INFORMATION TO USERS

This dissertation was produced from a microfilm copy of the original document. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the original submitted.

The following explanation of techniques is provided to help you understand markings or patterns which may appear on this reproduction.

1. The sign or “target” for pages apparently lacking from the document photographed is “Missing Page(s)”. If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting thru an image and duplicating adjacent pages to insure you complete continuity.

2. When an image on the film is obliterated with a large round black mark, it is an indication that the photographer suspected that the copy may have moved during exposure and thus cause a blurred image. You will find a good image of the page in the adjacent frame.

3. When a map, drawing or chart, etc., was part of the material being photographed the photographer followed a definite method in “sectioning” the material. It is customary to begin photoing at the upper left hand corner of a large sheet and to continue photoing from left to right in equal sections with a small overlap. If necessary, sectioning is continued again—beginning below the first row and continuing on until complete.

4. The majority of users indicate that the textual content is of greatest value, however, a somewhat higher quality reproduction could be made from “photographs” if essential to the understanding of the dissertation. Silver prints of “photographs” may be ordered at additional charge by writing the Order Department, giving the catalog number, title, author and specific pages you wish reproduced.

University Microfilms
300 North Zeeb Road
Ann Arbor, Michigan 48106
A Xerox Education Company
RABER, Neal Clifford, 1944-
ON OSTROM'S FINITE HYPERBOLIC PLANES.
The Ohio State University, Ph.D., 1972
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan
ON OSTROM'S FINITE
HYPERBOLIC PLANES

DISSERTATION
Presented in Partial Fulfillment of the Requirements
for the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Neal Clifford Raber, B.S., M.S.

The Ohio State University
1972

Approved by

[Signature]
Department of Mathematics
PLEASE NOTE:

Some pages may have indistinct print.

Filmed as received.

University Microfilms, A Xerox Education Company
ACKNOWLEDGMENTS

I wish to express my sincere gratitude to my adviser, Professor Arno Cronheim, for his many useful suggestions regarding this dissertation and for his time, patience, and concern throughout its preparation. I also wish to thank Professor Jill Yaqub for her help. Finally, I wish to thank my family, especially my wife, for their encouragement and understanding.
VITA

August 24, 1944 . . . Born - Lakewood, Ohio
1966 .............. B.S. in Ed., Kent State University, Kent, Ohio
1966-1968 ........ NDEA Fellow, The Ohio State University, Columbus, Ohio
1968 .............. M.S., The Ohio State University, Columbus, Ohio
1968-1969 ........ NDEA Fellow, The Ohio State University, Columbus, Ohio
1969-1972 ........ Teaching Associate, Department of Mathematics, The Ohio State University, Columbus, Ohio
1971-1972 ........ Lecturer, Department of Mathematics, Ohio Northern University at Riverside Methodist Hospital Nursing School, Columbus, Ohio

FIELDS OF STUDY

Major Field: Mathematics

Studies in Algebra. Professor A. Cronheim
Studies in Analysis. Professor B. Baishanski
Studies in Topology. Professor N. Levine
Studies in Geometry. Professors J. Yaqub, A. Cronheim
Studies in Combinatorics. Professor Ray-Chaudhuri
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>VITA</td>
<td>iii</td>
</tr>
<tr>
<td>Chapter</td>
<td></td>
</tr>
<tr>
<td>I.  INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. CONSTRUCTION OF THE PROJECTIVE PLANE</td>
<td>10</td>
</tr>
<tr>
<td>III. INVOLUTIONS</td>
<td>37</td>
</tr>
<tr>
<td>IV. MORE COMBINATORIAL PROPERTIES</td>
<td>59</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>75</td>
</tr>
</tbody>
</table>
CHAPTER I
INTRODUCTION

Under proper assumptions, the undefined concepts point, line, and on are sufficient for the development of a class of geometries including that of the hyperbolic plane, first studied in Bolyai's and Lobachevsky's non-euclidean geometry.

Models for these geometries are supplied by strictly convex regions in a real affine plane—briefly, regions as represented by a sheet of paper or the interior of a conic. In such a model, the term point means an interior point of the region, and line means a segment joining two points of the rim. (Because of the strict convexity of the region, all other points of the segment are in the interior of the region.) Two lines are said to meet if there exists a point (i.e., a point of the region) which is on both lines. Any two nonmeeting lines are called parallel.

The fundamental difference between hyperbolic geometry and affine geometry is that in the former each point is on more than one line parallel to a given line not on that point.
The subject of finite projective planes has been investigated from various points of view, but the subject of finite hyperbolic planes is rather undeveloped. For an account of the historical development of the subject, see Blumenthal and Menger [1].

In 1962, L. M. Graves [6] proposed the set of axioms for hyperbolic planes (also called Bolyai-Lobachevsky or B-L planes) listed below. These axioms were the first set of axioms for hyperbolic planes that admitted finite models.

A1. The plane $\mathcal{P}$ is a finite collection of elements called points.

A2. There are certain distinguished subsets of the plane $\mathcal{P}$, called lines.

A3. There are at least two points on each line.

A4. Two distinct points $A$ and $B$ are on one and only one line.

A5. The plane $\mathcal{P}$ contains at least four points, no three of which are on a line.

A6. If a subset of $\mathcal{P}$ contains three points not on a line, and contains all the lines through any pair of its points, then that subset contains all the points of $\mathcal{P}$.

A7. Through each point $X$ not on a line $L$ there pass at least two lines not meeting $L$. 
T. G. Ostrom [10] gives an example of a finite plane satisfying Graves' axioms. The example is based on his results on ovals in finite projective planes [9]. A summary of the results and a description of the hyperbolic plane follows.

A projective plane \( \mathcal{W} \) is said to be of order \( n \) if each line contains exactly \( n+1 \) points. (For a discussion of finite projective planes and its properties, see Dembowski [3].) We restrict ourselves to the case when \( n \) is odd and \( n \geq 9 \). A set \( \mathcal{O} \) of \( n+1 \) points is said to be an oval if no three points of \( \mathcal{O} \) are collinear. A line which contains exactly one point of \( \mathcal{O} \) is said to be a tangent line. A line which contains two points of \( \mathcal{O} \) is called a secant, and a line which contains no points of \( \mathcal{O} \) is called a nonsecant.

The points of \( \mathcal{W} \) occur in three disjoint classes: (1) the points of \( \mathcal{O} \), all of which are on exactly one tangent line; (2) the exterior points, which are on exactly two tangents; and (3) the interior points, which are on no tangent lines [Qvist, 11]. It can easily be shown [Ostrom, 9] that there are \( \frac{1}{2}n(n-1) \) interior points; no tangent contains any interior points; each secant contains exactly \( \frac{1}{2}(n-1) \) interior points; and each nonsecant contains exactly \( \frac{1}{2}(n+1) \) interior points.
If \( \mathcal{P} \) is the set of interior points of \( \Pi \) and the lines of \( \mathcal{P} \) are the set theoretic intersections of the secants and nonsecants of \( \Pi \) with \( \mathcal{P} \), then \( \mathcal{P} \) is a hyperbolic plane satisfying Graves' axioms. The lines of \( \mathcal{P} \) with \( \frac{1}{2}(n-1) \) points will be called secants and those with \( \frac{1}{2}(n+1) \) points will be called nonsecants.

Since \( \mathcal{P} \) is derived from a projective plane \( \Pi \), it is natural to ask how much is known about \( \Pi \) from within the incidence structure of \( \mathcal{P} \). Can it be determined from \( \mathcal{P} \) alone, when \( n+1 \) nonintersecting lines of \( \mathcal{P} \) are in the same pencil on a point of \( \Pi \)? Hence, can \( \Pi \) be reconstructed from within the incidence structure of \( \mathcal{P} \)?

In the case that \( \Pi \) is a finite desarguesian projective plane of odd order \( n \), \( \Pi \) is coordinatized by a field \( \mathcal{F} \) of order \( n \), and Segre [12] has shown that every oval in such a plane is a conic \( \mathcal{C} \), i.e. the set of points whose coordinates satisfy a nondegenerate equation of the second degree. The group of projective collineations which fix a conic \( \mathcal{C} \) in a desarguesian projective plane of odd order \( n \) is isomorphic to the orthogonal group of order \( (n+1)n(n-1) \) [Edge, 4]. This group has been shown to be generated by the involutory collineations fixing \( \mathcal{C} \) whose center is a point \( P \) not on \( \mathcal{C} \) and whose axis is the polar of \( P \) [Buekenhout, 2]. It is natural to ask, then, if \( \mathcal{P} \) is derived from a
conic in a finite desarguesian projective plane $\pi$ of order $n$ what is known about these involutions within the incidence structure of $\pi$.

The object of this dissertation is to investigate Ostrom's example of a finite hyperbolic plane in the case that $\pi$ is a finite desarguesian projective plane of order $n$ and $C$ a conic in $\pi$, in order to answer the questions posed above. Throughout the paper $n > 9$ except in the second part of Chapter II where we restrict $n$ to be greater than 17; and $\mathcal{P}$ is the hyperbolic plane derived from the desarguesian plane $\pi$ containing the conic $C$. $\mathcal{F}$ will be the coordinatizing field of order $n$. Then $\mathcal{F}$ is the union of three disjoint sets: the $\frac{1}{2}(n-1)$ elements of $\mathcal{F}$ which are squares, the $\frac{1}{2}(n-1)$ elements which are nonsquares, and zero.

In Chapter II, incidence properties of $\pi$ with respect to interior points are established. This is done in order to characterize in $\mathcal{P}$ polars of points and pencils of lines of $\mathcal{P}$ that intersect at points of $C$ in $\pi$ and at exterior points of $\pi$. With these characterizations the plane $\pi$ is reconstructed from $\mathcal{P}$.

In Chapter III, incidence properties in $\pi$ with respect to interior points of nonsecants and their images under the involutions fixing $C$ are determined. From these properties, it is shown that these involutions are
determined within the incidence structure of $P$. The chapter is concluded with an investigation of secants and their images under these involutions in the case $n > 17$.

In Chapter IV, further interesting properties of $P$ are determined and alternate methods of reconstructing $\Pi$ from $P$ are shown.

It is interesting to notice that many of the properties are analogous to properties in the real hyperbolic plane, for example Theorems 2.1, 3.9, and 4.1(1).

Since $\Pi$ is desarguesian, by an appropriate choice of coordinatization, we may assume without loss of generality that the conic $C$ in $\Pi$ from which $P$ is derived consists of the following points in homogeneous coordinates: $\{(x,y,z) \mid f(x,y,z) = xy + z^2 = 0\}$. (For a discussion of homogeneous coordinates in projective planes, see Maxwell's.) These points include the affine points $(x,y,1)$ where $xy = -1$ and the points $(0,1,0)$ and $(1,0,0)$ on $l_\infty$, i.e. the line with equation $z = 0$. We will say $C$ is associated with the form $f(x,y,z) = xy + z^2$.

First, we will establish a characterization of exterior points and interior points within the field $F$. This theorem is due to Qvist [11].
Theorem 1.1 If \( \mathcal{C} \) is a conic in a finite desarguesian projective plane \( \Pi \) of odd order associated with the form \( f(x,y,z) = xy + z^2 \), then \( P = (x_1, y_1, z_1) \) is an exterior point or an interior point according as \( f(P) = f(x_1, y_1, z_1) \) is a square or a nonsquare in the coordinatizing field \( \mathcal{F} \).

Proof: Since \( \mathcal{C} \) consists of all points \((x,y,z)\) such that \( f(x,y,z) = 0 \), \( \mathcal{C} \) has homogeneous, parametric point coordinate representation \((-r^2, 1, r)\) except for the point \((1,0,0)\). If \([u_1, u_2, u_3]\) are the homogeneous coordinates of a tangent line to \( \mathcal{C} \), then the equation 
\[-u_1 r^2 + u_2 + u_3 r = 0\]
has only one solution for \( r \); so its discriminant \( u_2^2 + 4u_1u_2 \) must equal zero, i.e. \( -u_3^2 = 4u_1u_2 \). Therefore the parametric coordinate representation of a tangent line is \([-k^2, 1, 2k]\) including the tangent at \((1,0,0)\) which has line coordinates \([0,1,0]\).

