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DERIVATION OF A CORRECTION TO THE NUCLEAR OPTICAL MODEL

POTENTIAL DEPTH FOR A SPIN-DEPENDENT NUCLEON-NUCLEON INTERACTION

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Murrell Elroy Lewis, B.S.

The Ohio State University
1971

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I express my sincere gratitude to Professor Bernard Mulligan whose ineffable qualities, both as research adviser and as friend, have made the completion of this work a reality.

I dedicate this achievement, as well as all others to date, with all my love to my mother and to the memory of my father.
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Studies in Nuclear Theory. Professors Bernard Mulligan and Bernard Margolis
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Introduction

The optical (pseudopotential) model arose in nuclear physics in an attempt to circumvent the perennially intractable problems associated with many-body interactions. The atomic nucleus, whether in situ or interacting with its environment, consists, in general, of a many-nucleon system whose internal structure and external influence is mediated via a predominately two-body nuclear force. The elucidation of the nature of this fundamental nucleon-nucleon (N-N) interaction has dominated both theoretical and experimental nuclear physics.

The key tool in extracting empirical knowledge of the nuclear force has been the analysis of the data obtained from processes involved in free N-N, as well as nucleon-nucleus (N-\pi) scattering experiments. In the latter case, the scattered nucleon interacts with a many nucleon system in a complex manner which belies any attempt at precise theoretical formulation. It was at this point that the optical model (among others) was introduced in an attempt to justify experimental results with a theory applicable over a broad spectrum of scattering energies.

Feshbach, Porter and Weisskopf [1] introduced a one-body complex potential \( V_0 + iV_1 \) to "masquerade" (hence the origin of the often used term "pseudopotential") as the actual nuclear potential which interacts with the incident nucleon. This conceptually simple innovation describes N-\pi scattering in analogy with the scattering of light through an optical medium, refraction and absorption processes.
having their counterparts in the scattering and absorption of the incident particles by the real and imaginary parts, respectively, of the complex (optical) potential. By assuming structures and magnitudes for the $V_0$, $V_1$, one attempts to account for the scattering data, obtained experimentally. Theoretical justification for this introduction of the optical model was given by Feshbach [2-4].

During the course of early attempts to fit the optical model parameters to the known data [1,5-7], it was found that the depths of both the real and the imaginary parts of the potential had to be functions of the energy [8,9]. In addition, it was found that a spin-orbit term should be added to the optical potential [10,11], and that the real and the imaginary parts of the potential do not necessarily have the same structure, especially near the nuclear surface, at low energies [12,13].

The variation with energy of the optical potential depths immediately posed the question as to the origin of this behavior, and its quantitative relation to the N-N potential. Tang, Lemmer, Wyatt, and Green [14] suggested that this characteristic behavior is associated with the nonlocal nature of the N-$\pi$ potential. This point of view found support in the later work of Perey and Buck [15]. Gersten [16] has obtained explicit expressions for a local potential approximately equivalent to that of Perey and Buck in the case of elastic neutron scattering. He examines the dependence on neutron energy, mass of the nucleus, and the parameter of non-locality.

As work continued, increasing difficulties arose in the attempts
to fit the depth and the other parameters to the then existent data. The amplitude of the oscillations in the differential scattering cross sections at large scattering angles appeared to be a major factor in this problem. Feshbach proposed [2,17] that this particular difficulty might be corrected by considering the nonlocal character of the N- potential. In addition to this development, the question arose as to whether the discrepancies in the fits of the N- scattering data could be corrected by the introduction of additional parameters into the optical potentials, and how these parameters could be related to the actual nonlocal potential.

Eventually, interest began to focus on the techniques and their justification, used in the transition from the nonlocal potential to the local, energy dependent, optical potential. The problem centered on two fundamental questions: (a) Why should a local potential fit the data arising from the nonlocal N- potential. (b) How could one develop a consistent technique with which to obtain the "best" local optical potential from the two-body N-N interaction. Work had been done on the extraction of optical model parameters from N-N data [18-20], but these methods were not amenable to the isolation of the successive correction terms involved in the calculations.

Clearly, what was needed was a detailed theoretical examination of the optical model. This program was carried out by Watson, et al. [21-25], culminating in the work of Kerman, McManus and Thaler [26] for N- scattering at high, non-relativistic energies. The latter collaborators calculated the depth of the local N- potential at energies above ~ 100 MeV in terms of the N-N scattering data. As an
expression for the non-local $N-\gamma$ potential they obtained

$$\mathcal{V} = (A-1)\langle \phi_0, t\phi_0 \rangle$$

(1)

where $\langle \phi_0, t\phi_0 \rangle$ represents the matrix element for the free two-body scattering matrix $t$, taken between the nuclear ground states, and $A$ is the nucleon number for a given nucleus.

Kidwai and Rook [27] have applied a formula, first suggested by Greenlees et al. [28], in a modified form to calculate the magnitude and energy variation of the real part of the $N-\gamma$ optical potential at somewhat lower energies. The Greenlees approach had involved the nucleon density weighted integral of all the individual internucleon potentials in the nucleus. Since this potential would be energy-independent, and the presence of a hard core in the interaction would cause the integral to diverge; the $N-N$ potential was replaced by the $t$-matrix in the spirit of Kerman et al. Whereas the latter had used this method at high energies, Kidwai and Rook point out that it is valid for scattering energies less than $\sim 50$ MeV. Recently, Feshbach and Hufner [29] have also treated this whole question in a systematic manner.

Two basic assumptions are utilized in the derivation of (1): (a) Strong correlations among the target nucleons permit the inelastic scattering of the incident particle, accompanied by an excitation of the target nucleus, and subsequent inelastic scattering which returns the nucleus to the ground state. The neglect of these effects at sufficiently high energies, where they are presumably small, constitutes
the multiple scattering approximation \([21,22]\). (b) When the incoming particle is scattered by a target nucleon, that scattering process is modified by the presence of the other nucleons. These modifications are also due to correlations among the nucleons, and are also presumably small at high energies. Their neglect constitutes the impulse approximation \([30-32]\), and is expressed by the presence of the free, two-body scattering (transition) matrix \(t\) in Eq. (1).

Simplifying assumptions are then utilized to convert the non-local potential (1) into an energy dependent local potential,

\[
\mathcal{U}_{\text{local}} = -4\pi \rho_o F(r) \frac{k^2}{m} \langle f(o) \rangle_{av}
\]

where \(\rho_o\) is the density at the center of the nucleus, \(F(r)\) is the potential shape factor, \(m\) is the free nucleon mass, and \(\langle f(o) \rangle_{av}\) is the center-of-mass N-N scattering amplitude in the forward direction, suitably averaged over the spins and the isospins of the nucleons in the nucleus.

At this point, we come to the subject of major importance to the present work; for we are concerned with the cavalier attitude with which the transition is made from (1) to (2) under the aegis of "simplifying assumptions." Mulligan \([33]\) calculated an energy-dependent local potential which took into account the first moment of the non-locality. He used a simple, central two-body force, calculated the \(t\)-matrix in the impulse approximation, neglecting pair correlations, and obtained considerable improvement in the theoretical fit to the
phenomenological optical potential \([8,34]\), as compared with that by Kerman et al. [26] who neglected this nonlocality.

Mulligan's calculation involved a first-order derivative of the \(t\)-matrix as its primary factor. Reading [35] obtained a similar derivative in his evaluation of an effective local optical potential to describe \(N-\pi\) scattering in terms of soft-core two-body potentials. Reading and Mackellar [36] investigated the transition from the nonlocal potential to the energy-dependent local potential in the light of Mulligan's work. Their method differed from his in that they used a two-body force which provided a better fit to the forward scattering amplitude in the energy region under consideration. Also, they suggest a method for taking into account second-order corrections to the impulse approximation. This approach avoids the approximations characteristic of the perturbation expansion utilized by Mulligan, but results in the necessity of the numerical evaluation of integrals which involve products of the two-body \(N-N\) potential with those of Bessel functions and the radial wave functions. Their nonlocal correction had an opposite sign to that found by Mulligan, thus invalidating this source, in their case, for the discrepancy between Watson's theory and experiment.

In later work, Reading [37] has questioned the use of an approximation method for the calculation of a local potential to describe scattering from a set of \(N\) nonoverlapping potentials. He indicates that the approximation procedure destroys the independence of the scattering on the on-energy-shell (OES) behavior of \(t\), and is inappropriate to the problem at hand. He suggests that the procedure
be obviated by the construction of a local two-body t-matrix which leads directly to a local potential.

Returning to the discussion of Mulligan's correction term, we note that the derivative involved depends upon the calculation of the off-energy-shell (FES) t-matrix. This latter problem has been the subject of much concerted effort because of its importance in determining the exact nature of the N-N interaction. Since our knowledge of the N-N force is achieved empirically through scattering phenomena which can be described by the t-matrix, a complete determination of this quantity would unambiguously determine the N-N potential. The difficulty lies in the fact that this complete determination of the t-matrix would involve its calculation both OES and FES. It is the FES calculation of the t-matrix which is involved in Mulligan's correction, and it is the determination of this quantity with which we are concerned in this work.

The N-N t-matrix OES is fairly well known within the elastic scattering region \( E_{lab} \lesssim 400 \text{ MeV} \) [38-40]. This information has been theoretically justified for various potential models: separable potentials [41,42], local potentials with core regions described by the boundary-condition model [43-45], and local potentials with soft-cores [46,47].

The N-N t-matrix FES is involved in the study of nuclear matter, p-p bremsstrahlung, three-body problem, etc. The development of the Faddeev equations [48] has given impetus to the three-body nuclear problem. The solutions of these equations express the properties of three-body states in terms of FES two-body t-matrix elements. Because
of the activity in this field, there have been developed several approaches to the computation of the FES t-matrix for singular core-interaction.

Van Leeuwen and Reiner [49] perform an explicit calculation of the t-matrix with a general complex argument for potentials consisting of a chain of rectangular wells. Detailed results are shown for the hard sphere interaction and the Herzfeld potential. They compare their approach to the resolvent method, and indicate the care which is necessary when a hard core is present in the interaction. It is pointed out that no direct physical meaning can be given to a generalized t-matrix with a complex argument which does not equal the scattering energy. This leads to the necessity of introducing quantities other than the OES phase shifts. The salient point appears to be the analytic behavior of the matrix elements $\langle k\ell|E\ell \rangle$ in the $E$-plane. The contributions to the t-matrix due to the regions $r < a$ (hard core radius) and $r > r_o$ (cut-off radius) can be found for each interaction, but the remaining part necessitates additional information about the matrix elements $\langle k\ell|E\ell \rangle$.

Kowalsky and Feldman in a series of papers [50-52] studied the half-off-shell (HFES) t-matrix for a hard-shell (infinite delta-function) core interaction and an arbitrary outside potential. In order to study the influence of FES effects on the optical potential for N-$\pi$ scattering, they presented a method which allowed them to calculate the N-N t-matrix, via the reactance matrix, in terms of an interaction potential and a scattering amplitude. The singular integral equations for the partial-wave amplitudes of the reactance
matrix are reduced to a Fredholm form which contains the scattering amplitude parametrically. They show that the iteration solution of these Fredholm equations is generally unreliable; however, the zeroth order iteration approximates the exact solution quite well near the energy shell. The replacement of the kernels of these integral equations by separable functions is discussed and the validity of the approximation is illustrated by a simple example. The requirement that the solutions of the (exact) Fredholm equations be consistent with the original singular integral equations yields a solution for the scattering amplitude in terms of the resolvent kernels of the Fredholm equations.

The preceding method is extended to constitute a complete and unified Fredholm formalism for the various integral equations which occur in the momentum-space formulation of two-body (potential) scattering problems. This unified treatment is made possible by the essential simplification that results with the demonstration of the formal similarity of the scattering integral equations with or without the presence of a hard core interaction. As a result, one has an apparatus which is useful in performing two-body calculations which occur in the investigation of high-energy N-\(\pi\) scattering. In particular, the technique may be used to obtain solutions for the integral equations satisfied by the t-matrices which appear in multiple-scattering theories. No assumptions are made concerning the nature of the two-body potentials other than that their partial wave amplitudes are suitably well behaved (Coulomb potential is excluded), and that they satisfy Hermiticity and invariance properties associated with the N-N
interaction, with the possibility of a hard core included.

