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ON FINITE GROUPS WHOSE SYLOW 2-SUBGROUPS ARE
THE DIRECT PRODUCT OF A DIHEDRAL AND A SEMI-
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The Ohio State University, Ph.D., 1972
Mathematics

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DIHEDRAL AND A SEMI-DIHEDRAL GROUP

DISSERTATION
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The Ohio State University
1971

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1. Introduction.

The purpose of this paper is to classify all finite fusion-simple groups which have a Sylow 2-subgroup that is the direct product of a dihedral and a semi-dihedral group. (A group $G$ is fusion-simple if $G$ has no subgroup of index 2 and $O(G) = Z(G) = 1$.)

We prove the following

Theorem. Let $G$ be a finite fusion-simple group with a Sylow 2-subgroup that is the direct product of a dihedral and a semi-dihedral group. Then $G$ has a normal subgroup of odd index of the form $L_1 \times L_2$ where

$$L_1 \cong A_7, \text{PSL}(2, q_1), q_1 \text{ odd, } q_1 \geq 5, \text{ or } Z_2 \times Z_2,$$

$$L_2 \cong M_{11}, \text{PSL}(3, q_2), q_2 \equiv -1 \pmod{4}, \text{ or } \text{PSU}(3, q_2),$$

$$q_2 \equiv 1 \pmod{4}.$$ 

In order to prove our theorem we use ideas developed in [6]. In particular, we assume that a group $G$ is a minimal counterexample to the theorem. We then show that $G$ has an involution fusion pattern compatible with the conclusion of the theorem. Next, we select an arbitrary elementary abelian $s$ group $A$ of $G$ of order 16. For suitable four-groups $X$ and $Y$ in $A$ such that $A = X \times Y$, we establish the following assertion:
If for $a \in A^\#$, one sets
\[ \theta(c_G(a)) = \langle c_G(a) \cap o(c_G(x)) \cap o(c_G(y)) \mid x \in X^\#, y \in Y^\# \rangle, \]
then $\theta$ is an $A$-signalizer functor on $G$ in the sense of Goldschmidt [4].

If $\theta$ is nontrivial, we conclude that $W_A = \langle \theta(c_G(a)) a \in A^\# \rangle$ is of odd order. We then show that the normalizer in $G$ of $W_A$ is a strongly imbedded subgroup of $G$ and this produces a contradiction.

We shall use the following definitions which are slight restrictions of those in [2].

A finite group $G$ is an SD-group if a Sylow $2$-subgroup of $G$ is a semi-dihedral group and $G$ contains precisely one conjugacy class of involutions and precisely one class of elements of order $4$.

A finite group $G$ is a $Q$-group if a Sylow $2$-subgroup of $G$ is a semi-dihedral group and $G$ contains precisely two conjugacy classes of involutions and precisely one class of involutions.

A finite group $G$ is a D-group if a Sylow $2$-subgroup of $G$ is a semi-dihedral group and $G$ contains precisely two classes of elements of order $4$ or if a Sylow $2$-subgroup of $G$ is a dihedral group and $G$ contains at most two classes of involutions.

Let $H$ be a group in which $O_p(H) \neq 1$, $p$ an odd prime, and let $R$ be a $p$-group in $H$ such that:

(a) $R \cap O_p', _p(H)$ is a Sylow $p$-subgroup of $O_p', _p(H)$.

(b) Either $R$ is normal in a Sylow $p$-subgroup of $H$ or $RK/K$ contains $O_p(H/K)$ for every normal subgroup $K$.
of H.

Under these conditions, we say that H is p-stable with respect to R provided for any nontrivial subgroup P of R such that 

\[ O_p(H)P \text{ is normal in } H, \]

we have

\[ AC_H(P)/C_H(P) \subseteq O_p(N_H(P)/C_H(P)) \]

for every subgroup A of R such that [P, R, R] = 1.

If G is an SD-group in which \( O(G) = 1 \), then by [2] G contains a normal subgroup M of odd index and M is a simple SD-group isomorphic to \( M_{11} \), the Mathieu group on 11 letters, \( L_2(q), q \equiv -1 (\text{mod 4}), \) or \( U_3(q), q \equiv 1 (\text{mod 4}) \). Let y be an involution in M contained in the four-group \( Y \), let T be a dihedral group of order 8 in M containing \( Y \), and let \( p \) be an odd prime dividing the order of M.

(i) If \( p \nmid |C_M(y)| \) and if \( P_i \) is a maximal Y-invariant p-subgroup of M, \( i = 1, 2 \), then \( P_1 \sim P_2 \) in \( N_M(Y) \).

(ii) If \( p \mid |C_M(y)| \) and if \( P_i \) is a maximal T-invariant p-subgroup of M, \( i = 1, 2 \), then \( P_1 \sim P_2 \) in \( N_M(T) \).

(iii) If \( M \cong M_{11} \), then \( C_M(y) = GL(3, 2) \). If \( M \cong L_2(q) \), then \( C_M(y) \cong GL(2, q)/Z \) where Z is a subgroup of order \( d = (3, q-1) \), and Z lies in the center of \( GL(2, q) \). If \( M \cong U_3(q) \), then \( C_M(y) \cong GU(2, q)/Z \) where Z is a subgroup of order \( d = (3, q+1) \) and Z lies in the center of \( GU(2, q) \).
(iv) If \( p \mid |C_M(y)| \) and if \( P_i \) is a maximal \( Y \)-invariant 
p-subgroup of \( M \) such that \([P_i, Y] \neq 1, \ i = 1,2,\) 
then \( P_1 \sim P_2 \) in \( N_M(Y) \). There is a unique maximal 
\( Y \)-invariant p-subgroup \( P \) of \( M \) such that \( Y \)
centralizes \( P \).

Next, we state the principle results of [6].

Theorem. Let \( G \) be a finite group with a nonabelian Sylow 
2-subgroup which is a direct produce of two dihedral groups.

If \( O^2(G) = G \) and \( O(G) = 1 \), then

(i) \( G' = L_1 \times L_2 \), where \( L_1 \cong A_7, \ L_2(q_1), \ q_1 \) odd and 
greater than 3, and \( L_2 \cong A_7, L_2(q_2), \ q_2 \) odd and 
greater than 3, or \( Z_{2^n}, \ n \) greater than 0,

(ii) \( G/G' \) is of odd order and rank at most 2.

Our notation is standard (see [5]) and includes the following:

\( O(G) \) the maximal normal odd order subgroup of \( G \).
\( Z^*(G) \) the preimage of \( G \) of \( Z(G/O(G)) \).
\( O^2(G) \) the smallest normal subgroup \( X \) of \( G \) such that 
\( G:X \) is a power of 2.
\([x,y] \) \( x^{-1}y^{-1}xy \)
\( X \sim Y \) the subset \( X \) of \( G \) is conjugate in \( G \) to the 
subset \( Y \).

We shall also utilize the bar convention \( \bar{\cdot} \) for denoting 
homomorphic images.
2. Preliminary lemmas

The following lemmas are direct consequences of the structures of SD-groups, Q-groups, and D-groups and for this reason their proofs are omitted.

**Lemma 2.1.** Let $H$ be a group in which $O(H) = 1$ and which contains a normal simple SD-group $M$ of odd index. Let $Y$ be a four-group in $M$ and let $p$ be an odd prime. Then the following statements are true:

(i) If $P$ is a maximal $Y$-invariant $p$-subgroup of $H$, then $PM$ is a maximal $Y$-invariant $p$-subgroup of $M$.

(ii) If $p 
mid |C_M(Y)|$ and if $P_1$ and $P_2$ are maximal $Y$-invariant $p$-subgroups of $H$, then $P_1 \sim P_2$ in $N_H(Y)$.

(iii) If $p \mid |C_M(Y)|$ and if $P_1$ and $P_2$ are maximal $Y$-invariant $p$-subgroups such that $Y$ centralizes neither $P_1$ nor $P_2$, then $P_1 \sim P_2$ in $N_H(Y)$.

**Lemma 2.2.** Let $H$ be a Q-group in which $O(H) = 1$. Let $L = O^2(H)$ and let $Y$ be a four-group in $L$. If $p$ is an odd prime, then the following statements hold:

(i) If $P$ is a maximal $Y$-invariant $p$-subgroup of $H$, then $PL$ is a maximal $Y$-invariant $p$-subgroup of $L$.

(ii) If $p \nmid |C_L(Y)|$ and if $P_1$ and $P_2$ are maximal $Y$-invariant $p$-subgroups of $H$, then $P_1 \sim P_2$ in $N_L(Y)$. 

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(iii) If \( p \mid |C_H(Y)| \) and if \( P_1 \) and \( P_2 \) are maximal \( Y \)-invariant \( p \)-subgroups of \( H \) neither of which is centralized by \( Y \), then \( P_1 \sim P_2 \) in \( N(Y) \).

**Lemma 2.3.** Let \( H \) be a \( D \)-group in which \( O(H) = 1 \) and let \( Y \) be a four-group in \( H \). If \( P_1 \) and \( P_2 \) are maximal \( Y \)-invariant \( p \)-subgroups of \( H \) for some odd prime \( p \), the \( P_1 \sim P_2 \) in \( N(Y) \).

**Lemma 2.4.** Let \( M \) be a simple \( SD \)-group. If \( Y \) is a four-group in \( M \), then \( M = \langle Y, C(y)^t | y \in Y^2 \rangle \).

**Lemma 2.5.** Assume that the group \( M \) is isomorphic to \( M_{11} \) and \( p = 3 \) or that \( M \) is isomorphic to \( L_2(q) \equiv -1 \pmod{4} \) and \( p \) divides \( q \) and is prime. Let \( Y \) be a four-group in \( M \) and let \( S \) be a subgroup of \( N(Y) \) isomorphic to \( S_4 \). If \( P \) is a maximal \( Y \)-invariant \( p \)-subgroup of \( M \), then \( M = \langle P, S \rangle \).

**Lemma 2.6.** Let \( H \) be an \( SD \)-group in which \( O_p(H) = 1 \), \( p \) an odd prime, and let \( M = O^{2'}(H)O(H) \). Set \( \overline{H} = H/O(H) \) and assume the following conditions are satisfied:

1. \( H = RM \) where \( R \) is a maximal \( Y \)-invariant \( p \)-subgroup of \( H \) for some four-group \( Y \) in \( M \).
2. \( Y \) does not centralize any Sylow \( p \)-subgroup of \( O(H) \).
3. If \( M \cong M_{11} \), then \( p = 3 \) and if \( M \cong L_2(q) \), \( p \) does not divide \( q \).

Then \( H \) is \( p \)-stable with respect to \( R \) and \( H = N(Z(J(R)))O(H) \).
Proof. If $H$ is $p$-stable with respect to $R$, then ii) implies that $H$ is $p$-constrained and then the extended form of Glauberman's ZJ-Theorem [2] implies that $N(Z(J(R)))$ covers $\overline{H}$. Thus, we need only show that $H$ is relatively $p$-stable.

By the maximality of $R$, $R \cap O(H)$ is a Sylow $p$-subgroup of $O(H)$ and by our assumptions $Y$ does not centralize $R \cap O(H)$. If $O_p'(H) \not\subseteq O(H)$, then $Y \subseteq O_p'(H)$ and it follows that $Y$ centralizes $R$, a contradiction. It follows that $R \cap O_p'(H)$ is a Sylow $p$-subgroup of $O_p'(H) \subseteq O(H)$.

Clearly, condition (b) of the definition of relative $p$-stability holds for $H$ and thus, only the last condition need be verified.

We can assume without loss of generality that $O_p'(H) = 1$.

Let $P$ be a nontrivial normal subgroup of $H$ contained in $R$. Suppose that $A$ is a subgroup of $R$ such that $[P,A,A] = 1$, but that $AC(P) \not\subseteq O_p'(H/C(P))$. As in the proof of Proposition 2.6.1 of [2], we can find an $H$-invariant section $\overline{P}_1$ of $P$ which is an elementary abelian $p$-group on which $H$ acts irreducibly and $A \not\subseteq C(\overline{P}_1)$.

Setting $V = \overline{P}_1$, we may reduce to the following situation:

(i) $H = RM$ where $M/O(H)$ is a simple SD-group, $Y$ is a four-group in $M$, and $R$ is a maximal $Y$-invariant $p$-subgroup of $H$.

(ii) $O_p'(H) = 1$ and $H$ acts faithfully and irreducibly on the vector space $V$ over $GF(p)$.

(iii) $A$ is a nontrivial subgroup of $R$ and $[V,A,A] = 1$. 
(iv) If $M/O(H) \simeq M_{11}$, then $p \neq 3$ and if $M/O(H) \simeq L_3(q)$, then $p \nmid q$.

(v) $O_p^r(H) \leq O(H)$.

Set $B = A \cap O(H)$. If $B = 1$, then $[V, B, B] = 1$ and we contradict Theorem B of Hall and Higman. Thus $B = 1$ and $A \not\leq O(H)$.

In $\overline{H} = H/O(H)$ we see easily that $M \not\leq M_{11}$ and $M \not\leq L_3(3)$ since in these cases $C(y)$ is solvable for any $y \in Y^\#$.

Since $U_3(5)$ contains a single class of subgroups isomorphic to $A_5$, $U_3(5)$ is $3$-stable and so we can assume that $M \not\leq U_3(5)$.

Let $E$ be the preimage in $H$ of $O^{2'}(C_M(y))$ and set $C = RE$. Let $K$ be the semi-direct product of $C$ and $V$. Then $RV$ is a maximal $Y$-invariant $p$-subgroup of $K$ and $K$ is a $Q$-group. We also have that $C_K(V) = V$ and $O(K) \cap C \leq RO(H)$.

We can then find a dihedral group $D$ order 8 in $E$ normalizing $RV$. By Proposition 2.6.1 of [2], we have that $K$ is $p$-stable with respect to $RV$. It follows that $A \leq O_p^r(K/C_K(V))$ and so $A \leq O(K) \cap C \leq RO(H)$. Since $[E, A]$ is a normal subgroup of odd order in $\overline{E}$, we see that $\overline{A}$ centralizes $\overline{E}$ and in particular, $\overline{A}$ centralizes $\overline{Y}$.

Select $z \in Y - \langle y \rangle$ and let $F$ in $C_M(z)$ correspond to $E$. Working in $AFV$, we conclude as above that $\overline{A}$ centralizes $\overline{F}$. By Lemma 2.4 we see that $\overline{A}$ centralizes $\overline{M}$ and so $A \leq O(H)$, a contradiction. This proves the lemma.

Lemma 2.7. Let $L$ be a simple SD-group and let $Y$ be a four-group in $L$. If $W$ is a subgroup of $L$ of odd order such
that \(N(Y) \subseteq N(W)\), then \(W \subseteq C(Y)\).

Proof. This lemma is an easy consequence of the structure of \(L\).

We close this section by stating a useful result of 6. First, we introduce a definition. Let \(A\) be an elementary abelian group of order 16 acting on a group \(K\) of odd order. Suppose that \(A = X \times Y\) where \(X\) and \(Y\) are four-groups. We say that \(K\) is \((X,Y)\)-generated if \(K = \langle K_{x,y} \mid x \in X^\#, y \in Y^\# \rangle\) where \(K_{x,y}\) is a normal subgroup in \(C_K(\langle x,y \rangle)\) for all \(x \in X^\#, y \in Y^\#\). An \(A\)-invariant subgroup \(F\) of \(K\) is said to be \((X,Y)\)-generated if \(F = \langle F \cap K_{x,y} \mid x \in X^\#, y \in Y^\# \rangle\). The next result gives sufficient conditions for every \(A\)-invariant subgroup of \(K\) to be \((X,Y)\)-generated.

Proposition 2.1 of 6. Suppose that \(A\) and \(K\) are given as above and assume that the following conditions hold for all \(x, x' \in X^\#, y, y' \in Y^\#\):

(a) \(C_{K_{x,y}}(x') \subseteq K_{x',y}\) and \(C_{K_{x,y}}(y') \subseteq K_{x,y'}\).

(b) Every element in \(C_K(\langle x,y \rangle)\) inverted by the involutions in both \(X = \langle x \rangle\) and \(Y = \langle y \rangle\) lies in \(K_{x,y}\).

