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RATE OF CONVERGENCE OF HERMITE INTERPOLATION

BASED ON THE ROOTS OF CERTAIN

JACOBI POLYNOMIALS

DISSERTATION

Presented in Partial Fulfillment of the Requirements
for the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

by

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The Ohio State University
1972

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INTRODUCTION

Let us consider the interval \([-1, 1]\), and let \(S_n = \{x_k\}_{k=1}^n\), where

\[1 > x_1 > x_2 > \ldots > x_{n-1} > x_n > -1,\]

denote a set of \(n\) distinct points of this interval. Let \(f_k, k = 1, \ldots, n\) and \(f'_k, k = 1, \ldots, n\), be two sets of arbitrary numbers.

Suppose we desire to construct a polynomial \(H_n(x)\) of lowest possible degree which satisfies the conditions

\[H_n(x_k) = f_k, \quad H'_n(x_k) = f'_k, \quad k = 1, \ldots, n.\]

Then as is well known

\[H_n(x) = \sum_{k=1}^n f_k A_{kn}(x) + \sum_{k=1}^n f'_k B_{kn}(x),\]

and \(H_n(x)\) is the Hermite Interpolation polynomial corresponding to \(S_n\), where

\[A_{kn}(x) = v_k(x) \frac{1^2(x)}{k^2},\]
\[B_{kn}(x) = (x - x_k) \frac{1^2(x)}{k^2},\]
\[L_k(x) = \frac{w_n(x)}{(x - x_k)w'_n(x_k)} ,\]
\[w_n(x) = c(x - x_1)(x - x_2) \ldots (x - x_n), \quad c \neq 0\]
\[v_k(x) = 1 - \frac{w''(x_k)}{w'(x_k)} (x - x_k).\]

(See Szegö [10], Chapter 14 or Natanson [7], Chapter 1, §6).
$l_k(x)$ are the basic Lagrange interpolating polynomials and $x_k$ are the nodes of interpolation. We shall say that $S_n$ is normal if $v_k(x) > 0$ in $[-1, 1]$, and that $S_n$ is strictly normal if $v_k(x) > 0$ in $[-1, 1]$.

If $f$ is a given function continuous on $[-1, 1]$ and we set

$$f_k = f(x_k), f_k' = 0, k = 1, \ldots, n$$

we shall denote the interpolating polynomial $H_n(x)$ by $H_n(f, x)$ and we have

$$H_n(f, x) = \sum_{k=1}^{n} f(x_k) A_{kn}(x).$$

We shall study the difference $H_n(f, x) - f(x)$, where $x$ is a fixed point of $[-1, 1]$ which is not a node $x_k$.

L. Fejér [2] considered the convergence of this interpolation process when the nodes $x_k$ are the roots of the Tchebyshev polynomial of the first kind

$$T_n(x) = \cos(n \arccos x)$$

$$x_k = \cos \frac{2k - 1}{2n} \pi, k = 1, \ldots, n.$$ 

Fejér [2] showed that in this case

$$\lim_{n \to \infty} H_n(f, x) = f(x)$$

uniformly on $[-1, 1]$, where $f(x)$ is a continuous function.

As to the rapidity of convergence in this case T. Popoviciu [8] has shown that

$$||H_n(f) - f|| \leq 2v_f \left(\frac{1}{\sqrt{n}}\right).$$

E. Moldovan [4] and later O. Shisha and B. Mond [9] have established
the following result

\[ \|H_n(f) - f\| \leq C_w \left( \frac{\log n}{n} \right), \quad n \geq 4, \]

where \( C \) is a positive constant.

Here \( \|f\| = \max \{|f(x)| : -1 \leq x \leq 1\} \); and \( w_f \) is the modulus of continuity of \( f \) defined by

\[ w_f(\delta) = \sup_{\|x' - x''\| \leq \delta} \{|f(x') - f(x'')|, x', x'' \in [-1, 1]\}. \]

R. Bojanic [1] has given a more precise estimate of the rate of convergence of the sequence \( \|H_n(f) - f\| \) in terms of the sequence \( \{w_f(1/k)\}_{k=1}^n \). He has proved that

\[ \|H_n(f) - f\| \leq \frac{C}{n} \sum_{k=1}^{n} w_f(1/k), \]

where \( C \) is a positive constant.

The main aim of this work is to investigate the rate of convergence of the sequence

\[ |H_n(f, x) - f(x)|, \quad n = 1, 2, \ldots; x \in [-1, 1], \]

assuming that the interpolation polynomial \( H_n(f, x) \) is based on the roots of the \( n \)-th degree Jacobi polynomial \( P_n^{(\alpha, \beta)}(x), (\alpha, \beta > -1) \).

In order to obtain a bound for the rate of convergence of this process we shall extend the method which Bojanic has used to obtain the inequality (*)

Our main result is the inequality

\[ |H_n(f, x) - f(x)| \leq \frac{C(\alpha, \beta)}{(1-x)^{\alpha} + \frac{1}{3}(1+x)^{\beta}} \left( \frac{1}{n} \sum_{k=1}^{n} w_f(1/k) \right), \]

valid for \( -\frac{1}{2} \leq \alpha \leq \frac{1}{2}, \quad -\frac{1}{2} \leq \beta \leq \frac{1}{2}, \quad x \text{ in } (-1, 1) \) and \( n \geq N(\alpha, \beta) \).
We also have the following inequalities valid for $-\frac{1}{2} \leq \alpha \leq 0$, $-\frac{1}{2} \leq \beta \leq 0$:

$$|H_n(f, x) - f(x)| \leq \frac{C(\alpha, \beta)}{n^{2q}} \sum_{k=1}^{n} w_f(1/k), \quad -1 + \varepsilon \leq x \leq 1,$$

and

$$H_n(f, x) - f(x) \geq \frac{C(\alpha, \beta)}{n^{2q}} \sum_{k=1}^{n} w_f(1/k), \quad -1 \leq x \leq 1 - \varepsilon,$$

where $\varepsilon > 0$ and $n \geq N(\alpha, \beta)$. In the above inequalities and what follows $C(\alpha, \beta)$ will always denote a positive constant.

We note that for $\alpha = \beta = -\frac{1}{2}$ the Jacobi polynomial is the Tchebyshev polynomial of the first kind, and $(\ast)$ follows.

We observe that if the last two inequalities are combined we obtain

$$|H_n(f, x) - f(x)| \leq \frac{C(\alpha, \beta)}{n^{2q}} \sum_{k=1}^{n} w_f(1/k), \quad -1 \leq x \leq 1,$$

where $q = \min(\alpha, |\beta|)$, $-\frac{1}{2} \leq \alpha < 0$, $-\frac{1}{2} \leq \beta < 0$ and $n \geq N(\alpha, \beta)$.

If $f \in \text{Lip} \gamma$, $0 < \gamma < 1$, we have $w_f(\delta) \leq M_f \delta^\gamma$ and so

$$\sum_{k=1}^{n} w_f(1/k) \leq M_f \sum_{k=1}^{n} \frac{1}{k^{\gamma}} \leq M_f \int_{0}^{n} \frac{dt}{t^\gamma} = \frac{M_f n^{1-\gamma}}{1-\gamma}.$$

Hence

$$\frac{1}{n^{2q}} \sum_{k=1}^{n} w_f(1/k) \leq \frac{M_f}{1-\gamma} \frac{1}{n^{2q+\gamma-1}} \rightarrow 0 \quad (n \rightarrow \infty),$$

if $0 \leq 1 - 2q < \gamma < 1$. Therefore we obtain the following corollary:

If $-\frac{1}{2} \leq \alpha < 0$, $-\frac{1}{2} \leq \beta < 0$ and if $f \in \text{Lip} \gamma$, where $0 \leq 1 - 2q < \gamma < 1$ and $q = \min(\alpha, |\beta|)$, then
Thus in this case the sequence \(\|H_n(f) - f\|\) converges uniformly to \(f\) on \([-1, 1]\).

If \(f \in \text{Lip}_1\), we have \(w_f(\delta) \leq M_f \delta\) and so
\[
\frac{1}{n^{2q}} \sum_{k=1}^{n} w_f(1/k) \leq \frac{M_f \log n}{n^{2q}} \to 0(n \to \infty).
\]
Hence in this case we have uniform convergence on \([-1, 1]\) also.

We recall that Szegö ([10] Theorem 14.6) has proved that the Hermite interpolating polynomial based on the roots of \(p_n^{(\alpha, \beta)}(x)\), \(\alpha > -1\), \(\beta > 1\), of a function \(f\) continuous on \([-1, 1]\) converges uniformly over every interval \([-1+\varepsilon, 1-\varepsilon]\), \(\varepsilon > 0\). For \(\alpha < 0\) it converges uniformly over the interval \([-1+\varepsilon, 1]\) and for \(\beta < 0\) it converges uniformly over the interval \([-1, 1-\varepsilon]\). The process is divergent at \(x = 1\) if \(f\) is merely continuous and \(\alpha \geq 0\), and is divergent at \(x = -1\) if \(\beta > 0\).

In Chapter 1 we have collected all the results about Jacobi polynomials which were needed for the proof of our main theorem.

Our main result will be proved in Chapter 2. The method of proof consists of first observing that
\[
|H_n(f, x) - f(x)| \leq \sum_{k=1}^{n} w_f(|x_k - x|) |v_k(x)| L^2_k(x),
\]
where
\[
v_k(x) = 1 - \frac{(\alpha + \beta + 2)x + \beta - \alpha}{1 - x_k^2} (x - x_k)
\]
Setting \( x = \cos \theta \) for \( \theta \) in \([0, \pi]\), \( x_k = \cos \theta_k \) for \( \theta_k \) in \((0, \pi)\), \( k = 1, \ldots, n \) and observing that

\[
\left| v_k(\cos \theta) \right| \leq \frac{C_1(\alpha, \beta) |\cos \theta - \cos \theta_k| + 2 \sin^2 \left( \frac{\theta + \theta_k}{2} \right)}{\sin^2 \theta_k}
\]

we find that

\[
\left| H_n(f, \cos \theta) - f(\cos \theta) \right| \leq \sum_{k=1}^{n} \left[ C_1(\alpha, \beta) |\cos \theta - \cos \theta_k| + \right.
\]

\[
+ \sin^2 \left( \frac{\theta + \theta_k}{2} \right) \right] w_f(|\theta - \theta_k|) \left( \frac{u_k(\cos \theta)}{\sin \theta_k} \right)^2
\]

The next step in the proof consists of expressing \( \frac{u_k(\cos \theta)}{\sin \theta_k} \) in terms of the function

\[(***) \quad U_n(\theta) = \left( \sin \frac{\theta}{2} \right)^{\alpha + \frac{1}{2}} \left( \cos \frac{\theta}{2} \right)^{\beta + \frac{1}{2}} p(\alpha, \beta)(\cos \theta) .
\]

Using (***) and the fact that \( 2\alpha + 1 > 0, 2\beta + 1 > 0 \) we find that

\[
\left( \sin \frac{\theta}{2} \right)^{2\alpha + 1} \left( \cos \frac{\theta}{2} \right)^{2\beta + 1} \left| H_n(f, \cos \theta) - f(\cos \theta) \right| \leq \]

\[
\leq C_2(\alpha, \beta) \sum_{k=1}^{n} w_f(|\theta - \theta_k|) \left( \frac{U_n(\theta)}{(\theta - \theta_k) U_n'(\theta_k)} \right)^2
\]

Now in most proofs of the convergence of a process like this one, the sum on the right hand side of the inequality above is usually estimated by splitting it into two parts. In one part \( |\theta - \theta_k| \leq \delta \), while in the other part \( |\theta - \theta_k| > \delta \), \( \delta > 0 \). In order to obtain more precise estimates it is necessary to split the sum above into \( 4n + 2 \) parts as follows

\[
\sum_{k=1}^{n} w_f(|\theta - \theta_k|) \psi^2_{nk}(\theta) \leq \sum_{k=1}^{n} w_f(|\theta - \theta_k|) \psi^2_{nk}(\theta)
\]
\[ + \sum_{r=1}^{4n+1} \left( \sum_{k} w_f(|\theta - \theta_k|) \psi_{nk}^2(\theta) \right) \]

where
\[ \psi_{nk}(\theta) = \frac{U_n(\theta)}{(\theta - \theta_k)U_n^{'}(\theta_k)} \]
\[ \sum_{k} \text{ is a sum over those } k \text{ satisfying } 0 \leq |\theta - \theta_k| \leq \frac{\pi}{4n+2} \text{ and } \sum_{r} \text{ is a sum over those } k \text{ satisfying } r\pi/(4n+2) \leq |\theta - \theta_k| \leq (r+1)\pi/(4n+2) \text{ for } r = 1, \ldots, 4n+1. \]

Now from Theorems 1.1, 1.2 and 1.3 in Chapter 1 it follows that there exists constants \( M_1(\alpha, \beta) \) and \( M_2(\alpha, \beta) \) such that
\[ |\psi_{nk}^2(\theta)| \leq \begin{cases} M_1(\alpha, \beta) \quad 0 \leq |\theta - \theta_k| \leq \pi/(4n+2) \\ M_2(\alpha, \beta) \quad r\pi/(4n+2) \leq |\theta - \theta_k| \leq (r+1)\pi/(4n+2) \end{cases} \]

Hence
\[ \sum_{k=1}^{n} w_f(|\theta - \theta_k|) \psi_{nk}(\theta) \leq M_1(\alpha, \beta) w_f(\pi/4n+2) v_c(\{k : 0 \leq |\theta - \theta_k| \leq \pi/4n+2\}) + \\
+ M_2(\alpha, \beta) \sum_{r=1}^{4n+1} \frac{1}{2} w_f((r+1)\pi/(4n+2)) v_c(\{k : \frac{r\pi}{4n+2} < |\theta - \theta_k| \leq \frac{(r+1)\pi}{4n+2}\}) \]

where \( v_c(E) \) is the number of elements of the set \( E \).

Using the properties of the zeroes of the Jacobi polynomials, in particular the inequality
\[ \frac{2k-1}{2n+1} \pi \leq \theta_k \leq \frac{2k\pi}{2n+1} \quad k = 1, \ldots, n \]
we find in Lemma 1.1 that $v_c(E) \leq 2$ for the sets appearing in the above sum. Hence
\[
\sum_{k=1}^{n} w_f(|\theta - \theta_k|) \psi_{nk}^2(\theta) \leq M_3(\alpha, \beta) \left( \frac{\varphi(\pi/4n+2)}{n} \sum_{r=1}^{4n+1} \frac{1}{r^2} w_f((r+1)\pi/4n+2) \right)
\]

Next since
\[
\sum_{r=1}^{m-1} \frac{1}{r^2} w_f((r+1)\pi/m) \leq \frac{8\pi}{m} \int_{\pi/m}^{\pi} \frac{w_f(t)}{t^2} \, dt
\]
and
\[
w_f(\pi/m) \leq \frac{\pi}{m-1} \int_{\pi/m}^{\pi} \frac{w_f(t)}{t^2} \, dt
\]
we see that for $m = 4n+2$
\[
\sum_{k=1}^{n} w_f(|\theta - \theta_k|) \psi_{nk}^2(\theta) \leq \frac{M_4(\alpha, \beta)}{n} \int_{\pi/4n+2}^{\pi} \frac{w_f(t)}{t^2} \, dt \leq \frac{M_5(\alpha, \beta)}{n} \sum_{k=1}^{n} w_f(1/k).
\]
Hence we have
\[
\left| H_n(f, \cos \theta) - f(\cos \theta) \right| \leq \frac{C_3(\alpha, \beta)}{(\sin \frac{\theta}{2})^{2\alpha+1} (\cos \frac{\theta}{2})^{2\beta+1}} \left( \frac{1}{n} \sum_{k=1}^{n} w_f(1/k) \right),
\]
which is our result.