If \( P = (x_1, y_1, z_1) \) is an interior point, it lies on no tangents, and thus \(-k^2 x_1 + y_1 + 2kz_1 = 0\) has no solutions \( k \) in \( \mathcal{F} \); so its discriminant \( 4z_1^2 + 4x_1 y_1 \) is a nonsquare in \( \mathcal{F} \). Thus \( f(P) = x_1 y_1 + z_1^2 \) is a nonsquare in \( \mathcal{F} \). If \( P \) is an exterior point it is on two tangents, and therefore the discriminant is a square in \( \mathcal{F} \), i.e. \( f(P) \) is a square in \( \mathcal{F} \). //
If $A$ is the matrix over $\mathbb{F}$,

$$
A = \begin{pmatrix}
0 & 2^{-1} & 0 \\
2^{-1} & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

and $P = (x, y, z)$ is a point of $\Pi$, then $f(P)$ can be represented by the following determinant:

$$
f(x, y, z) = xy + z^2 = |X^TAX| \quad \text{where} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
$$

If $g$ were any other form associated with the conic $\mathcal{C}$ by a change of coordinatization, let $B$ be the matrix associated with $g$ and let $M$ be the matrix such that

$$
X = MY \quad \text{where} \quad Y = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}
$$

and $(x_1, y_1, z_1)$ is the new coordinatization of $P$. Then $B = M^TAM$, and $g(P) = |Y^TBY| = |Y^TM^TAMY| = |(MY)^TAMY| = |X^TAX| = f(P)$. Therefore if $f$ and $g$ are forms associated with $\mathcal{C}$ by a change of coordinatization, and $P$ is a point of $\Pi$, $f(P) = g(P)$.

If $h$ is any form associated with $\mathcal{C}$, such that $h = af$ where $a \in \mathbb{F}$, then if $P$ is a point of $\Pi$, $h(P) = af(P)$. If $a$ is a square in $\mathbb{F}$, Theorem 1.1 is true for $h$, but if $a$ is a nonsquare in $\mathbb{F}$, $h(P)$ is a
square if and only if P is an interior point and h(P) is a nonsquare if and only if P is an exterior point.

Under any form associated with a conic ℂ in a finite desarguesian projective plane, the form values of interior points are of one type, square or nonsquare, and the form values of exterior points are the opposite type. Throughout this paper without loss of generality, we will associate with ℂ the form \( f(x,y,z) = xy+z^2 \).

Theorem 1.1 is a generalization of the "power of a point with respect to a circle" in the real Euclidean plane:

If \( F(x,y) = (x-x_0)^2+(y-y_0)^2-r^2 = 0 \) is the equation of a circle and \( P_1 = (x_1,y_1) \) is a point in the plane, then \( F(x_1,y_1) = (x_1-x_0)^2+(y_1-y_0)^2-r^2 \) is called the power of the point \( P_1 \) with respect to the circle \( F(x,y) = 0 \). Then \( F(x_1,y_1) \) is positive, zero, or negative according as \( P_1 \) lies outside (exterior), on, or inside (interior) the circle.
CHAPTER II
CONSTRUCTION OF THE PROJECTIVE PLANE

Theorem 2.1 If \( C \) is a conic in a finite desarguesian projective plane \( \Pi \) with coordinatizing field \( \mathcal{F} \), and \( A, B, \) and \( C \) are three distinct points on \( C \), then any line \( l \) that is not on \( A, B, \) or \( C \) intersects the secants \( AB, AC, \) and \( BC \) in exactly three exterior points or exactly one exterior point.

Proof: Since the group of collineations fixing \( C \) is sharply triply transitive on the points of \( C \) and since this group also leaves the set of interior points and the set of exterior points invariant, it is sufficient to prove the theorem for the points \( A, B, \) and \( C \) on \( C \) with homogeneous coordinates \((1,-1,1)\), \((1,0,0)\), and \((0,1,0)\), respectively. (See Figures 1 and 2.) Thus \( BC \) is the line \( l_\infty \) with equation \( z = 0 \), \( AB \) is the line with affine equation \( y = -1 \), and \( AC \) is the line with affine equation \( x = 1 \).

Let \( P \) be any point of \( BC \) not equal to \( B \) or \( C \), then \( P \) has coordinates \((1,m,0)\), \( m \neq 0 \). By Theorem 1.1,
P is exterior if and only if \( f(1,m,0) = m \) is a square and P is interior if and only if \( m \) is a nonsquare in \( \mathcal{F} \). If \( l \) is any line on \( P \) and not on \( A \), \( B \), or \( C \) then \( l \) has affine equation \( y = mx+b \) such that \( m+b \neq -1 \). Let \( Q \) be the point of intersection of \( l \) and \( AB \), and \( R \) the point of intersection of \( l \) and \( AC \). Then \( Q \) has coordinates \((-m^{-1}(b+1),-1,1)\) and \( R \) has coordinates \((1,m+b,1)\).

Now \( f(R) = f(1,m+b,1) = m+b+1 = m(m^{-1}+m^{-1}b+1) = m \cdot f(-m^{-1}(b+1),-1,1) = m \cdot f(Q) \). So if \( P \) is exterior, \( m \) is a square and \( f(R) \) and \( f(Q) \) are either both squares or both nonsquares in \( \mathcal{F} \), i.e. \( R \) and \( Q \) are either both exterior points or both interior points by Theorem 1.1. If \( P \) is interior, \( m \) is a nonsquare and exactly one of \( f(R) \) and \( f(Q) \) is a square in \( \mathcal{F} \) and the other is a nonsquare, i.e. exactly one of \( R \) and \( Q \) is exterior and the other is interior.

Thus \( l \) intersects \( AB \), \( AC \), and \( BC \) in exactly three exterior points or exactly one exterior point. By the triple transitivity of the collineations fixing \( \mathcal{C} \), we have the result for any three distinct points on \( \mathcal{C} \).

Theorem 2.1 is similar to Pasch's Axiom in the Euclidean plane: If \( P_1, P_2, P_3 \) are noncollinear points then each line meeting one of the three lines \( P_iP_j \) in
a point between $P_i$ and $P_j$ is either on $P_k$ or meets one of the other two lines, say $P_iP_k$, in a point that is between $P_i$ and $P_k$ and meets the third line either not at all (parallel) or in a point that is not between $P_j$ and $P_k$.

This statement is the most famous of the postulates introduced by Moriz Pasch in 1882 in order to fill the gaps in Euclid's Elements concerning linear and planar order.

**Theorem 2.2** If $AB$ and $AC$ are two distinct secants to a conic $C$ in a finite desarguesian projective plane $\Pi$ where $A, B, C$ are points on $C$, $P$ is a point not on $C$, $AB$, $AC$ or $BC$, and $K_i$, $i=1,2,3,4$ are the following sets of lines:

$K_1 = \{ l \mid P \in l, l \text{ intersects } AB \text{ and } AC \text{ in interior points}\}$

$K_2 = \{ l \mid P \in l, l \text{ intersects } AB \text{ and } AC \text{ in exterior points}\}$

$K_3 = \{ l \mid P \in l, l \text{ intersects } AB \text{ in an exterior point and } AC \text{ in an interior point}\}$

$K_4 = \{ l \mid P \in l, l \text{ intersects } AB \text{ in an interior point and } AC \text{ in an exterior point}\}$
then
\[ |K_i| = \frac{n-1}{4} + t_i, \quad |t_i| \leq 1 \quad i=1,2,3,4 \]

where \( n \) is the order of the coordinatizing field \( \mathbb{F} \) of \( \mathbb{V} \).

**Proof:** The \( \frac{n-1}{2} \) lines on \( P \) which intersect \( BC \) in interior points, except for at most the one line on \( A \), intersect \( AB \) and \( AC \) in one interior point and one exterior point, by Theorem 2.1. These lines belong to \( K_3 \) or \( K_4 \), and since any line on \( P \) that intersects exactly one of \( AB \) and \( AC \) in an interior point and the other in an exterior point must intersect \( BC \) in an interior point by Theorem 2.1, we have

(1) \[ |K_3| + |K_4| = \frac{n-1}{2} + s_1 \quad \text{where} \quad s_1 \text{ is } 0 \text{ or } -1. \]

The \( \frac{n-1}{2} \) lines on \( P \) which intersect \( AC \) in interior points belong to \( K_1 \) or \( K_3 \) except at most the one line on \( B \). Thus we have

(2) \[ |K_1| + |K_3| = \frac{n-1}{2} + s_2 \quad \text{where} \quad s_2 \text{ is } 0 \text{ or } -1. \]

The \( \frac{n-1}{2} \) lines on \( P \) which intersect \( AB \) in interior points belong to \( K_1 \) or \( K_4 \) except at most the line on \( C \). Thus we have

(3) \[ |K_1| + |K_4| = \frac{n-1}{2} + s_3 \quad \text{where} \quad s_3 \text{ is } 0 \text{ or } -1. \]
The $\frac{n-1}{2}$ lines on $P$ which intersect $AB$ in exterior points belong to $K_2$ or $K_3$ except for at most the one line on $C$. Thus we have

$$|K_2| + |K_3| = \frac{n-1}{2} + s_4$$

where $s_4$ is 0 or -1.

Since the line $PC$ intersects $AB$ in an interior point or an exterior point, exactly one of $s_3$ and $s_4$ is -1 and the other is 0.

Adding (1) and (2) and subtracting (3) yields:

$$|K_3| = \frac{n-1}{4} + \frac{1}{2}(s_1+s_2-s_3)$$

and $|s_1+s_2-s_3| \leq 2$.

Subtracting (5) from (1) yields:

$$|K_4| = \frac{n-1}{4} + \frac{1}{2}(s_1-s_2+s_3)$$

and $|s_1-s_2+s_3| \leq 2$.

Subtracting (5) from (2) yields:

$$|K_1| = \frac{n-1}{4} + \frac{1}{2}(s_2-s_1+s_3)$$

and $|s_2-s_1+s_3| \leq 2$.

Subtracting (5) from (4) yields:

$$|K_2| = \frac{n-1}{4} + \frac{1}{2}(2s_4+s_3-s_1-s_2)$$

and $|2s_4+s_3-s_1-s_2| \leq 2$ since exactly one of $s_4$ and $s_3$ is -1 and the other is 0.

Letting $t_1 = \frac{1}{2}(s_2-s_1+s_3)$, $t_2 = \frac{1}{2}(2s_4+s_3-s_1-s_2)$, $t_3 = \frac{1}{2}(s_1+s_2-s_3)$, and $t_4 = \frac{1}{2}(s_1-s_2+s_3)$, we have the result from (5), (6), (7), and (8).
Theorem 2.2 yields a geometric approach to a theorem of Jacobsthal [Hasse, p. 149, 7] (in the case \( n \) is a prime) counting the distribution of square and nonsquare values of a linear expression in one variable over a finite field \( \mathcal{F} \) of order \( n \).

**Corollary 2.3** If \( b \) is a nonzero element of a finite field \( \mathcal{F} \) and \( K_i^j \), \( i=1,2,3,4 \) are the following sets:

- \( K_1^1 = \{ m \mid m \text{ and } m+b \text{ are nonsquares in } \mathcal{F} \} \)
- \( K_2^2 = \{ m \mid m \text{ and } m+b \text{ are squares in } \mathcal{F} \} \)
- \( K_3^3 = \{ m \mid m \text{ is a square and } m+b \text{ is a nonsquare in } \mathcal{F} \} \)
- \( K_4^4 = \{ m \mid m \text{ is a nonsquare and } m+b \text{ is a square in } \mathcal{F} \} \)

then \( |K_i^j| = \frac{n-1}{4} + t_i^j \), \( |t_i^j| \leq 1 \) \( i=1,2,3,4 \).

**Proof:** In the desarguesian projective plane coordinatized by \( \mathcal{F} \), let \( \mathcal{C} \) be the conic associated with the form \( f(x,y,z) = xy+z^2 \), and let \( P \), \( A \), \( B \), and \( C \) be the points with coordinates \((0,b-1,1)\), \((0,1,0)\), \((1,0,0)\), and \((1,-1,1)\) respectively. Then \( AB \) is \( l_\infty \), i.e. the line with equation \( z = 0 \); \( AC \) is the line with affine equation \( x = 1 \); and \( BC \) is the line with affine equation \( y = -1 \). Since \( b \neq 0 \), \( P \) is not on \( BC \); and \( P \) is also not on \( AB \), \( AC \), or \( \mathcal{C} \). (See Figure 3.)

If \( l_m \) is any line on \( P \) and not on \( A \), then \( l_m \) has affine equation \( y = mx+(b-1) \) and \( l_m \) intersects \( AB \)
Figure 3
at the point \((1,m,0)\) and \(AC\) at the point \((1,m+b-1,1)\).

Since \(f(1,m,0) = m\) and \(f(1,m+b-1,1) = m+b\), by Theorem 1.1, \(m\) and \(m+b\) are both nonsquares in \(J\) if and only if \(l_m\) intersects \(AB\) and \(AC\) in interior points.

Therefore \(m \in \mathcal{K}_1\) if and only if \(l_m \in \mathcal{K}_1\) where \(\mathcal{K}_1\) is the set defined in Theorem 2.2. Thus \(|\mathcal{K}_1| = |\mathcal{K}_1^*|\), and by similar arguments \(|\mathcal{K}_i| = |\mathcal{K}_i^*|, i=2,3,4\) where \(\mathcal{K}_i, i=2,3,4\) are the sets defined in Theorem 2.2. From the result of Theorem 2.2, we have \(|\mathcal{K}_i^*| = \frac{n-1}{4} + t_i\), \(|t_i| \leq 1\) \(i=1,2,3,4\).