(c) Every element in \([C_K(x),Y] \cap C_K(y)\) inverted by the involutions in \(Y = \langle y \rangle\) lies in \(K_{x,y}\) and every element in \([C_K(y),X] \cap C_K(x)\) inverted by the involutions in \(X = \langle x \rangle\) lies in \(K_{x,y}\).

(d) If \(P\) is an \((X,Y)\)-generated \(p\)-subgroup of \(K\) where \(p\) is a prime, then every \(A\)-invariant subgroup of \(P\) is
$(X,Y)$-generated.

Then under these conditions every $A$-invariant subgroup of $X$ is $(X,Y)$-generated.
3. Fusion of involutions.

In this and in all succeeding sections we shall assume that $G$ is a minimal counter-example to our theorem. We let $S = S_1 \times S_2$ be a Sylow 2-subgroup of $G$ where $S_1$ is a dihedral group and $S_2$ is a semi-dihedral group. We let $z_i$ be an involution in the center of $S_i$, $i = 1, 2$. We also let $r_1$ and $s_1$ be two involutions which generate $S_1$ and if $S_1$ is abelian, we set $z_1 = r_1$. We let $s_2$ be an involution in $S_2 - Z(S_2)$ and we choose $v_2$ to be an element of maximal order in $S_2$ such that $s_2 = v_2^{-1}z_2$ and hence, $S_2 = \langle s_2, v_2 \rangle$.

If we set $S_i^* = \langle r_i e_1, s_i e_2 | e_1, e_2 \in Z(S_2) \rangle$, $S_2^* = \langle s_2 e_3, v_2 e_4 | e_3, e_4 \in Z(S_1) \rangle$, then $S = S_1^* \times S_2^*$ and $S_i^* \simeq S_i$ for $i = 1, 2$. Also every decomposition of $S$ as a direct product of a dihedral group with a semi-dihedral group is of this form for suitable $e_i$, $i = 1, 2, 3, 4$.

We have that $S_2$ has one conjugacy class of four-groups and that $S_1$ has one class if it is abelian and two otherwise. If $A$ is an elementary abelian subgroup of $S$ of order 16, then $A = (A \cap S_1) \times (A \cap S_2)$ and $A \simeq Z(S)$. Also $S$ has one or two conjugacy classes of elementary abelian subgroups of order 16, according as $S_1$ is abelian or nonabelian.

We shall say that $G$ has product fusion if it is possible to choose the factors $S_1^*, S_2^*$ in such a way that the following conditions hold:
(a) the involutions in $S_i^*$ are conjugate in $G$ for $i = 1, 2$;
(b) the involutions in $S - (S_1^* \cup S_2^*)$ are conjugate in $G$;
(c) the elements of order four in $S_2^*$ are conjugate in $G$;
(d) $G$ has exactly three conjugacy classes of involutions.

Since $G$ satisfies the hypotheses of our theorem, we have that $\theta_2^2(g) = G$ and $Z^*(g) = 1$. Our first goal in this section will be to show that $G$ must have product fusion.

Lemma 3.1. If $S_1$ is nonabelian, then $N_G(S) = S C_G(S)$.
If $S_1$ is abelian, then there is a 3-element in $N_G(S)$ which acts nontrivially on $Z(S)$ and $[N_G(S) : S C_G(S)] = 3$.

Proof. We first assume that $S_1$ is nonabelian. Then $\Omega_1(S)$ is the direct product of two nonabelian dihedral groups and is of index 2 in $S$. It follows that every element of odd order in $N_G(S)$ stabilizes the chain $S \supseteq \Omega_1(S) \supseteq 1$ and hence, every element of odd order must centralize $S$. This proves the first part of the lemma.

Next, assume that $S_1$ is abelian. Then $S/Z(S)$ is a dihedral group and by considering the chain $S \supseteq Z(S) \supseteq Z(S) \cap S' \supseteq 1$, we see that $S$ admits a single nontrivial odd order automorphism which is of order 3. If $[N_G(S) : S C_G(S)] = 1$, then no element in $G$ acts nontrivially on $Z(S)$. In this case Glauberman's $Z^*$-Theorem gives a contradiction. The second part of the lemma now follows directly from this.
Lemma 3.2. Suppose that $S_1$ is nonabelian and let $A$ and $B$ be representatives of the two conjugacy classes of elementary abelian subgroups of order 16 in $S$. We then have that $A$ is not conjugate to $B$ in $G$.

Proof. Suppose, by way of contradiction, that $A$ is conjugate to $B$ in $G$. Then by Alperin's Fusion Theorem [1] we can find $C$ and $D$ in $S$ such that $C \sim A$, $D \sim B$ in $S$ and such that $C$ and $D$ are contained in a Sylow 2-subgroup $T$ of $G$ and $N_S(S \cap T)$ is a Sylow 2-subgroup of $N_G(S \cap T)$ with $C^y = D$ for some $y \in N_G(S \cap T)$. Let $W$ be the normal closure of $C$ in $N_G(S \cap T)$. Since $C = (C \cap S_1) \times (C \cap S_2)$, we have $W = (W \cap S_1) \times (W \cap S_2)$ where $W \cap S_i$ is a dihedral group for $i = 1, 2$. If $g$ is of odd order in $N_G(S \cap T)$, we have from the structure of $W$ that $C^g = C$. It follows that $N_G(S \cap T) = N_{N_G(S \cap T)}(C) N_S(S \cap T)$. But then $D = C^y$ is conjugate to $C$ in $S$, a contradiction. This proves the lemma.

Lemma 3.3. If $S_1$ is nonabelian, then, relabeling if necessary, we have:

(i) The involutions in $S_2$ are conjugate in $C_G(z_1)$.

(ii) The involutions in $S_1$ are conjugate in $N_G(Q_1(S_2)')$.

(iii) The elements of order four in $S_2$ are conjugate in $C_G(z_1)$.

(iv) If $A$ is an elementary abelian subgroup of order 16 in $S$ and if $X = A \cap S_1$, $Y = A \cap S_2$, then
\[
N_G(A)/C_G(A) = S_3 \times S_3 \quad \text{(where } S_3 \text{ is the symmetric group on 3 letters), both } X \text{ and } Y \text{ are normal in } N_G(A), \text{ and the involutions in } X, \text{ in } Y, \text{ and in } A - (X \cup Y) \text{ are conjugate in } N_G(A).
\]

Proof. By Burnside's result and by Lemma 3.1 we have that the involutions in \(Z(S)\) are mutually non-conjugate in \(G\).

Let \(y\) be an involution in \(S_2 - Z(S_2)\). By Thompson's lemma \(y\) is conjugate in \(G\) to some involution \(t\) in \(S_1\langle v_2 \rangle\). Choose \(t\) such that a Sylow 2-subgroup of \(C_G(t)\) has maximal order. Then \(C_S(y)\) is a Sylow 2-subgroup of \(C_G(y)\) or \(C_S(t)\) is a Sylow 2-subgroup of \(C_G(t)\). Suppose that \(t\) is not contained in \(Z(S)\). Then for some \(g \in G\), we have \(C_S(t)^g \subseteq C_S(y)\) or \(C_S(y)^g \subseteq C_S(t)\). In either case it follows that \(z_1 \sim z_2\), a contradiction. Thus we have that \(y\) is conjugate to an involution in \(Z(S)\). But then \(\langle y, z_1 \rangle\) is the center of some Sylow 2-subgroup of \(G\) and so either \(y \sim z_2\) or \(yz_1 \sim z_2\). Replacing \(S_1\) by \(\langle s_2z_1, v_2 \rangle\) if necessary, we have the involutions in \(S_2\) are conjugate in \(C_G(z_1)\).

Next, let \(x\) be an involution in \(S_1 - Z(S_1)\). Again, by Thompson's lemma we have that \(x\) is conjugate to an involution in \(\langle r_1 s_1 \rangle S_2\). In particular, \(x\) is conjugate to an involution in \(Z(S)\). But then \(\langle u, z_2 \rangle\) is the center of some Sylow 2-subgroup of \(G\) and so \(u \sim z_1\) or \(uz_2 \sim z_1\). Replacing \(S_1\) by \(\langle r_1 e_1, s_1 e_2 \rangle\) for suitable \(e_1, e_2\) in \(Z(S_2)\) if necessary, we have that the involutions in \(S_1\) are conjugate in \(C_G(z_2)\).
Now let \( A = X \times Y \) be as in (iv). If \( a, b \in A \) and \( a \sim b \) in \( G \), then by Lemma 3.2 it follows that \( a \sim b \) in \( N_G(A) \). If \( xy \in A \) with \( x \in X^\# \), \( y \in Y^\# \), then by Thompson's lemma it follows that \( xy \) is conjugate to an involution in \( Z(S) \) if \( xy \notin Z(S) \).

We have already shown that the involutions in \( X \), in \( Y \), and in \( X \subseteq U Y \subseteq \) are conjugate in \( G \) and hence, in \( N_G(A) \). Since the involutions in \( Z(S) \) are mutually non-conjugate and since \( N_G(A)/C_G(A) \) is isomorphic to a subgroup of \( GL(4,2) \), it follows that (iv) holds.

By the preceding paragraph we conclude that no involution in \( S_2 \) is conjugate to an involution in \( S-\overline{S}_2 \). Again let \( u \) be an involution in \( S_1 - Z(S_1) \) and let \( T \) be a Sylow 2-subgroup of \( C_G(u) \) containing \( C_S(u) = \langle u, z_1 \rangle \times S_2 \). Then for some \( g \in G \), we have that \( \Omega_1(S_2)^g \subseteq T \). Since no involution in \( S_2 \) is conjugate to an involution in \( S-\overline{S}_2 \), it follows that \( g \in N_G(\Omega_1(S_2)^1) \). To complete the proof of the lemma we need to show (iii). Let \( \langle w \rangle \) be the cyclic group of order 4 in \( \Omega_1(S_2)^1 \) and let \( v \) be an element of order 4 in \( S_2 - \langle v_2 \rangle \). By Harada's Extended Transfer Theorem we have that \( v \) is conjugate to an element of order 4 in \( S_1 \Omega_1(S_2) \). It follows that \( v \sim wu \) where \( u \in S_1 \) and \( u^2 = 1 \). If \( u = 1 \), then we are done and so we can assume that this is not the case. By the preceding paragraph we can assume that \( u = z_1 \), since \( \langle w \rangle \) char \( \Omega_1(S_2)^1 \). If \( T \) is a Sylow 2-subgroup of \( C_G(v) \) containing \( C_S(v) \), we have that \( \langle z_1 \rangle = Z(T) \cap T^1 \). It follows that \( v \sim wz_1 \) in \( C_G(z_1) \). Thus we have that \( vz_1 \sim w \) in \( C_G(z_1) \). Replacing \( S_2 \)
by $\langle s_2, v_2z_1 \rangle$ if necessary, we conclude that the elements of order $4$ in $S_2$ are conjugate in $C_G(z_1)$. This completes the proof of the lemma.

**Lemma 3.4.** If $S_1$ is abelian, then, relabeling if necessary, we have:

(i) The involutions in $S_1$ are conjugate in $C_G(S_2)$.
(ii) The involutions in $S_2$ are conjugate in $C_G(S_1)$.
(iii) The elements of order $4$ in $S_2$ are conjugate in $C_G(S_1)$.
(iv) If $A$ is an elementary abelian subgroup of order $16$ in $S$ and if $X = A \cap S_1$, $Y = A \cap S_2$, then $N_G(A)/C_G(A) \cong S_2 \times Z_2$; both $X$ and $Y$ are normal in $N_G(A)$, and the involutions in $X$, in $Y$, and in $A - (X \cup Y)$ are conjugate in $N_G(A)$.

**Proof.** Let $g$ be a $3$-element in $N_G(S)$ which acts nontrivially on $Z(S)$ and which exists by Lemma 3.1. We may then relabel so that $S_1 = [Z(S), g]$ and $S_2 = C_S(g)$ and so (i) holds.

By Thompson's lemma every involution in $S$ is conjugate to an involution in $Z(S)$. Also by Burnside's lemma we have that $z_1$, $z_2$, and $z_1z_2$ are mutually non-conjugate in $G$. Let $A = X \times Y$ be as in (iv). If $a, b \in A^#$ and $a \sim b$ in $N_G(A)$ since $S$ has but one conjugacy class of elementary abelian subgroups of order $16$ when $S_1$ is abelian. We also have that $g \in N_G(A)$ and that $Y = C_A(g)$ and since $N_G(A)/C_G(A)$ is isomorphic to a subgroup of $GL(4, 2)$, we conclude that the involutions in $Y$ are conjugate in $C_{N_G(A)}(X)$ and that both (ii) and (iv) hold.
Next, let \( v \) be an element of order 4 in \( S_2 - \Omega_1(S_2) \). By Harada's theorem \( v \sim wx \) where \( w \) is an element of order 4 in \( \Omega_1(S_2) \) and \( x \in S_1 \). By the above we can find a 3-element \( g \) in \( C_G(v) \) which acts nontrivially on \( C_G(v) \). It follows that there exists a 3-element in \( C_G(wx) \) which acts nontrivially on \( S_1 \times \langle wx \rangle \). This forces \( x \) to be 1 and we have \( v \sim w \) in \( N_G(S_1) \). Since \( C_G(S_1)(S_2) \) covers \( N_G(S_1)/C_G(S_1) \), we conclude that \( v \sim w \) in \( C_G(S_1) \). This completes the proof of the lemma.

**Proposition 3.5.** The group \( G \) has product fusion. The involutions in \( S_1 \) are conjugate in \( N_G(\Omega_1(S_2)) \), and hence, in \( C_G(Z(S_2)) \). The involutions in \( S_2 \) are conjugate in \( C_G(Z(S_1)) \) and the elements of order 4 in \( S_2 \) are conjugate in \( C_G(Z(S_1)) \).

**Proof.** This lemma is a direct consequence of Lemmas 3.3 and 3.4.

Our next goal is to determine the structures of the centralizers of involutions in \( G \). We first prove

**Lemma 3.6.** Let \( C = C_G(z_1) \). If \( \overline{C} = C/\text{o}(C) \), then \( \overline{C} = \overline{S}_1 \times \overline{C}_1 \) where \( \overline{C}_1 \) has a normal subgroup \( \overline{C}_0 \) of odd index such that

\[
\overline{S}_2 \leq \overline{C}_0 \quad \text{and} \quad \overline{C}_0 \cong M_{11}, L_3(q_2), \quad q_2 = -1(\text{mod } 4), \quad \text{or} \quad U_3(q_2), \quad q_2 = 1(\text{mod } 4).
\]

**Proof.** Set \( \overline{C}_1 = o^2(\overline{C}) \). We claim that \( \overline{S}_2 \) is a Sylow 2-subgroup of \( \overline{C}_1 \). It follows by Proposition 3.5 that \( \overline{S}_2 \leq \overline{C}_1 \).

Set \( \overline{T}_1 = \overline{S}_1 \cap \overline{C}_1 \). Then \( \overline{T} = \overline{T}_1 \times \overline{S}_2 \) is a Sylow 2-subgroup of \( \overline{C}_1 \). Suppose that \( \overline{T}_1 \) is noncyclic and let \( \overline{t} \) be an involution in \( \overline{T}_1 - \langle \overline{z}_1 \rangle \). By Thompson's lemma \( \overline{t} \) is conjugate in \( \overline{C}_1 \).
to an involution in \( (z_1)^S_2 \). It follows that \( \bar{t} \) is conjugate in \( C \) to \( z_1 \), a contradiction. Next suppose that \( \bar{T}_1 \) is cyclic and nontrivial. Let \( \langle \bar{t} \rangle = \bar{T}_1 \). By Harada's theorem \( \bar{t} \) is conjugate to an element in \( \langle \bar{t}^2 \rangle S_2 \). But this forces \( z_1 \) to be conjugate to an involution in \( Z(S) = \langle z_1 \rangle \), a contradiction. Therefore \( S_2 \) is a Sylow 2-subgroup of \( C_1 \) as asserted.

If we now set \( \bar{C}_0 = O^{2'}(C_1) \), then \( \bar{C}_0 \) is a simple SD-group.

The lemma is now a direct consequence of the main result of [2], once we have shown that \( S_1 \) centralizes \( C_1 \). To see this let \( t \) be an arbitrary involution in \( S_1 \) and let \( c_1 \) be the preimage of \( C_1 \) in \( C \). Then \( \langle t \rangle \times S_2 \) is a Sylow 2-subgroup of \( \langle t \rangle C_1 \) and since \( G \) has product fusion, we have that \( t \) is isolated in \( \langle t \rangle C_1 \).