The proof of the other two inequalities follows along the same lines as above.

Concerning the inequality
\[
\left| H_n(f, x) - f(x) \right| \leq \frac{C(\alpha, \beta)}{(1-x)^{\alpha+\frac{1}{2}} (1+x)^{\beta+\frac{1}{2}}} \left( \frac{1}{n} \sum_{k=1}^{n} w_f(1/k) \right)
\]
we show that it cannot be essentially improved. We consider the class $C_\lambda(\Omega)$ of continuous functions on $[-1, 1]$ defined by
\[
f \in C_\lambda(\Omega) \iff \sup_{0 < h \leq 2} \frac{w_f(h)}{\Omega(h)} \leq \lambda
\]
or, equivalently, by \( f \in C_M(\Omega) \) if and only if
\[
|f(x') - f(x'')| \leq \Omega(|x' - x''|), x', x'' \in [-1, 1].
\]
Here, \( \Omega \) is a non-decreasing, continuous and subadditive function on \([0, 2]\) with \( \Omega(0) = 0 \).

We have then the following result:
\[
\sup_{f \in C_M(\Omega)} |H_n(f, x) - f(x)| \geq \frac{c(\alpha, \beta)}{n} \sum_{k=n_0(\alpha, \beta)}^{n} \Omega(1/k), \quad n \geq n_0(\alpha, \beta), x \in [-1, 1].
\]
This result is proved in Chapter III.
CHAPTER I

SOME PROPERTIES OF THE n-th DEGREE JACOBI POLYNOMIAL

1.1 The Jacobi polynomial \( P_n^{(\alpha, \beta)}(x) \) on \([-1, 1]\) satisfies the following equation, with \( y = P_n^{(\alpha, \beta)}(x) \)

\[
(1.1.1) \quad (1-x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x] y' + n(n+\alpha+\beta+1)y = 0.
\]

The normalization of \( P_n^{(\alpha, \beta)}(x) \) is affected by

\[
P_n^{(\alpha, \beta)}(1) = \binom{n+\alpha}{n}.
\]

We also have, since

\[
P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\alpha, \beta)}(-x),
\]

that

\[
P_n^{(\alpha, \beta)}(-1) = (-1)^n \binom{n+\alpha}{n}.
\]

(For more details on Jacobi polynomials see Szegö [10] Chapter 4.)

The zeroes \( x_k \), \( k = 1, \ldots, n \) of \( P_n^{(\alpha, \beta)}(x) \) are simple and are in \((-1, 1)\). In discussing these zeroes we shall for \( x_k = \cos \theta_k \), use the following enumeration:

\[
+1 > x_1 > x_2 > \ldots > x_n > -1; \quad 0 < \theta_1 < \theta_2 < \ldots < \theta_n < \pi.
\]

We shall consider in this work only the case \( |\alpha| \leq \frac{1}{2}, \ |\beta| \leq \frac{1}{2} \).

From Theorem 6.21.2 in Szegö [10] we have

\[
(1.1.2) \quad \frac{2k - 1}{2n + 1} \pi \leq \theta_k \leq \frac{2k}{2n + 1} \pi.
\]
The proof of our main theorem is based on a precise evaluation of the following sum:

\[
\sum_{k=1}^{n} \frac{\Omega(\theta - \theta_k^*)}{(\theta - \theta_k^*)^2} \left( \frac{u_n(\theta)}{u_n^*(\theta)} \right)
\]

We shall split this sum into 4n + 2 parts which are characterized by the fact that in each part \(|\theta - \epsilon_k|\) satisfies one of the following inequalities:

\[
0 \leq |\theta - \epsilon_k| \leq \pi/4n + 2 ,
\]
\[
r\pi/4n + 2 < |\theta - \epsilon_k| \leq (r+1)\pi/4n + 2 , \quad r = 1, \ldots, 4n + 1 .
\]

More precisely if we define \(E_r(n, \theta)\) by

\[
E_0(n, \theta) = \{k : 0 \leq |\theta - \epsilon_k| \leq \pi/4n + 2\}
\]
\[
E_r(n, \theta) = \{k : r\pi/4n + 2 < |\theta - \epsilon_k| \leq (r+1)\pi/4n + 2\} , \quad r = 1, \ldots, 4n + 1
\]

then we have the following result:

**Lemma 1.1:** For \(0 \leq \theta \leq \pi\), we have

\[
\{1, 2, \ldots, n\} \subseteq \bigcup_{r=0}^{4n+1} E_r(n, \theta)
\]

and

\[
\nu_c(E_r(n, \theta)) \leq 2 , \quad r = 0, 1, \ldots, 4n + 1 ;
\]

where \(\nu_c(E)\) is the number of elements in the set \(E\).

**Remark:** Since \(\{1, 2, \ldots, n\} \subseteq \bigcup_{r=0}^{4n+1} E_r(n, \theta)\), if \(\{\alpha_k(\theta)\}_{k=1}^{n}\) is any sequence of positive numbers, we have that
\[
\sum_{k=1}^{n} A_k(\theta) \leq \sum_{r=0}^{4n+1} \left( \sum_{k \in E_r(n, \theta)} A_k(\theta) \right).
\]

Now suppose that for all \( k \) in \( E_r(n, \theta) \) \( 0 \leq A_k(\theta) \leq \delta_r \). Then
\[
\sum_{k=1}^{n} A_k(\theta) \leq \sum_{r=0}^{4n+1} \left( \sum_{k \in E_r(n, \theta)} \delta_r \right) = \sum_{r=0}^{4n+1} \nu_c(E_r(n, \theta))
\]

where \( \nu_c(E) \) is the number of elements in the set \( E \).

**Proof of Lemma 1.1:** Since
\[
[0, \pi) = [0, \pi/4n+2] \cup \left( \bigcup_{r=1}^{4n+1} (\pi r/4n+2, (r+1)\pi/4n+2) \right)
\]
and \( |\theta - \theta_k| \in [0, \pi) \), it follows that for every \( k \) in \( \{1, 2, \ldots, n\} \) there exists an \( r \) in \( \{0, 1, \ldots, 4n+1\} \) such that \( k \in E_r(n, \theta) \). We note that some of the sets \( E_r(n, \theta) \) may be empty.

For the second part of the Lemma there are three cases to consider.

**Case i:** Let \( 0 \leq \theta \leq \theta_1 \). Then \( |\theta - \theta_k| = \theta_k - \theta \), \( k = 1, \ldots, n \) and we have by (1.1.2)

\[
(1.1.3) \quad \frac{2k}{2n+1} \pi > \theta_k > \theta_k - \theta \geq \theta - \theta_1 > \frac{2k-3}{2n+1} \pi, \quad k = 1, \ldots, n
\]

and
\[
\frac{2\pi}{2n+1} \geq \theta_1 > \theta_1 - \theta \geq 0.
\]

Now suppose that \( k \in E_r(n, \theta) \), where \( r \) is a non-negative integer. If \( r \neq 0 \), we have by the definition of \( E_0(n, \theta) \)
\[
\frac{2k-3}{2n+1} \pi \leq \theta_k - \theta \leq \frac{\pi}{4n+2}
\]
and so $2k - 3 < \frac{1}{2}$. Since $k$ is an integer, we must have $2k - 3 \leq 0$, from which it follows that $k = 1$.

Suppose next that $0 < r \leq 4n + 1$. Then from the definition of $E_r(n, \theta)$ and (1.1.3) we have

$$\frac{r \pi}{4n + 2} < \theta \leq \frac{2k\pi}{2n + 1}$$

and

$$\frac{2k - 3}{2n + 1} \pi < \theta \leq \frac{(r + 1)\pi}{4n + 2}$$

Hence $r < 4k$ and $4k - 6 < r + 1$, which means that $r + 1 \leq 4k \leq r + 6$. For a fixed $r > 0$, this is true for at most two values of $k$. Hence $\nu_c(E_r(n, \theta)) \leq 2$, for $r = 0, \ldots, 4n + 1$.

Case ii: Let $\theta \leq \theta \leq \pi$. Then $|\theta - \theta_k| = \theta - \theta_k$, $k = 1, \ldots, n$ and we have by (1.1.2)

$$(1.1.4) \quad \frac{2(n - k) + 2}{2n + 1} \pi \geq \theta - \theta_k \geq \theta - \theta_k \geq \frac{2(n - k) - 1}{2n + 1} \pi$$

for $k = 1, \ldots, n - 1$, and $\frac{2\pi}{2n + 1} \geq \theta - \theta_n \geq \theta - \theta_n \geq 0$.

Now suppose that $k \in E_r(n, \theta)$, where $r$ is a non-negative integer. If $r = 0$, we have by the definition of $E_0(n, \theta)$

$$\frac{2(n - k) - 1}{2n + 1} \pi \leq \theta - \theta_k \leq \frac{\pi}{4n + 1}$$

Hence $2(n - k) - 1 \leq \frac{1}{2}$. Since $2(n - k)$ is an integer, we must have $2(n - k) \leq 1$ which means that $n - \frac{1}{2} \leq k \leq n$. From this inequality and the fact that $k$ is an integer, it follows that $k = n$.

Suppose next that $0 \leq r \leq 4n + 1$. Then from the definition of $E_r(n, \theta)$ and (1.1.4) we have
\[ \frac{\pi}{4n+2} < \theta - \theta_k \leq \frac{2(n-k)+2}{2n+1} \pi \]

and

\[ \frac{2(n-k)-1}{2n+1} \pi < \theta - \theta_k \leq \frac{(r+1)\pi}{4n+2} . \]

Hence \( r < 4(n-k) + 4 \) and \( 4(n-k) - 2 < r + 1 \), and so \( r = 4 \leq 4(n-k) < r + 3 \). For a fixed \( r > 0 \) this is true for at most two values of \( k \), since \( n \) is fixed. Hence \( \nu_c(E_r(n, \theta)) \leq 2 \), \( r = 0, 1, \ldots, 4n+1 \).

**Case iii:** Let \( \theta_{i-1} \leq \theta < \theta_i \) for some \( i \) (\( 2 \leq i < n \)). Then if \( k \leq i-1 \), \( |\theta - \theta_k| = \theta - \theta_k \) and so \( \theta_{i-1} - \theta_k \leq \theta - \theta_k < \theta_i - \theta_k \). From (1.1.2) we have

(1.1.5) \[ \frac{2(i-k)-3}{2n+1} \pi < \theta - \theta_k \leq \frac{2(i-k)+1}{2n+1} \pi \]

Now suppose \( k \in E_r(n, \theta) \), where \( r \) is a non-negative integer. If \( r = 0 \), we have by the definition of \( E_0(n, \theta) \)

\[ \frac{2(i-k)-3}{2n+1} \pi < |\theta - \theta_k| \leq \frac{\pi}{4n+2} \]

and so \( 2(i-k) - 3 < \frac{3}{2} \) that is \( 2(i-k) \leq 3 \). Hence \( 0 \leq i-k \leq 3/2 \) or \( i-3/2 \leq k \leq i-1 \) and so \( k = i-1 \).

Next if \( 0 < r \leq 4n+1 \) then from (1.1.5) and the definition of \( E_r(n, \theta) \) we have

\[ \frac{\pi}{4n+2} < \theta - \theta_k \leq \frac{2(i-k)+1}{2n+1} \pi , \text{ and } \frac{2(i-k)-3}{2n+1} \pi < \]

\[ < \theta - \theta_k \leq \frac{(r+1)\pi}{4n+2} . \]

Hence \( r \leq 4(i-k) + 2 \) and \( 4(i-k) - 6 < r + 1 \) and so \( r - 1 \leq 4(i-k) \leq r + 6 \). For a fixed \( r > 0 \), this holds for at most 2 values of \( k \).
Hence $v_c(E_x(n, \theta)) \leq 2$, for $r = 0, 1, \ldots, 4n+1$.

Next if $k \geq i$, we have $|\theta - \theta_k| = \theta_k - \theta$ and so $\theta_k - \theta_i \leq \theta_k - \theta \leq \theta_i - 1$. By (1.1.2) we have

$$\frac{2(k-i)-1}{2n+1} \pi < \theta_k - \theta \leq \frac{2(k-i)+3}{2n+1} \pi.$$

Clearly by an analysis similar to the one for $k \leq i-1$ we also have $v_c(E_x(n, \theta)) \leq 2$, for $r = 0, 1, \ldots, 4n+1$; and the Lemma is completely proved.
1.2 In the proof of our main theorem we found it necessary to introduce the function

\[ U_n(\theta) = \left( \sin^2 \frac{\theta}{2} \right)^{\alpha} + \frac{1}{2} \left( \cos^2 \frac{\theta}{2} \right)^{\beta} + \frac{1}{2} p_n (\alpha, \beta) (\cos \theta), \quad 0 < \theta < \pi \]

In this section we derive inequalities for \( U_n(\theta) \) and its derivative \( U'_n(\theta) \). Our first result is

**Theorem 1.1:** For \( 0 < \theta < \pi \), \( |\alpha| \leq \frac{1}{2} \), \( |\beta| \leq \frac{1}{2} \), there exists a positive constant \( C_1(\alpha, \beta) \) such that

\[ |U_n(\theta)| \leq \frac{C_1(\alpha, \beta)}{\sqrt{n}}, \quad \text{for every } n = 1, 2, \ldots \]

**Remark:** This result is a more precise version of an asymptotic estimate by G. Szego, for \( |\alpha| \leq \frac{1}{2} \), \( |\beta| \leq \frac{1}{2} \). From the definition of \( U_n(\theta) \), Theorem 1.1 shows that

\[ \left| p_n (\alpha, \beta) (\cos \theta) \right| \leq \frac{C_1(\alpha, \beta) n^{-\frac{3}{2}}}{\left( \sin^2 \frac{\theta}{2} \right)^{\alpha + \frac{1}{2}} \left( \cos^2 \frac{\theta}{2} \right)^{\beta + \frac{1}{2}}}, \quad 0 < \theta < \pi. \]

We recall that by (7.32.5) in Szego [10]

\[ p_n (\alpha, \beta) (\cos \theta) = \theta^{-\alpha - \frac{1}{2}} \ell(n^{-\frac{3}{2}}) \] if \( \frac{c}{n} \leq \theta \leq \frac{\pi}{2} \)

where \( \alpha, \beta \) are arbitrary and real and \( c \) is a positive constant, \( n \to \infty \).

