(x, y, \pm h/2) &= T_{xz}(x, y, \pm h/2) = T_{yz}(x, y, \pm h/2) = 0
\end{align*}
\]  

(6.2)

The generalized Hooke's law for the stress-strain relations in the case of orthotropic material is
The nine elastic coefficients $K_{ij}$ are specified functions of $z$ but do not vary in the $x, y$ directions.

In order to account for transverse shear deformation and normal stress $\sigma_z$ effects in the heterogeneous plate theory to be established, we follow the same assumption of displacement field as before, namely,

\[
\begin{align*}
U(x, y, z) &= U_0(x, y) + \frac{z}{3} U_1(x, y) + \frac{z^2}{2} U_2(x, y) + \frac{z^3}{6} U_3(x, y) \\
V(x, y, z) &= V_0(x, y) + \frac{z}{3} V_1(x, y) + \frac{z^2}{2} V_2(x, y) + \frac{z^3}{6} V_3(x, y) \\
W(x, y, z) &= W_0(x, y) + \frac{z}{3} W_1(x, y) + \frac{z^2}{2} W_2(x, y).
\end{align*}
\]

The stress and moment resultants, each per unit length, are defined in the usual way, i.e.,
\[
\begin{align*}
(N_x, N_y, N_{xy}, Q_x, Q_y) &= \int_{A} \left( \sigma_x, \sigma_y, \tau_{xy}, \tau_{xx}, \tau_{yy} \right) \, d\gamma \\
(M_x, M_y, M_{xy}) &= \int_{A} \left( \sigma_x, \sigma_y, \tau_{xy} \right) \gamma \, d\gamma
\end{align*}
\]

(6.5)

For linear theory, the strain-displacement relations are those of equations (2.16).

The stress and moment resultants are obtained by substituting equations (6.4) into (2.16) and using (6.3) and (6.5). They are

\[
N_x = A_{11} \frac{\partial u_2}{\partial x} + A_{12} \frac{\partial v_2}{\partial y} + A_{13} w_1 + B_{11} \frac{\partial u_2}{\partial x} + B_{12} \frac{\partial v_2}{\partial y} + B_{13} w_2 + \frac{1}{2} \left( C_{11} \frac{\partial u_2}{\partial x} + C_{12} \frac{\partial v_2}{\partial y} \right) + \frac{1}{6} \left( D_{11} \frac{\partial u_2}{\partial x} + D_{12} \frac{\partial v_2}{\partial y} \right)
\]

(6.6)

\[
N_y = A_{12} \frac{\partial u_1}{\partial x} + A_{22} \frac{\partial v_1}{\partial y} + A_{23} w_1 + B_{12} \frac{\partial u_1}{\partial x} + B_{22} \frac{\partial v_1}{\partial y} + B_{23} w_2 + \frac{1}{2} \left( C_{12} \frac{\partial u_1}{\partial x} + C_{22} \frac{\partial v_1}{\partial y} \right) + \frac{1}{6} \left( D_{12} \frac{\partial u_1}{\partial x} + D_{22} \frac{\partial v_1}{\partial y} \right)
\]

(6.7)

\[
N_{xy} = A_{61} \left( \frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right) + B_{66} \left( \frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right) + \frac{1}{2} \left( C_{61} \frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right)
\]

(6.8)

\[
M_x = B_{11} \frac{\partial u_2}{\partial x} + B_{12} \frac{\partial v_2}{\partial y} + B_{13} w_1 + C_{11} \frac{\partial u_2}{\partial x} + C_{12} \frac{\partial v_2}{\partial y} + C_{13} w_2 + \frac{1}{2} \left( D_{11} \frac{\partial u_2}{\partial x} + D_{12} \frac{\partial v_2}{\partial y} \right) + \frac{1}{6} \left( F_{11} \frac{\partial u_2}{\partial x} + F_{12} \frac{\partial v_2}{\partial y} \right)
\]

(6.9)
\[ M_{xy} = B_{12} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + C_{44} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{1}{2} \left( D_{11} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) + \frac{1}{6} \left( F_{12} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) \]  

(6.10)

\[ M_{x} = B_{12} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + B_{22} \left( \frac{\partial v}{\partial y} \right) + C_{11} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + C_{22} \left( \frac{\partial v}{\partial y} \right) + C_{23} W \]  

+ \frac{1}{2} \left( D_{11} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) + \frac{1}{6} \left( F_{11} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \]  

(6.11)

\[ Q_{x} = A_{11} \left( u_{L} + \frac{\partial w}{\partial x} \right) + B_{55} \left( u_{L} + \frac{\partial w}{\partial x} \right) + \frac{1}{2} C_{55} \left( u_{L} + \frac{\partial w}{\partial x} \right) \]  

(6.12)

\[ Q_{y} = A_{44} \left( v_{L} + \frac{\partial w}{\partial y} \right) + B_{44} \left( v_{L} + \frac{\partial w}{\partial y} \right) + \frac{1}{2} C_{44} \left( v_{L} + \frac{\partial w}{\partial y} \right) \]  

(6.13)

The constants \( A_{ij} \), \( B_{ij} \), \( C_{ij} \), \( D_{ij} \) and \( F_{ij} \) are defined by the following integrals for a continuously heterogeneous plate,

\[
\begin{bmatrix}
(A_{ij}, B_{ij}, C_{ij}) = \int_{z_l}^{z_h} K_{ij}(\lambda, \xi, \eta^2) \, d\xi \\
(D_{ij}, F_{ij}) = \int_{z_l}^{z_h} K_{ij}(\xi^2, \eta^3) \, d\xi
\end{bmatrix} \quad (\lambda, \xi = 1, 2, 3, 4, 5, 6) \]

(6.14)

For a \( n \)-layer laminated plate, the integration limits depend on the thickness of each layer and the constants equal to the summation of all layers.

A direct substitution of equation (6.6) - (6.13) into (2.14) and (2.15) yields the following set of equilibrium equations in terms of displacement components.
\[
(A_{11} \frac{\partial^2}{\partial x^2} + A_{46} \frac{\partial^2}{\partial y^2}) u_x + (A_{12} + A_{66}) \frac{\partial V_x}{\partial x} + (B_{11} \frac{\partial^2}{\partial x^2} + B_{46} \frac{\partial^2}{\partial y^2}) u_z + (B_{12} + B_{66}) \frac{\partial V_z}{\partial x} + \frac{1}{6}(D_{12} \frac{\partial^2}{\partial y^2}) u_x + \frac{1}{6}(D_{12} + D_{66}) \frac{\partial^2 V_z}{\partial x \partial y} = 0
\]

(6.15)

\[
(A_{12} + A_{66}) \frac{\partial^2 u_x}{\partial x \partial y} + (A_{46} \frac{\partial^2}{\partial x^2} + A_{12} \frac{\partial^2}{\partial y^2}) V_x + (B_{12} + B_{66}) \frac{\partial^2 u_z}{\partial x \partial y} + (B_{46} \frac{\partial^2}{\partial x^2} + B_{22} \frac{\partial^2}{\partial y^2}) V_z + D_{12} \frac{\partial^2 u_x}{\partial x \partial y} + \frac{1}{6}(D_{12} + D_{66}) \frac{\partial^2 V_z}{\partial x \partial y} = 0
\]

(6.16)

\[
(A_{11} \frac{\partial^2}{\partial x^2} + A_{46} \frac{\partial^2}{\partial y^2}) V_x + A_{55} \frac{\partial^2 u_x}{\partial x \partial y} + A_{44} \frac{\partial^2 u_y}{\partial y^2} + (B_{55} \frac{\partial^2}{\partial x^2} + B_{44} \frac{\partial^2}{\partial y^2}) V_z + B_{56} \frac{\partial^2 u_x}{\partial x \partial y} + B_{44} \frac{\partial^2 V_z}{\partial y^2}
\]

\[
+ \frac{1}{2} (C_{56} \frac{\partial^2}{\partial x^2} + C_{44} \frac{\partial^2}{\partial y^2}) V_x + \frac{1}{2} (C_{45} \frac{\partial^2}{\partial x} + C_{56} \frac{\partial^2}{\partial y^2}) + \rho = 0
\]