Recently, Baranger et al. [53] have emphasized the fundamental advantages accruing to the utilization of the transition matrix rather than working directly with the N-N potential. They point out that, whereas it is common procedure to postulate a N-N interaction involving parameters which are then determined by fitting the two-nucleon data, it is the t-matrix, not the N-N potential, which is closely related to experiment. In particular, the OES t-matrix elements are expressible directly in terms of the scattering phase shifts. Further, they show that, in the case of an uncoupled partial wave without bound states, the symmetric part of the half-shell t-matrix may be specified arbitrarily. With this additional information, the entire FES transition matrix is determined. Their method thus permits a comparison of the effects due to the different FES behavior of the t-matrix to those effects determined by given OES fits to the N-N data.

Brander [54] started from the conventional definition of the transition operator in formal scattering theory to derive a modified definition which can be used when the interaction contains a hard core. When expressed in the Chew formulation [30,31] of the impulse approximation, this definition is exactly equivalent to the one used by Kowalsky and Feldman, but in the Watson formulation [21,52] there is a difference between the two definitions. He shows that this difference is small, except at high energies, and then proceeds to derive a formal integral equation for the complete off-energy-shell (CFES) t-matrix for the case of a hard core and an arbitrary outside potential, utilizing the two-potential formalism of Gell-Mann and Goldberger
Laughlin and Scott [56] developed a method for calculating FES two-body reaction matrix elements for a local potential with a hard core. An analytic expression is presented for the matrix elements \( t(k',k,s) \) in the special case of a hard core potential without any subsidiary interaction.

Fuda [57] presented a method for treating hard-core potentials within the framework of the Lippmann-Schwinger scattering theory formalism [58]. Using this method, a procedure was developed for constructing separable expansions of the t-matrix, arising from potentials which contain hard cores.

Loman et al. [59,60] have investigated the HFES t-matrix for the boundary condition model. Kim and Tubis [61] consider two-body interactions with an outside local potential and a core region described by the boundary condition model. They derive an explicit integral equation for the CFES t-matrix.

Kujawski [62] has used the optical potential to study the presence of two-body correlations in high-energy (1 - GeV) \( N-N \) scattering. He makes use of Mulligan's procedure to obtain an effective equivalent local potential, and shows that the correction term to be considered here is important in his case.

In this work, we obtain an expression for Mulligan's correction factor through an examination of the FES scattering amplitude in a manner analogous to that in which the OES scattering amplitude in integral form is reduced to a partial wave analysis. During the course of the derivation, certain auxiliary "phase shifts" are introduced in
addition to those which appear in the standard OES case.

We now proceed to carry out this program. In Chapter 1, we
calculate the derivative of the FES scattering amplitude (which is
proportional to the t-matrix), $\frac{d}{dk} M(K,k,\cos\theta)$, evaluated OES, in a
form which is explicitly dependent on the aforementioned "phase
shifts," for the case when spins are neglected. Chapter 2 will be
devoted to finding expressions for the phase shifts in analytic form.
In Chapter 3, these results will be applied to the special case of a
square well potential and orbital angular momentum $\ell = 0$. Finally,
in Chapter 4, we will exhibit the results for a general (well-behaved)
spin-dependent N-N interaction in a form applicable to N-$\alpha$ scattering
from an even-even, $J,T = 0$ nucleus.
Chapter I. Phase Shift Formulation of $\frac{d}{dk} M(K,k,\cos \theta)_{|K=k}$

We are interested in the matrix $M(K,k,\cos \theta)$, given by

$$M(K,k,\cos \theta) = \frac{1}{4\pi} \int \exp(-ikr'\cos \omega) \sqrt{\gamma} \psi(K,k,\cos \theta',r') d\tilde{r}'$$

(3)

where the variables are related to the direction of motion $\hat{k}$ of the incident particle as shown below in Figure 1.

![Figure 1. Relationship Between Variables](image)

$K^2 = \frac{mE}{\hbar^2} - k^2$. $\psi(K,k,\cos \theta',r')$ is the solution of the integral equation

$$\psi(K,k,\cos \theta,r) = e^{i k r \cos \theta} - \frac{1}{4\pi} \int e^{i kr'} \sqrt{\gamma} \psi(K,k,\cos \theta',r') d\tilde{r}'$$

(4)

with $R = [r^2 + (r')^2 - 2rr' \cos \alpha]^{\frac{1}{2}}$.

Specifically, we wish to obtain

$$\bar{M}(k,\theta) \equiv \left\{ \frac{d}{dk} M(K,k,\cos \theta) \right\}_{K=k}$$

(5)

Utilizing Eq.(3), $\bar{M}(k,\theta)$ may be expressed in integral form.
\[ M(k, \theta) = \frac{i}{4\pi} \int r' \cos \alpha e^{-ikr' \cos \alpha} \nabla(r') \Psi(k, k, \cos \theta, r') \, dr' \]

\[ -\frac{1}{4\pi} \int e^{-ikr' \cos \alpha} \nabla(r') \left[ \frac{d}{dk} \Psi(K, k, \cos \theta, r') \right] \, dr' \]

We now define,

\[ \Psi(\vec{r}) \equiv \Psi(k, k, \cos \theta, r) = e^{ik \cdot \vec{r}} - \frac{1}{4\pi} \int \frac{e^{ik \cdot \vec{r}}}{R} \nabla \Psi(\vec{r}') \, d\vec{r}' \]

\[ \Psi'(\vec{r}) \equiv \frac{d \Psi(\vec{r})}{dk} = i r \cos \theta e^{i k \cdot \vec{r}} - \frac{1}{4\pi} \int e^{ik \cdot \vec{r}} \nabla \Psi(\vec{r}') \, d\vec{r}' - \frac{1}{4\pi} \int \frac{e^{ik \cdot \vec{r}}}{R} \nabla \Psi'(\vec{r}') \, d\vec{r}' \]

\[ \Psi'_1(\vec{r}) \equiv \frac{d \Psi(K, k, \cos \theta, r)}{dk} \bigg|_{K=k} = i r \cos \theta e^{i k \cdot \vec{r}} \]

\[ + \frac{i}{4\pi} \int e^{ik \cdot \vec{r}} \nabla \Psi(\vec{r}') \, d\vec{r}' - \frac{1}{4\pi} \int \frac{e^{ik \cdot \vec{r}}}{R} \nabla \Psi'_1(\vec{r}') \, d\vec{r}' \]
where we have used the fact that \( \frac{dK}{dk} \bigg|_{K=k} = -1 \).

Using these definitions, we may write \( \tilde{M}(k, \theta) \) in the form

\[
\tilde{M}(k, \theta) \equiv M_B + M_C
\]

(10)

where

\[
M_B \equiv \frac{i}{4\pi} \int r' \cos \alpha e^{-ikr' \cos \alpha} \nabla \psi (\mathbf{r}') d\mathbf{r}'
\]

(11)

and

\[
M_C \equiv -\frac{1}{4\pi} \int e^{-ikr' \cos \alpha} \nabla' \psi (\mathbf{r}') d\mathbf{r}'
\]

(12)

The problem of finding \( \tilde{M}(k, \theta) \) is now reduced to an evaluation of \( M_B \) and \( M_C \).

We commence by expressing \( M_C \) as a coefficient in the asymptotic expansion of Eq.(9). Expanding \( R \) for large \( r \), and retaining terms through \( \frac{1}{r} \), we have

\[
R = r \left[ 1 - \frac{r'}{r} \cos \alpha + \frac{1}{2} \left( \frac{r'}{r} \right)^2 \left( 1 - \cos^2 \alpha \right) + \ldots \right]
\]

(13)

Using Eq.(13),

\[
e^{ikR} = e^{ikr} e^{-ikr' \cos \alpha} \left[ 1 + \frac{i k (r')^2}{2r} \left( 1 - \cos^2 \alpha \right) + \ldots \right]
\]

(14)
Substituting Eqs. (13) and (14) into Eq. (9), we have

\[
\psi_1'(\hat{r}) \rightarrow i r \cos \theta e^{ik\cdot\hat{r}} - i M_A e^{ikr} + (M_c + M_d) \frac{e^{ikr}}{r}
\]

(15)

where

\[
M_D = -\frac{k}{8\pi} \int (r')^2 e^{-ikr'\cos \alpha} (1 - \cos^2 \alpha) V \psi(\hat{r}') d\hat{r}'
\]

(16)

and

\[
M_A = -\frac{1}{4\pi} \int e^{-ikr'\cos \alpha} V \psi(\hat{r}') d\hat{r}'
\]

(17)

is the OES scattering amplitude which may be derived using the asymptotic expansion of Eq. (7) and expressed in terms of the usual phase shifts \(\delta_l\) as

\[
M_A = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1) \left[ e^{2i\delta_l} - 1 \right] P_l(\cos \theta)
\]

(18)

We must now obtain the appropriate asymptotic form for \(\psi_1'(\hat{r})\), independently of its integral form in Eq. (9). We accomplish this by obtaining and solving the differential equation for \(\psi_1'\). By adding Eqs. (8) and (9), we get

\[
\tilde{\delta}(\hat{r}) = 2i r \cos \theta e^{ik\cdot\hat{r}} - \frac{1}{4\pi} \int \frac{e^{ikR}}{R} V \tilde{\delta}(\hat{r}') d\hat{r}'
\]

(19)
where $\zeta(\vec{r}) = \psi'(\vec{r}) + \psi'_1(\vec{r})$. Recognizing that $-\frac{1}{4\pi} \frac{e^{ikR}}{R}$ is the Green's function for the operator $(\nabla^2 + k^2)$, we obtain

$$
(\nabla^2 + k^2) \zeta(\vec{r}) = (\nabla^2 + k^2) 2i r \cos \theta e^{i \vec{k} \cdot \vec{r}} + V \zeta(\vec{r})
$$

$$
= -4k e^{i \vec{k} \cdot \vec{r}} + V \zeta(\vec{r})
$$

(20)

Since $\psi(\vec{r})$ satisfies the usual Schroedinger equation,

$$
\nabla^2 \psi(\vec{r}) + k^2 \psi(\vec{r}) = V \psi(\vec{r})
$$

(21)

we can differentiate with respect to $k$ directly and obtain,

$$
\nabla^2 \psi'(\vec{r}) + k^2 \psi'(\vec{r}) = V \psi'(\vec{r}) - 2k \psi(\vec{r})
$$

(22)

Subtracting Eq.(22) from Eq.(20), we finally obtain a differential equation for $\psi'_1(\vec{r})$,

$$
\nabla^2 \psi'_1(\vec{r}) + k^2 \psi'_1(\vec{r}) = -4k e^{i \vec{k} \cdot \vec{r}} + 2k \psi(\vec{r}) + V \psi'_1(\vec{r})
$$

(23)

As Eq.(23) is inhomogeneous, with the inhomogeneous term depending on $\psi(\vec{r})$, it is important to note that the $\psi(\vec{r})$ has been completely specified by Eq.(7). The solution so specified may be expanded as
where $\chi_\ell(r)$ is the solution of the differential equation,

$$
\frac{d^2 \chi_\ell(r)}{dr^2} - \frac{\ell(\ell+1)}{r^2} \chi_\ell(r) + k^2 \chi_\ell(r) = V \chi_\ell(r)
$$

subject to the boundary condition $\chi_\ell(0) = 0$ and Eq. (7). The phase shifts in Eq. (24) are identical with those of Eq. (18). Under these conditions, $\chi_\ell(r)$ is a real function. If we expand $\psi_\ell(r)$ in partial waves,

$$
\psi_\ell(r) = \sum_{\ell=0}^{\infty} (2\ell + 1) i^\ell \frac{\chi_\ell(r)}{r} P_\ell(\cos \theta)
$$

and utilize the expansion,

$$
e^{i\mathbf{k} \cdot \mathbf{r}} = \sum_{\ell=0}^{\infty} (2\ell + 1) i^\ell j_{\ell}(kr) P_\ell(\cos \theta)
$$

the partial wave representation of Eq. (23) assumes the form,

$$
\frac{d^2 \chi_\ell'(r)}{dr^2} - \frac{\ell(\ell+1)}{r^2} \chi_\ell'(r) + k^2 \chi_\ell'(r) = -4kr j_{\ell}(kr) + 2ke^{i\delta_\ell} \chi_\ell(r) + V \chi_\ell'(r)
$$

This equation is important in describing the effect of the FES scattering process.
Equation (28) may be solved [Appendix A] immediately in the region \( r > r_o \), where \( r_o \) is some point beyond which we may assume \( V = 0 \). In this region the normalization of \( \chi_{\ell}(r) \), consistent with Eq. (7), gives

\[
\chi_{\ell}(r) = \left[ \cos \delta_{\ell} j_{\ell}(kr) - \sin \delta_{\ell} n_{\ell}(kr) \right]
\]

and a particular solution of the inhomogeneous equation for \( \chi'_{\ell}(r) \) is

\[
\rho \chi'_{\ell}(r) = r^2 \left[ (2 - e^{i\delta_{\ell}} \cos \delta_{\ell}) j_{\ell}(kr) + (e^{i\delta_{\ell}} \sin \delta_{\ell}) n_{\ell}(kr) \right]
\]