The proof of Theorem 1.1 is based on the fact that the function \( U_n(\theta) \) satisfies the following differential equation

(1.2.1) \[ U''_n(\theta) + \left( \frac{\frac{3}{2} - \alpha^2}{4 \sin^2 \frac{\theta}{2}} + \frac{\frac{1}{2} - \beta^2}{4 \cos^2 \frac{\theta}{2}} \right) \left( n + \frac{\alpha + \beta + 1}{2} \right)^2 U_n(\theta) = 0. \]

(see Szego [10], page 66)
Starting from this equation, Szego ([10] page 208) has deduced that $U_n(\theta)$ satisfies the integral equation

\[(1.2.2) \quad \theta^{-\frac{\alpha}{2}} U_n(\theta) = 2^{-\frac{1}{2}} N^{-\frac{\alpha}{2}} \frac{\Gamma(n+\alpha+1)}{n!} J_\alpha(N\theta) + \]

\[+ \rho(\alpha) \int_0^\theta \frac{J_\alpha(N\theta) J_\alpha(Nt) - J_\alpha(N\theta) J_\alpha(Nt)}{\alpha} t^{\frac{1}{2}} f(t) U_n(t) dt \]

where

\[\rho(\alpha) = \begin{cases} \pi & , \alpha \neq 0 \\ -1 & , \alpha = 0 \end{cases}, \]

\[f(t) = \frac{\beta^2 - \frac{1}{4}}{4 \cos^2 \frac{t}{2}} + \left(\frac{3}{2} - \alpha^2\right)(1/t^2 - 1/(4 \sin^2 t/2)) , \]

\[N = n^{\alpha/2} + \beta + 1 \]

and $J_\alpha(x)$ is the Bessel function of order $\alpha$ (see Watson [12] Chapter 3).

From this integral equation we shall obtain all necessary results for the proof of Theorem 1.1. The following is true:

**Lemma 1.2:** For $n=1, 2, \ldots$

\[\binom{n+\alpha}{n} \leq \begin{cases} \frac{3 \alpha^2}{2} & , 0 \leq \alpha \leq \frac{1}{2} \\ (n+1)^2 & , -\frac{1}{2} \leq \alpha \leq 0 \end{cases} . \]

**Proof:** We have for any

\[\binom{n+\alpha}{n} = \frac{(n+\alpha)(n+\alpha-1)\ldots(\alpha+1)}{n!} \]

\[= \prod_{k=1}^{n} \frac{1+\alpha/k}{k} . \]

Hence for $0 \leq \alpha \leq \frac{1}{2}$, we have
\[
\log\left( \frac{n+\alpha}{n} \right) = \sum_{k=2}^{n} \log\left(1 + \frac{\alpha}{k} \right).
\]

But \( \log(1+x) \leq x \), for \( x \geq 0 \), therefore
\[
\sum_{k=2}^{n} \log\left(1 + \frac{\alpha}{k} \right) \leq \sum_{k=2}^{n} \frac{\alpha}{k} = \alpha \sum_{k=2}^{n} \frac{1}{k} \leq \alpha \log n.
\]

Hence
\[
\frac{n+\alpha}{n} \leq (\alpha+1)n^\alpha \leq \frac{3}{2} n^\alpha.
\]

Next let \( -\frac{1}{2} \leq \alpha \leq 0 \). Then
\[
\log\left( \frac{n+\alpha}{n} \right) = \sum_{k=1}^{n} \log\left(1 + \frac{\alpha}{k} \right) = \sum_{k=1}^{n} \log\left(1 - \frac{|\alpha|}{k} \right).
\]

Now since \( \alpha = -|\alpha| \leq 0 \), and \( \log(1-x) \leq -x, x \geq 0 \) we have
\[
\log\left( \frac{n+\alpha}{n} \right) \leq -|\alpha| \sum_{k=1}^{n} \frac{1}{k} = |\alpha| \sum_{k=1}^{n} \frac{1}{k}.
\]

But \( \sum_{k=1}^{n} \frac{1}{k} \geq \log(n+1) \), and \( \alpha \leq 0 \). Therefore
\[
\log\left( \frac{n+\alpha}{n} \right) \leq \alpha \log(n+1) = \log(n+1)^\alpha.
\]

Hence
\[
\frac{n+\alpha}{n} \leq (n+1)^\alpha.
\]

**Lemma 1.3:** For \( |\alpha| \leq \frac{1}{2}, |\beta| \leq \frac{1}{2}, n = 1, 2, \ldots \)
\[
\frac{1}{2} \leq \frac{n^{-\alpha} \Gamma(n+\alpha+1)}{n!} \leq \frac{3}{2},
\]
where \( \Gamma(x) \) is the Gamma function.

**Proof:** If \( n \) is a natural number and \( 0 \leq s \leq 1 \), then
\[ (n+1)^{s-1} \leq \frac{\Gamma(n+s)}{\Gamma(n+1)} \leq n^{s-1} \]

(see Mitrinović [3], page 286). Hence

\[ (1.2.3) \quad (n+1)^{s-1} \leq \frac{\Gamma(n+s)}{n!} \leq n^{s-1} \]

and

\[ (1.2.4) \quad (n+s)(n+1)^{s-1} \leq \frac{\Gamma(n+s+1)}{n!} \leq (n+s)n^{s-1} \]

Next since \( N = n + \frac{\alpha + \beta + 1}{2} \), clearly \( n \leq N \leq n+1 \). Therefore if \( -\frac{1}{2} \leq \alpha \leq 0 \) and \( s = \alpha + 1 \), then by (1.2.3)

\[ \frac{1}{2} \leq \left( \frac{n}{n+1} \right) \leq (n+1)^{\alpha} \leq \frac{N^{\alpha} \Gamma(n+\alpha+1)}{n!} \leq \alpha N^{-\alpha} \leq \left( \frac{n+1}{n} \right)^{\left| \alpha \right|} \leq \frac{3}{2} \]

If \( 0 \leq \alpha \leq 1 \) then by (1.2.4)

\[ \frac{1}{2} < \frac{n+\alpha}{n+1} = \left( \frac{n+\alpha^2}{n+1} \right)^{\alpha-1} \leq \frac{N^{\alpha} \Gamma(n+\alpha+1)}{n!} \leq (n+\alpha)n^{\alpha-1}n^{-\alpha} = \]

\[ = \frac{n+\alpha}{n} \leq \frac{3}{2} \]

Lemma 1.4: For \( x > 0 \) and \( |\alpha| \leq \frac{1}{2} \) we have

\[ x^{\left| \alpha \right|} |J_{\alpha}(x)| \leq \sqrt{\frac{2}{\pi}} \]


Lemma 1.5: Let \( |\alpha| \leq \frac{1}{2} \), \( |\beta| \leq \frac{1}{2} \) and

\[ f(t) = \frac{2^\beta - \frac{1}{2} t^\beta}{4\cos^2 \frac{t}{2}} + \left( \frac{1}{2} - \frac{1}{2} \right)^{\left( \frac{1}{2} - \frac{1}{2} \right)^2} \]

where \( 0 < t < \pi \). Then for \( 0 < t < \pi/2 \) we have

\[ |f(t)| \leq \frac{1}{\frac{1}{2}} \]
Proof: Let \( 0 < t \leq \frac{\pi}{2} \). Since \( \left| \sin \frac{t}{2} \right| \geq \frac{1}{2} \), we have \( \frac{1}{t^2 \sin \frac{t}{2}} \leq \frac{\pi^2}{t^4} \). Also \( \left| \sin \frac{t}{2} \right| \leq \frac{1}{4} \) and \( \left| \sin \frac{t}{2} - \frac{t}{2} \right| \leq \frac{1}{48} \). Therefore

\[
\left| \frac{1}{t^2} - \frac{1}{4 \sin \frac{t}{2}} \right| = \left| \frac{\sin^2 \frac{t}{2} - \frac{t}{4}}{t^2 \sin^2 \frac{t}{2}} \right| \leq \frac{\pi^2}{t^4} \left| \sin \frac{t}{2} - \frac{t}{2} \right| \left| \sin \frac{t}{2} + \frac{t}{2} \right| \leq \frac{\pi^2}{t^4} \left| \frac{t}{2} \right| \leq \frac{\pi^2}{48} \cdot t \right| = \frac{\pi^2}{48}.
\]

Therefore

\[
|f(t)| = \left| \frac{\beta^2 - \frac{1}{4 \cos^2 \frac{t}{2}}}{} + (\frac{1}{t^2} - \alpha^2) \left( \frac{1}{t^2} - \frac{1}{4 \sin^2 \frac{t}{2}} \right) \right| \leq \left| \frac{\beta^2 - \frac{1}{4 \cos^2 \frac{t}{2}}}{} \right| + \left| \frac{1}{t^2} - \alpha^2 \right| \cdot \left| \frac{1}{t^2} - \frac{1}{4 \sin^2 \frac{t}{2}} \right| \leq \left| \beta^2 - \frac{1}{4 \cos^2 \frac{t}{2}} \right| + \frac{\pi^2}{48} |{1}-\alpha^2| \leq \frac{1}{8} + \frac{\pi^2}{192} \leq \frac{3}{16} < \frac{1}{5}.
\]

With these auxiliary results we are now in a position to give a proof of Theorem 1.1.

Proof of Theorem 1.1:

Let \( 0 < \theta \leq \frac{\pi}{2} \) and \( M_n = \max_{0 < t \leq \pi/2} |U_n(t)| \). Since \( |\alpha| \leq \frac{1}{4} \), \( |\beta| \leq \frac{1}{4} \), we have from the definition of \( U_n(\theta) \),

\[
M_n \leq \max_{0 \leq t \leq \pi/2} \left| p_n^{(\alpha, \beta)}(\cos t) \right| \leq \binom{n+\alpha}{n}.
\]
(see the remark following Theorem 7.32.1 in Szegö [10]). By Lemma 1.2 we have

\[ M_n \leq \frac{3n^\alpha}{2}, \quad n = 1, 2, \ldots \]

From (1.1.2) and Lemma 1.5 it follows that

\[ |u_n(\theta)| \leq 2^{-\frac{1}{2}}N^{-\frac{1}{2}} \frac{2}{\pi} \Gamma(n+\alpha+1) |\Lambda_\alpha(N\theta)| + \frac{1}{2} \]

\[ + \left| \frac{\rho(\alpha)}{5} \right| M_n \int_0^\theta \left| J_\alpha(N\theta) J_\alpha(N\theta) - J_\alpha(N\theta) J_\alpha(N\theta) \right| t^{\frac{1}{2}} dt. \]

Using Lemma 1.3 and Lemma 1.4 we have

\[ |u_n(\theta)| \leq 2^{-\frac{1}{2}}N^{-\frac{1}{2}} \sqrt{\frac{2}{\pi}} + \left| \frac{\rho(\alpha)}{5} \right| M_n \int_0^\theta \left( \sqrt{\frac{2}{\pi}} - \frac{1}{N} \right)^2 dt \leq \]

\[ \leq \frac{3N^{-\frac{1}{2}}}{2\sqrt{\pi}} + \frac{\left| \rho(\alpha) \right| M_n}{5} \frac{\pi}{N} \frac{\pi}{2} = \frac{3N^{-\frac{1}{2}}}{2\sqrt{\pi}} + \frac{\left| \rho(\alpha) \right| M_n}{5N}. \]

Hence if \( 0 < |\alpha| \leq \frac{1}{2}, \) since \( |\rho(\alpha)| \leq \frac{\pi}{4|\alpha|} \) we have from (1.2.6)

\[ |u_n(\theta)| \leq \frac{3N^{-\frac{1}{2}}}{2\sqrt{\pi}} + \frac{\pi}{10|\alpha|} \frac{M_n}{n}, \]

and if \( \alpha = 0, \) since \( |\rho(\alpha)| = 1, \) we have

\[ |u_n(\theta)| \leq N^{-\frac{1}{2}} + \frac{2}{5} \frac{M_n}{N}. \]

From the inequalities satisfied by \( M_n, \) we have for \( 0 < |\alpha| \leq \frac{1}{2}, \)

\[ |u_n(\theta)| \leq \frac{3N^{-\frac{1}{2}}}{2\sqrt{\pi}} + \frac{3\pi}{20|\alpha|} \frac{n^\alpha}{N} \]

\[ \leq \frac{3}{2\sqrt{\pi}N} + \frac{1}{2|\alpha|} \frac{1}{n^{1-\alpha}} \]

\[ \leq \frac{1}{\sqrt{n}} \left( \frac{3}{2\sqrt{\pi}} + \frac{1}{2|\alpha|n^{\frac{1}{2}-\alpha}} \right) \]
\[ \leq \frac{1}{\sqrt{n}} \left( \frac{3}{2 \sqrt{n}} + \frac{1}{2|\alpha|} \right) \]

since \( \frac{1}{2} \alpha \geq 0 \). If \( \alpha = 0 \) then we have

\[ |u_n(\theta)| \leq n^{-\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \left( 1 + \frac{2}{3 \sqrt{n}} \right) \leq \frac{3}{2 \sqrt{n}} \]

Hence, for \( 0 < \theta < \frac{\pi}{2} \), we have

\[ (1.2.7) \quad \left( \sin \frac{\theta}{2} \right)^{\alpha + \frac{1}{2}} \left( \cos \frac{\theta}{2} \right)^{\beta + \frac{1}{2}} |p_n(\alpha, \beta)(\cos \theta)| = |u_n(\theta)| \leq \frac{c_1(\alpha)}{\sqrt{n}} \]

where

\[ c_1(\alpha) = \begin{cases} \frac{3}{2 \sqrt{n}} + \frac{1}{2|\alpha|} , & 0 < |\alpha| \leq \frac{1}{2} \\ \frac{3}{2} , & \alpha = 0 \end{cases} \]

Since \( p_n(\alpha, \beta)(x) = (-1)^n p_n(\alpha, \beta)(-x) \) (see Szego [10] page 58), we have

\[ p_n(\alpha, \beta)(\cos \theta) = (-1)^n p_n(\alpha, \beta)(-\cos \theta) \]

and so

\[ u_n(\theta) = \left( \sin \frac{\theta}{2} \right)^{\alpha + \frac{1}{2}} \left( \cos \frac{\theta}{2} \right)^{\beta + \frac{1}{2}} p_n(\alpha, \beta)(\cos \theta) \]

\[ = \left( \cos \frac{\pi - \theta}{2} \right)^{\alpha + \frac{1}{2}} \left( \sin \frac{\pi - \theta}{2} \right)^{\beta + \frac{1}{2}} p_n(\alpha, \beta)(-\cos(\pi - \theta)) \]

\[ = (-1)^n \left( \sin \frac{\pi - \theta}{2} \right)^{\beta + \frac{1}{2}} \left( \cos \frac{\pi - \theta}{2} \right)^{\alpha + \frac{1}{2}} p_n(\alpha, \beta)(\cos(\pi - \theta)) \]

Thus for \( \frac{\pi}{2} \leq \theta \leq \pi \), we deduce from (1.2.7)

\[ |u_n(\theta)| \leq \frac{c_1(\beta)}{\sqrt{n}} \]

Hence for \( 0 < \theta < \pi \) there exists a positive constant \( c_1(\alpha, \beta) \ll \infty \) such that

\[ |u_n(\theta)| \leq \frac{c_1(\alpha, \beta)}{\sqrt{n}} , \quad n = 1, 2, \ldots \]
Our next result concerns the derivative of $U_n(\theta)$ at $\theta = \theta_k$. We have the following

**Theorem 1.2:** For $0 < \theta_k < \pi$, $|\alpha| \leq \frac{1}{2}$, $|\beta| \leq \frac{1}{2}$ there exists positive constants $c_2(\alpha, \beta)$ and $n_0(\alpha, \beta)$ both finite such that

$$|U_n'(\theta_k)| \geq c_2(\alpha, \beta)^{n^{\frac{1}{2}}}, n \geq n_0(\alpha, \beta).$$

**Remark:** From the definition of $U_n(\theta)$ we see that

$$|U_n'(\theta_k)| = 2 \left( \sin \frac{\theta_k}{2} \right)^{\alpha+3/2} \left( \cos \frac{\theta_k}{2} \right)^{\beta+3/2} p_n(\alpha, \beta)(\cos \theta_k)$$

and so from Theorem 1.2 we have

$$|p_n'(\alpha, \beta)'(\cos \theta_k)| \geq \frac{c_2(\alpha, \beta)^{n^{\frac{1}{2}}}}{\left( \sin \frac{\theta_k}{2} \right)^{\alpha+3/2} \left( \cos \frac{\theta_k}{2} \right)^{\beta+3/2}}, n \geq n_0(\alpha, \beta)$$

G. Szego ([10] Theorem 8.9.1) has shown that

$$|p_n'(\alpha, \beta)'(\cos \theta_k)| \sim k^{3/2} n^{\alpha+2}, 0 < \theta_k < \frac{\pi}{2}$$

in the sense that the ratio of these two expressions remains between certain positive bounds depending on $\alpha$ and $\beta$. Our result is a more precise version of this fact.