(6.17)

\[
(B_{11} \frac{\partial^2}{\partial x^2} + B_{46} \frac{\partial^2}{\partial y^2}) u_z + (B_{12} + B_{66}) \frac{\partial V_z}{\partial x} + (C_{11} \frac{\partial^2}{\partial x^2} + C_{46} \frac{\partial^2}{\partial y^2}) A_{55} \frac{\partial u_z}{\partial x} + (C_{45} \frac{\partial^2}{\partial x^2} + C_{56} \frac{\partial^2}{\partial y^2}) A_{12} \frac{\partial u_z}{\partial y} + C_{12} \frac{\partial V_z}{\partial y} + \frac{1}{2} (D_{12} \frac{\partial^2}{\partial y^2} + D_{66} \frac{\partial^2}{\partial y^2} - 2 B_{66}) u_z + \frac{1}{2} (D_{12} + D_{66}) \frac{\partial^2 V_z}{\partial x \partial y} + \frac{1}{2} (2 C_{15} - C_{55}) \frac{\partial V_z}{\partial x} + \frac{1}{6} (F_{12} \frac{\partial^2}{\partial x^2} + F_{46} \frac{\partial^2}{\partial y^2} - 3 C_{55}) u_z + \frac{1}{6} (F_{12} + F_{46}) \frac{\partial^2 V_z}{\partial x \partial y} = 0
\]

(6.18)
Equations (6.15) - (6.19) show that the extensional and flexural problems are generally coupled in a heterogeneous orthotropic plate. There are eleven unknown displacement components in the five equilibrium equations. These unknowns are to be determined simultaneously from eleven differential equations, namely, five equations (6.15) - (6.19) and six other equations from the results of satisfying the face boundary conditions at $z = h/2$. These six equations are to be derived in a later section.

6.3 Governing Equilibrium Equations of Symmetrically Heterogeneous Orthotropic Plate

Let the plate be made of a heterogeneous orthotropic material whose elastic properties are even functions of the $z$ coordinate (i.e., symmetric variation about the middle plane). According to the definition of equation (6.14) for such a plate

$$B_{ij} = D_{ij} = 0$$ (6.20)
The set of equilibrium equations (6.15) - (6.19) become uncoupled. The extensional system has two equations, namely,

\[
(A_{11} \frac{\partial^2}{\partial x^2} + A_{44} \frac{\partial^2}{\partial y^2}) u_0 + (A_{12} + A_{66}) \frac{\partial^2 v_0}{\partial x \partial y} + A_{11} \frac{\partial v_0}{\partial x} + \frac{1}{2} (C_{12} \frac{\partial^2}{\partial x^2} + C_{66} \frac{\partial^2}{\partial y^2}) u_2
\]

\[
+ \frac{1}{2} (C_{12} + C_{66}) \frac{\partial^2 v_0}{\partial x \partial y} = 0
\]  

\[
(\frac{\partial}{\partial x} + A_{66} \frac{\partial}{\partial y}) \frac{\partial^2 u_0}{\partial x \partial y} + (A_{44} \frac{\partial^2}{\partial x^2} + A_{22} \frac{\partial^2}{\partial y^2}) v_0 + A_{22} \frac{\partial v_0}{\partial y} + \frac{1}{2} (C_{12} + C_{66}) \frac{\partial^2 u_0}{\partial x \partial y}
\]

\[
+ \frac{1}{2} (C_{66} \frac{\partial^2}{\partial y^2} + C_{12} \frac{\partial^2}{\partial x \partial y}) v_2 = 0
\]  

and the flexural system has three equations, namely,

\[
(A_{11} \frac{\partial^2}{\partial x^2} + A_{44} \frac{\partial^2}{\partial y^2}) w_0 + A_{15} \frac{\partial^2 u_0}{\partial x \partial y} + A_{45} \frac{\partial^2 v_0}{\partial y \partial y} + \frac{1}{2} (C_{15} + C_{45}) \frac{\partial^2 w_0}{\partial x \partial y}
\]

\[
+ \frac{1}{2} (C_{15} \frac{\partial^2}{\partial y} + C_{45} \frac{\partial^2}{\partial y^2}) + p = 0
\]  

\[
-A_{55} \frac{\partial^2 w_0}{\partial y^2} + (C_{15} \frac{\partial^2}{\partial x^2} + C_{45} \frac{\partial^2}{\partial y^2} - A_{55}) u_1 + (C_{15} + C_{45}) \frac{\partial^2 v_0}{\partial x \partial y} + \frac{1}{2} (2C_{15} - C_{45}) \frac{\partial^2 w_0}{\partial x \partial y}
\]

\[
+ \frac{1}{6} (F_{11} \frac{\partial^2}{\partial x^2} + F_{44} \frac{\partial^2}{\partial y^2} - 3 C_{55}) u_3 + \frac{1}{6} (F_{15} + F_{45}) \frac{\partial^2 v_0}{\partial x \partial y} = 0
\]  

\[
-A_{44} \frac{\partial^2 w_0}{\partial y^2} + (C_{15} + C_{45}) \frac{\partial^2 u_1}{\partial x \partial y} + (C_{45} \frac{\partial^2}{\partial x^2} + C_{45} \frac{\partial^2}{\partial y^2} - A_{44}) v_1 + \frac{1}{2} (2C_{15} - C_{45}) \frac{\partial^2 w_0}{\partial y^2}
\]

\[
+ \frac{1}{6} (F_{15} + F_{45}) \frac{\partial^2 u_1}{\partial x \partial y} + \frac{1}{6} (F_{45} \frac{\partial^2}{\partial x^2} + F_{55} \frac{\partial^2}{\partial y^2} - 3 C_{44}) v_3 = 0
\]
It is interesting to note that the equations of equilibrium (6.21) - (6.22) and (6.23) - (6.25) for symmetrically heterogeneous orthotropic plates have the same form as equations (2.33) - (2.34) and (2.35) - (2.37) for the corresponding homogeneous plates. The difference is only in the constants of functional operators.

The six equations from the results of satisfying the face boundary conditions at $z = \pm h/2$ are also uncoupled. This will be seen later.

6.4 Governing Equilibrium Equations of Symmetrically Heterogeneous Isotropic Plate

In the case of isotropy and thickness heterogeneity which is symmetric about the middle plane (longitudinal properties are homogeneous), the constants of functional operators in equations (6.21) - (6.25) are further simplified. The elastic coefficients $K_{ij}$ in equation (6.3) have the following relations

\[
K_{12} = K_{13} = K_{23} = \frac{E \nu}{(1 + \nu)(1 - 2\nu)} = \lambda 
\]

(6.26)

\[
K_{44} = K_{55} = K_{66} = \frac{E}{2(1 + \nu)} = G
\]

(6.27)

\[
K_{11} = K_{22} = K_{33} = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)} = \lambda + 2G
\]

(6.28)

It should be noted that Young's modulus $E$ and Poisson's ratio $\nu$ (or $\lambda$ and $G$) are functions of the $z$ coordinate only.

Define the constants $A_{ij}$, $C_{ij}$, and $F_{ij}$ as follows

\[
A_{12} = A_{13} = A_{23} = \frac{E^h}{\int \lambda \, d\bar{z}} = \lambda
\]

(6.29)
\[ A_{44} = A_{65} = A_{66} = \int_{-l/2}^{l/2} G_1 \, d\bar{z} = G_1 \]
\[ A_{11} = A_{12} = A_{33} = \int_{-l/2}^{l/2} (\lambda + 2G_1) \, d\bar{z} = \lambda + 2G_1 \]
\[ C_{12} = C_{13} = C_{33} = \int_{-l/2}^{l/2} \lambda \, z^2 \, d\bar{z} = \lambda_1 \]
\[ C_{44} = C_{55} = C_{66} = \int_{-l/2}^{l/2} G_1 \, z^2 \, d\bar{z} = G_1 \]
\[ C_{11} = C_{22} = C_{55} = \int_{-l/2}^{l/2} (\lambda + 2G_1) \, z^2 \, d\bar{z} = \lambda_1 + 2G_1 \]
\[ F_{12} = \int_{-l/2}^{l/2} \lambda \, z^4 \, d\bar{z} = \lambda_2 \]
\[ F_{66} = \int_{-l/2}^{l/2} G_1 \, z^4 \, d\bar{z} = G_2 \]
\[ F_{11} = F_{22} = \int_{-l/2}^{l/2} (\lambda + 2G_1) \, z^4 \, d\bar{z} = \lambda_2 + 2G_2 \]

where \( \lambda_i \) and \( G_i \) (\( i = 0, 1, 2 \)) are constants.