The result of Corollary 2.3 easily generalizes to the same result for \(m\) and the linear expression \(am+b\) over \(J\) where \(a \neq 0\) and \(b \neq 0\). It is necessary to consider if \(a\) is a square or a nonsquare in \(J\) and then apply Corollary 2.3 to \(m\) and the expression \(m+a^{-1}b\).

**Definition 2.4** (a) If \(l_1\) and \(l_2\) are two secants to an oval \(\mathcal{O}\) in a projective plane \(\pi\), then an interior point \(P\) not on \(l_1\) or \(l_2\) will be said to be a special point with respect to \(l_1\) and \(l_2\) if and only if exactly two secants on \(P\) do not intersect either \(l_1\) or \(l_2\) in interior points and every other line on \(P\) intersects \(l_1\) in an interior point if and only if it does not intersect \(l_2\) in an interior point. If \(l_3\) is a secant not equal
to \( l_1 \) or \( l_2 \), \( l_3 \) will be said to be a special line with respect to \( l_1 \) and \( l_2 \) if and only if every interior point on \( l_3 \) is a special point with respect to \( l_1 \) and \( l_2 \).

(b) If \( l_1 \) and \( l_2 \) are two secants to an oval \( \mathcal{O} \) in a projective plane \( \Pi \) of order \( n \) and \( P \) is a point not on \( l_1 \) or \( l_2 \), then \( P \) will be said to be a normal point with respect to \( l_1 \) and \( l_2 \) if and only if

\[
|R_i| = \frac{n-1}{4} + t_i, \quad |t_i| \leq 1
\]

where \( R_i \) for \( i=1,2,3,4 \) are the following sets:

\[
R_1 = \{ l \mid P \in l, \text{ } l \text{ intersects } l_1 \text{ and } l_2 \text{ in interior points} \}
\]

\[
R_2 = \{ l \mid P \in l, \text{ } l \text{ intersects } l_1 \text{ and } l_2 \text{ in exterior points} \}
\]

\[
R_3 = \{ l \mid P \in l, \text{ } l \text{ intersects } l_1 \text{ in an exterior point and } l_2 \text{ in an interior point} \}
\]

\[
R_4 = \{ l \mid P \in l, \text{ } l \text{ intersects } l_1 \text{ in an interior point and } l_2 \text{ in an exterior point} \}
\]

Theorem 2.5 If \( l_1 \) and \( l_2 \) are two secants to a conic \( \mathcal{C} \) in a finite desarguesian projective plane \( \Pi \) of order \( n \) \((n > 9)\) that intersect at a point \( A \) on \( \mathcal{C} \), then there exists one and only one special line with respect to \( l_1 \) and \( l_2 \).
Proof: Let \( l_1 \) and \( l_2 \) be the secants \( AB \) and \( AC \) respectively where \( B \) and \( C \) are two distinct points on \( \mathcal{C} \). By Theorem 2.1, the secant \( BC \) is such that any line that intersects \( BC \) in an interior point intersects exactly one of \( l_1 \) and \( l_2 \) in an interior point and the other in an exterior point, except a line on \( A \). Thus for any interior point \( P \) on \( BC \), any line on \( P \) except \( BC \) and \( PA \) intersects \( l_1 \) in an interior point if and only if it intersects \( l_2 \) in an exterior point. So \( BC \) is a special line \( l_3 \) with respect to \( l_1 \) and \( l_2 \).

\( BC \) is unique, because by Theorem 2.2 if \( n > 9 \), \( \frac{n - 5}{4} > 1 \) and any point \( Q \) not on \( AB \), \( AC \), \( BC \), or \( \mathcal{C} \) is on at least three lines that either intersect \( AB \) and \( AC \) in two interior points or no interior points. Thus no interior point off \( BC \) could belong to a special line with respect to \( l_1 \) and \( l_2 \).

Theorem 2.2 shows that if \( l_1 \) and \( l_2 \) are two secants to a conic \( \mathcal{C} \) in a desarguesian projective plane \( \mathcal{P} \) of order \( n \) (\( n > 9 \)) that intersect on \( \mathcal{C} \), then every point not on \( l_1 \), \( l_2 \), \( \mathcal{C} \), or the special line \( l_3 \) with respect to \( l_1 \) and \( l_2 \) is a normal point with respect to \( l_1 \) and \( l_2 \). Therefore in \( \mathcal{P} \), if two secants intersect on \( \mathcal{C} \) in \( \mathcal{P} \), all of the points of \( \mathcal{P} \) not on
these two secants are either special points or normal points; and all the special points are on the special line with respect to these two secants.

Lemma 2.6 Let \( C \) be the conic associated with the form \( f(x,y,z) = xy+z^2 \) in the desarguesian projective plane \( \mathcal{P} \) coordinatized by a field \( \mathcal{F} \). If \( l \) is a line on the point \((1,m,0)\) on \( l_{\infty} \) where the affine points \((x,y,1)\) on \( l \) are given by the equation \( y = mx+b \), \( b \in \mathcal{F} \), then \( l \) is a nonsecant, a secant, or a tangent to \( C \) according as \( b^2-4m \) is a nonsquare, a square, or zero respectively.

Proof: If \( m = 0 \) and \( b = 0 \), then \( b^2-4m = 0 \) and \( l \) is the line with affine equation \( y = 0 \). Then \( l \) does not intersect \( C \) at any affine points, but \( l \) intersects \( C \) on \( l_{\infty} \) at the point \((1,0,0)\). Thus \( l \) is a tangent to \( C \). If \( m = 0 \) and \( b \neq 0 \), then \( b^2-4m = b^2 \) is a square in \( \mathcal{F} \), and \( l \) is the line with affine equation \( y = b \). Then \( l \) is a secant to \( C \), intersecting \( C \) at the point \((1,0,0)\) on \( l_{\infty} \) and at the affine point \((-b^{-1},b,1)\).

If \( m \neq 0 \), \( l \) only intersects \( C \) in affine points. The affine points \((x,y,1)\) on \( C \) satisfy the equation \( xy+1 = 0 \). The intersection of these points with the
points on $1$ are all those with abscissa satisfying the equation $mx^2 + bx + 1 = 0$. Therefore $1$ is a nonsecant, a tangent, or a secant to $C$ according as this equation has $0$, $1$, or $2$ solutions in $\mathcal{F}$, i.e. according as its discriminant $b^2 - 4m$ is a nonsquare, zero, or a square in $\mathcal{F}$.

\begin{flushright}
//
\end{flushright}

Remark 2.7 Any line on the point $(0,1,0)$ of $1_{\infty}$ has affine equation $x = c$, $c \in \mathcal{F}$; and is a secant if and only if $c \neq 0$. For if $c = 0$, it intersects $C$ only at the point $(0,1,0)$; and if $c \neq 0$, it intersects $C$ at the points $(0,1,0)$ and $(c,-c^{-1},1)$. Therefore $x = 0$ is a tangent to $C$ and $x = c$, $c \neq 0$, is a secant to $C$.

Theorem 2.8 Let $C$ be a conic in a desarguesian projective plane $\pi$ coordinatized by a field $\mathcal{F}$ of order $n$, $P$ a point not on $C$, and $l$ the polar of $P$.

(1) In the case $n \equiv 1 \pmod{4}$, if $P$ is interior (exterior) then any secant on $P$ intersects $l$ in an interior (exterior) point and any nonsecant on $P$ intersects $l$ in an interior (exterior) point;

(2) In the case $n \equiv 3 \pmod{4}$, if $P$ is interior (exterior) then any secant on $P$ intersects $l$ in an exterior (interior) point and any nonsecant intersects $l$ in an interior (exterior) point.
Proof: Without loss of generality, we may assume $\mathcal{C}$ is the conic associated with the form $f(x,y,z) = xy+z^2$. Since the group of projectives fixing $\mathcal{C}$ is transitive on the exterior points and also on the interior points, it is sufficient to prove the theorem in the case $P$ is on the line $l_\infty$. Thus let $P = (1,m,0)$, and then the polar of $P$, line $l$, has line coordinates $[m,1,0]$, i.e. $l$ is the line whose affine points satisfy the equation $y = -mx$.

Any line $l(b)$ on $P$ is such that its affine points satisfy the equation $y = mx+b$, $b \in \mathcal{F}$ except the line $l_\infty$. The point of intersection of a line $l(b)$ on $P$ and the line $l$ is the point $A(b) = (-b,bm,2m)$. The point of intersection of $l_\infty$ and $l$ is the point $B = (1,-m,0)$. $f(A(b)) = -b^2m+4m^2 = -m(b^2-4m)$ and $f(B) = -m$. $f(A(b))$ is a square according as $-m(b^2-4m)$ is a square.

In the case $n \equiv 1 \pmod{4}$, $-1$ is a square in $\mathcal{F}$, and if $P$ is interior (exterior), by Lemma 2.6 any secant $l(b)$ on $P$ is such that $b^2-4m$ is a square in $\mathcal{F}$. Any such secant intersects $l$ at the point $A(b)$ and since $P$ is an interior (exterior) point, $f(P) = m$ is a nonsquare (square) by Theorem 1.1. Hence, $-m$ is a nonsquare (square) in $\mathcal{F}$, and thus $f(A(b))$ is a nonsquare (square). So by Theorem 1.1, $A(b)$ is an interior
(exterior) point. The secant \( l_\infty \) intersects \( l \) at \( B \) and if \( P \) is interior (exterior), \( f(B) = -m \) is a non-square (square) in \( \mathcal{F} \), so \( l_\infty \) intersects \( l \) in an interior (exterior) point by Theorem 1.1. If \( l(b) \) is a nonsecant, by Lemma 2.6 \( b^2 - 4m \) is a nonsquare in \( \mathcal{F} \), and therefore if \( P \) is interior (exterior), \( f(A(b)) \) is a square (nonsquare) and \( A(b) \) is an exterior (interior) point.

In the case \( n \equiv 3 \pmod{4} \), \( -1 \) is not a square in \( \mathcal{F} \) and the argument is similar to the above. //

Theorem 2.8 is related to a theorem of Ostrom [Ostrom, 9].

Lemma 2.9 Let \( h(x) \) and \( k(x) \) be two quadratics over a field \( \mathcal{F} \) of order \( n \) which have no roots in \( \mathcal{F} \). If for each \( u \) in \( \mathcal{F} \), there exists a square \( t_u \) in \( \mathcal{F} \) such that \( h(u) = t_u k(u) \), then there exists a square \( t \) in \( \mathcal{F} \) such that \( h(x) = tk(x) \).

Proof: Let \( u_1, u_2, \ldots, u_n \) be the elements of \( \mathcal{F} \) and \( t_1, t_2, \ldots, t_n \) the corresponding squares in \( \mathcal{F} \) such that \( h(u_i) = t_i k(u_i) \), \( i=1,2,\ldots,n \). Since the number of squares in \( \mathcal{F} \) is \( \frac{n-1}{2} \) and since \( h(x) \) and \( k(x) \) have
no roots in \( \mathcal{F} \), there must exist a square \( t \) in \( \mathcal{F} \) that occurs at least three times among the squares \( t_i \) \( i=1,2,...,n \).

Consider the polynomial \( h(x)-tk(x) \) over \( \mathcal{F} \). It is of degree at most two. Since there are at least three distinct roots of this polynomial, \( h(x)-tk(x) = 0 \), i.e. \( h(x) = tk(x) \) for all \( x \) in \( \mathcal{F} \). //

**Theorem 2.10** If \( 
\) is a conic in a desarguesian projective plane \( \Pi \) coordinatized by a field \( \mathcal{F} \) of order \( n \), \( P \) an interior point of \( \Pi \) and \( l \) the polar of \( P \), then \( l \) is the unique nonsecant intersected by the secants and nonsecants on \( P \) in the manner described in (1) and (2) of Theorem 2.8.

**Proof:** As in Theorem 2.8, we let \( \mathcal{C} \) be the conic associated with the form \( f(x,y,z) = xy+z^2 \) and \( P = (1,m,0) \) be a point on \( \infty \). Since \( P \) is an interior point, \( f(P) = m \) is a nonsquare by Theorem 1.1. Also the affine points of \( l \) satisfy the equation \( y = -mx \), and \( l \) is a nonsecant.