**Proof of Theorem 1.2:** Differentiating (1.2.2) with respect to $\theta$ gives

$$-\frac{1}{2} \theta^{-3/2} U_n(\theta) + \theta^{-\frac{1}{2}} U_n'(\theta) = 2^{-\frac{1}{2}} N^{-\alpha} \Gamma(n+\alpha+1) N_{n-1}^\alpha(N\theta) +$$

$$+ p(\alpha) \sum_{\alpha} (J_n(\alpha \theta) J_{n+\alpha}(\alpha \theta) - J_{n+\alpha}(\alpha \theta) J_n(\alpha \theta)) \frac{1}{2} \theta^{-\frac{1}{2}} f(t) U_n(t) dt.$$
Multiplying (1.2.9) by $\theta J_{-\alpha}(N\theta)$ gives

\[(1.2.11) \quad -\frac{1}{2} \theta J_{-\alpha}(N\theta) U_n(\theta) + \theta J_{-\alpha}(N\theta) U_n'(\theta) =
\]
\[= 2^{-\frac{1}{2}} N^{-\frac{\alpha}{2}} \frac{1}{\pi \rho(\alpha)} + \int_0^\theta \frac{\alpha}{\pi \rho(\alpha)} J_{-\alpha}(N\theta) J_{-\alpha}(N\theta) U_n(\theta) +
\]
\[+ \rho(\alpha) \int_0^\theta N\theta J_{-\alpha}(N\theta) J_{-\alpha}(N\theta) J_{-\alpha}(N\theta) J_{-\alpha}(N\theta) t^\frac{1}{2} f(t) U_n(t) dt.
\]
Subtracting (1.2.10) from (1.2.11) we have

\[(1.2.12) \quad -\frac{1}{2} \theta J_{-\alpha}(N\theta) U_n(\theta) + \theta J_{-\alpha}(N\theta) U_n'(\theta) =
\]
\[= 2^{-\frac{1}{2}} N^{-\frac{\alpha}{2}} \frac{1}{\pi \rho(\alpha)} + \int_0^\theta \frac{\alpha}{\pi \rho(\alpha)} J_{-\alpha}(N\theta) J_{-\alpha}(N\theta) J_{-\alpha}(N\theta) J_{-\alpha}(N\theta) t^\frac{1}{2} f(t) U_n(t) dt.
\]
Replacing $\theta$ by $\theta_k$ in (1.2.12) we obtain for $0 < \theta_k \leq \frac{\pi}{2}$

\[(1.2.13) \quad \theta_k^2 J_{-\alpha}(N\theta_k) U_n'(\theta_k) = 2^{-\frac{1}{2}} N^{-\frac{\alpha}{2}} \frac{1}{\pi \rho(\alpha)} + \int_0^\theta \frac{\alpha}{\pi \rho(\alpha)} J_{-\alpha}(N\theta) J_{-\alpha}(N\theta) J_{-\alpha}(N\theta) J_{-\alpha}(N\theta) t^\frac{1}{2} f(t) U_n(t) dt.
\]
Hence since $|U_n(t)| \leq M_n$, $|f(t)| \leq \frac{1}{5}$ we have

\[
\theta_k^2 |J_{-\alpha}(N\theta_k)| |U_n'(\theta_k)| \geq 2^{-\frac{1}{2}} N^{-\frac{\alpha}{2}} \frac{1}{\pi \rho(\alpha)} - \frac{M_n}{5} \int_0^\theta |J_{-\alpha}(N\theta)| t^\frac{1}{2} dt.
\]
Applying Lemma 1.4 and Lemma 1.5 to the above inequality it follows that

\[(1.2.14) \quad \theta_k^2 |J_{-\alpha}(N\theta_k)| |U_n'(\theta_k)| \geq \frac{1}{2 \sqrt{2}} |\rho(\alpha)| - \frac{M_n}{5} \sqrt{\frac{2}{\pi N}} \sqrt{\frac{\pi}{2}}.
\]
But by Theorem 1.1 we have that

\[ M_n \leq \frac{c_1(\alpha)}{\sqrt{n}} \quad , \quad n = 1, 2, \ldots \]

Therefore using Lemma 1.1 in (1.2.14) it follows that

\[ (1.2.15) \quad \sqrt{n} \frac{2}{\pi} |u_n'(\theta_k)| \geq \frac{1}{2\sqrt{2} |\rho(\alpha)|} \frac{c_1(\alpha)}{5n} \sqrt{\frac{n}{2}} \]

Hence

\[ (1.2.16) \quad |u_n'(\theta_k)| \geq \sqrt{\frac{\pi}{4 |\rho(\alpha)|}} - \frac{c_1(\alpha) \pi}{10\sqrt{n}} \]

\[ \geq \sqrt{n} \left( \frac{\sqrt{\pi}}{4 |\rho(\alpha)|} - \frac{c_1(\alpha) \pi}{10n} \right) \]

If \( n > \sqrt{\pi} |\rho(\alpha)| c_1(\alpha) \), we have \( \frac{c_1(\alpha) \pi}{10} < \frac{\sqrt{\pi}}{10 |\rho(\alpha)|} \); therefore using this we have from (1.2.16)

\[ (1.2.17) \quad |u_n'(\theta_k)| \geq \delta \frac{\sqrt{\pi}}{\rho(\alpha)} n^{\frac{k}{2}} \quad , \quad \text{if} \quad n \geq n_0(\alpha) = \sqrt{\pi} |\rho(\alpha)| c_1(\alpha) \]

Next by (1.2.8) we have

\[ |u_n'(\theta_k)| = (-1)^n (-\sin(\pi - \theta_k))(\cos \frac{\pi - \theta_k}{2})^{\alpha + \frac{1}{2}} (\sin \frac{\pi - \theta_k}{2})^{\beta + \frac{1}{2}} \]

\[ P_n(\beta, \alpha)'(\cos(\pi - \theta_k)) \]

\[ = (-1)^n + 1 2 \left( \sin \frac{\pi - \theta_k}{2} \right)^{\beta + 3/2} (\cos \frac{\pi - \theta_k}{2})^{\alpha + 3/2} \]

\[ P_n(\beta, \alpha)'(\cos(\pi - \theta_k)) \]

Thus for \( \frac{\pi}{2} \leq \theta_k < \pi \), we deduce from (1.2.17)

\[ |u_n'(\theta_k)| \geq \delta \frac{\sqrt{\pi}}{\rho(\beta)} n^{\frac{k}{2}} \quad , \quad \text{if} \quad n \geq n_0(\beta) \]

Hence for \( 0 < \theta_k < \pi \), there exists positive constants \( c_1(\alpha, \beta) \) and
\( n_0(\alpha, \beta) \) both finite, such that
\[
|u'_n(e_k)| \geq c_2(\alpha, \beta) n^{\frac{1}{2}}, \ n \geq n_0(\alpha, \beta),
\]
and the theorem is proved.
1.3 We shall now give bounds for the expression

\[ \psi_n(\theta) = \frac{U_n(\theta)}{(\theta - \theta_k)U_n'(\theta_k)} \]

assuming that \(0 < |\theta - \theta_k| \leq \frac{\pi}{4n+2}\) and \(\frac{r\pi}{4n+2} < |\theta - \theta_k| \leq \frac{(r+1)\pi}{4n+2}\), \(r=1,2,...,4n+1\). Since

\[ U_n'(\theta_k) = \lim_{\epsilon \to \theta_k} \frac{U_n(\theta) - U_n(\theta_k)}{\theta - \theta_k} \]

we have that \(\psi_n(\theta_k) = 1\), \(\psi_n(\theta_j) = 0\), \(j \neq k\). Thus the function \(\psi_n(\theta)\) has the same properties as the basic Lagrange interpolation polynomial \(l_k(x)\). We have the following

**Theorem 1.3:** For \(|\alpha| \leq \frac{1}{2}\), \(|\beta| \leq \frac{1}{2}\) and \(0 < \theta < \pi\), there exists positive constants \(K_1(\alpha, \beta), K_2(\alpha, \beta)\) and \(n_0(\alpha, \beta)\) all finite such that for \(n \geq n_0(\alpha, \beta)\) we have

\[
|U_n(\theta)| \leq \begin{cases} K_1(\alpha, \beta), & \text{for } 0 \leq |\theta - \theta_k| \leq \frac{\pi}{4n+2} \\text{and} \\ K_2(\alpha, \beta), & \text{for } \frac{r\pi}{4n+2} < |\theta - \theta_k| \leq \frac{(r+1)\pi}{4n+2} \end{cases} \]

for \(r=1,2,...,4n+1\).

**Proof:** We shall assume that \(\theta \neq \theta_k\), since

\[ \frac{U_n(\theta)}{(\theta - \theta_k)U_n'(\theta_k)} = 1 \text{ if } \theta = \theta_k \].

Hence suppose first that \(0 < |\theta - \theta_k| \leq \frac{\pi}{4n+2}\).
Using Taylor's formula with remainder we have

\begin{equation}
U_n(\theta) = U_n(\theta_k) + (\theta - \theta_k)U'_n(\theta_k) + (\theta - \theta_k)^2U''_n(\epsilon_k),
\end{equation}

where \(0 < \epsilon_k < \theta_k\) or \(\theta_k < \epsilon_k < \theta\). Since \(U_n(\theta_k) = 0\), it follows by (1.3.3) that

\begin{equation}
\frac{U_n(\theta)}{(\theta - \theta_k)U'_n(\theta_k)} = 1 + (\theta - \theta_k) \frac{U''_n(\epsilon_k)}{U'_n(\theta_k)}.
\end{equation}

Now by (1.2.1) we have

\begin{equation}
U''_n(\epsilon_k) + g(\epsilon_k)U'_n(\epsilon_k) = 0,
\end{equation}

where

\begin{equation}
g(t) = \frac{-\alpha^2}{4\sin \frac{2t}{2}} + \frac{\beta^2}{4\cos \frac{2t}{2}} + (n + \alpha + \beta + 1)^2/2.
\end{equation}

From (1.3.5) we have

\[|U''_n(\epsilon_k)| \leq |g(\epsilon_k)| |U'_n(\epsilon_k)|,
\]

and so by Theorems 1.1 and 1.2, it follows that for \(n \geq n_0(\alpha, \beta)\)

\begin{equation}
\left| \frac{U_n(\theta)}{(\theta - \theta_k)U'_n(\theta_k)} \right| \leq 1 + |\theta - \theta_k| \frac{c_1(\alpha, \beta)}{c_2(\alpha, \beta)} \frac{1}{n} \cdot \left( \max_{0 < |t - \theta_k| \leq \pi/4n + 2} |g(t)| \right).
\end{equation}

We shall need an estimate for

\[0 < \max_{0 < |t - \theta_k| \leq \pi/4n + 2} |g(t)|.
\]

Suppose first that \(\theta_k - \frac{\pi}{4n + 2} \leq t < \theta_k\), then we have
\[ |g(t)| \leq \frac{1}{16} \left( \frac{1}{\sin^2 \left( \frac{(\theta - \frac{\pi}{2n+1})}{2} \right)} + \frac{1}{\cos^2 \frac{\theta_k}{2}} \right) + (n+1)^2. \]

Now since \( 0 < \theta_k < \theta , \frac{2k-1}{2n+1} \pi < \theta_k \leq \frac{2k\pi}{2n+1} \), we have

\[
\sin \left( \frac{\theta_k - \frac{\pi}{2n+1}}{2} \right) \geq \frac{1}{n} \left( \frac{\theta_k - \frac{\pi}{2n+1}}{2} \right) = \frac{\theta_k}{n} - \frac{1}{4n+2} \geq \frac{2k-1}{2n+1} - \frac{1}{4n+2} = \frac{4k-3}{4n+2},
\]

and

\[
\cos \frac{\theta_k}{2} = \sin \left( \frac{\pi - \theta_k}{2} \right) \geq \frac{\pi - \theta_k}{\pi} \geq 1 - \frac{2k-1}{2n+1} = \frac{2(n-k)+1}{2n+1}.
\]

Hence

\[
|g(t)| \leq \frac{1}{16} \left( \left( \frac{4n+2}{4k-3} \right)^2 + \left( \frac{2n+1}{2(n-k)31} \right)^2 \right) + (n+1)^2,
\]

that is

\[
|g(t)| \leq \frac{5}{16} (2n+1)^2 + (n+1)^2 \leq 3(n+1)^2.
\]

We find likewise that \( |g(t)| \leq 3(n+1)^2 \) for \( \theta_k < t < \theta_k + \frac{\pi}{2n+2} \)

\[
\left| \frac{\tilde{U}_n(\theta)}{(\theta - \theta_k)^{\frac{\theta_k}{u_n(\theta_k)}}} \right| \leq 1 + \frac{3}{2} |\theta - \theta_k| \frac{c_1(\alpha, \beta)}{c_2(\alpha, \beta)} \frac{(n+1)^2}{n}
\]

\[
\leq 1 + \frac{3c_1(\alpha, \beta)}{2c_2(\alpha, \beta)} \frac{(n+1)^2}{n(4n+2)}
\]

and (1.3.1) follows.