The governing equilibrium equations (6.21) - (6.25) take the following forms after taking notice of (6.29) - (6.37)
\[ \begin{align*}
\left[ (\lambda_0 + 2G_0) \frac{\partial^2 u_0}{\partial x^2} + G_0 \frac{\partial^2 u_0}{\partial y^2} \right] u_0 + (\lambda_0 + G_0) \frac{\partial^2 v_0}{\partial x \partial y} + \lambda_0 \frac{\partial^2 w_0}{\partial y^2} + \frac{1}{2} \left[ (\lambda_1 + 2G_1) \frac{\partial^2 u_1}{\partial x^2} + G_1 \frac{\partial^2 u_1}{\partial y^2} \right] u_1 \\
+ \frac{1}{2} (\lambda_1 + G_1) \frac{\partial^2 v_0}{\partial x \partial y} = 0 
\end{align*} \]  

(6.38)

\[ \begin{align*}
(\lambda_0 + G_0) \frac{\partial^2 w_0}{\partial x^2} + \left[ G_0 \frac{\partial^2}{\partial x^2} + (\lambda_0 + 2G_0) \frac{\partial^2}{\partial y^2} \right] v_0 + \lambda_0 \frac{\partial^2 w_1}{\partial y^2} + \frac{1}{2} (\lambda_1 + G_1) \frac{\partial^2 u_1}{\partial x \partial y} \\
+ \frac{1}{2} \left[ G_1 \frac{\partial^2}{\partial x^2} + (\lambda_1 + 2G_1) \frac{\partial^2}{\partial y^2} \right] v_0 = 0 
\end{align*} \]  

(6.39)

\[ \begin{align*}
G_0 \left( \nabla^2 w_0 + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} \right) + \frac{G_1}{2} \left( \nabla^2 w_2 + \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} \right) + p = 0 
\end{align*} \]  

(6.40)

\[ \begin{align*}
-G_0 \frac{\partial^2 w_0}{\partial x^2} + \left[ (\lambda_1 + 2G_1) \frac{\partial^2}{\partial x^2} + G_1 \frac{\partial^2}{\partial y^2} - G_0 \right] u_1 + (\lambda_1 + G_1) \frac{\partial^2 v_0}{\partial x \partial y} + (\lambda_1 + G_1) \frac{\partial^2 v_1}{\partial x \partial y} \\
+ \frac{1}{6} \left[ (\lambda_1 + 2G_1) \frac{\partial^2}{\partial x^2} + G_1 \frac{\partial^2}{\partial y^2} - 3G_0 \right] u_3 + \frac{1}{6} (\lambda_1 + G_1) \frac{\partial^2 v_3}{\partial x \partial y} = 0 
\end{align*} \]  

(6.41)

\[ \begin{align*}
-G_0 \frac{\partial^2 w_0}{\partial y^2} + (\lambda_1 + G_1) \frac{\partial^2 u_1}{\partial x \partial y} + \left[ G_1 \frac{\partial^2}{\partial x^2} + (\lambda_1 + 2G_1) \frac{\partial^2}{\partial y^2} - G_0 \right] v_1 + (\lambda_1 + G_1) \frac{\partial^2 w_1}{\partial x \partial y} \\
+ \frac{1}{6} (\lambda_1 + G_1) \frac{\partial^2 u_3}{\partial x \partial y} + \frac{1}{6} \left[ G_1 \frac{\partial^2}{\partial x^2} + (\lambda_1 + 2G_1) \frac{\partial^2}{\partial y^2} - 3G_0 \right] v_3 = 0 
\end{align*} \]  

(6.42)

For homogeneous and isotropic plates, $\lambda$ and $G$ are constants,

hence

\[
\begin{align*}
\lambda_0 &= h \lambda \\
\lambda_1 &= h^3 \lambda / 12 \\
\lambda_2 &= h^5 \lambda / 80 \\
G_0 &= h^3 G \\
G_1 &= h^3 G / 12 \\
G_2 &= h^5 G / 80
\end{align*} \]  

(6.43)
Substitution of equation (6.43) into (6.38) - (6.42) yields the set of equations (2.33) - (2.37).

6.5 Boundary Conditions

Application of the face boundary conditions at \( z = \pm h/2 \), i.e., equations (6.2), yields a set of six differential equations. These six equations along with the equations of equilibrium constitute the system of governing differential equations for the plate problem.

For heterogeneous orthotropic plate, the following six differential equations are the results from satisfying the face boundary conditions at \( z = \pm h/2 \).

\[
\begin{align*}
K_{11}' \frac{\partial u_1}{\partial x} + K_{23}' \frac{\partial v_1}{\partial y} + \frac{A}{4} \left( K_{11} \frac{\partial u_1}{\partial x} + K_{23'} \frac{\partial v_1}{\partial y} \right) + K_{15}' \frac{\partial w_1}{\partial x} + \frac{A}{8} \left( K_{15} \frac{\partial u_2}{\partial x} + K_{25'} \frac{\partial v_2}{\partial y} \right) \\
+ \frac{A}{2} K_{15} \frac{\partial w_2}{\partial x} + \frac{A}{4} \left( K_{15} \frac{\partial u_2}{\partial x} + K_{25} \frac{\partial v_2}{\partial y} \right) &= P \\
K_{11}^- \frac{\partial u_1}{\partial x} + K_{23}^- \frac{\partial v_1}{\partial y} - \frac{A}{2} \left( K_{11}^- \frac{\partial u_1}{\partial x} + K_{23}^- \frac{\partial v_1}{\partial y} \right) + K_{15}^- \frac{\partial w_1}{\partial x} + \frac{A}{8} \left( K_{15}^- \frac{\partial u_2}{\partial x} + K_{25}^- \frac{\partial v_2}{\partial y} \right) \\
- \frac{A}{2} K_{15}^- \frac{\partial w_2}{\partial x} - \frac{A}{4} \left( K_{15}^- \frac{\partial u_2}{\partial x} + K_{25}^- \frac{\partial v_2}{\partial y} \right) &= 0 \\
\left( \frac{\partial w_1}{\partial x} + u_1 \right) + \frac{A}{8} \left( \frac{\partial w_1}{\partial x} + u_1 \right) &= 0 \\
\frac{\partial w_1}{\partial x} + u_1 &= 0
\end{align*}
\]
where $K_{ij}^+$ and $K_{ij}^-$ indicate the elastic coefficients at the top and bottom surfaces of the plate. It can be seen from equations (6.44) - (6.49) that all displacement components are coupled. From equations (6.15) - (6.19) and (6.44) - (6.49) the eleven displacement components can be determined simultaneously.

If the elastic coefficients of the material are symmetric about the middle plane of the plate, then $K_{ij}^+ = K_{ij}^-$. Two systems of differential equations are thus obtained.

$$
\left( \frac{2w_1}{2y} + v_1 \right) + \frac{K_{11}^+}{8} \left( \frac{2w_1}{2y} + v_3 \right) = 0 \quad (6.48)
$$

$$
\frac{2w_1}{2y} + v_2 = 0 \quad (6.49)
$$

$K_{11}^+ \frac{2u_1}{2x} + K_{12}^+ \frac{2v_1}{2y} + K_{13}^+ w_1 + \frac{K_{11}^+}{8} \left( K_{11}^+ \frac{2u_1}{2x} + K_{23}^+ \frac{2v_3}{2y} \right) = \frac{P}{2} \quad (6.50)
$$

$$
\frac{2w_1}{2x} + u_2 = 0 \quad (6.51)
$$

$$
\frac{2w_1}{2y} + v_2 = 0 \quad (6.52)
$$

$$
K_{11}^+ \frac{2u_1}{2x} + K_{12}^+ \frac{2v_1}{2y} + K_{13}^+ w_2 + \frac{K_{11}^+}{24} \left( K_{11}^+ \frac{2u_3}{2x} + K_{23}^+ \frac{2v_3}{2y} \right) = \frac{P}{h} \quad (6.53)
$$

$$
\left( \frac{2w_1}{2x} + u_1 \right) + \frac{K_{11}^+}{8} \left( \frac{2w_1}{2x} + u_3 \right) = 0 \quad (6.54)
$$

$$
\left( \frac{2w_1}{2y} + v_1 \right) + \frac{K_{11}^+}{8} \left( \frac{2w_1}{2y} + v_3 \right) = 0 \quad (6.55)
$$
For symmetric isotropic material, we have

\[ K_{13}^+ = K_{23}^+ = \lambda^+ \]  