Now Claubermerman's theorem yields that \( \bar{t} \) centralizes \( C_1 \). Since \( S_1 = O_1(S_1) \), we conclude that \( S_1 \) centralizes \( C_1 \).

We shall retain the notation of this lemma. Henceforth, we let \( C = C_G(z_1) \) and we let \( C_i \) denote the preimage in \( C \) of \( C_i \), \( i = 0,1 \).

Lemma 3.7. If \( D = C_G(z_2) \) and \( D = D/0(D) \), then \( D \) has a normal subgroup \( D_0 \) of odd index of the form \( D_1 \times D_2 \) where \( D_1 \) and \( D_2 \) have the following structures:

(i) \( S_1 \leq D_1 \) and \( D_1 \simeq A_7 \), \( \text{PSL}(2,q_1) \), \( q_1 \) odd, \( q_1 \geq 5 \), or \( Z_2 \times Z_2 \);

(ii) \( S_2 \leq D_2 \) and \( D_2 \simeq \text{SL}^\pm(2,3) \) if \( c_0/0(c) \simeq M_{11} \), \( \text{SL}^\pm(2,q_2) \) if \( c_0 \simeq L_3(q_2) \), or \( \text{SU}^\pm(2,q_2) \) if \( c_0 \simeq U_3(q_2) \).
Also both $\overline{D}_1$ and $\overline{D}_2$ are normal in $\overline{D}$.

Proof. Set $V = \langle s_2 y_2, v_2 \rangle$, so that the index of $V$ in $S_2$ equals 2 and $V$ is a generalized quaternion group. Then we have that $s_2$ is not conjugate in $D$ to any involution in $S_1 \times V$. From the structure of $C_0$ the elements of order 4 in $V$ are conjugate in $C_0 \cap D$. By Proposition 3.5 the involutions in $S_1$ are conjugate in $D$. It follows that $D$ contains a subgroup $E$ of index 2 such that $S_1 \times V$ is a Sylow 2-subgroup of $E$.

Set $\tilde{E} = D/Z^*(D)$. Then $\tilde{E}$ is a fusion-simple group and $\tilde{S}_1 \times \tilde{V}$ is a direct product of two dihedral groups. Furthermore, $\tilde{V}$ is nonabelian and thus we can apply Theorem A* of [6] to conclude that $\tilde{E}$ has a normal subgroup of odd index of the form $\tilde{L}_1 \times \tilde{L}_2$ where

(i) $\tilde{S}_1 \subset \tilde{L}_1$ and $\tilde{L}_1 \cong A_7$, $L_2(q_1)$, $q_1$ odd, $q_1 \geq 5$, or $Z_2 \times Z_2$;

(ii) $\tilde{V} \subset \tilde{L}_2$ and $\tilde{L}_2 \cong A_7$, $L_2(q_2')$, $q_2$ odd, $q_2' \geq 5$.

Also both $\tilde{L}_1$ and $\tilde{L}_2$ are normal in $\tilde{E}$. By considering the preimage in $D$ of $\tilde{L}_2(\tilde{s}_2)$, we see that $L_2 \not\cong A_7$.

Now let $\overline{L}_1$ and $\overline{L}_2$ be the preimages in $\overline{D}$ of $\tilde{L}_1$ and $\tilde{L}_2$ respectively. We have that $\overline{S}_1 \times \langle \overline{s}_2 \rangle$ is a Sylow 2-subgroup of $\overline{L}_1$ and so $\overline{L}_1$ has a normal subgroup $\overline{D}_1$ of index 2 such that $\overline{S}_1 \subset \overline{D}_1$. If $\overline{S}_1 = \overline{D}_1$, then $S_1$ is a four-group and since $G$ has product fusion, $\overline{D}_1 < \overline{D}$. If $\overline{D}_1$ is simple, then $\overline{D}_1 \cong \overline{L}_1$ and $\overline{D}_1$ char $\overline{L}_1$. Again, we have $\overline{D}_1 < \overline{D}$. By a result of Schur [ ] we have that $\overline{L}_2 = SL(2, q_2')$. Moreover, $\overline{s}_1$ centralizes $\overline{L}_2$ and so
\( L_2 = L_2 \cap C_0 \). From this it follows that \( q_2^1 = 3 \) if \( C_0/0(C) \approx M_{11} \) or \( q_2^1 = q_2 \) if \( C_0/0(C) = L_j(q_2) \) or \( U_j(q_2) \).

We have \( \overline{s}_1 \times \langle \overline{s}_2 \rangle \) is a Sylow 2-subgroup of \( \langle \overline{s}_2 \rangle \overline{D}_1 \) and since \( G \) has product fusion, \( \overline{s}_2 \) is isolated in \( \langle \overline{s}_2 \rangle \overline{D}_1 \). It follows by Glauberman's theorem that \( \overline{s}_2 \) centralizes \( \overline{D}_1 \). Set \( \overline{D}_2 = \langle \overline{s}_2 \rangle \overline{L}_2 \). Then \( \overline{D}_2 \) centralizes \( \overline{D}_1 \) and \( \overline{D}_1 \cap \overline{D}_2 = 1 \). Also \( \overline{D}_2 \cong SL(2,3) \) if \( C_0/0(C) \approx M_{11} \), \( SL(2,q_2) \) if \( C_0/0(C) \approx U_j(q_2) \). Moreover, \( \overline{D}_2 \) char \( C_0(\overline{D}_1) \) and so \( \overline{D}_2 \triangleleft \overline{D} \). Set \( \overline{D}_0 = \overline{D}_1 \times \overline{D}_2 \). This completes the proof of the lemma.

Henceforth we let \( D = C_G(z_1) \) and we let \( D_i \) denote the pre-image in \( D \) of \( \overline{D}_1 \), \( i = 0,1,2 \). We also find it convenient to fix some further notation. We let \( A \) denote a fixed elementary abelian subgroup of order 16 in \( S \). Set \( X = A \cap S_1 \) and \( Y = A \cap S_2 \). Also let \( x_i \) denote the involutions in \( X \) and \( y_i \) denote the involutions in \( Y \), \( i = 1,2,3 \). Finally, we let \( x_1 = z_1 \) and \( y_1 = z_2 \).

We now have

**Proposition 3.8.** If \( \overline{C}_G(X) = C_G(x)/0(C_G(x)) \), \( \overline{C}_G(Y) = C_G(y)/0(C_G(y)) \), \( \overline{C}_G(x)/0(C_G(x)) \), \( M = O^2(C_G(x)) \), and \( N = O^2(C_G(y)) \), then \( \overline{C}_G(X) = X \times M \), \( \overline{C}_G(Y) = Y \times N \), and \( M \) and \( N \) contain characteristic subgroups of odd index, \( \overline{M}_0 \) and \( \overline{N}_0 \) respectively such that

1. \( \overline{s}_2 \subseteq \overline{N}_0 \) and \( \overline{N}_0 \cong C_0/0(C) \);
2. \( \overline{s}_1 \subseteq \overline{N}_0 \) and \( \overline{N}_0 \cong D_2/0(D) \).

**Proof.** This proposition is a direct consequence of Lemmas 3.6.
and 3.7.

We shall retain the notation of this proposition and also we shall let \( M_0 \) and \( N_0 \) denote the preimages in \( C_G(x) \) and \( C_G(y) \) of \( M_0 \) and \( N_0 \) respectively. We note that \( O(C_G(x)) = O(M) = O(M_0) \) and \( O(C_G(y)) = O(N) = O(N_0) \).

Lemma 3.9. If \( B = C_G(z_1 z_2) \) and \( \overline{B} = B/0(B) \), then
\[
\overline{B} = \overline{S}_1 \times \overline{B}_1 \quad \text{where} \quad \overline{B}_1 \quad \text{has a normal subgroup} \quad \overline{B}_0 \quad \text{of odd index such that} \quad \overline{S}_2 = \overline{B}_0 \quad \text{and} \quad \overline{B}_0 \approx D_2/O(D).
\]

Proof. We first show that \( z_1 \) is isolated in \( B \). Suppose, on the contrary, that \( z_1 \sim t \) in \( B \) where \( t \in S - \langle z_1 \rangle \).

Since \( G \) has product fusion, we have \( t \in S_1 \). But then
\[
z_2 = z_1 z_2 \sim t z_1 z_2 \sim z_1 z_2,
\]
a contradiction. It follows that \( \overline{B} = C_B(z_1) = C_B(z_2) \) and this lemma is now a direct consequence of Lemmas 3.6 and 3.7.

Henceforth, we shall let \( B = C_G(z_1 z_2) \) and \( B_i \) shall denote the preimage in \( B \) of \( \overline{B}_i \), \( i = 0,1 \).

In this section we study the subgroup structure of $G$ to the extent needed to enable us to construct a suitable signalizer functor on $G$. In this section $H$ will denote a proper subgroup of $G$. Moreover, since we are primarily concerned with the subgroups of $G$ which contain $A = X \times Y$, we shall assume that $A \subseteq H$.

In order to study the abstract structure of $H$, we can assume without loss of generality that $H \cap S$ is a Sylow 2-subgroup of $H$.

We first prove

Lemma 4.1. If $H$ has an isolated involution, then either $C_H(x)$ or $C_H(y)$ covers $H/0(H)$ for some $x \in X^\#$ or $y \in Y^\#$.

Proof. Set $T = H \cap S$. Then we must have $Z(T) \subseteq A$. Thus if $z$ is an isolated involution in $H$, we can assume that $z \in A$. By Glauberman’s theorem $C_H(z)$ covers $H/0(H)$. Suppose that $z \notin X \cup Y$. Then $z = xy$ with $x \in X^\#$, $y \in Y^\#$. By Lemma 3.9 both $x$ and $y$ are isolated in $C_G(z)$. In particular, they are isolated in $C_H(z)$ and it follows that both $C_H(x)$ and $C_H(y)$ cover $H/0(H)$ in this case.

Lemma 4.2. If $H$ contains no isolated involution and $\overline{H} = H/0(H)$, then $O^2(\overline{H})$ has a normal subgroup of odd index of the form $\overline{L}_1 \times \overline{L}_2$ where both $\overline{L}_1$ and $\overline{L}_2$ are normal in $H$ and have the following structures:
(i) $\overline{S}_1 \cap \overline{L}_1$ is a Sylow 2-subgroup of $\overline{L}_1$ and $\overline{L}_1 \cong A_7$, $L_2(r_1)$, $r_1$ odd, $r_1 \geq 5$, or $Z_2 \times Z_2$.

(ii) $\overline{S}_2 \cap \overline{L}_2$ is a Sylow 2-subgroup of $\overline{L}_2$ and $\overline{L}_2 \cong M_{11}$, $L_3(r_2)$, $r_2 = -1 \mod 4$, $U_3(r_2)$, $r_2 = 1 \mod 4$, $A_7$, $L_2(r_2)$, $r_2$ odd, $r_2 \geq 5$, or $Z_2 \times Z_2$.

Proof. Set $K = \sigma^{(2)}(H)$ and $T = S \cap K$ so that $\overline{K} = \sigma^{(2)}(H)$ and $T$ is a Sylow 2-subgroup of $K$. Also set $T_i = T \cap S_i$, $i = 1, 2$.

Suppose first that $T_1 \cap X = 1$. Then we must have $T_1 = 1$. But $H \cap S$ covers $H/K$ and it follows that $z_1$ is isolated in $H$, contrary to our assumptions. Thus we have $T_1 \cap X \neq 1$. A similar argument gives that $T_2 \cap Y \neq 1$.

If $T_i$ is cyclic, $i = 1, 2$ or if $T_2$ is generalized quaternion, then it follows that $K$ contains an isolated involution which is not the case. Thus $T_1$ is a dihedral group and $T_2$ is a dihedral or a semi-dihedral group.

Suppose that $T \neq T_1 \times T_2$. Since $T_2 \triangleleft T$ and $[N_S(T_2):T_2C_T(T_2)] \leq 2$, we conclude that $T_1 \times T_2 = T_2C_T(T_2)$ is of index 1 or 2 in $T$. Thus this index must be 2.

If there are involutions in $T - T_1 \times T_2$, let $t = t_1t_2$ be one where $t_1$ is an involution in $S_i$, $i = 1, 2$. If there are none, let $t = t_1t_2$ be an element of order 4 in $T - T_1 \times T_2$ where $t_1$ is an involution in $S_1$ and $t_2$ is an element of order 4 in $S_2$. Either by Thompson's lemma or by Harada's theorem $t$ is conjugate in $K$ to some element $u$ in $T_1T_2$.
We have that $E = C_T(t)$ is abelian of type $(2,2,2)$ or $(2,4)$ and $C_T(u)$ contains an abelian subgroup of type $(2,2,2,2)$ or $(2,2,4)$. It follows that $E$ is contained in an abelian subgroup of $K$ of type $(2,2,2,2)$ or $(2,2,4)$. We can then conclude that for some $k \in C_K(<z_1,z_2>)$, we have $t^k$ is contained in $T_1T_2$. By section 3 we see that $C_K(<z_1,z_2>)$ has a normal subgroup of index 2 which contains $T_1T_2$ and this is a contradiction. Therefore $T = T_1 \times T_2$ is a Sylow 2-subgroup of $K$.

If $T$ is abelian, then the structure of $K$ is determined by [10]. Since $G$ has product fusion, $K \cong L_2(16)$ and consequently, $K'$ is of odd order and has the asserted structure. Suppose that $T$ is nonabelian. Then we are in a position to apply either Theorem A* of [6] or we utilize the fact that we are working in a minimal counter-example to our theorem. In either case $K$ is fusion-simple and our lemma is proved.

Lemma 4.3. If $H$ contains no isolated involutions, if $J = O^2(H)A$, and if $\overline{H} = H/O(H)$, then $\overline{J}$ contains a normal subgroup of odd index of the form $\overline{F}_1 \times \overline{F}_2$ where $\overline{F}_1$ and $\overline{F}_2$ have the following structures:

1. $\overline{F}_1 \cap \overline{F}_2$ is a Sylow 2-subgroup of $\overline{F}_1$ and $\overline{F}_1 \cong A_7$, $L_2(r_1)$, $PGL(2,r_1)$, $r_1$ odd, or $Z_2 \times Z_2$.

2. $\overline{F}_2 \cap \overline{F}_2$ is a Sylow 2-subgroup of $\overline{F}_2$ and $\overline{F}_2 \cong M_{11}$, $L_3(r_2)$, $r_2 \equiv -1(\text{mod } 4)$, $U_3(r_2)$, $r_2 \equiv 1(\text{mod } 4)$, $A_7$, $L_2(r_2)$, $PGL(2,r_2)$, $r_2$ odd, or $Z_2 \times Z_2$.

Also both $\overline{F}_1$ and $\overline{F}_2$ are normal in $\overline{J}$. 

Proof. By the preceding lemma \( K = O^2(\overline{H}) \) has a normal subgroup of odd index of the form \( \overline{L}_i \times \overline{L}_j \) where \( \overline{S}_i \cap \overline{L}_i \) is a Sylow 2-subgroup of \( \overline{L}_i \) and \( \overline{L}_j \) has the structure specified in that lemma, \( i = 1, 2 \). Since \( G \) has product fusion, Glauberman's theorem yields that \( \overline{X} \) centralizes \( \overline{L}_2 \) and \( \overline{Y} \) centralizes \( \overline{L}_1 \).

Set \( \overline{F}_1 = \overline{L}_1 \overline{X} \) and \( \overline{F}_2 = \overline{L}_2 \overline{Y} \). Then \( \overline{F}_i \) has the structure asserted in the conclusion of this lemma and \( \overline{S}_i \cap \overline{F}_i \) is a Sylow 2-subgroup of \( \overline{F}_i \), \( i = 1, 2 \). Moreover, \( \overline{F}_1 \) and \( \overline{F}_2 \) centralize each other and \( \overline{F}_1 \cap \overline{F}_2 = 1 \). In order to obtain the conclusion of the lemma it is sufficient to show that \( \overline{F}_1 \trianglelefteq \overline{J} \), \( i = 1, 2 \). If \( \overline{X} \leq \overline{L}_1 \) and \( \overline{Y} \leq \overline{L}_2 \), then this follows from the fact that \( \overline{L}_1 \trianglelefteq \overline{J} \), \( i = 1, 2 \).