Suppose there exists another nonsecant \( l' \) such that \( l' \) also satisfies the intersection properties of Theorem 2.8 with respect to secants and nonsecants on \( P \). Since \( l' \) is a nonsecant, by Remark 2.7 the affine points
on \( l' \) can only satisfy an equation of the form \( y = Mx + B \) where \( M, B \in \mathcal{J}, \) and \( M \neq m \) since \( l' \) cannot be on \( P. \) Also by Lemma 2.6, since \( l' \) is a nonsecant, \( M \neq 0. \)

The affine points of any line \( l(b) \) on \( P \) (except \( l_\infty \)) satisfy an equation of the form \( y = mx + b, \ b \in \mathcal{J}. \)

The point of intersection of \( l(b) \) and \( l' \) is the point \( X(b) = (b-B, Mb-Bm, M-m) \). The point of intersection of \( l(b) \) and \( l \) is the point \( Y(b) = (-b, b_m, 2m) \). The form values of these points of intersection are

\[
\begin{align*}
\ell(X(b)) &= (Mb-Bm)(b-B) + (M-m)^2 \\
&= Mb^2 - Bmb - BMb + B^2m + (M-m)^2 \\
&= Mb^2 - (BM+Bm)b + B^2m + (M-m)^2
\end{align*}
\]

and

\[
\ell(Y(b)) = -b^2m + 4m^2.
\]

Since \( l \) and \( l' \) both satisfy the properties of Theorem 2.8, any line \( l(b) \) on \( P \) intersects \( l \) in an interior point if and only if it intersects \( l' \) in an interior point. In terms of the form values by Theorem 1.1 we have that \( \ell(X(b)) \) is a nonsquare in \( \mathcal{J} \) if and only if \( \ell(Y(b)) \) is a nonsquare. But \( \ell(X(b)) \) and \( \ell(Y(b)) \) are two quadratics in \( b \) that have no roots in \( \mathcal{J} \) because \( l \) and \( l' \) contain no points of \( \mathcal{C} \); so \( \ell(X(b)) \) is a square in \( \mathcal{J} \) if and only if \( \ell(Y(b)) \) is a square. Thus for each \( b \) in \( \mathcal{J}, \) there exists a square
t_b in \( I \) such that \( f(X(b)) = t_b f(Y(b)) \). By Lemma 2.9, \( f(X(b)) = tf(Y(b)) \) where \( t \) is a square in \( I \). We obtain the equations \( M = -mt \), \( (M-m)^2 + B^2 m = 4m^2 t \), and \( BM + Bm = 0 \). The first and third equations give us \( mB(1-t) = 0 \), i.e. \( t = 1 \) or \( B = 0 \) since \( m \) is a nonsquare.

If \( t = 1 \), then \( M = -m \) and \( (-2m)^2 + B^2 m = 4m^2 \). Thus \( B = 0 \) and \( \ell' \) is the line \( \ell \). If \( B = 0 \), then \( (M-m)^2 = 4m^2 t \) and since \( M = -mt \), \( (mt-m)^2 = 4m^2 t \) or \( (t+1)^2 = 4t \) or \( (t-1)^2 = 0 \). Therefore \( t = 1 \) and \( \ell' \) is the line \( \ell \).

Therefore \( \ell \), the polar of \( P \), is unique with respect to Theorem 2.8. //

Theorem 2.11 If \( E \) is an exterior point to an oval \( \mathcal{O} \) in a projective plane \( \mathcal{P} \) of order \( n \) and if \( \ell \) is any secant not equal to the polar of \( E \) that does not intersect any of the nonsecants on \( E \) in an interior point, then \( \ell \) is on \( E \).

Proof: Suppose \( \ell \) is not on \( E \). Since a nonsecant contains no points of \( \mathcal{O} \) and since the nonsecants on \( E \) do not intersect \( \ell \) in an interior point, the nonsecants on \( E \) intersect \( \ell \) in exterior points. Since
each of these nonsecants intersects \( l \) in a distinct exterior point, this determines \( \frac{n-1}{2} \) exterior points on \( l \).

Because \( E \) is an exterior point, there are two tangents on \( E \), each of which contains \( n \) exterior points. Since \( l \) is not the polar of \( E \), one of the tangents on \( E \) must intersect \( l \) in an exterior point distinct from the \( \frac{n-1}{2} \) exterior points already determined on \( l \) by nonsecants on \( E \). Therefore \( l \) contains at least \( \frac{n-1}{2} + 1 = \frac{n+1}{2} \) exterior points. This is a contradiction since \( l \) is a secant and contains therefore only \( \frac{n-1}{2} \) exterior points.

Thus \( l \) is on \( E \).  //

**Theorem 2.12** Let \( P \) be a point on an oval \( \mathcal{O} \) in a finite projective plane \( \Pi \) of order \( n \) and let \( l_i \), \( i=1,2,...,\frac{n+3}{2} \) be distinct secants on \( P \). If \( l \) is any secant of \( \mathcal{O} \) that does not intersect any of the secants \( l_i \), \( i=1,2,...,\frac{n+3}{2} \) in an interior point, then \( l \) is on \( P \).

**Proof:** Suppose \( l \) is not on \( P \). \( l \) can intersect at most two of the lines \( l_i \), \( i=1,2,...,\frac{n+3}{2} \) on \( \mathcal{O} \), and since the lines \( l_i \), \( i=1,2,...,\frac{n+3}{2} \) are all on \( P \), \( l \) intersects these lines in at least \( \frac{n-1}{2} \) distinct exterior points.
Since there exists a tangent $t$ to $O'$ at $P$, and $l$ is not on $P$, $l$ must intersect $t$ in another exterior point. Therefore $l$ contains at least $\frac{n-1}{2} + 1 = \frac{n+1}{2}$ exterior points. This is a contradiction since $l$ is a secant.

Therefore $l$ is on $P$.  

\textbf{Theorem 2.13} If $P$ is an exterior point to an oval $O'$ in a finite projective plane $\Pi$ of order $n$ and if $l$ is a nonsecant such that $l$ does not intersect any of the secants on $P$ in an interior point, then $l$ is on $P$.

\textbf{Proof:} Suppose $l$ is not on $P$. Since $l$ contains no points of $O'$, the two tangents on $P$ and the $\frac{n-1}{2}$ non-secants on $P$ must intersect $l$ in $\frac{n-1}{2} + 2 = \frac{n+3}{2}$ distinct exterior points. This is a contradiction since $l$ contains only $\frac{n+1}{2}$ exterior points.

Therefore $l$ is on $P$.  

\textbf{Remark 2.14} In the hyperbolic plane $\mathcal{P}$, Theorem 2.12 shows that $\frac{n+3}{2}$ secants of the $n$ secants on a point $P$ of the oval is enough to determine the complete pencil of lines of $\mathcal{P}$ that are on $P$. Any secant of $\mathcal{P}$ that does not intersect these $\frac{n+3}{2}$ lines is in the pencil. Theorem 2.13 shows that the secants on an exterior
point \( P \) already determines the complete pencil of lines of \( \mathcal{P} \) that are on \( P \). Any nonsecant of \( \mathcal{P} \) that does not intersect in \( \mathcal{P} \) any of the secants on an exterior point is also on this exterior point.

In the rest of this chapter \( \mathcal{P} \) (as described in Chapter I) will be the hyperbolic geometry determined from a conic \( \mathcal{C} \) in a finite desarguesian projective plane \( \mathcal{W} \) of order \( n \). It will be shown that the projective plane \( \mathcal{W} \) can be reconstructed wholly from within the incidence structure of \( \mathcal{P} \). To accomplish this it will be necessary to determine the exterior points of \( \mathcal{W} \) and the points on \( \mathcal{C} \) and to determine the pencils of lines on such points. It will also be necessary to determine the tangents to \( \mathcal{C} \).

(1) Reconstruction of the Exterior Points

In determining the pencils of lines in \( \mathcal{P} \) that are on an exterior point \( E \) in \( \mathcal{W} \), the \( \frac{n-1}{2} \) nonsecants of such a pencil will be determined first. A nonsecant in \( \mathcal{P} \) is distinguished as a line containing \( \frac{n+1}{2} \) points. A secant in \( \mathcal{P} \) is a line containing \( \frac{n-1}{2} \) points.

Let \( l_i \) be any secant in \( \mathcal{P} \); then \( l_i \) contains the points \( P_i, i=1,2,\ldots,\frac{n-1}{2} \). These are interior points in \( \mathcal{W} \). By Theorem 2.8 and Theorem 2.10 the polar
of $P_i$, the nonsecant $b_i$, in $P$ is the unique line intersected in $P$ only by all secants of $P_i$ if $n \equiv 1 \pmod{4}$ or intersected in $P$ only by all nonsecants on $P_i$ if $n \equiv 3 \pmod{4}$. Thus by this property, the polars of the $\frac{n-1}{2}$ points on $l_1$ can be distinguished in $P$. The pole of $l_1$ in $P$ is an exterior point $E_1$, and since $P$ is desarguesian and $P_i$, $i=1,2,...,\frac{n-1}{2}$, lie on the polar of $E_1$, the lines $b_i$, $i=1,2,...,\frac{n-1}{2}$, belong to the pencil on the exterior point $E_1$ of $P$.

Let $L_1 = \{b_i | b_i$ is the polar of a point $P_i$ on $l_1$ in $P\}$. This set consists of $\frac{n-1}{2}$ nonsecants of $P$ in a pencil on an exterior point $E_1$ of $P$. By Theorem 2.11, if $b$ is a secant in $P$ not equal to $l_1$ such that $b$ does not intersect in $P$ any of the nonsecants in $L_1$, then $b$ also belongs to the same pencil as the lines in $L_1$. Therefore let $L_1' = \{b | b \neq l_1, b$ a secant in $P, b$ does not intersect in $P$ any line in $L_1\}$, so $L_1'$ consists of $\frac{n-1}{2}$ secants in the pencil on the exterior point $E_1$ of $P$; and $L_1 \cup L_1'$ consists of all lines in $P$ on an exterior point $E_1$ in $P$.

In this manner, we can determine the $\frac{(n+1)n}{2}$ sets $L_i \cup L_i'$, $i=1,2,...,\frac{(n+1)n}{2}$ of lines in $P$, each of which make up a pencil, except for tangents, on an exterior point of $P$. Therefore we can add to $P$ the points
$E_i^* = (n+1)n/2$ where $E_i^*$ is defined to be the point of intersection of the lines in $L_i \cup L_i$. These points correspond to the $(n+1)n$ exterior points of $\pi$.

(2) **Reconstruction of the Points on $\xi$**

In constructing the points on the conic $\xi$, it is necessary to determine $n+1$ sets of $n$ secants in $P$ such that each of these sets of $n$ secants consists of a pencil of lines on a point of the conic $\xi$ in $\pi$.

If $s_1$ and $s_2$ are two secants of $P$ that do not intersect in $P$ and do not belong to a pencil on a point $E_i^* = (n+1)n/2$, then $s_1$ and $s_2$ must intersect at a point $Q_1$ on $\xi$ in $\pi$. By Theorem 2.5, there exists a unique secant $s$ that is a special line with respect to $s_1$ and $s_2$. This line can be distinguished in $P$, for it is the unique line $s$ in $P$ such that if $P$ is a point of $P$ on $s$ every line on $P$ not equal to $s$, except exactly one line intersects $s_1$ in $P$ if and only if it does not intersect $s_2$ in $P$.

In $P$, the secant $s$ contains $n-1/2$ points. On each of these points, there is exactly one line besides $s$ that does not intersect $s_1$ or $s_2$ in $P$. Each of these $n-1/2$ lines is necessarily on $Q_1$ in $\pi$. Along with $s_1$ and $s_2$ this determines $n+3/2$ secants of $P$ which must intersect at a point $Q_1$ on $\xi$ in $\pi$. By
Remark 2.14 this determines in $\mathcal{P}$, the pencil of secants of $\mathcal{P}$ that are on $Q_1$ in $\Pi$. Thus we can determine in $\mathcal{P}$ the secants $s_3, s_4, \ldots, s_n$ such that $s_i$, $i=1,2,\ldots,n$ are the secants of $\mathcal{P}$ that are in a pencil on a point $Q_1$ on $\mathcal{C}$ in $\Pi$. If we add to $\mathcal{P}$ the point $Q_1^*$ where $Q_1^*$ is defined to be the point of intersection of the secants $s_i$, $i=1,2,\ldots,n$; $Q_1^*$ corresponds to $Q_1$ in $\Pi$.

Continuing in this manner we can determine in $\mathcal{P}$ the $n+1$ sets of $n$ secants $S_i$, $i=1,\ldots,n+1$, each of which consists of $n$ secants belonging to the same pencil on a point of $\mathcal{C}$ in $\Pi$. The points $Q_i^*$, $i=1,\ldots,n+1$ are added to $\mathcal{P}$ where $Q_i^*$ is defined to be the point of intersection of the secants in $S_i$. The points $Q_i^*$, $i=1,\ldots,n+1$ correspond to the points on $\mathcal{C}$ in $\Pi$.