(6.56)

\[ K_{33}^+ = \lambda^+ + 2G^+ \]

Substituting equations (6.56) into (6.50) - (6.55) we then get

\[ \lambda^+ \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) + \left( \lambda^+ + 2G^+ \right) w_t + \frac{4}{B} \lambda^+ \left( \frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 v_y}{\partial y \partial x} \right) = \frac{P}{h} \]  

(6.57)

\[ \frac{\partial^2 w_x}{\partial x} + u_t = 0 \]  

(6.58)

\[ \frac{\partial^2 w_y}{\partial y} + v_t = 0 \]  

(6.59)

\[ \lambda^+ \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) + \left( \lambda^+ + 2G^+ \right) w_t + \frac{1}{24} \lambda^+ \left( \frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 v_y}{\partial y \partial x} \right) = \frac{P}{h} \]  

(6.60)

\[ \left( \frac{\partial^2 w_x}{\partial x} + u_t \right) + \frac{4}{B} \left( \frac{\partial w_x}{\partial x} + u_t \right) = 0 \]  

(6.61)

\[ \left( \frac{\partial^2 w_y}{\partial y} + v_t \right) + \frac{4}{B} \left( \frac{\partial w_y}{\partial y} + v_t \right) = 0 \]  

(6.62)

Equations (6.57) - (6.62) reduce to (2.40) and (2.41) for the homogeneous isotropic plate if \( \lambda^+ \) and \( G^+ \) are replaced by \( \lambda \) and \( G \), respectively.

The edge boundary conditions are the same as in the case of homogeneous isotropic plate. Five boundary conditions must be prescribed at
each edge. For simply supported, clamped, and free edge boundary conditions, see equations (2.47) - (2.49).

6.6 Some Closed Form Solutions for Coupled and Uncoupled Heterogeneous Rectangular Plates Having Simply Supported Edges

As pointed out in the previous sections, the important difference between a heterogeneous plate and a homogeneous plate is the existence of coupling between flexure and extension. In order to ascertain the behavior of a coupled heterogeneous plate, we seek solutions to some simple boundary value problems. In particular we choose rectangular orthotropic plates, with thickness heterogeneity, for which closed form solutions can be obtained in the case of special, simple support boundary conditions.

Solutions to the uncoupled heterogeneous plates are similar to that of homogeneous plates. In order to see the difference, we solve a symmetrically heterogeneous isotropic plate subjected to a uniformly distributed load.

6.6.1 Closed Form Solution for a Heterogeneous Orthotropic Rectangular Plate Having Simply Supported Edges

Consider a rectangular simply supported heterogeneous orthotropic plate of dimensions $a$ and $b$ under a sinusoidally distributed load (see fig. 3.2)

$$p(x,y) = p_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{b}$$

(6.63)

where $p_0$ is a constant.
The edge boundary conditions to be satisfied by the present theory are

at $x=\pm a/2$ : $v_0 = w_0 = v_1 = N_x = N_y = 0$ ;

(6.64)

at $y=\pm b/2$ : $u_0 = w_0 = u_1 = N_y = N_y = 0$.

(6.65)

The system of governing differential equations is (6.21) - (6.25) and (6.44) - (6.49) where flexure and extension are coupled.

Assume

$u_0 = \alpha_0 \sin \frac{\pi x}{a} \cos \frac{\pi y}{b}$

$\nu_0 = \alpha_1 \cos \frac{\pi x}{a} \sin \frac{\pi y}{b}$

$w_0 = \alpha_2 \cos \frac{\pi x}{a} \cos \frac{\pi y}{b}$

$u_1 = \alpha_3 \sin \frac{\pi x}{a} \cos \frac{\pi y}{b}$

$v_1 = \alpha_4 \cos \frac{\pi x}{a} \sin \frac{\pi y}{b}$

$w_1 = \alpha_5 \cos \frac{\pi x}{a} \cos \frac{\pi y}{b}$

(6.66)

$u_2 = \alpha_6 \sin \frac{\pi x}{a} \cos \frac{\pi y}{b}$

$v_2 = \alpha_7 \cos \frac{\pi x}{a} \sin \frac{\pi y}{b}$

$w_2 = \alpha_8 \cos \frac{\pi x}{a} \cos \frac{\pi y}{b}$

$u_3 = \alpha_9 \sin \frac{\pi x}{a} \cos \frac{\pi y}{b}$

$v_3 = \alpha_{10} \cos \frac{\pi x}{a} \sin \frac{\pi y}{b}$

where $\alpha_i$ are constants. Equations (6.66) satisfy all boundary conditions.
in (6.64) and (6.65). Substituting equations (6.66) into (6.21) - (6.25) and (6.44) - (6.49), we get the following system of equations

\[
\begin{align*}
\left( \frac{A_{11} + A_{12}}{a^3} \right) \varphi_0 + \frac{A_{13} + A_{14}}{a^3} \varphi_1 + \frac{A_{15}}{a^3} \varphi_2 + \frac{1}{2} \left( \frac{C_{11}}{a^3} + \frac{C_{12}}{b^3} \right) \varphi_3 + \frac{C_{15} + C_{24}}{2ab} \varphi_4 &= 0 \\
\frac{A_{21}}{ab} \varphi_0 + \left( \frac{A_{22} + A_{23}}{b^3} \right) \varphi_1 + \frac{A_{24}}{b^3} \varphi_2 + \frac{1}{2} \left( \frac{C_{22}}{a^3} + \frac{C_{23}}{b^3} \right) \varphi_3 &= 0 \\
\left( \frac{A_{31} + A_{32}}{a^3} \right) \varphi_0 - \frac{A_{33}}{a^3} \varphi_2 - \frac{A_{34}}{b^3} \varphi_3 + \frac{1}{2} \left( \frac{C_{33}}{a^3} + \frac{C_{34}}{b^3} \right) \varphi_4 - \frac{C_{35}}{2ab} \varphi_5 &= 0 \\
\frac{C_{45}}{2b^3} \varphi_0 &= \frac{p_2}{\pi^2} \\
\frac{A_{55}}{a^3} \varphi_0 - \left( \frac{C_{55}}{a^3} + \frac{C_{56}}{b^3} \right) \varphi_2 - \frac{A_{54}}{b^3} \varphi_3 - \frac{2C_{55} - C_{56}}{2ab} \varphi_4 - \frac{\frac{1}{2} \left( \frac{A_{61} + A_{62}}{a^3} \right) + \frac{A_{63}}{b^3} + \frac{3C_{66}}{a^3}}{6ab} \varphi_5 &= 0 \\
\frac{A_{64}}{b^3} \varphi_2 - \frac{C_{64}}{a^3} \varphi_3 - \left( \frac{C_{66}}{a^3} + \frac{C_{67}}{b^3} + \frac{A_{67}}{a^3} \right) \varphi_4 - \frac{2C_{66} - C_{67}}{2ab} \varphi_5 &= 0 \\
\frac{A_{65}}{a^3} \varphi_2 - \frac{C_{65}}{a^3} \varphi_3 - \left( \frac{C_{66}}{a^3} + \frac{C_{67}}{b^3} + \frac{A_{67}}{a^3} \right) \varphi_4 - \frac{2C_{66} - C_{67}}{2ab} \varphi_5 &= 0 \\
\frac{C_{67}}{6ab} \varphi_2 - \frac{1}{a^3} \left( \frac{A_{61} + A_{62}}{a^3} \right) \varphi_3 - \frac{3C_{67}}{b^3} \varphi_5 &= 0
\end{align*}
\]
The constants \( \alpha_0 \ldots \alpha_{10} \) can be determined by solving the eleven simultaneous algebraic equations (6.67) - (6.77). The stress and moment resultants can be obtained by direct substitution of equations (6.66) into (6.6) - (6.13).

In this case also, for any other transverse loading, solutions can be obtained by a Fourier series analysis.
6.6.2 Closed Form Solution for a Symmetrically Heterogeneous: Isotropic Rectangular Plate Having Simply Supported Edges

Flexure and extension are uncoupled in the case of a symmetrically heterogeneous isotropic plate. The system of differential equations for extensional problem are equations (6.38), (6.39) and (6.57) - (6.59) involving displacement components \( u_0, v_0, w_1, u_2, \) and \( v_2 \). For flexural problem, we have equations (6.40) - (6.42) and (6.60) - (6.62) involving displacement components \( w_0, u_1, v_1, w_2, u_3, \) and \( v_3 \).