Finally if \( \frac{r\pi}{4n+2} < |\theta - \theta_k| \leq \frac{(r+1)\pi}{4n+2} \), then we have for \( n \geq n_0(\alpha, \beta) \) by Theorems 1.1 and 1.2

\[
\left| \frac{\hat{U}_n(\theta)}{(\theta - \theta_k)^{\frac{\theta_k}{u_n(\theta_k)}}} \right| \leq \frac{c_1(\alpha, \beta)}{c_2(\alpha, \beta)} \frac{1}{n|\theta - \theta_k|} = \frac{K_2(\alpha, \beta)}{n|\theta - \theta_k|},
\]

and Theorem 1.3 is completely proved.
CHAPTER II

UPPER BOUND FOR THE RATE OF CONVERGENCE

2.1 Let $x_k = x_k^{(n)}$ where $k = 1, 2, \ldots, n$ be the roots of the $n$-th degree Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$. If $f$ is a function continuous on $[-1, 1]$ then the Hermite interpolation polynomial for $f$ based on the roots of $P_n^{(\alpha, \beta)}(x)$ is defined by

$$H_n(f, x) = \sum_{k=1}^{n} f(x_k)v_k(x)\frac{x_k^2}{(x-x_k)^2}$$

where

$$v_k(x) = 1 - \frac{P_n^{(\alpha, \beta)}(x_k)}{P_n^{(\alpha, \beta)}(x_k)} (x-x_k)$$

$$= \frac{1-x_k}{1-x_k^2} + \frac{x_k-x}{1-x_k^2} ((\alpha+\beta+1)x_k+\alpha-\beta)$$

$$l_k(x) = \frac{P_n^{(\alpha, \beta)}(x)}{(x-x_k)P_n^{(\alpha, \beta)}'(x_k)}$$

As is well known the polynomial $H_n(f, x)$ satisfies the conditions

$$H_n(f, x_k) = f(x_k), \quad k = 1, \ldots, n$$

$$H_n'(f, x_k) = 0, \quad k = 1, \ldots, n$$

From (2.1.1) we have clearly

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Suppose now that \( \Omega \) is an increasing, subadditive and continuous function on the set \( \{ x : x \geq 0 \} \) with \( \Omega(0) = 0 \). Let \( C_M(\Omega) \) be the class of continuous functions on \([-1,1]\) defined by \( f \in C_M(\Omega) \) if and only if
\[
|f(x') - f(x'')| \leq M\Omega(|x' - x''|)
\]
for all \( x', x'' \) in \([-1,1]\). We note that if \( w_f \) is the modulus of continuity of \( f \) then \( f \in C_M(\Omega) \) if and only if
\[
\sup_{0 \leq h \leq 2} \frac{w_f(h)}{\Omega(h)} \leq M.
\]

We have the following

**Theorem 2.1:** Let \( f \) be a continuous function on \([-1,1]\) and let \( H_n(f,x) \) be the corresponding Hermite interpolation polynomial based on the roots of the Jacobi polynomial \( P_n^{(\alpha,\beta)}(x) \), \( \alpha > -1 \), \( \beta > -1 \). If \( |\alpha| \leq \frac{1}{2} \), \( |\beta| \leq \frac{1}{2} \), then for all \( f \) in \( C_M(\Omega) \) there exists positive constants \( C(\alpha,\beta) \) and \( N(\alpha,\beta) \) both finite such that
\[
|H_n(f,x) - f(x)| \leq \frac{C(\alpha,\beta)}{(1-x)^{\alpha+\frac{1}{2}}(1+x)^{\beta+\frac{1}{2}}} \left( \frac{1}{n} \sum_{k=1}^{n} \Omega(1/k) \right),
\]
valid for \( x \in (-1,1) \) and \( n \geq N(\alpha,\beta) \).

**Proof:** From (2.1.4) we have
\[
H_n(f,x) - f(x) = \sum_{k=1}^{n} (f(x_k) - f(x))v_k(x) l_k^2(x)
\]
(2.1.5) \[
|H_n(f,x) - f(x)| \leq \sum_{k=1}^{n} |f(x_k) - f(x)| |v_k(x)| l_k^2(x).
\]
Hence for \( f \in C_M(\Omega) \) we have from (2.1.5)
(2.1.6) \[ |H_n(f, x) - f(x)| \leq M \sum_{k=1}^{n} \Omega(|x_k - x| |v_k(x)| |l_k^2(x)|). \]

Writing \( x = \cos \theta \), \( x_k = \cos \theta_k \) such that \( 0 < \theta_1 < \theta_2 < \ldots < \theta_n < \pi \), and observing that
\[
|x_k - x| = |\cos \theta_k - \cos \theta| = 2 \left| \sin \left( \frac{\theta - \theta_k}{2} \right) \sin \left( \frac{\theta + \theta_k}{2} \right) \right| 
\leq |\theta - \theta_k|,
\]
we find from (2.1.6) that

(2.1.7) \[ |H_n(f, \cos \theta - f(\cos \theta)| \leq M \sum_{k=1}^{n} \Omega(|\theta - \theta_k| |v_k(\cos \theta)| |l_k^2(\cos \theta)|. \]

Next by (2.1.2) we see that
\[
|v_k(\cos \theta)| = \frac{|1 - \cos \theta \cos \theta_k + (\alpha - \beta + (\alpha + \beta + 1)\cos \theta_k)(\cos \theta - \cos \theta_k)|}{\sin^2 \theta_k} \leq \frac{|1 - \cos \theta \cos \theta_k + (|\alpha - \beta| + |1 + \alpha + \beta|) \cos \theta - \cos \theta_k|}{\sin^2 \theta_k} \leq \frac{2 \sin^2 \left( \frac{\theta + \theta_k}{2} \right) + c_1(\alpha, \beta)|\cos \theta - \cos \theta_k|}{\sin^2 \theta_k}
\]
where \( c_1(\alpha, \beta) = (|\alpha - \beta| + |1 + \alpha + \beta|) \).

Hence

(2.1.8) \[ |H_n(f, \cos \theta - f(\cos \theta)| \leq M \sum_{k=1}^{n} \Omega(|\theta - \theta_k|) (2 \sin^2 \left( \frac{\theta + \theta_k}{2} \right) + c_1(\alpha, \beta)|\cos \theta - \cos \theta_k| \frac{l_k(\cos \theta)}{\sin \theta_k} \right)^2. \]

Next we shall express \( l_k(\cos \theta) \) in terms of the function
\[
u_n(\theta) = \left( \sin \frac{\theta}{2} \right)^{\alpha + \frac{1}{2}} \left( \cos \frac{\theta}{2} \right)^{\beta + \frac{1}{2}} r_n(\alpha, \beta)(\cos \theta).
\]
We have
\[ U_n'(\theta_k) = -2\left(\sin\frac{\theta_k}{2}\right)^{\alpha+3/2} \left(\cos\frac{\theta_k}{2}\right)^{\beta+3/2} P_{n}^{(\alpha, \beta)}(\cos \theta_k) \]
and so from (2.1.3)
\[ l_k(\cos \theta) = \frac{P_{n}^{(\alpha, \beta)}(\cos \theta)}{(\cos \theta - \cos \theta_k)P_{n}^{(\alpha, \beta)}(\cos \theta_k)} = \]
\[ = -\sin \theta_k \left(\frac{\sin \frac{\theta_k}{2}}{\sin \frac{\theta}{2}}\right)^{\alpha+\frac{1}{2}} \left(\frac{\cos \frac{\theta_k}{2}}{\cos \frac{\theta}{2}}\right)^{\beta+\frac{1}{2}} \frac{U_n(\theta)}{\sin(\theta - \theta_k)U_n'(\theta_k)} \]

(2.1.9) \[ \frac{l_k(\cos \theta)}{\sin \theta_k} = \frac{-1}{2\sin(\frac{\theta + \theta_k}{2})} \left(\frac{\sin \frac{\theta_k}{2}}{\sin \frac{\theta}{2}}\right)^{\alpha+\frac{1}{2}} \left(\frac{\cos \frac{\theta_k}{2}}{\cos \frac{\theta}{2}}\right)^{\beta+\frac{1}{2}} \frac{U_n(\theta)}{\sin(\frac{\theta - \theta_k}{2})U_n'(\theta_k)} \]

Using this in (2.1.8) we have
\[ (2.1.10) \left(\sin \frac{\theta}{2}\right)^{2\alpha+1} \left(\cos \frac{\theta}{2}\right)^{2\beta+1} \left| H_n(f, \cos \theta) - f(\cos \theta) \right| \leq \]
\[ \leq \frac{M}{2} \sum_{k=1}^{n} \Omega(|\theta - \theta_k|) \left(\sin \frac{\theta_k}{2}\right)^{2\alpha+1} \left(\cos \frac{\theta_k}{2}\right)^{2\beta+1} \frac{U_n(\theta)}{\sin(\theta - \theta_k)U_n'(\theta_k)} \]
\[ + c_1(\alpha, \beta) \frac{M}{4} \sum_{k=1}^{n} \Omega(|\theta - \theta_k|) \left(\sin \frac{\theta_k}{2}\right)^{2\alpha+1} \left(\cos \frac{\theta_k}{2}\right)^{2\beta+1} \frac{\cos \theta - \cos \theta_k}{\sin^2\left(\frac{\theta + \theta_k}{2}\right)} \]
\[ \cdot \left(\frac{U_n(\theta)}{\sin(\theta - \theta_k)U_n'(\theta_k)}\right)^2 \]

We will show that
(2.1.11) \[
\frac{|\cos \theta - \cos \theta_k|}{\sin^2\left(\frac{\theta + \theta_k}{2}\right)} \leq 4, \text{ for } k = 1, \ldots, n.
\]

First we have
\[
\frac{|\cos \theta - \cos \theta_k|}{\sin^2\left(\frac{\theta + \theta_k}{2}\right)} = \frac{2|\sin\left(\frac{\theta - \theta_k}{2}\right)|}{\sin\left(\frac{\theta + \theta_k}{2}\right)} \leq \frac{|\theta - \theta_k|}{\sin\left(\frac{\theta + \theta_k}{2}\right)}.
\]

If \(0 \leq \frac{\theta + \theta_k}{2} \leq \frac{\pi}{2}\) then since \(\sin x \geq \frac{2x}{\pi}\) for \(x\) in \([0, \pi/2]\) it follows that
\[
\frac{|\theta - \theta_k|}{\sin\left(\frac{\theta + \theta_k}{2}\right)} \leq \frac{\theta + \theta_k}{\sin\left(\frac{\theta + \theta_k}{2}\right)} \leq \pi < 4.
\]

On the other hand, if \(\frac{\pi}{2} \leq \frac{\theta + \theta_k}{2} < \pi\) let \(\alpha = \pi - \theta, \alpha_k = \pi - \theta_k\). Then \(\alpha - \alpha_k = \theta_k - \theta\) and \(\alpha + \alpha_k = 2\pi - (\theta + \theta_k)\). Hence
\[
\frac{|\theta - \theta_k|}{\sin\left(\frac{\theta + \theta_k}{2}\right)} = \frac{|\alpha - \alpha_k|}{\sin\left(\frac{\alpha + \alpha_k}{2}\right)} \leq \frac{\alpha + \alpha_k}{\sin\left(\frac{\alpha + \alpha_k}{2}\right)} \leq \pi < 4; \text{ since in this case}
\]
\(0 < \frac{\alpha + \alpha_k}{2} < \frac{\pi}{2}\).

Using the inequality (2.1.11) and observing that \(\sin^2\left(\frac{\theta - \theta_k}{2}\right) \geq \frac{1}{\pi^2} (\theta - \theta_k)^2\), we find that

(2.1.12) \[
\left(\sin\frac{\theta}{2}\right)^{2\alpha + 1}\left(\cos\frac{\theta}{2}\right)^{2\beta + 1} \left|H_n(f, \cos \theta) - f(\cos \theta)\right| \leq
\]
\[
\leq (1 + c_1(\alpha, \beta))M \sum_{k=1}^{n} \Omega(|\theta - \theta_k|)\left(\sin\frac{\theta_k}{2}\right)^{2\alpha + 1}\left(\cos\frac{\theta_k}{2}\right)^{2\beta + 1}\left(\frac{U_n(\theta)}{(\theta - \theta_k)U_n'(\theta_k)}\right)^2
\]
\[
\leq c_2(\alpha, \beta) \sum_{k=1}^{n} \Omega(|\theta - \theta_k|)\left(\frac{U_n(\theta)}{(\theta - \theta_k)U_n'(\theta_k)}\right)^2,
\]

since \(2\alpha + 1 \geq 0, 2\beta + 1 \geq 0\).

From the remark following Lemma 1.1 and from (2.1.12) it follows
that

\[(2.1.13) \quad (\sin \frac{\theta}{2})^{2\alpha+1} (\cos \frac{\theta}{2})^{2\beta+1} |H_n(f, \cos \theta) - f(\cos \theta)| \]

\[\leq c_2(\alpha, \beta) \sum_{r=0}^{4n+1} \Omega(\theta - \theta_k) \left( \sum_{k \in E_r(n, \theta)} \frac{U_n(\theta)}{(\theta - \theta_k)U_n'(\theta_k)} \right)^2 . \]

Now by Lemma 1.1 each of the sets \(E_r(n, \theta)\) has at most two elements, and \(|\theta - \theta_k| < \frac{r+1}{4n+2}\), for \(k \in E_r(n, \theta)\) therefore it follows from (2.1.13) that

\[(2.1.14) \quad (\sin \frac{\theta}{2})^{2\alpha+1} (\cos \frac{\theta}{2})^{2\beta+1} |H_n(f, \cos \theta) - f(\cos \theta)| \leq \]

\[\leq c_2(\alpha, \beta) \sum_{r=0}^{4n+1} \Omega \left( \frac{r+1}{4n+2} \right) \sup_{k \in E_r(n, \theta)} \left\{ \left( \frac{U_n(\theta)}{(\theta - \theta_k)U_n'(\theta_k)} \right)^2 \right\} \]

\[\leq 2c_2(\alpha, \beta) \sum_{r=0}^{4n+1} \Omega \left( \frac{r+1}{4n+2} \right) \sup_{k \in E_r(n, \theta)} \left( \frac{U_n(\theta)}{(\theta - \theta_k)U_n'(\theta_k)} \right)^2 \]

By Theorem 1.3 there exists constants \(M_1(\alpha, \beta), M_2(\alpha, \beta)\) both positive and finite such that for \(r = 0\) we have

\[\left| \frac{U_n(\theta)}{(\theta - \theta_k)U_n'(\theta_k)} \right| \leq M_1(\alpha, \beta), \quad 0 < \theta < \pi, \quad n \geq N(\alpha, \beta) ; \]

while for \(1 \leq r \leq 4n+1\) we have

\[\left| \frac{U_n(\theta)}{(\theta - \theta_k)U_n'(\theta_k)} \right| \leq \frac{M_2(\alpha, \beta)}{n} \leq \frac{N(\alpha, \beta)}{n} \left( \frac{4n+2}{r\pi} \right), \quad 0 < \theta < \pi, \quad n \geq N(\alpha, \beta) . \]

For the last inequality we used the fact that \(\frac{r\pi}{4n+2} < \left| \theta - \theta_k \right| \).