Suppose then that \( \overline{X} \not\leq \overline{L}_1 \). If \( \overline{L}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \), then \( \overline{W} = \overline{K} \cap \overline{S}_2 \) is a four-group and \( \overline{W} = \overline{L}_2 \). It follows that \( \overline{J} = N_\overline{j}(\overline{W}) \). From Proposition 3.8 we conclude that \( \overline{F}_1 \) char \( C_{\overline{j}}(\overline{W}) \trianglelefteq \overline{J} \) and so \( \overline{F}_1 \trianglelefteq \overline{J} \).

If \( \overline{L}_2 \) is a simple group, then \( C_{\overline{j}}(\overline{L}_2) \cap \overline{L}_2 = 1 \). Since \( C_{\overline{j}}(\overline{L}_2) \) is a D-group, we have \( \overline{F}_1 \) char \( C_{\overline{j}}(\overline{L}_2) \) and that \( \overline{F}_1 \trianglelefteq \overline{J} \).

Suppose now that \( \overline{Y} \not\leq \overline{L}_2 \). Then \( \overline{S}_2 \cap \overline{L}_2 \) is a dihedral group and so \( \overline{Y}(\overline{S}_2 \cap \overline{L}_2) \) is also a dihedral group. Arguing as in the preceding paragraph, we can conclude that \( \overline{F}_2 \trianglelefteq \overline{J} \) as well. This completes the proof.

We now find it convenient to fix some more notation. We set \( p_2 = 3 \) if \( M_0/O(M) \cong M_{11} \), we set \( p_2 \) to be the prime which divides \( q_2 \) if \( M_0/O(M) \cong L_3(q_2) \), and we do not define \( p_2 \) if \( M_0/O(M) \cong U_3(q_2) \). Also we let \( \mathfrak{S}_2 \) denote the set of all odd primes which divide the order of \( C_{M_0/O(M)}(Y) \).
Our next goal is to prove some results on the transitivity of maximal $A$-invariant $p$-subgroups under conjugation by the elements in $N_g(A)$ where $p$ is an odd prime.

Lemma 4.4. Suppose that $p$ is an odd prime and that $p \not\in \mathbb{S}_2$. If $P_1$ and $P_2$ are maximal $A$-invariant $p$-subgroups of $H$, then $P_1 \sim P_2$ in $N_H(A)$.

Proof. Any maximal $A$-invariant $p$-subgroup of $H$ contains a Sylow $p$-subgroup of $O(H)$ and any two $A$-invariant Sylow $p$-subgroups of $O(H)$ are conjugate in $C_{O(H)}(A)$. It is immediate from this that it will suffice to prove that $\overline{P}_1 \sim \overline{P}_2$ in $N_\overline{H}(A)$ where $\overline{H} = H/O(H)$.

If $H$ has no isolated involution and if $J = O^2(H)A$, then by the preceding lemma $\overline{J}$ has a normal subgroup of odd index of the form $\overline{F}_1 \times \overline{F}_2$ where $\overline{F}_1 \cap \overline{F}_2$ is a Sylow 2-subgroup of $\overline{F}_1$ and $\overline{F}_1$ has the structure asserted in the conclusion of that lemma, $i = 1,2$.

If $F_i$ denotes the pre-image in $H$ of $\overline{F}_i$, $i = 1,2$, then $F_1 = F_1 \cap N_0$ and $F_2 = F_2 \cap N_0$. Since $p \not\in \mathbb{S}_2$, we have that $p$ does not divide the order of $C_{\overline{F}_2}(\overline{\gamma})$.

Set $\overline{U}_i = \overline{F}_i \cap \overline{F}_1$, $\overline{V}_i = \overline{F}_i \cap \overline{F}_2$, and $\overline{R}_i = C_{\overline{F}_i}(\overline{A})$. Then by Lemmas 2.1, 2.2, or 2.3 and by the structures of $\overline{F}_1$ and $\overline{F}_2$, $U_1V_1 \sim U_2V_2$ in $N_{\overline{F}_1\overline{F}_2}(A)$. Also by the maximality of $P_i$ we have that $\overline{R}_i$ is a Sylow $p$-subgroup of $N_J(\overline{U}_i\overline{V}_i) \cap C_J(A)$, $i = 1,2$.

Since $\overline{F}_1 = \overline{U}_1\overline{V}_1\overline{R}_1$, $i = 1,2$, we can conclude as in the proof of Lemma 2.1 that $\overline{F}_1 \sim \overline{F}_2$ in $N_\overline{J}(A)$.

Next we consider the case that $H$ contains an isolated involv-
tion. Suppose first that some \( x \in X^\# \) is isolated. For definiteness, let \( x = x_1 \). Then we have \( \overline{J} = X \times \overline{J} \cap C_1 \). Set \( F = O^2(\overline{J} \cap C_1) \). Then \( \overline{S_2} \cap F \) is a Sylow 2-subgroup of \( F \) and since \( p \not\in S_2 \), \( p \) does not divide the order of \( C_F(\overline{Y}) \). Since \( F \) is an SD-group, a Q-group, a D-group, or a 2-group, we have by Lemmas 2.1, 2.2, and 2.3 that \( \overline{P}_1 \sim \overline{P}_2 \) in \( N_{\overline{J} \cap C_1}(\overline{Y}) \) and hence, in \( N_{\overline{Y}}(A) \).

If no involution in \( X \) is isolated in \( H \), then by Lemma 4.1 we have that some \( y \in X^\# \) is isolated in \( H \). For definiteness, let \( y = y_1 \). Set \( J_1 = J \cap D_1 \), \( J_2 = J \cap D_2 \), and \( E = O(C_J(A)) \).

Since \( J = J \cap D \), we have \( \overline{J}_1 \) and \( \overline{J}_2 \) are normal in \( J \). Also \( J_1 \cap J_2 \) has odd order and so \( \overline{J}_1 \overline{J}_2 = \overline{J}_1 \times \overline{J}_2 \). Now let \( K = J_1 J_2 \overline{E}_0(H) \) and \( \overline{K} = K / 0(K) \). Also set \( \overline{F}_1 = O^2(\overline{J}_1) \), \( i = 1, 2 \).

As above we have that \( p \) does not divide \( C_{\overline{F}_2}(\overline{Y}) \). Since \( \overline{P}_i = (\overline{P}_1 \cap \overline{F}_1 \times \overline{P}_1 \cap \overline{F}_2)(\overline{P}_1 \cap \overline{F}) \), \( i = 1, 2 \), we conclude that \( \overline{P}_1 \sim \overline{P}_2 \) in \( N_{\overline{K}}(A) \). This completes the proof of the lemma.

Proposition 4.5. Suppose that \( p \) is an odd prime and that \( p \not\in S_2 \). If \( P_1 \) and \( P_2 \) are maximal \( A \)-invariant \( p \)-subgroups of \( G \), then \( P_1 \sim P_2 \) in \( N_G(A) \).

Proof. Suppose that the proposition is false and choose \( P_1 \) and \( P_2 \) such that they violate it and such that the order of \( R = P_1 \cap P_2 \) is maximal subject to this. Set \( K = N_G(R) \). Without loss we can assume that \( K \cap S \) is a Sylow 2-subgroup of \( K \). If \( R \not\leq 1 \), then \( K \) is a proper subgroup of \( G \) and the preceding lemma is applicable. This leads to a contradiction by a standard argument.
In any case, we have \( P_i \neq 1, i = 1,2 \). Considering the action of \( A \) on \( P_i \), we see then that \( C_{P_i}(T_i) \neq 1 \) for some maximal subgroup \( T_i \) of \( A, i = 1,2 \). Setting \( H = C_G(T) \) where \( T = T_1 \cap T_2 \neq 1 \) and hence, \( C_{P_i}(T) \neq 1, i = 1,2 \), we can assume without loss that \( S \cap H \) is a Sylow 2-subgroup of \( H \). Again, the preceding lemma is applicable. We let \( Q_i \) be a maximal \( A \)-invariant \( p \)-subgroup of \( G \) containing \( C_{P_i}(T) \) as well as a maximal \( A \)-invariant \( p \)-subgroup \( R_i \) of \( H, i = 1,2 \). Then \( Q_i \cap P_i \neq 1, i = 1,2 \)
and \( Q_i \) contains \( R_j \) for some \( u \in N_H(A) \) by the preceding lemma. These conditions together with our maximal choice of \( P_1 \cap P_2 = R \) now force \( R \neq 1 \) and the lemma is proven.

Lemma 4.6. If \( p \in S_2 \) and if \( P \) is a maximal \( A \)-invariant \( p \)-subgroup of \( G \), then \( P \cap M_0 \) covers a maximal \( Y \)-invariant \( p \)-subgroup of \( M_0/O(M) \).

Proof. Let \( \Gamma \) be the set of all maximal \( A \)-invariant \( p \)-subgroups \( P^* \) of \( G \) such that \( P^* \) covers a maximal \( Y \)-invariant \( p \)-subgroup of \( M_0/O(M) \). Suppose, by way of contradiction, that there exist maximal \( A \)-invariant \( p \)-subgroups of \( G \) not contained in \( \Gamma \). Among those not in \( \Gamma \), select the subset \( \mathcal{J} \) consisting of all subgroups \( P_1 \) such that the order of \( P_1 \cap M_0 \) is maximal. For each group \( P_1 \in \mathcal{J} \), let \( \Gamma(P_1) \) denote the subset of \( \Gamma \) which consists of all groups \( P_2 \) such that \( P_2 \supset P_1 \cap M_0 \). It is clear that \( \Gamma(P_1) = \emptyset \) for each \( P_1 \in \mathcal{J} \). We now consider the set of all pairs of groups \((P_1, P_2)\) where \( P_1 \in \mathcal{J} \) and \( P_2 \in \Gamma(P_1) \) and among these we choose a pair \((P_1, P_2)\) such that the order of \( R = P_1 \cap P_2 \) is
Suppose that $R = 1$. Since $P_1 \cap M_0 \leq R$, $P_1 \cap M_0 = 1$. Since $P_1 \neq 1$, $C_{P_1}(x) \neq 1$ for some $x \in X$. But a maximal A-invariant $p$-subgroup of $C_g(x)$ covers a maximal $Y$-invariant $p$-subgroup of $M_0/0(M)$ and so we can find some $P_3 \in \Gamma$ containing $C_{P_1}(x)$. Since $P_1 \cap M_0 = 1$, we have $P_3 \in \Gamma(P_1)$. Since $P_1 \cap P_3 \neq 1$, we have contradicted the maximality of $R$. Thus we have that $R \neq 1$.

Set $H = N_G(R)$. Without loss we can assume $S \cap H$ is a Sylow 2-subgroup of $H$. Also set $K = O^2(H)A$. Now let $Q_1 = N_{P_1}(R)$ and let $V_1$ be a maximal A-invariant $p$-subgroup of $G$ containing $Q_1$ and a maximal A-invariant $p$-subgroup $U_1$ of $H$, $i = 1, 2$.

Suppose that $V_1 \in \Gamma$; since $V_1 \supseteq R \supseteq P_1 \cap M_0$, we have $V_1 \in \Gamma(P_1)$. Since $V_1 \cap P_1 \supseteq Q_1 \supseteq R$, we have a contradiction. It follows that $V_1 \notin \Gamma$. Since $V_1 \cap M_0 \supseteq P_1 \cap M_0$, we see that $V_1 \in X$.

Suppose next that $V_2 \notin \Gamma$. Since $V_2 \supseteq R \supseteq P_1 \cap M_0$, it follows that $V_2 \notin X$ and also that $V_2 \cap M_0 = P_1 \cap M_0$. From this it follows that $P_2 \in \Gamma(V_2)$ and since $P_2 \cap V_2 \supseteq R$, we again have a contradiction. Thus we have that $V_2 \in \Gamma$.

Now suppose that $H$ has no isolated involution and set $K = O^2(H)A$ and $\overline{H} = H/0(H)$. Also let $\overline{F} = \overline{F_1} \times \overline{F_2}$ be the normal subgroup of odd index in $\overline{K}$ which satisfies the conclusion of Lemma 4.3 and retain the notation of that lemma. Finally, let $F_1$ denote
the pre-image in H of $F_j$, $i = 1,2$. Then we see that $F_2 = F_2 \cap M_0$.

By the maximality of $U_1$ we have $U_1 \cap F_2$ is a maximal $Y$-invariant $p$-subgroup of $F_2$. Since $U_1 \cap M_0 \subset R \cap M_0$ and the order of $U_1 \cap M_0$ equals the order of $P_1 \cap M_0$, we conclude that $U_1 \cap M_0 \subset R \subset O(H)$. Since $U_1 \cap F_2 = U_1 \cap F_2 \cap M_0$, it follows that $U_1 \cap F_2 = 1$. But now the structure of $F_2$ forces $U_1 \cap F_2 = 1$ also. It follows then that $U_1 = (U_1 \cap F_1) \circ U_1$ for $i = 1,2$ and we also have that $U_1 \cap F_1$ is a maximal $X$-invariant $p$-subgroup of $F_1$. As in previous arguments we then have that $U_2^k = U_1$ for some $k \in N_K(A)$. Since $k$ normalizes $M_0$ and $v_2^k \supseteq R \supseteq P_1 \cap M_0$, it follows that $v_2^k \in \Gamma(P_1)$. This again contradicts the maximality of $R$, since $v_2^k \cap P_1 \supseteq R$.

We can assume then that $H$ contains an isolated involution.

First suppose that $x$ is isolated in $H$ for some $x \in X$. Then we have that $K = X \times K \cap M$ and that $U_1$ is a maximal $Y$-invariant $p$-subgroup of $K \cap M$, $i = 1,2$. Since $K \cap M_0 \subset K \cap M$, we see that $U_1 = U_1 \cap (K \cap M_0, A)$, $i = 1,2$. But as above we conclude that $U_1 \cap K \cap M_0 = 1$ and hence, that $U_2 \cap K \cap M_0 = 1$ also. Then $U_1$ is a Sylow $p$-subgroup of $O_K(A)$, $i = 1,2$ and it follows easily that $U_1 \sim U_2$ in $N_K(A)$. But this leads to the same contradiction as above.

Finally, we consider the case that an involution $y$ in $Y$ is isolated in $H$. For definiteness, we let $y = y_1$. Set $K_1 = K \cap D$.
Lemma 4.7. Suppose that \( p \in S_g \) and that \( P_i \) is a maximal \( A \)-invariant \( p \)-subgroup of \( G \), \( i = 1, 2 \). If \( P_i \cap P_j = 0(M) \) covers a maximal \( Y \)-invariant \( p \)-subgroup of \( M_{Q(A)} \), then \( P_i \) is a maximal \( A \)-invariant \( p \)-subgroup of \( M_{Q(A)} \).

Proof. Suppose that \( P_i \cap P_j = 0(M) \) and \( P_i \) is a maximal \( A \)-invariant \( p \)-subgroup of \( M_{Q(A)} \) and \( P_j \) is a maximal \( Y \)-invariant \( p \)-subgroup of \( M_{Q(A)} \), then \( P_i \) is a maximal \( A \)-invariant \( p \)-subgroup of \( M_{Q(A)} \) and \( P_j \) is a maximal \( Y \)-invariant \( p \)-subgroup of \( M_{Q(A)} \).

Set \( H = N_{g}(R) \) and let \( U \) be a maximal \( A \)-invariant \( p \)-subgroup of \( M_{Q(A)} \) and \( P_1 \) such that the order of \( R = P_1 \cap P_2 \) is maximal. Then we see that \( R \) covers a maximal \( Y \)-invariant \( p \)-subgroup of \( M_{Q(A)} \) and \( U \) contains \( P_i \). This leads to a contradiction. A standard argument and proves the lemma.

Proposition 4.8. Suppose that \( p \cap p = 0(M) \) and \( P_i \) is a maximal \( A \)-invariant \( p \)-subgroup of \( G \), \( i = 1, 2 \). If \( P_1 \cap P_2 = 0(M) \) and \( P_1 \cap P_2 = 0(M) \), then \( P_i \) is a maximal \( A \)-invariant \( p \)-subgroup of \( M_{Q(A)} \), then \( P_i \) is a maximal \( A \)-invariant \( p \)-subgroup of \( M_{Q(A)} \).