(3) Reconstruction of the Tangent Lines

The constructions in (1) and (2) have completed $\mathcal{P}$ to the desarguesian projective plane $\Pi$ except for tangent lines. Therefore in order to complete $\mathcal{P}$ to $\Pi$, we only have to characterize the tangent lines. Each tangent line contains one point of $\mathcal{C}$ and $n$ exterior points in $\Pi$. Thus we must only determine which $n$ of the points $E_i^*$ are on a tangent at a point $Q_i^*$. 
Let \( S_i = \{ s_{ij}^i | j=1,2,\ldots,n \} \) \( i=1,2,\ldots,n+1 \) be as in construction (2). In construction (1), for each secant \( s_{ij}^i \) in \( R \) a point \((E_j^i)^*\) was added to \( R \) corresponding to the pole in \( \Pi \) of \( s_{ij}^i \), an exterior point \( E_j^i \). Since \( \Pi \) is desarguesian and the \( n \) secants in the set \( S_i \) intersect at a point \( Q_i \) on \( \mathcal{C} \) in \( \Pi \), the poles of these secants, i.e. \( \{ E_j^i | j=1,2,\ldots,n \} \), lie on the polar of \( Q_i \) in \( \Pi \). But the polar of \( Q_i \) is the tangent \( t_i \) to \( \mathcal{C} \) at \( Q_i \). The points of the tangent \( t_i \) to \( \mathcal{C} \) at \( Q_i \) in \( \Pi \) are the points in the set \( \{ E_j^i | j=1,2,\ldots,n \} \cup \{ Q_i \} \).

Therefore if we add to \( R \) the \( n+1 \) lines \( t_i^* \), \( i=1;\ldots,n+1 \) defined in the following way:

\[
t_i^* = \{(E_j^i)^* | j=1,2,\ldots,n\} \cup \{ Q_i^* \}
\]

these lines correspond to the tangents to \( \mathcal{C} \) in \( \Pi \).

Beginning then with the hyperbolic plane \( R \), if we extend the set of points in \( R \) and the set of lines in \( R \) in the manner described in constructions (1), (2), and (3), we have the plane \( \Pi \). Thus we have the following theorem:
Theorem 2.15 If $\Pi$ is a finite desarguesian projective plane and $\mathcal{P}$ is the hyperbolic plane derived from a conic $\mathcal{C}$ in $\Pi$ in the manner of Ostrom, then the incidence structure of $\Pi$ is completely determined within the incidence structure of $\mathcal{P}$; hence, the embedding of $\mathcal{P}$ into $\Pi$ is unique.
CHAPTER III
INvolutions

In the finite desarguesian projective plane \( \Pi \) coordinatized by a field \( \mathcal{F} \), let \( \mathcal{C} \) be the conic associated with the form \( f(x,y,z) = xy+z^2 \) and \( P = (1,m,0) \), \( m \neq 0 \), be a point on \( l_{\infty} \). The polar of \( P \) is the line \( l \) whose affine points satisfy the equation \( y = -mx \).

If \( w \) is the collineation induced on \( \Pi \) by the non-singular matrix \( W \),
\[
W = \begin{pmatrix}
0 & -m & 0 \\
-m^{-1} & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
then \( w \) is involutory with center \( P \) and axis \( l \), and \( w \) fixes \( \mathcal{C} \). Hence \( w \) fixes the set of interior points and the set of exterior points. For any line \( l_1 \) whose affine points satisfy the equation \( y = Ax+B \), \( A \neq 0 \), \( w(l_1) = l_2 \) is the line whose affine points satisfy the equation \( y = A^{-1}m^2x+A^{-1}mB \). If the affine points of \( l_1 \) satisfy the equation \( y = u \), \( u \neq 0 \) then the affine points of \( l_2 \) satisfy the equation \( x = -m^{-1}u \).
Definition 3.1 If $P$ is a point not on an oval $\mathcal{O}$ of a projective plane $\Pi$, the line $l_1$ is said to be associated with $l_2$ with respect to the lines on $P$ if and only if for every line $l'$ on $P$, $l'$ intersects $l_1$ in an interior point if and only if $l'$ intersects $l_2$ in an interior point.

Note: Such a line $l_1$ associated with $l_2$ cannot be on $P$, and $l_1$ is a secant or a nonsecant if and only if $l_2$ is a secant or a nonsecant, respectively.

Theorem 3.2 If $\mathcal{C}$ is a conic in a finite desarguesian projective plane $\Pi$ coordinatized by a field $\mathcal{F}$ of order $n$, $P$ a point not on $\mathcal{C}$, and $l$ the polar of $P$, then for any nonsecant $l_1$, not equal to $l$, there exists one and only one nonsecant $l_2$ associated with $l_1$ with respect to lines on $P$.

Proof: Without loss of generality, let $\mathcal{C}$ be the conic associated with the form $f(x,y,z) = xy+z^2$. Since the group of projectivities fixing $\mathcal{C}$ is transitive on interior points and on exterior points, we may also assume $P = (1,m,0)$ on $l_\infty$ where $m \neq 0$ since $P$ is not on $\mathcal{C}$. 
Let \( w \) be the involutory collineation with center \( P \) and axis \( \ell \), and let \( \ell_2 = w(\ell_1) \). If \( \ell' \) is any line on \( P \), \( X_1 \) the point of intersection of \( \ell' \) and \( \ell_1 \) and \( X_2 \) the point of intersection of \( \ell' \) and \( \ell_2 \), then \( w(\ell') = \ell' \) and since \( w(\ell_1) = \ell_2 \) \( w(X_1) = X_2 \). Because \( w \) fixes both the set of interior points and the set of exterior points as well as the points on \( \ell \), \( X_1 \) and \( X_2 \) are either both exterior, both interior, or both on \( \ell \). Thus any line \( \ell' \) on \( P \) intersects \( \ell_1 \) in an interior point if and only if it intersects \( \ell_2 \) in an interior point. So \( \ell_2 \) is a line associated with \( \ell_1 \) with respect to lines on \( P \).

Since \( \ell_1 \) is a nonsecant, its point of intersection with \( \ell_\infty \) is not at either the point \((1,0,0)\) or \((0,1,0)\) because these points are on \( \ell \). Therefore the affine points of \( \ell_1 \) satisfy an equation of the form \( y = Ax + B \) where \( A \neq 0 \), and \( A \neq m \) because \( \ell_1 \) is not on \( P \). Since \( w \) is induced on \( \Pi \) by the matrix \( W \) given above, the affine points on \( \ell_2 = w(\ell_1) \) satisfy the equation \( y = m^2A^{-1}x+mA^{-1}B \).

In order to show the uniqueness of \( \ell_2 \), let \( \ell_3 \) be any nonsecant of \( \Pi \) associated with \( \ell_1 \) with respect to lines on \( P \). Since \( \ell_3 \) is a nonsecant, its affine points must satisfy an equation of the form \( y = Cx + D \) where \( C \neq 0, m \).
If \( l(b) \) is any line on \( P \), its affine points satisfy an equation of the form \( y = mx+b \). The intersection of \( l_1 \) and \( l(b) \) is the point \( P_1 = (B-b, Bm-Ab, m-A) \) and

\[
f(P_1) = (B-b)(Bm-Ab) + (m-A)^2
\]

\[
= Ab^2 - (Bm+AB)b + (m-A)^2 + B^2m
\]

The intersection of \( l_3 \) and \( l(b) \) is the point \( P_3 = (D-b, Dm-Cb, m-C) \) and

\[
f(P_3) = (D-b)(Dm-Cb) + (m-C)^2
\]

\[
= Cb^2 - (Dm+CD)b + (m-C)^2 + D^2m
\]

Since \( l(b) \) intersects \( l_1 \) in an interior point if and only if it intersects \( l_3 \) in an interior point and since \( l_1 \) and \( l_3 \) are nonsecants, \( l(b) \) also intersects \( l_1 \) in an exterior point if and only if it intersects \( l_1 \) in an exterior point; and neither \( l_1 \) nor \( l_3 \) contains a point of \( C \). In terms of the form values of the points of intersection, by Theorem 1.1 \( f(P_1) \) and \( f(P_3) \), considered as quadratics in \( b \), are such that \( f(P_1) \) is a nonsquare or a square if and only if \( f(P_3) \) is a nonsquare or a square, respectively. Therefore, for all \( b \) in \( \mathcal{F} \), there exists a square \( t_b \) in \( \mathcal{F} \) such that \( f(P_3) = t_b f(P_1) \); so by Lemma 2.9, \( f(P_3) = tf(P_1) \) for some square \( t \) in \( \mathcal{F} \).
We obtain the following equations: \( C = tA \), \((Dm+CD) = t(Bm+AB)\), and \((m-C)^2+D^2m = t[(m-A)^2+B^2m]\).

From the first, we obtain \( t = CA^{-1} \) and replacing \( t \) with this expression in the second and third equations yields the equations \( D = A^{-1}(m+C)^{-1}C(Bm+AB) \) and \( A(m-C)^2+AD^2m = C(m-A)^2+CB^2m \). Substituting for \( D \) in the last equation yields

\[
A(m-C)^2 + Am[A^{-1}(m+C)^{-1}C(Bm+AB)]^2 = C(m-A)^2 + CB^2m
\]
or
\[
A(m-C)^2 + A^{-1}(m+C)^{-2}[mC^2(Bm+AB)^2] = C(m-A)^2 + CB^2m.
\]

Multiplying by \( A(m+C)^2 \) yields

\[
A^2(m-C)^2(m+C)^2 + mC^2(Bm+AB)^2 = AC(m-A)^2(m+C)^2 + AB^2mC(m+C)^2
\]
or the following fourth degree equation in \( C \):

\[
A^2C^4 + [2mA^2-m^2A-A^3m^2-ABm^2]C^3
\]
\[
+ [B^2m^3+2m^2A^2+A^2B^2m-2m^3A-2mA^3]C^2
\]
\[
+ [2A^2m^3-A^4m^2-A^3m^2ABm^2]C + A^2m^4 = 0.
\]

Since \( l_1 \) itself and \( l_2 \) are two lines associated with \( l_1 \) with respect to lines on \( P \), \( C = A^{-1}m^2 \) and \( C = A^{-1}m^2 \) are two solutions of this equation. The fourth degree polynomial has the factor \((C-A)(C-A^{-1}m^2)\) = \( A^{-1}[(C-A)(CA-m^2)]\) and the other factor yields the quadratic equation in \( C \):

\[
(*)
A^2C^2 + [2mA^2-ABm^2]C + A^2m^2 = 0.
\]
The discriminant of this quadratic is

\[ A^2B^2m^2 - 4A^3B^2m^2 = (ABm)^2(B^2-4A) \, . \]

By Lemma 2.6, since \( l_1 \) is a nonsecant, \( B^2-4A \) is a nonsquare in \( \mathcal{F} \); hence the discriminant of (*) is a nonsquare in \( \mathcal{F} \) so the quadratic (*) in \( C \) has no solutions in \( \mathcal{F} \). Therefore \( C = A \) or \( C = A^{-1}m^2 \) and \( D = B \) or \( D = A^{-1}Bm \), respectively. So \( l_3 \) is \( l_1 \) or \( l_2 \). //

Theorem 3.2 not only shows that for a point \( P \) that there exists exactly one line associated with any nonsecant \( l_1 \) not on \( P \) and not equal to the polar of \( P \), but that the unique line associated with \( l_1 \) is the image of \( l_1 \) under the involution fixing \( \mathcal{C} \) with center \( P \) and axis the polar of \( P \).

Theorem 3.3 If \( \mathcal{C} \) is a conic in a finite desarguesian projective plane \( \mathcal{T} \) coordinatized by a field \( \mathcal{F} \) of order \( n \) and \( \mathcal{P} \) is the hyperbolic plane derived from \( \mathcal{C} \) in the manner of Ostrom, then the collineations of \( \mathcal{P} \) induced by the involutory homologies of \( \mathcal{T} \) fixing \( \mathcal{C} \) are completely determined within the incidence structure of \( \mathcal{P} \).
Proof: Let \( \zeta \) be an involutory homology of \( \Pi \) fixing \( \mathcal{C} \), \( P \) the center of \( \zeta \) where \( P \) is a point not on \( \mathcal{C} \) and \( l \) is the polar of \( P \).

(1) \( P \) is a point of \( \mathcal{P} \)

If \( P \) is a point of \( \mathcal{P} \), \( P \) is an interior point of \( \Pi \) and \( l \) is a nonsecant. By Theorem 2.8 and Theorem 2.10, \( l \) is the unique line of \( \mathcal{P} \) intersected in \( \Phi \) by all secants on \( P \) if \( n \equiv 1 \pmod{4} \) or intersected in \( \Phi \) by all nonsecants on \( P \) if \( n \equiv 3 \pmod{4} \). \( l \) is fixed pointwise by \( \zeta \) and the only other fixed lines are those on \( P \).

If \( l_1 \) is any nonsecant not equal to \( l \) and not on \( P \), let \( l_2 = \zeta(l_1) \). Then by Theorem 3.2, \( l_2 \) is the unique line associated with \( l_1 \) with respect to lines on \( P \). Therefore in \( \Phi \), \( l_2 \) is the unique line with the property that if \( l' \) is any line on \( P \), \( l' \) intersects \( l_2 \) if and only if it intersects \( l_1 \) in \( \Phi \).