Assume a rectangular plate of dimensions \( a \) and \( b \) under a uniformly distributed load \( P \) on the top surface (see fig. 3.1). The problem of extension and flexure can be solved separately following the same procedure for the homogeneous isotropic plate problem.

We consider the extensional problem first. Assuming \( \nabla^2 w_1 = 0 \), then we get

\[
\begin{align*}
  u_1 &= -\frac{2w_1}{a x} \\
  v_1 &= -\frac{2w_1}{b y} \\
  w_1 &= \frac{1}{\lambda^2 + 2\alpha^2} \left[ -\lambda^2 \left( \frac{2u_1}{a^2} + \frac{2v_1}{b^2} \right) + \frac{P}{2} \right]
\end{align*}
\]

A direct substitution of equations (6.78) - (6.80) into (6.38) and (6.39) yields

\[
\left[ \left( \lambda^2 + 2\alpha^2 \right) - \frac{\lambda^2 + 2\alpha^2}{\lambda^2 + 2\alpha^2} \right] \frac{2}{a^2} + \frac{2}{b^2} \right] u_0 + \left( \lambda^2 + 2\alpha^2 \right) \frac{2v_0}{\alpha^2} = 0
\]
A solution satisfying equations (6.81), (6.82) and also the boundary conditions at edges \(x=0, a\) (i.e., \(v_0 = N_x = 0\)) is

\[
U_0 = P \left[ \sum_{m=1}^{\infty} Y_1(y) \cos \kappa_m x - \frac{\kappa_m}{4\left(\lambda_0^* \gamma + \lambda_0^* \gamma + 2 \lambda_0 \gamma^*\right)} (x - \frac{a}{2}) \right] \quad \text{(6.83)}
\]

\[
V_0 = P \sum_{m=1}^{\infty} Y_2(y) \sin \kappa_m x \quad \text{(6.84)}
\]

where, as before, \(\kappa_m = \frac{m\pi}{a}\), \(Y_1(y)\) and \(Y_2(y)\) take the following forms

\[
Y_1(y) = \left( A_1 \cosh \kappa_m y + A_2 \sinh \kappa_m y \right) + \frac{\lambda_0^* \gamma + \lambda_0^* \gamma + 2 \lambda_0 \gamma^*}{\lambda_0^* \gamma + 2 \lambda_0 \gamma^* + 2 \lambda_0 \gamma^*} \left( A_2 \sinh \kappa_m y + A_4 \cosh \kappa_m y \right) \quad \text{(6.85)}
\]

\[
Y_2(y) = \left( A_1 \sinh \kappa_m y + A_3 \cosh \kappa_m y \right) + \frac{\lambda_0^* \gamma + \lambda_0^* \gamma + 2 \lambda_0 \gamma^*}{\lambda_0^* \gamma + 2 \lambda_0 \gamma^* + 2 \lambda_0 \gamma^*} \left( A_2 \sinh \kappa_m y + A_4 \cosh \kappa_m y \right) \quad \text{(6.86)}
\]

The constants \(A_1 \ldots A_4\) are to be determined from the boundary conditions at the edges \(y= \pm b/2\), namely, \(u_0 = N_y = 0\). Hence, the displacements components \(u_0, v_0, u_2, v_2\) and \(w_1\) are completely determined. All other related quantities can be obtained accordingly. The assumption \(\sqrt{\kappa} = 0\) can be proved directly from equation (6.80).

As can be seen from equations (6.83) - (6.86), the general form of solution for the symmetrically heterogeneous isotropic plate problem is the same as the homogeneous isotropic plate problem. The only difference is those constants which involve elastic coefficients.
Next, we consider the flexural problem. Again, from the assumption $\nabla^2 w_2 = \frac{P(S_1 + S_2)}{h^2}$ ($S_1$ and $S_2$ are constants) we obtain

$$u_3 = -\frac{2w_1}{2x} - \frac{6}{A_1} (u_1 + \frac{2w_2}{2x}) \quad (6.87)$$

$$v_3 = -\frac{2w_1}{2y} - \frac{6}{A_1} (v_1 + \frac{2w_2}{2y}) \quad (6.88)$$

$$w_3 = \frac{1}{\lambda^* + 2g^*} \left[ \frac{\lambda^*}{24} P(S_1 + S_2) - \frac{2\lambda^*}{3} (\frac{2u_1}{2x} + \frac{2v_1}{2y}) + \frac{\lambda^*}{3} \nabla^2 w_3 + \frac{P}{A_1} \right] \quad (6.89)$$

The three equilibrium equations in terms of $w_0$, $u_1$, and $v_1$ are

$$\frac{2w_1}{2x} + \frac{2v_1}{2y} + \nabla^2 w_3 = -\frac{P}{L_1} \quad (6.90)$$

$$(-L_1 + L_2 \frac{2u_1}{2y} + L_3 \frac{2v_1}{2y}) u_1 + L_4 \frac{2v_1}{2y} + (-L_1 \frac{2u_1}{2x} + L_3 \frac{2v_1}{2x}) \nabla^2 w_3 = 0 \quad (6.91)$$

$$L_4 \frac{2u_1}{2x} + (-L_1 + L_2 \frac{2u_1}{2y} + L_3 \frac{2v_1}{2y}) v_1 + (-L_1 \frac{2v_1}{2y} + L_3 \frac{2v_1}{2y}) \nabla^2 w_3 = 0 \quad (6.92)$$

where the constants $L_1$ are

$$L_1 = G_{10} - \frac{4}{3A^*} G_{11}$$

$$L_2 = (\lambda_2 + 2G_{11}) - \frac{2\lambda_1 \lambda^*}{3(\lambda^* + 2G_{11})} - \frac{4}{3A^*} (\lambda_1 + 2G_{12})$$

$$L_3 = G_{11} - \frac{4}{3A^*} G_{12} \quad (6.93)$$

$$L_4 = (\lambda_2 + G_{11}) - \frac{2\lambda_1 \lambda^*}{3(\lambda^* + 2G_{11})} - \frac{4}{3A^*} (\lambda_1 + G_{12})$$

$$L_5 = \frac{\lambda_1 \lambda^*}{3(\lambda^* + 2G_{11})} - \frac{4}{3A^*} (\lambda_1 + 2G_{12})$$
A solution satisfying the equilibrium equations and also the boundary conditions at the edges $x=0,a$ (i.e., $w_0 = y_1 = H_x = 0$) is

$$u_i = P \left[ (a-2x) S_i + \sum_{m=1}^{\infty} Y_0(y) \cos \alpha_m x \right] \quad (6.94)$$

$$v_i = P \sum_{m=1}^{\infty} Y_0(y) \sin \alpha_m x \quad (6.95)$$

$$w_i = P \left[ \left( -\frac{1}{2} L_1 - S_i \right) (a-x) + \sum_{m=1}^{\infty} Y_0(y) \sin \alpha_m x \right] \quad (6.96)$$

where

$$Y_3(y) = I_m \left( B_1 \sinh I_m y + B_2 \cosh I_m y \right) + \left( B_3 \cosh \alpha_m y + B_5 \sinh \alpha_m y \right)$$

$$+ y \left( B_4 \cosh \alpha_m y + B_6 \sinh \alpha_m y \right) - \frac{4}{a} \cdot \frac{1}{L_1 - L_5} \cdot \frac{1}{\alpha_m^2} \quad (6.97)$$

$$Y_4(y) = \alpha_m \left( B_1 \cosh I_m y + B_2 \sinh I_m y \right) + \left( B_3 \sinh \alpha_m y + B_5 \cosh \alpha_m y \right)$$

$$+ \frac{1}{\alpha_m} \left( B_4 \cosh \alpha_m y + B_6 \sinh \alpha_m y \right) + y \left( B_7 \sinh \alpha_m y + B_8 \cosh \alpha_m y \right) \quad (6.98)$$

$$Y_5(y) = -\frac{1}{\alpha_m} \left( B_7 \cosh \alpha_m y + B_8 \sinh \alpha_m y \right) + \frac{2(L_1 - L_5)}{L_1} \left( B_9 \sinh \alpha_m y + B_{10} \cosh \alpha_m y \right)$$

$$- \frac{y}{\alpha_m} \left( B_4 \cosh \alpha_m y + B_6 \sinh \alpha_m y \right) + \frac{4}{a} \cdot \frac{1}{L_1 - L_5} \cdot \frac{1}{\alpha_m^2} \quad (6.99)$$

$$S_i = \frac{1}{12} \cdot \frac{L_1^2}{\lambda_x + 2 \lambda_1} \cdot \left( \frac{\lambda_x + 2 \lambda_1}{L_1} \right) - \frac{1}{24} \cdot \frac{\lambda_x + 2 \lambda_1}{L_1} - \frac{2 \lambda_x + 2 \lambda_1 - \frac{\lambda^2}{L_1}}{L_1} - \frac{\lambda^2 / \lambda_x}{L_1} - \frac{\lambda_1}{L_1} - (\lambda_x + 2 \lambda_1) \quad (6.100)$$
The constants $B_1 \ldots B_6$ are to be determined from the six boundary conditions at the edges $y=\pm b/2$.