Finally, let \( F_i = K_i \cdot E_i(A) \) and let \( i = 1, 2 \). Then we have that \( p_i \leq E_i(A) \) and that \( F_i = K_i \cdot E_i(A) \) and \( i = 1, 2 \). Again, we see that \( p_i \) is a maximal \( A \)-invariant \( p \)-subgroup of \( M_{Q(A)} \) and this leads to the same contradiction as above. This how-
If \([P \cap M_0, Y] \neq O(M)\), then \(P \sim P_2\) in \(N_G(A)\).

Proof. Let \(Q_i\) be a maximal \(A\)-invariant \(p\)-subgroup of \(M_0\), 
\(i = 1, 2\) such that \([Q_1, Y] \subset O(M)\) and \([Q_2, Y] \neq O(M)\) and let \(P_i\) be a maximal \(A\)-invariant \(p\)-subgroup of \(G\) containing \(Q_i\), 
\(i = 1, 2\).

Suppose that \(P\) is a maximal \(A\)-invariant \(p\)-subgroup of \(G\) and that \([P \cap M_0, Y] \subset O(M)\). Since \(P \cap M_0\) covers a maximal \(Y\)-invariant \(p\)-subgroup of \(M_0/O(M)\) by Lemma 4.6, we have \((P \cap M_0)^n \subset P_i \cap M_0\) for some \(m \in N_{M_0}(A)\). Then \((P_1 \cap M_0)^n \subset (P \cap M_0)^m\) covers a maximal \(Y\)-invariant \(p\)-subgroup of \(M_0/O(M)\). By the preceding lemma \(P \sim P_1\) in \(N_G(A)\) and hence, \(P \sim P_2\) in \(N_G(A)\).

If \([P \cap M_0, Y] \neq O(M)\), we can apply a similar argument to conclude that \(P \sim P_2\) in \(N_G(A)\).

We shall say that a proper subgroup \(H\) of \(G\) covers \(M_0/O(M)\) if \(H \cap M_0\) covers \(M_0/O(M)\). Similarly, we shall say that \(H\) covers \(N_0/O(N)\) if \(H \cap N_0\) covers \(N_0/O(N)\). We now prove some results concerning \(p\)-local subgroups of \(G\) which cover \(M_0/O(M)\) and \(N_0/O(N)\).

Lemma 4.9. Suppose that a Sylow \(p\)-subgroup of \(O(C)\) is non-trivial. If \(P\) is a maximal \(A\)-invariant \(p\)-subgroup of \(G\), then there is a \(p\)-local subgroup \(K\) of \(G\) containing \(PA\) and covering \(M_0/O(M)\).

Proof. Let \(Q\) be an \(A\)-invariant Sylow \(p\)-subgroup of \(O(C)\). Then \(N_0(Q)\) contains \(A\) and covers \(C_0/O(C)\) and hence, covers \(M_0/O(M)\). Let \(R\) be a maximal \(A\)-invariant \(p\)-subgroup of \(N_0(Q)\) so
that \( R \) covers a maximal \( Y \)-invariant \( p \)-subgroup of \( M_0/0(M) \). If \( p \in S_2 \), we choose \( R \) such that \( \overline{R} \sim \overline{M_0/0(M)} \) in \( \overline{M_0} \) where \( \overline{M_0} = M_0/0(M) \).

Now among all \( p \)-local subgroups of \( G \) containing \( RA \) and covering \( M_0/0(M) \) choose \( K \) such that an \( A \)-invariant \( p \)-subgroup \( U \) of \( K \) has maximal order subject to containing \( R \). Suppose that \( U \) is not a maximal \( A \)-invariant \( p \)-subgroup of \( G \). Then there exists an \( A \)-invariant \( p \)-subgroup of \( G \) which properly contains and normalizes \( U \). We denote this subgroup by \( V \).

By Lemmas 4.1, 4.2, and 4.3 \( O^2(K)A \) has a normal subgroup \( L \) containing \( O(K) \) such that \( \overline{L} = \overline{L \cap M_0} \sim M_0/0(M) \) in \( \overline{K} = K/0(K) \).

Set \( F = LUA \) and let \( \overline{F} = F/0(F) \). By the maximality of \( U \) we have that \( V_0 = U \cap O(F) \) is a Sylow \( p \)-subgroup of \( O(F) \). Also \( \overline{F} = \overline{L \cap M_0 \cap U} \times \overline{X} \) and \( N_L \cap M_0 (V_0) \) covers \( L \cap M_0 \) by the Frattini argument and hence, covers \( M_0/0(M) \).

Now \( C_V(x) \not\subset U \) for some \( x \in X^# \). Suppose \( x \) centralizes \( V_0 \).

Then \( C_V(x) \supset U \) and we can find a maximal \( A \)-invariant \( p \)-subgroup of \( C_G(x) \) containing \( C_V(x) \) such that this subgroup and \( A \) are contained in a \( p \)-local subgroup of \( C_G(x) \) which covers \( M_0/0(M) \). However, this contradicts the maximality of \( U \) and our choice of \( K \).

Thus we can assume that \( [V_0, x] \not\subset L \). Set \( V_1 = [V_0, x] \). Then \( V_1 \) is normalized by \( U C_V(x) \), by \( A \), and by \( N_L \cap M_0 (V_0) \). But then \( N_G(V_1) \) contradicts our choice of \( K \). This contradiction then proves our lemma since \( U \) must be a maximal \( A \)-invariant \( p \)-subgroup of \( G \) and \( U \sim P \) in \( N_G(A) \).
Lemma 4.10. Suppose that a Sylow $p$-subgroup of $O(D)$ is non-trivial. Then there exists a $p$-local subgroup $H$ of $G$ containing $A$ and a maximal $A$-invariant $p$-subgroup $P$ of $G$ such that $H$ covers $N_0/O(N)$.

Proof. Let $V$ be an $A$-invariant Sylow $p$-subgroup of $O(D)$. Then $N_D(V)$ contains $A$ and covers $D_1/O(D)$ and hence, covers $N_0/O(N)$. Now among all $p$-local subgroups of $G$ containing $A$ and covering $N_0/O(N)$ choose $H$ such that an $A$-invariant $p$-subgroup $P$ of $H$ has maximal order. Suppose that $P$ is not a maximal $A$-invariant $p$-subgroup of $G$. Then there is an $A$-invariant $p$-subgroup $U$ of $G$ containing $P$ properly and normalizing $P$. Then $C_U(y) \not\subset P$ for some $y \in Y$. We can now argue as in the preceding lemma to obtain a contradiction to our choice of $H$. It follows that $P$ is a maximal $A$-invariant $p$-subgroup of $G$ and the lemma is proved.

We conclude this section with a result needed in the next.

Lemma 4.11. Suppose $p \in \mathfrak{S}_2$ and $R$ is an $A$-invariant $p$-subgroup such that $R = \langle R \cap O(G)(x) \mid x \in X \rangle$. Then $R$ is contained in maximal $A$-invariant $p$-subgroups $P$ and $Q$ of $G$ such that $[P \cap M_0, Y] \subseteq O(M)$ and $[Q \cap M_0, Y] \not\subseteq O(M)$.

Proof. Let $U$ be a maximal $A$-invariant $p$-subgroup of $G$ containing $R$. By the preceding Lemma 4.9 there is a $p$-local subgroup $K$ of $G$ containing $UA$ and covering $M_0/O(M)$. Then $O^2(K)A$ has a normal subgroup $L$ containing $O(K)$ such that $L = L \cap M_0 \cong M_0/O(M)$ in $K = K/O(K)$. Let $J = UA$ and set $J = J/O(J)$. Then $J = L \cap M_0 U X$ and for $x \in X$ we have that $[R \cap O(G)(x), L \cap M_0]$
is a normal subgroup of odd order in \( \overline{L} = L \cdot M_0 \). It follows that \( R \cdot O(C_G(x)) \) centralizes \( \overline{L} \) and so we conclude that \( R \cap O(C_G(x)) \leq O(J) \). It follows that \( R \leq O(J) \). Since \( J \) covers \( M_0/O(M) \), we have that maximal A-invariant p-subgroups of \( J \) cover maximal Y-invariant p-subgroups of \( M_0/O(M) \). Moreover, we can find maximal A-invariant p-subgroups \( Q_1 \) and \( Q_2 \) of \( J \) such that \( O(J) \cap U \leq Q_1 \cap Q_2 \) and such that \( [Q_1 \cap M_0, Y] \leq O(M) \) and \( [Q_2 \cap M_0, Y] \not\leq O(M) \).

It is now only necessary to choose \( P_i \) to be a maximal A-invariant p-subgroup of \( G \) containing \( Q_i \), \( i = 1, 2 \) in order to obtain the conclusion of our lemma.
5. An A-signalizer functor.

Our main goal in this section is to show that if for \( a \in A^# \) we set
\[
\theta(\sigma_G(a)) = \langle \sigma_G(a) \cap o(\sigma_G(x)) \cap o(\sigma_G(y)) \mid x \in X^#, y \in Y^# \rangle
\]
then \( \theta \) is an A-signalizer functor on \( G \). In order to do this we must show that \( \theta(\sigma_G(a)) \) has odd order for all \( a \in A^# \) and that \( \theta \) satisfies the balance condition:
\[
\theta(\sigma_G(a)) \cap \sigma_G(b) = \theta(\sigma_G(b)), \ a, b \in A^#.
\]
We shall use Proposition 2.1 of [6] to show this.

We are retaining the following notation of the preceding sections: \( B, C, D, B_1, C_1, D_0, D_1, D_2, M, M_0, N, N_0, C_0, B_0 \).

We first prove the following useful lemma.

Lemma 5.1. The following conditions hold for all \( x, x' \in X^#, y, y' \in Y^# \)
\[
\begin{align*}
(i) \ & c_G(x) \cap o(c_G(x')) = o(c_G(x)) \\
(ii) \ & c_G(y) \cap o(c_G(x)) \cap o(c_G(y')) = o(c_G(y)) \\
(iii) \ & c_G(xy) \cap o(c_G(x')) = o(c_G(xy)) \\
(iv) \ & [D \cap o(c_G(x)), D_2] = o(D) \text{ and in particular }\ D \cap o(c_G(x)) \text{ centralizes } \overline{D} \text{ in } \overline{D} = D/\sigma(D)
\end{align*}
\]

Proof. Choose \( x \in X - < x_1 > \) and set \( R = C \cap o(c_G(x)) \). If \( C = C/\sigma(C) \) then \( [R, C_1] \) is a normal subgroup of odd order in \( C_1 \) since \( C_1 = C_1 \cap \overline{M} \). It follows that \( R \) centralizes \( C_1 \) and so \( R = o(C) \). Now (i) follows easily from this.

Choose \( y \in Y - < y_1 > \) and set \( Q = D \cap o(C) \cap o(c_G(y)) \).
If $\overline{D} = D/O(D)$, then $[\overline{Q}, \overline{D}_1]$ is a normal subgroup of odd order in $\overline{D}_1$, $i = 1, 2$ since $\overline{D}_1 = \overline{D}_1 \cap \overline{N}$ and $\overline{D}_2 = \overline{D}_2 \cap \overline{M}$. From this it follows that $\overline{Q}$ centralizes both $\overline{D}_1$ and $\overline{D}_2$ and hence, $R \subseteq O(D)$. Then (ii) follows easily from this.

Next, choose $x$ in $X - \langle x_1 \rangle$ and set $\overline{P} = B \cap O(C_G(x))$. If $\overline{B} = B/O(B)$, then $[\overline{P}, \overline{B}_1]$ is a normal subgroup of odd order in $\overline{B}_1$ since $\overline{B}_1 = \overline{B}_1 \cap \overline{M}$. We conclude that $R \subseteq O(B)$ and (iii) follows easily from this.

If $\overline{D} = D/O(D)$, then $[\overline{D} \cap O(C), \overline{D}_2]$ is a normal subgroup of odd order in $\overline{D}_2$. It follows that $\overline{D} \cap O(C)$ centralizes $\overline{D}_2$ and (iv) is a consequence of this.

**Lemma 5.2**. Let $E$ be an $A$-invariant subgroup of $G$ of odd order such that $AE \subseteq H \cap K$ where $H$ is a proper subgroup of $G$ covering $N_0/O(N)$ and $K$ is a proper subgroup of $G$ covering $M_0/O(M)$. Then $E \subseteq O(H) \cap O(K)$ if and only if $E = \langle E \cap O(C_G(x)) \cap O(C_G(y)) \mid x \in X^#, y \in Y^# \rangle$.

**Proof.** Without loss of generality we can assume that $H = O^2(H)A$ and $K = O^2(K)A$. Set $\overline{H} = H/O(H)$ and $\overline{K} = K/O(K)$. By Lemmas 4.1, 4.2, and 4.3 we have that $\overline{H}$ has a normal subgroup of odd index of the form $\overline{F}_1 \times \overline{F}_2$ where $\overline{F} \subseteq \overline{F}_1$, $\overline{F} \subseteq \overline{F}_2$, and $\overline{F}_1 \cong N_0/O(N)$ and $\overline{K}$ has a normal subgroup of odd index of the form $\overline{L}_1 \times \overline{L}_2$ where $\overline{L} \subseteq \overline{L}_1$, $\overline{L} \subseteq \overline{L}_2$, and $\overline{L}_2 \cong M_0/O(M)$. Let $\overline{F}_i$ be the pre-image in $H$ of $\overline{F}_i$, $i = 1, 2$ and let $\overline{L}_j$ be the pre-image in $K$ of $\overline{L}_j$, $j = 1, 2$. We then have $\overline{F}_1 = \overline{F}_1 \cap \overline{N}_0$, $\overline{F}_2 = \overline{F}_2 \cap \overline{M}_0$, $\overline{L}_1 = \overline{L}_1 \cap \overline{N}_0$, and $\overline{L}_2 = \overline{L}_2 \cap \overline{M}_0$. 

First assume that $E \subseteq O(H) \cap O(K)$ and let $R = E \cap C \cap D$.

Set $\overline{C} = C/0(C)$. Since $\overline{C} = \overline{C}_0 = \overline{C}_0 \cap M_0 = \overline{C}_0 \cap N_0 \cap L_2$, it follows that $[\overline{R}, \overline{C}_0]$ is a normal subgroup of odd order in $\overline{C}_0$ and thus, that $\overline{R}$ centralizes $\overline{C}_0$. We conclude that $R \subseteq O(C)$. Now set $\overline{D} = D/0(D)$. Then $\overline{D}_1 = \overline{D}_1 \cap N_0 \cap F_1$ and $\overline{D}_2 = \overline{D}_2 \cap C$ and so we see as above that $\overline{R}$ centralizes both $\overline{D}_1$ and $\overline{D}_2$. It follows that $R \subseteq O(D)$. From this we easily conclude that $C_E(<x,y>) \subseteq O(C_G(x)) \cap O(C_G(y))$ for all $x \in X^\#$, $y \in Y^\#$. Since $E = \langle C_E(<x,y>) \mid x \in X^\#, y \in Y^\# \rangle$, the "only if" part of the lemma is proved.

Next, assume that $E = \langle E \cap O(C_G(x)) \cap O(C_G(y)) \rangle$. Set $R = O(C) \cap O(D)$. We then see that $\overline{R}$ centralizes $\overline{F}_1$ and $\overline{F}_2$ in $\overline{H}$ and that $\overline{R}$ centralizes $\overline{L}_1$ and $\overline{L}_2$ in $\overline{K}$. It follows that $R \subseteq O(H) \cap O(K)$ and we easily conclude that $E \subseteq O(H) \cap O(K)$. This completes the proof of the lemma.

We now select an arbitrary $a \in A$ and set $K = \theta(C_G(a))$. By Lemma 5.1 $K$ has odd order. If for $x \in X^\#$ and $y \in Y^\#$ we set $K_{x,y} = K \cap O(C_G(x)) \cap O(C_G(y))$, then $K_{x,y} \not\subseteq C_K(x,y)$ and $K = \langle K_{x,y} \mid x \in X^\#, y \in Y^\# \rangle$. We shall show that every $A$-invariant subgroup of $K$ is $(X,Y)$-generated with respect to the subgroups $K_{x,y}$.

**Lemma 5.3.** For all $x, x' \in X^\#$, $y, y' \in Y^\#$ we have

$C_{K_{x,y}}(x') \subseteq K_{x',y}$ and $C_{K_{x,y}}(y') \subseteq K_{x,y}'$.