This completely determines the collineation induced by \( \zeta \) on \( \Phi \), because for any point \( Q \) of \( \Phi \), \( Q \) is the point of intersection of two nonsecants of \( \Phi \) and thus the image of \( Q \) under \( \zeta \) is the intersection of the images under \( \zeta \) of the nonsecants on \( Q \).

(2) \( P \) is not a point of \( \Phi \)

If \( P \) is not a point of \( \Phi \), then \( P \) is an exterior point of \( \Pi \) and \( l \) is a secant. Thus
1 is an element of \( P \) containing \( \frac{n-1}{2} \) points. Beginning with 1 in \( P \) we can determine the \( n-1 \) lines of \( P \) that intersect at \( P \) in \( \Pi \), as in the construction of exterior points in Chapter II. Let these lines be \( s_i \), \( i=1,2,\ldots,n-1 \). These lines are fixed by the collineation of \( P \) induced by \( \xi \) and 1 is fixed pointwise.

If \( l_1 \) is any nonsecant not equal to \( s_i \), \( i=1,2,\ldots,n-1 \) and \( l_2 \) is a line in \( P \) such that any line \( s_i \), \( i=1,2,\ldots,n-1 \) intersects \( l_1 \) in \( P \) if and only if it intersects \( l_2 \) in \( P \), then by Theorem 3.2 \( l_2 \) is the unique line associated with \( l_1 \) with respect to lines on \( P \) in \( \Pi \). Thus \( l_2 \) is the image of \( l_1 \) under the collineation induced by \( \xi \) on \( P \).

This completely determines the collineation, because any point \( Q \) of \( P \) is the point of intersection of two nonsecants of \( P \).

Remark 3.4 By the methods of construction in Chapter II, the pencil of lines on any point of \( \Pi \) can be determined within the incidence structure of \( P \). By Theorem 3.3 the image of any line of \( P \) under an involutory homology of \( \Pi \) fixing the conic \( \mathcal{C} \) can be determined within the incidence structure of \( P \). Since any point \( Q \) of \( \Pi \) is on at least two lines of \( P \), the image of \( Q \) under
any involutory homology of \( \Pi \) fixing the conic \( \mathcal{C} \) is the intersection of the images of these lines in \( \mathcal{P} \). Hence, the involutory homologies of \( \Pi \) fixing \( \mathcal{C} \) are completely determined within the incidence structure of \( \mathcal{P} \).

It was not necessary to consider secants in order to establish Theorem 3.3, but it is of interest to consider the combinatorial properties of secants in \( \mathcal{P} \) with respect to a point \( P \) in \( \mathcal{P} \) and its polar \( l \).
For the rest of this chapter, \( \mathcal{P} \) is a finite desarguesian projective plane of order \( n \) where \( n > 17 \).

**Theorem 3.5** Let \( \mathcal{C} \) be a conic in a finite desarguesian projective plane \( \mathcal{P} \) of order \( n \) and \( A, B, C, D \) be four distinct points on \( \mathcal{C} \) such that the secants \( AB \) and \( CD \) intersect at a point \( E \).

If \( l \) is any line not on \( A, B, C, \) or \( D \) but on a diagonal point other than \( E \) of the complete quadrangle determined by \( A, B, C, \) and \( D \) then \( l \) intersects \( AB \) in an interior point if and only if \( l \) intersects \( CD \) in an interior point.

**Proof:** Let \( X \) be a diagonal point of the quadrangle \( ABCD \) where \( X \neq E \). Without loss of generality, we
may assume that \( X \) is the point of intersection of the lines \( AD \) and \( BC \). (See Figure 4.) Let \( l \) be any line on \( X \), not \( AD \) or \( BC \) and let the points of intersection of \( l \) and the lines \( AB \), \( CD \), and \( BD \) be \( P \), \( Q \), and \( R \), respectively. If \( P = Q = E \), the result is trivial. If \( P \neq Q \), consider the two triangles, each made up of three secants, \( ABD \) and \( CBD \).

If \( X \) and \( R \) are interior points, then by Theorem 2.1 \( P \) and \( Q \) must be exterior points. If \( X \) and \( R \) are exterior points, by Theorem 2.1, \( P \) and \( Q \) must be exterior points. If \( X \) is an interior point and \( R \) is an exterior point, by Theorem 2.1, \( P \) and \( Q \) are interior points. Finally if \( X \) is an exterior point and \( R \) is an interior point, \( P \) and \( Q \) must be interior points.

In each case \( l \) intersects \( AB \) in an interior point if and only if it intersects \( CD \) in an interior point. 

Lemma 3.6 If \( \mathcal{F} \) is a field of order \( n \), \( n > 17 \), there does not exist a quadratic over \( \mathcal{F} \) of the form \( f(x) = c(x-a)^2 - b \), \( b, c \neq 0 \) such that \( f(x) \) is a nonsquare in \( \mathcal{F} \) if and only if \( x \) is a nonsquare in \( \mathcal{F} \).
Proof: Suppose there does exist \( f(x) = c(x-a)^2 - b \); \( b, c \neq 0 \) such that \( f(x) \) is a nonsquare in \( \mathcal{F} \) if and only if \( x \) is a nonsquare in \( \mathcal{F} \), and hence \( f(x) \) is zero or a square in \( \mathcal{F} \) if \( x \) is a square in \( \mathcal{F} \).

(1) \( a \neq 0 \)

By Corollary 2.3, there are at least \( \frac{n-5}{4} \) values of \( x \) which are nonsquares in \( \mathcal{F} \) such that \(-x+2a\) is a square in \( \mathcal{F} \). Since \( f(x) = f(2a-x) \), there is at least one square \( s \) in \( \mathcal{F} \) such that \( f(s) \) is a nonsquare. This is a contradiction.

(2) \( a = 0 \), i.e. \( f(x) = cx^2 - b \)

If \( n \equiv 3 \pmod{4} \), then \(-1\) is a nonsquare in \( \mathcal{F} \). But if \( x \) is a nonsquare in \( \mathcal{F} \), \( f(x) \) is a nonsquare in \( \mathcal{F} \); and since \( f(x) = f(-x) \), there is at least one square \( s \) in \( \mathcal{F} \) such that \( f(s) \) is a nonsquare. This is a contradiction.

If \( n \equiv 1 \pmod{4} \), then \(-1\) is a square in \( \mathcal{F} \). Since \( f(x) \) is zero or a square in \( \mathcal{F} \) if \( x \) is a square in \( \mathcal{F} \) and since there are \( \frac{n-1}{2} \) squares in \( \mathcal{F} \) and \( f(x) \) is zero for at most two squares there are at least \( \frac{n-5}{2} \) square values of \( x \) in \( \mathcal{F} \) such that \( cx^2 - b = y^2 \) for some \( y \neq 0 \) in \( \mathcal{F} \). Therefore the equation \( cx^2 - y^2 = b \) has at least \( (n-5)+2 = n-3 \) solutions \((x, y)\) in \( \mathcal{F} \) where \( x \) is a square in \( \mathcal{F} \). But then the number \( N_1 \) of solutions \((x, y)\) of the
equation \( cx^4 - y^2 = b \) is at least \( 2(n-3) \). But the number \( N_1 \) of solutions \((x, y)\) of an equation \( cx^4 - y^2 = b \) over a finite field \( \mathcal{F} \) of order \( n \) where \( n \equiv 1 \pmod{4} \) is such that \( N_1 \leq n + 3\sqrt{n} \) \([\text{Weil, 13}]\). Since \( n > 20 \), \( 2n - 6 > n + 3\sqrt{n} \) and therefore we have a contradiction.

Therefore there is no quadratic of the form
\[
f(x) = c(x-a)^2 - b \quad b, c \neq 0
\]
such that \( f(x) \) is a nonsquare in \( \mathcal{F} \) if and only if \( x \) is a nonsquare in \( \mathcal{F} \).

If \( f(x) = ax^2 + bx + c \) is any quadratic over a field \( \mathcal{F} \) of order \( n \) where \( a \neq 0 \) and \( b^2 - 4ac \neq 0 \), then \( f(x) \) can be written in the form
\[
f(x) = a(x+b(2a)^{-1})^2 - (b^2 - 4ac)(4a)^{-1}
\]
and \( a, (b^2 - 4ac)(4a)^{-1} \neq 0 \). So we have the following corollary:

**Corollary 3.7** If \( \mathcal{F} \) is a field of order \( n \), \( n > 17 \), there does not exist a quadratic over \( \mathcal{F} \) of the form \( f(x) = ax^2 + bx + c \) where \( a \neq 0 \) and \( b^2 - 4ac \neq 0 \) such that \( f(x) \) is a nonsquare in \( \mathcal{F} \) if and only if \( x \) is a nonsquare in \( \mathcal{F} \).
Corollary 3.8 If $\mathcal{F}$ is a field of order $n$, $n > 17$, and $f(x) = ax + b$, $a \neq 0$ is a linear expression in $x$ over $\mathcal{F}$, then there does not exist a quadratic expression in $x$ over $\mathcal{F}$, $g(x) = Ax^2 + Bx + C$, $A \neq 0$, such that $f(x)$ is a nonsquare in $\mathcal{F}$ if and only if $g(x)$ is a nonsquare in $\mathcal{F}$.

Proof: Suppose there does exist a quadratic $g(x) = Ax^2 + Bx + C$, $A \neq 0$, such that $g(x)$ is a nonsquare in $\mathcal{F}$ if and only if $f(x) = ax + b$, $a \neq 0$, is a nonsquare in $\mathcal{F}$. Hence $g(x)$ is a square or zero in $\mathcal{F}$ if and only if $f(x)$ is a square or zero in $\mathcal{F}$.

Let $t = ax + b = f(x)$, then $g(x)$ is a quadratic expression in $t$, i.e. $g(x) = h(t) = A_1 t^2 + B_1 t + C_1$ where $A_1 \neq 0$ and $B_1^2 - 4A_1 C_1 \neq 0$ since $h(t)$ is a nonsquare in $\mathcal{F}$ for some values of $t$ and $h(t)$ is a square in $\mathcal{F}$ for some values of $t$.

But, $h(t)$ is a nonsquare in $\mathcal{F}$ if and only if $t$ is a nonsquare in $\mathcal{F}$. This is a contradiction to Corollary 3.7.

Thus no such $g(x)$ exists in $\mathcal{F}$.  //

Corollary 3.8 can also be proven with the application of a formula of Hasse and Weil [Eichler p.306, 5] concerning the number of solutions in a finite field.
of order \( n \) of an equation of the form \( y^2 = h(x) \) where \( h(x) \) is a cubic polynomial. For if there would exist a linear expression \( f(x) = ax + b \) and a quadratic expression \( g(x) = Ax^2 + Bx + C \) such that \( f(x) \) is a nonsquare if and only if \( g(x) \) is a nonsquare, then the equation

\[(*)\]
\[y^2 = h(x) = f(x) \cdot g(x)\]

would have at least \( 2n-3 \) solutions \((x, y)\). But the number \( N_1 \) of solutions of \((*)\) is such that

\[|N_1-(n+1)| \leq 2\sqrt{n} \quad \text{or} \quad N_1 \leq n+1+2\sqrt{n} . \]

If \( n \geq 11 \), then \( 2n-3 \geq n+1+2\sqrt{n} \) and thus such an \( f(x) \) and \( g(x) \) could not exist.

**Theorem 3.9** Let \( \mathcal{C} \) be a conic in a finite desarguesian projective plane \( \pi \) of order \( n > 17 \), \( P \) a point not on \( \mathcal{C} \) and \( l \) the polar of \( P \). For any secant \( l_1 \) not equal to \( l \) and not on \( P \) the following is true:

(1) if \( P \) is an interior point, there exists one and only one secant \( l_2 \) associated with \( l_1 \) with respect to lines on \( P \)

or (2) if \( P \) is an exterior point, there exists exactly three other secants \( l_2, l_3, \) and \( l_4 \) associated with \( l_1 \) with respect to lines on \( P \) .
Proof: First we coordinatize \( \mathcal{C} \) in the following way:

Let \( l_1 \) be the line \( YZ \) where \( Y \) and \( Z \) are points of \( \mathcal{C} \). Since \( l_1 \) is not the polar of \( P \), at least one of the lines \( PY \) or \( PZ \) is a secant to \( \mathcal{C} \). Without loss of generality, assume \( PY \) is the secant \( l' \) and let the point \( X \) be the other point of \( \mathcal{C} \) on \( l' \). If \( X = (1,0,0) \), \( Y = (0,1,0) \) and the point of intersection of the tangents to \( \mathcal{C} \) at \( X \) and \( Y \) is \( O = (0,0,1) \), and \( Z = (1,-1,1) \) then \( \mathcal{C} \) is coordinatized by a field \( \mathcal{F} \) of order \( n \) and \( l' \) is the line \( l_\infty \), \( P = (1,m,0), m \neq 0 \), and the affine points on \( l_1 \) satisfy the equation \( x = 1 \). (See Figure 5.) \( \mathcal{C} \) is the conic associated with the form \( f(x,y,z) = xy+z^2 \).