It is noted that equations (6.97) - (6.101) reduce to (3.35), (3.42), and (3.43) if the plate is homogeneous isotropic.
CHAPTER VII
SUMMARY AND RECOMMENDATIONS FOR FUTURE WORK

A linear theory based on the power series expansion of displacement components with respect to the plate thickness has been developed for the equilibrium of homogeneous and heterogeneous thick elastic plates. The effects of transverse shear and normal stresses have been included in the analysis. The system of governing differential equations, directly derived from the three-dimensional elasticity theory, was established both in the Cartesian and polar coordinates. In the case of homogeneous isotropic plates, two systems of differential equations, one fourth order and one sixth order, governing the extensional and flexural problems respectively, were obtained. On the other hand, in the case of heterogeneous orthotropic plates a coupled differential equation of tenth order was resulted. Extensive numerical calculations and comparisons with the Reissner, Donnell-Lee thick plate theories and Love's exact solution of a moderately thick circular plate indicated a good agreement in certain boundary value problems.

Because of the high power displacements assumption in the present theory, the extension to other areas of interest is rather limited. However, the following several problems are worth investigating:

1. Stress concentration problem. The bending problem of an infinite plate with a circular hole first solved by Goodier (ref. 34) is based on the classical thin plate theory. It was
found that within this theory the stress concentration factor did not depend on the plate thickness and the size of the hole.

Solution of the problem based on Reissner's plate theory was obtained by Reissner (ref. 3). In his theory the stress concentration factor was found to be a function of the ratio of plate thickness to hole diameter. Application of the present theory to the similar problem is suggested.

2. Thermal stress problem. Thermal stresses in thin plates are generally considered to be well established (refs. 35, 36, and 37), on the other hand investigations pertaining to thick plates appear to be very limited. A three-dimensional series solution for elastic plates subjected to general temperature distribution was obtained by Lee (ref. 38). It is possible to obtain a solution by using the present theory if the temperature distribution is assumed to be expressible in the form

\[ T(x,y,z) = \sum_{k=0}^{n} z^k T_k(x,y) \]

3. Free vibration and buckling problems. For a rectangular thick plate having simply supported edges, a solution based on the present displacement assumption to the free vibration and buckling problems is possible. The method used by Srinivas and Rao (ref. 18) offers a good possibility.
APPENDIX A

REISSNER'S THEORY ON THE BENDING OF
A THICK ELASTIC PLATE

In the article (ref. 2) Reissner considered a plate of constant thickness h, acted upon by a normal force of variable intensity \( p(x,y) \) on the top surface of the plate. It is assumed that body force is absent.

The law of the stress distribution through the plate thickness is given by the equalities

\[
\sigma_x = \frac{M_x}{h/6} \cdot \frac{3}{h/2} \quad \text{(A.1)}
\]

\[
\sigma_y = \frac{M_y}{h^3/6} \cdot \frac{3}{h/2} \quad \text{(A.2)}
\]

\[
\tau_{xy} = \frac{M_{xy}}{h^3/6} \cdot \frac{3}{h/2} \quad \text{(A.3)}
\]

The law of the distribution of the remaining stresses is determined from the equilibrium equations of the three-dimensional problem in the theory of elasticity,

\[
\tau_{x\gamma} = \frac{Q_x}{2h/3} \left[ 1 - \left( \frac{3}{h/2} \right)^2 \right] \quad \text{(A.4)}
\]

\[
\tau_{y\gamma} = \frac{Q_y}{2h/3} \left[ 1 - \left( \frac{3}{h/2} \right)^2 \right] \quad \text{(A.5)}
\]

\[
\sigma_\gamma = \frac{3}{4} \cdot p \left[ \frac{3}{h/2} - \frac{1}{3} \left( \frac{3}{h/2} \right)^3 + \frac{2}{3} \right] \quad \text{(A.6)}
\]
The shear stress resultants and the intensity of the normal loading \( p \) are related by the equation

\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -p
\]  
(A.9)

The total energy of the plate having displacement boundary conditions is

\[
\Pi = \frac{1}{2E} \iint \left[ \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2V (\sigma_x u_x + \sigma_y u_y + \sigma_z u_z) + 2(1+\nu) (\tau_{xy}^2 + \tau_{xy}^2) + \tau_{yy}^2 \right] dx dy dz - \iint (\sigma_n u_n + \tau_{ns} u_s + \tau_{ny} w) dS dy
\]  
(A.10)

where \( u_n \) and \( u_s \) are displacement components parallel to the plane of the plate in normal and tangential direction, and \( w \) is the displacement component normal to the plane of the plate.

By virtue of equations (A.1) - (A.6) and after carrying out the integration with respect to \( z \), the total energy expression (A.10) is brought to the form

\[
\Pi = \frac{1}{2E} \iint \left\{ \frac{12}{h^3} \left[ (M_x + M_y)^2 + 2(1+\nu)(M_x^2 - M_x M_y) + \frac{3}{5}(1+\nu)(Q_x^2 + Q_y^2) \right] 
- \frac{M_x}{h} N_p (M_x + M_y) + \int_{-h/2}^{h/2} \sigma_z^2 dy \right\} dx dy 
- \int \frac{M_z}{h^2} \int_{-h/2}^{h/2} u_n \frac{3}{h^2} dS
+ \frac{M_{nx}}{h^1/2} \int_{-h/2}^{h/2} u_s \frac{3}{h/2} dS
+ \frac{Q_n}{2h/3} \int_{-h/2}^{h/2} w [1 - (\frac{3}{h/2})^2] dS
\]  
(A.11)
Furthermore, Reissner assumed the displacement components $u_n$, $u_s$ and $w$ as below:

\[ u_n = \overline{u}_n(s) \cdot \gamma \] \hspace{1cm} (A.12)

\[ u_s = \overline{u}_s(s) \cdot \gamma \] \hspace{1cm} (A.13)

\[ w = \overline{w}(s) \] \hspace{1cm} (A.14)

Castigliano's theorem of least work is then applied to $\pi$. The variation of $\pi$ according to equation (A.11) is made equal to zero in such a way that the equilibrium equation (A.9) remains satisfied. Using equations (A.12)-(A.14) and a Lagrange multiplier $\lambda$ then we have

\[
\delta \left\{ \frac{b}{E_h^3} \int \int \left[ (M_x + M_y)^2 + 2(1+\nu)(M_x^2 - M_x M_y) + \frac{k^2}{5} (1+\nu) (Q_x^2 + Q_y^2) \right.ight.
\]
\[ - \frac{k^2}{5} \nu \lambda (M_x + M_y) \bigg] dx dy - \int (M_n \overline{u}_n + M_n \overline{u}_s + Q_n \overline{w}) ds 
\]
\[ + \int \int \lambda(x,y) \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \rho \right) dx dy \right\} = 0 \hspace{1cm} (A.15)
\]

Carrign out the variations of equation (A.15), we then get

\[
\int \int \left\{ \frac{12}{E_h^3} \left( M_x - \nu M_y - \frac{k^2}{5} (1+\nu) \frac{\partial Q_x}{\partial x} - \frac{k^2}{10} \nu \lambda \right) \delta M_x \right. \
\]
\[ + \left. \frac{12}{E_h^3} (M_y - \nu M_x - \frac{k^2}{5} (1+\nu) \frac{\partial Q_y}{\partial y} - \frac{k^2}{10} \nu \lambda \right) \delta M_y \right. \
\]
\[ + \left. \frac{12}{E_h^3} (2(1+\nu) M_{xy} - \frac{k^2}{5} (1+\nu) \frac{\partial Q_y}{\partial x} + \frac{\partial Q_x}{\partial y}) + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \delta M_{xy} \right\} dx dy 
\]
\[ - \int \left\{ \overline{u}_n + \frac{\partial w}{\partial n} - \frac{12(1+\nu)}{5E_h^3} Q_n \right\} \delta M_n + \left[ \overline{u}_s + \frac{\partial w}{\partial s} - \frac{12(1+\nu)}{5E_h^3} Q_s \right] \delta M_s \right. \
\]
\[ - (\overline{w} - w) \delta Q_n \bigg] ds = 0 \hspace{1cm} (A.16)\]
Equation (A.16) yields the differential equations of Reissner's theory and the natural boundary conditions.