Thus, for \( r > r_o \), the general solution to Eq. (28) is \( \rho \chi'_{\ell}(r) \) plus \( h \chi'_{\ell}(r) \), the general solution to the homogeneous equation

\[
\frac{d^2 h \chi'_{\ell}(r)}{dr^2} - \frac{\ell(\ell+1)}{r^2} h \chi'_{\ell}(r) + \rho^2 h \chi'_{\ell}(r) = 0
\]

Asymptotically, the general solution to Eq. (31) may be written as

\[
h \chi'_{\ell}(r) \xrightarrow{r \to \infty} C'_{\ell} \sin (kr - \ell \frac{\pi}{2} + \beta_{\ell})
\]

where \( C'_{\ell} \) and \( \beta_{\ell} \), in general, may be complex. Finally, the asymptotic form, to terms of \( O(\frac{1}{r}) \), of the function \( \psi'_{\ell}(r) \) is given by

\[
\psi'_{\ell}(r) \xrightarrow{r \to \infty} \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell P_{\ell}(\cos \theta) \left[ \frac{C'_{\ell}}{r} \sin (kr - \ell \frac{\pi}{2} + \beta_{\ell}) \right]
\]

(continued on next page)
\[+(2-e^{i\delta_2 \cos \delta}) \left[ \frac{1}{k} \cos (kr - \frac{\pi}{2} \ell) - \frac{\ell(\ell-1)}{2k^2} \sin (kr - \frac{\pi}{2} \ell) \right] \]

\[+(e^{i\delta_2 \sin \delta}) \left[ \frac{1}{k} \sin (kr - \frac{\pi}{2} \ell) + \frac{\ell(\ell-1)}{2k^2} \cos (kr - \frac{\pi}{2} \ell) \right] \]

\[(33)\]

If Eq.(33) is expressed in exponential form and compared with Eq.(15), we find, in terms of the as yet undetermined phase shifts \( \beta_\ell \),

\[C'_\ell = -\frac{(\ell+1)}{k^2} e^{i\beta_\ell} \]

and

\[M_c + M_d = \frac{1}{4ik^2} \sum_{\ell=0}^{\infty} (2\ell + 1) \left\{ \ell(\ell-1) \left[ e^{2i\delta_2} - 1 \right] - 2(\ell+1) \left[ e^{2i\delta_2} - 1 \right] \right\} \mathcal{P}(\cos \theta) \]

\[(35)\]

We now proceed to find \( M_d \). Let us define a function \( \Psi(\kappa, k, \cos \theta, r) \) which satisfies the inhomogeneous integral equation

\[\Psi(\kappa, k, \cos \theta, r) = e^{i\kappa r \cos \theta} - \frac{1}{4\pi} \int \frac{e^{i\kappa R}}{R} \nabla \Psi(\hat{r}') d\hat{r}' \]

\[\Psi(\hat{r}') \]

\[(36)\]

where \( \Psi(\hat{r}') \) is defined in Eq.(7). We assume \( \kappa \) is independent of \( k \).

Differentiating Eq.(36) with respect to \( \kappa \) gives

\[\frac{d}{d\kappa} [\Psi(\kappa, k, \cos \theta, r)] = i\cos \theta e^{i\kappa r \cos \theta} - i \frac{1}{4\pi} \int e^{i\kappa R} \nabla \Psi(\hat{r}') d\hat{r}' \]

\[(37)\]
After this differentiation, we set \( \kappa = k \) and define

\[
\Psi' \equiv \frac{d}{\kappa} \Psi(\kappa, k, \cos \theta, r) \bigg|_{\kappa = k}
\]  

Equation (37) becomes

\[
\Psi' = i r \cos \theta e^{ikr \cos \theta} - \frac{i}{4\pi} \int e^{ikr} \nabla \Psi(\mathbf{r}') d\mathbf{r}'
\]

Substituting Eq. (14) into Eq. (39), we have

\[
\Psi' \rightarrow \infty i r \cos \theta e^{ikr \cos \theta} + i M e^{ikr} - M_0 e^{ikr}
\]

In order to obtain the asymptotic form of \( \Psi' \) in terms of phase shifts, we need the differential equation for \( \Psi' \). From Eq. (36) the differential equation for \( \Psi(\kappa, k, \cos \theta, r) \) assumes the form

\[
\nabla^2 \Psi(\kappa, k, \cos \theta, r) + \kappa^2 \Psi(\kappa, k, \cos \theta, r) = \nabla \Psi(\mathbf{r})
\]

Differentiating Eq. (41) with respect to \( \kappa \) and then setting \( \kappa = k \), gives

\[
\nabla^2 \Psi' + k^2 \Psi' = -2k \Psi(\mathbf{r})
\]

This differential equation can now be integrated using the same techniques as that involved in the solution of Eq. (23). We first expand the \( \Psi' \) as
Using Eq. (24), the partial wave representation of Eq. (42) assumes the form,

$$\Psi'(r) = \sum_{l=0}^{\infty} (2l+1) i^l \frac{\xi_l(r)}{r} P_l(\cos \theta)$$

(43)

Assuming \( \chi_l(r) \) to have the form given in Eq. (29), we can obtain a particular solution for Eq. (44), valid for \( r > r_0 \), having the form,

$$\frac{d^2 \xi_l(r)}{dr^2} - \frac{\ell(\ell+1)}{r^2} \xi_l(r) + k^2 \xi_l(r) = -2k e^{i\delta_2} \chi_l(r)$$

(44)

Adding Eq. (45) to the homogeneous solution, finite at \( r = 0 \), gives

$$\Psi' \xrightarrow{r \to \infty} \sum_{l=0}^{\infty} (2l+1) i^l P_l(\cos \theta) \left\{ \frac{D_l'}{r} \sin (kr - \frac{\pi}{2}l + \gamma_l) \\ + (e^{i\delta_0} \cos \xi_2) \left[ \frac{1}{k} \cos (kr - \frac{\pi}{2}l) - \frac{\ell(l+1)}{2k^2r} \sin (kr - \frac{\pi}{2}l) \right] \\ - (e^{i\delta_0} \sin \xi_2) \left[ \frac{1}{k} \sin (kr - \frac{\pi}{2}l) + \frac{\ell(l+1)}{2k^2r} \cos (kr - \frac{\pi}{2}l) \right] \right\}$$

(46)
\[ D'_\ell = -\frac{(l+1)}{k^2} e^{i\gamma_\ell} \]  

(47)

and

\[ M_D = \frac{1}{4ik^2} \sum_{\ell=0}^{\infty} (2l+1) \left\{ 2(l+1) \left[ e^{2i\gamma_\ell} - 1 \right] + \ell(\ell-1) \left[ e^{2i\gamma_\ell} - 1 \right] \right\} \mathcal{P}_\ell(\cos \theta) \]  

(48)

In order to find an expression involving \( M_B \), we expand \( \psi(r) \), as given by Eq. (7), asymptotically, to terms of \( O\left(\frac{1}{r^2}\right) \),

\[ \psi \xrightarrow{r \to \infty} e^{ikr} + M_A \frac{e^{ikr}}{r} + \left[ M_B + M_D \right] \frac{i e^{ikr}}{r^2} \]  

(49)

By comparing Eq. (49) with the asymptotic expansion of \( \psi \) as given by Eqs. (24) and (29), we obtain,

\[ M_B + M_D = \frac{1}{4ik^2} \sum_{\ell=0}^{\infty} (2l+1) \ell (\ell+1) \left[ e^{2i\Delta_\ell} - 1 \right] \mathcal{P}_\ell(\cos \theta) \]  

(50)

In accordance with Eq. (10), we can combine Eqs. (35), (48), and (50) to find,

\[ \mathbf{M}(k, \theta) = \frac{1}{2ik^2} \sum_{\ell=0}^{\infty} (2l+1) \left\{ \ell e^{2i\beta_\ell} - (l+1) \left[ e^{2i\beta_\ell} - 1 \right] - 2(l+1) \left[ e^{2i\gamma_\ell} - 1 \right] \right\} \mathcal{P}_\ell(\cos \theta) \]  

(51)
Chapter II. Determination of the Phase Shifts from a N-N Potential

In the preceding section, we derived an expression for $\tilde{M}$ which is dependent on the "phase shifts" $\beta$, $\gamma$, and $\delta$. We now proceed to find expressions for $\beta$, $\gamma$, and $\delta$ which will enable us to evaluate $\tilde{M}(k, \theta)$ for a given N-N potential.

Let us commence with an examination of Eq. (28). As may be seen, the solution to this equation will, in general, be complex. We can separate Eq. (28) into its real and imaginary parts by introducing

$$\chi^i = \omega^i + i \zeta^i.$$  

In place of the complex Eq. (28) we obtain the real equations,

$$\frac{d^2 \omega^i}{dr^2} - \frac{l(l+1)}{r^2} \omega^i + k^2 \phi^i = -4k j_l(kr) + 2k \cos \delta \chi^i(r) + \nabla \phi^i$$

(52)

$$\frac{d^2 \zeta^i}{dr^2} - \frac{l(l+1)}{r^2} \zeta^i + k^2 \zeta^i = 2k \sin \delta \chi^i(r) + \nabla \zeta^i$$

(53)

We must obtain solutions of Eqs. (52,53) for all $r$, i.e., both inside the potential where $r < r_o$, $V \neq 0$ and outside the potential where $r > r_o$, $V = 0$. Since the inhomogeneous parts of Eqs. (52,53) are dependent on the $\chi^i(r)$, we must first determine these functions for all $r$. The $\chi^i$ satisfy Eq. (25) which, since it is homogeneous, can be integrated to give $\chi^i(r)$ for all $r$ with arbitrary normalization and subject to the boundary condition $\chi^i(0) = 0$. 

24
Outside of $V$ (where $r > r_o$) the $\bar{\chi}_k(r)$ assume the analytic form,

$$\bar{\chi}_k(r) = kr [A_k j_k(kr) - B_k n_k(kr)]$$

which is correct only if $A_k$ and $B_k$ are related. We proceed to find the relation between $A_k$ and $B_k$. If we take any two points $r = a, b > r_o$, we have\(^1\)

1. $\bar{\chi}_k(a) = ka [A_k j_k(ka) - B_k n_k(ka)]$
2. $\bar{\chi}_k(b) = kb [A_k j_k(kb) - B_k n_k(kb)]$

These equations must hold for all $A_k$ and $B_k$, independent of the normalization of $\bar{\chi}_k$. The relationship between $A_k$ and $B_k$ may be obtained by multiplying Eq. (55) by $\bar{\chi}_k(b)$ and Eq. (56) by $\bar{\chi}_k(a)$, then subtracting the resultant equations, which gives

$$A_k = \frac{[a \bar{\chi}_k(b)n_k(ka) - b\bar{\chi}_k(a)n_k(kb)]}{[a \bar{\chi}_k(b)j_k(ka) - b\bar{\chi}_k(a)j_k(kb)]} B_k$$

---

\(^1\) We have used the expressions relating the function at two points rather than the logarithmic derivative at one point, because this lends itself better to numerical integration by the Fox-Goodwin method, for which $d\chi_k/\text{d}r$ is not normally calculated.
Utilizing this relation in Eq. (54), we can find, regardless of the normalization chosen for \( \tilde{\chi}_\ell(r) \), a \( B_\ell \) for which the resultant equation holds.

In order to determine \( \delta_\ell \), we want that \( \chi_\ell(r) \) consistent with Eq. (29). This may be achieved by setting \( B_\ell = \sin \delta_\ell \) and \( A_\ell = \cos \delta_\ell \). This leads to the result that

\[
\cot \delta_\ell = \frac{[a \tilde{X}_\ell(b) j_\ell(kb) - b \tilde{X}_\ell(a) j_\ell(kb)]}{[a \tilde{X}_\ell(b) j_\ell(ka) - b \tilde{X}_\ell(a) j_\ell(kb)]}
\]

(58)

from which we may determine \( A_\ell \) and \( B_\ell \).

Now let us look at Eq. (52). By numerical integration, subject to the boundary condition that \( \tilde{\phi}'_\ell(0) = 0 \), we obtain a particular solution, \( \tilde{\phi}'_\ell(r) \).

A general solution of Eq. (52) may be expressed as \( \tilde{\phi}'_\ell + C_\ell \chi_\ell \) where \( C_\ell \) is, in general, an arbitrary constant. In our case, we shall show that \( C_\ell \) is determined by the asymptotic form of \( \tilde{\phi}'_\ell \).