**Proof.** By Lemma 5.1 $C_{K_{x,y}}(x') \subseteq O(C_G(x'))$ and $C_{K_{x,y}}(y') \subseteq O(C_G(y'))$. Therefore, $C_{K_{x,y}}(x') \subseteq K_{x',y}$ and $C_{K_{x,y}}(y') \subseteq K_{x,y}'$. The lemma follows.
Lemma 5.4. Every element in $C_K(<x,y>)$ inverted by the involutions in both $X - <x>$ and $Y - <y>$ lies in $K_{x,y}$.

Proof. Suppose that $k \in C_K(x,y)$ and that $k$ is inverted by the involutions in both $X - <x>$ and $Y - <y>$. By (i) of Lemma 5.1 $k \in O(C_G(x))$ and then by (iv) of the same lemma $k \in O(C_G(y))$. It follows that $k \in K_{x,y}$.

Lemma 5.5. The elements in $[C_K(x),Y]^* \cap C_K(y)$ inverted by the involutions in $Y - <y>$ lie in $K_{x,y}$. The elements in $[C_K(y),X]^* \cap C_K(x)$ inverted by the involutions in $X - <x>$ lie in $K_{x,y}$.

Proof. For definiteness let $x = x_1$ and $y = y_1$. We then have that $[K \cap C, Y] = O_C(0(C_G(a))) \subset O(C^G(a))$ which is abelian in $C = C/O(C)$. It follows that $[K \cap C, Y]^* \subset O(C)$. If $g \in [K \cap C,Y]^* \cap D$ and $g$ is inverted by $y_2$, then $g \in O(D)$ by (iv) of Lemma 5.1. Thus we have $g \in K_{x,y}$.

We also see that $[K \cap D,X]$ is an $X$-invariant subgroup of $D_1$ of odd order and so it is abelian by the structure of $D_1$ in $D = D/O(D)$. It follows that $[K \cap D,X]^* \subset O(D)$. If $g \in [K \cap D,X]^* \cap C$ and $g$ is inverted by $x_2$, then $g \in O(C)$ since $g$ is of odd order and so $g \in K_{x,y}$.

Lemma 5.6. If $R$ is an $(X,Y)$-generated $p$-subgroup of $K$ for some prime $p$, then every $A$-invariant subgroup of $R$ is $(X,Y)$-generated.

Proof. We assume that $R \neq 1$, otherwise the lemma is trivial.
By Lemmas 4.9, 4.10, and 4.11 we can find a maximal A-invariant p-subgroup $P$ of $G$ containing $R$ and we can find $p$-local subgroups $H$ and $K$ of $G$ containing $PA$ such that $H$ covers $N_0/O(N)$ and $K$ covers $M_0/O(M)$. Now by Lemma 5.2 we have $R \subseteq O(H) \cap O(K)$ and by the same lemma we conclude that every A-invariant subgroup of $R$ is $(X,Y)$-generated.

**Proposition 5.7.** We have that $\theta$ is an A-signalerizer functor on $G$ and that the group $W = \langle \theta(C_G(a)) \mid a \in A^# \rangle$ is of odd order.

**Proof.** Since $\theta(C_G(a)), a \in A^#$, is of odd order, we need only verify the balance condition. Choose $a,b \in A^#$ and set $K = \theta(C_G(a))$. Then Lemmas 5.3--5.6 show that $K$ satisfies conditions (a)--(d) of Proposition 2.1 of [ ]. It follows by that proposition that every A-invariant subgroup of $K$ is $(X,Y)$-generated. Since $K \cap C_G(b)$ is A-invariant, we conclude that $K \cap C_G(b) = \langle K_{x,y} \cap C_G(b) \mid x \in X^#, y \in Y^# \rangle$. Thus $\theta$ is an A-signalerizer functor on $G$ and the second part of the lemma is a consequence of the main result of Goldschmidt's paper [4].
6. A strongly imbedded subgroup.

In this section we will show that \( N_G(W) \) is a strongly imbedded subgroup of \( G \) if \( W \neq 1 \) where \( W \) is the group defined in Proposition 5.7. We retain the notation of the preceding sections and we set \( G^* = N_G(W) \).

If \( H \) is a proper subgroup of \( G \) containing \( A \) and covering \( N_0/O(N) \), then by Lemmas 4.1, 4.2, and 4.3 we conclude that \( H \) has a normal subgroup \( F \) containing \( O(H) \) such that \( X \leq F \) and \( \bar{F} = F \cap N_0 \cong N_0/O(N) \) in \( \bar{H} = H/O(H) \). Similarly, if \( K \) is a proper subgroup of \( G \) containing \( A \) and covering \( M_0/O(M) \), then \( K \) has a normal subgroup \( L \) containing \( O(K) \) such that \( Y \leq L \) and \( \bar{L} = L \cap M_0 \cong M_0/O(M) \). We shall use these facts several times in this section. We also note here that \( W = \langle O(G(x)) \cap O(G(y)) \mid x \in X^#, \ y \in Y^# \rangle \).

Lemma 6.1. We have that \( \langle N_G(A), O(M), O(N), O(G(y)) \mid y \in Y^# \rangle \subseteq G^* \).

Proof. Since the subgroups \( O(G(x)) \cap O(G(y)) \), \( x \in X^# \) and \( y \in Y^# \), are permuted among themselves by \( N_G(A) \), we see that \( N_G(A) \subseteq G^* \).

Let \( E \) be an \( A \)-invariant Sylow \( p \)-subgroup of \( O(M) \) and let \( E = E_1E_2E_3E_4 \) be the \( Y \)-decomposition of \( E \). If \( g \in E_1 \) then \( \bar{g} \)
centralizes $\overline{Y}$ in $\overline{D} = D/0(D)$. Since $g$ is inverted by $y_2$, we see that $g \in O(D)$ and it follows that $g \in W$. We then see that $E_i \subseteq G^*$, $i = 1, 2, 3$. Since $E_0 \subseteq N_G(A)$, we conclude that $E \subseteq G^*$. It follows that $0(M) \subseteq G^*$. Recalling that if $g \in O(N) \cap O(C_G(x))$ where $x \in X^#$, then $g \in O(C_G(y))$ for all $y \in Y^#$, we may use a similar argument to show that $0(N) \subseteq G^*$.

Now let $R$ be an $A$-invariant Sylow-$p$-subgroup of $O(D)$. This time we let $R = R_0 R_1 R_2 R_3$ be the $X$-decomposition of $R$. Suppose that $g \in R_1$. Since $g$ is of odd order and is inverted by $x_2$, we see that $g \in O(C)$. It follows that $R_i \subseteq W \subseteq G^*$, $i = 1, 2, 3$. Now suppose that $g \in R_0$ and is inverted by $y_2$. If $M = M/0(M)$, then we see that $\overline{g} \in M_0 \cap O(D) \subseteq Z(C_M(\overline{y}_1))$. Since $y_2$ inverts $g$, we conclude that $g \in O(M)$. It now follows that $R_0 \subseteq G^*$ and thus, that $R \subseteq G^*$. It then follows that $O(G(y)) \subseteq G^*$, $y \in Y^#$.

Now set $W_{x,y} = O(C_G(x)) \cap O(C_G(y))$, $x \in X^#$, $y \in Y^#$. Then $W = \langle W_{x,y} \mid x \in X^#, y \in Y^# \rangle$ and so $W$ is $(X,Y)$-generated. We then have

Lemma 6.2. Every $A$-invariant subgroup of $W$ is $(X,Y)$-generated.

Proof. We again use Proposition 2.1 of [6]. Condition (iv) follows by the proof of Lemma 5.6 and conditions (i) and (ii) follow by the proofs of Lemmas 5.3 and 5.4. Thus we need only verify condition (iii) to prove our lemma.

Suppose that $u \in [C_W(x), Y]^* \cap C_G(y)$ where $x \in X^#$, $y \in Y^#$ and suppose that $u$ is inverted by the involutions in $Y - \langle y \rangle$. For
definiteness let \( x = x_1 \) and \( y = y_1 \). We then have \([C \cap W, Y]\) is a subgroup of odd order in \( C_0 \) which is normalized by \( N_{C_0}(A) \). It follows by Lemma 2.7 that \([C \cap W, \overline{Y}] \leq C_{C_0}(\overline{Y})\) and so is abelian in \( \overline{C} = C/O(C) \) and thus \([C \cap W, Y]' \leq O(C)\). Since \( y_2 \) inverts \( u \), we have by Lemma 5.1 that \( u \in O(D) \) and so \( u \in W_{x,y} \).

Now suppose that \( u \in [D \cap W, X]' \cap C \) and that \( u \) is inverted by the involutions in \( X - \langle x_1 \rangle \). Since the order of \( u \) is odd, \( u \in O(C) \). Since \([D \cap W, X]\) is an \( X \)-invariant subgroup of \( D_1 \) of odd order in \( D = D/O(D) \), it is abelian. It follows that \([D \cap W, X]' \leq O(D) \) and that \( u \in W_{x_1,y_1} \). Thus condition (iii) of proposition 2.1 of [6] is verified and this lemma is a direct consequence of that proposition.

We now introduce a concept defined in [6]. We then prove a result which gives a sufficient condition for the existence of a \( p \)-local subgroup \( J \) of \( G \) which covers both \( N_0/O(N) \) and \( M_0/O(M) \) and which contains \( A \). Let \( H \) be a subgroup of \( G \) which contains \( A \). We say that \( H \) is \((X, p)\)-constrained if \( X \) does not centralize any Sylow \( p \)-subgroup of \( O(H) \) and we say that \( H \) is \((Y, p)\)-constrained if \( Y \) does not centralize any Sylow \( p \)-subgroup of \( O(H) \).

We recall that \( p_2 = 3 \) if \( M_0/O(M) \cong M_{11} \) and \( p_2 \) divides \( q_2 \) if \( M_0/O(M) \cong L_3(q_2) \). We then have

Lemma 6.3. Let \( p \) be an odd prime such that \( p \neq p_2 \). If \( C \) is \((Y,p)\)-constrained and if \( D \) is \((X,p)\)-constrained, then for some
maximal A-invariant p-subgroup $P$ of $G$ we have $N_G(Z(J(P)))$

Proof. By our assumptions Sylow $p$-subgroups of both $O(C)$ and

$O(D)$ are nontrivial. By Lemma 4.10 we can find a $p$-local sub-

group $H$ of $G$ containing $A$ and covering $N_O/O(N)$ such that $H$

also contains a maximal A-invariant $p$-subgroup $P$ of $G$. By the

proof of that lemma we can assume that $P$ contains an A-invariant

Sylow $p$-subgroup $R_1$ of $O(D)$. Then $H$ contains a normal subgroup

$F$ such that $X_0(H) \subseteq F$ and $F = F \cap N_O$ in $H = H/O(H)$ and

$F = N_O/O(N)$. Without loss we can assume that $H = FP$. Let $H_1 = FP$

and let $Q = P \cap O(H_1)$. Since $R_1 \subseteq Q$, we conclude that $H$

is $(X,p)$-constrained and so $O_p^*(H_1) \subseteq 0(H_1)$. As in section 5 of [6],

we have that $H_1$ is $p$-stable with respect to $P$ and by the extended

from Glauberman's $ZJ$-theorem we have that $N_{H_1}^*(Z(J(P)))$

covers $H_1/O(H_1)$ and hence, covers $N_O/O(N)$.

By Lemma 4.9 we can find a $p$-local subgroup $K$ of $G$ con-

taining $PA$ and covering $M_O/O(M)$. Then $K$ has a normal subgroup

$L$ containing $X_0(K)$ such that $L = L \cap M_O \subseteq M_O/O(M)$ in $K = K/O(K)$.

Without loss of generality we can assume that $K = LPA$. Since $P$ is

a maximal A-invariant $p$-subgroup of $G$, we can also assume that $P$

contains an A-invariant Sylow $p$-subgroup $R_2$ of $O(C)$. Let

$V = P \cap O(K) = P \cap O(LP)$. Then $R_2 \subseteq V$. Suppose that $O_p^*(LP) \neq$

$\neq O(LP)$. Then $O_p^*(LP)$ has even order and so we can assume that

$Y \subseteq O_p^*(LP)$. But then $[Y, R_2] \subseteq R_2 \cap O_p^*(LP) = 1$ and this
contradicts our assumption that $C$ is $(Y, p)$-constrained. We may now apply Lemma 2.6 to conclude that $N_{L^p}(Z(J(P)))$ covers $L^p/O(L^p)$ and hence covers $M_0/O(M)$. This then completes the proof of the lemma.

The next proposition incorporates many ideas found in Section 5 of [6].

Proposition 6.4. Let $p$ be a prime divisor of the order of $W$. If $R$ is an $A$-invariant Sylow $p$-subgroup of $W$, then $N_{\mathfrak{g}}(R)$ covers $M_0/O(M)$.

Proof. We assume, by way of contradiction, that the proposition is false. The proof is then broken into a number of steps.

By Lemma 6.2 $R$ is $(X,Y)$-generated and so by Lemmas 4.9, 4.10, and 4.11 we can find a maximal $A$-invariant $p$-subgroup $P$ of $G$ containing $R$ and we can find $p$-local subgroups $H$ and $K$ of $M_0/O(M)$ containing $PA$ such that $H$ covers $N_0/O(N)$ and $K$ covers $M_0/O(M)$. As we have seen previously, $H$ contains a normal subgroup $F$ such that $O_X(H) \subseteq F$ and $\overline{F} = F \cap N_0 \cong N_0/O(N)$ in $\overline{H} = H/O(H)$ and $K$ contains a normal subgroup $L$ such that $O_Y(K) \subseteq L$ and $\overline{L} = L \cap M_0 \cong M_0/O(M)$ in $\overline{K} = K/O(K)$. Without loss of generality we can assume that $H = FPA$ and that $K = LPA$. We may also choose $H$ and $K$ such that the orders of $O_P(H)$ and $O_P(K)$ are maximal. If $Q = P \cap O(H)$ and $V = P \cap O(K)$, then we have $Q$ is a Sylow $p$-subgroup of $O(H)$ and by the maximality of $O_P(H)$, $Q \leq H$, and we have $V$ is a Sylow $p$-subgroup of $O(K)$ and also $V \leq K$. By Lemma 5.2 we conclude that $R = Q \cap V$ is a Sylow $p$-subgroup of
(a) We have $RA$ is not contained in any proper subgroup $J$ of $G$ such that $J$ covers both $M_0/O(M)$ and $N_0/O(N)$.

Proof. Suppose there is such a $J$. Then by Lemma 5.2 we have $R \subseteq O(J)$ and $O(J) \subseteq W$ and so $R$ is a Sylow $p$-subgroup of $O(J)$. It follows that $N_J \cap M_0(R)$ covers $(J \cap M_0)O(J)/O(J)$ by the Frattini argument and hence, covers $M_0/O(M)$, contrary to our assumption.

(b) We have that $p \neq p_2$.

Proof. Set $E = N_{M_0}(R)$. If $p = p_2$, then $E$ contains a maximal $Y$-invariant $p$-subgroup $P \cap M_0$ of $M_0$ since $R \not\subseteq P$. By the Frattini argument $N_{M_0}(W)W = EW$ and so in $M_0 = M_0/O(M)$, $E$ contains a subgroup $S \cong S_4$. By Lemma 2.5 we conclude that $M_0 = E$, a contradiction. This proves (b).

We shall retain the notation $E = N_{M_0}(R)$.

(c) We have that $Y$ does not centralize $V$.

Proof. If $Y$ centralizes $V$, then $C_L \cap M_0(V)$ covers $I/O(K)$ and hence, covers $M_0/O(M)$. Since $R \subseteq V$, this is a contradiction.

(d) We have that $X$ centralizes $Q$ and that $V \not\subseteq Q$.

Proof. Suppose that $X$ does not centralize $Q$. Then $H$ is $(X,p)$-constrained and as in the proof of Lemma 6.3, we conclude that $N_H(Z(J(P)))$ covers $F/O(H)$ and hence, covers $N_0/O(N)$. Since $Y$ does not centralize $V$ by (c), we see that $K$ is $(Y,p)$-constrained and since $p \neq p_2$ by (b), we also conclude that $O(H) \cap O(K)$.
$N_K(Z(J(P)))$ covers $L/0(K)$ and hence, covers $M/0(M)$. Since $RA \leq N_G(Z(J(P)))$, this contradicts (a). Thus $X$ centralizes $Q$.