The corresponding three boundary conditions to be prescribed along the edge of the plate are

\[
M_n = \overline{M}_n \quad \text{or} \quad \frac{2w}{2n} - \frac{12(1+\nu)}{5Eh} Q_n = -\overline{U}_n', \quad (A.17)
\]

\[
M_{ns} = \overline{M}_{ns} \quad \text{or} \quad \frac{2w}{\partial s} - \frac{12(1+\nu)}{5Eh} Q_s = -\overline{U}_s', \quad (A.18)
\]

\[
Q_n = \overline{Q}_n \quad \text{or} \quad W = \overline{w}, \quad (A.19)
\]

The double integral in equation (A.16) is equivalent to three differential equations. They are, if the first two are solved for \( M_x \) and \( M_y \)

\[
M_x = \frac{h^2}{5(1-\nu)} \left( \frac{\partial Q_y}{\partial x} + \nu \frac{\partial Q_x}{\partial y} \right) - \frac{\nu h^2}{10(1-\nu)} \rho = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (A.20)
\]

\[
M_y = \frac{h^2}{5(1-\nu)} \left( \frac{\partial Q_x}{\partial y} + \nu \frac{\partial Q_y}{\partial x} \right) - \frac{\nu h^2}{10(1-\nu)} \rho = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (A.21)
\]

\[
M_{xy} = \frac{h^2}{10} \left( \frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} \right) = -(1-\nu) D \frac{\partial^2 w}{\partial x \partial y} \quad (A.22)
\]

By means of equation (A.9), (A.20) - (A.22) are changed into

\[
M_x = \frac{h^2}{5} \left( \frac{\partial Q_y}{\partial x} + \frac{\nu h^2}{10(1-\nu)} \right) \rho = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (A.23)
\]

\[
M_y = \frac{h^2}{5} \left( \frac{\partial Q_x}{\partial y} + \frac{\nu h^2}{10(1-\nu)} \right) \rho = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (A.24)
\]
Substituting of these expressions in equations (A.7) and (A.8) yields, if one observes (A.9), the result

\[ M_{xy} = \frac{E}{10} \left( 2 \frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} \right) = - (1 - \nu) D \frac{\partial^5 w}{\partial x \partial y^4} \]  

(A.25)

Equations (A.9), (A.26), and (A.27) may be solved simultaneously for \( Q_x, Q_y, \) and \( w \). By differentiating equation (A.26) with respect to \( x \) and (A.27) with respect to \( y \), adding and using (A.9) gives

\[ D \nabla^4 W = \rho - \frac{E}{10} \frac{2 - \nu}{1 - \nu} \nabla^2 \rho \]  

(A.28)

It is readily seen that particular integrals to equations (A.9), (A.26), and (A.27) are given by

\[ Q_{x1} = - D \frac{\partial}{\partial x} \nabla^2 W - \frac{E}{10} \frac{2 - \nu}{1 - \nu} \frac{\partial^2 \rho}{\partial x^2} \]  

(A.29)

\[ Q_{y1} = - D \frac{\partial}{\partial y} \nabla^2 W - \frac{E}{10} \frac{2 - \nu}{1 - \nu} \frac{\partial^2 \rho}{\partial y^2} \]  

(A.30)

and with

\[ Q_x = Q_{x1} + Q_{x2} \]  

(A.31)

\[ Q_y = Q_{y1} + Q_{y2} \]  

(A.32)

then \( Q_{x2} \) and \( Q_{y2} \) must satisfy the homogeneous system of equations. The homogeneous equations will be satisfied by
where the stress function \( \phi \) is governed by the equation

\[
\nabla^2 \phi - \frac{10}{h^4} \phi = 0
\]

Having established two differential equations, one of which, equation (A.28), is of the fourth order and the other, equation (A.35), of the second order, we now are able to satisfy three conditions, instead of only two, on the edge of the plate.

For a rectangular plate having simply supported edges subjected to a uniformly distributed load \( P \) (see fig. 3.1) the results for \((Q_x)_{x=0}, (M_x)_{\text{max}}, (w_0)_{\text{max}}, \) and \((\sigma_x)_{\text{max}}\) are

\[
(Q_x)_{x=0} = \frac{4Pa^2}{\pi^2} \sum_{m=1,3} \frac{1}{m^2} \left( 1 - \frac{\cosh m \pi b/a}{\cosh \mu} \right)
\]

\[
(M_x)_{\text{max}} = \frac{Pa^2}{8} \left[ 1 - \frac{3}{2} \sum_{m=1,3} (-1)^{(m-1)/2} \frac{4 + (1 - \nu) m \pi (b/a) \tanh \mu}{m^3 \cosh \mu} \right]
\]

\[
+ \frac{2^\nu}{5\pi} \frac{Pa^2}{D} \left[ \sum_{m=1,3} (-1)^{(m-1)/2} \frac{1}{m \cosh \mu} \right] \left( \frac{a}{l} \right)^2
\]

\[
(w_0)_{\text{max}} = \frac{4}{\pi^2} \frac{Pa^4}{D} \sum_{m=1,3} (-1)^{(m-1)/2} \frac{1}{m^2} \left[ 1 - (1 + \frac{h_m}{4} \cdot \frac{b}{a} \cdot \tanh \mu)/\cosh \mu \right]
\]

\[
+ \frac{4}{\pi^2} \frac{Pa^4}{D} \sum_{m=1,3} (-1)^{(m-1)/2} \frac{1}{m^3} \left( 1 - \text{sech} \mu \right) \left( \frac{a}{l} \right)^2
\]
\[ (Q_x)_{\text{max}} = \frac{3}{4} \frac{P_0 a^5}{h^5} \left[ 1 - \frac{3}{\pi^2} \sum_{m=1,3}^{\infty} (-1)^{(m-1)/2} \left( 4 + (1-\nu) \frac{b}{a} \frac{1}{\tanh \mu} \right) / m^2 \cosh \mu \right] \]

\[ + \frac{12\nu}{5\pi} \frac{P_0 a^5}{h^5} \sum_{m=1,3}^{\infty} (-1)^{(m-1)/2} \frac{1}{m \cosh \mu} \left( \frac{D}{a^2} \right)^2 \]  

(A.39)

where \( \alpha = \frac{mv}{a} \), \( \mu = \frac{mv}{2} \frac{b}{a} \) and \( m=1,3,5, \ldots \) only.

For a square plate having simply supported edges subjected to a sinusoidally distributed load \( p=p_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} \) (see fig. 3.2) the corresponding quantities are

\[ (Q_x)_x = \frac{a_x}{c} = \frac{P_0 a}{2\pi} \cos \frac{\pi y}{a} \]  

(A.40)

\[ (M_x)_{\text{max}} = \frac{1+\nu}{4\pi^2} \frac{P_0 a^5}{h^5} \left( 1 + \frac{\pi^2}{5} \frac{2-\nu}{1+\nu} \frac{a_x}{a^2} \right) \]  

(A.41)

\[ (W_x)_{\text{max}} = \frac{P_0 a^5}{4\pi^2 D} \left( 1 + \frac{\pi^2}{5} \frac{2-\nu}{1+\nu} \frac{a_x}{a^2} \right) \]  

(A.42)

\[ (\sigma_x)_{\text{max}} = \frac{3(1+\nu)}{2\pi^2} \frac{P_0 a^5}{h^5} \left( 1 + \frac{\pi^2}{5} \frac{2-\nu}{1+\nu} \frac{a_x}{a^2} \right) \]  

(A.43)
Donnell and Lee developed a theory (refs. 14 and 15) suitable for thick elastic plates by starting with expressions suggested by classical thin plate theory for the stresses as functions of the loading, and calculating additional terms so as to satisfy more and more exactly the theory of elasticity. Expressions are thus obtained for stresses and displacements in the form of infinite series involving higher and higher derivatives of the loading.