Using these results in (2.1.14) we have
We shall need the following properties of the function $\Omega$:

For $\lambda > 0$, $h > 0$ we have

$$(2.1.16) \quad \Omega(\lambda h) \leq (1 + \lambda) \Omega(h) .$$

**Lemma 1.6**: For $m > 2$ we have

$$\frac{\pi}{m} \int_{\pi/m}^{\pi} \frac{\Omega(t)}{t^2} \, dt \leq \sum_{r=1}^{m-1} \frac{1}{r} \Omega\left(\frac{r+1}{m}\pi\right) \leq \frac{8\pi}{m} \int_{\pi/m}^{\pi} \frac{\Omega(t)}{t^2} \, dt$$


Next since $\Omega$ is an increasing function we have for $t \in \left[\frac{\pi}{4n+2}, \pi\right]$

$$\Omega\left(\frac{\pi}{4n+2}\right) \leq \Omega(t) .$$

Hence

$$\Omega\left(\frac{\pi}{4n+2}\right) \int_{\pi/4n+2}^{\pi} \frac{1}{t^2} \, dt \leq \int_{\pi/4n+2}^{\pi} \frac{\Omega(t)}{t^2} \, dt ,$$

and so

$$(2.1.17) \quad \Omega\left(\frac{\pi}{4n+2}\right) \leq \frac{\pi}{4n+2} \int_{\pi/4n+2}^{\pi} \frac{\Omega(t)}{t^2} \, dt .$$

Using (2.1.17) and Lemma 1.6, with $m = 4n+2$, in (2.1.15) we obtain

$$(2.1.18) \quad \left(\sin \frac{\theta}{2}\right)^{2\alpha+1} \left(\cos \frac{\theta}{2}\right)^{2\beta+1} \left|H_n(f, \cos \theta) - f(\cos \theta)\right| \leq$$
Next we have
\[ \int_{\pi/4n+2}^{\pi} \frac{\Omega(t)}{t^2} \, dt = \frac{1}{\pi} \int_1^{4n+2} \Omega\left(\frac{\pi}{t}\right) \, dt. \]

But by (2.1.16)
\[ \Omega\left(\frac{\pi}{t}\right) \leq (1+\pi) \Omega\left(\frac{1}{t}\right), \]
and so we have

\[ (2.1.19) \quad \int_{\pi/4n+2}^{\pi} \frac{\Omega(t)}{t^2} \, dt \leq 2 \int_1^{4n+2} \Omega\left(\frac{1}{t}\right) \, dt. \]

But
\[ \int_1^{4n+2} \Omega\left(\frac{1}{t}\right) \, dt \leq \int_1^{n} \Omega\left(\frac{1}{t}\right) \, dt + \sum_{k=1}^{n} \int_{k}^{(k+1)n} \Omega\left(\frac{1}{t}\right) \, dt, \]

and
\[ \int_{kn}^{(k+1)n} \Omega\left(\frac{1}{t}\right) \, dt = \int_{0}^{n} \Omega\left(\frac{1}{v+kn}\right) \, dv \leq c \int_{1}^{n} \Omega\left(\frac{1}{t}\right) \, dt, \quad c > 0. \]

Therefore we see from (2.1.19) that

\[ (2.1.20) \quad \int_{\pi/4n+1}^{\pi} \frac{\Omega(t)}{t^2} \, dt \leq 10c \int_{1}^{n} \Omega\left(\frac{1}{t}\right) \, dt. \]

Using (2.1.20) in (2.1.18) we obtain
\[ (\sin \frac{\theta}{2})^{2x+1} \left(\cos \frac{\theta}{2}\right)^{2\beta+1} |H_n(f, \cos \theta) - f(\cos \theta)| \leq \frac{C_6(\alpha, \beta)}{n} \int_{1}^{n} \Omega\left(\frac{1}{t}\right) \, dt \]
\[ \leq \frac{C_6(\alpha, \beta)}{n} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right), \]
and so
(2.1.21) \[ |H_n(f, \cos \theta) - f(\cos \theta)| \leq \frac{C_{6}(\alpha, \beta)}{(\sin \frac{\theta}{2})^{2\alpha + 1} (\cos \frac{\theta}{2})^{2\beta + 1}} \left( \frac{1}{n} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right) \right). \]

Finally since \(2\cos^{2} \frac{\theta}{2} = 1 + x, 2\sin^{2} \frac{\theta}{2} = 1 - x\), \( \theta = \cos \theta \) we have

(2.1.22) \[ |H_n(f, x) - f(x)| \leq \frac{C(\alpha, \beta)}{(1 - x)^{\alpha + \frac{1}{2}}(1 + x)^{\beta + \frac{1}{2}}} \left( \frac{1}{n} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right) \right), \]

valid for \( x \) in \((-1, 1)\), \( |\alpha| \leq \frac{1}{2}, |\beta| \leq \frac{1}{2} \) and \( n > N(\alpha, \beta) \), and the theorem is completely proved.

2.2 In this section we consider the case \( x \in [-1 + \varepsilon, 1] \) or \( x \in [-1, 1 - \varepsilon] \) for some \( \varepsilon > 0 \).

We know from the remarks following Theorem 7.32.1 in Szego [10] that

(2.2.1) \[ \max_{0 \leq x \leq 1} |P_{n}(\alpha, \beta)(x)| \leq \left( \frac{n + \alpha}{n} \right), \]

(2.2.2) \[ \max_{-1 \leq x \leq 0} |P_{n}(\alpha, \beta)(x)| \leq \left( \frac{n + \beta}{n} \right) \]

Using these results we shall prove the following

**Theorem 2.2:** Under the hypotheses of Theorem 2.1, there exists positive constants \( C(\alpha, \beta) \) and \( N(\alpha, \beta) \) both finite such that given \( \varepsilon > 0 \) we have

\[ |H_n(f, x) - f(x)| \leq C(\alpha, \beta) n^{2\alpha} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right), \quad -1 + \varepsilon \leq x \leq 1 \]

and

\[ |H_n(f, x) - f(x)| \leq C(\alpha, \beta) n^{2\beta} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right), \quad -1 \leq x \leq 1 - \varepsilon \]
valid for $n \geq N(\alpha, \beta)$, $|\alpha| \leq \frac{1}{2}$, $|\beta| \leq \frac{1}{2}$.

**Proof:** Since

$$U_n(\theta) = \left(\sin^\theta + \frac{1}{2}\right) \left(\cos^\theta + \frac{1}{2}\right) p^{(\alpha, \beta)}(\cos \theta),$$

we have from (2.1.13)

$$(2.2.3) \quad |H_n(f, \cos \theta) - f(\cos \theta)| \leq$$

$$\leq C_2(\alpha, \beta) \sum_{k=0}^{4n+1} \left(\sum_{k \in E_r(n, \theta)} \frac{\Omega(\theta - \theta_k)}{(\theta - \theta_k)^2} \left(\frac{p^{(\alpha, \beta)}(\cos \theta)}{u_n(\theta_k)^2}\right)^2\right).$$

First let $0 < \theta < \frac{\pi}{4n+2}$. Then since $|\theta - \theta_k| = \theta_k - \theta > \theta_k - \theta > 0$, there is no $k \in \{1, \ldots, n\}$ such that $0 < |\theta - \theta_k| < \frac{\pi}{2n+1}$. Hence $E_0(n, \theta) = \emptyset$ and so we obtain from (2.2.3) for $0 < \theta < \frac{\pi}{4n+2}$

$$|H_n(f, \cos \theta) - f(\cos \theta)| \leq C_2(\alpha, \beta) \max_{0 \leq \theta < \pi/4n+2} \left(\frac{p^{(\alpha, \beta)}(\cos \theta)}{u_n(\theta_k)^2}\right)^2$$

$$\cdot \sum_{r=1}^{4n+1} \left(\sum_{k \in E_r(n, \theta)} \frac{\Omega(\theta - \theta_k)}{(\theta - \theta_k)^2 u_n(\theta_k)^2}\right).$$

From (2.2.1) and Lemma 1.2 the above inequality reduces to

$$(2.2.4) \quad |H_n(f, \cos \theta) - f(\cos \theta)| \leq$$

$$\leq C_3(\alpha, \beta) n^2 \sum_{r=1}^{4n+1} \left(\sum_{k \in E_r(n, \theta)} \frac{\Omega(\theta - \theta_k)}{(\theta - \theta_k)^2 u_n(\theta_k)^2}\right).$$

Using Theorem 1.2 and Lemma 1.1 we find that
\[
\sum_{r=1}^{4n+1} \left( \sum_{k \in \mathbb{E}_r(n, \theta)} \frac{\Omega(\theta - \sigma_k)}{((\theta - \sigma_k)U^t_n(\sigma_k))^2} \right) \leq C_4(\alpha, \beta)n \sum_{r=1}^{4n+1} \frac{1}{r^2} \Omega\left(\frac{(r+1)\pi}{4n+2}\right)
\]
for \( n \geq N(\alpha, \beta) \). Hence by (2.2.4) we have

\[
|H_n(f, \cos \theta) - f(\cos \theta)| \leq C_5(\alpha, \beta)n^{2\alpha + 1} \sum_{r=1}^{4n+1} \frac{1}{r^2} \Omega\left(\frac{(r+1)\pi}{4n+2}\right)
\]

By arguments similar to those used in the proof of Theorem 2.1 we find from (2.2.5) for \( 0 \leq \theta \leq \frac{\pi}{4n+2} \) that

\[
|H_n(f, \cos \theta) - f(\cos \theta)| \leq C_6(\alpha, \beta)n^{2\alpha} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right), \quad n \geq N(\alpha, \beta).
\]

Now given \( \epsilon > 0 \), let \( \frac{\pi}{2} < \theta < \pi \) where \( \cos \theta = -1 + \epsilon \). We find from (2.1.21), for \( \frac{\pi}{4n+2} < \theta \leq \theta_\epsilon \) that

\[
|H_n(f, \cos \theta) - f(\cos \theta)| \leq \frac{C_7(\alpha, \beta)}{\left(\sin \left(\frac{\theta}{2}\right)\right)^{2\alpha + 1} \left(\cos \left(\frac{\theta}{2}\right)\right)^{2\beta + 1}} \left(\frac{1}{n} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right)\right)
\]

Now since \( \frac{\pi}{4n+2} \leq \theta \leq \theta_\epsilon \), we have that

\[
\sin \frac{\theta}{2} > \sin \left(\frac{\pi/2}{4n+2}\right) > \frac{1}{4n+2}
\]

and \( \cos \frac{\theta}{2} \geq \cos \frac{\theta_\epsilon}{2} = \frac{1 - \cos \theta_\epsilon}{2} \). Hence from (2.2.7) we have

\[
|H_n(f, \cos \theta) - f(\cos \theta)| \leq C_8(\alpha, \beta)n^{2\alpha} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right).
\]

Combining this last inequality with (2.2.6) we find that for \( 0 \leq \theta \leq \theta_\epsilon \)

\[
|H_n(f, \cos \theta) - f(\cos \theta)| \leq C_9(\alpha, \beta)n^{2\alpha} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right)
\]

and so our first inequality is proved.

For the second inequality we consider \( \theta_n + \frac{\pi}{4n+2} < \theta \leq \pi \). Since...
\[ |\theta - \theta_k| = \theta - \theta_k > \theta_n + \frac{\pi}{4n+2} - \theta_n = \frac{\pi}{4n+2} \text{, there is no } k \in \{1, \ldots, n\} \]
such that \[ 0 \leq |\theta - \theta_k| \leq \frac{\pi}{4n+2} \text{.} \] By arguments similar to those used in the proof of our first inequality we see that for \[ \theta_n + \frac{\pi}{4n+2} < \theta \leq \pi \]
\[
|H_n(f, \cos \theta) - f(\cos \theta)| \leq C_{10}(\alpha, \beta)n^{2\beta} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right) \]
valid for \[ n \geq N(\alpha, \beta) \]

Now given \( \epsilon > 0 \), let \( \epsilon < \frac{\pi}{2} \) where \( 1 - \epsilon = \cos \theta_\epsilon \). We have from (2.1.21) for \( \theta_\epsilon \leq \theta \leq \theta_n + \frac{\pi}{4n+2} \) that

\[
|H_n(f, \cos \theta) - f(\cos \theta)| \leq \left( \frac{C_{11}(\alpha, \beta)}{(\sin \frac{\theta}{2})^{2\alpha+1}} \left( \cos \frac{\theta}{2} \right)^{2\beta+1} \left( \frac{1}{n} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right) \right) \right) \; \left( \frac{1}{1 - \cos \theta} \right) \]

Since \( \theta_\epsilon \leq \theta \leq \theta_n + \frac{\pi}{4n+2} \) we find that \( \sin \frac{\theta}{2} > \sin \frac{\theta_\epsilon}{2} = \sqrt{1 - \cos \theta_\epsilon} \)

and \( \cos \frac{\theta}{2} = \sin \frac{\pi - \theta}{2} \geq \frac{\pi - \theta}{2} \geq \frac{(\pi - \theta_n - \pi/4n+2)}{\pi} \geq \frac{1}{4n+2} \). Hence by (2.2.9) we have

\[
|H_n(f, \cos \theta) - f(\cos \theta)| \leq C_{12}(\alpha, \beta)n^{2\beta} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right) \]

Combining this last inequality with (2.2.8) we obtain for \( \theta_\epsilon \leq \theta \leq \pi \)

\[
|H_n(f, \cos \theta) - f(\cos \theta)| \leq C_{13}(\alpha, \beta)n^{2\beta} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right) \text{, } n \geq N(\alpha, \beta) \]

and our second inequality is proved.
CHAPTER III
LOWER BOUND FOR THE RATE OF CONVERGENCE

3.1 Let $\Delta_n(\Omega, x) = \sup_{f \in C_M(\Omega)} \left| H_n(f, x) - f(x) \right|$. From Theorem 2.1 follows that if $-\frac{1}{2} \leq \alpha, \beta \leq \frac{1}{2}$ there exists positive constants $C(\alpha, \beta)$ and $N(\alpha, \beta)$ such that for every $x \in (-1, 1)$ and $n \geq N(\alpha, \beta)$ we have

$$
\Delta_n(\Omega, x) \leq \frac{C(\alpha, \beta)}{(1-x)\alpha + \frac{1}{2}(1+x)\beta + \frac{1}{2}} \left( \frac{1}{n} \sum_{k=1}^{n} \Omega(\frac{1}{k}) \right)
$$