If $V \leq Q$, then $R = V$ and so $R \nleq K$, a contradiction.

(e) We have that $X$ centralizes $V$ and $P$.

Proof. Suppose that $V_1 = [V, X] \neq 1$. Since $X$ centralizes $Q$, we have that $C_H(Q)$ covers $F/0(H)$ and also that $V_1 \leq C_R(Q)$. Moreover, we have $V_1 \neq Q$. Since $C_P(X)$ covers $F$ in $K$, we have that $[P, X] = V_1$ and so $V_1$ is a Sylow $p$-subgroup of $C_F(x)$ for some $x \in X^#$ in $H$. Then $C_{V_1}(X) \leq Q$. Since $V_1 \triangleleft P$ and $R$ centralizes $X$, we see that $R$ normalizes $C_{V_1}(X)$. But $L \cap M_0$ and $C_H(Q)$ both normalize $C_{V_1}(X)$ and this contradicts (a), if $C_{V_1}(X) \neq 1$. It follows that $V_1 \cap Q = 1$.

Set $P_1 = C_P(X)$ and so $P = P_1V_1$. Since $V_1 \neq 1$, we have $F = L_2(q_1)$, $q_1$ odd and $q_1 \geq 5$. It follows that $F \cap P_1 = 1$. If $P_1 = 1$, then $P_1 = Q$ and $R = C_Y(X)$ is normal in $L \cap M_0$, a contradiction. Thus $P_1 \neq 1$ and so $C_F(N_1(P_1)) \cong L_2(q)$ where $q^P = q_1$ and $C_{P_1}(V_1) = 1$ by the proof of (v) of Lemma 2.4.2 of [2]. It follows that $C_{P_1}(V_1) = Q$.

Set $V^* = C_Y \cap P_1(V_1)$. Then we have $V^* \triangleleft V_1P_1 = P$. Also we have $V_1 \cong V_1$ since $V_1 \cap Q = 1$. We claim that $V^* \neq 1$. Now $Y$ centralizes $V_1$ and since $V = C_Y(V_1)$, we conclude that for all $y \in Y^#$, $d$ does not centralize $C_Y(X)$. We can also find a 3-element $u \in L \cap M_0$ which permutes the involutions in $Y$ cyclically and so $\langle u \rangle Y$ acts on $C_Y(X)$. Now as in the proof of Lemma 5.10
in [6] we conclude that $C_V(x)$ contains an elementary abelian subgroup of order $p^3$. Since $C_V(x)$ normalizes $V_1$ and $V_1$ is cyclic, we see that $V^* = C_{C_V(x)}(V_1) \neq 1$ as asserted. Now $L \cap M_0$ normalizes $V^*$ and since $V^* \in Q$, $C_H(Q)$ centralizes $V^*$, and $R$ normalizes $V^*$, we have contradicted (a). This forces $V_1 = 1$ and (e) is proved.

(f) We have $F \cong L_2(q_1)$, $q_1$ odd, $q_1 \geq 5$ and $F \cap \bar{F} = 1$ in $H = H/\mathfrak{O}(H)$.

Proof. To prove (f) it is sufficient to show that $F \not\cong A_7$, since we already have that $F \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Suppose then that $F \cong A_7$. Then $\bar{V} = \bar{F}$ is of order 3 in $\bar{F}$. Without loss of generality we can assume that $S_1 \subset C_H(Q)$ and we also have that $S_1 \cong D_8$. It follows that $S_1$ normalizes $C_G(x)$ and centralizes $M/\mathfrak{O}(M)$. We can also assume that $S_1$ acts on $\bar{F}$ in $\bar{F}$ and thus that $S_1$ normalizes $P$. By our maximal choice of $O_p(K)$ we must have that $V$ is a Sylow $p$-subgroup of $O(M)$ and so $S_1$ also normalizes $V = P \cap O(M)$. We then have $C_x(S_1) \subset \mathfrak{O} \subset C_V(S_1)$ and so $R = C_V(S_1)$. Since $C_G(S_1) \cap N_{\mathfrak{O}}(V)$ covers $M/\mathfrak{O}(M)$ and normalizes $R$, we have a contradiction. This proves (f).

Now as in the proof of Lemma 5.13 of [6], we can find an $A$-invariant $t$-subgroup $T^*$ where $t$ is an odd prime distinct from $p$ such that $T^* \subset C_F(O_x(x))$ for some $x \in \bar{x}^\#$. $T^*$ is permutable with $P$, and $[T^*,x] = T^*$. For definiteness let $x = x_1$. We also have $C_P(T^*) = 1$ in $\bar{H}$ and so $C_P(T^*) \subset Q$. Since $T^* = [T^*,x] \subset C_H(Q)$ and so $Q = C_P(T^*)$. 


As we have seen above, $V$ is a Sylow $p$-subgroup of $O(M)$ and for the same reason we have $V$ is a Sylow $p$-subgroup of $O(C_G(x))$ for all $x \in X^\#$. Since $T^* = [T^*, X]$, we have $T^* \subseteq O(C)$ and so we can find an $A$-invariant Sylow $t$-subgroup $T$ of $O(C)$ containing $T^*$ and permutable with $P$. Then $T$ is also permutable with $V$ since $VT = PT \cap O(C)$. Thus we see that $VT$ is a Hall-subgroup of $O(C)$. We set $I = [T, X]$ and see that $I = [TV, X] \trianglelefteq TV$ and $I \neq 1$ since $T^* \subseteq I$.

(g) We have that $C_V(I) = 1$.

Proof. Set $V_1 = C_V(I)$ and assume that $V_1 \neq 1$. Since $N_C(VT)$ covers $C_o/O(C)$, we have $J = N_{M_0}((VT)$ covers $M_o/O(M)$. Also we have $I \trianglelefteq J$ and $V \subseteq J$ and since $V$ is a Sylow $p$-subgroup of $O(M)$, $V$ is a Sylow $p$-subgroup of $O(J)$. By the Frattini argument $J_1 = N_J(V)$ covers $J/O(J)$ and hence, covers $M_o/O(M)$. We also have $V_1 \trianglelefteq J_1$. Since $T^* \subseteq I$ and $Q = C_P(T^*)$, we have $V_1 \subseteq Q$ and so $C_H(V_1)$ covers $N_o/O(N)$. Since $R$ normalizes $I$, this contradicts (a). Thus $V_1 = 1$ and this proves (g).

(h) If $I = I_1I_2I_3$ is that $X$-decomposition of $I$, then $I_1 \subseteq O(C_G(y_1))$ and $X$ does not centralize $I_1$ for each $i = 1, 2, 3$.

Proof. First we show that $Y$ does not centralize $I$. Set $J = IVY$. If $Y \subseteq C_J(I)$, then $[V, Y] \subseteq C_J(I)$ because $C_J(I) \triangleleft J$. Since $[V, Y] \neq 1$, this contradicts (g). Thus $Y$ does not centralize $I$. Since $N_{M_0}(I)$ covers $M_o/O(M)$ we can find a 3-element which cyclically permutes the involutions in $Y$ and which is contained in $N_{M_0}(I)$. This element then cyclically permutes $I_1$. 


\( i = 1, 2, 3 \). Since \( I_i' \subset C(y_i) \), we conclude that \( I_i' \subset C_0(y_i) \), \( i = 1, 2, 3 \) by Lemma 5.1. If \( X \) centralizes \( I_i' \), \( i = 1, 2, 3 \), then 
\[ [I, X] \subset I_0 = C_1(Y) \]. Since \( I = [I, X] \), this is a contradiction. It follows that \( X \) does not centralize \( I_i' \) for each \( i = 1, 2, 3 \).

(i) There is a maximal \( A \)-invariant \( t \)-subgroup \( U \) of \( G \) permutable with \( V \) and containing \( I \) and there is a \( t \)-local subgroup \( J \) of \( G \) covering \( M_0/O(M) \) and containing \( UVA \). If \( t \in S_2 \), and if \( U^* \) is any maximal \( A \)-invariant \( t \)-subgroup of \( G \), then a conjugate of \( U^* \) by a suitable element in \( N_G(A) \) has the properties of the preceding sentence.

Proof. Let \( T_0 \) be a maximal \( A \)-invariant \( t \)-subgroup of \( C \) containing \( T \) such that \( T_0 \) is permutable with \( V \). If \( t \in S_2 \), we can choose \( T_0 \) such that \( [T_0, Y] \subset C(0) \) or we can choose \( T_0 \) such that \( [T_0, Y] \notin C(0) \). Since \( T_0 \) covers a maximal \( Y \)-invariant \( t \)-subgroup of \( M_0/O(M) \), we see by Proposition 4.8 that in order to prove (i) it is sufficient to show that \( V \) is permutable with a maximal \( A \)-invariant \( t \)-subgroup \( U \) of \( G \) containing \( T_0 \) such that \( UVA \) is contained in a \( t \)-local subgroup of \( G \) which covers \( M_0/O(M) \).

Now \( N_G(I) \) contains \( T_0VA \) and covers \( M_0/O(M) \). Among all \( t \)-local subgroups of \( G \) containing \( T_0VA \) and covering \( M_0/O(M) \) choose \( J \) such that an \( A \)-invariant \( t \)-subgroup \( T_1 \) of \( J \) containing \( T_0 \) and permutable with \( V \) has maximal order and relative to this, choose \( J \) such that an \( A \)-invariant \( t \)-subgroup of \( J \) containing \( T_1 \) has maximal order.

We first show that \( T_1 \) is a maximal \( A \)-invariant \( t \)-subgroup of...
Without loss of generality we can assume that $J = O^2(J)A$. By Lemmas 4.1, 4.2, and 4.3 $J$ has normal subgroups $L_1$ and $L_2$ such that $X_0(J) \subseteq L_1$, $X_0(J) \subseteq L_2$, and in $\overline{J} = J/O(J)$ we have $\overline{L_1L_2} = \overline{L_1} \times \overline{L_2}$ is of odd index and $\overline{L_2} = \overline{L_2} \cap M_0 \cong M_0/O(M)$. Since $V \subseteq O(M)$, we see that $\overline{V}$ centralizes $\overline{L_2}$ and so $\overline{V} = C_{\overline{V}}(A)$. Let $T_2^0 T_1$ where $T_2$ is a maximal $A$-invariant $t$-subgroup of $J$.

Since $L_1$ is a 2-group or $L_1 \cong L_2(q)$ or $\text{PGL}(2,q)$, $q$ odd, we conclude that $[\overline{T_2}, \overline{X}]$ is a characteristic subgroup of $C_{\overline{L_1}}(\overline{x})$ for some $x \in X^L$. It follows that $\overline{V}$ normalizes $[\overline{T_2}, \overline{X}]$ and so $V \subseteq J_0$ where $J_0 = L_2 \cdot \text{O}(C_J(A))A$. If $J_0 = J_0/O(J_0)$, then $X$ centralizes $O^2(J_0)$ and since $V \subseteq O(M)$, we conclude that $V \subseteq O(J_0)$.

We claim that $V$ is a Sylow $p$-subgroup of $O(J_0)$. Suppose $V$ is contained in an $A$-invariant $p$-subgroup $V_1$ of $O(J_0)$. Since $X$ centralizes $P$, we conclude that $X$ centralizes every $A$-invariant $p$-subgroup of $G$ and in particular, $X$ centralizes $V_1$. We then have that $V_1$ centralizes $(J_0 \cap M_0)O(M)/O(M) = M_0/O(M)$ and hence, $V_1 \subseteq O(M)$. It follows that $V_1 = V$ and that $V$ is a Sylow $p$-subgroup of $O(J_0)$. We then conclude that $V$ is permutable with a conjugate $T_2^j$ of $T_2$ containing $T_1$ where $j \in N_{J_0}(A)$ and by our maximal choice of $T_1$ we have $T_1^j = T_2^j = T_2$. It follows that $T_1$ is a maximal $A$-invariant $t$-subgroup of $J$ as asserted. We can now assume without loss that $J = L_2T_2^0VA$.

Suppose that $T_1$ is contained in an $A$-invariant $t$-subgroup $U$ of $G$. We shall show that $U$ must equal $T_1$ and this will then show that $T_1$ is a maximal $A$-invariant $t$-subgroup of $G$ and
complete the proof of (i). Assume, by way of contradiction, that
\( T_1 \) is properly contained in \( U \); we may choose \( U \) such that \( T_1 \triangleleft U \). Then \( C_U(x) \neq T_1 \) for some \( x \in X^2 \). Since \( T_0 \triangleleft T_1 \) and \( T_0 \) is a
maximal \( \Delta \)-invariant \( t \)-subgroup of \( C \), we conclude that \( x \neq x_1 \).
Since \( X \) centralizes \( J = J/0(J) \), we see that \( I \subseteq O(J) \). Set
\( I_0 = T_1 \cap O(J) \) and set \( I_1 = [I_0, x] \). Since \( I \subseteq I_1 \), we have
\( I_1 \neq 1 \). By the Frattini argument \( N_U(I_0) \) covers \( J \) and so
\( N_U(I_0) \) covers \( M_0/0(M) \). We also have \( [T_1, x] = [T_1, x] = [I_0, x] = I_1 \) and so \( V \) normalizes \( I_1 \). Since \( I_1 = [T_1, C_U(x), x] \), we have
\( I_1 \triangleleft T_1, C_U(x) \). Finally, we have \( I_1 \triangleleft N_U(I_0) \) and since \( T_1 \subseteq T_1, C_U(x) \), we have contradicted our original choice of \( J \). This
contradiction completes the proof of (i).

(J) There is a maximal \( \Delta \)-invariant \( t \)-subgroup \( U \) of \( G \) con­taining \( I \) and permutable with \( V \) such that \( N_G(Z(J(U))) \) contains
\( UVA \) and covers \( M_0/0(N) \) and such that \( UVA \) is contained in a \( t \)-
local subgroup \( J \) of \( G \) which covers \( M_0/0(M) \).

Proof. By (h) we see that \( X \) does not centralize any Sylow
\( t \)-subgroup of \( O(D) \) and so by Lemma 4.10 we can find a \( t \)-local
subgroup \( H_0 \) of \( G \) containing \( A \) and a maximal \( \Delta \)-invariant \( t \)-
subgroup \( U_0 \) of \( G \). Without loss we can assume that \( H_0 = F_0 U_0 A \)
where \( F_0 \triangleleft H_0 \) and \( F_0 \triangleleft x_0(H_0) \) and \( \overline{F} = \overline{F_0} \cap N_0 = N_0/0(N) \) in
\( \overline{H_0} = H_0/0(H_0) \). By Propositions 4.5 and 4.8 a conjugate \( U \) of
\( U_0 \) by an element in \( N_G(A) \) satisfies the conclusions of (i). With­
out loss of generality we can assume that \( U = U_0 \). If \( I = I_0 1_1 1_2 1_3 \)
is the \( \gamma \)-decomposition of \( I \), we have by (h) that \( I_1 \subseteq O(C_G(y_1)) \)
and hence, $I'_1 \subseteq O(H_0)$. Also by (h) we conclude that $H_0$ is $(X, t)$-constrained. Arguing as in Lemma 6.3, we conclude that

$N_{H_0}(Z(J(U)))$ covers $\overline{H_0}$ and hence, covers $M_0/O(M)$. To complete the proof it remains to show that $V$ normalizes $Z(J(U))$. Since $I \subseteq U$ and $C_{V}(I) = 1$, $O_{p}(UV) = 1$. By Glauberman's $ZJ$-theorem we have $Z(J(U))$ is normal in $UV$ and this completes the proof.

(k) We have $t = p_2$.

Proof. Let $U$ and $J$ be as in the conclusion of (j). We assume, by way of contradiction, that $t \neq p_2$. As we have seen before, $J$ has a normal subgroup $L_2$ containing $YO(J)$ such that

$\overline{L_2} = \overline{L_2} \cap M_0 \cong M_0/O(M)$ in $\overline{J} = J/0(J)$. Without loss of generality we can assume that $J = L_2UVA$. Since $I \subseteq O(C)$, we conclude that $I \subseteq O(J)$. In the proof of (h) we have seen that $Y$ does not centralize $I$ and it follows that $J$ is $(Y, t)$-constrained. Since we are assuming that $t \neq p_2$, we can now argue as in Lemma 6.3 to conclude that $N_{J}(Z(J(U)))$ covers $J$ and hence, covers $M_0/O(M)$. Since $V$ normalizes $Z(J(U))$ and $R \subseteq V$, we have contradicted (a). This proves (k).