Let $T(x,y)$ and $B(x,y)$ represent the normal unit pressure at the top and bottom surfaces, respectively, as shown in figure B.1, and let

\[
d(x,y) = T - B \tag{B.1}
\]
\[
s(x,y) = T + B \tag{B.2}
\]

The governing differential equations are

\[
\nabla^4 \vartheta = d \tag{B.3}
\]
\[
\nabla^4 S = s \tag{B.4}
\]

The stress resultants are given as

\[
N_x = -\frac{1}{2} \nu \mu \frac{d^3}{dz^3} \nabla^2 S + \ldots \tag{B.5}
\]
\[
N_{y y} = \frac{1}{2} \nu \mu \frac{d^3}{dz^3} \nabla^2 S + \ldots \tag{B.6}
\]
\[
Q_x = -\frac{2}{\nu} \nabla^2 \vartheta + \ldots \tag{B.7}
\]
Figure B.1 - Coordinate system and loading.
The expressions for \( N_y \), \( Q_y \), and \( M_y \) are obtained from \( N_x \), \( Q_x \), and \( M_x \) by interchanging \( x \) and \( y \). The displacement components \( u^* \), \( v^* \), and \( w^* \) at the middle surface of the plate are

\[
M_x = \left( \frac{\partial^4 \theta}{\partial y^4} + \frac{\partial^2 \theta}{\partial x^2 \partial y^2} \right) + \frac{1}{10} \nu E h^3 \frac{\partial^2 \theta}{\partial x^2 \partial y^2} + \frac{1}{4200} \nu E h^4 \frac{\partial^4 \theta}{\partial x^4} + \cdots \quad (B.8)
\]

\[
M_{xy} = -(1 - \nu) \frac{\partial^2 \theta}{\partial x \partial y} + \frac{1}{10} \nu E h^3 \frac{\partial^2 \theta}{\partial x^3 \partial y} + \frac{1}{4200} \nu E h^4 \frac{\partial^4 \theta}{\partial x^4} + \cdots \quad (B.9)
\]

The stress components \( \sigma_x \) ... \( \tau_{xy} \) are given as equations (2) in reference 14. Minor changes of notation have to be made to conform more closely with the present coordinate system and loadings.

The derivation of the above equations is lengthy and tedious. Details of the derivation are beyond the scope of this appendix.

For a rectangular plate having simply supported edges subjected to a uniformly distributed load \( P \) (see fig. 3.1) the results for \( (Q_x)_{x=0} \), \( (M_x)_{\text{max}} \), \( (w_0)_{\text{max}} \), and \( (\sigma_x)_{\text{max}} \) are

\[
(Q_x)_{x=0} = \frac{4PA}{h^4} \left( \sum_{m=1}^{\infty} \frac{1}{m^4} \left( 1 - \frac{\cosh d_x}{\cosh \mu} \right) \right) \quad (B.13)
\]

\[
+ \frac{\pi^2 (137 - 157 \nu)}{16800} PA \sum_{m=1}^{\infty} \left[ \frac{m^3 \cosh d_x}{(1 + \frac{\nu}{4} \frac{m^4 h^4}{4\pi})} \right] \frac{1}{\cosh \mu} \left( \frac{h}{a} \right)^4 \cdots
\]
\[(M_{x})_{\text{max}} = \frac{Pa^2}{8} \left[ 1 - \frac{9}{\pi^2} \sum_{m = 1,3}^{\infty} (-1)^{(m-1)/2} \frac{4 + (1-\nu)\pi (b/a) \tanh \mu}{m^3 \cosh \mu} \right] \]

\[+ \frac{2 \nu}{5 \pi} \frac{Pa^2}{D} \left[ \sum_{m = 1,3}^{\infty} (-1)^{(m-1)/2} \frac{1}{m \cosh \mu} \right] \left( \frac{A}{a} \right)^2 \]

\[+ \frac{\pi (227-157 \nu)}{16800} \frac{Pa^2}{D} \left[ \sum_{m = 1,3}^{\infty} (-1)^{(m-1)/2} \left( \frac{40 \nu \pi b + 10 (1-\nu) \pi m \nu \tanh \mu}{40 + 8 + \nu \pi m \nu (4/a)^2} \right) / \cosh \mu \right] \left( \frac{A}{a} \right)^4 + \ldots \]

(B.14)

\[(W_{x})_{\text{max}} = \frac{4}{\pi^2} \frac{Pa^2}{D} \left[ \sum_{m = 1,3}^{\infty} (-1)^{(m-1)/2} \frac{1}{m^3} \left[ 1 - \left(1 + \frac{4m \pi b}{a} \tanh \mu \right) \sqrt{\cosh \mu} \right] \right] \]

\[+ \frac{1}{\pi (1-\nu)} \frac{Pa^2}{D} \sum_{m = 1,3}^{\infty} (-1)^{(m-1)/2} \left\{ \frac{8 - 3 \nu}{10 \pi} \cdot \frac{1 - \text{sech} \mu}{m^3} \right\} \left( \frac{A}{a} \right)^2 \]

\[- \frac{227 - 157 \nu}{16800} \cdot \frac{1}{m} \left[ 1 - \left(1 + \frac{40 \nu \pi b + 10 (1-\nu) \pi m \nu \tanh \mu}{40 + 8 + \nu \pi m \nu (4/a)^2} \right) / \cosh \mu \right] \left( \frac{A}{a} \right)^4 \] + \ldots \]

(B.15)

\[(G_{x})_{\text{max}} = \frac{3}{4} \frac{Pa^2}{Ka^2} \left[ 1 - \frac{9}{\pi^2} \sum_{m = 1,3}^{\infty} (-1)^{(m-1)/2} \left( 4 + (1-\nu) \pi m \nu \frac{b}{a} \tanh \mu \right) / m^3 \cosh \mu \right] \]

\[+ \pi \frac{Pa^2}{Ka^2} \sum_{m = 1,3}^{\infty} (-1)^{(m-1)/2} \left[ \frac{4}{5 \pi m^2} \left( 1 - \frac{1 - \nu}{\cosh \mu} \right) \right] \left( \frac{A}{a} \right)^2 \]

\[+ \frac{227 - 157 \nu}{2800} \cdot \frac{m}{\cosh \mu} \left[ 1 + \frac{40 \nu \pi m + 10 (1-\nu) \pi m \nu \tanh \mu}{40 + 8 + \nu \pi m \nu (4/a)^2} \right] \left( \frac{A}{a} \right)^4 + \ldots \]

(B.16)

where \( \nu_m = \frac{m \pi}{a} \), \( \mu = \frac{m \pi b}{a} \), and \( m = 1,3,5, \ldots \) only.

For a square plate having simply supported edges subjected to
a sinusoidally distributed load \( p = p_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} \) (see fig. 3.2) the corresponding quantities are

\[
(Q_{x})_{x=0/2} = -\frac{P_o a}{2\pi} \cos \frac{\pi y}{a} \tag{B.17}
\]

\[
(M_{y})_{max} = \frac{1+\nu}{4\pi^2} P_o a^4 \left[ 1 + \frac{\pi^2}{5(1+\nu)} \left(\frac{a}{h}\right)^2 - \frac{7\pi^2}{1050(1+\nu)} \left(\frac{a}{h}\right)^4 + \ldots \right] \tag{B.18}
\]

\[
(W_{y})_{max} = \frac{P_o a^4}{4\pi^4 D} \left[ 1 + \frac{\pi^2(3-2\nu)}{20(1-\nu)} \left(\frac{a}{h}\right)^2 - \frac{\pi^2(327-157\nu)}{16800(1-\nu)} \left(\frac{a}{h}\right)^4 + \ldots \right] \tag{B.19}
\]

\[
(\sigma_{y})_{max} = \frac{3(1+\nu)}{2\pi^2} \frac{P_o a^4}{h^2} \left[ 1 + \frac{\pi^2}{15} \left(\frac{a}{h}\right)^2 - \frac{71\pi^4}{3150} \left(\frac{a}{h}\right)^4 + \ldots \right] \tag{B.20}
\]

It is noted that the edge boundary conditions to be satisfied by this theory are (see fig. 3.1)

at \( x=0 \), \( a \)

\( w_0 = M_x = v_0 = N_x = 0 \)

at \( y=\pm b/2 \)

\( w_0 = M_y = u_0 = N_y = 0 \)
REFERENCES


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