Outside of \( V \), \( \tilde{\phi}'_\ell + C_\ell \chi_\ell \) assumes the analytic form,

\[
\phi'_\ell(r) = \tilde{\phi}'_\ell + C_\ell \chi_\ell(r) \equiv r^{-2} \left[ j_{\ell-1}(kr)(2 - \cos^2 \delta_\ell) + \eta_{\ell-1}(kr) \sin \delta_\ell \cos \delta_\ell \right] + kr \left[ F'_2 j'_\ell(kr) - G'_2 \eta'_\ell(kr) \right] = f'_\ell(r) + kr \left[ F'_2 j'_\ell(kr) - G'_2 \eta'_\ell(kr) \right]
\]

(59)

where \( f'_\ell(r) \) is a particular solution, the real part of Eq. (30), of the inhomogeneous equation and \( kr[F'_2 j'_\ell(kr) - G'_2 \eta'_\ell(kr)] \) is the general solution.
of the homogeneous equation.

Since we have one arbitrary constant in the left member of Eq. (59) and two arbitrary constants in the right member, this identity can hold if and only if \( F'_l \) is a function of \( G'_l \) (or \( G'_l \) of \( F'_l \)). We relate \( F'_l \) and \( G'_l \) so that Eq. (59) holds for all \( r > r_o \) by means of the method used above to relate \( A'_l \) and \( B'_l \). We evaluate Eq. (59) at two points \( r = a, b > r_o \), eliminate \( C'_l \), and obtain,

\[
F'_l' = \frac{[\bar{\varphi}'_l(a) - \bar{f}'(a)]X'_l(b) - [\bar{\varphi}'_l(b) - \bar{f}'(b)]X'_l(a)}{[KA'_l(ka)X'_l(b) - KB'_l(kb)X'_l(a)]} + \frac{[\alpha \eta_k'k(a)X'_l(b) - \beta \eta_k'k(b)X'_l(a)]}{[\alpha K_0(ka)X'_l(b) - \beta K_0(kb)X'_l(a)]} G'_l
\equiv Q'^_l + R'_l G'_l
\]

(60)

Utilizing this result, we may express Eq. (59) in the form

\[
\phi'_l(r) = \overline{\varphi}'_l(r) + C'_l X'_l(r) = \bar{f}'(r) + r \bar{j}(kr) Q'^_l + kr [\bar{j}(kr) R'_l - \eta(kr)] G'_l
\]

(61)

We see that for any \( C'_l \) we may choose a \( G'_l \), or, conversely, for any \( G'_l \), a \( C'_l \), such that relation (61) holds.

We may proceed in a like manner with the solution to the differential Eq. (53), so that
\[(69) \quad \mathcal{N}[(\alpha)^{\gamma} u_3 - R(\alpha^2) u_3] + \mathcal{O}(\alpha^2) + (1) \mathcal{G} \equiv (1) \chi \frac{\beta}{\alpha} + (1) \frac{\gamma_3}{2} = (1) \gamma_5 \]

\begin{align*}
\mathcal{N}^5 R + \mathcal{O} & \equiv \\
\mathcal{N} \left[ \frac{(\alpha) \chi (\beta) \gamma_5 \gamma_4 - (\beta) \chi (\alpha^2) \gamma_5 \gamma_4}{(\alpha) \chi (\beta) \gamma_5 \gamma_4 - (\beta) \chi (\alpha^2) \gamma_5 \gamma_4} \right] + \\
\mathcal{N} \left[ \frac{(\alpha) \chi (\beta) \gamma_5 \gamma_4 - (\beta) \chi (\alpha^2) \gamma_5 \gamma_4}{(\alpha) \chi [(\beta) \gamma_5 \gamma_4 - (\beta) \gamma_5 \gamma_4]} \right] & = \gamma_5
\end{align*}

\text{We find}

\[(62) \quad [\chi, \mathcal{N} - (\alpha^2)] \chi + \mathcal{G} = [(\alpha^2) u_3, \mathcal{N} - (\alpha^2)] \chi + \mathcal{G}
\]

\[
[\chi, \mathcal{N} \mathcal{O}(\alpha^2 u_3) - \chi, \mathcal{O}(\alpha^2 u_3)] \chi \equiv (1) \chi \frac{\beta}{\alpha} + \frac{\gamma_3}{2} = (1) \gamma_5
\]
We wish to express Eqs. (61) and (64) in terms of $\beta_{\ell}$. This may be accomplished by letting $\beta_{\ell} = \beta^R_{\ell} + i\beta^I_{\ell}$ and writing Eq. (33) explicitly in terms of its real and imaginary components. We find

$$\Psi'_{\ell} \xrightarrow{r \to \infty} \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{r} \left[ f'(r \to \infty) - \frac{1}{2k^2} \left[ 1 + \exp(-2\beta^I_{\ell}) \cos 2\beta^R_{\ell} \right] \right]$$

$$\times (l+1) \sin(kr - \ell \frac{\pi}{2}) - \frac{1}{2k^2} \left[ \exp(-2\beta^I_{\ell}) \sin 2\beta^R_{\ell} \right](l+1)$$

$$\times \cos(kr - \ell \frac{\pi}{2})$$

$$+ i \left[ g'(r \to \infty) - \frac{1}{2k^2} \left[ \exp(-2\beta^I_{\ell}) \sin 2\beta^R_{\ell} \right](l+1) \right]$$

$$\times \sin(kr - \ell \frac{\pi}{2}) - \frac{1}{2k^2} \left[ 1 - \exp(-2\beta^I_{\ell}) \cos 2\beta^R_{\ell} \right](l+1) \cos(kr - \ell \frac{\pi}{2}) \right] P_{\ell}(k, r)$$

(65)

where $f'(r \to \infty)$ and $g'(r \to \infty)$ denote the asymptotic forms for $f'(r)$ and $g'(r)$, respectively. From this result, we immediately obtain expressions for $\phi'_{\ell}$ and $\zeta'_{\ell}$ in terms of $\beta^R_{\ell}$, $\beta^I_{\ell}$ which when compared with the asymptotic forms of Eqs. (61) and (64) lead to the relations

$$Q^\phi_{\ell} + R_{\ell} G'_{\ell} = -\frac{1}{2k^2} \left[ 1 + \exp(-2\beta^I_{\ell}) \cos 2\beta^R_{\ell} \right](l+1)$$

(66)

$$G'_{\ell} = -\frac{1}{2k^2} \left[ \exp(-2\beta^I_{\ell}) \sin 2\beta^R_{\ell} \right](l+1)$$

(67)
We may eliminate \( G'_l \) between Eqs. (66,67) and \( N'_l \) between Eqs. (68,69) to obtain the equations

\[
Q^s_l + R_l N'_l = -\frac{1}{2k^2} [\exp(-2\beta^r_l)\sin 2\beta^R_l] (l+1) \tag{68}
\]

\[
N'_l = -\frac{1}{2k^2} [1 - \exp(-2\beta^I_l)\cos 2\beta^R_l] (l+1) \tag{69}
\]

Equations (70,71) constitute two equations in the two unknowns \( \beta^R_l \) and \( \beta^I_l \). Hence we see that the phase shifts are uniquely determined and may be expressed as follows:

\[
\tan 2\beta^R_l = \frac{2k^2 [Q^s_l - R_l Q^\phi_l] - 2(l+1)R_l}{2k^2 [Q^\phi_l + R_l Q^s_l] - (l+1)(1-R_l^2)}, \tag{72}
\]
\[
\exp(-2\beta^I_l) = \frac{(l+1) + 2k^2 Q^*}{(l+1)[R\sin 2\beta^R_l - \cos 2\beta^R_l]}
\]

\[
= \frac{(l+1)R - 2k^2 Q^\frac{3}{2}}{(l+1)[R\cos 2\beta^R_l + \sin 2\beta^R_l]}
\]

(73)

Furthermore, from a knowledge of \(\beta^R_l\) and \(\beta^I_l\), we can determine the \(G'_l\) and \(N'_l\) from Eqs. (67) and (69) which will ensure the wave functions having the correct asymptotic forms.

We can apply this same procedure to the complex Eq. (44) in order to obtain analogous expressions for the \(Y_l\):

\[
\tan 2\gamma^R_l = \frac{2k^2 [Q^\frac{3}{2} - R\cos \gamma^R_l] - 2(l+1)R}{2k^2 [Q^\frac{3}{2} + R\cos \gamma^R_l] - (l+1)(1-R^2)}
\]

(74)

\[
\exp(-2\gamma^I_l) = \frac{(l+1) + 2k^2 Q^*}{(l+1)[R\sin 2\gamma^R_l - \cos 2\gamma^R_l]}
\]
In Eqs. (74, 75), the symbols $Q_{\ell}^\phi$, $Q_{\ell}^\zeta$ are as defined in Eqs. (60, 63) except that they are functions of $\tilde{\phi}_{\ell}$, $\tilde{\zeta}_{\ell}$ rather than $\tilde{\phi}'_{\ell}$, $\tilde{\zeta}'_{\ell}$. $R_{\ell}$ is as defined in Eq. (60).
Chapter III. Application to a Square Well Potential When $\ell=0$.

It is possible to obtain an explicit expression for the $\beta_{\ell}$, $\gamma_{\ell}$, and $\delta_{\ell}$ in the case where $V$ is a square well potential. We shall consider here only the case where $\ell=0$. This evaluation has been included in order to provide additional insight into the technique presented in the previous section.

Let

$$V(r) = -V_0 \quad 0 \leq r \leq r_o$$

$$= 0 \quad r > r_o$$

(76)

where $r_o$ has been taken to be the range of the potential. As we have pointed out, the solution to Eq. (28) will, in general, be complex. Therefore, let us utilize the above potential and let us split this equation into its real and imaginary parts for the special case $\ell=0$ and all $r < r_o$. We have

$$\frac{d^2 \phi_0(<)}{d r^2} + k^2 \phi_0(<) = -4k r j_0(kr) + 2k \cos \delta_0 \chi_0(<) - V_0 \phi_0(<) \tag{77}$$

and

$$\frac{d^2 \zeta_0(<)}{d r^2} + k^2 \zeta_0(<) = 2k \sin \delta_0 \chi_0(<) - V_0 \zeta_0(<) \tag{78}$$

where $\phi_0(<)$, $\zeta_0(<)$, $\chi_0(<)$ have been used to represent $\phi_o(x)$, $\zeta_o(x)$, $\chi_o(x)$.
for values of $r < r_o$.

Let us first look at the equation satisfied by $\phi_o'(r)$, for which we must obtain those solutions that are valid both inside and outside of $V$. In order to do this, we must determine $\chi_o(r)$ in these regions. For $r < r_o$, the equation for $\chi_o$ assumes the form,

$$\frac{d^2 \chi_o(r)}{dr^2} + K^2 \chi_o(r) = 0$$

(79)

where $K^2 = k^2 + V_o$ and the solution,

$$\chi_o(r) = C \sin Kr$$

(80)

where $C$ is an arbitrary constant. Outside of the potential (with obvious change in notation) we obtain

$$\chi_o(r) = A \sin kr + B \cos kr$$

(81)

Through application of the boundary conditions at $r = r_o$ which require the continuity of $\chi_o$ and $d\chi_o/dr$, we may express $A$ and $B$ in terms of $C$. Thus,

$$A = C\left[ \sin kr_o \sin Kr_o + \frac{K}{k} \cos kr_o \cos Kr_o \right] \equiv CR$$

(82)
\[ B = C \left[ \cos kr, \sin Kr_1 - \frac{K}{k} \sin kr \cos Kr_1 \right] \equiv CR_2 \]  

(83)

and we may write

\[ \chi_{\ell}(r) = C \left[ R_1 \sin kr + R_2 \cos kr \right] \]

(84)

In order to obtain the correct normalization, we observe that in Eq.(29) the asymptotic form for \( \chi_{\ell} \) was chosen so that, with \( \ell=0 \), we have

\[ \chi_0(\infty) \equiv \chi_{0}(>) \xrightarrow{r \to \infty} \frac{\cos \delta_0}{k} \sin kr + \frac{\sin \delta_0}{k} \cos kr \]

(85)

From a comparison of Eq.(85) with Eq.(84), we may conclude that

\[ C = \frac{\cos \delta_0}{kR_1} \quad \text{or} \quad \frac{\sin \delta_0}{kR_2} \]

(86)

and Eq.(77) becomes

\[ \frac{d^2 \phi(r)}{dr^2} + k^2 \phi(r) = -4 \sin kr + D_1 \sin Kr \]

(87)

where \( D_1 = 2kC \cos \delta_0 \). The general solution to Eq.(87) may be written
\[ \phi_0'(\langle) = -\frac{4}{K^2-k^2} \sin kr - \frac{D_1 r}{2K} \cos Kr + E_1' \sin Kr \]

(88)

where \( E_1' \sin Kr \) is the homogeneous part.