In this chapter we shall give a lower bound for this expression assuming that $-\frac{1}{2} \leq \alpha < 0, -\frac{1}{2} \leq \beta < 0$. The reason for this restriction on $\alpha, \beta$ is the fact that $v_k(x) > 0$ when $\alpha, \beta < 0$; and in this case the polynomial $H_n(f, x)$ has a non-negative value for all $x$ in $[-1, 1]$ whenever $f$ is a non-negative function on $[-1, 1]$. We shall prove that there exists $C \in [-1, 1]$ and positive numbers $C_2(\alpha, \beta)$ and $N(\alpha, \beta)$ such that for $n \geq N(\alpha, \beta)$ we have

$$
\Delta_n(\Omega, c) \geq \frac{C_2(\alpha, \beta)}{n} \sum_{k=n(\alpha, \beta)}^{n} \Omega(\frac{1}{k})
$$

Let $-\frac{1}{2} \leq \alpha, \beta < 0$. On $[-1, 1]$ let $g(x) = M \Omega(|x - c|)$, where $c = \cos \frac{\pi}{n}, c \in [-1, 1], c \neq x_k$. The choice of the point $c$ will depend on $\alpha$ and $\beta$. It will be such that $p_n(\alpha, \beta)(c) \neq 0$ for all $n = 1, 2, \ldots$ By definition

$$
\left| g(x') - g(x'') \right| = M \left| \Omega(|x' - c| - \Omega(|x'' - c|) \right| \leq
$$

$$
\leq M \Omega(|x' - x''|)
$$
and so $g \in C_M(\Omega)$. Thus
\[
\Delta_n(\Omega, c) = \sup_{f \in C_M(\Omega)} \left| H_n(f, c) - f(c) \right| \geq \left| H_n(g, c) - g(c) \right| = H_n(g, c) .
\]

Now by definition we have
\[
(3.1.1) \quad H_n(g, c) = M \sum_{k=1}^{n} \Omega(|x_k - c|) v_k(c) \left( \frac{p_n^{(\alpha, \beta)}(c)}{(c - x_k)p_n^{(\alpha, \beta)}'(x_k)} \right)^2 .
\]

Since $\alpha, \beta < 0$, and $v_k(x)$ is linear, it is easy to see that
\[
v_k(c) \geq \min\{v_k(-1), v_k(1)\} \geq \min\{-\alpha, -\beta\} = q > 0 .
\]

Hence from (3.1.1) we have
\[
(3.1.2) \quad H_n(g, c) \geq M \sum_{k=1}^{n} \Omega(|x_k - c|) \left( \frac{p_n^{(\alpha, \beta)}(c)}{(c - x_k)p_n^{(\alpha, \beta)}'(x_k)} \right)^2 .
\]

Now $0 < |x_k - c| = |\cos \theta_k - \cos \psi| < \theta_k - \psi$ and so using the fact that if $0 < h_1 < h_2$ then
\[
\frac{2\Omega(h_1)}{h_1} \geq \frac{\Omega(h_2)}{h_2}
\]
(see Lemma 3.5) it follows from (3.1.2) that
\[
H_n(g, c) \geq \frac{Mq}{2} \sum_{k=1}^{n} \frac{\Omega(|\theta_k - \psi|)}{(\psi - \theta_k)^2} \left( \frac{p_n^{(\alpha, \beta)}(\cos \psi)}{p_n^{(\alpha, \beta)'}(\cos \theta_k)} \right)^2 .
\]

Hence
\[
(3.1.3) \quad \Delta_n(g, c) \geq \frac{Mq}{2} \frac{p_n^{(\alpha, \beta)}(\cos \psi)}{\left(\psi - \theta_k\right)^2} \sum_{k=1}^{n} \frac{\Omega(|\theta_k - \psi|)}{p_n^{(\alpha, \beta)'}(\cos \theta_k)^2} .
\]

We need a lower bound for $|p_n^{(\alpha, \beta)}(\cos \psi)|$ and an upper bound for $|p_n^{(\alpha, \beta)'}(\cos \theta_k)|$. 
3.2 In this section we shall derive certain bounds needed for our
theorem. We have the following results:

**Lemma 3.1**: There exists \( \Psi \in (0, \pi) \), and constants \( M_1(\alpha, \beta) \), \( N(\alpha, \beta) \)
such that
\[
|P_n^{(\alpha, \beta)}(\cos \Psi)| \geq \sqrt{\frac{n}{\pi}} M_1(\alpha, \beta) > 0, \text{ for } n \geq N(\alpha, \beta),
\]
where \( \alpha, \beta > -1 \).

**Proof**: From formula (8.21.10) in Szegö [10] we have
\[
P_n^{(\alpha, \beta)}(\cos \theta) \approx \sqrt{\frac{n}{\pi}} \frac{\cos(N\theta + \gamma)}{(\sin \frac{\theta}{2})^\alpha (\cos \frac{\theta}{2})^\beta}, \quad n \to \infty
\]
where \( N = n + \frac{\alpha + \beta + 1}{2} \), \( \gamma = -\frac{(\alpha + \frac{1}{2})\pi}{2} \), \( 0 < \theta < \pi \).

Hence for \( n \geq N(\alpha, \beta) \) we have
\[
(3.2.1) \quad |P_n^{(\alpha, \beta)}(\cos \theta)| \geq \sqrt{\frac{n}{\pi}} \frac{|\cos(N\theta + \gamma)|}{(\sin \frac{\theta}{2})^\alpha (\cos \frac{\theta}{2})^\beta}.
\]
Suppose first that \( \alpha \neq \beta \) and let \( \Psi = \frac{\pi}{2} \). Then
\[
N\Psi + \gamma = (n + \alpha + \beta + 1) \frac{\pi}{2} - 2\alpha + 1 - \frac{\pi}{2} = n\frac{\pi}{2} + (\beta - \alpha) \frac{\pi}{4}.
\]
Hence
\[
|\cos(N\Psi + \gamma)| = \left| \cos \left( \frac{n\pi}{2} + \frac{\beta - \alpha \pi}{4} \right) \right|
\]
\[
= \left| \cos \frac{n\pi}{2} \cos \frac{\beta - \alpha \pi}{4} - \sin \frac{n\pi}{2} \sin \frac{\beta - \alpha \pi}{4} \right|
\]
Therefore
\[
|\cos(N\Psi + \gamma)| = \begin{cases} 
\left| \sin \frac{\beta - \alpha \pi}{4} \right|, & n \text{ odd} \\
\left| \cos \frac{\beta - \alpha \pi}{4} \right|, & n \text{ even},
\end{cases}
\]
and so
\[ |\cos(N\gamma + \gamma)| > \min \left( \sin\left(\frac{\beta - \alpha}{4}\right), \cos\left(\frac{\beta - \alpha}{4}\right) \right) = c_1(\alpha, \beta) > 0. \]

Next suppose that \( \alpha = \beta \), and let \( \gamma = \frac{\pi}{3} \). Then

\[ N\gamma + \gamma = \left( n + \frac{2\alpha + 1}{2} \right) \frac{\pi}{3} - \frac{2\alpha + 1}{2} \cdot \frac{\pi}{2} = \frac{n\pi}{3} - \frac{2\alpha + 1}{12} \pi. \]

Hence

\[ |\cos(N\gamma + \gamma)| = |\cos n\frac{\pi}{3} \cos \frac{2\alpha + 1}{12} \pi + \sin n\frac{\pi}{3} \sin \frac{2\alpha + 1}{12} \pi| \]

\[ \geq \cos \frac{\pi}{12} |\cos n\frac{\pi}{3}| - \sin \frac{\pi}{12} \sin \frac{n\pi}{3} | \]

\[ \geq \frac{1}{2} \cos \frac{\pi}{12} \sqrt{3} \sin \frac{n\pi}{2} = c_2(\alpha, \beta) > 0. \]

From (3.3.1) it follows that, for \( n > N(\alpha, \beta) \) we have

\[ |p_n^{(\alpha, \beta)}(\cos \gamma)| \geq \frac{1}{n} \sqrt{\frac{\pi}{n}} \cdot \begin{cases} c_1(\alpha, \beta) \left(\sqrt{2}\right)^{\alpha + \beta + 1} + 1, & \alpha \neq \beta \\ c_2(\alpha, \beta) \left(\sqrt{3}\right)^{\alpha + \frac{1}{2}}, & \alpha = \beta \end{cases} \]

and the result follows.

**Lemma 3.2:** There exists a constant \( M_2(\alpha, \beta) \) such that

\[ |p_n^{(\alpha, \beta)}(\cos \theta_k)| \leq \frac{M_2(\alpha, \beta) \frac{n}{\theta_k}}{\alpha + 7/2 \theta_k \beta + 7/2 n} \geq 2. \]

**Proof:** By (4.5.7) in Szegö [10] we have

\[ (2n + \alpha + \beta)(1 - x^2) \frac{d}{dx}_n p_n^{(\alpha, \beta)}(x) = -n((2n + \alpha + \beta)x + \beta - \alpha)p_n^{(\alpha, \beta)}(x) + + 2(n + \alpha)(n + \beta)p_{n-1}^{(\alpha, \beta)}(x). \]

Therefore since \( p_n^{(\alpha, \beta)}(x_k) = 0 \), we have

\[ p_n^{(\alpha, \beta)}(x_k) = \frac{2(n + \alpha)(n + \beta)}{2n + \alpha + \beta} \cdot \frac{p_{n-1}^{(\alpha, \beta)}(x_k)}{1 - x_k^2}. \]
and so

\[ |P_n^{(\alpha, \beta)}(\cos \theta_k)| = \frac{2(n+\alpha)(n+\beta)}{2n+\alpha+\beta} \frac{|P_{n-1}^{(\alpha, \beta)}(\cos \theta_k)|}{\sin^3 \theta_k} \]

Now \( U_{n-1}(\theta) = \left( \sin \frac{\theta}{2} \right)^{\alpha+\frac{1}{2}} \left( \cos \frac{\theta}{2} \right)^{\beta+\frac{1}{2}} P_{n-1}^{(\alpha, \beta)}(\cos \theta) \), with \( U_{n-1}(\theta_k) \neq 0 \). Therefore

\[ |P_n^{(\alpha, \beta)}(\cos \theta_k)| = \frac{2(n+\alpha)(n+\beta)}{2n+\alpha+\beta} \frac{|U_{n-1}(\theta_k)|}{\sin^3 \theta_k}\left( \sin \frac{\theta_k}{2} \right)^{\alpha+\frac{1}{2}} \left( \cos \frac{\theta_k}{2} \right)^{\beta+\frac{1}{2}}\]

\[ = \frac{2(n+\alpha)(n+\beta)}{8(2n+\alpha+\beta)} \frac{|U_{n-1}(\theta_k)|}{\sin^3 \frac{\theta_k}{2}}\left( \sin \frac{\theta_k}{2} \right)^{\alpha+\frac{7}{2}} \left( \cos \frac{\theta_k}{2} \right)^{\beta+\frac{7}{2}} \]

By Theorem 1.1 there exists \( C(\alpha, \beta) \) such that

\[ |U_{n-1}(\theta)| \leq \frac{c(\alpha, \beta)}{\sqrt{(n-1)}} \quad \text{for} \quad 0 < \theta < \pi, \quad n \geq 2. \]

Hence

\[ |P_n^{(\alpha, \beta)}(\cos \theta_k)| \leq \frac{\frac{2(n+\alpha)(n+\beta)c(\alpha, \beta)}{8(2n+\alpha+\beta)}}{\sin^3 \frac{\theta_k}{2}}\left( \sin \frac{\theta_k}{2} \right)^{\alpha+\frac{7}{2}} \left( \cos \frac{\theta_k}{2} \right)^{\beta+\frac{7}{2}} \]

and so

\[ |U_n^{(\alpha, \beta)}(\cos \theta_k)| \leq \frac{M_2(\alpha, \beta)n^{\frac{1}{2}}}{\sin^2 \theta_k} \left( \sin \frac{\theta_k}{2} \right)^{\alpha+\frac{7}{2}} \left( \cos \frac{\theta_k}{2} \right)^{\beta+\frac{7}{2}} \]

\[ \text{for } n \geq 2. \]

We note that from (3.2.2) follows

\[ |U_n^{(\alpha, \beta)}(\cos \theta_k)| \leq \frac{M_3(\alpha, \beta)n^{\frac{1}{2}}}{\sin^2 \theta_k} \]

Combining this last inequality with Theorem 1.2 we have for \( B(\alpha, \beta), \)

\[ A(\alpha, \beta) > 0, \]

\[ B(\alpha, \beta)n^{\frac{1}{2}} \leq |U_n^{(\alpha, \beta)}(\cos \theta_k)| \leq \frac{A(\alpha, \beta)n^{\frac{1}{2}}}{\sin^2 \theta_k}, \quad n \in \mathbb{N}(\alpha, \beta). \]
Lemma 3.3: Let \( j_n \) be such that \( \theta_{j_n} \leq \frac{\pi}{2} \leq \theta_{j_n+1} \). Then
\[
\frac{n}{2} - \frac{3}{4} < j_n < \frac{n}{2} + \frac{3}{4}
\]
Proof: Since \( \theta_{j_n} \leq \frac{\pi}{2} \leq \theta_{j_n+1} \), we have
\[
0 < \frac{\pi}{2} - \theta_{j_n} \leq \theta_{j_n+1} - \theta_{j_n}
\]
and
\[
0 < \theta_{j_n+1} - \frac{\pi}{2} < \theta_{j_n+1} - \theta_{j_n}
\]
From the inequalities satisfied by \( \theta_{j_n} \) and \( \theta_{j_n+1} \) we obtain
\[
0 < \frac{(2j_n - 1)\pi}{2} = \left( \frac{2n + 1}{2} - (2j_n - 1) \right) \frac{\pi}{2n+1} = \left( \frac{2n - 4j_n + 3}{2} \right) \frac{\pi}{2n+1},
\]
and
\[
0 < \frac{2j_n + 2 - \pi}{2} = \left( \frac{2j_n + 2 - 2n + 1}{2} \right) \frac{\pi}{2n+1} = \frac{(4j_n - 2n + 3)}{2} \frac{\pi}{2n+1}.
\]
Since \( \theta_{j_n+1} - \theta_{j_n} \leq \frac{3\pi}{2n+1} \), we have \( \frac{2n - 4j_n + 3}{2} < 3 \) and
\[
\frac{4j_n - 2n + 3}{2} < 3.
\]
From these inequalities we have \( n - 2j_n < \frac{3}{2} \) and \( 2j_n - n < \frac{3}{2} \), and the lemma follows.

Lemma 3.4: Let \( j_n \) be such that \( \theta_{j_n} \leq \frac{\pi}{3} \leq \theta_{j_n+1} \). Then
\[
\frac{n}{3} - \frac{5}{6} < j_n < \frac{n}{3} + \frac{2}{3}.
\]
Proof: In the same manner as in the proof of Lemma 3.3 we have
\[
0 < \frac{\pi}{2n+1} \left( \frac{2n+1}{3} - (2j_n - 1) \right) = \frac{\pi}{2n+1} \left( \frac{2n - 6j_n + 4}{3} \right) < \frac{3\pi}{2n+1}
\]
and
\[
0 < \frac{\pi}{2n+1} \left( 2j_n + 2 - \frac{2n + 1}{3} \right) = \frac{\pi}{2n+1} \left( \frac{6j_n + 5 - 2n}{3} \right) < \frac{3\pi}{2n+1}.
\]
These two inequalities give us Lemma 3.4.