As in Theorem 3.2, if \( w \) is the involutory collineation with center \( P \) and axis \( l \), \( l_2 = w(l_1) \) is a secant associated with \( l_1 \) with respect to lines on \( P \). \( l_2 \) is the line whose affine points satisfy the equation \( y = -m \). Let \( l_3 \) be any secant not equal to \( l \) and associated with \( l_1 \) with respect to lines on \( P \).

Suppose the affine points on \( l_3 \) satisfy the equation \( y = Mx+B, M \neq m \). If \( l(b) \) is any line on \( P \), the affine points on \( l(b) \) satisfy an equation of the form \( y = mx+b \). If \( P_1 \) is the point of intersection of \( l_1 \) and \( l(b) \), then \( P_1 = (1,m+b,1) \) and
Figure 5
If \( P_3 \) is the point of intersection of \( l(b) \) and \( l_3 \),
\[
P_3 = ((b-B)(M-m)^{-1},(Mb-Bm)(M-m)^{-1},1)
\]
\[
= (b-B,Mb-Bm,M-m)
\]
and
\[
f(P_3) = (Mb-Bm)(b-B) + (M-m)^2
\]
\[
= Mb^2 - (BM+Bm)b + (M-m)^2 + B^2m.
\]

Since \( l(b) \) intersects \( l_1 \) in an interior point if and only if it intersects \( l_3 \) in an interior point, in terms of the form values by Theorem 1.1, \( f(P_1) \) and \( f(P_3) \), considered as expressions in \( b \), are such that \( f(P_1) \) is a nonsquare in \( \mathcal{F} \) if and only if \( f(P_3) \) is a nonsquare in \( \mathcal{F} \). This is a contradiction to Corollary 3.8 unless \( M = 0 \). Therefore if \( l_3 \) is a secant associated with \( l_1 \), the affine points on \( l_3 \) must satisfy an equation of the form \( x = u \) or \( y = v \) where \( u \) and \( v \) are nonzero elements of \( \mathcal{F} \).

First, let the affine points on \( l_3 \) satisfy the equation \( x = u \), \( u \neq 0 \). Then \( P_3 \), the point of intersection of \( l_3 \) and \( l(b) \) is the point \( P_3 = (u, mu+b, 1) \) and \( f(P_3) = ub + mu^2 + 1 \). Thus \( f(P_1) \) and \( f(P_3) \) are two linear expressions in \( b \) such that \( f(P_1) \) is a nonsquare in \( \mathcal{F} \) if and only if \( f(P_3) \) is a nonsquare in \( \mathcal{F} \). But by Corollary 2.3, this can occur only if \( f(P_3) \) is a multiple of \( f(P_1) \), i.e. there exists a square \( t \) in \( \mathcal{F} \).
such that $f(P_3) = tf(P_1)$ for all $b$ in $\mathcal{F}$. From $ub+(mu^2+1) = t(b+m+1)$ we obtain the equations $t = u$ and $mu^2+1 = um+u$ which yields $u = 1$ or $u = m^{-1}$. Thus $x = 1$ or $x = m^{-1}$. Note that if $P$ is an interior point, $m$ is a nonsquare and $m^{-1}$ is a nonsquare, but then since $t = m^{-1}$ and $t$ is a square, $l_3$ can only be the secant $l_1$.

Now let the affine points on $l_3$ satisfy the equation $y = v$, $v \neq 0$, then $P_3 = (m^{-1}(v-b),v,1)$ and $f(P_3) = -m^{-1}vb+m^{-1}v^2+1$ and as above $f(P_3) = tf(P_1)$ where $t$ is a square in $\mathcal{F}$, so $t = -m^{-1}v$ and $m^{-1}v^2+1 = -v-m^{-1}v$, i.e. $v = -1$ or $v = -m$. Thus $y = -1$ or $y = -m$. Again note that if $P$ is an interior point $m$ is a nonsquare and $m^{-1}$ is a nonsquare, thus $v$ could not be $-1$ since $t = -m^{-1}v$ and $t$ is a square. Thus $l_3$ can only be $l_2$ if $P$ is an interior point.

Therefore if $P$ is an interior point, $l_2$ is the only line associated with $l_1$ with respect to lines on $P$. If $P$ is an exterior point, the secants $l_2$, $l_3$, and $l_4$ are associated with $l_1$ where $l_3$ and $l_4$ are the secants whose affine points satisfy the equations $x = m^{-1}$ and $y = -1$. //
It is easily seen that if $l_1$, $l_2$, $l_3$, and $l_4$ are the secants with affine equations $x = 1$, $y = -m$, $x = m^{-1}$, and $y = -1$ respectively that $P = (1, m, 0)$ is a diagonal point of the quadrangle determined by the points of $C$ on these lines. (See Figure 6.) By Theorem 2.1 and Theorem 3.5 if $P$ is exterior since it is a diagonal point, any line on $P$ must intersect $l_1$, $l_2$, $l_3$, and $l_4$ in the same kind of point. By Theorem 2.1, if $P$ is interior $P$ is a special point with respect to any two of the above secants that intersect on $C$, and therefore for any pair of these secants that intersect on $C$, any line on $P$ intersects one of a pair in an interior point if and only if it intersects the other in an exterior point. So we obtain the following corollaries:

**Corollary 3.10** If $C$ is a conic in a finite desarguesian projective plane $\Pi$ of order $n > 17$, $P$ an exterior point and $l$ the polar of $P$, then the $\frac{n^2-1}{2}$ secants not on $P$ and not equal to $l$ are partitioned into $\frac{n^2-1}{8}$ sets of four secants each such that each of these sets consists of all the secants associated with any one of its members with respect to lines on $P$ and for each set of four secants, $P$ is a diagonal point of the quadrangle on $C$ determined by these secants.
Figure 6
Note: There are \( \frac{(n+1)n}{2} \) secants in \( \Pi \), \( \frac{n-1}{2} \) secants on \( P \), and \( 1 \) is a secant. \( \frac{(n+1)n}{2} - \left( \frac{n-1}{2} + 1 \right) = \frac{n^2-1}{2} \).

**Corollary 3.11** If \( C \) is a conic in a finite desarguesian projective plane \( \Pi \) of order \( n > 17 \), \( P \) an interior point and \( 1 \) the polar of \( P \), then if \( l_1 \) and \( l_2 \) are two associated secants not on \( P \) there exists exactly two other associated secants \( l_3 \) and \( l_4 \) such that if \( l' \) is any line on \( P \) \( l' \) intersects \( l_1 \) and \( l_2 \) in interior points if and only if \( l' \) intersects \( l_3 \) and \( l_4 \) in exterior points and \( P \) is a diagonal point of the quadrangle on \( C \) determined by \( l_1 \), \( l_2 \), \( l_3 \), and \( l_4 \).

Corollary 3.10 also shows that the polar of \( P \), a secant not included in the partition, is the unique secant intersected by the lines on \( P \) in the manner described in (1) and (2) of Theorem 2.8. Thus Theorem 2.10 is also true when \( P \) is an exterior point.
CHAPTER IV
MORE COMBINATORIAL PROPERTIES

Theorem 4.1 If $C$ is a conic in a finite desarguesian projective plane $\Pi$ coordinatized by a field $\mathcal{F}$ of order $n$, and the two secants $AB$ and $CD$ where $A,B,C,D$ are on $C$ intersect at an exterior point $E$, then the other two diagonal points of the complete quadrangle determined by $A$, $B$, $C$, and $D$ consist of

(1) one interior point and one exterior point
if $n \equiv 3 \pmod{4}$
or (2) two interior points or two exterior points
if $n \equiv 1 \pmod{4}$.

Proof: Without loss of generality we may assume that $C$ is associated with the form $f(x,y,z) = xy+z^2$. Because the group of projectivities fixing $C$ is triply transitive on $C$, it will be sufficient to prove the result when $A = (0,1,0)$, $B = (1,0,0)$ and $C = (-1,1,1)$. Let $D$ be any other point on $C$, i.e. $D = (t,-t^{-1},1)$, and $X$ and $Y$ be the two diagonal points not equal to $E$. (See Figure 7.) Then $E = (1,(1+t^{-1})(-1-t)^{-1},0)$,
Figure 7(a)

Figure 7(b)
$X = (-1, -t^{-1}, 1)$, and $Y = (t, 1, 1)$. Thus $f(E) = -(1+t^{-1})(1+t)^{-1}$, $f(X) = t^{-1} + 1$, and $f(Y) = t + 1$, so $f(E)f(X)f(Y) = -(t^{-1} + 1)^2$. Since $E$ is an exterior point $f(E)$ is a square in $\mathcal{F}$, so $f(X)f(Y) = -k^2$ for some $k$ in $\mathcal{F}$.

If $n \equiv 3 \pmod{4}$, $-1$ is a nonsquare in $\mathcal{F}$, so $f(X)f(Y)$ is a nonsquare in $\mathcal{F}$, and therefore one of $f(X)$ and $f(Y)$ is a square and the other is a nonsquare. By Theorem 1.1, $X$ and $Y$ consist of one interior point and one exterior point.

If $n \equiv 1 \pmod{4}$, $-1$ is a square in $\mathcal{F}$, so $f(X)f(Y)$ is a square in $\mathcal{F}$, and therefore both $f(X)$ and $f(Y)$ are squares or both are nonsquares in $\mathcal{F}$. By Theorem 1.1, $X$ and $Y$ consist of two exterior points or two interior points. //

**Theorem 4.2** Let $A$, $B$, $C$, and $D$ be four distinct points on a conic $C$ in a finite desarguesian projective plane $\mathcal{P}$ of order $n$ such that the secants $AB$ and $CD$ intersect in an exterior point $E$. If $P$ is an interior point on $AD$ but not on $BC$ and $R_i$ $i=1,2,3,4$ are the following sets of lines:

$$R_i = \{ l \mid P \in l, \ l \text{ intersects } AB \text{ in an interior point and } CD \text{ in an exterior point} \}$$
\[ R_2 = \{ l \mid P \in l, \text{ } l \text{ intersects } AB \text{ in an exterior point and } CD \text{ in an interior point} \} \]

\[ R_3 = \{ l \mid P \in l, \text{ } l \text{ intersects } AB \text{ and } CD \text{ in exterior points} \} \]

\[ R_4 = \{ l \mid P \in l, \text{ } l \text{ intersects } AB \text{ and } CD \text{ in interior points} \} \]

then \[ |R_i| = \frac{n-1}{4} + t_i, \quad |t_i| \leq 1 \quad i=1,2,3,4. \] (See Figure 8.)

**Proof:** Since \( P \) is not on \( BC \), Theorem 2.2 applies to the distribution of lines on \( P \) with respect to \( AB \) and \( AC \). By Theorem 2.2 there are \( \frac{n-1}{4} + t_2 \) (\( |t_2| \leq 1 \)) lines on \( P \) that intersect \( AB \) and \( AC \) in exterior points. Considering the secant triangle \( ACD \), since \( P \) is interior by Theorem 2.1, each of these lines intersects \( CD \) in an interior point. Thus all these lines belong to \( R_2 \), and since any line that is in \( R_2 \) must intersect \( AC \) in an exterior point, \[ |R_2| = \frac{n-1}{4} + t_2. \]

By Theorem 2.2, there are \( \frac{n-1}{4} + t_3 \) (\( |t_3| \leq 1 \)) lines on \( P \) that intersect \( AB \) in an exterior point and \( AC \) in an interior point. Considering the triangle \( ACD \), since \( P \) is interior by Theorem 2.1, each of these lines must intersect \( CD \) in an exterior point. Thus all these lines belong to \( R_3 \), and since any line that is in \( R_3 \) must intersect \( AC \) in an interior point, \[ |R_3| = \frac{n-1}{4} + t_3. \] (This set contains the line \( PE \).)
By Theorem 2.2, there are \( \frac{n-1}{4} + t_4 \) \((|t_4| \leq 1)\) lines on \( \ell \) that intersect \( AB \) and \( AC \) in interior points. Considering triangle \( ACD \), by Theorem 2.1, each of these lines must intersect \( CD \) in an exterior point. Thus these lines belong to \( R_1 \) and since any line in \( R_1 \) must intersect \( AC \) in an interior point, \( |R_1| = \frac{n-1}{4} + t_4 \).