Outside the potential (where \( r > r_o \)), the equation for \( \phi_0' \) may be written

\[ \frac{d^2 \phi_0'(\rangle)}{dr^2} + k^2 \phi_0'(\rangle) = P_1' \sin kr + P_2' \cos kr \]

(89)

where \( P_1' = 2kC_1 \cos \delta_o - 4 \) and \( P_2' = 2kC_2 \cos \delta_o \). This equation has the general solution

\[ \phi_0'(\rangle) = \frac{r}{2k} \left[ P_2' \sin kr - P_1' \cos kr \right] + F' \sin kr + G' \cos kr \]

(90)

where \( F' \sin kr + G' \cos kr \) is the homogeneous part.

We must now match \( \phi_0'(\langle) \) and \( \phi_0'(\rangle) \) at \( r = r_o \) and obtain \( F' \) and \( G' \) in terms of \( E_1' \). We find

\[ F' = \frac{P_1'}{2k} \cos^2 kr_o - \frac{P_2'}{2k} \left[ r_o + \frac{\cos kr_o \sin kr_o}{k} \right] - \frac{D_1 \cos Kr_o}{2K} \left[ \frac{\cos kr_o}{k} + r_o \sin kr_o \right] + \frac{D_2 \cos kr_o \sin Kr_o}{2k} - \frac{4}{K^2-k^2} + E_1' \left[ \sin kr_o \sin Kr_o + \frac{K \cos kr_o \cos Kr_o}{k} \right] \]

(91)
and

\[
G' = \frac{P_2'}{2k} \sin^2 kr_0 + \frac{P_1'}{2k} \left[ r_0 - \frac{\sin kr_0 \cos kr_0}{k} \right] - \frac{D_2 \cos Kr_0}{2K} \left[ r_0 \cos kr_0 \\
- \frac{\sin kr}{k} \right] - \frac{D_2}{2k} \sin kr_0 \sin Kr_0 + E_1' \left[ \cos kr_0 \sin Kr_0 - \frac{K \sin kr_0 \cos kr_0}{k} \right]
\]

(92)

We now consider ζ'_0, the imaginary part of ζ'_0. Inside the potential, we have

\[
\frac{d^2 \zeta_0(\langle r \rangle)}{dr^2} + k^2 \zeta'_0(\langle r \rangle) = D_2 \sin Kr
\]

(93)

where $D_2 = 2kC \sin \delta_0$. The general solution to Eq. (93) may be written

\[
\zeta'_0(\langle r \rangle) = - \frac{D_2}{2K} r \cos Kr + E_2' \sin Kr
\]

(94)

where $E_2'$ is now taken to be the normalization constant for the homogeneous solution.

Outside the potential, we have

\[
\frac{d^2 \zeta'_0(\langle r \rangle)}{dr^2} + k^2 \zeta'_0(\langle r \rangle) = S_1' \sin kr + S_2' \cos kr
\]

(95)
where $S_1' = 2kC_1 \sin \delta_0$ and $S_2' = 2kC_2 \sin \delta_0$. The general solution to Eq. (95) may be written,

$$\zeta'_o(> \psi) = \frac{r}{2k} \left[ S_2' \sin kr - S_1' \cos kr \right] + L' \sin kr + N' \cos kr$$

(96)

Where $L' \sin kr + N' \cos kr$ is the homogeneous part of the solution.

Following the procedure used to determine $F'$ and $G'$, we now determine $L'$ and $N'$ by matching $\zeta'_o(<)$ and $\zeta'_o(>)$ at $r = r_o$. We find

$$L' = \frac{S_1'}{2k^2} \cos^2 kr - \frac{S_2'}{2k} \left[ r_o + \frac{\sin kr_0 \cos kr_o}{k} \right] - \frac{D_2}{2k} \cos Kr_o \left[ \frac{\cos kr_0}{k} + r_o \sin kr_o \right] + \frac{D_2 r_o}{2k} \cos kr_o \sin Kr_o + E_2' \left[ \frac{\sin kr_0 \sin kr_o + K \cos kr_0 \cos Kr_o}{k} \right]$$

(97)

and

$$N' = \frac{S_2'}{2k^2} \sin^2 kr_o + \frac{S_1'}{2k} \left[ r_o - \frac{\sin kr_0 \cos kr_o}{k} \right] - \frac{D_2}{2k} \cos Kr_o \left[ \frac{r_o \cos kr_0 - \sin kr_o}{k} \right] - \frac{D_2 r_o}{2k} \sin kr_o \sin Kr_o + E_2' \left[ \cos kr_0 \sin kr_o - \frac{K \sin kr_0 \cos Kr_o}{k} \right]$$

(98)
We know from Eq. (32) that, asymptotically, the homogeneous part of 
\( \chi'_0 \) has the form

\[
\chi'_0 \xrightarrow{r \to \infty} -\frac{1}{2k} \left\{ \exp(-2\beta^r_0)\sin(kr+2\beta^r_0) + \sin kr \right\} - i\left\{ \exp(-2\beta^i_0)\cos(kr+2\beta^r_0) - \cos kr \right\}
\]

(99)

where we have taken \( \beta_0 = \beta^R_0 + i\beta^I_0 \). In terms of the constants \( F', G', L', \) and \( N' \), we may write

\[
\chi'_0 \xrightarrow{r \to \infty} F'\sin kr + G'\cos kr + i(L'\sin kr + N'\cos kr)
\]

(100)

By comparing Eqs. (99) and (100), we obtain

\[
F' = -\frac{1}{2k^2} \left\{ \exp(-2\beta^r_0)\cos 2\beta^r_0 + 1 \right\}
\]

(101)

\[
G' = -\frac{1}{2k^2} \exp(-2\beta^i_0)\sin 2\beta^r_0
\]

(102)

\[
L' = -\frac{1}{2k^2} \exp(-2\beta^i_0)\sin 2\beta^r_0
\]

(103)

\[
N' = -\frac{1}{2k^2} \left[ 1 - \exp(-2\beta^i_0)\cos 2\beta^r_0 \right]
\]

(104)

For convenience, we shall utilize the notation:
\[ T = \sin^2 k_0 \; ; \; W = \sin k_0 \sin K_0 \; ; \; Y = \sin k_0 \cos K_0 \]

\[ U = \sin k_0 \cos k_0 \; ; \; X = \cos k_0 \sin K_0 \; ; \; Z = \cos k_0 \cos K_0 \]

\[ V = \cos^2 k_0 \]

in Eqs. (91,92) and Eqs. (97,98) and write:

\[ \frac{P'}{2k^2} V - \frac{P'}{2k} \left[ r_0 + \frac{U}{k} \right] - \frac{D_z}{2k} \left[ \frac{Z}{k} + r_0 Y \right] + \frac{D_z r_0}{2k} X - \frac{4}{K^2 - k^2} \]

\[ + E'_z \left[ W + \frac{K}{k} Z \right] = - \frac{1}{2k^2} \left[ \exp(-2 \beta^i) \cos 2 \beta^g + 1 \right] \]

\[ \frac{P'}{2k^2} T + \frac{P'}{2k} \left[ r_0 - \frac{U}{k} \right] - \frac{D_z}{2k} \left[ r_0 Z - \frac{Y}{k} \right] - \frac{D_z r_0}{2k} W \]

\[ + E'_z \left[ X - \frac{K}{k} Y \right] = - \frac{\exp(-2 \beta^i)}{2k^2} \sin 2 \beta^g \]

\[ \frac{S'}{2k^2} V - \frac{S'}{2k} \left[ r_0 + \frac{U}{k} \right] - \frac{D_z}{2k} \left[ \frac{Z}{k} + r_0 Y \right] + \frac{D_z r_0}{2k} X \]

\[ + E'_z \left[ W + \frac{K}{k} Z \right] = - \frac{\exp(-2 \beta^i)}{2k^2} \sin 2 \beta^g \]
\[ \frac{S'_3}{2k^3} T + \frac{S'_4}{2k} \left[ \frac{r_0 - U}{k} \right] - \frac{D_z}{2K} \left[ \frac{Z - Y}{k} \right] - \frac{D_{\phi}}{2K} \mathcal{W} \]

\[ + E'_2 \left[ X - \frac{K}{k} Y \right] = - \frac{1}{2k^2} \left[ 1 - \exp \left( -2 \beta^T \right) \cos 2\beta^R \right] \]

(109)

We proceed to eliminate \( E'_1 \) between Eqs. (106) and (107), then \( E'_2 \) between Eqs. (108) and (109), obtaining the relations:

\[ Q'_1 \equiv \left[ X - \frac{K}{k} Y \right] \left[ \frac{1}{2k} \left\{ P'_1 V - P'_2 U + 1 \right\} + \frac{1}{2k} \left\{ (D_X - P'_2) r_0 - \frac{D_z}{K} Z \right\} - \frac{D_{\phi}}{2K} Y \right] - \left[ \mathcal{W} + \frac{K}{k} Z \right] \left[ \frac{1}{2k} \left\{ P'_2 T - P'_1 U \right\} + \frac{1}{2k} \left\{ (P'_1 - D, W) r_0 + \frac{D_z}{K} Y \right\} - \frac{D_{\phi}}{2K} r_0 Z \right] \]

\[ = \exp \left( -2 \beta^T \right) \left[ \left\{ \mathcal{W} + \frac{K}{k} Z \right\} \sin 2\beta^R - \left\{ X - \frac{K}{k} Y \right\} \cos 2\beta^R \right] \]

(110)
\[ Q_2 = \left[ x - \frac{k}{k} y \right] \left[ \frac{1}{2k} \left( s' V - s' U \right) + \frac{1}{2k} \left( D_z X - S_z \right) r_o \right] - \frac{D_z Z}{2k} \left( W + \frac{k}{k} Z \right) \left[ \frac{1}{2k} \left( S_z T - S_z \right) U + 1 \right] + \frac{1}{2k} \left( S_z - D_z W \right) r_o - \frac{D_z}{2k} Z \rceil \]

\[ = - \exp \left( -2 \beta_o^I \right) \left[ \left( W + \frac{k}{k} Z \right) \cos 2 \beta_o^R + \left( x - \frac{k}{k} y \right) \sin 2 \beta_o^R \right] \]

By dividing Eq. (110) by Eq. (111), we may then solve for \( \tan 2 \beta_o^R \):

\[ \tan 2 \beta_o^R = \frac{Q_2 \left( x - \frac{k}{k} y \right) - Q_2' \left( W + \frac{k}{k} Z \right) }{Q_2 \left( W + \frac{k}{k} Z \right) + Q_2' \left( x - \frac{k}{k} y \right) } \]

and by utilizing Eq. (110) or Eq. (111) we may obtain \( \exp (-2 \beta_o^I) \)

\[ \exp (-2 \beta_o^I) = \frac{2 Q_2' \frac{k}{2} }{\left( \left( W + \frac{k}{k} Z \right) \sin 2 \beta_o^R - \left( x - \frac{k}{k} y \right) \cos 2 \beta_o^R \right) } \]

or
\[
\exp(-2\beta_o^1) = \frac{-2Q_k^2}{\left[\left\{W + \frac{K^2}{\kappa}\right\}\cos 2\beta_o^R + \left\{\chi - \frac{K}{\kappa}\right\}\sin 2\beta_o^R\right]}
\]

(114)

We may now proceed to find expressions which determine \(\gamma_o\). If we let \(\xi_o = \phi_o + i\zeta_o\), then we may split Eq.(44) into its real and imaginary parts for the case when \(l=0\) and for all \(r < r_o\).

\[
\frac{d^2\phi_o(<)}{dr^2} + k^2\phi_o(<) = -2k \cos \xi_o \chi_o(<)
\]

(115)

\[
\frac{d^2\zeta_o(<)}{dr^2} + k^2\zeta_o(<) = -2k \sin \xi_o \chi_o(<)
\]

(116)

Utilizing Eq.(80) for \(\chi_o(<)\), we may rewrite Eq.(116) as

\[
\frac{d^2\phi_o(<)}{dr^2} + k^2\phi_o(<) = -D_1 \sin Kr
\]

(117)

The general solution to Eq.(117) may be written

\[
\phi_o(<) = \frac{D_1}{K^2 - k^2} \sin Kr + E_1 \sin kr
\]

(118)

where \(E_1 \sin kr\) is the homogeneous part.
Outside the potential (where \( r > r_0 \)), the equation for \( \phi_0 \) may be written

\[
\frac{d^2 \phi_0}{dr^2} + k^2 \phi_0 = P_1 \sin kr + P_2 \cos kr
\]

(119)

where \( P_1 = -2kC_1 \cos \delta_0 \), and \( P_2 = -2kC_2 \cos \delta_0 \). This equation has the solution

\[
\phi_0 = \frac{1}{2k} \left[ P_2 \sin kr - P_1 \cos kr \right] + F \sin kr + G \cos kr
\]

(120)

where \( F \sin kr + G \cos kr \) is the homogeneous part.