Lemma 3.5: Let \( 0 < h_1 < h_2 \). Then
\[
\frac{2\Omega(h_1)}{h_1} \geq \frac{\Omega(h_2)}{h_2}.
\]
3.3 Theorem 3.1: Let \(-\frac{1}{2} \leq \alpha < 0, -\frac{1}{2} \leq \beta < 0\). Then there exist positive numbers \(N(\alpha, \beta), C(\alpha, \beta)\) and \(C \in [-1, 1]\) such that

\[
\Delta_n(\Omega, c) = \sup_{f \in \mathcal{C}_M(\Omega)} |h_n(f, c) - f(c)| \geq \frac{C(\alpha, \beta)}{n} \sum_{k=N(\alpha, \beta)}^{n} \Omega \left( \frac{1}{k} \right), n \geq N(\alpha, \beta).
\]

Proof:

\text{case i: If } \alpha \neq \beta, \text{ let } \Psi = \frac{\pi}{2}. \text{ By (3.1.3), Lemma 3.1 and Lemma 3.2 we have for } n \geq N(\alpha, \beta)

\[
(3.3.1) \quad \Delta_n(\Omega, c) \geq \frac{M_3(\alpha, \beta)}{n^2} \sum_{k=1}^{n} \frac{\Omega(|\theta_k - \pi/2|)}{(\pi/2 - \theta_k)^2} \left( \sin \frac{\theta_k}{2} \right)^{2\alpha + 7} \left( \cos \frac{\theta_k}{2} \right)^{2\beta + 7}.
\]

Now if \(0 < \frac{\theta_k}{2} < \frac{\pi}{4}\) we have \(1 > \cos \frac{\theta_k}{2} \geq \sqrt{2} / 2\) and \(0 < \sin \frac{\theta_k}{2} \leq \sqrt{2} / 2\); while if \(\frac{\pi}{4} < \frac{\theta_k}{2} < \frac{\pi}{2}\) we have \(1 > \sin \frac{\theta_k}{2} \geq \sqrt{2} / 2\) and \(0 < \cos \frac{\theta_k}{2} \leq \sqrt{2} / 2\).

Using these inequalities and the fact that each term of the sum in (3.3.1) is positive we have

\[
\Delta_n(\Omega, c) \geq \frac{M_4(\alpha, \beta)}{n^2} \sum_{\frac{\pi}{4} \leq \theta_k \leq \frac{\pi}{2}} \frac{\Omega(\theta_k - \pi/2)}{(\theta_k - \pi/2)^2} \left( \cos \frac{\theta_k}{2} \right)^{2\beta + 7}.
\]

Next let \(j_n\) be such that \(\theta_{j_n} < \frac{\pi}{2} \leq \theta_{j_n + 1}\). Then from the inequality above, since \(\cos \frac{\pi - \theta_k}{2} \geq \frac{\pi - \theta_k}{\pi}\),

\[
(3.3.2) \quad \Delta_n(\Omega, c) \geq \frac{M_4(\alpha, \beta)}{n^2} \sum_{k=j_n+1}^{n} \frac{\Omega(\theta_k - \pi/2)}{(\theta_k - \pi/2)^2} \left( \frac{\pi - \theta_k}{\pi} \right)^{2\beta + 7}.
\]

From the inequality satisfied by \(\theta_k\) we have

\[
\theta_k - \frac{\pi}{2} \leq \theta_k - \theta_{j_n} \leq \frac{2k\pi}{2n+1} - \frac{2j_n - 1}{2n+1} \pi = (2(k - j_n) + 1) \frac{\pi}{2n+1}.
\]
Hence by Lemma 3.5 we have

\[
\frac{2\Omega(\theta_k - \pi/2)}{(\theta_k - \pi/2)^2} \geq \Omega\left(\frac{\pi}{2n+1} \frac{(2(k - j_n) + 1)}{(2(k - j_n) + 1)\pi}\right)^2
\]

\[
\geq \left(\frac{4n}{6(k - j_n)\pi}\right)^2 \Omega\left(\frac{\pi}{2n+1} \frac{(2(k - j_n) + 1)}{(2(k - j_n) + 1)\pi}\right)
\]

Also \(\frac{\pi - \theta_k}{\pi} \geq \frac{2n+1 - 2k}{2n+1}\). Therefore using these inequalities in (3.3.2) we have

(3.3.3) \(\Delta_n(\Omega, c) \geq \)

\[
\geq M_5(\alpha, \beta) \sum_{k = j_n + 1}^{n} \frac{1}{(2(k - j_n))^2} \Omega\left(\frac{(2(k - j_n) + 1)\pi}{2n+1}\right) \left(\frac{2(n - k) + 1}{2n+1}\right)^{2\beta + 7}
\]

where \(n_1 = \left\lfloor \frac{7n}{8} \right\rfloor\) and \([x]\) is the greatest integer less than or equal to \(x\).

Now by Lemma 3.1 we have \(\frac{n}{2} - \frac{3}{4} < j_n < \frac{n}{2} + \frac{3}{4}\), and so

\[
\frac{2(n - k) + 1}{2n+1} \geq \frac{2(n - [7n/8] + 1)}{2n+1} \geq \frac{n/4 + 1}{2n+1} \geq \frac{1}{12}
\]

Using this in (3.3.3) we obtain

(3.3.4) \(\Delta_n(\Omega, c) \geq M_6(\alpha, \beta) \sum_{r = 1}^{n} \frac{1}{(2\pi r)^2} \Omega\left(\frac{(2r + 1)\pi}{2n+1}\right)\)

where \(r = k - j_n\).

Now suppose \(t \in \left[\frac{2\pi}{m}, \frac{(2\pi + 1)\pi}{m}\right]\), then we have by the monotonicity of \(\Omega\)
\[
\frac{\Omega(t)}{t^2} \leq \frac{\Omega((2r+1)\pi)/m}{t^2}
\]

and so
\[
\int_{2r\pi/m}^{(2r+1)\pi/m} \frac{\Omega(t)}{t^2} \, dt \leq \Omega\left(\frac{(2r+1)\pi}{m}\right) \int_{2r\pi/m}^{(2r+1)\pi/m} \frac{dt}{t^2} \leq \frac{m}{\pi(2r)^2} \Omega\left(\frac{(2r+1)\pi}{m}\right).
\]

Therefore
\[
\frac{1}{(2r)^2} \Omega\left(\frac{(2r+1)\pi}{m}\right) \geq \frac{\pi}{m} \int_{2r\pi/m}^{(2r+1)\pi/m} \frac{\Omega(t)}{t^2} \, dt.
\]

Using these inequalities in (3.3.4) we have

\[
(3.3.5) \quad \Delta_n(\Omega, c) \geq \frac{M_7(\alpha, \beta)}{n} \sum_{r=1}^{n_1-j_n} \int_{2r\pi/(2n+1)}^{(2r+1)\pi/(2n+1)} \frac{\Omega(t)}{t^2} \, dt =
\]

\[
= \frac{M_7(\alpha, \beta)}{n} \int_{2\pi/(2n+1)}^{(2(n_1-j_n)+1)\pi/(2n+1)} \frac{\Omega(t)}{t^2} \, dt.
\]

Now by Lemma 3.1 \(\left[\frac{7n}{8}\right] - j_n \geq \frac{7n}{8} - 1 - \frac{n}{2} - \frac{3}{4} \). Therefore \(\frac{3n-2}{8} < \left[\frac{7n}{8}\right] - j_n \)

and so \(\frac{3n}{4} - 3 < 2\left[\frac{7n}{8}\right] - j_n + 1 \). We have then

\[
\left(2\left[\frac{7n}{8}\right] - j_n + 1\right) \frac{\pi}{2n+1} > \left(\frac{3n}{4} - 3\right) \frac{\pi}{2n+1} = \left(\frac{3n-12}{2n+1}\right) \frac{\pi}{4} \geq \frac{3\pi}{44}.
\]

Hence we have from (3.3.5)

\[
\Delta_n(\Omega, c) \geq \frac{M_7(\alpha, \beta)}{n} \int_{2\pi/(2n+1)}^{\pi/15} \frac{\Omega(t)}{t^2} \, dt
\]

\[
\geq \frac{M_8(\alpha, \beta)}{n} \int_{15}^{n} \Omega\left(\frac{1}{t}\right) \, dt
\]

\[
\geq \frac{M_8(\alpha, \beta)}{n} \sum_{k=15}^{n} \Omega\left(\frac{1}{k}\right), \quad n \geq 15.
\]
case ii: Let \( \alpha = \beta \) and \( \psi = \frac{\pi}{3} \). Then by (3.1.3) we have

\[
\Delta_n(\Omega, c) \geq \frac{M_9(\alpha, \beta)}{n^2} \sum_{k=1}^{n} \frac{\Omega(\theta_k - \pi/3)}{(\theta_k - \pi/3)^2} \left( \sin \frac{\theta_k}{2} \right)^{2\alpha + 7} \left( \cos \frac{\theta_k}{2} \right)^{2\alpha + 7}
\]

\[
\geq \frac{M_9(\alpha, \beta)}{n^2} \sum_{\pi/3 \leq \theta_k < \pi} \frac{\Omega(\theta_k - \pi/3)}{(\theta_k - \pi/3)^2} \left( \sin \frac{\theta_k}{2} \right)^{2\alpha + 7} \left( \cos \frac{\theta_k}{2} \right)^{2\alpha + 7}
\]

If \( \frac{\pi}{3} \leq \theta_k < \pi \) then \( \frac{\pi}{6} \leq \frac{\theta_k}{2} \leq \frac{\pi}{2} \) and so we have \( 1 > \sin \frac{\theta_k}{2} > \frac{1}{2} \) and \( 0 < \cos \frac{\theta_k}{2} < \sqrt{\frac{3}{2}} \). Therefore from (3.3.6) we have

\[
\Delta_n(\Omega, c) \geq \frac{M_{10}(\alpha, \beta)}{n^2} \sum_{\pi/3 \leq \theta_k < \pi} \frac{\Omega(\theta_k - \pi/3)}{(\theta_k - \pi/3)^2} \left( \cos \frac{\theta_k}{2} \right)^{2\alpha + 7}
\]

Next let \( j_n \) be such that \( \theta_{j_n} < \frac{\pi}{3} < \theta_{j_n + 1} \). Then since \( \cos \frac{\theta_k}{2} \geq \frac{\pi - \theta_k}{\pi} \) the above inequality gives

\[
\Delta_n(\Omega, c) \geq \frac{M_{10}(\alpha, \beta)}{n^2} \sum_{k=j_n+1}^{n} \frac{\Omega(\theta_k - \pi/3)}{(\theta_k - \pi/3)^2} \left( \frac{\pi - \theta_k}{\pi} \right)^{2\alpha + 7}
\]

Now \( \theta_k - \frac{\pi}{3} < \theta_k - \theta_{j_n} \leq \frac{2k\pi}{2n+1} - \frac{2j_n - 1}{2n+1} \pi = \frac{\pi}{2n+1} (2(k - j_n) + 1) \), \( k = j_n + 1, \ldots, n \). Hence by Lemma 3.5 we have

\[
\frac{2\Omega(\theta_k - \pi/3)}{(\theta_k - \pi/3)^2} \geq \left( \frac{4n}{6(k - j_n)\pi} \right)^2 \Omega \left( \frac{\pi}{2n+1} (2(k - j_n) + 1) \right).
\]

Also \( \frac{\pi - \theta_k}{\pi} \geq \frac{2n + 1 - 2k}{2n+1} \). Therefore (3.3.7) reduces to

\[
\Delta_n(\Omega, c) \geq \frac{M_{11}(\alpha, \beta)}{n^2} \sum_{k=j_n+1}^{n} \frac{1}{(2(k - j_n))^2} \Omega \left( \frac{(2(k - j_n) + 1)\pi}{2n+1} \right) \left( \frac{2(n-k)+1}{2n+1} \right)^{2\alpha + 7}
\]
By Lemma 3.4 \( \frac{8n}{9} \frac{n}{3} - \frac{2}{3} \leq \frac{8n}{9} - j_n \leq \frac{8n}{9} - \frac{n}{3} + \frac{5}{6} \). Also

\[
\frac{2(n - k) + 1}{2n + 1} \geq \frac{2(n - (8n/9)) + 1}{2n + 1} \geq \frac{2n/9 + 1}{2n + 1} = \frac{2n - 9}{9(2n + 1)} > \frac{1}{9}
\]

Therefore we obtain from (3.3.8)

\[
(3.3.9) \quad \Delta_n(\Omega, c) \geq \frac{M_{12}(\alpha, \beta)}{n} \sum_{k = j_n + 1}^{n_2 - j_n} \frac{1}{(2(k - j_n))^{2n + 1}} \Omega \left( \frac{2(2 - j_n) + 1)\pi}{2n + 1} \right) \left( \frac{2n - k + 1}{2n + 1} \right)^{2\alpha + 7}
\]

By an analysis similar to that for \( \psi = \frac{\pi}{2} \) we have from (3.3.9)

\[
\Delta_n(\Omega, c) \geq \frac{M_{12}(\alpha, \beta)}{n} \int_{2\pi/(2n + 1)} \frac{\Omega(t)}{t^2} \, dt
\]

Now by Lemma 3.2 \( \left[ \frac{8n}{9} \right] - j_n \geq \frac{8n}{9} - 1 - \frac{n}{3} - \frac{2}{3} = \frac{5n}{9} - \frac{5}{3} \). Therefore

\[
(2\left[ \frac{8n}{9} \right] - j_n + 1) \frac{\pi}{2n + 1} > \left( \frac{10n}{9} - \frac{10}{3} + 1 \right) \frac{\pi}{2n + 1} = \left( \frac{10n}{9} - \frac{7}{3} \right) \frac{\pi}{2n + 1} \geq \frac{\pi}{7}
\]

for \( n \geq 3 \). Hence

\[
\Delta_n(\Omega, c) \geq \frac{M_{12}(\alpha, \beta)}{n} \int_{2\pi/(2n + 1)}^{\pi/7} \frac{\Omega(t)}{t^2} \, dt , \quad n \geq 3 .
\]

From this it follows that

\[
\Delta_n(\Omega, c) \geq \frac{M_{12}(\alpha, \beta)}{n} \int_7^{2n + 1/2} \Omega \left( \frac{\pi}{t} \right) \, dt
\]

\[
\frac{M_{12}(\alpha, \beta)}{n} \int_7^n \Omega \left( \frac{1}{t} \right) \, dt
\]

\[
\frac{M_{13}(\alpha, \beta)}{n} \sum_{k = 7}^{n} \Omega \left( \frac{1}{k} \right) , \quad n \geq 7 ,
\]
and the theorem is completely proved.
REFERENCES