Again, let $U$ and $J$ be as in the conclusion of (j) and set $J^* = N_{M}(Z(J(U)))A$ so that $J^*$ covers $M_0/O(M)$. Set $Z = 0(J) \cap O(J^*)$ and so by Lemma 5.2 $R$ is a Sylow $p$-subgroup of $Z$. Let $U_0$ be an $A$-invariant Sylow $t$-subgroup of $N_{UZ}(R)$ so that $UZ = U_0Z$. Since $U$ covers a maximal $Y$-invariant $t$-subgroup of $M_0/O(M)$ and $\overline{U_0} = \overline{U}$ in $\overline{J}$, we conclude that $U_0$ covers a maximal $Y$-invariant $t$-subgroup of $M_0/O(M)$. Set $\overline{M_0} = M_0/O(M)$ and recall that
E = N_{M_0}(R). As in the proof of (a) we see that \( E \) contains a subgroup \( S \cong S_4 \). Since \( t = p_2 \) and \( U_0 \cap M_0 \) is a \( Y \)-invariant Sylow \( t \)-subgroup of \( M_0 \), we have by Lemma 2.7 that \( E = M_0 \), contrary to our original assumption. This contradiction proves our proposition.

Lemma 6.5. We have \( G^* = N_G(W) \) contains \( C_G(X) \).

Proof. We have \( C_G(X) = X \times M \) and \( M = N_M(A)M_0 \). By Lemma 6.1 we see that \( N_M(A) \) and \( O(M) \) are contained in \( G^* \). Let \( R \) be an \( A \)-invariant Sylow \( p \)-subgroup of \( W \). By the preceding proposition we have \( M_0 = N_{M_0}(R)O(M) \). It follows that \( R \) is a \( G^* \) subgroup of \( W \) for all \( m \in M_0 \) and this implies that \( M_0 \subseteq G^* \). This proves the lemma.

Lemma 6.6. We have \( O(C_G(y)) \subseteq O(G^*)O(C_A(A)) \) for all \( y \in \mathcal{Y}^\# \).

Proof. Since \( N_g(A) \subseteq G^* \), it will be sufficient to show that \( O(D) \subseteq O(G^*)O(C_A(A)) \). Set \( G_0 = O^2(G^*)A \) and \( \overline{G}_0 = G_0/O(G_0) \). Then by Lemmas 4.1, 4.2, and 4.3 \( G_0 \) has normal subgroups \( L_1 \) and \( L_2 \) such that \( X_0(G_0) \subseteq L_1, Y_0(G_0) \subseteq L_1, \overline{L}_1 = \overline{L}_2 \cap N_0 \) and \( \overline{L}_1 \) is a 2-group or \( \overline{L}_1 \cong A_7, L_2(q) \), or \( SL(2, q) \), \( q \) odd and \( \overline{L}_2 = \overline{L}_2 \cap M_0 \cong M_0/O(M) \). Then \( O(D) \) centralizes \( \overline{L}_1 \) and \( O(D) \cap \overline{L}_2 \subseteq Z(C_{\overline{L}_2}(\overline{Y})) \) so that \( O(D) \) also centralizes \( \overline{Y} \). Since \( X \subseteq \overline{L}_1 \), we conclude that \( C_{O(D)}(A) \) covers \( O(D) \) and the lemma follows from this.

Lemma 6.7. If \( g \in C_G(Y) \), then \( W^g \subseteq O(G^*)O(C_A(A)) \). Also we have \( O(G^*) = W_{G_0}(G^*)(Y) \).

Proof. Since \( W = \langle W \cap O(C_G(y)) \mid y \in \mathcal{Y}^\# \rangle \), we see that \( W^g \subseteq \langle O(C_G(y)) \mid y \in \mathcal{Y}^\# \rangle \) and so \( W^g \subseteq O(G^*)O(C_A(A)) \) by the preceding
Let $E$ be an $A$-invariant Sylow $p$-subgroup of $W$. Since $M \leq G^*$, we have $C_E(x) \leq O(C_G(x))$ for all $x \in X^\#$. Let $F = C_E(\langle x, y \rangle)$ for some $x \in X^\#$, $y \in Y^\#$. Also let $F^*$ denote the set of elements in $F$ which are inverted by the involutions in $Y - \langle y \rangle$. By lemma 5.1 we have $F^* \leq O(C_G(y))$ and hence, $F^* \leq W$. We then have $F \leq WC_0(G^*)(Y)$ and it follows that $E \leq WC_0(G^*)(Y)$. The lemma follows immediately from this.

**Lemma 6.8.** If $y \in Y^\#$, then $C_G(y)W$ is a group and $O(C_G(y)W) = O(C_G(y))W$.

**Proof.** For definiteness let $y = y_\perp$. Using Lemma 3.7 we see that $D = N_D(A)(D_2 \cap N_0)O(D)(D_1 \cap N_0)$ and so if $d \in D$, then $W^d \leq O(G^*)O(C_G(A))$. Thus we have $[W, D] \leq O(G^*)O(C_G(A)) = WC_0(G^*)(Y)O(C_G(A))$ by the preceding lemma. It follows that $[W, D]$ is of odd order and is contained in $WD$. Since $W[W, D]D = WD$, we conclude that $WD$ is a group. We then see that $W \lhd WD$ and so $W \leq O(WD)$. Since $O(D)$ is also contained in $O(WD)$, we have $D \cap O(WD) = O(D)$ and it follows that $O(WD) = WO(D)$. This completes the proof.

**Lemma 6.9.** If $g \in C_G(Y)$, then $W^g \leq O(G^*)$.

**Proof.** Let $G_0$, $\overline{G_0}$, $L_1$ and $L_2$ be as in the proof of Lemma 6.6 and let $\bar{O}$ denote the intersection of the groups $WO(C_G(y_i))$, $i = 1, 2, 3$. Then $\bar{O}$ is of odd order in $G_0$ and $\overline{L_1 \cap N_0} = \overline{L_1}$. 
Now \( C_{L_2}(y_i) \) contains a subgroup \( J_1 \) such that \( \overline{J}_1 \cong SL^\pm(2,3) \) if \( \overline{L}_2 \cong M_{11} \), \( \overline{J}_1 \cong SL^\pm(2,q_2) \) if \( \overline{L}_2 \cong L_3(q_2) \), or \( \overline{J}_1 \cong SU^\pm(2,q_2) \) if \( \overline{L}_2 \cong U_3(q_2) \), \( i = 1,2,3 \). We then have \([\overline{J}_1, \overline{0}] \) is a normal subgroup of odd order in \( \overline{J}_1 \), because \( \overline{J}_1 \) char \( C_{L_2}(y_i) \) and it follows that \( \overline{0} \) centralizes \( \overline{J}_1 \), \( i = 1,2,3 \). Since \( \overline{L}_2 = \langle \overline{J}_1 \mid i=1,2,3 \rangle \) by lemma 2.4, we conclude that \( \overline{0} \) centralizes \( \overline{L}_2 \). It follows that \( 0 \in \mathcal{O}(G^*) \) and since \( W^G \subseteq 0 \) by the preceding lemma, this lemma is proved.

Lemma 6.10. If \( N^* = NW \), then \( N^* \) is a group and \( \mathcal{O}(N^*) = \mathcal{W}(N) \).

Proof. By the previous lemma we have \([W,N] \subseteq \mathcal{O}(G^*) \) and by Lemma 6.7 \( \mathcal{O}(G^*) = \mathcal{W}(\mathcal{O}(G^*) \cap N) \). This lemma then follows by a proof similar to that of Lemma 6.8.

Lemma 6.11. If \( Z = \mathcal{O}(N^*) \cap \mathcal{O}(G^*) \), then \( Z \) contains \( W \) and \( Z \) is normal in \( N^* \).

Proof. We first show that \( \mathcal{O}(N^*) = WC_0(N)(X) \). Let \( E \) be an \( A \)-invariant Sylow \( p \)-subgroup of \( \mathcal{O}(N) \). If \( E = E_0 E_1 E_2 E_3 \) is the \( X \)-decomposition of \( E \), then \( E'_i \) has odd order and so \( E'_i \in \mathcal{O}(G_i(x_i)) \), \( i = 1,2,3 \). We then see that \( E'_i \) centralizes \( D_0 \) in \( D = D/0(D) \) and so \( E'_i \subseteq 0(D) \), \( i = 1,2,3 \). It follows that \( E'_i \subseteq \mathcal{W} \), \( i = 1,2,3 \). Thus \( \mathcal{O}(N) \subseteq WC_0(N)(X) \) and so \( \mathcal{O}(N^*) = WC_0(N)(X) \).

Let \( W^* \) be the normal closure of \( W \) in \( N^* \) and set \( N^*/W^* \). Also set \( J_0 = N_0 \mathcal{O}(N^*) \). We then have \( \mathcal{O}(J_0) = \mathcal{O}(N^*) \) and \( \mathcal{O}(J_0)/\mathcal{O}(J_0) \cong N_0/\mathcal{O}(N) \). But \( \mathcal{O}(N^*) = WC_0(N)(X) = C_0(N)(X) \) and so
c_0 \langle O(N^*) \rangle \) covers \( J_0 / O(J_0) \). It follows that \( c_0 \langle \bar{Z} \rangle \) also covers \( J_0 / O(J_0) \) and since \( Z \triangleleft O(N^*) \), we conclude that \( \bar{Z} \triangleleft \bar{J}_0 \). Since \( W^* \leq Z \), we have \( Z \triangleleft J_0 \). Since \( N^* = N_{N^*}(A)N_0O(N^*) \), we have \( Z \triangleleft N^* \) and the lemma is proved.

Lemma 6.12. We have \( Z = W \) and so both \( M \) and \( N \) normalize \( W \).

Proof. Let \( G_0 \) and \( L_2 \) be as in the proof of Lemma 6.6. Set \( \bar{L}_2 = L_2/W \). Then \( O(\bar{L}_2) = O(G^*) = \overline{W_0(G^*)}(Y) = \overline{C_0(G^*)(Y)} \) and \( L_2/O(L_2) \cong M_0/O(M) \). Then \( C_{\bar{L}_2}(O(G^*)) \) covers \( L_2/O(L_2) \) and \( C_0(\bar{Z}) \) also covers \( L_2/O(L_2) \). Since \( Z \triangleleft O(G^*) \) and \( W \leq Z \), we conclude that \( Z \triangleleft L_2 \). Now \( N_G(Z) \) contains \( L_2 \) which covers \( M_0/O(M) \), contains \( N^* \) which covers \( N_0/O(N) \), and contains \( A \) and so by Lemma 5.2 we conclude that \( Z \leq W \). Thus \( Z = W \) and the lemma follows from this.

Proposition 6.13. We have \( W = 1 \) and so for all \( x \in X^# \), \( y \in Y^# \) we have \( O(C_G(x)) \cap O(C_G(y)) = 1 \).

Proof. Suppose \( W \neq 1 \). Since \( D = N_D(A)(D_2 \cap M_0)N_0O(D) \), we have \( D \leq G^* \) and so \( C_G(y) \leq G^* \) for all \( y \in Y^# \). We also see that \( C_0(C)(y) \leq G^* \) for all \( y \in Y^# \) and thus, \( O(C) \leq G^* \). Since \( S \leq D \) and since \( C = SMO(C) \), we have \( C \leq G^* \) and it follows that \( C_G(x) \leq G^* \) for all \( x \in X^# \). Acting on \( O(B) \) with \( Y \), we conclude that \( O(B) \leq G^* \) and it follows that \( B \leq G^* \). We now see that \( C_G(a) \leq G^* \) for all \( a \in A^# \) and since every involution in \( S \) is conjugate in \( G^* \) to an involution in \( A \), we conclude that \( G^* \supset C_G(z) \) for every involution \( z \in G^* \). Thus \( G^* \) is a strongly imbedded subgroup of \( G \).
and by a well known argument it follows that $G$ has only one conjugacy class of involutions, a contradiction. Therefore $W = 1$ and the proposition is proved.
7. The proof of the main theorem.

In this section we show that our minimal counter-example $G$ satisfies the conclusion of our main theorem. This contradiction then proves that theorem. We retain the notation of the preceding sections.

**Lemma 7.1.** We have $B = C \cap D$.

**Proof.** By Lemma 3.9 it is sufficient to show that $x_1 = z_1$ centralizes $O(B)$. Suppose that this is not the case. Since $O(B) = \langle C_0(B) \rangle$, for some $x \in X^*$, $y \in Y^*$ there is an element $g$ in $C_0(B)$ such that $g \neq 1$ and $g$ is inverted by $x_1$ and hence, by $y_1$. Since $g$ is of odd order, $g \in O(G(x))$ and so by Lemma 5.1 $g \in O(G(y))$ because $g$ is inverted by $y_1$. This contradicts Proposition 6.13. Thus $B \subseteq C$ and it follows that $B = C \cap D$.

**Lemma 7.2.** The order of $G$ equals the order of $CD$ and so $G = CD$.

**Proof.** For $z = x_1, y_1$, and $x_1y_1$ let $J(z)$ be the set of all ordered pairs $(u, v)$ such that $u \sim x_1$ and $v \sim y_1$ in $G$ and $z \in \langle uv \rangle$. By a result of Thompson (proven in [7]) we have

$$[G:C][G:D] = [G:C]n(x_1) + [G:D]n(y_1) + [G:B]n(x_1y_1)$$

where $n(z)$ denotes the order of $J(z)$, $z = x_1, y_1, x_1y_1$.

We claim that $n(x_1) = n(y_1) = 0$. Suppose first that $u \sim x_1, v \sim y_1$ in $G$ and that $x_1 \in \langle uv \rangle$. Then both $u$ and $v$ are
contained in \( C \). If \( \overline{C} = C/o(C) \), then \( \overline{C} = \overline{S_1} \times \overline{C_1} \) where \( \overline{C_1} \) is as in Lemma 3.6. We then see that \( \overline{u} \in \overline{S_1} \) and \( \overline{v} \in \overline{C_1} \). Since \( (\overline{u} \overline{v})^k = \overline{x_1} \) for some integer \( k \), we must have \( k \) odd and it follows that \( \overline{u} \overline{v} = \overline{x_1} \). It follows that \( \overline{v} \in \overline{S_1} \) and this is a contradiction. Thus \( n(x_1) = 0 \). Next, suppose that \( u \sim x_1 \) and \( v \sim y_1 \) in \( G \) and that \( y_1 \in \langle uv \rangle \). If \( \overline{D} = D/o(D) \), then \( \overline{u} \in \overline{D_1} \) and \( \overline{v} \in \overline{D_2} \) and it follows that \( \overline{u} \in \overline{D_2} \), a contradiction. Thus \( n(y_1) = 0 \).

Now suppose that \( u \sim x_1 \) and \( v \sim y_1 \) in \( G \) and that \( x_1 y_1 \in \langle uv \rangle \). We claim that \( u = x_1 \) and \( v = y_1 \). If \( \overline{B} = B/o(B) \), then arguing as above we have \( \overline{u} \in \overline{S_1} \) and \( \overline{v} \in \overline{B_1} \) and it follows that \( \overline{u} \overline{v} = \overline{x_1 y_1} \). We then see that \( \overline{u} = \overline{x_1} \) and \( \overline{v} = \overline{y_1} \). Since \( B = C \cap D \), we conclude that \( u = x_1 \) and \( v = y_1 \). It follows that \( n(x_1 y_1) = 1 \) and that \( |G| = |C| |D|/|B| = |CD| \).

We are now in a position to complete the proof of our main theorem. Let \( F \) be the normal closure in \( G \) of \( x_1 \) and let \( L \) be the normal closure in \( G \) of \( y_1 \). By the preceding lemma \( F \subseteq D \) and \( L \subseteq C \). It follows that \( F \subseteq D_1 \) and that \( F/o(F) \cong D_1/o(D) \). We also have \( L \subseteq C_0 \) and \( L/o(L) \cong C_0/o(C) \). Since \( o(G) = 1 \), we have \( o(F) = o(L) = 1 \) and since \( F \cap L \) has odd order, we see that \( FL = F \times L \). Since the index of \( FL \) in \( G \) is odd, we conclude that \( G \) satisfies the conclusions of our theorem and this is contrary to our choice of \( G \). This then proves our main theorem.
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