We must now match \( \phi_0 (<) \) and \( \phi_0 (>\) at \( r = r_0 \) in order to find \( F \) and \( G \) in terms of \( E_1 \). We obtain

\[
F = \frac{P_1}{2k} \cos^2 kr + \frac{P_2}{2k} \left[ r_o + \frac{\sin kr_o \cos kr_o}{k} \right]
\]

\[
+ \frac{D_1}{K^2 - k^2} \left[ \sin kr_o \sin Kr_o + \frac{K \cos kr_o \cos Kr_o}{k} \right] + E_1
\]

(121)

\[
G = \frac{P_2}{2k} \sin^2 kr + \frac{P_1}{2k} \left[ r_o - \frac{\sin kr_o \cos kr_o}{k} \right]
\]

\[
+ \frac{D_1}{K^2 - k^2} \left[ \cos kr_o \sin Kr_o - \frac{K}{k} \sin kr_o \cos Kr_o \right]
\]

(122)
We now consider \( \zeta_0 \), the imaginary part of \( \xi_0 \). Inside the potential, we have

\[
\frac{d^2 \zeta_0(<)}{dr^2} + k^2 \zeta_0(<) = -D_2 \sin Kr
\]

(123)

The general solution to Eq. (123) may be written

\[
\zeta_0(<) = \frac{D_2}{K^2 - k^2} \sin Kr + E_2 \sin kr
\]

(124)

Outside the potential (where \( r > r_o \)), the equation for \( \zeta_0 \) may be written

\[
\frac{d^2 \zeta_0(>)}{dr^2} + k^2 \zeta_0(>) = S_1 \sin kr + S_2 \cos kr
\]

(125)

where \( S_1 = -2kCR_1 \sin \delta_0 \), and \( S_2 = -2kCR_2 \sin \delta_0 \). This equation has the solution

\[
\zeta_0(>) = \frac{r}{2k} \left[ S_2 \sin kr - S_1 \cos kr \right] + L \sin kr + N \cos kr
\]

(126)

where \( L \sin kr + N \cos kr \) is the homogeneous part.

We must now match \( \zeta_0(<) \) and \( \zeta_0(>) \) at \( r = r_o \) in order to find \( L \) and \( N \) in terms of \( E_2 \). We obtain
\[ L = \frac{S_1}{2k} \cos^2 kr_0 - \frac{S_2}{2k} \left[ r_0 + \frac{\sin kr_0 \cos kr_0}{k} \right] \]

\[ + \frac{D_2}{K^2-k^2} \left[ \sin kr_0 \sin Kr_0 + \frac{K \cos kr_0 \cos Kr_0}{k} \right] + E_z \]

(127)

\[ N = \frac{S_2}{2k} \sin^2 kr_0 + \frac{S_1}{2k} \left[ r_0 - \frac{\sin kr_0 \cos kr_0}{k} \right] \]

\[ + \frac{D_2}{K^2-k^2} \left[ \cos kr_0 \sin Kr_0 - \frac{K \sin kr_0 \cos Kr_0}{k} \right] \]

(128)

We know from Eq. (46) that, asymptotically, the homogeneous part of \( \xi_0 \) has the form

\[ \xi_0 \to \infty \rightarrow -\frac{1}{2k} \left\{ \left[ \exp(-2\gamma^I_0) \sin(kr + 2\gamma^R_0) + \sin kr \right] \right. \]

\[ \left. -i \left[ \exp(-2\gamma^I_0) \cos(kr + 2\gamma^R_0) - \cos kr \right] \right\} \]

(129)

where we have taken \( \gamma_0 = \gamma^R_0 + i\gamma^I_0 \). In terms of the constants \( F, G, L \) and \( N \), we may write
\[ F \sim e^{i k r} \left( F \sin kr + G \cos kr + i (L \sin kr + N \cos kr) \right) \]

(130)

By comparing Eqs. (129) and (130), we obtain

\[
F = - \frac{1}{2k^2} \left[ \exp \left( -2\gamma^I_0 \right) \cos 2\gamma^R_0 + 1 \right]
\]

(131)

\[
G = - \frac{\exp \left( -2\gamma^I_0 \right)}{2k^2} \sin 2\gamma^R_0
\]

(132)

\[
L = - \frac{\exp \left( -2\gamma^I_0 \right)}{2k^2} \sin 2\gamma^R_0
\]

(133)

\[
N = - \frac{1}{2k^2} \left[ 1 - \exp \left( -2\gamma^I_0 \right) \cos 2\gamma^R_0 \right]
\]

(134)

Utilizing the notation (105), we may rewrite Eqs. (122) and (128) as

\[
Q_1 = - \frac{P_2}{2k^2} T - \frac{P_1}{2k} \left[ r_0 - \frac{U}{k} \right] - \frac{D_1}{k^2} \left[ \lambda - \frac{\kappa}{k} \gamma \right] = \exp \left( -2\gamma^I_0 \right) \sin 2\gamma^R_0
\]

(135)
\[
Q_2 = \left( \frac{S_2 + 1}{2k^2} \right) T + \frac{S_1}{2k} \left[ x - \frac{U}{k} \right] + \frac{D_2}{k^2 - k^2} \left[ x - \frac{k}{k} \right] = \frac{\exp(-2\gamma_i^R)}{2k^2} \cos 2\gamma_o^R
\]  

(136)

By dividing Eq. (135) by Eq. (136), we may obtain an expression for \( \tan 2\gamma_o^R \)

\[
\tan 2\gamma_o^R = \frac{Q_1}{Q_2}
\]  

(137)

and by utilizing Eq. (135) or Eq. (136) we may obtain \( \exp(-2\gamma_o^I) \)

\[
\exp(-2\gamma_o^I) = \frac{2Q_1 k^2}{\sin 2\gamma_o^R}
\]  

or

\[
\exp(-2\gamma_o^I) = \frac{2Q_2 k^2}{\cos 2\gamma_o^R}
\]  

(138)
Chapter IV. Scattering Matrix Formulation of $\frac{d}{dk} M(K,k,\cos\theta)\big|_{K=k}$

For the Spin-Dependent Case.

We now wish to rederive the results of Chapter I, i.e., Eq.(51), in the physically realistic case where the nucleons each have spin $\frac{1}{2}$. In order to effect this derivation in an efficient manner, we can commence with Eq.(23), rewritten as follows:

$$\nabla^2 \langle \vec{r} | \Psi' \rangle + k^2 \langle \vec{r} | \Psi' \rangle = -4ke^{ik \cdot \vec{r}} \chi_S^{S_z}$$

$$+ 2k \langle \vec{r} | \Psi \rangle + \sqrt{2} \langle \vec{r} | \Psi' \rangle$$

where the wave functions $\langle \vec{r} | \psi \rangle$ etc. now include spin and $\chi_S^{S_z}$ is a singlet-triplet (ST) spinor. We utilize the partial wave expansions:

$$\langle \vec{r} | \psi \rangle = \sqrt{\frac{4\pi}{J \ell' \ell}} \chi_{(r)}^{J \ell} \chi_S^{S_z} \mathcal{Y}_{(\theta, \phi)}^{\ell' \ell} \langle J S_z S_m l m l' S_z' \rangle$$

where the wave functions $\chi_S^{S_z}$ etc. now include spin and $\chi_S^{S_z}$ is a singlet-triplet (ST) spinor. We utilize the partial wave expansions:

$$\langle \vec{r} | \psi \rangle = \sqrt{4\pi} \sum_{J, \ell, \ell'} \hat{\chi}_{ \ell}^{J \ell'} \chi_{(r)}^{J \ell} \chi_S^{S_z} \mathcal{Y}_{(\theta, \phi)}^{J \ell' \ell} \langle J S_z S_m l m l' S_z' \rangle$$
\[
\langle \hat{\tau} | \psi' \rangle = \sqrt{4\pi} \sum_{J, \ell, \ell'} \hat{\ell} \hat{\ell} \, \chi^{J,S}_{\ell \ell'} \, \mathcal{Y}^{S_z}_{\ell \ell'}(\theta, \phi) \langle JS_z Sl \mid LOSS_z \rangle
\]

(142)

\[
e^{i \mathbf{k} \cdot \mathbf{r}} \chi^S_s = \sqrt{4\pi} \sum_{\ell, J} \hat{\ell} \hat{\ell} \, j_{\ell}(kr) \mathcal{Y}^{S_z}_{\ell \ell'}(\theta, \phi) \langle JS_z Sl \mid LOSS_z \rangle
\]

(143)

\[
\nabla \langle \hat{\tau} | \psi' \rangle = \sqrt{4\pi} \sum_{J, \ell, \ell', L} \hat{\ell} \hat{\ell} \hat{\ell} \hat{\ell} \langle \ell | \nabla(\mathbf{r}) | \ell' \rangle \chi^{J,S}_{\ell \ell'} \mathcal{Y}^{S_z}_{\ell \ell'}(\theta, \phi) \langle JS_z Sl \mid LOSS_z \rangle
\]

(144)

where \( \ell = (2J+1)^{1/2} \), \( \mathcal{Y}^{S_z}_{J}(\theta, \phi) = \chi^{S}_s(\theta, \phi) \otimes \chi^{S}_s(\theta, \phi) \) are the generalized spherical harmonics, and \( \langle JS_z Sl \mid LOSS_z \rangle \) represent the relevant Clebsch-Gordan coefficients. If we now substitute these expansions into Eq.\,(140), we obtain

\[
\sum_{J, \ell, \ell'} \hat{\ell} \hat{\ell} \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} - \frac{\ell(\ell+1)}{r^2} + k^2 \right] \chi^{J,S}_{\ell \ell'} \mathcal{Y}^{S_z}_{\ell \ell'}(\theta, \phi) \langle JS_z Sl \mid LOSS_z \rangle
\]

(continued on next page)
\[
\sum_{J,L'J',L} \hat{c}_L \langle l' | \chi_{J,SS_{z}}^{(r)} | l' \rangle \chi_{J,L'}^{(r)} \langle JS_{z}SL | LOSS_{z} \rangle \\
+ 2k \sum_{J,L'J',L} \hat{c}_L \left[ \chi_{J,L'}^{(r)} \langle JS_{z}SL | LOSS_{z} \rangle - 2r_{J,L'}^{(Kr)} \langle JS_{z}SL | LOSS_{z} \rangle \right]
\]

(145)

We now multiply Eq. (145) by \( \psi_{J',L''}^{S_{z}} \) and obtain by orthogonality

\[
\sum_{L,L'} \hat{c}_L \left[ \frac{d^2}{dr^2} - \frac{\lambda' (\lambda' + 1)}{r^2} + k^2 \right] \delta_{L''L'} \chi_{J,L'}^{(r)} \langle JS_{z}SL | LOSS_{z} \rangle = \sum_{L',L} \hat{c}_L \langle l'' | \chi_{J,L'}^{(r)} | l' \rangle \chi_{J,L'}^{(r)} \langle JS_{z}SL | LOSS_{z} \rangle \\
+ 2k \sum_{L,L'} \hat{c}_L \left[ \chi_{J,L'}^{(r)} \delta_{L''L'} - 2r_{J,L'}^{(Kr)} \delta_{L''L'} \right] \langle JS_{z}SL | LOSS_{z} \rangle
\]

(146)

The sums over \( L, L \) can be removed by multiplying Eq. (146) by \( \langle SS_{z}L'O|L'SJS_{z} \rangle \), summing over \( S_{z} \) and setting \( L' = L \) with the result:
The function $\chi_{l,\ell}^{J,S}(r)$ in the region $r > r_0$, where $\langle \ell'' | \mathbf{J}_S(r) | \ell' \rangle \equiv 0$ is subject to the normalization:

$$\chi_{l,\ell''}^{J,S}(r) = \sum_{\ell'} A_{l,\ell}\, r^l \left[ \delta_{\ell',\ell''} \, j_{\ell''}(kr) + \chi_{\ell',\ell''}^{J,S} \, n_{\ell''}(kr) \right]$$

with

$$\sum_{\ell'} A_{l,\ell'}^{J,S} \left[ \delta_{\ell',\ell'} + i \chi_{\ell',\ell'}^{J,S} \right] = \delta_{l,l'}$$

A particular solution of the inhomogeneous equation for $\chi_{l,\ell''}^{J,S}(r)$ may be found [Appendix A]:

$$p^l \chi_{l,\ell''}^{J,S} = r^2 \left[ (2 - \sum_{\ell'} A_{l,\ell'}^{J,S} \, \delta_{\ell',\ell''}) \, j_{\ell''}(kr) - \sum_{\ell'} A_{l,\ell'}^{J,S} \, \chi_{\ell',\ell''}^{J,S} \, n_{\ell''}(kr) \right]$$

Thus, for $r > r_0$, the general solution to Eq. (147) is $p^l \chi_{l,\ell''}^{J,S}(r)$ plus $\chi_{l,\ell''}^{J,S}(r)$, the general solution to the homogeneous equation

$$\frac{d^2}{dr^2} \chi_{l,\ell''}^{J,S} - \frac{\ell''(\ell''+1)}{r^2} \chi_{l,\ell''}^{J,S} + k^2 \chi_{l,\ell''}^{J,S} = 0$$
Asymptotically, the general solution to Eq. (150) may be written as

\[
\chi^{\text{JS}}_{l',l''} \rightarrow \sum_{l',l''} C_{l' \rightarrow l''}^{\text{JS}} \left[ \delta_{l' \rightarrow l''} \sin (k r - \frac{\pi}{2} l') - Y_{l' \rightarrow l''}^{\text{JS}} \cos (k r - \frac{\pi}{2} l'') \right]
\]

(151)

where \( C_{l' \rightarrow l''}^{\text{JS}} \) and \( Y_{l' \rightarrow l''}^{\text{JS}} \), in general, may be complex. Finally, the asymptotic form, to terms of \( O \left( \frac{1}{r} \right) \), of the function \( \langle \hat{r} | \psi' \rangle \) is given by

\[
\langle \hat{r} | \psi' \rangle \rightarrow \frac{1}{\sqrt{4\pi}} \sum_{l',l''} \chi^{\text{JS}}_{l',l''} \left\{ \frac{C_{l' \rightarrow l''}^{\text{JS}}}{r} \delta_{l' \rightarrow l''} \sin (k r - \frac{\pi}{2} l') \right. \\
- Y_{l' \rightarrow l''}^{\text{JS}} \cos (k r - \frac{\pi}{2} l') \left] + (2 - A_{l' \rightarrow l''}^{\text{JS}}) \left[ \frac{1}{k} \cos (k r - \frac{\pi}{2} l') + \right. \\
- \frac{l' (l' - 1)}{2kr} \sin (k r - \frac{\pi}{2} l') \right] - A_{l' \rightarrow l''}^{\text{JS}} X_{l' \rightarrow l''}^{\text{JS}} \left[ \frac{1}{k} \sin (k r - \frac{\pi}{2} l') + \right.
\left. \frac{l' (l' - 1)}{2kr} \cos (k r - \frac{\pi}{2} l') \right] \right\} \psi^{\text{JS}}_{l',l''}(\theta, \phi) \langle J S_z S_l | l'0 S_z \rangle
\]

(152)

If Eq. (152) is expressed in exponential form and compared with the spin-dependent form of Eq. (15):

\[
\langle \hat{r} | \psi' \rangle \rightarrow \frac{1}{i r \cos \theta} e^{i k \cdot \hat{r}} \chi^{S_z}_{S_z'} - \sum_{S_z'} M_{A}^{S_z, S_z'} \chi^{S_z'}_{S_z} e^{i k r}
\]

(continued on next page)
\[ + \sum_{S_z'} (M^S_{c} S_z' + M^S_{d} S_z') \chi^S_{s'} e^{i k r} \]

where

\[ M^S_{A} = \frac{\sqrt{4\pi}}{2i k} \sum_{J',\ell',\ell''} \hat{J}^J (-i)^{\ell''} \left[ S_{j,\ell'}^{j,\ell''} - \delta_{\ell'\ell''} \right] \langle SS_z' S_{\ell} S_{\ell'}' | \ell' \ell'' S J S_z \rangle \times \langle J S_z S \ell \ell' l o s s_z \rangle Y^{S_z - S_z'}_{\ell''} \]

we find

\[ \sum_{J''} C^{J}_J \left[ \delta_{\ell''\ell} + i Y^{J}_{\ell''} \right] = - \frac{(L+1)}{k} \delta_{\ell''\ell} \]

(154)

Rewriting Eq. (154) in matrix notation, we see that

\[ C = - \frac{(L+1)}{k} \left[ I + i Y \right]^{-1} \]

(155)

In addition, we have

\[ M^S_{c} S_z' + M^S_{d} S_z' = \frac{\sqrt{4\pi}}{4 i k} \sum_{J,\ell,\ell'} \hat{J}^J (-i)^{\ell'} \left[ \ell' (\ell' - 1) \left( S_{j,\ell'}^{j,\ell} - \delta_{\ell'\ell} \right) \right] \]

\[ - 2 (\ell' + 1) \left[ S_{j,\ell'}^{j,\ell} - \delta_{\ell'\ell} \right] \langle SS_z' \ell' S_{\ell} S_{\ell'}' | \ell' \ell'' S J S_z \rangle \times \langle J S_z S \ell \ell' l o s s_z \rangle Y^{S_z - S_z'}_{\ell''} \]

(156)
We proceed to examine Eq. (42), rewritten in the form:

\[ \nabla^2 \langle \hat{r} | \Psi' \rangle + k^2 \langle \hat{r} | \Psi' \rangle = -2k \langle \hat{r} | \Psi \rangle \]

(157)

Utilizing the same procedure as that used to transform Eq. (140) into Eq. (147), we may express Eq. (157) as

\[ \left[ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2 \right] \xi_{l,l'}^{J,S} = 2k \chi_{l,l'}^{J,S} \]

(158)

Continuing, we may find a particular solution for Eq. (158) [Appendix A], so that

\[ p \xi_{l,l'}^{J,S} = r^2 \sum_{l'} A_{l,l'}^{J,S} \left[ \delta_{l,l'} j_{l-1}(kr) + X_{l,l'}^{J,S} \eta_{l-1}(kr) \right] \]

(159)

This equation, combined with the general solution to the homogeneous equation

\[ \frac{d^2}{dr^2} \xi_{l,l'}^{J,S} - \frac{l(l+1)}{r^2} \xi_{l,l'}^{J,S} + k^2 \xi_{l,l'}^{J,S} = 0 \]

(160)
gives, for \( r > r_0 \), the general solution to Eq.(157).

Asymptotically, \( h_{s,L,L'}(r) \) may be written as

\[
\lim_{r \to \infty} h_{s,L,L'}(r) = \sum_{L'} \frac{D_{s,L,L'}^{JS}}{r} \left[ \delta_{L,L'} \sin( kr - \frac{\pi}{2} L') - \frac{Z_{L,L'}^{JS}}{r} \cos( kr - \frac{\pi}{2} L') \right]
\]

(161)

where \( D_{s,L,L'}^{JS} \) and \( Z_{L,L'}^{JS} \), in general, may be complex. Therefore, the asymptotic form, to terms of \( O \left( \frac{1}{r} \right) \), of the function \( \langle r' | \Psi' \rangle \) is given by

\[
\langle r' | \Psi' \rangle \to \infty \sqrt{4 \pi} \sum_{J,L,L'} \delta_{J,L} \left\{ \frac{D_{s,L,L'}^{JS}}{r} \left[ \delta_{L,L'} \sin( kr - \frac{\pi}{2} L') - \frac{Z_{L,L'}^{JS}}{r} \cos( kr - \frac{\pi}{2} L') \right] - \frac{Z_{L,L'}^{JS}}{2 kr} \sin( kr - \frac{\pi}{2} L') \right\} + \frac{\delta_{L,L'}^{JS}}{2 kr} \sin( kr - \frac{\pi}{2} L') + \frac{\delta_{L,L'}^{JS}}{2 kr} \cos( kr - \frac{\pi}{2} L') \right\}
\]

(162)

If Eq.(162) is expressed in exponential form and compared with
the spin-dependent form of Eq. (40),

\[
\langle \tilde{r} | \psi' \rangle \propto \lim_{r \to \infty} i r \cos \theta \ e^{i k \cdot \tilde{r}} \ \chi_{s}^{s_{z}} + i \sum_{S_{z}'} M_{A}^{S_{z} S_{z}'} \chi_{s}^{S_{z}'} e^{i k r}
\]

\[
- \sum_{S_{z}'} M_{D}^{S_{z} S_{z}'} \chi_{s}^{S_{z}'} \frac{e^{i k r}}{r}
\]  

we find

\[
\sum_{l'' \ell''} D_{l'' \ell''}^{J} [ \delta_{l'' \ell''} + i \sum_{l'' \ell''} D_{l'' \ell''}^{J} ] = - \frac{(l + 1) \ell}{k^2} \delta_{l' \ell'}
\]  

(163)

In matrix notation, Eq. (164) becomes

\[
D = - \frac{(l + 1)}{k^2} [ I + i Z ]^{-1}
\]  

(164)

We also obtain

\[
M_{D}^{S_{z} S_{z}'} = \frac{\sqrt{4 \pi}}{4 i k^2} \sum_{J, l, \ell, l'} \hat{l} i^J (-i)^{l'} \left\{ 2 \left( l' + 1 \right) \left[ S_{l''}^{J S_{z}'} - \delta_{l'' \ell'} \right] \\
+ \ell' (\ell' - 1) \left[ S_{l''}^{J S_{z}'} - \delta_{l'' \ell'} \right] \right\} \langle S S_{z} S_{z} S_{z} - S_{z} S_{z} - S_{z} S_{z} | l' S_{J} S_{J} S_{z} \rangle \times \langle J S_{z} S_{z} l | l' 0 S S_{z} \rangle \gamma_{l' \ell'}^{S_{z} S_{z}'}
\]  

(166)
We may obtain an expression involving $M^S_z S'_z$ from Eq. (49), rewritten as

$$
\langle \hat{r} | \psi \rangle \xrightarrow{r \to \infty} e^{i \hat{k} \hat{r}} \chi^S_z + \sum_{S'_z} M^S_z S'_z \chi^S_z \frac{e^{i \hat{k} \hat{r}}}{r}
$$

$$
+ \sum_{S'_z} i \left( M^S_z S'_z + M^S_z S'_z \right) \chi^S_z \frac{e^{i \hat{k} \hat{r}}}{r^2}
$$

(167)

Comparing (167) with the asymptotic expansion of $\langle \hat{r} | \psi \rangle$, obtained from Eq. (141) and normalization (148), we find

$$
M^S_z S'_z + M^S_z S'_z = \frac{\sqrt{4\pi}}{4i\kappa} \sum_{J, \ell, \ell'} \hat{J} \hat{J} e^{i(\ell-1)\ell'}(\ell' + 1) \left[ S^J_{\ell\ell'} - S^\ell_{\ell'} \right] \times
$$

$$
\langle S S'_z, S_z - S'_z | \ell' S J S_z \rangle \langle J S_z S \ell \ell \ell OS S_z \rangle Y^S_{\ell' \ell'}(\hat{\theta}, \hat{\phi})
$$

(168)

From definition (10), we write

$$
\overline{M}^S_z S'_z(k, \theta) \equiv M^S_z S'_z + M^S_z S'_z
$$

(169)

and combine Eqs. (156), (166), and (168) to obtain

$$
\overline{M}^S_z S'_z(k, \theta) = \frac{\sqrt{4\pi}}{2i\kappa} \sum_{J, \ell, \ell'} \hat{J} \hat{J} e^{i(\ell-1)\ell'} \left[ \ell' \left[ S^J_{\ell\ell'} - S^\ell_{\ell'} \right] \right] +
$$

(continued on next page)
\[-(l' + 1)[S^S_{l' l'} - \delta_{l' l'}] - 2(l' + 1)[S^S_{l' l'} - \delta_{l' l'}] \times \\
\times \langle S S'_{l} S_{z} S_{z} - S'_{l} S_{ll} S S'_{l} \rangle \langle S S'_{l} S_{ll} S_{ll} S S'_{l} \rangle \delta_{l, l'} n^S_{l' - S'} \]

(170)

Equation (170) gives us the desired result for a spin-dependent interaction. The \( \overline{M}_{z} z_{z}(k, \theta) \), expressed in S-T representation, is determined, via the scattering process, by the \( S^JS_{l' l'}(\delta) \), \( S^JS_{l' l'}(\beta) \), and \( S^JS_{l' l'}(\gamma) \). The notation differentiating the three matrices originates from the analogous role they play to the \( \exp 2i\delta_{l}, \)
\( \exp 2i\beta_{l}, \) and \( \exp 2i\gamma_{l} \) in the spinless case, given by Eq.(51).

It is evident that once the matrices \( S^JS_{l' l'} \) are obtained, then the \( \overline{M}_{z} z_{z}(k, \theta) \) are determined. As has been seen, the matrices \( S^JS_{l' l'} \)
were introduced to replace quantities of the form \( \frac{1 - ix}{1 + ix} \), \( \frac{1 - iy}{1 + iy} \), and \( \frac{1 - iz}{1 + iz} \). The X, Y, and Z matrices are readily found from the asymptotic behavior of the functions \( \chi_{J, l, l'}(r) \), \( \chi'_{J, l, l'}(r) \), and \( \xi_{J, l, l'}(r) \). The inverses \( [1 + iX]^{-1} \), \( [1 + iY]^{-1} \), and \( [1 + iZ]^{-1} \) exist and are determined by the normalization conditions for \( A, C, \) and \( D. \)

Our final task is to express \( \overline{M}_{z} z_{z}(k, \theta) \) in a form which describes the N-\( \Xi \) scattering process for the case where the target comprises an even-even nucleus with \( J = 0, \) and \( T = 0. \) Since \( \overline{M}(k, \theta) \) is evaluated OES, we may utilize the standard procedure to express \( \overline{M}_{z} z_{z}(k, \theta) \) as an explicit function of the spin and isotopic spin of the target nucleons, which when averaged results in
\[ \langle \mathbf{M}(k,\theta) \rangle = \langle \mathbf{A}(k,\theta) \rangle + \langle \mathbf{C}(k,\theta) \rangle \mathbf{\hat{n}} \cdot \mathbf{N} \]  \hspace{1cm} (171)

where

\[ \langle \mathbf{A}(k,\theta) \rangle = \frac{1}{4} (3 \mathbf{A}_1 + \mathbf{A}_0) \]  \hspace{1cm} (172)

\[ \langle \mathbf{C}(k,\theta) \rangle = \frac{1}{4} (3 \mathbf{C}_1 + \mathbf{C}_0) \]  \hspace{1cm} (173)

\[ A'_o = \sum_{\ell (\text{odd}) = 1}^{\infty} \frac{1}{16 \ell^2} \left\{ 5 (2 \ell + 1) [ \ell \Delta_{\ell, \ell} - (\ell + 1) B_{\ell, \ell} + \right. \]

\[ -2 (\ell + 1) \Gamma_{\ell, \ell} ] + 4 (2 \ell + 3) [(\ell + 1) \Delta_{\ell, \ell + 1} + \right. \]

\[ - (\ell + 2) B_{\ell, \ell + 1} - 2 (\ell + 2) \Gamma_{\ell, \ell + 1} ] + 4 (2 \ell - 1) [(\ell - 1) \Delta_{\ell, \ell - 1} \]

\[ - \ell \Gamma_{\ell, \ell - 1} - 2 \ell \Gamma_{\ell, \ell - 1} ] \right\} P_{\ell} (\theta) \]  \hspace{1cm} (174)
\[ C' = \sum_{\ell = 1}^{\infty} \frac{1}{4i|\ell|} \left\{ \frac{(2\ell+1)}{\ell} \left[ (\ell) \Delta_{\ell,\ell-1} - \ell B_{\ell,\ell-1} \right] \right\} \]

\[ -2 \ell \Gamma_{\ell,\ell-1} + \frac{(2\ell+1)}{\ell(\ell+1)} \left[ \Delta_{\ell,\ell} - (\ell+1) B_{\ell,\ell} - 2(\ell+1) \Gamma_{\ell,\ell} \right] \]

\[ + \frac{1}{(\ell+1)} \left[ (\ell+1) \Delta_{\ell,\ell+1} - (\ell+2) B_{\ell,\ell+1} - 2(\ell+2) \Gamma_{\ell,\ell+1} \right] \]

\[ \times P_{\ell}(\theta) \]

(175)

and

\[ \Delta_{\ell,\ell'} \equiv S_{\ell,\ell'}^{(s)} - \delta_{\ell,\ell'} \]

(176)

\[ B_{\ell,\ell'} \equiv S_{\ell,\ell'}^{(s)} - \delta_{\ell,\ell'} \]

(177)
\[ \Gamma_{l,l'} \equiv S_{l l'}^{js} - \delta_{l l'} \]
Appendix A: Green's Function Solution of Equations (28), (44),
(147), and (158)

(a) Spin-Independent Case.

Both Eq. (28) and Eq. (44) may be solved using a Green's function
defined by

$$\frac{d^2 G(r|r')}{dr^2} + \left[ k^2 - \frac{\ell(\ell+1)}{r^2} \right] G(r|r') = -\delta(r-r')$$

(179)

where

$$G(r|r') = -krr' \begin{cases} j_\ell(kr) \eta_\ell(kr') & ; r \leq r' \\ j_\ell(kr') \eta_\ell(kr) & ; r \geq r' \end{cases}$$

(180)

We may now write $\chi'_\ell(r)$ in terms of $G(r|r')$ as follows:

$$\chi'_\ell(r) = krj_\ell(kr) + 4k \int_0^\infty G(r|r') r' j_\ell(kr') dr'$$

$$-2ke^{i\delta_\ell} \cos \delta_\ell \int_0^\infty G(r|r') r' j_\ell(kr') dr' +$$

(continued on next page)
\[+ 2ke^{i\delta} \sin \delta \int_0^\infty G(r| r') r' \eta_{l}(kr') \, dr'\]

We proceed to evaluate the integrals in Eq. (181):
\[I_1 = \int_0^\infty G(r| r') r' j_{l}(kr') \, dr' = -kr \eta_{l}(kr) \int_0^r (r')^2 [j_{l}(kr')^2 \, dr'\]
\[= -kr \int_0^r (r')^2 j_{l}(kr') \eta_{l}(kr') \, dr'\]

(182)

\[I_2 = \int_0^\infty G(r| r') r' \eta_{l}(kr') \, dr' = -kr \eta_{l}(kr) \int_0^r (r')^2 [j_{l}(kr) \eta_{l}(kr')] \, dr'\]
\[= -kr \int_0^r (r')^2 [\eta_{l}(kr')]^2 \, dr'\]

(183)

We have
\[I_1' = \int_0^r (r')^2 [j_{l}(kr')]^2 \, dr' = \lim_{a \to 0} \int_0^r (r')^2 [j_{l}(kr')]^2 \, dr'\]
\[= \lim_{a \to 0} \frac{(r')^3}{4} \left[ 2 j_l^2(kr') - j_{l+1}(kr') j_{l-1}(kr') - j_{l-1}(kr') j_{l+1}(kr') \right]_a^r\]
\[= \frac{r^3}{2} \left[ j_l^2(kr) - j_{l+1}(kr) j_{l-1}(kr) \right] - \lim_{a \to 0} \frac{a^3}{2} \left[ j_l^2(ka) - j_{l+1}(ka) j_{l-1}(ka) \right]\]

(184)

We may evaluate the limit in Eq. (184) by using the following expansions for \(j_{l}(kr)\) and \(n_{l}(kr)\):
\[
\lim_{kr \to 0} j_{\ell}(kr) = \frac{1}{1 \cdot 3 \cdot 5 \cdots (2 \ell + 1)} (kr)^\ell
\]
(185)

\[
\lim_{kr \to 0} \eta_{\ell}(kr) = \frac{-1 \cdot 1 \cdot 3 \cdot 5 \cdots (2 \ell - 1)}{(kr)^{\ell+1}}
\]
(186)

We find

\[
I'_1 = \frac{r^3}{2} \left[ j_{\ell}^2(kr) - j_{\ell+1}(kr) j_{\ell-1}(kr) \right]
\]
(187)

Next we evaluate

\[
I''_1 \equiv \int_{r}^{\infty} (r')^2 [j_{\ell}(kr') \eta_{\ell}(kr')] dr' = \lim_{b \to \infty} \int_{r}^{b} (r')^2 [j_{\ell}(kr') \eta_{\ell}(kr')] dr'
\]
(188)

In order to evaluate the limit in Eq. (188), we utilize the following expansions for \( j_{\ell}(kr) \) and \( \eta_{\ell}(kr) \)

\[
j_{\ell}(kr) \xrightarrow{kr \to \infty} \frac{1}{kr} \cos[kr - \frac{\pi}{2} (\ell+1)]
\]
(189)

\[
\eta_{\ell}(kr) \xrightarrow{kr \to \infty} \frac{1}{kr} \sin[kr - \frac{\pi}{2} (\ell+1)]
\]
(190)

and find that
\[ I_1'' = C_1''(\infty) - \frac{r^3}{4} \left[ 2 j_l(kr) n_l(kr) - j_{l+1}(kr) n_{l+1}(kr) - j_{l-1}(kr) n_{l-1}(kr) \right] \]

(191)

where

\[ C_1''(\infty) \equiv \lim_{b \to \infty} \frac{b}{2k^3} \sin 2[kb - \frac{\pi}{2} l] \]

(192)

Proceeding with a similar evaluation of the remaining integrals, we obtain

\[ I_2' \equiv \int_0^r (r')^2 \left[ j_l(kr') n_l(kr') \right] dr' = C_2' + \]

\[ + \frac{r^3}{4} \left[ 2 j_l(kr) n_l(kr) - j_{l+1}(kr) n_{l+1}(kr) - j_{l-1}(kr) n_{l-1}(kr) \right] \]

(193)

where

\[ C_2' \equiv - \frac{(2l+1)}{4k^3} \]

(194)

and

\[ I_2'' \equiv \int_r^\infty (r')^2 \left[ n_l(kr') \right]^2 dr' = C_2''(\infty) + \]

\[ - \frac{r^3}{2} \left[ n_l^2(kr) - n_{l+1}(kr) n_{l-1}(kr) \right] \]

(195)
where
\[
C_2''(\infty) \equiv \lim_{b \to \infty} \frac{b}{2k} \cos 2[kb - \frac{\pi}{2}]
\]
(196)

We now substitute these results for the integrals in Eq. (181), and drop all multiples of \( r_j^k(\kappa r) \) and \( r_n^k(\kappa r) \), since they are simply solutions of the homogeneous equation. We obtain a particular solution in the form

\[
\tilde{\chi}(r) = r^2 \left[ (2 - e^{i\delta} \cos \delta_j) j_{\frac{\kappa - 1}{2}}(\kappa r) + (e^{i\delta} \sin \delta_j) n_{\frac{\kappa - 1}{2}}(\kappa r) \right]
\]
(197)

which is identical to Eq. (30). A similar procedure applied to Eq. (44) leads to Eq. (45).

(b) Spin-Dependent Case.

In order to obtain a particular solution for Eq. (147) for \( r > r_o \), we utilize a Green's function similar to that given in Eq. (180):

\[
G_{\kappa}(r | r') = -krr' \begin{cases} 
  j_{\frac{\kappa - 1}{2}}(\kappa r) n_{\frac{\kappa - 1}{2}}(\kappa r') & ; r \leq r' \\
  j_{\frac{\kappa - 1}{2}}(\kappa r) n_{\frac{\kappa - 1}{2}}(\kappa r) & ; r \geq r'
\end{cases}
\]
(198)

Therefore
\[ \chi'_{j, \ell}^{(r)} = kr j_{(kr)} + 4k \int_{0}^{\infty} G_{j}^j(r r') r' j_{(kr')} dr' + \]
\[ -2k \sum_{L'} A_{j}^{js} \delta_{L' L''} \int_{0}^{\infty} G_{j}^j(r r') r' j_{(kr')} dr' \]
\[ -2k \sum_{L'} A_{j}^{js} X_{L' L''}^{js} \int_{0}^{\infty} G_{j}^j(r r') r' \eta_{(kr')} dr' \]

(199)

The integrals in Eq. (199) have the same form as those involved in the spinless case, so that we may proceed to evaluate \( \chi'_{j, \ell, \ell''}^{(r)} \)
and, after eliminating all multiples of the solutions to the homogeneous equation, to obtain

\[ p \chi'_{j, \ell}^{(r)} = r^2 [ (2 - \sum_{L'} A_{j}^{js} \delta_{L' L''}) j_{(kr)} - \sum_{L'} A_{j}^{js} X_{L' L''}^{js} \eta_{(kr)}] \]

(200)

In a similar manner we find

\[ p \xi_{j, \ell}^{(r)} = r^2 \sum_{L'} A_{j}^{js} \left[ \delta_{L' L} j_{(kr)} + X_{L' L}^{js} \eta_{(kr)} \right] \]

(201)
Bibliography


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