By Theorem 2.2, there are \( \frac{n-1}{4} + t_4 \) \((|t_4| \leq 1)\) lines that intersect \( AB \) in an interior point and \( AC \) in an exterior point, and by Theorem 2.1 each of these lines must intersect \( CD \) in an interior point. Thus these lines belong to \( R_4 \) and since any line in \( R_4 \) must intersect \( AC \) in an exterior point, \( |R_4| = \frac{n-1}{4} + t_4 \).

\[ \]

Remark 4.3 The result of Theorem 4.2 is true for any two secants intersecting at an exterior point and any interior point \( P \) on a diagonal of the quadrangle except a diagonal point. Therefore if two secants intersect at an exterior point, every interior point on a diagonal of the quadrangle determined by the four points on \( \mathcal{Q} \), except a diagonal point, is a normal point with respect to the two secants.
Theorem 4.4 Let $A$, $B$, $C$, and $D$ be four distinct points on a conic $\mathcal{C}$ in a finite Desarguesian projective plane $\pi$ of order $n$ and $P$ a point not on a diagonal of the quadrangle $ABCD$.

(1) If $P$ is an interior point, there exists a line on $P$ that intersects exactly one of the secants $AB$ and $CD$ in an interior point and the other in an exterior point.

(2) If $P$ is an exterior point, there exists a line on $P$ that intersects both secants $AB$ and $CD$ in interior points or both in exterior points.

Proof: Consider the lines $PA$, $PB$, and $PC$. At most two of these lines are tangents to $\mathcal{C}$. Without loss of generality, assume that $PA$ is not a tangent. Let $X$ be the other point of intersection of $PA$ and $\mathcal{C}$.

(See Figure 9.) Then $X$ is not equal to $B$, $C$, or $D$ since $P$ is not on a diagonal of $ABCD$. Let $Y$ be the point of intersection of the lines $XB$ and $CD$. Let $Z$ be the point of intersection of the lines $PY$ and $AB$. $Z \neq A$, for otherwise $Y$ and $X$ would be on $PA$, and $B = A$, but $B \neq A$. $Z \neq B$, for otherwise $P$ would be on $XB$ and $B \neq A$.

$P$ is a point on the side $AX$ of the secant triangle $ABX$ and $Y$ and $Z$ are on the sides $BX$ and $AB$, respectively.
By Theorem 2.1, if \( P \) is an interior point, exactly one of \( Y \) and \( Z \) is an exterior point and the other is an interior point. Thus the line \( PY \) satisfies (1).

By Theorem 2.1, if \( P \) is an exterior point, both \( Y \) and \( Z \) are interior points or both are exterior points. Thus the line \( PY \) satisfies (2).

**Remark 4.5** If \( A, B, C, \) and \( D \) are four different points on a conic \( \mathcal{C} \) and \( P \) an interior point not on a diagonal of the quadrangle \( ABCD \), then by Theorem 3.5, there exists a line on \( P \) that intersects \( AB \) and \( CD \) in two interior points or two exterior points. This line is the line on \( P \) and on a diagonal point of the quadrangle other than the diagonal point on \( AB \) and \( CD \).

The combinatorial properties established in this chapter make it possible to determine in \( \mathcal{P} \) when two secants intersect on \( \mathcal{C} \) in \( \mathcal{P} \) rather than at an exterior point of \( \mathcal{P} \). Since this is possible, the chapter will be concluded with a method of reconstructing in \( \mathcal{P} \) the pencil on a point of \( \mathcal{C} \) in \( \mathcal{P} \) without first determining the pencils of lines on exterior points of \( \mathcal{P} \) as was done in Chapter II.
If \( n \equiv 3 \pmod{4} \), in Theorem 4.7 it is shown that two non-intersecting secants in \( \mathcal{P} \) intersect at an exterior point in \( \mathcal{W} \) if and only if there exists a diagonal point in \( \mathcal{P} \) with respect to these two secants.

For all \( n > 9 \), it was shown in Theorem 2.5 that if two secants of \( \mathcal{W} \) intersect on the conic \( \mathcal{C} \), then there exists a unique special line with respect to these secants. In Theorem 4.8 it is shown that such a line does not exist when the secants intersect at an exterior point of \( \mathcal{W} \).

**Definition 4.6** If \( l_1 \) and \( l_2 \) are two secants to an oval \( \mathcal{O} \) in a finite projective plane \( \mathcal{W} \) the interior point \( P \) is called a diagonal point with respect to \( l_1 \) and \( l_2 \) if and only if every line on \( P \) intersects \( l_1 \) in an interior point if and only if it intersects \( l_2 \) in an interior point.

**Note:** \( P \) cannot be a point on \( l_1 \) or \( l_2 \).

**Theorem 4.7** If \( n \equiv 3 \pmod{4} \) and if \( l_1 \) and \( l_2 \) are two secants to a conic \( \mathcal{C} \) in a finite desarguesian projective plane \( \mathcal{W} \) of order \( n \) \((n > 9)\) that intersect at an exterior point \( E \) of \( \mathcal{W} \), then there exists
in \( \Pi \) a diagonal point with respect to \( l_1 \) and \( l_2 \).

If \( l_1 \) and \( l_2 \) intersect at a point on \( \mathcal{C} \), then there exists no point in \( \Pi \) which is a diagonal point with respect to \( l_1 \) and \( l_2 \).

**Proof:** If \( l_1 \) and \( l_2 \) intersect at an exterior point \( E \) of \( \Pi \), by Theorem 4.1 since \( n \equiv 3 \pmod{4} \) exactly one of the diagonal points of the quadrangle determined by the four points of \( \mathcal{C} \) on \( l_1 \) and \( l_2 \) is an interior point. By Theorem 3.5 this point is a diagonal point with respect to \( l_1 \) and \( l_2 \).

If \( l_1 \) and \( l_2 \) intersect at a point on \( \mathcal{C} \), by Theorem 2.5, there exists a special line \( l_3 \) with respect to \( l_1 \) and \( l_2 \) and each of the interior points on this line is a special point with respect to \( l_1 \) and \( l_2 \). By Theorem 2.2, every interior point of \( \Pi \) not on \( l_1 \), \( l_2 \), or \( l_3 \) is a normal point with respect to \( l_1 \) and \( l_2 \). Therefore there is no diagonal point with respect to \( l_1 \) and \( l_2 \) in \( \Pi \). //

Therefore in the case \( n \equiv 3 \pmod{4} \) the existence of a diagonal point in \( \mathcal{C} \) shows that two non-intersecting secants intersect at an exterior point of \( \Pi \). In the case \( n \equiv 1 \pmod{4} \), Theorem 3.5 still shows that if two secants intersect at an exterior point of \( \Pi \), the diagonal points of the quadrangle
on \( C \) determined by these secants have the property of Definition 4.6 but by Theorem 4.1, these points are not necessarily interior points, and therefore might not belong to \( \mathcal{P} \).

The following theorem shows that for all \( n > 9 \), two non-intersecting secants in \( \mathcal{P} \) intersect on a point of \( C \) in \( \Pi \) if and only if there exists a special line in \( \mathcal{P} \) with respect to these two secants.

**Theorem 4.8** If \( l_1 \) and \( l_2 \) are two secants to a conic \( C \) in a finite Desarguesian projective plane \( \Pi \) of order \( n \) \( (n > 9) \) that intersect at an exterior point \( E \) of \( \Pi \), then there does not exist in \( \Pi \) a special line with respect to \( l_1 \) and \( l_2 \).

**Proof:** Let \( l_1 \) be the line \( AB \) and \( l_2 \) the line \( CD \) where \( A, B, C, \) and \( D \) are four distinct points on \( C \). Consider the quadrangle \( ABCD \). Let \( X \) be the diagonal point on \( AC \) and \( BD \), and let \( Y \) be the diagonal point on \( AD \) and \( BC \). (See Figure 7.)

Suppose that there exists in \( \Pi \) a special line \( l \) with respect to \( l_1 \) and \( l_2 \). If \( l \) is not on \( E \), and since \( l \) is special there exists on each of the \( \frac{n-1}{2} \) interior points of \( l \) exactly one secant not equal to \( l \) that does not intersect either \( l_1 \) or \( l_2 \) in an interior point. Each of these lines must be on \( E \),
because a line on \( E \) does not intersect \( l_1 \) or \( l_2 \) in an interior point. Along with \( l_1 \) and \( l_2 \) this determines \( \frac{n-1}{2} \) secants on \( E \), which is impossible since \( E \) is an exterior point and thus is on \( \frac{n-1}{2} \) secants. Therefore \( l \) has to be a line on \( E \).

Let \( l \) be any secant on \( E \) not equal to \( EX \) or \( EY \). Since \( l \) is a special line and since every interior point not equal to \( X \) or \( Y \) on the four diagonals of \( ABCD \) is a normal point by Remark 4.3, there exists an interior point \( P \) on \( l \) and not on \( XY \) or any of the four diagonals of \( ABCD \) different from \( l_1 \) and \( l_2 \). By Theorem 3.5, the two distinct lines \( PX \) and \( PY \) are such that they either intersect both \( l_1 \) and \( l_2 \) in interior points or neither \( l_1 \) nor \( l_2 \) in interior points, since \( X \) and \( Y \) are diagonal points of \( ABCD \). If one of the lines \( PX \) and \( PY \) intersect both \( l_1 \) and \( l_2 \) in interior points, \( P \) cannot be a special point on \( l \). If both the lines \( PX \) and \( PY \) do not intersect \( l_1 \) or \( l_2 \) in interior points, \( P \) is on three lines, \( PX \), \( PY \), and \( l \), that do not intersect \( l_1 \) or \( l_2 \) in interior points, and thus \( P \) cannot be a special point. Hence \( l \) is not a special line unless \( l \) is either the line \( EX \) or the line \( EY \).

Let \( l \) be the line \( EY \). By Remark 4.3, each interior point not equal to \( X \) on a diagonal of \( ABCD \) on
the point $X$ is a normal point with respect to $l_1$ and $l_2$. Thus the two diagonals on $X$ do not intersect $l$ in special points. Hence the $\frac{n-1}{2}$ special points on $l$ determine $\frac{n-1}{2}$ distinct lines on $X$, none of which is a diagonal line of $ABCD$. Since these $\frac{n-1}{2}$ lines are on $X$, by Theorem 3.5 each line intersects both $l_1$ and $l_2$ in interior points or neither $l_1$ nor $l_2$ in interior points. But since each of these $\frac{n-1}{2}$ lines is on a special point of $l$, none of them can intersect both $l_1$ and $l_2$ in interior points.

Since a line not equal to $l$ on a special point of $l$ that does not intersect $l_1$ or $l_2$ in an interior point is a secant, these $\frac{n-1}{2}$ lines on $X$ are secants. But along with the diagonals on $X$, this determines $\frac{n+3}{2}$ secants on $X$, which is a contradiction since $X$ is a point not on $c$. Similarly if $l$ is the line $EX$, there is a contradiction.

Therefore there is no special line with respect to $l_1$ and $l_2$ in $\pi$. //

Remark: In the case $n \equiv 3 \pmod{4}$, it is much easier to show that the lines $EX$ and $EY$ cannot be special lines. For by Theorem 4.1, one of $X$ and $Y$ is an interior point. Without loss of generality assume $X$ is an interior point. Then $EX$ cannot be a special
line because it contains the interior point $X$ which is not a special point by Theorem 3.5. $EY$ is not a special line because it is the polar of $X$ and thus is a nonsecant [Ostrom 9].

Since special lines are defined in terms of interior points, their existence or nonexistence with respect to two lines in the hyperbolic plane $\mathcal{P}$ can be determined within the incidence structure of $\mathcal{P}$. This allows the following construction of the points on $\mathcal{C}$ without first constructing the exterior points:

Reconstruction of the Points on $\mathcal{C}$ -- Method 2

In constructing the points on the conic $\mathcal{C}$, as in Chapter II, we would like to determine $n+1$ sets of $n$ secants such that each of these sets of $n$ secants consists of a pencil on a point of the conic $\mathcal{C}$ in $\Pi$.

By Theorem 2.5, if $l_1$ and $l_2$ are two secants in $\mathcal{P}$ that do not intersect in $\mathcal{P}$ but intersect on $\mathcal{C}$ in $\Pi$, then there exists one and only one secant $l_3$ in $\mathcal{P}$ such that $l_3$ is a special line with respect to $l_1$ and $l_2$. By Theorem 4.8,
if \( l_1 \) and \( l_2 \) are two secants in \( \mathcal{P} \) that do not intersect in \( \mathcal{P} \) but intersect off \( \ell \) in \( \Pi \) at an exterior point, then there does not exist a special line in \( \mathcal{P} \) with respect to \( l_1 \) and \( l_2 \).

Therefore the existence of such a special line \( l_3 \) in \( \mathcal{P} \) with respect to the secants \( l_1 \) and \( l_2 \) shows that \( l_1 \) and \( l_2 \) intersect on \( \ell \) in \( \Pi \).

Now the construction continues as in Chapter II.
BIBLIOGRAPHY


Many papers related to the topic of this dissertation are cited in the following paper: