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SOME MODELS FOR MULTISTAGE
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DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

by

Nicholas George Gionis, B.S., M.S.E.E.

* * * * *

The Ohio State University
1972

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I am indebted to my adviser, Dr. William T. Morris, whose guidance and contributions added immeasurably to this research. Dr. Morris introduced me to management theories and techniques of decision making and during the period of study under him, the need for multistage decision making models became apparent. The published works of and a private conversation with Dr. Amnon Rapoport stimulated my interest in a particular approach to multistage decision making models. Dr. Albert B. Bishop furnished me with the tools necessary to develop my models while Dr. James A. Wise introduced me to the work of Dr. Rapoport. Dr. R. E. Wendell made several valuable recommendations which led to an improved dissertation. My sincere appreciation and thanks go to each of the above named individuals.

Finally I wish to acknowledge the encouragement, support, and patience of my wife Argie. I am deeply grateful to her and to our son Kyle for their understanding during the period of this research.
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CHAPTER I
INTRODUCTION

Over the years much effort has been expended for research in the area of human decision making. The need for both normative and descriptive decision making models is apparent. It is perhaps true that the primary reason for retaining managers in business, economic, and military systems is to make decisions. One of the earliest models involved maximization of expected value. However, considering the fact that many people purchase insurance even though the seller makes a profit and many people buy lottery tickets even though the seller makes a profit caused some to search for models which explain these actions more satisfactorily than the expected-value model. Examination of the St. Petersburg paradox led Daniel Bernoulli to propose that people maximized some function which not only included expected value but also risk. His model is known as the expected-utility maximization model.

This dissertation is concerned with multistage or sequential decision making under risk which contrasts with single-stage decision making where single decisions or a series of independent decisions are made. The importance of
a multistage decision making theory is obvious upon examination of the tasks of a decision maker in real world systems. A single decision is seldom made in isolation and series of decisions are seldom made independently of each other. Nevertheless many theoretical studies of human decision making are concerned with single decisions or a series of independent decisions. Such static decision making theory can only serve as a first approximation to a realistic multistage process. In reality any decision is imbedded in a sequence of other decisions (Ray, 1963).

What is meant by a multistage or sequential decision making problem under risk? What distinguishes it from a nonsequential problem? In sequential problems, at least one decision will not be made until one or more of the stochastic parameters of the system have actually been observed. For example, the amount to invest during the second quarter of a fiscal year may not be decided upon until the returns from the first quarter are received. To make the best possible decision, it is necessary to observe values of the stochastic parameters before making the decision. This is certainly true if the outcomes at each stage are continuous, i.e., an infinite number of possible outcomes exist, or if the possible values of the stochastic parameters are unknown. However, in a discrete situation where the possible values of the stochastic parameters are known, a decision tree may be constructed where each branch indicates a
possible outcome from each stage. In this manner, it is possible to calculate the optimum decisions for the entire process contingent on the outcomes of the stochastic variables at each stage. Obviously if each stage contains many possible outcomes, the tree would grow very large rapidly and may become impractical.

Hadley (1964, page 175) states, "In solving sequential decision stochastic programming problems, one is normally interested only in the optimal initial decision. When it is time to make the second decision, things may have changed enough that it is desirable to revise the data used in the problem and solve it all over again. This will be done each time a decision is made." This author does not fully concur with Hadley's comment. For a discrete situation, it may be possible to solve the entire problem in the beginning using a decision tree and taking into account changes in capital and changes in probabilities. This procedure will be illustrated in Chapter II. Essentially it will give the optimal investment decision for the present stage contingent upon where the decision maker is located in the process, i.e., depending on the outcomes at the previous stages and the parameter values for the subsequent stages. Only in the case of exogenous changes may either complete or partial re-solving be required.

Proposed Multistage Investment Process

In the multistage process of this paper, the decision
maker begins with some initial capital and is required to allocate it over \( j \), \((j=1,2,\ldots,h)\), alternatives. Alternatives are defined as the prospective opportunities that are available for investment. These alternatives may be the various products under consideration in new product development, the stocks available in portfolio selection, or the various industries a corporation is considering for diversification. The alternatives have \( i \), \((i=1,2,\ldots,m)\), mutually exclusive outcomes, each of which occurs with probability \( p_{ijk} \), where \( p_{ijk} \geq 0 \) and \( \sum_{i=1}^{m} p_{ijk} = 1 \). Outcomes are defined as the possible values of the random variables of the stochastic investment process with which the decision maker is involved. The outcomes may be the rates of return on investment or the dividends from stocks. It is not required that all of the available capital be allocated at each stage. There are \( k \), \((k=1,2,\ldots,n)\), stages, which are defined as the units into which the investment process is divided. These stages may be quarters of a fiscal year or phase points in the development of new products. At a given stage \( k \), the decision maker may withhold part of his capital from investment, i.e.,
\[
\sum_{j=1}^{h} d_{jk} \leq x_k
\]
where \( d_{jk} \) is the amount allocated to alternative \( j \) at stage \( k \) and where \( x_k \) is the actual capital available at stage \( k \). If outcome \( i \) obtains at a given stage, the capital gain from alternative \( j \) is \( d_{jk} r_{ijk} \), where \( r_{ijk} \) is the
rate of return per unit invested from outcome i, alternative j, stage k.

Since at a given stage all the alternatives obtain, the total actual capital available for investment at the next stage is \( d_{ijk} \) for all outcomes i that obtain summed over all the alternatives plus \( \sum_{j=1}^{h} d_{jk} \), the stage investment, plus \( x_k - \sum_{j=1}^{h} d_{jk} \), the amount not invested. Note that all alternatives obtain, but only one outcome per alternative may obtain. The total expected capital available for investment at the next stage is \( \sum_{j=1}^{h} \sum_{i=1}^{m} d_{ijk} p_{ijk} \), the expected capital gain, plus \( \sum_{j=1}^{h} d_{jk} \), the stage investment, plus \( x_k - \sum_{j=1}^{h} d_{jk} \), the amount not invested. Since the actual capital available for stage k is not known until the previous stage has taken place, the actual capital available for stage k is assumed to be the expected value of capital from the previous stage. If this assumption is not made, then both sums stated above are expected values. For the purpose of this paper, this assumption will be made. That is, the amount of capital available at each stage, except the first stage, will be the expected value of capital available at the output of the preceding stage. The use of this expected value in place of the random variables will be discussed further in Chapter II.
Criterion

The criterion used in the model development of this paper is the minimization of risk, where risk is measured by the variance of the final capital, subject to a constraint on the expected value of capital at the last stage. The constraint on expected value takes the form of an expectation or aspiration level for the entire process. The initial capital for the n-stage process is $x_n$. The decision maker aspires to multiply his initial capital by some factor $K$, ($K \geq 1$), which may be called his expectation level. Note that this constraint states that the expected value of capital at the conclusion of the n-stage process is equal to $Kx_n$. In addition, it should be noted that this constraint states nothing about an expectation level at each stage of the process.

In summary, in the sequential process of this paper, the decision maker is not required to invest all of his capital at each stage (since one of the alternatives may be a cash holding, this assumption should offer no problem), the rates of return are stochastic variables, and the distributions of returns are assumed independent.

Scope

The theory presented herein is normative theory and although under some circumstances it may also be descriptive, this can only be a hypothesis as far as this paper is concerned. Several multistage stochastic decision making
processes will be presented. Information obtained during the sequence of stages allows for revision of the probabilities. Models employing such revision are known as adaptive decision making models. The models to be presented in this paper are:

1. Two outcome, multistage model - Chapter II.
2. Multiple outcome, multistage model - Chapter III.
3. Multiple outcome, multiple alternative, multistage model - Chapter IV.

The first two models will include adaptive decision making and each chapter will include a numerical example. For the two-outcome, multistage model and for the multiple outcome, multistage model, explicit solutions (expressions) for the optimal investment policy will be derived, whereas for the multiple outcome, multiple alternative, multistage model, only a computational procedure will be derived.

This chapter will include a brief description of some related literature and the final section will present a brief description of the optimization techniques that will be used in Chapters II, III, and IV. The techniques to be used include dynamic programming and convex programming. Chapter V will present some comparisons of other models to the models proposed in this paper. Chapter VI will apply the models developed in Chapters II, III, and IV to bidding and new product development. Chapter VII will include a summary and some recommended research topics.
Related Literature

The multistage process, the criterion, and the constraints presented in this paper are similar to both the multistage betting game (MBG) proposed by Rapoport (1970) and to portfolio analysis proposed by Markowitz (1959). However, there are differences, particularly with the process in Rapoport's MBG, which will be discussed below. In the portfolio problem, the decision maker is assumed to allocate all of his capital over the h securities (called alternatives in the formulation of this paper) at each stage, all the alternatives obtain at each stage, the returns \( r_{ijk} \) are stochastic variables, and the distributions of returns may be correlated. In the MBG, the decision maker is not required to allocate all of his capital over the h alternatives which are mutually exclusive, each alternative has only one outcome, only one alternative may obtain at each stage, and the rate of return for each alternative is a stochastic variable that is fixed over the stages of the process.

Markowitz suggests two criteria in the portfolio selection problem - maximization of expected utility and minimization of risk subject to a constraint on the expected value of return. He suggests three solution techniques for portfolio analysis - a dynamic programming approach to maximization of expected utility, single period utility analysis, and efficient set analysis. Efficient set analysis, the major topic of Markowitz's monograph, involves the use of critical lines,
where a point on these lines minimizes the variance among portfolios with the same expected return, and the use of efficient sets, where membership in these sets implies that there exist no portfolios with expectation-variance combinations such that the expectation is greater while the variance is equal to or less than the variance of the efficient portfolio or the variance is less while the expectation is greater than or equal to the expectation of the efficient portfolio. Markowitz prefers efficient set analysis, in which the computations are done by quadratic programming, to the maximization of expected utility. He shows that whenever an investor's utility function can be reasonably approximated be a quadratic utility function, one of the portfolios which minimizes variance for some value of expected value of return provides almost the maximum obtainable expected utility. In other words, both methods, efficient set analysis and expected utility maximization, give approximately the same solution.

Likewise in the multistage investment processes considered in this paper, two approaches, expected utility maximization and minimization of variance for some value of expected return, will give almost the same results for a single stage process. This equivalence will be discussed in Chapter V. Why use one technique rather than the other? Markowitz (1959) states on page 291 of his monograph: "The investor, the investment manager, or someone else who can
speak definitively concerning the objectives of the investor must answer questions concerning preferences among probability distributions of return. If the respondent has little or no experience with matters such as the theory of rational behavior, he may consider such questions a queer way to go about selecting a portfolio. Afterward, when the optimum portfolio is finally produced, we can tell the investor only that his portfolio is best because it best suits his utility function. This approach will probably have less immediate meaning and intuitive appeal for him than an analysis in which the investor is shown combinations of 'risk' and 'return' and is then asked to pick carefully the combination which best suits his needs. Choosing a combination of risk and return is a more natural procedure than expressing attitudes toward risk in terms of a utility function . . . " The multistage processes introduced in this paper, particularly the multiple outcome, multiple alternative, multistage process of Chapter IV, may be called a multistage generalization of Markowitz's model.

Rapoport (1970) assumes that the expectation level is the same for each stage in his MBG model. Recall that the criterion of this paper makes no reference to individual stage expectation levels.

Two processes, multistage betting games (Rapoport, 1970) and portfolio selection (Markowitz, 1959), have been described thus far and have been compared to the processes
of this paper. There are obviously many additional models which incorporate expectation and risk in multistage investment processes. The factors that differentiate these models from each other are the investment process itself, the objective function, the various constraints, and the manner in which risk is measured. Several models will be discussed below with emphasis on the similarities and differences to the models and methodologies proposed in this paper.

Markowitz's monograph has served as a standard for portfolio selection, with many researchers extending and refining his original model. His efficient set may consist of many portfolios and, in the absence of any information about the investor's preference structure, it cannot be determined which portfolio is preferred. Baumol (1963) has devised a method which reduces the size of Markowitz's efficient set by investigating a confidence interval about the expected value of return. Consider, for example, the following two portfolios (Baumol, 1963):

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<tr>
<td>Expected Value (E)</td>
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</tr>
<tr>
<td>Standard Deviation ((\sigma))</td>
<td>2</td>
</tr>
<tr>
<td>(E + \sigma)</td>
<td>10</td>
</tr>
<tr>
<td>(E - \sigma)</td>
<td>6</td>
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On the basis of Markowitz's \((E, \sigma)\) criterion, a choice between portfolios A and B is not possible. However, by investigating Baumol's \((E, k\sigma)\) criterion, portfolio B is superior since the
lower confidence limit for the investor's return, \( E - \sigma \)
(where \( k=1 \)), is greater than portfolio A's upper confidence
limit, \( E + \sigma \). An increase in \( E \) may more than counter-
balance an increase in \( \sigma \). Despite greater variability in the
return from the portfolio with the larger expected value, it
may be considered relatively safe because the lower confi-
dence limit is relatively high. There remains the problem
of choosing a value for \( k \). If the distribution of returns
is known, then the probability that the return is lower than
the lower confidence limit may be computed. If the distribu-
tion is not known, then Chebyshev's inequality may assist in
choosing a value of \( k \). The investor's attitude toward risk
manifests itself in the value of \( k \). It can be shown that
Baumol's efficient set is a subset of Markowitz's efficient
set.

Markowitz's model, which does not assume stochastic
independence among securities, involves quadratic programming
problems which are solved by the critical line method.
However, the analysis requires a large number of comparisons,
the number of which depends on the size and complexity of the
variance-covariance matrix of the securities. Sharpe's
(1963) diagonal model reduces the computational effort
required in such comparisons and evidence exists that a
large part of the security interrelationships are captured,
i.e., the existence of interrelationships among the securi-
ties is not overlooked. The returns of the various
securities are related only through common relationships with some basic underlying factor such as the level of the stock market, the Gross National Product, or some other factor thought to be the single most important influence on the returns from securities. Sharpe assumes that the covariance of the stochastic portion of the common relationship is zero for any two securities. It is therefore possible to write the variance of the returns of the securities in terms of the common relationship with the only non-zero terms of the variance-covariance matrix on the main diagonal. This means that the covariance terms are zero, the variance-covariance matrix is less complex, and the computational effort required for solving the problem is greatly reduced.

In the Markowitz formulation, the expected value and variance of return from any security are subjective estimates based on one's forecast of environmental factors such as future business conditions, monetary and fiscal policies, and economic and political developments. Mao and Sarndal (1966) suggest that a model of portfolio selection is clearly superior if it forces the analyst to make explicit his views regarding these factors and their subsequent revision. They reformulate the Markowitz model by applying Bayesian analysis to the subjective estimates. A similar Bayesian probability revision is incorporated in the models contained in this paper.

Cord (1964) developed a model for prescribing allocation
of funds to investment projects when the returns are uncertain. He assumes that investment funds are limited, the sum of the cost of the projects exceeds the investment budget, the returns are stochastic, and the investments are ranked according to interest rate of return, which is defined as that rate of interest which makes the present value of the cash proceeds expected from an investment equal to the cost of the investment. Cord maximizes the total return on investment subject to the following constraints: the total funds available for investment will not be exceeded and the average variance for total investment will not exceed some preassigned level. It can be shown that this criterion is equivalent to the minimization-of-risk-subject-to-a-preassigned-expected-value criterion of this paper. However, Cord applies his criterion to a capital budgeting process rather than to a portfolio selection process. There are two major differences in these processes. In capital budgeting, a project requires a given amount of funds and it is either totally funded or not funded at all, whereas partial funding occurs in portfolio selection. Consequently the decision variables in capital budgeting are either one or zero. Outlays for capital budgeting may stretch out over more than one budget period, while in portfolio selection the funds are realized at the conclusion of each period. Cord uses the knapsack or flyaway-kit dynamic programming technique to solve his allocation problem.
A slightly different approach to capital budgeting by Klevorick (1966) assumes that gross and net cash inflows are stochastic while cash outlays for projects are deterministic. A given amount of funds are available for the projects in each period and returns from a project are not available for use in succeeding periods. Klevorick maximizes an expected multiperiod utility function subject to the opportunities available and some budget constraints. The utility function used is management's perception of the utility to owners of consumption alternatives available in different periods.

Two papers (Naslund and Whinston, 1962; Agnew, Agnew, Rasmussen, and Smith, 1969) maximize expected value subject to probabilistic risk constraints. The technique used to solve this optimization problem was developed by Charnes and Cooper (1959) and is appropriately called Chance Constrained Programming.

The criteria for the Naslund and Whinston model is the maximization of expected gain in the stock market subject to a risk constraint that losses must be stochastically less than a specified amount and a capital constraint that stipulates that invested capital should be below a limit which varies according to accumulated capital gains.

The Agnew, Agnew, Rasmussen, and Smith criterion of optimization is maximization of expected gain. The objective is to determine an optimal portfolio which maximizes
\[ E \left[ \sum_{i=1}^{n} (R_i + d_i)x_i \right], \]

where \( R_i \) is the percentage increase in value for asset \( i \) over the period based on market values at the beginning of the period, \( d_i \) denotes the dividend or fixed percentage due from asset \( i \) during the period, and \( x_i \) is the decision variable.

A generalized risk chance constraint specifies that the end of period gain will fall below a given value \( \gamma \) with a probability not to exceed \( \alpha \), i.e.,

\[ \Pr \left[ \sum_{i=1}^{n} (R_i + d_i)x_i < \gamma \right] < \alpha. \]

Both of these chance constrained problems may be written in the general form of

\[
\begin{align*}
\text{maximize} & \quad E \left[ \sum_{j=1}^{n} c_jx_j \right] \\
\text{subject to:} & \quad \Pr \left[ \sum_{j=1}^{n} a_{ij}x_j \leq b_i \right] \geq \alpha_i, \quad i=1,2,\ldots,m \\
& \quad \text{all } x_j \geq 0.
\end{align*}
\]

where there are \( i \) chance constraints and where \( c_j \), \( a_{ij} \), and \( b_i \) are all normally distributed random variables. This problem can be converted to an equivalent nonlinear deterministic problem (Kataoka, 1963) which is often a convex program (when \( \alpha_i \geq 0.5 \)).

A very nice feature of chance constrained programming concerns dual variables which reflect how the expected value varies as a person alters his views on the degree of risk to
which he is willing to expose himself (Naslund and Whinston, 1962). By altering the constraints, the exact implications of changes in the objective function are demonstrated.

A somewhat different approach to sequential investments under uncertainty is contained in a paper by Kaufman (1963). His analysis gives decision rules for deciding when to accept or reject investment prospects that appear sequentially. His decision rule is to accept the current deal only if the sum of its payoff and the expected value of having a reduced amount of capital (reduced by the amount invested in the current deal) is greater than the expected value of having the total budget to invest in the remaining deals. Kaufman introduces eight versions of his model where the payoffs, the investments required by a deal, and the number of deals are either certain or random variables.

Kaufman's decision rules depend only on expected value and, in general, it has been illustrated that expectation-dependent decision rules are not sufficient to describe and/or prescribe a completely satisfactory policy. For example, at a given stage the expected value may be very promising and, in accordance with Kaufman's rule, the decision is made to invest. However, one of the possible outcomes may mean disaster for the firm. In other words, risk (or variance of return) should also be considered. A decision rule which substitutes expected utility and utility
of payoff for expected value of payoff and payoff may be more valid.

An excellent survey paper by Weingartner (1966) investigates various computational techniques of capital budgeting problems and emphasizes the often neglected project interrelationships and interdependencies.

Optimization Techniques

Dynamic Programming for Multistage Decision Making Problems

A multistage decision making process, one with more than one stage, contrasts with single stage decision making where the decision maker makes a single decision or a series of independent decisions. Upon analysis of the tasks of a decision maker in the real world, the importance of sequential or multistage decision making becomes quite obvious. His job may be characterized by the responsibility of having to make a sequence of interrelated decisions where the outcomes of earlier decisions affect and serve as inputs to later decisions. The stages may be discrete or continuous, although in this paper they will be restricted to the discrete category.

At the beginning of the first stage, the decision maker obtains the initial state or condition of the system and makes his first decision, which results in a new state for the beginning of the next stage. The new state depends on three factors - the initial state, the decision, and one or more stochastic variables. The objective is to make a
sequence of decisions which optimizes some criterion, which has been determined by the decision maker, for the entire process. In general, the optimum thus obtained differs from one obtained by optimizing the criterion function for each stage independent of every other stage and of the system as a whole.

Dynamic programming is an iterative technique for finding optimal decisions for multistage decision processes. It takes a multistage decision process containing many interdependent variables and converts it into a series of single stage problems, each containing only a few variables (Nemhauser, 1966). It is based on Bellman's (1957) dynamic programming principle of optimality which states that "an optimal set of decisions has the property that whatever the first decision is, the remaining decisions must be optimal with respect to the outcome which results from the first decision." Once the dynamic programming formulation is achieved, some optimization technique must be employed in order to determine a solution. One such technique is described in the next section.

Definition of Terms

(1) Stage. A stage is an arbitrary unit in time or space or anything else into which the process is divided. Stages may occur naturally or may be a mathematical artifact which allow the decision maker to make decisions one at a time. By breaking
an entire problem into an equivalent series of subproblems, a difficult problem may be solved more easily.

(2) State. Each stage has two types of states - an input and an output. The output of one stage serves as the input to the succeeding stage. As the names imply, the input state describes the system at the beginning of the stage and the output state describes the system at the end of the stage. In this paper, the state variables are scalars rather than vectors and are denoted by $x_k$ where $k$ indicates the number of stages remaining. For an $n$-stage system, $x_n$ is the initial state of the system and $x_0$ is the final state. Except for the initial state, the state variables are controlled by the decision maker by means of his decisions.

(3) Decisions. Decisions, denoted by $d_k$, control the operation and nature of each stage $k$ and of the entire system.

(4) Stage Transformation. State transformations or transitions or couplings express the output state as a function of the input state, and stage decisions, and the random variables.

(5) Criterion Function. This function serves as a device for evaluating the different decisions in an optimization problem. It normally depends on the
state variables and the decisions for the entire process. Examples include profit, cost, variance, or expected value.

(6) Policy. The policy consists of the set of decisions. A policy that optimizes the criterion function is called an optimal policy.

(7) Stage Returns. A return is a scalar quantity that measures the performance of a stage and is a function of the stage input and stage decision, i.e., \( r = r(x, d) \). The stage returns are somehow combined to give the system return function.

Figure 1 illustrates a multistage decision process. The (\(n-2\))nd state variable, \( x_{n-2} \), is a function of \( x_{n-1} \) and \( d_{n-1} \). The functional relationship is the (\(n-1\))st stage transformation.

Figure 1. Dynamic Programming Formulation of a Multistage Decision Process
Dynamic Programming Procedures

The organization and content of the following material follow closely that of Rapoport (1967) except for the notation, which has been changed somewhat to coincide with the notation used later in this paper.

Consider an n-stage decision making process where the state of the system for stage k is \( x_k \), the decision is \( d_k \), and the transformation function is \( t_k(x_k, d_k) \). As stated earlier, stage \( k \) denotes that \( k \) stages remain in the process. The transformation equation is written as

\[
x_{k-1} = t_k(x_k, d_k), \quad k = n, n-1, \ldots, 1.
\]

Assume that the total return for this process is

\[
Z = \sum_{k=1}^{n} g_k(x_k, d_k)
\]

where \( g_k(x_k, d_k) \) is the return from stage \( k \). Let the criterion be the minimization of the total return function of the process, i.e.,

\[
f_n(x_n) = \min_{d} \sum_{k=1}^{n} g_k(x_k, d_k).
\]

This equation gives the minimum of the total return function beginning in stage \( n \) and using an optimal policy. Applying the principle of optimality yields the following:

\[
f_n(x_n) = \min_{d} [g_n(x_n, d_n) + f_{n-1}(x_{n-1})].
\]

Recall that \( f_{n-1}(x_{n-1}) \) is optimal with respect to the outcome that results from the first decision, i.e., \( x_{n-1} \). Also recall that \( x_{n-1} = t_n(x_n, d_n) \), therefore \( f_n(x_n) \) may be written as
\[ f_n(x_n) = \min_{d} \left[ g_n(x_n, d_n) + f_{n-1}(x_n, d_n) \right] \]

Continuing in a similar fashion,
\[ f_{n-1}(x_{n-1}) = \min_{d} \left[ g_{n-1}(x_{n-1}, d_{n-1}) + f_{n-2}(x_{n-2}) \right] \]
\[ \vdots \]
\[ f_1(x_1) = \min_{d} \left[ g_1(x_1, d_1) \right] \]

Note how the equations given above decompose the \( n \)-stage process into the current stage and all remaining stages.

Procedurally solve for \( f_1(x_1) \) in terms of \( x_1 \) and \( d_1 \), then for \( f_2(x_2) \), where
\[ f_2(x_2) = \min_{d} \left[ g_2(x_2, d_2) + f_1(x_1) \right] \]

Continue until the first stage, stage \( n \), where \( d_n \) can be solved for explicitly since \( x_n \) is specifically known. Then by backsolving, \((x_{n-1}, d_{n-1}), (x_{n-2}, d_{n-2}), \ldots, (x_2, d_2), (x_1, d_1)\) can be determined. These solutions yield the optimal \( n \)-stage return, \( f_n(x_n) \), and the optimal decisions \( d_k^* \), \( k=1,2,\ldots,n \). The case presented above is deterministic whereas the case that follows is stochastic.

In the stochastic case, the decision \( d_k \) does not determine a unique transformation, \( t_k(x_k, d_k) \), but rather a distribution of outcomes depending on a stochastic variable \( e \) with a known probability distribution \( h(e) \). The set of transformations is written as
\[ x_{k-1} = t_k(x_k, d_k, e_k) \]

The stochastic variables are assumed independent and identically distributed with finite means and variances.
This assumption is necessary in order to solve a multistage decision problem by dynamic programming. However, this is not the primary reason for assuming independence. There exist many important investment processes for which independence exists or may be assumed with little adverse effect on the final optimal solution. Nevertheless since in many processes alternative and stage dependencies do exist, it will be recommended that a subsequent study be performed to apply the methodologies of this paper to the case where dependencies may not be disregarded. An approach such as that by Sharpe (1963) may be useful for such a future study. Recall that Sharpe's diagonal model assumes that the random variables \( r \), the rates of return on investment, are related only through some common relationships with some basic underlying factor and that the covariance between alternatives and stages, when \( r \) is written in terms of this common relationship, is zero. The methodologies of this paper may be applied to this modified problem while avoiding the complexities of the covariance terms. Determining the common factor to be used so that a major portion of the dependencies between the alternatives and stages are captured is an empirical problem that must be solved. This modified model would be a worthwhile enrichment of the model of this paper and in the case of independence, it would degenerate into the model presented in this paper.
The recursive equations are written as

\[ f_n(x_n) = \min_D E \left[ \sum_{k=1}^{n} g_k(x_k, d_k, e_k) \right] \]

where \( E \) is the expected-value operator. Likewise,

\[ f_n(x_n) = \min_D E \left[ g_n(x_n, d_n, e_n) + f_{n-1}(t_n(x_n, d_n, e_n)) \right] \]

\[ \ldots \]

\[ f_1(x_1) = \min_D E \left[ g_1(x_1, d_1, e_1) \right]. \]

This decomposition assumes two sufficient conditions - separability and monotonicity (Nemhauser, 1967, Chapter II). The problems worked in this paper are decomposable.

It can be seen that apart from the introduction of expected values, the stochastic problem is the same as the deterministic problem, provided the probability distribution \( h(e) \) is known or can be assumed.

**Convex Programming**

At each stage of a dynamic programming problem, an optimization must be performed. In some of the models that follow, the stage optimization may be simply performed by differential calculus. However, in the multi-outcome, multi-alternative, multistage model, the stage optimization is performed using convex programming, and therefore a few brief remarks concerning this technique will be made. The reference for these remarks is Chapter III of the text *Nonlinear Programming* by Kuenzi and Krelle (1966).

Let \( F(x) \) and \( f_j(x) \), \( j=1,2,\ldots,m \), be convex functions of
n variables, \((x_1, x_2, \ldots, x_n) = \bar{x}'\). A convex program has the following form:

\[
\begin{align*}
F(\bar{x}) \text{ is to be minimized under the constraints} \\
&f_j(\bar{x}) \leq 0, \quad j = 1, 2, \ldots, m; \\
&\bar{x} \geq \bar{0}.
\end{align*}
\]

A function is said to be convex over a convex set \(X\) in \(E^n\) if for any two points \(x_1\) and \(x_2\) in \(X\) and for all \(\lambda\),

\[
0 \leq \lambda \leq 1,
\]

\[
f[\lambda x_2 + (1-\lambda)x_1] \leq \lambda f(x_2) + (1-\lambda) f(x_1).
\]

The function \(F(\bar{x})\) to be minimized is called the objective function and the functions \(f_j(\bar{x})\) are the constraints. The Kuhn and Tucker theorem allows generalization of Lagrange's classical multiplier method to the determination of extrema under constraints that may include inequalities. A generalized Lagrange function \(\Phi\) is constructed by introducing \(m\) new variables, \(\lambda_1, \ldots, \lambda_m\), called Lagrange multipliers. \(\Phi\), a function of \((m+n)\) variables \((\bar{x}, \bar{\lambda})\), is written as:

\[
\Phi(\bar{x}, \bar{\lambda}) = F(\bar{x}) + \sum_{j=1}^{m} \lambda_j f_j(\bar{x})
\]

If \(F(\bar{x})\) and \(f_j(\bar{x})\) are differentiable functions, then a vector \(\bar{x}\) which satisfies the following Kuhn-Tucker conditions represents a solution to convex programming problem Pl.
\[
\begin{align*}
(\frac{\partial f}{\partial x_1}) & \geq 0, \\
(\frac{\partial f}{\partial x_1}) & = 0 \quad \text{for } i=1,2,\ldots,n \\
x_i & \geq 0
\end{align*}
\]

\[
\begin{align*}
(\frac{\partial f}{\partial \lambda_j}) & \leq 0, \\
(\frac{\partial f}{\partial \lambda_j}) & = 0 \quad \text{for } j=1,2,\ldots,m \\
\lambda_j & \geq 0
\end{align*}
\]

If, however, \(f_j(x)\) is linear, as it is in the multi-outcome, multi-alternative, multistage model, then a constraint of the form \(f_j(x) = 0\) is also possible. That is, the convex programming problem may be stated as:

\[
F(x) \text{ is to be minimized under the constraints}
\]
\[
f_j(x) = 0, \quad j=1,2,\ldots,m;
\]
\[
x \geq \bar{0}.
\]

The linear equality constraints may be written as:

\[
f_j(x) \leq 0 \text{ and } -f_j(x) \leq 0.
\]

Replacement of the linear equality constraints by the two equivalent linear inequality constraints transforms the convex programming problem stated directly above into the convex programming problem stated at the beginning of this section. The Kuhn-Tucker conditions reduce to the following form:
\[
\begin{align*}
\left( \frac{\partial f}{\partial x_i} \right) x, \lambda & \geq 0 \\
\frac{x_i \left( \frac{\partial f}{\partial x_i} \right) x, \lambda = 0 }{ } & \text{for } i=1,2,\ldots,n \tag{4} \\
x_i & \geq 0 \\
\left( \frac{\partial f}{\partial x_j} \right) x, \lambda = 0 & \text{for } j=1,2,\ldots,m.
\end{align*}
\]

This is true since \( \left( \frac{\partial f}{\partial x_i} \right) x, \lambda \) is merely \( f_j(x) \), which is identically zero. Likewise \( \lambda_j \left( \frac{\partial f}{\partial x_j} \right) x, \lambda_j = 0 \) since the term in parentheses is zero. The multiplier \( \lambda_j \) is now no longer restricted with respect to sign.

In order to obtain the optimum (minimum in this case) solution, equations (4) must be solved for \( \hat{x} \). Since this involves solving \((m+n)\) equations, which are not necessarily linear, for \( \hat{x} \) and \( \hat{\lambda} \), considerable work may be required.
CHAPTER II

TWO-OUTCOME MULTISTAGE DECISION MAKING MODEL

The first investment process to be considered in this paper is a two-outcome multistage decision making process, where the amount of capital invested at each stage is determined by the decision maker (DM). Each of DM's decisions regarding the outcome of a risky event affects the size of his capital on subsequent occasions and therefore determines the amount of capital subsequently available for investment.

Consider the following sequential investment process: An investor is presented with a sequence of \( n \) investment opportunities where each has two possible outcomes. One of the outcomes will be called success and the other failure. The opportunities will be the stages in the dynamic programming formulation and the outcomes are assumed probabilistically independent over the stages. At each stage \( i \), DM is allowed to invest a quantity of capital \( d_i \) subject to the restriction that \( 0 \leq d_i \leq x_i \), where \( x_i \), \( i=1,2,\ldots,n-1 \), denotes the expected amount of capital available for the \( i \)th stage and \( x_n \) denotes the amount of his initial capital. If the outcome of the risky event at stage \( i \) is successful, DM has an expected amount of capital available for the next stage equal to \( r_i d_i + d_i + (x_i - d_i) \), where \( r_i \) is the rate
of return (dollar return per dollar invested) for success at stage $i$. The first term above is the capital gain from the investment, the second term is the capital invested, and the third term is the expected capital remaining in cash. Likewise if the outcome is not successful, $DM$ has an expected amount of capital available for the next stage equal to 

$$-s_i d_i + d_i + (x_i - d_i),$$

where $-s_i, s_i \geq 0$, is the rate of return for failure at stage $i$. The amount not invested is considered a cash reserve, although with slight modification it could be invested at some interest rate. The expected amount of capital available for investment at stage $(i - 1)$ is 

$$p_i[r_i d_i + d_i + (x_i - d_i)] + q_i[-s_i d_i + d_i + (x_i - d_i)]$$

or 

$$x_i + d_i(p_i r_i - q_i s_i),$$

where $p_i$ is the probability of success at stage $i$, that is, that the rate of return will be $r_i$, and $q_i, q_i = 1-p_i$, is the probability of failure at stage $i$, that is, that the rate of return will be $-s_i$. Knowing the values of $p_i, q_i, r_i, s_i$, and $x_n$, how should $DM$ invest? Before this question is answered, two important matters will be discussed. The first concerns the use of expected value of capital as the amount of capital available at each stage, except the first stage where actual capital is used. Expected value of capital is used because the actual capital that will be available at the following stage is not known with certainty until the present stage has taken place. However under certain restrictions, the use of expected values in place of independent random variables will give
correct results with much computational savings. These restrictions are satisfied by the processes presented in this paper. More will be said about the replacement of random parameters by their expected value later in this chapter.

The second matter concerns the restriction on the amount of capital $d_1$ invested at each stage. This restriction, $0 \leq d_1 \leq x_1$, states that the capital to be invested is less than the expected value of capital. It may be asked what will occur if the amount prescribed to be invested is less than the expected value of capital but greater than the actual capital available. This is a legitimate concern if the optimal investment policy obtained from the process was actually used at all stages. However, as will be discussed later in this chapter, only the initial optimal investment will be made and the constraint, $0 \leq d_n \leq x_n$, restricts $d_n$ to be less than or equal to the actual initial capital, $x_n$. The remaining investment variables, $d_{n-1}, d_{n-2}, \ldots, d_2, d_1$, are used only for planning purposes. At the conclusion of this initial stage, stage $n$, the amount of capital available for stage (n-1) is known with certainty. The entire problem will then be re-solved for the (n-1)-stage process with the restriction that $0 \leq d_{n-1} \leq x_{n-1}$, where $x_{n-1}$ is the actual initial capital for a (n-1)-stage process. This re-solving will take place at each stage throughout the entire process; therefore, for the decision variable that is to be
implemented, it will be true that the optimal investment will be less than or equal to the actual capital available for investment at that stage.

Returning to the question of how the DM should invest, the problem is to choose a criterion, and an investment policy that is optimal with respect to this criterion and that prescribes to the DM the amount to invest at each stage. In stochastic processes, the criterion must be defined to recognize that the state of the process is a random variable. Thus, in a stochastic process, the DM might desire to (Murphy, 1965):

(1) achieve a maximum expected value with a given permissible level of risk due to the uncertainty of the process, or

(2) achieve a given expected target value with a minimum level of risk.

Many processes can be formulated in terms of one of the criteria stated above while in other processes, less conventional objectives or criteria may be required. In any case, an important task that a DM must face is to establish a criterion which incorporates some strategy to handle the uncertainty of the stochastic process. Next, the DM must determine what should enter into the evaluation of his criterion. If, for example, his criterion is the minimization of risk while achieving a given expected value, then the variables that measure expected value and risk must be
identified and measured. These variables become part of the state function of the process. In many economic problems, the state function will have only one element - utility (Murphy, 1965). Several criteria have been chosen and the optimal investment policies have been determined for this two-outcome investment process.

**Expected Value Maximization**

It can be shown (Appendix A, Part 1) that if the DM wishes to maximize the expected value of his capital at the end of n stages, then his policy is

\[
d_i = \begin{cases} 
  x_i & \text{if } p > 1/2 \\
  0 & \text{if } p < 1/2 
\end{cases}
\]

where \( x_i \) is the actual capital available at stage \( i \) and \( i=1,2,\ldots,n \). This policy states that the DM must invest all the capital available at the stages whose probability of success is greater than 1/2. If both outcomes at a stage have a probability of success of 1/2, then he may invest any portion of his capital. However, even though this policy does maximize the expected value, it also increases the DM's probability of loss of his entire capital. For a one-stage process, this policy results in a probability of bankruptcy of \( 1-p \); for a two-stage process in which the probability of success remains constant, the probability of bankruptcy is \( 1-p^2 \); for an \( n \)-stage process, the probability of bankruptcy is \( 1-p^n \). As \( n \) increases, the probability of bankruptcy approaches one quickly. For example, if \( n = 5 \) and \( p = 3/4 \),
the probability of bankruptcy is 0.7627. Since survival is probably highly desirable, the DM will not likely use the expected-value-maximization criterion. He will use some criterion which prescribes investment of capital in such a way as to maintain the expected capital from his investments at a satisfactory level while keeping the probability of bankruptcy low (Rapoport and Jones, 1970). One approach which includes consideration of risk, which is measured by the variance of the returns from the investment, is maximization of expected utility of capital.

This criterion has been investigated by many, including Bellman and Kalaba (1957, 1958), Breiman (1961), Kelly (1956), and Murphy (1965), for a logarithmic utility function. Bernoulli suggested that the same proportionate addition to an amount of wealth carries to an individual the same absolute addition of utility at all values of absolute level of wealth to which this addition is accruing (Rapoport and Jones, 1970). This suggestion is equivalent to postulating that utility is equal to the logarithm to any base of the amount of wealth.

Expected Utility Maximization

Define \( f_n(x_1) \) as the maximum expected value of the logarithm of the capital after \( n \geq 1 \) stages, where \( x_1 \) is the initial capital available for investment. The rate of return for success is identical at each stage and is denoted by \( r \), the rate of return for failure is identical at each stage
and is denoted by \(-s\), the capital available for stage \((i-1)\) is \(x_i + r d_i\) if successful at the preceding stage and it is \(x_i - s d_i\) if not successful at the preceding stage. In addition, it is assumed that \(p > q\) and \(r\) and \(s > 0\). It can be shown (Appendix A, Part 2) that

\[
f_n(x_i) = \log x_i + n[p \log p + q \log q + p \log \frac{r+s}{s} + q \log \frac{r+s}{r}]
\]

and that the optimal investment policy is given by

\[
d_i = \begin{cases} 
0 & \text{if } pr-qs < 0 \\
\frac{pr-qs}{rs} x_i & \text{if } 0 \leq pr-qs \leq rs \\
x_i & \text{if } pr-qs > rs.
\end{cases}
\]

An examination of the optimal investment policy \(d_i\) reveals support for one's intuition. It is intuitively logical that as the probability of success increases, \(d_i\) should increase; as the probability of failure increases, \(d_i\) should decrease; as the rate of return for success increases, \(d_i\) should increase; and as the rate of return for failure increases, \(d_i\) should decrease. The first two logical statements are obviously true by examination of the numerator in the expression for \(d_i\). Writing \(d_i\) as \((p/s - q/r)x_i\) reveals the last two logical statements. As \(r\) increases, \(q/r\) decreases, the term in parentheses increases, and \(d_i\) increases. As \(s\) increases, \(p/s\) decreases, the term in parentheses decreases, and \(d_i\) decreases.

Bellman and Kalaba (1957) also investigated the policy for maximization of expected utility of capital where a power
utility function of the form \( Cx^{M+1} \) was assumed with \(-1 < M < 0\) and \( C = 1/(M+1) \). In Appendix A, Part 3, it is shown that

\[
f_n(x_1) = Cx_1^{M+1}w^n,
\]

where

\[
w = \left[ \frac{r+s}{s(qs)^{1/M} + r(pr)^{1/M}} \right]^{M+1} \left[ \frac{p(qs)(M+1)/M + q(pr)(M+1)/M}{s(qs)^{1/M} + r(pr)^{1/M}} \right]^{M+1}
\]

and that the optimal investment policy is

\[
d_i = \begin{cases} 
0 & \text{if } (qs)^{1/M} - (pr)^{1/M} < 0 \\
\frac{1}{s(qs)^{1/M} + r(pr)^{1/M}} x_i & \text{if } 0 \leq (qs)^{1/M} - (pr)^{1/M} \leq s(qs)^{1/M} + r(pr)^{1/M} \\
x_i & \text{if } (qs)^{1/M} - (pr)^{1/M} > r(pr)^{1/M} + s(qs)^{1/M}.
\end{cases}
\]

Intuition again dictates that the optimal investment policy \( d_i \) should increase as the probability of success \( p \) increases and as the rate of return for success \( r \) increases, while it should decrease as the rate of return for failure increases. It is necessary to rewrite the expression for \( d_i \) in order to observe these implications.

\[
d_i = \frac{1}{s^{M+1}} - r^{M+1} \left( \frac{1}{1-p} \right)^{M+1} x_i
\]

As \( p \) increases, \( \left( \frac{1}{1-p} \right)^{M} \) decreases, the numerator of \( d_i \)
increases, the denominator decreases, and $d_i$ increases. Observe that $(M+1)/M > 1/M$ and $s(M+1)/M > s^{1/M}$. As $s$ increases, $s^{1/M}$ has a greater percentage decrease than $s(M+1)/M$, therefore both numerator and denominator decrease, but the numerator decreases more, and $d_i$ decreases as expected. As $r$ increases, the numerator increases, the denominator decreases, and $d_i$ increases. An investor might also be interested in the observation that the optimal policy $d_i$ is less sensitive to a change in $s$ than to a change in $p$. Therefore in order to increase the optimal policy $d_i$, it would be more profitable to increase $p$ rather than decrease $s$.

Another approach suggested by Murphy (1965), the approach with which this paper primarily deals, establishes as the criterion the attainment of a given or target expected value of capital after the $n$ stages while minimizing the risk (measured by the variance) of achieving this expected value of capital.

**Two-Outcome Multistage Investment Process: p Known**

The following notation and definitions will be used in development of this model:

- $i$ subscript denoting the stage number
- $p_i$ probability of success for the investment at stage $i$
- $q_i = 1 - p_i$ probability of failure for the investment at stage $i$
\( r_i, r_i > 0 \) \hspace{1cm} rate of return from the successful investment at stage \( i \)

\( -s_i, s_i > 0 \) \hspace{1cm} rate of return from the unsuccessful investment at stage \( i \)

\( x_n = L \) \hspace{1cm} initial capital available for investment

\( x_i, i=1,2,\ldots,n-1 \) \hspace{1cm} expected value of capital available for investment at stage \( i \)

\( x_0 = KL \) \hspace{1cm} expected value of capital available at the conclusion of the process

\( K \) \hspace{1cm} expectation or target level

\( d_i \) \hspace{1cm} investment or decision variable at stage \( i \)

\( \sigma_i \) \hspace{1cm} variance of capital at stage \( i \); stage return, i.e., the dynamic programming measure of performance of a stage

Figure 2 illustrates this multistage process.

\[ \begin{aligned}
\text{Stage } n & \quad \rightarrow \quad \text{Stage } n-1 \\
\text{Stage } n-1 & \quad \rightarrow \quad \text{Stage } n-2 \\
\text{Stage } n-2 & \quad \rightarrow \quad \text{Stage } 2 \\
\text{Stage } 2 & \quad \rightarrow \quad \text{Stage } 1 \\
\text{Stage } 1 & \quad \rightarrow \quad \text{Stage } 0 = KL
\end{aligned} \]

Figure 2. Dynamic Programming Formulation of a Multistage Decision Process

Theorem: The variance of the final capital \( V \) is equal to the sum of the stage variances. \( V = \sum_{i=1}^{n} \sigma_i \).

(See Appendix B for proof)
The random variables in this process are the stage investment returns (not to be confused with the stage returns of the dynamic programming formulation which are the \( v_i \)'s). Since the capital at each stage is a function of the rate of return, which is a stochastic variable, then the stage capital is also a stochastic variable.

An example of the expected value of capital at the input to stage \( i \) is

\[
x_i = p_{i+1}(x_{i+1} + r_{i+1}d_{i+1}) + q_{i+1}(x_{i+1} - s_{i+1}d_{i+1})
\]

\[
= x_{i+1} + d_{i+1}(p_{i+1}r_{i+1} - q_{i+1}s_{i+1}).
\]

As an example of stage variance, recall

\[
V(x_i) = E(x_i^2) - [E(x_i)]^2.
\]

Therefore,

\[
v_i = p_i(x_i + r_id_i)^2 + q_i(x_i - s_id_i)^2 - [p_i(x_i + r_id_i) + q_i(x_i - s_id_i)]^2
\]

\[
= d_i^2(r_i + s_i)^2 p_i q_i.
\]

Since the expected final capital is constrained to equal KL, the target value, then it is true that

\[
x_0 = KL = x_1 + d_1(p_1r_1 - q_1s_1)
\]

or

\[
d_1 = \frac{KL-x_1}{p_1r_1-q_1s_1}.
\]

As will be evident in the analysis that follows, the target expectation level \( K \) is not unrestricted. Suppose, for example, that \( p_1r_1 \) is approximately equal to \( q_1s_1 \). For a
given value of $K$, $d_1$ would be very large and perhaps exceed $x_1$. However $d_1$ is restricted to be less than or equal to $x_1$. The answer to this problem is that $K$ is not specified initially. The constraints $0 \leq d_i \leq x_i$ will restrict $K$ to a specific range of values. Therefore if $p_1 r_1$ is approximately equal to $q_1 s_1$, $KL$ will be approximately equal to $x_1$ and the restrictions on $d_1$ will not be violated. In addition, only investments with favorable expectations, i.e., those where $p_1 r_1 > q_1 s_1$, will be considered. Under this assumption, $d_i$ cannot be negative.

The criterion for this $n$-stage, two-outcome investment process may be stated as

$$\text{minimize} \quad \sum_{i=1}^{n} d_i^2 (r_i + s_i)^2 p_i q_i$$

subject to:

$$x_n = L$$

$$x_i = x_{i+1} + d_{i+1} (p_i r_{i+1} - q_{i+1} s_{i+1})$$

$$i = 0, 1, 2, \ldots, n-1$$

$$x_0 = KL$$

$$0 \leq d_i \leq x_i.$$

This optimization problem may be solved using Lagrange multipliers and the Kuhn-Tucker conditions. Since the objective function is convex and differentiable and the constraints are linear or convex and differentiable, convex programming may be used to solve for the optimal solution. This problem will be solved both by dynamic programming and by convex programming later (Appendix C). In some cases one technique
is simpler, while in other cases the other technique is simpler. In convex programming, all the variables are found simultaneously from the Kuhn-Tucker conditions, while in dynamic programming one variable is found at a time. Dynamic programming does not preclude the use of convex programming or some other optimization technique. Rather the question involving which technique to use is whether to use some optimization technique for the whole problem directly or to decompose the problem by dynamic programming and solve a series of subproblems (Nemhauser, 1966). Some problems cannot be decomposed and therefore dynamic programming cannot be used.

Dynamic Programming Solution

Let \( f_n(x_n) \) denote the minimum variance of capital after \( n \geq 1 \) stages, where \( x_n \) is the initial capital available for investment. It will be shown that

\[
f_n(x_n) = (KL - x_n)^2 \left[ \frac{n \sum_{j=1}^{n} \left( \pi_{t}^{j} q_{t}^{j} (r_{t} + s_{t})^{2} \right)}{\sum_{j=1}^{n} \left( \sum_{t=1}^{n} \pi_{t}^{j} q_{t}^{j} (r_{t} + s_{t})^{2} \right)} \right]
\]

and that the optimal investment policy is given by

\[
d_{i,n} = (KL - x_n) \left\{ \frac{n \sum_{j=1}^{n} \left( \sum_{t=1}^{n} \pi_{t}^{j} q_{t}^{j} (r_{t} + s_{t})^{2} (r_{i}p_{i} - s_{i}q_{i}) \right)}{\sum_{j=1}^{n} \left( \sum_{t=1}^{n} \pi_{t}^{j} q_{t}^{j} (r_{t} + s_{t})^{2} \right)} \right\}
\]
where \( i=1,2,...,n \). \( d_{i,n} \) denotes the decision, i.e., the amount to invest at stage \( i \) of an \( n \)-stage process. Dynamic programming and the principle of mathematical induction will be used in the derivation of \( f_n(x_n) \) and \( d_{i,n} \).

Stage 1: Let \( f_1(x_1) \) denote the minimum one-stage return.

\[
f_1(x_1) = \text{minimum} \left[ d_1^2 (r_1 + s_1)^2 p_1 q_1 \right]
\]

subject to: \( 0 \leq d_1 \leq x_1 \)

\[
d_1 = \frac{KL-x_1}{p_1 r_1 - q_1 s_1}
\]

This stage does not require optimization since it is required that \( d_1 = (KL-x_1)/(p_1 r_1 - q_1 s_1) \). Therefore

\[
f_1(x_1) = (r_1 + s_1)^2 p_1 q_1 \left[ \frac{KL-x_1}{p_1 r_1 - q_1 s_1} \right]^2.
\]

The constraint, \( 0 \leq d_1 \leq x_1 \), implies that

\[
0 \leq (KL-x_1)/(p_1 r_1 - q_1 s_1) \leq x_1 \text{ or }
\]

\[
0 \leq KL-x_1 \leq x_1 (p_1 r_1 - q_1 s_1) \text{ or }
\]

\[
x_1 / L \leq K \leq (x_1 / L)(p_1 r_1 - q_1 s_1 + 1).
\]

Similar constraints on \( K \) will be obtained at each stage. These constraints will impose a restriction on the range of values that \( K \) may assume.

Stage 2: \( f_2(x_2) = \text{minimum} \left[ d_2^2 (r_2 + s_2)^2 p_2 q_2 + f_1(x_1) \right] \)

subject to: \( 0 \leq o_2 \leq x_2 \)
Substituting (4) into (6) gives

\[
f_2(x_2) = \min_{0 \leq d_2 \leq x_2} \left\{ d_2^2 (r_2 + s_2)^2 p_2 q_2 + \frac{(r_1 + s_1)^2 q_1}{r_1 - q_1 s_1} \left[ \frac{KL-x_1}{p_1 r_1 - q_1 s_1} \right]^2 \right\}.
\]

The transformation equation between stages 1 and 2 is

\[ x_1 = x_2 + d_2 (p_2 r_2 - q_2 s_2). \]

Therefore,

\[
f_2(x_2) = \min_{0 \leq d_2 \leq x_2} \left\{ d_2^2 (r_2 + s_2)^2 p_2 q_2 + \frac{(r_1 + s_1)^2 q_1}{r_1 - q_1 s_1} \left[ KL-x_2 - d_2 (p_2 r_2 - q_2 s_2) \right]^2 \right\}.
\]

Let

\[ Q_2 = d_2^2 (r_2 + s_2)^2 p_2 q_2 + \frac{(r_1 + s_1)^2 q_1}{r_1 - q_1 s_1} \left[ KL-x_2 - d_2 (p_2 r_2 - q_2 s_2) \right]^2. \]

For the time being \( f_2(x_2) \) will be considered an unconstrained minimization problem and differential calculus will be used to find the \( d_2 \) which minimizes \( f_2(x_2) \).

\[
\frac{dQ_2}{dd_2} = 2d_2^2 (r_2 + s_2)^2 p_2 q_2 - \frac{(r_1 + s_1)^2 q_1}{(r_1 - q_1 s_1)^2} \left[ KL-x_2 - d_2 (p_2 r_2 - q_2 s_2) \right] (p_2 r_2 - q_2 s_2)
\]

Setting \( \frac{dQ_2}{dd_2} \) equal to zero and solving gives

\[
d_2 = \frac{(KL-x_2) p_1 q_1 (r_1 + s_1)^2 (p_2 r_2 - q_2 s_2)}{p_1 q_1 (r_1 + s_1)^2 (p_2 r_2 - q_2 s_2)^2 + p_2 q_2 (r_2 + s_2)^2 (p_1 r_1 - q_1 s_1)^2}. \quad (8)
\]

It can be shown that \( \frac{d^2 Q_2}{dd_2^2} \geq 0 \) and therefore (8) minimizes
\[ f_2(x_2) \]. Substituting (8) into (7) and simplifying gives
\[ f_2(x_2) = \frac{(KL-x_2)^2p_1q_1p_2q_2(r_1+s_1)^2(r_2+s_2)^2}{p_1q_1(r_1+s_1)^2(p_2r_2-q_2s_2)^2 + p_2q_2(r_2+s_2)^2(p_1r_1-q_1s_1)^2}. \] (9)

The constraint \( 0 < d_2 < x_2 \) simplifies to

\[ \frac{x_2}{L} \leq k \leq \frac{x_2}{L} \]
\[ [1+ \frac{p_1q_1(r_1+s_1)^2(p_2r_2-q_2s_2)^2 + p_2q_2(r_2+s_2)^2(p_1r_1-q_1s_1)^2}{p_1q_1(r_1+s_1)^2(r_2p_2-q_2s_2)^2}]. \] (10)

Stage 3: \( f_3(x_3) = \text{minimum} \left[ d_3^2(r_3+s_3)^2p_3q_3 + f_2(x_2) \right] \quad 0 \leq d_3 \leq x_3 \)

The details will be omitted, but it can be shown that the \( d_3 \) which minimizes \( f_3(x_3) \) is

\[ d_3 = (KL-x_3) \left\{ \frac{(p_3r_3-q_3s_3)^2}{\sum_{t=1}^{3} [(p_{t}r_{t}-q_{t}s_{t})^2]} \frac{\pi}{\sum_{j=1}^{3} p_{j}q_{j}(r_{j+s_{j}})^2} \right\} \]. (11)

and

\[ f_3(x_3)= (KL-x_3)^2 \left\{ \frac{\sum_{t=1}^{3} [(p_{t}r_{t}-q_{t}s_{t})^2]}{\sum_{t=1}^{3} [(p_{t}r_{t}-q_{t}s_{t})^2]} \frac{\pi}{\sum_{j=1}^{3} p_{j}q_{j}(r_{j+s_{j}})^2} \right\}. \] (12)

The constraint \( 0 \leq d_3 \leq x_3 \) simplifies to
\[
\frac{x_3}{L} \leq K \leq \frac{x_3}{L}
\] (13)

At this point it seems appropriate to assume the inductive hypothesis. For some integer \( N \geq n \) assume

\[
d_N = (KL-x_N)^2 \frac{\sum_{t=1}^{N-1} \left( \frac{\left(p_t r_t - q_t s_t\right)^2}{\pi} p_j q_j (r_j + s_j)^2 \right)}{\sum_{t=1}^{N} \left( \frac{\left(p_t r_t - q_t s_t\right)^2}{\pi} p_j q_j (r_j + s_j)^2 \right)}
\] (14)

and

\[
f_N(x_N) = (KL-x_N) \frac{\sum_{j=1}^{N} \left( \frac{\pi}{\pi} p_j q_j (r_j + s_j)^2 \right)}{\sum_{t=1}^{N} \left( \frac{\left(p_t r_t - q_t s_t\right)^2}{\pi} p_j q_j (r_j + s_j)^2 \right)}
\] (15)

From expressions (14) and (15), the following may be written:

\[
d_{N+1} = (KL-x_{N+1})^2 \frac{\sum_{t=1}^{N+1} \left( \frac{\left(p_t r_t - q_t s_t\right)^2}{\pi} p_j q_j (r_j + s_j)^2 \right)}{\sum_{t=1}^{N+1} \left( \frac{\left(p_t r_t - q_t s_t\right)^2}{\pi} p_j q_j (r_j + s_j)^2 \right)}
\] (16)

and

\[
f_{N+1}(x_{N+1}) = (KL-x_{N+1}) \frac{\sum_{j=1}^{N+1} \left( \frac{\pi}{\pi} p_j q_j (r_j + s_j)^2 \right)}{\sum_{t=1}^{N+1} \left( \frac{\left(p_t r_t - q_t s_t\right)^2}{\pi} p_j q_j (r_j + s_j)^2 \right)}
\] (17)
Starting with the inductive hypotheses, (14) and (15), (16) and (17) must be proved.

Stage \((N+1)\):

\[
 f_{N+1}(x_{N+1}) = \text{minimum}_{0 \leq d_{N+1} \leq x_{N+1}} \left[ d_{N+1}^2 (r_{N+1} + s_{N+1})^2 p_{N+1} q_{N+1} + f_N(x_N) \right] \tag{18}
\]

The derivative of the term within brackets in (18), with respect to \(d_{N+1}\), set equal to zero is

\[
 2d_{N+1}(r_{N+1} + s_{N+1})^2 p_{N+1} q_{N+1} - 2z[KL - x_{N+1} - d_{N+1}(p_{N+1} r_{N+1} - q_{N+1} s_{N+1})]
\]

\[
 [p_{N+1} r_{N+1} - q_{N+1} s_{N+1}] = 0, \tag{19}
\]

where

\[
 z = \frac{\sum_{j=1}^{N} \pi p_{j} q_{j} (r_{j} + s_{j})^2}{\sum_{t=1}^{N} \left[ (p_{t} r_{t} - q_{t} s_{t})^2 \sum_{j=1,j \neq t}^{N} \pi p_{j} q_{j} (r_{j} + s_{j})^2 \right]}
\]

Equation (19) may be solved for \(d_{N+1}\).

\[
d_{N+1} = (KL - x_{N+1}) \sum_{j=1}^{N} \pi p_{j} q_{j} (r_{j} + s_{j})^2 (p_{N+1} r_{N+1} - q_{N+1} s_{N+1})
\]

\[
 \left\{ \begin{array}{c}
 (r_{N+1} + s_{N+1})^2 p_{N+1} q_{N+1} \sum_{t=1}^{N} \left[ (p_{t} r_{t} - q_{t} s_{t})^2 \sum_{j=1,j \neq t}^{N} \pi p_{j} q_{j} (r_{j} + s_{j})^2 \right] \\
 + (p_{N+1} r_{N+1} - q_{N+1} s_{N+1})^2 \sum_{j=1}^{N} \pi p_{j} q_{j} (r_{j} + s_{j})^2
\end{array} \right\} \tag{20}
\]
Close examination reveals the fact that (20) may be written as

\[ d_{N+1} = (KL-x_{N+1}) \frac{(p_{N+1}r_{N+1}-q_{N+1}s_{N+1})^{N}}{N+1} \prod_{j=1}^{N} p_{j}q_{j}(r_{j}+s_{j})^{2} \],

(21)

which is identically (16) and was to be proved. Substituting (15) and (21) into (18) yields (17) and completes the inductive proof. It has been shown that (14) and (15) are true for \( n=1 \) and for \( n=N+1 \) having assumed the inductive hypothesis. It is therefore true that (14) and (15) are true for all positive integers \( n \). That is,

\[ d_{n} = (KL-x_{n}) \frac{\prod_{j=1}^{n-1} p_{j}q_{j}(r_{j}+s_{j})^{2}}{\sum_{t=1}^{n} [(p_{t}r_{t}-q_{t}s_{t})^{2} \prod_{j=1}^{n} p_{j}q_{j}(r_{j}+s_{j})^{2}]} \],

and

\[ f_{n}(x_{n}) = (KL-x_{n})^{2} \frac{\prod_{j=1}^{n} p_{j}q_{j}(r_{j}+s_{j})^{2}}{\sum_{t=1}^{n} [(p_{t}r_{t}-q_{t}s_{t})^{2} \prod_{j=1}^{n} p_{j}q_{j}(r_{j}+s_{j})^{2}]} \].

Decision \( d_{n} \) denotes the optimal investment at the \( n \)th stage of an \( n \)-stage process. However, one is interested in more than just the optimal investment at stage \( n \); he is interested in the optimal investment at each stage, that is, the optimal
investment policy. The technique for solving for the remaining optimal investments, \( d_1, d_2, \ldots, d_{n-1} \), involves a technique called backsolving. Recall that \( x_{n-1} = x_n + d_n(p_n r_n - q_n s_n) \). Having determined \( d_n \) and knowing \( x_n \), the initial capital available for investment, it is possible to solve for \( x_{n-1} \) which is required for the solution of \( d_{n-1} \). These terms, \( d_{n-1} \) and \( x_{n-1} \), when substituted into \( x_{n-2} = x_{n-1} + d_{n-1}(p_{n-1} r_{n-1} - q_{n-1} s_{n-1}) \) give \( x_{n-2} \), which in turn allows for solution of \( d_{n-2} \). This process of backsolving continues until all decision variables \( d \) have been determined. The result is (2).

Consider a two-stage process where the optimal investment at stage 1 is given by

\[
d_{1,2} = \frac{(KL-L)(p_1 r_1 - q_1 s_1)p_2 q_2 (r_2 + s_2)^2}{(p_1 r_1 - q_1 s_1)^2 p_2 q_2 (r_2 + s_2)^2 + (p_2 r_2 - q_2 s_2)^2 p_1 q_1 (r_1 + s_1)^2}
\]

and the optimal investment at stage 2 is given by

\[
d_{2,2} = \frac{(KL-L)(p_2 r_2 - q_2 s_2)p_1 q_1 (r_1 + s_1)^2}{(p_1 r_1 - q_1 s_1)^2 p_2 q_2 (r_2 + s_2)^2 + (p_2 r_2 - q_2 s_2)^2 p_1 q_1 (r_1 + s_1)^2}
\]

Intuitively it seems that if \( p_1 \) increases, the investment at stage 1 increases. Unfortunately this may not be the case. The reason for this apparent counterintuitive result concerns the criterion of this model. It may be possible to either decrease or not change \( d_{1,2} \) and by changing \( d_{2,2} \), achieve the expected value goal at less variance of capital than by increasing \( d_{1,2} \). Consider the following example: Assume
\[ r_1 = 3, \ s_1 = 1, \ p_1 = 6/10, \ q_1 = 4/10, \ r_2 = 1, \ s_2 = 1, \]
\[ p_2 = 3/4, \ q_2 = 1/4, \] and \( KL - L = 10 \). The optimal investment at stage 1 is 4.32. Now assume \( p_1 \) increases to 7/10 (\( q_1 = 3/10 \)). The optimal investment at stage 1, with the other parameters remaining constant, is 4.13. This result is counter to the intuitive implication stated above that the investment at stage 1 increases as \( p_1 \) increases.

Consider the following ratio:

\[
R = \frac{d_{1,2}}{d_{1,2} + d_{2,2}}
\]

\[
= \frac{(p_1r_1 - q_1s_1)p_2q_2(r_2 + s_2)^2}{(p_1r_1 - q_1s_1)p_2q_2(r_2 + s_2)^2 + (p_2r_2 - q_2s_2)p_1q_1(r_1 + s_1)^2}
\]

\[
= \frac{p_2q_2(r_2 + s_2)^2}{p_2q_2(r_2 + s_2)^2 + (p_2r_2 - q_2s_2)(r_1 + s_1)^2} \cdot \frac{p_1q_1}{p_1r_1 - q_1s_1}
\]

As \( p_1 \) (\( p_1 > 1/2 \)) increases, \( p_1q_1 \) decreases and \( p_1r_1 - q_1s_1 \) increases. This results in a decrease in the denominator of \( R \) and consequently an increase in \( R \). Therefore if \( p_1 \) increases, the percentage of the total investment that is invested at stage 1 increases.

An intuitive, and obvious, implication is that as the expectation factor \( K \) increases, both investment decisions increase by the same multiplicative factor.

In summary, (1) is the minimum variance of capital after \( n \) stages and (2) denotes the optimal investment policy. It
should be recalled that the restrictions at each stage, given by equations (5), (10), (13), and continuing similarly for all \( n \) stages, must be satisfied. These restrictions limit the target value \( K \) that may be achieved.

In order to compare the optimal policy of this model with those of the expected value and expected utility maximization models previously mentioned, it is necessary to let all \( p_i = p, q_i = q, r_i = r, \) and \( s_i = s \). The results are

\[
d_{i,n} = \frac{KL-L}{n(pr-qs)}, \text{ where } L = x_n
\]

and

\[
f_n(x_n) = \frac{(KL-L)^2 \cdot pq(r+s)^2}{n(pr-qs)^2}.
\]

Does the model prescribe an optimal policy which supports a decision maker's intuition? It is important to keep in mind that the objective of this model is to achieve an expected value goal under minimum variance of capital - not to maximize the expected value of capital. Without examination of the policy prescribed by the model, some intuitive implications will be discussed. As the expectation level \( K \) increases, the optimal investment \( d_i \) must increase. As the number of stages increases, the time available to achieve an expected value goal increases and the investment required at each stage decreases. As the probability of success and/or the rate of return for success increase, the expected value for a constant investment increases. However, if the expected value is kept constant and \( p \) and/or \( r \) are increased,
the optimal investment decreases. The opposite occurs, i.e., the optimal investment increases, when q and/or s are increased. Each of these intuitive implications is supported by the model and its optimal investment policy (22).

Replacement of Random Parameters by Their Expected Values

In the model just developed, the amount of capital available at each stage after the first stage is the expected amount of capital available from the previous stage. In general stochastic optimization problems, replacing random variables by their expected value will lead to erroneous results. Hadley (1964, page 180) states that: "A method which has been employed frequently to find approximate solutions to stochastic programming problems is to replace all random parameters by their expected values and solve the resulting deterministic programming problem." At first glance it appears as if this approximate method is exactly the method used in this paper, and it is. However, it has been shown by Tou (1963) that in the case of quadratic stage returns and linear transformations, the independent random variables can be replaced by their expected values and the resulting deterministic problem solved to give proper results at a great savings in computational effort.

A simple verification of this very important result by Tou will now be presented. Consider a two-stage process in which \(p_1=p_2, q_1=q_2, r_1=r_2,\) and \(s_1=s_2.\) A tree diagram of this process is shown in Appendix B. The variance of the
final capital is $d_1^2pq(r+s)^2 + d_2^2pq(r+s)^2$, a fact that was derived in the appendix. In addition it can be shown that the expected value of the final capital is $L + (pr-qs)(d_1+d_2)$:

<table>
<thead>
<tr>
<th>Final Outcomes</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L+rd_2+rd_1$</td>
<td>$p^2$</td>
</tr>
<tr>
<td>$L+rd_2-sd_1$</td>
<td>$pq$</td>
</tr>
<tr>
<td>$L-sd_2+rd_1$</td>
<td>$pq$</td>
</tr>
<tr>
<td>$L-sd_2-sd_1$</td>
<td>$q^2$</td>
</tr>
</tbody>
</table>

Expected Value = $p^2(L+rd_2+rd_1) + pq(L+rd_2-sd_1) + pq(L-sd_2+rd_1) + q^2(L-sd_2-sd_1)$

= $(p^2 + 2pq + q^2)L+rd_2(p^2+pq) - sd_2(pq+q^2)$

$+$ $rd_1(p^2+pq) - sd_1(pq+q^2)$

= $L + rpd_2 - sqd_2 + rpd_1 - sqd_1$

= $L + d_2(rp-sq) + d_1(rp-sq)$

= $L + (rp-sq)(d_1 + d_2)$

The optimization problem is

minimize $(d_1^2 + d_2^2)(r + s)^2pq$

subject to: $L + (rp-sq)(d_1+d_2) = KL$

where $KL$ is the target expected value of final capital. The Lagrange function is

$\bar{\Phi} = (d_1^2 + d_2^2)(r+s)^2pq + \lambda[L+(pr-qs)(d_1+d_2) - KL]$.

$\frac{\partial \bar{\Phi}}{\partial d_1} = 0 = 2d_1(r+s)^2pq + \lambda(pr-qs)$
\[ \frac{\partial \phi}{\partial d_2} = 0 = 2d_2(r+s)^2pq + \lambda(pq-qs) \]
\[ \frac{\partial \phi}{\partial \lambda} = 0 = d_1 + d_2 - \frac{KL-L}{pr-qs} \]

The first two equations imply \( d_1 = d_2 \). From the third equation, the following result is obtained:
\[ d_1 = d_2 = \frac{1}{2}(KL-L)/(pr-qs). \]

This result was obtained by observing only the final outcomes - no intermediate state variables were used.

For the dynamic programming solution in which the expected value of the rates of return is used in determination of the state variables, the decision variables \( d_1 \) and \( d_2 \) are obtained from (22):
\[ d_1 = d_2 = \frac{1}{2}(KL-L)/(pr-qs). \]

Of course this is identical to the solution obtained from the final outcomes method as stated by Tou.

**Decision Tree Example**

As was stated in Chapter I, except for the initial optimal decision, it is necessary to observe values of the stochastic parameters before making subsequent decisions. However, in a system with a finite number of discrete outcomes, where the possible values of the stochastic parameters are known, a decision tree may be constructed with each branch indicating a possible outcome at each stage. In this manner it is possible to calculate the optimum decisions
for the entire process contingent on the outcomes of the stochastic variables at the various stages. An example of a two-outcome, four-stage process will be worked to illustrate the tree-diagram technique. It is assumed that the probability distribution associated with the random variable remains constant during the entire process. Figure 3 illustrates the process. Recall that $x_4$ indicates the initial capital available for investment; $x_0$ is the target or goal expected value; $x_1, x_2,$ and $x_3$ are the expected values at stages 1, 2, and 3 respectively; $v_i, i=1,2,3,4,$ are the stage variances; and $d_i, i=1,2,3,4,$ are the optimal decisions at each stage.

It is certainly true that the value of $K$ is not totally unrestricted. Temporarily no value will be assigned in order to determine the range of $K$ values that are obtainable for the parameter values chosen. Equation 2 is used to determine the optimal policy. For example,


$$= \frac{(100K-100)(59.76)}{(382.51)}$$

$$= (K-1)(15.62)$$

Also $d_2 = (K-1)(5.23); d_3 = (K-1)(15.99); d_4 = (K-1)(18.13).$ In each case above, $d_i \geq 0$ implies $K \geq 1.$ Recall, however,
Figure 3. Two-Outcome, Four-Stage Investment Process
that it is also true that $d_i \leq x_i$. Therefore,

$$d_4 \leq x_4$$

$$(K-1)(18.13) \leq 100$$

$$K \leq 6.52.$$  

Also $d_3 \leq x_3$

$$(K-1)(15.99) \leq 100 + 2.2(K-1)(18.13)$$

$$K \geq -3.18.$$  

Likewise $d_2 \leq x_2$ and $d_1 \leq x_1$ imply that $K$ is greater than or equal to a negative number. This is a redundant constraint since $K \geq 1$. Therefore the allowable values of $K$ for this problem are

$$1 \leq K \leq 6.52.$$  

Now assume that $K = 3$. Table 1 contains the optimal policy, the stage expected value, the stage variance, and the cumulative expected value.

**Table 1.**

Summary of Results for Two-Outcome Four-Stage Investment Process

<table>
<thead>
<tr>
<th>Stage</th>
<th>Optimal Decisions $d_i$</th>
<th>Cumulative Expected Values $x_i$</th>
<th>Stage Variances $v_i$</th>
<th>Stage Expected Values $EV_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$31.24$</td>
<td>$260.96$</td>
<td>$1.6875 \ d_1^2$</td>
<td>$1.25 \ d_1$</td>
</tr>
<tr>
<td>2</td>
<td>$10.46$</td>
<td>$242.13$</td>
<td>$7.26 \ d_2^2$</td>
<td>$1.80 \ d_2$</td>
</tr>
<tr>
<td>3</td>
<td>$31.98$</td>
<td>$179.77$</td>
<td>$2.5725 \ d_3^2$</td>
<td>$1.95 \ d_3$</td>
</tr>
<tr>
<td>4</td>
<td>$36.26$</td>
<td>$100.00$</td>
<td>$2.56 \ d_4^2$</td>
<td>$2.20 \ d_4$</td>
</tr>
</tbody>
</table>
A casual examination of Table 1 reveals one significant, and intuitive, fact. At stage 2 where the coefficient for the stage variance is high (7.26), the resultant optimal investment is low compared to the remaining optimal investments.

The optimal decision variable of primary importance in Table 1 is \( d_4 \). This amount will actually be invested at the first investment opportunity. The remaining optimal decisions may be used for future planning purposes, but it is highly unlikely that any of the values will actually be used. The reason for this is because at the conclusion of this first stage, i.e., stage 4, a new initial capital will be available, the investment process will then be a three-stage process, and a new optimal policy will be calculated based on the actual capital then available.

Thus the initial optimal decision, \( d_4 \), is $36.26. This investment is made and the outcome from this risky event occurs, i.e., a rate of return of 3 or a rate of return of -1 occurs. At the conclusion of this stage, stage 4, the DM either has $100 + 3($36.26) or $100 - $36.26. The DM is now faced with a two-outcome, three-stage process with either $208.78 or $63.74 initial capital. Assuming that his goal does not change, i.e., that he still desires to end up with $300 expected capital, a new optimal policy must be determined. A tree diagram of what has transpired to date is pictured as:
The tree diagram shows, at the node, the initial capital available of $100 and the optimal investment of $36.26. On one branch is the rate of return 3 with its associated probability, while on the other branch is the rate of return of -1 with its probability. At the terminal end of these branches is the capital available, which depends on the outcome of this risky event.

There now exists a three-stage investment process with initial capital of $208.78 and target of $300 expected capital. Using equation 2, the following optimal policy is derived:

\[
d_3 = (300 - 208.78) \frac{\sum_{j=1}^{3} p_j r_j (s_j + s_3)^2 (r_3 p_j - s_3 q_3)}{\sum_{j=1}^{3} (p_j r_j - s_j q_j)^2} \sum_{t=1}^{3} p_t q_t (r_t + s_t)^2
\]

\[
\]

Likewise \(d_2 = $7.93\) and \(d_1 = $23.72\). The new initial decision, \(d_3\), is $24.26, the possible rates of return are 3 and -0.5 and the decision tree is now the following:
Likewise assuming an initial capital of $63.74, the following optimal policy is determined:

\[ d_3 = \$62.85, \quad d_2 = \$20.53, \quad d_1 = \$61.43. \]

Recall that it must be true that \( d_3 \leq x_3 \), where \( x_3 \) is the expected capital available at the conclusion of stage 4. However, in the tree diagram-contingency method, \( x_3 \) is the actual amount available at the conclusion of stage 4. Therefore \( \$62.85 \leq \$63.74 \). Later in this example will be discussed the situation where \( d_i \nleq x_i \). The decision tree is now the following:
The next stage of the tree is completed in a similar manner except for the bottom branch where the initial capital is $32.31. The optimal decision for this branch is as follows:

\[
d_2 = (300 - 32.31) \frac{\sum_{t=1}^{2} p_t q_t (r_t + s_t)^2 (r_2 p_2 - s_2 q_2)}{\sum_{j=1, t \neq j}^{2} (r_j p_j - s_j q_j)^2}
\]

\[
d_2 = (267.69)(.1875)(9)(1.8)
\]

\[
(1.25)^2(.24)(30.25) + (1.8)^2(.1875)(9)
\]

\[
d_2 = $48.45
\]

Thus to achieve the stated goal, $48.45 must be invested at this stage. However, only $32.31 is available for investment. Therefore it becomes necessary to establish a new target if this point is reached. For the remaining initial values, the following optimal policies are derived:

for $281.56: \quad d_2 = $3.34
\quad d_1 = $9.96

for $252.29: \quad d_2 = $8.64
\quad d_1 = $25.76

for $196.65: \quad d_2 = $18.71
\quad d_1 = $55.81

and the tree diagram becomes the following:
Finally for the initial values of $294.92, $276.55, ..., $239.33, the following optimal policies result: $d_1 = \$4.06, $18.76, $22.81, $105.14, $10.52, $48.54 and the final tree diagram is shown in Figure 4.

In order to show that the goal of $300 expected capital has been achieved, the expected capital of the top two branches will be calculated.

Expected Value = .75(303.04) + .25(290.86) = 300

In summary, the contingency tree diagram gives the optimal investment for each stage at each node. As long as the parameters remain constant and the goal is not changed
Figure 4. Decision Tree for Two-Outcome Four-Stage Investment Process
during the process, it is possible to construct this tree
diagram prior to any actual investments. The next example
is very similar to the example just worked in the mechanics
employed; however, it is quite different conceptually.

**Adaptive Investment Process**

Murphy (1965, page 108) states: "Investors do not have
full knowledge of the probability function of the payoff
process. The perfect foresight investment service simply
does not exist." The adaptive process involves not only
risk in terms of the stochastic rates of return, but in
addition the probabilities associated with these rates are
unknown parameters.

Let us consider a two-outcome, multistage process in
which the investor knows that the values p and q remain
constant throughout the process, i.e., the probability
distribution is stationary, although he does not know what
the values are. In addition the other parameters of the
stochastic process remain fixed throughout the process. The
investor must learn the value of the parameter p of this
binomial process and simultaneously make decisions in an
effort to achieve a given expected capital under minimum
risk (variance of capital). This adaptive or probability
learning process is commonly solved using Bayesian analysis.
When faced with uncertainty, such as an unknown probability
of success, a common sense approach is to learn from experi­
ence. As the process unfolds, the investor should be able
to learn more and more about the unknown structure and the unknown parameter. For this binary process with a known value of \( p \), the probability of \( r \) successes out of \( n \) stages can be computed. A distribution is required to express the DM's prior uncertainty concerning \( p \). Morris (1968, page 134) recommends a distribution which "permits expression of a wide variety of prior attitudes" and "combines easily with a binomial sampling distribution according to the logic of Bayes' theorem." The beta family is such a distribution. The following development and notation is taken from Morris (1968, Chapter 10).

The prior probability of \( p \) may be written as

\[
PR(p) = \frac{(n_{pr}-1)!}{(r_{pr}-1)!} \frac{r_{pr}^{-1} p^{r_{pr}-1} (1-p)^{n_{pr}-r_{pr}-1}}{(n_{pr}-r_{pr}-1)!}.
\]

where \( n_{pr} \) and \( r_{pr} \) are the parameters which characterize the beta probability function. The values assigned to these parameters indicate the conviction with which prior attitudes are held. The parameter \( n_{pr} \) may be thought of as the prior number of stages and the parameter \( r_{pr} \) as the prior number of successes. The likelihood, given by the binomial distribution, is

\[
LK(r \text{ successes}|p, n \text{ trials}) = \binom{n}{r} p^r (1-p)^{n-r}.
\]

Combining the prior and likelihood distributions by Bayes' theorem yields the following posterior probability distribution of \( p \), given \( r \) successes in \( n \) trials:
\[ P_0(p) = \frac{n!}{r!(n-r)!} p^{r(1-p)^{n-r}} \frac{(n_{pr}-1)!}{(r_{pr}-1)!}(n_{pr}-r_{pr}-1)! \frac{r_{pr}^{r-1}(1-p)^{n_{pr}-r_{pr}-1}}{(n_{pr}-r_{pr}-1)!} \]

\[ f \frac{n!}{r!(n-r)!} p^{r(1-p)^{n-r}} \frac{(n_{pr}-1)!}{(r_{pr}-1)!}(n_{pr}-r_{pr}-1)! \frac{r_{pr}^{r-1}(1-p)^{n_{pr}-r_{pr}-1}}{(n_{pr}-r_{pr}-1)!} dp \]

\[ \Rightarrow \frac{(n_{pr}+n-1)!}{(r_{pr}+r-1)!}(n_{pr}-r_{pr}+n-r-1)! \frac{r_{pr}^{r-1}(1-p)^{n_{pr}+(r_{pr}+r)-1}}{(n_{pr}+(r_{pr}+r)-1)!} \]

Let \( r_{po} = r_{pr}+r \) and \( n_{po} = n_{pr}+n \). Then

\[ P_0(p) = \frac{(n_{po}-1)!}{(r_{po}-1)!}(n_{po}-r_{po}-1)! \frac{r_{po}^{r-1}(1-p)^{n_{po}-r_{po}-1}}{(n_{po}-r_{po}-1)!} \]

which is a beta distribution with a mean of \( r_{po}/n_{po} \), or \( (r_{pr}+r)/(n_{pr}+n) \).

In words, the above says that if the likelihood is binomial and the prior beliefs about \( p \) can be expressed by a beta, then the posterior distribution of \( p \) is beta whose mean is \( (r_{pr}+r)/(n_{pr}+n) \), where \( r_{pr} \) and \( n_{pr} \) are the parameters of the prior distribution and \( r \) and \( n \) are the parameters of the likelihood distribution. Experience is used to modify the prior mean of \( p \), \( r_{pr}/n_{pr} \), by the sample results, \( r \) successes out of \( n \) stages, to obtain the posterior mean of \( p \), \( (r_{pr}+r)/(n_{pr}+n) \).

Consider a four-stage binomial investment process with initial capital of $1000 and a final expected capital target of $1500. Each investment (stage) is binary, i.e., will result in success or failure. Having previously observed
this process, the DM decides that \( p \) is beta distributed with parameters \( r_{pr} = 6 \) and \( n_{pr} = 8 \). A summary of the data that the DM has is:

\[
L = \$1000; \quad K = 1.5; \quad m_{pr}(p) = \text{prior expected value of} \quad p = \frac{3}{4}; \quad r = 3; \quad s = 1; \quad p \sim \text{beta} (r_{pr} = 6, n_{pr} = 8).
\]

The optimal policy given by (22) is to invest

\[
\frac{1}{4}(KL-L)/(pr qs) \text{ at each stage. Therefore } d_1 = d_2 = d_3 = d_4 = \\
\frac{1}{4}(1500-1000)/(3/4 \times 3 - 1/4) = \$62.50, \text{ where } m_{pr}(p) \text{ was used for } p \text{ and } 1 - m_{pr}(p) \text{ was used for } q. \text{ More will be said about the use of these expected values in place of the random variables } p \text{ and } q.
\]

If the investment is successful, then the DM has \$1000 + 3(\$62.50) or \$1187.50 for the next (second) stage. If it is unsuccessful, then he has \$1000 - \$62.50 or \$937.50.

Since the DM was uncertain about \( p \), the outcome of the first stage will likely cause a revision in its probability distribution function. The expected value of the posterior distribution of \( p \) will equal \((r_{pr} + r)/(n_{pr} + n)\) or \((6 + 1)/(8 + 1)\) or \(7/9\) if successful and \(6/(8+1)\) or \(6/9\) if unsuccessful.

Consider the investor successful at the first stage. His capital is now \$1187.50 and he has three stages in which to achieve the \$1500 goal. The expected value of \( p \) is now \(7/9\). For the three-stage process, the optimal decision is to invest \((KL-L)/3(pr qs)\) at each stage. Therefore

\[
d_1 = d_2 = d_3 = (1500-1187.50)/3(7/9 \times 3 - 2/9) = \$49.34. \text{ Once again the expected values of the random variables } p \text{ and } q
are used in order to determine the optimal policy. More will be said about this later. Continuing in a similar fashion, Figure 5 is completed. There are two calculations that require some explanation.

Consider the DM who invested $62.50 at stage 4 and was unsuccessful. The probability of success $p$ becomes $6/9$. He then invests $112.50 at stage 3 and again is unsuccessful. The probability of success $p$ now becomes $6/10$ and the DM has $1000 - 62.50 - 112.50 or $825. The optimal policy with this initial capital and two stages remaining is

$$d_1 = d_2 = \frac{(1500-825)}{2(6/10 \times 3 - 4/10)} = 241.07,$$

which he invests at stage 2. If he is now successful, the capital available for the final stage, stage 1, is $825 + 3(241.07) or $1548.21. DM has achieved his goal (actually more than his goal) and, assuming no change of goal, he does not invest (or perhaps he invests the $48.21) at stage 1.

However, if the DM was not successful at stage 2, the capital available for the last stage is $825 - 241.07 or $583.93. The new optimal policy is therefore given by

$$d_1 = \frac{(1500-583.93)}{(6/11 \times 3 - 5/11)} = 719.77.$$

However, there is only $583.93 available for investment and therefore either no investment is made or there is a change in the original goal.

In the adaptive process presented, Bayesian analysis was used to revise the values of $p$ and $q$ as a result of the
Figure 5. Adaptive Decision Tree for Two-Outcome
Four-Stage Investment Process
success or failure at each stage of the investment process. It was assumed that \( p \) is beta distributed with parameters \( r_{pr} \) and \( n_{pr} \). The expected value of \( p, r_{pr}/n_{pr} \), was used as the value of \( p \) in determining the optimal investment at the first stage. At the conclusion of this stage, \( p \) was revised in accordance with Bayes' theorem as a result of the success or failure at the first stage. This new value of \( p \) is also beta distributed with parameters \( r_{pr} + r \) and \( n_{pr} + n \). The expected value of this posterior value of \( p, (r_{pr}+r)/(n_{pr}+n) \), is used to determine the optimal investment at the second stage. This posterior value of \( p \) is then used as a prior value and the consequences of the second stage result in a second revision of \( p \). This revision continues through all stages of the process and, at each stage, the expected value of \( p \) is used in determining the optimal investment. It may be asked why the expected value of \( p \) is used rather than some other statistic. A decision theoretic approach will be used to answer this question in Appendix E.

In summary, the decision tree diagram for the adaptive investment process is similar in construction to the risky investment process contingency tree. However, the adaptive tree involves an additional factor, probability learning, that is not present in the earlier case. Of what value to the decision maker is the probability learning that takes place in the adaptive process? Or said in another way, what is the value of learning? A method which illustrates the
value of learning involves comparison of the following ratios:

\[
\begin{array}{c|c|c}
\text{learning model: } & \text{ith stage investment} & \text{true p model: } \text{ith stage investment} \\
\hline
\text{true p model: } \text{ith stage investment} \\
\hline
\text{true p model: } \text{ith stage investment}
\end{array}
\]

and

\[
\begin{array}{c|c|c}
\text{no-learning model: } & \text{ith stage investment} & \text{true p model: } \text{ith stage investment} \\
\hline
\text{true p model: } \text{ith stage investment} \\
\hline
\text{true p model: } \text{ith stage investment}
\end{array}
\]

Consider the second stage investment illustrated in Figure 5. The investor begins with a prior expected value of the probability of success of \(6/8\) and revises this probability in light of the success or failure at the first stage. It may be said that he learns probability as a result of his experience at the first stage. In tree diagram I in Figure 6, no such learning takes place as a result of the outcome at the first stage. If the decision maker were successful at the first stage and if learning took place, the expected value of \(p\) becomes \((6+1)/(8+1)\) or \(7/9\); if he were unsuccessful \(p\) becomes \((6+0)/(8+1)\) or \(6/9\). Tree diagram II assumes the true value of \(p\) is \(7/9\) and tree diagram III assumes the true value is \(6/9\).

Consider the outcome of the first stage investment successful. The revised expected value of \(p\) is \(7/9\) and the optimal investment at the second stage is \$49.34. The no-learning model prescribes an optimal investment at the
Decision Tree I: No-Learning Model

Decision Tree II: True Value of \( p = \frac{7}{9} \)

Decision Tree III: True Value of \( p = \frac{6}{9} \)

Figure 6: Value-of-Learning Tree Diagram
second stage of $52.08. Assuming the true value of p as 7/9, the optimal investment is $50.90. The percent error between the optimal investment from the learning model and the true p-value model is calculated as follows:

$$\frac{|49.34 - 50.90|}{50.90} \times 100 = 3.06\%$$

Likewise the percent error for the no-learning model is

$$\frac{|52.08 - 50.90|}{50.90} \times 100 = 2.32\%.$$

Now consider the outcome of the first stage investment unsuccessful. The revised expected value of p is 6/9 and the optimal investment at the second stage of the learning model is $112.50. The no-learning model prescribes an optimal investment at the second stage of $93.75. Assuming the true value of p as 6/9, the optimal investment is $115. The percent error between the optimal investment from the learning model and the true p-value model is

$$\frac{|112.50 - 115|}{115} \times 100 = 2.17\%.$$ 

Likewise the percent error for the no-learning model is

$$\frac{|93.75 - 115|}{115} \times 100 = 18.48\%.$$
The prior expected percent error, i.e., assuming a prior expected probability of success of 6/8, for the learning model is
\[ \frac{3}{4}(3.06\%) + \frac{1}{4}(2.17\%) = 2.84\%. \]
Likewise the prior expected percent error for the no-learning model is
\[ \frac{3}{4}(2.32\%) + \frac{1}{4}(18.48\%) = 6.36\%. \]

The prior expected percent error for the no-learning model is greater than for the learning model. Thus there is a value of learning as indicated by this smaller prior expected percent error in the optimal investment prescribed by the learning model.
CHAPTER III

MULTIPLE OUTCOME MULTISTAGE DECISION MAKING MODEL

The two-outcome, multistage decision making model was selected as the first model to be developed in this paper because of its simplicity. None the less, the criterion and the optimization techniques used are also applicable to the models developed in this and in the next chapter. The first refinement will involve increasing the number of outcomes at each stage, i.e., at stage $j$ the possible rates of return are $r_{1j}, r_{2j}, \ldots, r_{mj}$ with corresponding probabilities of occurrence $p_{1j}, p_{2j}, \ldots, p_{mj}$. As in Chapter II, if outcome 1 obtains, the expected capital available for the next stage is $x_j + r_{1j}d_j$; if outcome 2 obtains, $x_j + r_{2j}d_j$ is available; etc. Only one outcome obtains at each stage.

M-Outcome, Two-State Process

The notation to be used in this chapter is identical to that of Chapter II except that $r$ and $p$ will have double subscripts, with the first subscript indicating the outcome and the second subscript indicating the stage. Figure 7 illustrates a two-stage process.
The expected value of capital from stages 1 and 2 are calculated in the usual manner.

\[ x_1 = p_{12}(L + r_{12}d_2) + p_{22}(L + r_{22}d_2) + \ldots + p_{m2}(L + r_{m2}d_2) \]
\[ = L + d_2 \sum_{i=1}^{m} p_{i2}r_{i2} \]

\[ x_0 = p_{11}(x_1 + r_{11}d_1) + p_{21}(x_1 + r_{21}d_1) + \ldots + p_{m1}(x_1 + r_{m1}d_1) \]
\[ = x_1 + d_1 \sum_{i=1}^{m} p_{i1}r_{i1} \]

Setting \( x_0 = KL \), the target expected value for the process, \( d_1 \) may be determined.
The variance of capital from stage 2 is calculated below:

\[ v_2 = p_{12}(L+r_{12}d^2_2)^2 + p_{22}(L+r_{22}d^2_2)^2 + \ldots \]

\[ + p_{m2}(L+r_{m2}d^2_2)^2 - [L + d^2_2 \sum_{i=1}^{m} p_{i2}r_{i2}]^2 \]

\[ = d^2_2 \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i2}p_{h2}(r_{i2} - r_{h2})^2. \]

Likewise, \[ v_1 = d^2_1 \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i1}p_{h1}(r_{i1} - r_{h1})^2. \]

The criterion for this multi-outcome, multistage process is written as

minimize \[ 2 \sum_{j=1}^{2} d^2_j \sum_{i=1}^{m} \sum_{h=1}^{m} p_{ij}p_{hj}(r_{ij} - r_{hj})^2 \]

subject to: \[ 0 \leq d_j \leq x_j, \quad j=1,2 \]

\[ x_2 = L \]

\[ x_j = x_{j+1} + d_{j+1} \sum_{i=1}^{m} p_{i,j+1}r_{i,j+1}, \quad j=0,1 \]

\[ x_0 = KL \]

The optimization problem stated above is solved by
dynamic programming and differential calculus (see Appendix D for the solution) with the following results:

\[
d_2 = (KL-L) \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i1}p_{h1}(r_{i1}-r_{h1})^2 \sum_{i=1}^{m} p_{i2}r_{i2} - \left[ \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i2}p_{h2}(r_{i2}-r_{h2})^2 \sum_{i=1}^{m} p_{i1}r_{i1} + \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i1}p_{h1}(r_{i1}-r_{h1})^2 \sum_{i=1}^{m} p_{i2}r_{i2} \right]
\]

and

\[
d_1 = (KL-L) \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i2}p_{h2}(r_{i2}-r_{h2})^2 \sum_{i=1}^{m} p_{i1}r_{i1} - \left[ \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i1}p_{h1}(r_{i1}-r_{h1})^2 \sum_{i=1}^{m} p_{i2}r_{i2} + \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i2}p_{h2}(r_{i2}-r_{h2})^2 \sum_{i=1}^{m} p_{i1}r_{i1} \right].
\]

By applying the principle of mathematical induction on the stage number \(j\), the following optimal allocation can be determined for an \(m\)-outcome, \(n\)-stage process:

\[
d_j = \frac{(KL-L) \left\{ \sum_{t=1}^{n} \left[ \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i1}p_{ht}(r_{it}-r_{ht})^2 \right] \sum_{i=1}^{m} p_{ij}r_{ij} \right\}}{\sum_{g=1}^{n} \left\{ \sum_{t=1}^{n} \sum_{i=1}^{m} \sum_{h=1}^{m} p_{it}p_{ht}(r_{it}-r_{ht})^2 \left[ \sum_{i=1}^{m} p_{ig}r_{ig} \right] \right\}}.
\]
The minimum variance for this process is
\[ f_n(x_n) = \frac{(KL-L)^2}{\sum_{j=1}^{n} \left( \sum_{i=1}^{m} p_{ij} r_{ij} \right)^2 \left( \sum_{i=1}^{m} \sum_{h=1}^{m} p_{ij} p_{hj} (r_{ij} - r_{hj})^2 \right)} \]
\[ \sum_{t=1}^{n} \left[ \sum_{i=1}^{m} \sum_{h=1}^{m} p_{it} p_{ht} (r_{it} - r_{ht})^2 \right] \left[ \sum_{i=1}^{m} p_{ij} r_{ij} \right]^2 \] (2)

It should be recalled that there exist constraints,
\[ 0 \leq d_i \leq x_i, \quad i=1,2,\ldots,n \], which limit the range of the values of K.

**Example -- Capital Investment**

Let us consider the following capital investment problem as an example of a multiple outcome, multistage investment process. A home construction company has completed the past fiscal year with a profit, after all expenses, of $100,000. It has the option of investing this capital at a fixed interest rate (no risk) or of hiring additional work crews. The company establishes an expected goal for the coming year of increasing the $100,000 to $K(100,000)$, where the value of $K(100,000)$ cannot be achieved with "no risk" investments. A decision is therefore made to hire additional work crews. Questions remain as to what is a reasonable value of $K$ to expect, and once having decided on $K$, how much should be
invested in additional home construction during each of the selling seasons - spring, summer, autumn, and winter. Construction takes place in the season preceding the selling season, i.e., a home sold in the spring will be constructed during the preceding winter. As could be expected, the rate of return on investments and the corresponding probabilities are not constant throughout the year. For the year in question, Table 2 gives the possible outcomes and corresponding probabilities:

Table 2
Data for Four-Outcome, Four-Stage Home Construction Example

<table>
<thead>
<tr>
<th>SPRING</th>
<th>SUMMER</th>
<th>AUTUMN</th>
<th>WINTER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of Return</td>
<td>Probability</td>
<td>Rate of Return</td>
<td>Probability</td>
</tr>
<tr>
<td>0.7</td>
<td>0.2</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3</td>
<td>0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>0.1</td>
<td>0.4</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>-0.2</td>
<td>0.1</td>
<td>-0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

For example, if the company invests $25,000 in additional home construction which is to be sold during the summer, the probability distribution of capital is:

Probability $= .2$ that the capital is $25000 + .8(25000)$ or $45,000.$
Probability = .4 that the capital is $25000 +
.6(25000) or $40,000.
Probability = .3 that the capital is $25000 +
.3(25000) or $32,500.
Probability = .1 that the capital is $25000 -
.1(25000) or $22,500.

Intuitively the company knows that it should invest most
heavily in homes to be sold during the summer months, next
most heavily in autumn homes, and least heavily in winter
homes. However, the exact amount to be invested during each
season is not known. The company therefore consults with an
analyst who, in addition to the information given in Table 2,
obtains from the company the fact that it wishes to achieve
its goal with minimum risk of final capital. By presenting
examples which utilize other measures of risk such as mini-
mizing the maximum loss, minimizing the probability of
maximum loss, or minimizing some combination of loss and
probability of loss, he convinces the company that the most
reasonable measure of risk to use is the variance of capital
at the conclusion of the next fiscal year. He points out
that over a wide range of investment processes, the use of
variance as the measure of risk leads to consistently
reasonable decisions. The criterion is:

\[
\text{minimize} \quad \sum_{j=1}^{4} \sum_{i=1}^{4} p_{ij} p_{hj} (r_{ij} - r_{hj})^2
\]
subject to: $0 \leq d_j \leq x_j$, $j=1,2,3,4$

$x_4 = $100,000

$x_j = x_{j+1} + d_{j+1} \sum_{i=1}^{m} p_{i,j+1} r_{i,j+1}$, $j=0,1,2,3$

$x_0 = $100,000K

Using the formula for the optimal allocation, equation (1), developed in the previous section, the following results are obtained:

\[
\begin{align*}
    d_1(\text{winter}) &= (K-1)(31,522) \\
    d_2(\text{autumn}) &= (K-1)(87,050) \\
    d_3(\text{summer}) &= (K-1)(100,725) \\
    d_4(\text{spring}) &= (K-1)(53,330)
\end{align*}
\]

(3)

Likewise the minimum variance from (2) is

\[f_4(x_4) = (K-1)^2(14.63 \times 10^8)\]

From the constraints, $0 \leq d_j \leq x_j$, $j=1,2,3,4$, the range of $K$ is found to be $1 \leq K \leq 2.19$. Figure 8, which shows the optimal investments and the minimum variance for allowable values of $K$, is presented to company management who must decide which expected capital-risk combination it prefers. It is then simply a matter of either picking the optimal investments off the graph or substituting in (3). For example, if management decides that a value of $K = 1.5$ results in the highest risk it is willing to accept ($V = 3.6575 \times 10^8$), then the optimal policy is $d_1(\text{winter}) = $15,761; $d_2(\text{autumn}) = $43,525; $d_3(\text{summer}) = $50,363; $d_4(\text{spring}) = $26,665.
Figure 8. Optimal Policy and Minimum Variance Versus K for Home Construction Example
It may be observed from the preceding figure that as \( K \) increases, each investment increases; however for this particular example, the investment in the selling season with the largest expected value increases most rapidly. In addition, for large values of \( K \), an equal incremental increase in \( K \) results in a larger change in the minimum variance than at the lower values of \( K \). The difference between the amounts invested in each selling season increases as \( K \) increases, however, the percentage of the total investment invested in each season remains constant for all values of \( K \).

The contingency decision tree method used in the preceding chapter is also applicable to the process of this chapter.

**Adaptive Investment Process**

There exists an adaptive multi-outcome, multistage process analogous to the two-outcome, multistage adaptive process. In this process the values of the stochastic parameters \( r_{ij} \) remain constant as do the corresponding probabilities \( p_{ij} \). However, the values of the probabilities are unknown to the DM. He must learn these values in this multinomial process and simultaneously make decisions in an effort to achieve a given expected capital under minimum variance of this capital. A Bayesian analysis will be used to solve this adaptive, probability learning, process.

In an effort to use similar notation to that used in the preceding chapter, some confusion concerning the use of
"r" may arise. The letter $r_i$, unless otherwise stated, denotes the number of times outcome $i$ occurs. The vector $ar{R}$ denotes a frequency vector whose components indicate the number of times each outcome occurs, i.e., $\bar{R} = (r_1, r_2, \ldots, r_m)$, where there are $m$ possible outcomes. The vector $\bar{P}$ denotes the probabilities associated with the outcomes. $\bar{P} = (p_1, p_2, \ldots, p_m)$. The sum of the $r_i$, $\sum_{i=1}^{m} r_i$, equals $n$, the number of stages or trials.

The likelihood probability of a frequency vector $\bar{R}$, given a probability vector $\bar{P}$ and $n$ stages, is calculated using the multinomial distribution.

$$L_K(\bar{R}|\bar{P},n) = \frac{n!}{r_1! \cdot r_2! \cdot \ldots \cdot r_m!} \cdot \frac{r_1}{p_1} \cdot \frac{r_2}{p_2} \cdot \ldots \cdot \frac{r_m}{p_m}$$

$$= \frac{n!}{\prod_{i=1}^{m} r_i! \cdot \prod_{i=1}^{n} p_i}$$

(4)

A distribution analogous to the beta, which was used as the prior distribution in the binomial process, is the Dirichlet (sometimes called the multinomial beta distribution). This distribution is given by

$$PR(\bar{P}) = \frac{\Gamma\left(\sum_{i=1}^{m} r_{pri}\right)}{\prod_{i=1}^{m} \Gamma(r_{pri})} \cdot \prod_{i=1}^{m} p_{i}^{-r_{pri}},$$

(5)

where $\bar{r}_{pr} = (r_{pri}, r_{pr2}, \ldots, r_{prm})$ represents the DM's prior convictions about $\bar{P}$. Since it is true that $\int_{\bar{P}}^{PR(\bar{P})} d\bar{P} = 1$, then
It can be shown that the prior mean of \( p_i \), \( m_{\text{pri}}(p_i) = \frac{r_{\text{pri}}}{\sum_{i=1}^{m} r_{\text{pri}}} \).

Combining the prior and likelihood distributions by Bayes' theorem yields the posterior distribution of \( \bar{F} \), given \( R \) and \( n \), i.e.,

\[
P_0(\bar{F}|\bar{R},n) = \frac{\int \text{L}(\bar{R}|\bar{F},n) \text{PR}(\bar{F}) \, d\bar{F}}{\int \text{L}(\bar{R}|\bar{F},n) \text{PR}(\bar{F}) \, d\bar{F}}.
\]

Substituting (4) and (5) into (7) yields

\[
P_0(\bar{F}|\bar{R},n) = \frac{\int \prod_{i=1}^{m} \frac{r_{\text{pri}}}{p_i} \, d\bar{F}}{\int \prod_{i=1}^{m} \frac{r_{\text{pri}}}{p_i} \, d\bar{F}}
\]
The expected value of the posterior distribution, $m_{po}(p_i)$, can be found by

$$m_{po}(p_i) = \frac{\int p_i P_0(\tilde{P}|R,n) d\tilde{P}}{\int p_i^{r_1+r_{pri}} p_2^{r_2+r_{pr2}} \ldots p_i^{r_i+r_{pri}} \ldots p_m^{r_m+r_{prm}} d\tilde{P}}$$

$$= \frac{\Gamma(r_1+r_{pri}) \Gamma(r_2+r_{pr2}) \ldots \Gamma(r_i+r_{pri}+1) \ldots \Gamma(r_m+r_{prm})}{\Gamma(r_1+r_{pr1}+r_2+r_{pr2}+\ldots+r_i+r_{pri}+1+\ldots+r_m+r_{prm})}$$

$$= \frac{\prod_{j=1}^{m} \Gamma(r_j+r_{pj}) \Gamma(r_{i}+r_{pri})}{\prod_{j=1}^{m} \Gamma(r_j+r_{pj}) \Gamma(r_{i}+r_{pri})} \frac{\prod_{j=1}^{m} \Gamma(r_j+r_{pj}) \Gamma(r_{i}+r_{pri})}{\prod_{j=1}^{m} \Gamma(r_j+r_{pj}) \Gamma(r_{i}+r_{pri})}$$

Using the functional relationship of the gamma function, $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$, the expression above simplifies to

$$m_{po}(p_i) = \frac{\frac{r_i + r_{pri}}{\sum_{i=1}^{m} r_i + \sum_{i=1}^{m} r_{pri}}} = \frac{r_i + r_{pri}}{n + n_{pr}}.$$
In words, the analysis says that if the likelihood is a multinomial distribution and the DM's prior beliefs about \( P \) can be expressed by a Dirichlet distribution, then the posterior distribution of \( P \) is a Dirichlet with a mean of \( p_i \) equal to \((r_i + r_{pri})/(n + n_{pr})\). \( r_{pri} \) and \( n_{pr} \) are parameters of the prior distribution and \( r_i \) and \( n \) are parameters of the likelihood distribution. Experience has been used to modify the prior mean by the sample results to obtain the posterior mean. An example, similar to the one worked for the two-outcome, multistage adaptive process, would illustrate how the \( p_{ij} \) change due to the outcome on the previous stage.

Note that this analysis is identical to the Bayesian analysis of Chapter II when \( m = 2 \).

A complete analysis of the example worked in this chapter would require the use of the tree diagrams that were used in Chapter II.
CHAPTER IV
MULTIPLE OUTCOME MULTIPLE ALTERNATIVE
MULTISTAGE DECISION MAKING MODEL

The process introduced in Chapter II involved two outcomes per alternative, one alternative per stage, and \( n \geq 1 \) stages per process. This process was enriched in Chapter III by increasing the number of outcomes from two to \( m \geq 1 \) per alternative. This chapter will present the second enrichment - the number of alternatives per stage will be increased from one to \( h \geq 1 \). General expressions for the optimal investment policy and the minimum variance were derived for the processes of Chapters II and III. However for the process of this chapter, only a computational procedure will be derived.

**M-Outcome, L-Alternative, N-Stage Process**

A typical stage for this process resembles Figure 9 on the following page. The notation to be used in this chapter is similar to that of Chapters II and III except for the number of subscripts on the variables. A single subscript \( k \) indicates the stage number; a double subscript \( jk \) indicates the alternative and the stage respectively; a triple subscript \( ijk \) indicates the outcome, the alternative, and
Figure 9. Stage $k$ of an $M$-Outcome, $i$-Alternative, $N$-Stage Process

the stage respectively. Starting with an initial capital of $\$L$, the objective is to achieve an expected value of capital of $K$ times $\$L$ under minimum risk (variance) of the final capital. The expected value of capital for a stage is equal to the sum of the expected values of capital for the $h$ alternatives. The alternative expected value of capital is
equal to the rate of return (dollar return per dollar invested) times the amount invested in the alternative times the probability that the rate of return will obtain, summed over all the outcomes. Notice a slight modification from previous processes in calculation of the expected value of capital. Whereas in previous models the capital from an investment included the capital gain and the investment itself, in this process the DM receives only the capital return from an investment and not the investment itself. Either approach is valid depending on the particular circumstances of the investment. This is simply a matter of how the rate of return \( r \) is defined, i.e., whether it includes the investment itself or not.

Let \( x_{jk} \) denote the expected value of capital from alternative \( j \), stage \( k \).

\[
x_{jk} = p_{1jk} r_{1jk} d_{jk} + p_{2jk} r_{2jk} d_{jk} + \cdots + p_{mjk} r_{mjk} d_{jk}
\]

\[
= d_{jk} \sum_{i=1}^{m} p_{ijk} r_{ijk}
\]

Let \( x_k \) denote the expected value of capital from stage \( k \).

\[
x_k = \sum_{j=1}^{h} d_{j,k+1} \sum_{i=1}^{m} p_{ij,k+1} r_{ij,k+1}
\]

(1)

The variance of capital from alternative \( j \), stage \( k \) is denoted by \( v_{jk} \) and
\[ v_{jk} = \sum_{i=1}^{m} p_{ijk} (r_{ijk} d_{jk})^2 - \left[ \sum_{i=1}^{m} p_{ijk} r_{ijk} \right]^2 \]

\[ = d_{jk}^2 \sum_{i=1}^{m} \sum_{t=1}^{n} p_{ijk} p_{tjk} (r_{ijk} - r_{tjk})^2. \]

The variance of capital from stage \( k \) is denoted by \( v_k \) and

\[ v_k = \sum_{j=1}^{h} d_{jk}^2 \sum_{i=1}^{m} \sum_{t=1}^{n} p_{ijk} p_{tjk} (r_{ijk} - r_{tjk})^2. \] (2)

The variance of the final capital for the entire process is denoted by \( V \) and

\[ V = \sum_{k=1}^{n} d_{jk}^2 \sum_{i=1}^{m} \sum_{t=1}^{n} p_{ijk} p_{tjk} (r_{ijk} - r_{tjk})^2. \]

In previous models the constraints at each stage have included

\[ 0 \leq \sum_{j=1}^{h} d_{jk} \leq x_k. \]

However, instead of having these inequality constraints at each stage, a slack alternative is added. Investment in this alternative is analogous to holding capital as cash. The amount of this investment is \( x_k - \sum_{j=1}^{h} d_{jk} \). The constraint \( \sum_{j=1}^{h} d_{jk} \leq x_k \) may then be written as a linear constraint \( \sum_{j=1}^{h+1} d_{jk} = x_k \), where the \((h+1)\)st alternative is the slack alternative. Since \( h \) is any integer greater than or equal
to one, it may be assumed that the h alternatives include the slack alternative.

The criterion for this multiple outcome, multiple alternative, multistage process is

$$\text{minimize } \sum_{k=1}^{n} \sum_{j=1}^{h} d_{jk}^2 \sum_{i=1}^{m} \sum_{t=1}^{m} p_{ijk}p_{tjk} (r_{ijk} - r_{tjk})^2$$

$$t > 1$$

subject to: $$x_0 = KL = \sum_{j=1}^{h} \sum_{i=1}^{m} d_{ijl} \sum_{i=1}^{m} p_{ijl} r_{ijl}$$

$$\sum_{j=1}^{h} d_{jk} = x_k, \quad k=1,2,\ldots,n$$

where $$x_k = \sum_{j=1}^{h} \sum_{i=1}^{m} d_{j,k+1} \sum_{i=1}^{m} p_{ij,k+1} r_{ij,k+1}$$

$$k = 1,2,\ldots,n-1$$ and $$x_n = L$$

$$d_{jk} \geq 0, \quad j=1,2,\ldots,h; \quad k=1,2,\ldots,n.$$}

The first constraint is the target expected value of final capital for the entire process; the second constraint implies that there is a limit on the amount of capital available at each stage for investment; and the third constraint is the nonnegativity constraint on the decision variables. The second constraint at stage n states that the amount invested at this stage is equal to the initial capital available. At the other stages, n-1, n-2, ..., 1, this constraint states that the amount invested at each stage is equal to the expected value of capital available from the previous stage.
This optimization problem will be solved by dynamic programming. Because of the lengthy and complex calculations involved in the solution, only an approximately optimal solution will be determined. However, it is theoretically possible to solve for and verify an exact optimal solution.

Example

A two-outcome, three-alternative, two-stage process will be worked to illustrate the computational procedure. Figure 10 illustrates this process. The indices have the following values: \( m = 2, h = 3, \) and \( n = 2. \) From (2) it follows that \( v_1 = 5.76d_{11}^2 + 0.0525d_{21}^2 \) and \( v_2 = 6.35d_{12}^2 + 2.94d_{22}^2. \) A sample calculation of \( v_2 \) follows:

\[
v_2 = \sum_{j=1}^{3} d_{j2}^2 \sum_{i=1}^{2} \sum_{t=1}^{2} p_{ij2}p_{tj2}(r_{ij2} - r_{tj2})^2
\]

Likewise it follows from (1) that \( x_1 = 4.35d_{12} + 3.6d_{22} + d_{32} \) and \( x_0 = 5.8d_{11} + 2.85d_{21} + d_{31}. \) A sample calculation follows.
Figure 10. Two-Outcome, Three-Alternative, Two-Stage Process
\[ x_1 = \frac{3}{\sum_{j=1}^{2} d_{j2}} \sum_{i=1}^{2} p_{ij2} r_{ij2} \]
\[ = d_{12}(p_{112}r_{112} + p_{212}r_{212}) + d_{22}(p_{122}r_{122} + p_{222}r_{222}) \]
\[ + d_{32}(p_{132}r_{132} + p_{232}r_{232}) \]
\[ = 4.35d_{12} + 3.6d_{22} + d_{32}. \quad (4) \]

Notice that the expressions for \( v_1 \) and \( v_2 \) do not include alternative 3, the riskless investment, as is mathematically as well as intuitively to be expected.

For this specific problem, criterion (3) is:

minimize \((6.35d_{12}^2 + 2.94d_{22}^2 + 5.76d_{11}^2 + 0.0525d_{21}^2)\)

subject to: \( 5.8d_{11} + 2.85d_{21} + d_{31} = 100K \)
\[ \frac{3}{\sum_{j=1}^{2} d_{jk}} = x_k, \; k=1,2 \]

where \( x_2 = 100 \) and
\[ x_1 = 4.35d_{12} + 3.6d_{22} + d_{32} \]
\[ d_{jk} \geq 0, \; j=1,2,3, \; \text{and} \; k=1,2. \]

The solution follows.

Stage 1:
\[ f_1(x_1) = \text{minimum} \left[ 5.76d_{11}^2 + 0.0525d_{21}^2 \right] \]
subject to:
\[ 5.8d_{11} + 2.85d_{21} + d_{31} = 100K \]
\[ d_{11} + d_{21} + d_{31} = x_1 \]
\[ d_{11}, d_{21}, d_{31} \geq 0. \]
The assumptions required for the solution of this stage 1 optimization problem by convex programming are satisfied. The generalized Lagrange function $\Phi$, is written as

$$\Phi_1 = 5.76d_{11}^2 + 0.0525d_{21}^2 + \lambda_1(5.8d_{11} + 2.85d_{21} + d_{31} - 100K)$$

$$+ \lambda_2(d_{11} + d_{21} + d_{31} - x_1).$$

The following Kuhn-Tucker conditions are both necessary and sufficient for a global minimum:

$$\frac{\partial \Phi_1}{\partial d_{11}} = 11.52d_{11} + 5.8\lambda_1 + \lambda_2 \geq 0$$

$$\frac{\partial \Phi_1}{\partial d_{21}} = 0.105d_{21} + 2.85\lambda_1 + \lambda_2 \geq 0$$

$$\frac{\partial \Phi_1}{\partial d_{31}} = \lambda_1 + \lambda_2 \geq 0$$

$$d_{11}\frac{\partial \Phi_1}{\partial d_{11}} = d_{11}(11.52d_{11} + 5.8\lambda_1 + \lambda_2) = 0$$

$$d_{21}\frac{\partial \Phi_1}{\partial d_{21}} = d_{21}(0.105d_{21} + 2.85\lambda_1 + \lambda_2) = 0$$

$$d_{31}\frac{\partial \Phi_1}{\partial d_{31}} = d_{31}(\lambda_1 + \lambda_2) = 0$$

$$d_{11} \geq 0$$

$$d_{21} \geq 0$$

$$d_{31} \geq 0$$
\[ \frac{\partial \phi_1}{\partial \lambda_1} = 5.8d_{11} + 2.85d_{21} + d_{31} - 100K = 0 \] \hspace{1cm} (14)

\[ \frac{\partial \phi}{\partial \lambda_2} = d_{11} + d_{21} + d_{31} - x_1 = 0 \] \hspace{1cm} (15)

An observation of Figure 10 reveals the smallest and largest values of K that are obtainable from this process. Obviously the smallest value equals 1 and is obtained by investing all capital at both stages in the slack alternative. The largest value of K is obtained by investing all capital in the most favorable alternative at each stage, i.e., the one with the largest \( \sum_{i=1}^{m} p_{ijk} \). At stage 2 this equals 4.35 and at stage 1 it equals 5.8. Consequently the largest value of K obtainable is \((5.8)(4.35)\) or 25.23. For the smallest value of K, K=1, the optimal policy is \( d_{12} = 0, \ d_{22} = 0, \ d_{32} = 100, \ d_{11} = 0, \ d_{21} = 0, \) and \( d_{31} = 100 \). For the largest value of K, K=25.23, \( d_{11} \) is obtained from (14).

\[ 5.8d_{11} = 100(25.23) \]
\[ d_{11} = 435 \]

From (4), \( d_{12} \) is obtained:

\[ 4.35d_{12} = 435 \]
\[ d_{12} = 100 \]

The optimal policy for K=25.23 is \( d_{12} = 100, \ d_{22} = 0, \ d_{32} = 0, \ d_{11} = 435, \ d_{21} = 0, \) and \( d_{31} = 0 \). These two optimal policies are intuitive and were found rather informally.
Later in this example, these policies will be formally determined.

These two cases discussed above are somewhat less interesting than other values of $K$ in the interval $(1, 25, 23)$. As will be illustrated, the optimal policy is piecewise linear in $K$ over certain intervals of $K$, therefore it is possible to calculate the optimal policy over the entire range of $K$. In general the Kuhn-Tucker conditions include equalities, inequalities, linear equations, and quadratic equations and therefore their solution may be no simple matter to obtain.

Case 1: Assume $d_{11} = d_{12} = 0$.

Stage 1:

\[
\min \left[ 0.0525d_{21}^2 \right] \quad \text{subject to:}
\]

\[
2.85d_{21} + d_{31} = 100K \\
d_{21} + d_{31} = x_1 \\
d_{21}, d_{31} \geq 0
\]

The generalized Lagrange function is

\[
\Phi_1 = 0.0525d_{21}^2 + \lambda_1 (2.85d_{21} + d_{31} - 100K) + \lambda_2 (d_{21} + d_{31} - x_1)
\]

The Kuhn-Tucker conditions are

\[
\frac{\partial \Phi_1}{\partial d_{21}} = 0.105d_{21} + 2.85\lambda_1 + \lambda_2 \geq 0
\]

\[
\frac{\partial \Phi_1}{\partial d_{31}} = \lambda_1 + \lambda_2 \geq 0
\]
\[\frac{\partial \phi_1}{\partial d_{21}} = d_{21}(0.105d_{21} + 2.85\lambda_1 + \lambda_2) = 0\]
\[\frac{\partial \phi_1}{\partial d_{31}} = d_{31}(\lambda_1 + \lambda_2) = 0\]
\[d_{21} \geq 0\]
\[d_{31} \geq 0\]
\[\frac{\partial \phi_1}{\partial \lambda_1} = 2.85d_{21} + d_{31} = 100K \quad (16)\]
\[\frac{\partial \phi_1}{\partial \lambda_2} = d_{21} + d_{31} = x_1 \quad (17)\]

Solving (16) and (17) simultaneously gives
\[d_{21} = \frac{100K - x_1}{1.85} = 54.05K - 0.5405x_1\]
and
\[f_1(x_1) = 0.0525(54.05K - 0.5405x_1)^2.\]

Stage 2:

\[f_2(x_2) = \text{minimum} \left[ 2.94d_{22}^2 + f_1(x_1) \right]\]
subject to:
\[d_{22} + d_{32} = 100\]
\[d_{22}, d_{32} \geq 0\]

The generalized Lagrange function is
\[\bar{\phi}_2 = 2.94d_{22}^2 + 0.0525 \left[ 54.05K - 0.5405(3.6d_{22} + d_{32}) \right]^2 + \lambda_3(d_{22} + d_{32} - 100).\]

The solution to the resulting Kuhn-Tucker conditions at Stage 2 is
\[
\begin{align*}
\text{d}_{22} &= 1.31K - 1.31 \\ 
\text{d}_{32} &= 101.31 - 1.31K \\
\lambda_3 &= 2.97K - 2.97.
\end{align*}
\]

Backsolving gives
\[
\begin{align*}
\text{d}_{21} &= 52.21K - 52.21 \\ 
\text{d}_{31} &= 148.8 - 48.8K \\
\lambda_1 &= 2.97 - 2.97K \\
\lambda_2 &= 2.97K - 2.97.
\end{align*}
\]

Satisfying the Kuhn-Tucker conditions results in an interval for K for this case of approximately (1, 2.95). Note that the decision variables are linear in K in this interval. For \(K = 1\), (18), (19), (20), and (21) give as the optimal policy \(d_{12} = 0, d_{22} = 0, d_{32} = 100, d_{11} = 0, d_{21} = 0\), and \(d_{31} = 100\). Notice that this solution is identical to the intuitive solution found earlier for \(K = 1\).

Case 2: Assume \(d_{31} = 0\).

Stage 1:
\[
f_1(x_1) = \min \left[ 5.76d_{11}^2 + 0.0525d_{21}^2 \right]
\]
subject to: \(5.8d_{11} + 2.85d_{21} = 100K\)
\(d_{11} + d_{21} = x_1\)
\(d_{11}, d_{21} \geq 0.\)

The generalized Lagrange function \(\bar{f}_1\) is
\[
\bar{f}_1 = 5.76d_{11}^2 + 0.0525d_{21}^2 + \lambda_1(5.8d_{11} + 2.85d_{21} - 100K)
+ \lambda_2(d_{11} + d_{21} - x_1).
\]

The following Kuhn-Tucker conditions result:
\[ 11.52d_{11} + 5.8\lambda_1 + \lambda_2 = 0 \]
\[ 0.105d_{21} + 2.85\lambda_1 + \lambda_2 = 0 \]
\[ 5.8d_{11} + 2.85d_{21} = 100K \]
\[ d_{11} + d_{21} = x_1. \]

Solving the last two Kuhn-Tucker conditions simultaneously gives
\[ d_{11} = 33.9K - 0.966x_1 \quad \text{and} \]
\[ d_{21} = 1.966x_1 - 33.9K. \]

Therefore
\[ f_1(x_1) = 5.76(33.9K - 0.966x_1)^2 + 0.0525(1.966x_1 - 33.9K)^2. \]

Stage 2:
\[ f_2(x_2) = \text{minimum} \ [6.35d_{12}^2 + 2.94d_{22}^2 + f_1(x_1)] \]
subject to: \[ d_{12} + d_{22} + d_{32} = 100 \]
\[ d_{12}, d_{22}, d_{32} \geq 0 \]

The generalized Lagrange function is
\[ \mathcal{L}_2 = 6.35d_{12}^2 + 2.94d_{22}^2 + 5.76 [33.9K - 0.966(4.35d_{12} + 3.6d_{22} + d_{32})]^2 \]
\[ + 0.0525 [1.966(4.35d_{12} + 3.6d_{22} + d_{32}) - 33.9K]^2 \]
\[ + \lambda_3 (d_{12} + d_{22} + d_{32} - 100). \]

Since it is assumed that \( d_{12}, d_{22}, \) and \( d_{32} \neq 0, \) the four Kuhn-Tucker condition equations are linear and when solved give
\[ d_{12} = 4.28K - 12.4 \]
\[ d_{22} = 7.17K - 20.8 \]
\[ d_{32} = 133 - 11.45K \]
\[ \lambda_3 = 16.2K - 47.1 \]
and back solving gives 

\[ d_{11} = 2.03K - 4.08 \]
\[ d_{21} = 30.95K + 8.3 \]
\[ d_{31} = 0 \]
\[ \lambda_1 = 16.2 - 6.8K \]
\[ \lambda_2 = 16.2K - 47.1. \]

Satisfying the Kuhn-Tucker conditions results in an interval for K for this case of approximately (2.95, 11.65).

Case 3: Assume \( d_{31} = d_{32} = 0 \).

Stage 1: Stage 1 is identical to stage 1 for Case 2.

Stage 2: 

\[ f_2(x_2) = \text{minimum} \left[ 6.35d_{12}^2 + 2.94d_{22}^2 + f_1(x_1) \right] \]

subject to: 
\[ d_{12} + d_{22} = 100 \]
\[ d_{12}, d_{22} \geq 0 \]

Solving the stage 2 optimization problem by convex programming gives 

\[ d_{12} = 11.41K - 96.7 \]
\[ d_{22} = 196.7 - 11.41K \]
\[ d_{32} = 0 \]
\[ \lambda_3 = 1104K - 12740 \]

and back solving gives 

\[ d_{11} = 25.65K - 278 \]
\[ d_{21} = 564 - 17.2K \]
\[ d_{31} = 0 \]
\[ \lambda_1 = 1105 - 100.5K \]
\[ \lambda_2 = 289K - 3220. \]
Satisfying the Kuhn-Tucker conditions results in an interval for $K$ for this case of approximately $(11.65, 17.25)$.

Case 4: Assume $d_{31} = d_{32} = d_{22} = 0$.

Stage 1: Stage 1 is identical to stage 1 for Cases 2 and 3.

Stage 2:
\[
\begin{align*}
  f_2(x_2) &= \text{minimum } \left[ 6.35d_{12}^2 + f_1(x_1) \right] \\
  \text{subject to: } &d_{12} = 100 \\
  &d_{12} \geq 0
\end{align*}
\]

Obviously $d_{12} = 100$ and $x_1 = 4.35(100) = 435$. Therefore
\[
\begin{align*}
  d_{11} &= 33.9K - 0.966(435) = 33.9K - 420.21 \\
  d_{21} &= 1.966(435) - 33.9K = 855.21 - 33.9K \\
  \lambda_1 &= 1670 - 133.6K \\
  \lambda_2 &= 384.5K - 4848 \\
  \lambda_3 &= 1670K - 22400.
\end{align*}
\]

Satisfying the Kuhn-Tucker conditions results in an interval for $K$ for this case of approximately $(17.25, 25.23)$.

Table 3 gives approximate values for the optimal policies and the minimum variance for the end points of the four regions of $K$. At the boundaries of the four cases, the minimum variances are only approximately equal as are the optimal policies. The discrepancies in the boundary values are attributed to the lengthy calculations. Figure 11 shows for all values of $K$ in the interval $(1, 25.23)$, the approximate optimal policy and the approximate minimum variance for this multistage process.
Table 3
Optimal Policy and Minimum Variance for
Two-Outcome, Three-Alternative,
Two-Stage Process

<table>
<thead>
<tr>
<th>CASE 1</th>
<th>CASE 2</th>
<th>CASE 3</th>
<th>CASE 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>K</td>
<td>1</td>
<td>2.9</td>
<td>2.9</td>
</tr>
<tr>
<td>d_{12}</td>
<td>0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>d_{22}</td>
<td>0</td>
<td>2.5</td>
<td>0.0</td>
</tr>
<tr>
<td>d_{32}</td>
<td>100</td>
<td>97.5</td>
<td>99.8</td>
</tr>
<tr>
<td>d_{11}</td>
<td>0</td>
<td>0.0</td>
<td>1.8</td>
</tr>
<tr>
<td>d_{21}</td>
<td>0</td>
<td>99.2</td>
<td>98.1</td>
</tr>
<tr>
<td>d_{31}</td>
<td>100</td>
<td>7.3</td>
<td>0.0</td>
</tr>
<tr>
<td>V</td>
<td>0</td>
<td>533</td>
<td>524</td>
</tr>
</tbody>
</table>

Variables \(d_{11}\) and \(d_{12}\) are the investments in the high expectation alternatives, \(d_{21}\) and \(d_{22}\) are the investments in the medium expectation alternatives, and \(d_{31}\) and \(d_{32}\) are the investments in the low expectation alternatives. Generally speaking, in the region of low expectation (below \(K=2\)), more capital is invested in alternatives 3; in the region of mid-expectation (between \(K=8\) and \(K=13\)), more capital is invested in alternatives 2; and in the region of high expectation, (above \(K=19\)), more capital is invested in alternatives 1. As the expectation level \(K\) increases, capital is shifted from the riskless, low expectation alternatives, to the riskier, high expectation alternatives. For \(K = 11.65\),
Figure 11. Optimal Policy and Minimum Variance Versus K for Two-Outcome, Three-Alternative Two-Stage Process
a middle value of $K$, consider the investments in alternatives 2 at both stages. Alternative 2 at stage 1 has a very low coefficient of variance (0.0525) as opposed to alternative 2, stage 2 (2.94), while the expectations are relatively close (2.85 versus 3.6). It is to be expected that more capital would be invested in alternative 2, stage 1 since more capital is expected to be available, yet the extreme difference is additionally attributed to the wide difference in variances.

In summary, this chapter has presented a computational procedure for a multiple outcome, multiple alternative, multistage process. Additional research may result in general expressions for the optimal policy and minimum variance similar to those derived in Chapters II and III. Solutions have been obtained for all values of $K$ between 1 and 25.23. It is therefore possible for an investor to choose a combination of expectation level $K$ and variance $V$ that he prefers. As $K$ increases, investment in the low expectation alternatives, $d_{31}$ and $d_{32}$, decreases; investment in the middle expectation, low variance alternatives, $d_{21}$ and $d_{22}$, and in the high expectation, high variance alternatives, $d_{11}$ and $d_{12}$, increases. However, investments in the middle expectation, low variance alternatives increase more rapidly until a value of $K = 11.65$ is reached. For values of $K$ larger than 11.65, the middle expectation, low variance investments decrease while investments in the high
expectation, high variance alternatives increase. As a general comment, for \( K \) less than 11.65, it may be said that variance dominates the determination of investments and for \( K \) greater than 11.65, expected value dominates. These results are intuitive. As the system is stressed, i.e., large values of \( K \) are chosen, the minimum variance increases rapidly. For small values of \( K \), changes in \( K \) effect the minimum variance much less than equal changes in \( K \) at larger values of \( K \).

The four cases presented above are not the only possible combinations of the \( d_{ij} \)'s for the various values of \( K \). There are sixty-four combinations of the \( d_{ij} \)'s, however, these four cases give the global minimum variance of all the possible cases.
CHAPTER V

COMPARISON WITH OTHER MULTISTAGE CRITERIA

The purpose of this chapter is to compare the model proposed in this paper with some of the common sequential or multistage decision making models. The chapter will conclude by investigating the question of equivalence between the model introduced in this paper and the expected utility maximization model. Equivalence is defined in this paper to mean that for equal expected value, two or more models prescribe optimal policies which result in equal variance of final capital.

Model Comparisons

The models to be compared with the model proposed in this paper (hereafter called the constrained minimization-of-variance model) are the expected utility (linear, logarithmic, and power utility functions) maximization models. These models have been chosen for comparison because of their broad application to multistage decision processes and because of their dependence on expected value and variance for determination of optimal policies. Comparisons will be made for two investment processes - a two outcome, one
alternative, two-stage process and a four alternative, one-stage process.

Recall from Appendix A, Part 1 that in order to maximize the expected value of capital after n stages, a decision maker should invest all his available capital on the alternative whose probability of success is greater than one half when the rate of return for success s and the rate of return for failure r are equal to one. When r and s are not equal to one, the optimal investment for stage i, \( d_i^* \), is

\[
d_i^* = \begin{cases} 
  x_i & \text{if } pr-qs > 0 \\
  0 & \text{otherwise}.
\end{cases}
\]

The maximum expected value is \( x(1+pr-qs)^n \), where x is defined as the initial capital available for investment and \( x_i \) is the capital available at stage i. It is assumed that p, q, r, and s remain constant throughout all stages for the comparisons of this chapter.

Define \( f_n(x) \) as the maximum of the expected value of the logarithm of capital after n stages, where x is the initial capital. The decision maker has \( x+rd \) at the end of one stage if successful and \( x-sd \) if unsuccessful, where d is the amount of the investment. It was shown in Appendix A, Part 2 that

\[
f_n(x) = \log x + n(p \log p + q \log q + p \log \frac{rs}{s} + q \log \frac{rs}{r})
\]

and the optimal investment policy is

\[
d_i^* = \begin{cases} 
  0 & \text{if } pr-qs < 0 \\
  \frac{pr-qs}{rs} x_i & \text{if } pr-qs \leq rs \\
  x_i & \text{if } pr-qs > rs.
\end{cases}
\]

The optimal investment policy and the maximum expected utility of capital after n stages for a power utility function of the form \( Cx^{M+1} \), where \( C = 1/(M+1) \) and \( -1 < M < 0 \), were derived in Appendix A, Part 3:

\[
f_n(x) = Cx^{M+1} \left[ \frac{r+s}{s(qs)^{1/(M+1)}} \right]^n \left[ \frac{p(qs)^{1/(M+1)} + q(pr)^{1/(M+1)}}{s(qs)^{1/(M+1)} + r(pr)^{1/(M+1)}} \right]^n
\]

and

\[
d_i^* = \begin{cases} 
  0 & \text{if } (qs)^{1/M} - (pr)^{1/M} < 0 \\
  \frac{(qs)^{1/M} - (pr)^{1/M}}{s(qs)^{1/M} - r(pr)^{1/M}} x_i & \text{if } 0 \leq (qs)^{1/M} - (pr)^{1/M} \leq s(qs)^{1/M} - r(pr)^{1/M} \\
  x_i & \text{if } (qs)^{1/M} - (pr)^{1/M} > s(qs)^{1/M} - r(pr)^{1/M}.
\end{cases}
\]

For the constrained minimization-of-variance model derived in Chapter II, for an expected value equal to K, \( K \geq 1 \), times the initial capital, i.e., \( Kx \), the optimal policy is

\[
d_i^* = \frac{k-1}{n(pr-qs)} x \text{ for all stages } i.
\]

For the given parameter values and the expected-value-of-capital goal, the optimal policy above results in minimum variance of capital.
Example 1

Assume the initial capital $x$ is 10, the number of stages $n$ is 2, the probability of success $p$ is $3/4$, and the rates of return for success and for failure, $r$ and $s$, are 1. The optimal policy for the expected-logarithm-of-capital-maximization model is

$$d_i^* = \frac{pr qs}{rs} x_i = \frac{1}{2} x_i.$$

At each stage, one half of the capital available will be invested. Tree diagram (a) in Figure 12 illustrates the possible values of capital for this process. The expected value of capital, $(.5625)(22.50) + (.1875)(7.50) + (.1875)(7.50) + (.0625)(2.50)$, is 15.625 and the variance of capital, $(.5625)(22.50)^2 + (.1875)(7.50)^2 + (.1875)(7.50)^2 + (.0625)(2.50)^2 - (15.625)^2$, is 62.11.

The optimal policy for the constrained minimization-of-variance model is

$$d_i^* = \frac{K-1}{n(pr qs)} x = \frac{1.5625-1}{2(3/4-1/4)} (10) = 5.625.$$

The value of $K$ is selected as 1.5625 so that the expected-value goal, 15.625, is equal to the expected value of capital which resulted from the logarithmic-utility-maximization model. Tree diagram (b) in Figure 12 illustrates the possible values of capital for this process. The expected value of capital, the goal for this model, is 15.625 and the
variance of capital, \((.5625)(21.25)^2 + (.1875)(10)^2 + (.1875)(10)^2 + (.0625)(-1.25)^2 - (15.625)^2\), is 47.46.

The optimal policy prescribed by the constrained minimization-of-variance model results in less variance of final capital for the same expected value than the variance of final capital prescribed by the expected-utility-maximization model. This fact will be discussed further at the end of this chapter.

Example 2

In this example assume the decision maker's utility for capital is described by a power function of the form \(2x^{1/2}\). The optimal policy resulting from this power function and from the values of the parameters assumed for example 1 is

\[
d^*_i = \frac{(1/4)^{-2} - (3/4)^{-2}}{(1/4)^{-2} + (3/4)^{-2}} x_i = 0.8x_i.
\]

At each stage 80% of the available capital will be invested. Tree diagram (c) in Figure 12 illustrates the possible values of capital for this process. The expected value and variance of capital for this model are 19.60 and 211.20 respectively.

For the constrained minimization of variance model, with \(K\) chosen to be 1.96 in order for the expected value of capital to be equal to that of the power utility model, the optimal policy is

\[
d^*_i = \frac{K-1}{n(pr-qs)} x = \frac{19.6-10}{2(3/4-1/4)} = 9.60.
\]
Tree diagram (d) in Figure 12 illustrates the possible values of capital for this process. The expected value and variance of capital for this model are 19.60 and 138.24 respectively. Once again the optimal policy prescribed by the constrained minimization-of-variance model results is less variance of final capital for the same expected value than the variance of final capital from the expected-utility-maximization model.

**Example 3**

This example involves a different multistage process from that of examples 1 and 2. Consider a four-proposal, one-stage process in which only one proposal may obtain. Define $p_i$ as the probability that proposal $i$ obtains, $r_i$ as the rate of return from proposal $i$, $x$ as the total budget available for investment, and $d_i$ as the amount of capital allocated to proposal $i$. Since only one proposal may obtain, it is true that $\sum_{i=1}^{4} p_i = 1$, and assume that $\sum_{i=1}^{4} d_i = x$. If proposal $i$ obtains, the amount of capital available is $r_i d_i$. The constrained minimization of variance criteria may be written as

$$\text{minimize } \left[ \sum_{i=1}^{4} p_i (r_i d_i)^2 - \left( \sum_{i=1}^{4} p_i r_i d_i \right)^2 \right]$$

subject to: $\sum_{i=1}^{4} p_i r_i d_i = Kx$

$\sum_{i=1}^{4} d_i = x$. 
(a) Expected-Logarithm-of-Capital-Maximization Model

(b) Constrained Minimization-of-Variance Model: $K = 1.5625$

(c) Expected-Power-of-Capital-Maximization Model

(d) Constrained Minimization-of-Variance Model: $K = 1.96$

Figure 12. Decision Trees for Two-Outcome, One-Alternative, Two-Stage Process
This optimization problem may be solved by convex programming. A closed form solution was obtained by Rapoport (1970).

If, on the other hand, the criterion is the maximization of the expected value of capital, it can be shown that the decision maker should invest his entire capital on the most favorable proposal, i.e., the one whose product $p_i r_i$ is largest. For maximization of the expected logarithm of capital, the optimal policy can be shown to be $d_i^* = p_i x$, $i = 1, 2, 3, 4$, and for maximization of the expected power of capital, the optimal policy can be shown to be

$$d_i^* = \frac{p_i}{\sum_{i=1}^{4} p_i r_i - (M+1)/M} x.$$

The power utility function is of the form $C x^M$, where $C = 1/M$ and $-1 < M < 0$.

Assume $x = $100,000, $r_i = 4$, $i = 1, 2, 3, 4$, $p_1 = 0.4$, $p_2 = 0.3$, $p_3 = 0.2$, and $p_4 = 0.1$.

The maximization-of-expected-value model prescribes that $d_1^* = $100,000, $d_2^* = d_3^* = d_4^* = 0$ and the resulting maximum expected value is $(0.4)(4)(100,000)$ or $160,000$. The constrained minimization-of-variance model with an expected value of capital goal of $160,000$ (i.e., $K = 1.6$), prescribes the identical optimal policy. The variance for both models for this policy is
The reason that the constrained minimization-of-variance model prescribes the same optimal policy, with the same minimum variance, as the maximization-of-expected-value model is because this policy is the only policy that can achieve this expected value. Also, at this value of expected capital, there is only one value of variance of capital possible. In other words, when the expected-capital goal equals the maximum expected capital possible, the two models prescribe the same policy. At this value of expected capital, the constrained minimization-of-variance model disregards variance in order to achieve the expected-value-of-capital goal. However, if the expectation level $K$ is decreased to 1.5, the optimal policy for the constrained minimization-of-variance model is $d_1^* = 70,000$, $d_2^* = 30,000$, $d_3^* = d_4^* = 0$ and the minimum variance of capital is $12,610 \times 10^6$. A decrease of 6.25% in expected value of capital decreases the variance of capital 67.2%.

The optimal policy for the logarithmic utility function is $d_1^* = 40,000$, $d_2^* = 30,000$, $d_3^* = 20,000$, and $d_4^* = 10,000$. This policy results in an expected value of capital of $120,000$ and a variance of capital of $1600 \times 10^6$. The constrained minimization-of-variance model with an expected value goal of $120,000$, prescribes the following optimal policy: $d_1^* = 36,250$, $d_2^* = 32,778$, $d_3^* = 25,833$, and
The optimal policy for the power utility function with $M = \frac{1}{2}$ is $d_1^* = \$53,333, d_2^* = \$30,000, d_3^* = \$13,333$ and $d_4^* = \$3,333$. This policy results in an expected value of capital of $\$133,332$ and a variance of capital of $5334 \times 10^6$.

The constrained minimization-of-variance model with an expected value goal of $\$133,332$, prescribes the following optimal policy: $d_1^* = \$48,134; d_2^* = \$37,029; d_3^* = \$14,819; d_4^* = 0$. The expected value of capital is $\$133,332$ and the variance is $4342 \times 10^6$.

Table 4 summarizes the results for example 3. Notice that the comparisons between the constrained minimization-of-variance model and the maximization-of-expected-utility models (linear, logarithmic, and power) are made for equal expected values of capital in order to make meaningful comparisons of the variances of capital. For each case, the constrained minimization-of-variance model results in smaller variance for the same expected capital except for the expected-value-maximization model. The largest expected value possible for the parameter values chosen is $\$160,000 (K = 1.6). Only at this largest value of K will the policy of the two models be the same.

The Question of Equivalence

It can be shown that the utility of any probability distribution of capital may be approximately expressed in
Table 4
Comparison of Optimal Policy and Variance for Expected Value, Expected Utility, and Constrained Minimization-of-Variance Models

<table>
<thead>
<tr>
<th>Model</th>
<th>$d_1^*$</th>
<th>$d_2^*$</th>
<th>$d_3^*$</th>
<th>$d_4^*$</th>
<th>Exp Capital</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max Expected Value</td>
<td>100000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>160000</td>
<td>38400x10^6</td>
</tr>
<tr>
<td>Constrained Min-of-Var</td>
<td>100000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>160000</td>
<td>38400x10^6</td>
</tr>
<tr>
<td>Max Expected Utility:Log</td>
<td>40000</td>
<td>30000</td>
<td>20000</td>
<td>10000</td>
<td>120000</td>
<td>1600x10^6</td>
</tr>
<tr>
<td>Constrained Min-of-Var</td>
<td>36250</td>
<td>32778</td>
<td>25833</td>
<td>5000</td>
<td>120000</td>
<td>1343x10^6</td>
</tr>
<tr>
<td>Max Expected Utility:Power</td>
<td>53333</td>
<td>30000</td>
<td>13333</td>
<td>3333</td>
<td>133332</td>
<td>5334x10^6</td>
</tr>
<tr>
<td>Constrained Min-of-Var</td>
<td>48134</td>
<td>37029</td>
<td>14819</td>
<td>0</td>
<td>133332</td>
<td>4342x10^6</td>
</tr>
</tbody>
</table>

Terms of the mean and variance of the distribution (Morris, 1964). For a risk averse decision maker who adheres to the expected utility maxim, it is sometimes assumed that for a given expected value, expected utility is maximized by minimizing the variance of a distribution of returns. Markowitz (1959, page 284) says "The portfolios which maximize any of the above approximations [approximations to risk-averse utility functions] minimize variance for some [some value of] mean." In other words, there exists concave utility functions which when maximized result in variance which is
almost equal to the variance that results when the variance of capital is minimized for a particular expected value.

For the models compared above, the constrained minimization-of-variance model prescribes an optimal policy which results in less variance of final capital for a given expected value than the expected-utility-maximization model. These results do not contradict the statements that maximizing expected utility for a particular expected value results in minimum variance or that the utility of any probability distribution may be expressed in terms of its mean and variance. Both of these statements are approximations that are exactly true only under special conditions. Tobin (1959) states that the expectation-variance criterion is valid if the utility function is quadratic or the distribution of returns is a two-parameter distribution. Neither of these two conditions are satisfied by the utility functions used in the comparisons of this chapter. Recall that equivalence is defined in this paper to mean that for a particular expected value, two models prescribe an optimal policy which results in equal variance of final capital. Obviously for the models compared, such an equivalence does not exist.

Examine more closely the constrained minimization-of-variance criterion used in Example 2. Consider the first stage.
The expected value of capital is \((\cdot .75)(15.625) + (\cdot .25)(4.375)\) or \$12.8125 and the variance is \((\cdot .75)(15.625)^2 + (\cdot .25)(4.375)^2 - (12.8125)^2\) or 23.73. Now consider the top half of tree diagram (b) at stage 2.

The expected value and variance of capital are \$18.44 and 23.73 respectively. For the bottom half,

the expected value and variance are \$7.19 and 23.73 respectively.

The variance at the three decision nodes are equal; however, the expected values of capital are not equal. A utility model would not prescribe an optimal policy which results in equal stage variances and different stage expected values of capital.

Does this lack of equivalence between these models invalidate utility theory or the theory contained in this paper? The answer to both parts of this question is no. Whether utility theory is dependent only on expectation and variance or whether it is dependent on these and additional
moments of the distribution of returns is an empirical question to which this paper does not address itself. It is the purpose of this paper to present models and methodologies for multistage decision making which do not include utility theory, which prescribe reasonable, consistent normative policies, and which give these policies with minimum computational effort.
CHAPTER VI
APPLICATIONS

The selection of areas for the application of the criteria of this paper is intended in no way to indicate a limitation of applicability but rather the author's preferences. In addition the selection of examples with the various constraints is a compromise between realism on the one hand and simplicity or freedom from complexity on the other hand. It is felt that these criteria are applicable to almost any probabilistic multistage decision process.

Bidding Models

Bidding is competition for the right to perform a service or to acquire property. Opposing bidders compete for these rights under rules established by the government and/or the party who puts up the rights for bid. Most services and property that the government acquires from private sources involve submission of sealed bids (Morris, 1968). The theory of bidding involves two points of view - the sponsor's as well as the bidder's. The sponsor's point of view concerns solicitation of bids that most fairly reflect the cost of performing the contract, while the bidder desires establishment of a bidding practice which results in
improvement of profits. This section deals with the latter point of view. Most businesses are involved in some form of bidding. In fact, the pricing of property or service can be conceived as bidding for customer dollars (Churchman, Ackoff, and Arnoff, 1957).

There are two sources of uncertainty involved in bidding decisions - the cost of performing the contract and the amount of the bid which, when submitted, results in winning the contract. Both of these sources of uncertainty must be considered in determination of the bid. Of the two common types of bidding situations that occur, closed bidding and open, or auction, bidding, only the former will be discussed. The model to be developed concerns closed bidding where the number of bidders is large and unknown. For the case of a small number of bidders, game theory techniques may be used. The assumption that the competitor's bids are independent of the firm's bid need not be made when game theory techniques are used.

Consider a case where the government solicits bids for a contract from many companies within the same industry. Each company submits a closed bid and the company submitting the lowest bid is awarded the contract. The number of companies that submit bids is unknown. In order to develop a bidding policy, the company must define its objective. The most common objective in bidding literature is maximization of total expected profit. There are additional
objectives such as achievement of at least a given percentage gain on investment (cost), obtaining contracts in order to meet overhead and keep the company operating, minimization of expected losses, or, since profit is a random variable, reduction of its variance when financial resources are limited. A firm's unique economic status will somehow determine the bidding objective(s) and constraints. In this paper a bidding model will be developed which has as its objective the minimization of the variance of profits subject to some constraints. A portion of the model development is patterned after Churchman, et. al. (1957), although the objective and the solution technique are totally different.

Let C be the estimated cost of fulfilling the contract. The actual cost can be determined only after performing the contract. However, by studying past estimated and actual costs, it is possible to determine the bias of the cost estimate. Let S be the ratio of true to estimated cost, h(S) be the density function of this ratio, and d be the amount bid for the contract. If this bid wins, the profit will be d-SC. Let P(d) be the probability that d will be the lowest bid and consequently win the contract. The expected profit is

\[ E(d) = \int_0^\infty P(d) (d-SC) h(S) dS, \]

where h(S)dS is the probability that the ratio of true to estimated cost is between S and S+dS. E(d) may be written as
The first integral is equal to one and the second integral is the expected value of \( S \). Therefore

\[
E(d) = P(d) (d - C'),
\]

where \( C' = C \int_0^\infty Sh(S) dS \) is the expected cost corrected for the bias of the cost-estimating procedures. Another assumption implicit in the development is that the cost of preparing and submitting the bid is negligible, and if the bid fails, the resultant profit is zero.

"Bids are often a matter of public record, so that it may be possible to gain an insight into a competitor's bidding strategy by studying the ratio of his bid to our cost estimate for past contracts" (Stark and Mayer, 1970). Consider competitor A. On past contracts on which he bid and on which one's firm made a cost estimate, the ratio \( r \) of his bid to one's cost estimate is determined and, if sufficient data is available, the density function, \( f(r) \), is also determined. If it is known exactly who is to bid, the probability of winning with a bid of \( d \) is easily determined. It is assumed that each competitor bids as he has done in the past, therefore the probability of being lower than A by bidding \( d \) is the area to the right of \( r = d/C \) under the graph of \( f(r) \) (see Figure 13). The probability of being the lowest bidder, among A and B, with a bid of \( d \) is the
product of the probability of being lower than A and the probability of being lower than B.

![Diagram](image)

**Figure 13. Bidding Behavior Patterns**

When it is unknown who the competitors will be, the concept of an average bidder is used. By combining all data of opposing bids to one's cost estimate, a single density function \( f_A(r) \) is obtained. Let \( f_A(r) \) be the density function of the ratio of the average competitor's bid to one's cost estimate. The probability of being lower than \( k \) average bidders is \( \left[ \int_{d/C}^{\infty} f_A(r)dr \right]^k \). Let \( g(k) \) be the probability of \( k \) competitors submitting bids. Then

\[
P(d) = \sum_{k=0}^{\infty} \left\{ g(k) \left[ \int_{d/C}^{\infty} f_A(r)dr \right]^k \right\}.
\]

It now remains a problem of fitting distributions to the data available for \( r \) and \( k \). Churchman, et. al. (1957) suggest a gamma distribution for \( f_A(r) \) and a Poisson distribution for the number of bidders, i.e.,

\[
f_A(r) = \frac{a^{(b+1)}}{b!} r^b e^{-ar} \quad \text{and}
\]
where a and b are constants of the gamma distribution and λ is the expected number of bidders. It then follows that

\[ P(d) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \lambda \int_{d/C}^{\infty} \frac{a^{(b+1)}}{b!} e^{-ar} dr \right]^k. \]

Since the summation is an exponential series, \( P(d) \) may be written as

\[ P(d) = e^{-\lambda} \lambda \int_{d/C}^{\infty} \frac{a^{(b+1)}}{b!} e^{-ar} dr. \]

By performing the indicated integration in the expression above, \( P(d) \) becomes

\[ P(d) = \exp \left\{ -\lambda \left[ 1 - \sum_{i=0}^{b} \frac{1}{i!} \left( \frac{ad}{C} \right)^i e^{-ad/C} \right] \right\}. \]

Notice that the summation above is simply a cumulative Poisson distribution and may be found in appropriate tables.

The expected profit \( E(d) \) now becomes

\[ E(d) = (d-C') \exp \left\{ -\lambda \left[ 1 - \sum_{i=0}^{b} \frac{1}{i!} \left( \frac{ad}{C} \right)^i e^{-ad/C} \right] \right\}. \]

For the expected profit maximization model, it is simply necessary to differentiate \( E(d) \) with respect to \( d \), set the derivative equal to zero, and solve for the optimal \( d \). The variance of profit can be determined using the equation
above for expected profit and the following equation for variance:

\[ V(d) = E(d^2) - [E(d)]^2. \]

\[
V(d) = (d-C')^2 \exp \left\{ -\lambda \left[ \sum_{i=0}^{b} \frac{1}{i} \left( \frac{\frac{ad}{c}}{i} \right)^i e^{-\frac{ad}{C}} \right] \right\} - \left\{ (d-C') \exp \left[ -\lambda \left[ \sum_{i=0}^{b} \frac{1}{i} \left( \frac{\frac{ad}{c}}{i} \right)^i e^{-\frac{ad}{C}} \right] \right]\right\}^2
\]

Consider the following example. The constants for the gamma distribution of \( r \) have been determined to be \( a=b=2 \).

The expected number of bidders is 4, the estimated cost of fulfilling the contract is \( $1000 \), and there is no bias in estimated cost. This means that the ratio of true to estimated cost from past data has a mean of one and consequently \( C = C' = $1000 \). Figure 14 illustrates the relationship between \( P(d), E(d), \) and \( V(d) \) and \( d \). Note that the \( E(d) \) versus \( d \) graph shows for any bid below estimated cost, the expected profit is negative, it increases as \( d \) increases until reaching a maximum at \( d = $1600 \), then it decreases as \( d \) continues to increase. Also as \( d \) increases, the probability of winning decreases and the variance of profit increases. The expected-profit-maximization model states that one should bid approximately \( $1600 \) in order to receive an expected profit of \( $50.28 \), a variance of profit of \( 27,640 \), and a probability of winning of 0.0838. Obviously an unconstrained minimization of variance model gives zero variance for a bid of \( d = $1000 \); however the expected profit also equals zero.
Figure 14. Expected Profit, Variance of Profit, and Probability of Winning Versus Amount Bid for Bidding Model
A more realistic approach might be to minimize the variance of profit subject to a given expected profit. For instance, if the constraint on expected profit were $43.80, then the minimum variance would be 11,222. A decrease in expected profit from $50.28 to $43.80, a 12.9% decrease, results in a 59.4% decrease in variance (from 27,640 to 11,222). It is certainly reasonable that a compromise such as this is a practical approach to decision making under risk.

A more interesting case concerns bidding on several contracts simultaneously. It will become evident that this bidding model is similar to the two-outcome, multistage investment process introduced earlier. The contracts correspond to the stages while success or failure on the contracts corresponds to the two outcomes. Consider the case of cost-independent, static bidding (Stark and Mayer, 1970). Cost-independence implies that there are no economies, such as the sharing of equipment or personnel in two construction projects at adjacent sites, to be realized by winning and executing any particular combination of contracts. Static bidding means that matters of timing are inconsequential, i.e., the sequence and duration of activities necessary for fulfilling contracts are not relevant. Each firm must decide on which contracts it wishes to bid and on the amount to bid. This decision may be further complicated by constraints on the bidding such as:

(1) the total amount bid on all contracts cannot exceed $D.
(2) the total estimated cost cannot exceed a given amount.

(3) the number of contracts won cannot exceed a given value.

(4) the probability of win must exceed some value before a bid can be submitted.

The total expected profit from \( n \) cost-independent contracts is

\[
E(\bar{d}) = \sum_{i=1}^{n} (d_i - C_i') P(d_i)
\]

and the total variance of profit is

\[
V(\bar{d}) = \sum_{i=1}^{n} \left\{ (d_i - C_i')^2 P(d_i) - [(d_i - C_i') P(d_i)]^2 \right\},
\]

where \( \bar{d} \) is a vector whose components are the bids made on the \( n \) contracts.

In the spirit of the criterion recommended earlier for multistage decision making, the criterion for simultaneous bidding on more than one contract, when restriction (1) from above is imposed, is

\[
\text{minimize } \sum_{i=1}^{n} \left\{ (d_i - C_i')^2 P(d_i) - [(d_i - C_i') P(d_i)]^2 \right\}
\]

subject to:

\[
\sum_{i=1}^{n} (d_i - C_i') P(d_i) = K \sum_{i=1}^{n} C_i'
\]  \hspace{1cm} (P1)

\[
\sum_{i=1}^{n} d_i \leq D
\]

\[
\bar{d} \geq \delta.
\]
In words, the objective is to minimize the variance of profits subject to: (1) the expected profit is some constant \( K \geq 0 \) times the total investment (i.e., the total cost of performing the \( n \) contracts), (2) the total bid on all contracts cannot exceed \( \$D \), and (3) the nonnegativity constraint on the bids.

If the sum of the optimal bids determined from the solution to \( P_1 \) with the second constraint deleted exceeds \( D \), then it may be assumed that the inequality constraint may be replaced by an equality constraint. The Lagrange function

\[
\Phi = \sum_{i=1}^{n} \left\{ (d_i - C_i')P(d_i) - \left[ (d_i - C_i')P(d_i) \right]^2 \right\}
\]

\[
+ \lambda_1 \left[ \sum_{i=1}^{n} (d_i - C_i')P(d_i) - K \sum_{i=1}^{n} C_i' \right] + \lambda_2 \left[ \sum_{i=1}^{n} d_i - D \right]
\]

may be solved for the optimal \( d_i^* \) by convex programming. However, when the number of contracts is large, it may be desirable to use dynamic programming, which is illustrated below by outlining the solution to an example problem.

Assume that our firm is interested in bidding for three cost-independent government contracts. The distribution for \( r \), the ratio of the average competitor's bid to our cost estimate, is gamma with \( a=b=2 \). This implies that the expected value of \( r \) equals 1.5 (\( \bar{r} = \frac{b+1}{a} \)). Past data indicate that there is no bias in the cost estimating procedure, i.e., \( C = C' \). For the first contract, the estimated cost of fulfilling the contract, \( C_1 \), is \( \$1000 \); for the second
contract. \( C_2 = \$3000 \); and for the third contract, \( C_3 = \$2000 \).

The expected number of firms, including one's firm, that will compete for the three contracts is 4. Considering the contracts individually, the following table shows for the maximum expected profit, the optimum bid \( d_i^* \) with the associated probability of winning and variance:

<table>
<thead>
<tr>
<th>Contract ( i )</th>
<th>( C_i' )</th>
<th>( d_i^* )</th>
<th>( E(d_i^*) )</th>
<th>( P(d_i^*) )</th>
<th>( V(d_i^*) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1000</td>
<td>$1600</td>
<td>$50.28</td>
<td>0.0838</td>
<td>27,640</td>
</tr>
<tr>
<td>2</td>
<td>$3000</td>
<td>$4800</td>
<td>$150.84</td>
<td>0.0838</td>
<td>248,760</td>
</tr>
<tr>
<td>3</td>
<td>$2000</td>
<td>$3200</td>
<td>$100.56</td>
<td>0.0838</td>
<td>110,560</td>
</tr>
</tbody>
</table>

The total expected profit is \$310.68 and the total variance is 386,960. As previously illustrated for the one contract situation, a small reduction in expected profit will reduce the variance of profit significantly. The firm decides that an expected profit of \$250 is the minimum acceptable expected profit. In addition, the government or some other creditor limits the total amount bid on the three contracts to \$8000 or less. Since the sum of the optimal bids (\$1600 + \$4800 + \$3200 = \$9600) without the government imposed total bid constraint exceeds \$8000, then the inequality constraint may be replaced by an equality constraint. Problem P1 may therefore be restated as:

\[
\text{minimize } \sum_{i=1}^{n} \left\{ (d_i - C_i')^2 P(d_i) - [(d_i - C_i')P(d_i)]^2 \right\}
\]
subject to: \[ \sum_{i=1}^{n} [(d_i - C_i)P(d_i)] \geq 250 \]
\[ d_1 + d_2 + d_3 = 8000 \]
\[ d_1, d_2, d_3 \geq 0. \]

It is possible to solve for all three \( d_i \) simultaneously. However, the simultaneous solution to the non-linear equations which would result from a Lagrange formulation is difficult. A simpler approach might be to delete the expected profit constraint, formulate a dynamic programming solution to the new problem, and check the resulting solution to see if it satisfies the expected profit constraint. If it does, then this solution is an optimal solution for the original problem; otherwise a more laborious solution technique must be used.

Basically this is an allocation problem. There are $8000 available for bidding and it must be determined what the optimal allocation is in order to minimize the variance of profit. The state variables \( x_i, i=0,1,2,3 \), are introduced and defined as the number of dollars available for bidding on the \( i \) remaining contracts. Since \( d_1 + d_2 + d_3 = 8000 \), replacing \( d_1 + d_2 + d_3 \) with \( x_3 \) results in \( x_3 = 8000 \).

Likewise, \( x_2 = x_3 - d_3 \), \( x_1 = x_2 - d_2 \), and \( x_0 = x_1 - d_1 \). Adding these four equations gives
\[ x_3 + x_2 + x_1 + x_0 = 8000 + x_3 - d_3 + x_2 - d_2 + x_1 - d_1 \]
or
\[ d_1 + d_2 + d_3 = 8000 - x_0. \]
Since \( d_1 + d_2 + d_3 = 8000 \), it follows that \( x_0 = 0 \) and consequently \( x_1 = d_1 \). Also since \( d \geq 0 \), it follows that \( d_1 = x_1 \geq 0 \),

\[
0 \leq d_2 \leq x_2,
\]

and

\[
0 \leq d_3 \leq x_3.
\]

Consequently the optimization problem may be restated as

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{n} \left[ (d_i - C'_i)^2 P(d_i) - [(d_i - C'_i)P(d_i)]^2 \right] \\
\text{subject to:} & \quad x_1 = x_2 - d_2, \quad x_1 = d_1 \geq 0 \\
& \quad x_2 = x_3 - d_3, \quad 0 \leq d_2 \leq x_2 \\
& \quad x_3 = 8000, \quad 0 \leq d_3 \leq x_3.
\end{align*}
\]

At stage 1, \( f_1(x_1) \) equals the minimum variance of profit resulting from an allocation of \( x_1 \) dollars among the last \( i \) contracts using an optimal policy with a bid of \( d_i \) on the \( i \)th contract. At stage 1

\[
f_1(x_1) = \text{minimum} \left\{ (d_1 - 1000)^2 P(d_1) - [(d_1 - 1000)P(d_1)]^2 \right\}
\]

subject to: \( d_1 = x_1 \).

At stage 2

\[
f_2(x_2) = \text{minimum} \left\{ (d_2 - 3000)^2 P(d_2) - [(d_2 - 3000)P(d_2)]^2 + f_1(x_1) \right\}
\]

subject to: \( 0 \leq d_2 \leq x_2 \)

\[
x_1 = x_2 - d_2,
\]

and similarly for the remaining stages. It is then possible to obtain the optimal allocation by backsolving. If this
solution satisfies the expected profit constraint, the original problem is solved; otherwise, some other optimization technique must be used to include the active expected profit constraint.

It should be noted that the solution to the optimization problem $P_1$ could conceivably result in values of $d_i$ which are less than $C_i'$ for some $i$. Likewise the optimal $d_i$ may result in $P(d_i)$, the probabilities of win, which are extremely small. It therefore may be reasonable to include constraints such as $d_i \geq C_i'$ and $P(d_i) \geq w_i$, where $0 \leq w_i \leq 1$.

There is an analogy between the simultaneous bidding model developed here and the two-outcome, multistage investment process described in Chapter II. In both cases there are two possible outcomes and in both cases the objective is to minimize the variance of final capital. A major difference concerns the expected value constraint. In the investment process, the capital available for subsequent stages depends on the success or failure at earlier stages. On the other hand in the bidding process, the capital available for investment in each contract, i.e., the cost of each contract, depends on the initial capital available and not on the outcome of individual contracts. A closer analogy between the bidding and investment processes is the multiple alternative, two-outcome, one stage process described in Chapter IV. The alternatives of the investment process correspond to the $n$ contracts of the bidding process.
"Capital budgeting literature has not yet given much consideration to the analysis of risk" (Hillier, 1963). Since the time this statement was made by Frederick S. Hillier, a considerable amount of literature has incorporated risk and expected value in capital investment (Hertz, 1964; Van Horne, 1966). Much of the literature deals with determination of the expected value of a prospective cash flow along with its variance or standard deviation. With this information, management is able to weigh the possible consequences of the proposed investment and make a sound decision regarding the proposal. However, even though management knows the expected value and variance of the cash flow, the choice of a "best" investment is not always clear cut, especially when many investments are being considered. Hertz (1964) determines the expected value and variance of return on the investment and uses these measures for comparing alternative opportunities. Van Horne (1966) proposes a method for selecting the best combination of investment opportunities. This method, which is similar to Markowitz's (1959) portfolio selection, involves the use of expected value-variance indifference curves which are obtained from management by a utility analysis. Although valid, this technique has the disadvantage of requiring the difficult task of utility measurement.

The technique proposed here for capital investment
involves the criterion introduced earlier in this paper and will be illustrated by an example. Consider capital investment in new product development. By a method to be discussed later, the number of new products to be developed has been set at $h$. Assume that the corporate staff has allotted $\$L$ for investment in the development of these $h$ products, that the expected-value-of-capital goal for this investment is $\$KL$, and that the goal is to be attempted under minimum risk conditions, where risk is measured by the variance of capital obtained from the new products. The probability distributions for the rates of return for the $h$ new products are obtained from the firm's data bank. The problem of determining the optimal allocation of the new product development budget is solved by convex programming.

Assume that the probability density for the rates of return for all $h$ products involves $m$ outcomes - hence an $h$ alternative, $m$ outcome optimization problem. The evolution of new products involves many stages which here will be condensed to the following four - basic development, production engineering, financial analysis, and market analysis (Magee, 1964). The problem is the allocation of new product development funds to each product at each of the stages so as to minimize the variance of capital subject to an expected-value-of-capital goal.

An interesting question concerns determination of the number of new products to consider. Perhaps there are only
h products under consideration or perhaps the firm's corporate staff has some procedure for limiting the number of products to h. Another procedure involves considering all new products, regardless of the number, defined by the firm. The procedures previously mentioned for determination of the optimal investment on each product at each stage are applied. Then only the products allocated an amount of capital for all stages which exceeds a given level will be developed. The procedures for determining the optimal investment policy will then be repeated for the products whose original investment capital exceeded the given level. This procedure prevents trying to develop a new product with an inadequate budget and limits the number of products to be developed.

These procedures should be dynamic. This means that the entire procedure should be repeated periodically in as much as it is possible that the values of the random variables will change during the various development stages. It may occur that a product being developed receives a budget inadequate for further development due to changes in the values of the random variables. At such time either the remaining products will continue in the development stages or a product previously deleted will be once again considered.

Example

By one of the previously mentioned methods, the ABC Company has decided to develop four new products. An
assumption in the material that follows is that the random variables, rate of return on investment for each product, are probabilistically mutually independent and the rates of return on investment at each stage in new product development for a given product are mutually independent. Table 5 gives the probability distributions for the four stages for each product. For this example only three values of the random variable \( r \), the rate of return on investment, have been used. However, \( r \) could take on any number of values or could even be described by a continuous probability density function.

Assume that the budget for new product development is \$10 million and that the expected-value-of-capital goal is \$12 million. The budget constraint may be written as

\[
\sum_{i=1}^{4} \sum_{j=1}^{4} d_{ij} = \$10,000,000,
\]

where \( d_{ij} \) is the amount allocated to product \( i \) at stage \( j \). The expected-value-of-capital goal constraint may be written as

\[
\sum_{i=1}^{4} \sum_{j=1}^{4} (EV)_{ij} d_{ij} = K \sum_{i=1}^{4} \sum_{j=1}^{4} d_{ij}
\]

where \( (EV)_{ij} \) is the expected value of the rate of return for product \( i \), stage \( j \). The objective for this optimization problem is to minimize the variance of capital subject to the budget and expected value constraints. The objective may be written as
Table 5
Probability Distribution for New Product Development Example

| Development Stages | Product 1 p | r | EV | Var. | | | Product 2 p | r | EV | Var. |
|--------------------|------------|---|----|-----|--|--|------------|---|----|-----|--|
| 1 - Basic Development | .2 | .95 | .6 | 1.1 | 1.08 | .0046 | | .2 | .9 | .6 | 1.1 | 1.08 | .0096 |
|                      | .2 | 1.15 | | | | | | .2 | 1.2 | |
| 2 - Production Engineering | .2 | .8 | .6 | 1.09 | 1.054 | .017944 | | .2 | .95 | .6 | 1.1 | 1.09 | .0064 |
|                      | .2 | 1.2 | | | | | | .2 | 1.2 | |
| 3 - Financial Analysis | .2 | .7 | .6 | 1.1 | 1.06 | .0384 | | .2 | .6 | .6 | 1.2 | 1.14 | .0864 |
|                      | .2 | 1.3 | | | | | | .2 | 1.5 | |
| 4 - Market Analysis | .2 | .6 | .6 | 1.2 | 1.12 | .0736 | | .2 | .7 | .6 | 1.2 | 1.14 | .0544 |
|                      | .2 | 1.4 | | | | | | .2 | 1.4 | |

|                   | Product 3 | | | |
|--------------------|-----------|---|----|-----|--|
| 1 - Basic Development | .2 | .8 | .6 | 1.3 | 1.22 | .0456 | | .2 | .9 | .6 | 1.4 | 1.36 | .0664 |
|                      | .2 | 1.4 | | | | | | .2 | 1.7 | |
| 2 - Production Engineering | .2 | .9 | .6 | 1.3 | 1.24 | .0304 | | .2 | .8 | .6 | 1.5 | 1.42 | .0616 |
|                      | .2 | 1.4 | | | | | | .2 | 1.8 | |
| 3 - Financial Analysis | .2 | .8 | .6 | 1.2 | 1.20 | .064 | | .2 | .9 | .6 | 1.4 | 1.38 | .0816 |
|                      | .2 | 1.6 | | | | | | .2 | 1.8 | |
| 4 - Market Analysis | .2 | .8 | .6 | 1.4 | 1.30 | .064 | | .2 | .8 | .6 | 1.4 | 1.36 | .1024 |
|                      | .2 | 1.5 | | | | | | .2 | 1.8 | |
minimize $\sum_{i=1}^{4} \sum_{j=1}^{4} V_{ij}d_{ij}^2$ ,

where $V_{ij}$ is the variance of the rate of return for product $i$, stage $j$. The optimization problem may be stated as

$$\sum_{i=1}^{4} \sum_{j=1}^{4} V_{ij}d_{ij}^2$$

subject to: $\sum_{i=1}^{4} \sum_{j=1}^{4} d_{ij} = 10,000,000$

$$\sum_{i=1}^{4} \sum_{j=1}^{4} (EV)_{ij}d_{ij} = 12,000,000$$

$d_{ij} \geq 0, \ i=1,2,\ldots,4; \ j=1,2,\ldots,4$.

This is a convex program and the solution techniques presented in Chapter I may be used to obtain the following solution:

Table 6

Solution for New Product Development Example

<table>
<thead>
<tr>
<th>Development Stages</th>
<th>Product 1</th>
<th>Product 2</th>
<th>Product 3</th>
<th>Product 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Basic Development</td>
<td>1,734,845</td>
<td>831,280</td>
<td>569,299</td>
<td>661,744</td>
</tr>
<tr>
<td>2 Production Engineering</td>
<td>258,649</td>
<td>1,447,587</td>
<td>938,440</td>
<td>838,400</td>
</tr>
<tr>
<td>3 Financial Analysis</td>
<td>140,931</td>
<td>181,550</td>
<td>365,492</td>
<td>569,955</td>
</tr>
<tr>
<td>4 Market Analysis</td>
<td>178,225</td>
<td>288,344</td>
<td>566,159</td>
<td>429,100</td>
</tr>
<tr>
<td><strong>Total Product Budget</strong></td>
<td><strong>2,312,650</strong></td>
<td><strong>2,748,761</strong></td>
<td><strong>2,439,390</strong></td>
<td><strong>2,499,199</strong></td>
</tr>
</tbody>
</table>
Some interesting observations may be made from the data in the preceding two tables. Consider, for example, stage 1 of product 1. The variance of returns from this stage is less than from the other stages while the expected value of the return is second largest of the four stages. Intuitively, therefore, the amount allocated to this stage should likely be larger than all other stages of product 1. Next observe that the expected value of return for products 1 and 2 at stage 1 is equal while the variance of return is smaller for product 1. Intuitively the amount allocated to product 1, stage 1 should be greater than to product 2, stage 1. Both of these intuitive observations are indeed true.

As previously mentioned, the optimization problem should be re-solved periodically and at any time a change in the values of the parameters occurs.

Although the examples presented in this chapter are not identical to the investment processes discussed in Chapters II, III, and IV, the methodology is the same and the processes are similar. Two areas were chosen and the methodology was applied. It is felt that the methodology can be applied to most probabilistic multistage decision processes.
CHAPTER VII

SUMMARY

Utility theory originated because, in general, individuals do not maximize expected value. One may ask if the theory of risky choice introduced in this paper is a replacement for utility theory. Such an ambitious project could hardly have been undertaken. Moreover, utility theory has not yet provided adequate, although there are several criticisms of its use in multistage decision making under risk which will be discussed below. The models introduced in this paper are an alternative to utility theory in the solution of multistage decision processes.

What specific criticisms of utility theory warrant search for an alternative theory of multistage decision making under risk? Several major criticisms exist. Utility theory is not able to adjust to changes in the status quo which result because of the outcomes on previous stages of an investment process. In order to use utility theory in multistage decision processes, an explicit, analytic expression for the utility function is required. One may ask how such an expression may be obtained. The utility function, such as a logarithmic function, may be assumed or the
reference contract method may be used to determine the function and some curve fitted to this empirically derived function. A final criticism concerns the invariability of utility over time. If it cannot be assumed that the utility will not change over time, some procedure for handling this variability must be devised. The existence of multiperiod utility functions, i.e., utility functions which are able to adjust for changes in time or in the status quo, is unknown. Markowitz (1959, page 278) says "To attempt to derive a representative utility function for consumption over time, if feasible at all, is nothing short of a major research project." For these reasons, the theory of this paper is presented as a utility-less theory of risky choice. Other risky choice theories which have avoided nonlinear utility include Markowitz's portfolio selection (1957) and Coombs and Huang's risk-preference portfolio theory (1970).

The models introduced in this paper prescribe to the investor or decision maker an optimal policy for each stage of the multistage process. However, only the initial optimal decision will be of immediate interest. When it is time to make the second decision, things may have changed enough to warrant revision of the data used in the problem and consequently the problem will be solved again. It may be possible by use of a decision tree to solve the entire problem in the beginning and take into account the possible changes in the capital available at each stage. In addition, probability
changes, due to success or failure at each stage, can be accounted for by Bayesian analysis prior to the commencement of the investment process. However, exogenous changes may require re-solving the problem at any intermediate stage. These changes introduce no particular difficulty in the application of this theory.

Recommended Research

As an Industrial Engineer, one may be interested in applications of the theory contained in this paper to areas in which decision models have traditionally utilized expected values. Such applications include inventory, resource allocation, purchasing, replacement, and bidding.

There are certain refinements and enrichments which need be made to this theory. These include consideration of stages which are not statistically independent. Markowitz (1959) and Farrar (1962) have included statistical dependence in their models and this theory requires similar refinement. An approach such as that by Sharpe (1963) may be useful for a future study. The terms risk and variance have been used almost synonymously. However, risk remains completely undefined with variance merely one possible measure of risk. Coombs and Huang (1970) suggest risk is a function of the probability of losing and the amount of the loss, while Markowitz (1959) mentions six measures of risk: standard deviation, semi-variance, expected value of loss, expected absolute deviation, probability of loss, and maximum loss.
Additional research involving other measures of risk for the models of this paper would be of interest.

As previously mentioned, the models of this paper are normative. Their descriptive value may only be ascertained by experimentation. Valid results from such experimentation would greatly strengthen this theory.

In conclusion, additional application, enrichment, and successful experimentation would reinforce and validate this theory and consequently increase its usefulness.
APPENDIX A

Part 1 Expected-Value Maximization Model

Part 2 Expected-Utility Maximization Model - Logarithmic Utility Function

Part 3 Expected-Utility Maximization Model - Power Utility Function
APPENDIX A

Part 1 Expected-Value Maximization Model

The following symbols are defined and will be used in the derivation of the expected-value maximization model:

- $p$: probability of success
- $q = (1-p)$: probability of failure
- $r$: rate of return if successful
- $-s$: rate of return if not successful
- $i$: stage number
- $x_i, i=1,2,...,n$: actual capital available at stage $i$
- $d_i, i=1,2,...,n$: investment at stage $i$
- $f_i(x_i)$: maximum expected value of capital after $i$ stages
- $f_i(x_1)$: maximum expected value of capital after $i$ stages in terms of the initial capital $x_1$

For a one-stage process, the investor is faced with the problem of maximizing the expected value of capital, i.e.,

$$f_1(x_1) = \max_{0 \leq d_1 \leq x_1} [p(x_1 + rd_1) + q(x_1 - sd_1)].$$

Let $Q_1 = p(x_1 + rd_1) + q(x_1 - sd_1)$. To find the maximum of $Q_1$, $d_1 = 0$ or $d_1 = x_1$ since $Q_1$ is a linear function of $d_1$. For $d_1 = 0$, $Q_1 = px_1 + qx_1 = x_1$. For $d_1 = x_1$, $Q_1 = px_1(l+r) + qx_1(1-s) = x_1(l+pr-qs)$. Therefore
if \( pr-qs > 0 \)
\[
d_1 = \begin{cases} 
  x_1 & \text{if } pr-qs > 0 \\
  0 & \text{if } pr-qs < 0 
\end{cases}
\]

and for \( pr-qs > 0 \),
\[
f_1(x_1) = p(x_1+rx_1) + q(x_1-sx_1) = x_1(l+pr-qs).
\]

Thus in order to maximize the expected value of capital, the decision maker invests \( x_1 \) if \( pr-qs \) is greater than zero. This inequality states that the expectation of the investment must be positive. One could hardly expect the DM to invest his capital if the odds were not favorable for a profit.

Assuming a favorable investment prospect (i.e., \( pr-qs > 0 \)) since DM invested all of his capital \( x_1 \) at stage 1, if he were successful, his capital available for investment at stage 2 is \( x_1+rx_1 \). If unsuccessful, then he has \( x_1-sx_1 \) available at the second stage. Denote either \( x_1+rx_1 \) or \( x_1-sx_1 \) by \( x_2 \).

At stage 2 of a two-stage process, the investor is again faced with the problem of maximizing the expected value of capital, i.e.,
\[
f_2(x_2) = \max_{0 \leq d_2 \leq x_2} \left[ p(x_2+rd_2) + q(x_2-sd_2) \right].
\]

The optimal policy is again to invest \( x_2 \) if \( pr-qs \) is greater than zero. Therefore
\[
d_2 = \begin{cases} 
  x_2 & \text{if } pr-qs > 0 \\
  0 & \text{if } pr-qs < 0 
\end{cases}
\]
and for $pr-qs > 0$,

$$f_2(x_2) = p(x_2 + rx_2) + q(x_2 - sx_2)$$

$$= x_2(1 + pr-qs).$$

In terms of the initial capital $x_1$,

$$f_2(x_1) = p(x_1 + rx_1)(1 + pr-qs) + q(x_1 - sx_1)(1 + pr-qs)$$

$$= x_1(1 + pr-qs)^2.$$

The principal of optimality yields the recursive relations

$$f_1(x_1) = x_1(1 + pr-qs) \quad (1)$$

and

$$f_n(x_1) = \max_{0 \leq d_1 \leq x_1} \left[p f_{n-1}(x_1 + rd_1) + q f_{n-1}(x_1 - sd_1)\right], \quad n \geq 2. \quad (2)$$

Theorem: For an $n \ (n \geq 1)$ stage process,

$$f_n(x_1) = x_1(1 + pr-qs)^n$$

and the optimal policy is

$$d_i = \begin{cases} x_i & \text{if } pr-qs > 0 \\ 0 & \text{if } pr-qs < 0. \end{cases}$$

Proof. The optimal policy is independent of $n$. The proof of the maximum expected value will be accomplished by mathematical induction. The proof has been completed for $n = 1$ (see (1) above). Assume the result holds for $n$, i.e.,

$$f_n(x_1) = x_1(1 + pr-qs)^n,$$

and prove it holds for $n+1$, i.e.,

$$f_{n+1}(x_1) = x_1(1 + pr-qs)^{n+1}.$$
From (2)

\[ f_{n+1}(x_1) = \max_{0 \leq d_1 \leq x_1} \left[ pf_n(x_1 + rd_1) + qf_n(x_1 - sd_1) \right] \]

\[ = \max_{0 \leq d_1 \leq x_1} \left[ p(x_1 + rd_1)(1 + pr - qs)^n + q(x_1 - sd_1)(1 + pr - qs)^n \right]. \]

The maximum occurs at \( d_1 = x_1 \). Therefore

\[ f_{n+1}(x_1) = p(x_1 + rx_1)(1 + pr - qs)^n + q(x_1 - sx_1)(1 + pr - qs)^n \]

\[ = x_1(l + pr - qs)^{n+1}. \]

**Part 2 Expected-Utility Maximization Model — Logarithmic Utility Function**

The symbols to be used in the following derivation are defined as they were in Part 1. For a one-stage process, the investor is faced with the problem of maximizing the expected logarithm of capital, i.e.,

\[ f_1(x_1) = \max_{0 \leq d_1 \leq x_1} \left[ p \log(x_1 + rd_1) + q \log(x_1 - sd_1) \right]. \]

Let \( Q_1 = p \log(x_1 + rd_1) + q \log(x_1 - sd_1) \). To find the maximum of \( Q_1 \), set \( dQ_1/dd_1 \) equal to zero and solve for \( d_1 \). The result is

\[ d_1 = \frac{pr - qs}{rs} x_1. \]

Since \( 0 \leq d_1 \leq x_1 \), then \( 0 \leq pr - qs \leq rs \). The second derivative of \( Q_1 \) with respect to \( d_1 \) is positive therefore \( Q_1 \) is everywhere concave and the optimal policy is
The only optimal solution from above that is economically interesting is \( d_1 = \frac{(pr-qs)/rs}{x_1} \). This can be illustrated by investigating the inequalities \( pr-qs > 0 \) and \( pr-qs < rs \). The first inequality states that before a DM will invest anything, the expectation must be nonnegative. It is obvious that an investor would not invest if the expectation were not favorable. It will be assumed that \( 0 \leq s \leq 1 \).

This means that for \( s = 1 \), it is possible for a total loss of capital when \( d_1 = x_1 \). By restricting \( s \) to this interval, it is not possible to lose more capital than one has. The inequality \( pr-qs > rs \) implies the following when \( s = 1 \):

\[
\begin{align*}
pr - (1-p)(1) &> r(1) \\
pr - 1 + p &> r \\
p(r+1) &> r+1 \\
p &> 1.
\end{align*}
\]

This limiting case for \( s \) means that an investor will fully invest, i.e., \( d_1 = x_1 \), only if the probability of success is greater than one (i.e., the probability of a total loss of capital is zero) (Murphy, 1965).
The maximum expected logarithm of capital when
\[ 0 \leq pr-qs \leq rs \] is

\[ f_1(x_1) = p \log(x_1 + r\frac{pr-qs}{rs} x_1) + q \log(x_1 - s\frac{pr-qs}{rs} x_1) = \log x_1 + R, \]

where \( R = p \log p + q \log q + p \log \frac{r+s}{s} + q \log \frac{r+s}{r} \).

If DM were successful at stage 1, the capital available at stage 2 is \( x_1 + r\frac{pr-qs}{rs} x_1 \) or \( x_1 \frac{r+s}{s} \). If he were not successful, then \( x_1 q \frac{r+s}{r} \) is available at stage 2.

For a two-stage process, the DM is again faced with the problem of maximizing the expected logarithm of capital, i.e.,

\[ f_2(x_2) = \max_{0 \leq d_2 \leq x_2} \left[ p \log(x_2 + rd_2) + q \log(x_2 - sd_2) \right] \]

where \( x_2 \), the capital available at stage 2, is either \( x_1 \frac{r+s}{s} \) or \( x_1 \frac{r+s}{r} \). The optimal policy is again

\[ d_2 = \begin{cases} 
0 & \text{if } pr-qs < 0 \\
\frac{pr-qs}{rs} x_2 & \text{if } 0 \leq pr-qs \leq rs \\
x_2 & \text{if } pr-qs > rs.
\end{cases} \]

The maximum expected logarithm of capital is

\[ f_2(x_2) = \log x_2 + R \]
or in terms of the initial capital \( x_1 \),

\[ f_2(x_1) = p \log x_1 \frac{r+s}{s} + q \log x_1 \frac{r+s}{r} + R = \log x_1 + 2R. \]
The principle of optimality yields the recursive relations

\[ f_1(x_1) = \log x_1 + R \]  

and

\[ f_n(x_1) = \max_{0 \leq d_1 \leq x_1} \left[ pf_{n-1}(x_1 + rd_1) + qf_{n-1}(x_1 - sd_1) \right], \quad n \geq 2. \]  

**Theorem:** For an \( n \) (\( n \geq 1 \)) stage process,

\[ f_n(x_1) = \log x_1 + nR \]

and the optimal policy is

\[ d_i = \begin{cases} 
0 & \text{if } pr - qs < 0 \\
\frac{pr - qs}{rs} x_i & \text{if } 0 \leq pr - qs \leq rs \\
x_i & \text{if } pr - qs > rs. 
\end{cases} \]

**Proof.** The optimal policy is independent of \( n \). The proof of the maximum logarithm of expected capital will be accomplished by mathematical induction. The proof has been completed for \( n = 1 \) (see (1) above). Assume the result holds for \( n \), i.e.,

\[ f_n(x_1) = \log x_1 + nR, \]

and prove it holds for \( n+1 \), i.e.,

\[ f_{n+1}(x_1) = \log x_1 + (n+1)R. \]

From (2)

\[ f_{n+1}(x_1) = \max_{0 \leq d_1 \leq x_1} \left[ pf_n(x_1 + rd_1) + qf_n(x_1 - sd_1) \right] \]

\[ = \max_{0 \leq d_1 \leq x_1} \left\{ \left[ p \log(x_1 + rd_1) + nR \right] + q \left[ \log(x_1 - sd_1) + nR \right] \right\}. \]
The maximum occurs at \( d_1 = \frac{pr-qs}{rs} x_1 \). Therefore

\[
f_{n+1}(x_1) = nR + p\log(x_1+r(pr-qs)/rs x_1) \\
+ q\log(x_1-s(pr-qs)/rs x_1)
= nR + \log x_1 + R \\
= \log x_1 + (n+1)R.
\]

**Part 3 Expected-Utility Maximization Model - Power Utility Function**

The symbols to be used in the following derivation are defined as they were in Part 1. For a one-stage process, the investor is faced with the problem of maximizing the expected value of some power of his capital, i.e.,

\[
f_1(x_1) = \text{maximum} \left[ pC(x_1+rd_1)^{M+1} + qC(x_1-sd_1)^{M+1} \right]_{0 \leq d_1 \leq x_1}.
\]

Let \( Q_1 = pC(x_1+rd_1)^{M+1} + qC(x_1-sd_1)^{M+1} \). To find the maximum of \( Q_1 \), set \( \frac{dQ_1}{dd_1} \) equal to zero and solve for \( d_1 \). The result is

\[
d_1 = \frac{1}{r(pr)^{\frac{1}{M}}} - \frac{1}{s(qs)^{\frac{1}{M}}} \ x_1 = Hx_1.
\]

Since \( 0 \leq d_1 \leq x_1 \), then \( 0 \leq (qs)^{\frac{1}{M}} - (pr)^{\frac{1}{M}} \leq r(pr)^{\frac{1}{M}} + s(qs)^{\frac{1}{M}} \).

The second derivative of \( Q_1 \) with respect to \( d_1 \) is positive for the parameter values of interest, therefore \( Q_1 \) is concave and the optimal policy is
The function $d_1$ is given by:

$$d_1 = \begin{cases} 
0 & \text{if } (qs)^{\frac{1}{M}} - (pr)^{\frac{1}{M}} < 0 \\
Hx_1 & \text{if } 0 \leq (qs)^{\frac{1}{M}} - (pr)^{\frac{1}{M}} \leq r(pr)^{\frac{1}{M}} + s(qs)^{\frac{1}{M}} \\
x_1 & \text{if } (qs)^{\frac{1}{M}} - (pr)^{\frac{1}{M}} > r(pr)^{\frac{1}{M}} + s(qs)^{\frac{1}{M}} 
\end{cases}$$

The first solution, $d_1 = 0$, is valid for $pr - qs < 0$ which, because of its negative expectation, is not economically interesting. Likewise for the last solution, $d_1 = x_1$, if $s$ equals 1, the decision maker can lose all of his capital if he is unsuccessful. The constraint on this solution, $(qs)^{1/M} - (pr)^{1/M} > r(pr)^{1/M} + s(qs)^{1/M}$, where $-1 < M < 0$, may be expressed as:

$$\frac{pr}{qs} > \left(\frac{1+r}{1-s}\right)^N,$$ where $0 < N < 1.$

Note that as $s$ approaches one, $(1+r)/(1-s)$ approaches infinity. In order for the inequality to hold, $p$ must approach one ($q$ approaches zero). In this limiting case, $DM$ would fully invest only if the probability of success were one (i.e., the probability of total loss were zero). Consequently only the second solution, $d_1 = Hx_1$, will be further considered.

The maximum expected value of some power of capital for $0 \leq (qs)^{1/M} - (pr)^{1/M} \leq r(pr)^{1/M} + s(qs)^{1/M}$ is

$$f_1(x_1) = pC(x_1 + rHx_1)^{M+1} + qC(x_1 - sHx_1)^{M+1}$$
If DM were successful at stage 1, the capital available at stage 2 is $x_1 + rHx_1$ and if he were not successful, $x_1 - sHx_1$ is available.

For a two-stage process, DM is again faced with the problem of maximizing the expected value of some power of his capital, i.e.,

$$f_2(x_2) = \text{maximum} \left\{ pC(x_2 + rd_2)^{M+1} + qC(x_2 - sd_2)^{M+1} \right\},$$

where $x_2$, the capital available at stage 2, is either $x_1 + rHx_1$ or $x_1 - sHx_1$. The optimal policy is again

$$d_2 = \begin{cases} 0 & \text{if } (qs)^{\frac{1}{M}} - (pr)^{\frac{1}{M}} < 0 \\ Hx_2 & \text{if } 0 \leq (qs)^{\frac{1}{M}} - (pr)^{\frac{1}{M}} < r(pr)^{\frac{1}{M} + s(qs)^{\frac{1}{M}}} \\ x_2 & \text{if } (qs)^{\frac{1}{M}} - (pr)^{\frac{1}{M}} > r(pr)^{\frac{1}{M} + s(qs)^{\frac{1}{M}}} \end{cases}$$

The maximum expected value of some power of capital is

$$f_2(x_2) = Cx_2^{M+1}W,$$

or in terms of the initial capital $x_1$,

$$f_2(x_1) = pC(x_1 + rHx_1)^{M+1}W + qC(x_1 - sHx_1)^{M+1}W$$

$$= Cx_1^{M+1}W^2.$$
The principle of optimality yields the recursive relations

\[ f_1(x_1) = Cx_1^{M+1} W \]  \hspace{1cm} (1)

and

\[ f_n(x_1) = \max_{0 \leq d_1 \leq x_1} \left[ pf_{n-1}(x_1 + rd_1) + qf_{n-1}(x_1 - sd_1) \right], \quad n \geq 2. \]  \hspace{1cm} (2)

Theorem: For an \( n \) (\( n \geq 1 \)) stage process,

\[ f_n(x_1) = Cx_1^{M+1} W^n \]

and the optimal policy is

\[ d_i = \begin{cases} 
0 & \text{if } (qs)^M - (pr)^M < 0 \\
\frac{1}{M} & \text{if } 0 \leq (qs)^M - (pr)^M \leq r(pr)^M + s(qs)^M \\
x_i & \text{if } (qs)^M - (pr)^M > r(pr)^M + s(qs)^M.
\end{cases} \]

Proof. The optimal policy is independent of \( n \). The proof of the maximum expected value of some power of capital will be accomplished by mathematical induction. The proof has been completed for \( n = 1 \) (see (1) above). Assume the result holds for \( n \), i.e.,

\[ f_n(x_1) = Cx_1^{M+1} W^n, \]

and prove it holds for \( n+1 \), i.e.,

\[ f_{n+1}(x_1) = Cx_1^{M+1} W^{n+1}. \]

From (2),
\[ f_{n+1}(x_1) = \max_{0 \leq d_1 \leq x_1} \left[ p f_n(x_1 + rd_1) + q f_n(x_1 - sd_1) \right] \]

\[ = \max_{0 \leq d_1 \leq x_1} \left[ p C (x_1 + rd_1)^{M+1} + q C (x_1 - sd_1)^{M+1} \right]. \]

The maximum occurs at \( d_1 = H x_1 \). Therefore

\[ f_{n+1}(x_1) = C w^n \left[ p (x_1 + r H x_1)^{M+1} + q (x_1 - s H x_1)^{M+1} \right] \]

\[ = C w^n x_1^{M+1} \]

\[ = C x_1^{M+1} w^{n+1}. \]
APPENDIX B

Proof: Variance of Final Capital Equals
Sum of Stage Variances
APPENDIX B

Proof: Variance of Final Capital Equals Sum of Stage Variances

Throughout this paper the various models have included in the criterion the minimization of the expected value of capital at the conclusion of the final stage. However, in the dynamic programming formulation, the sum of the stage variances has been minimized and it has been hypothesized that this sum equals the variance of the final capital. This will be proved for a two-stage, two-outcome process. The general proof for an n-stage, m-outcome process may be obtained by mathematical induction on n and on m.

Proof: Consider a two-outcome, two-stage decision process where the rate of return is r for the outcome called success and -s for the outcome called failure and these rates are constant for both stages as are the probabilities of success and failure. The tree diagram given below indicates the various outcomes and their associated probabilities.

Let V denote the variance of the final capital, \( v_1 \) denote the variance of capital at stage 1, and \( v_2 \) denote
the variance of capital at stage 2. The theorem states that \( V = v_1 + v_2 \). Throughout this proof, the fact that \( V(x) = E(x^2) + [E(x)]^2 \) will be used.

\[
\begin{align*}
v_2 &= p(L+rd_2)^2 + q(L-sd_2)^2 - [p(L+rd_2) + q(L-sd_2)]^2 \\
&= \frac{d_2^2}{2}pq(r + s)^2.
\end{align*}
\]

Now assume that the capital available at the input to stage 1 is \( x_1 \).

\[
\begin{align*}
v_1 &= p(x_1+rd_1)^2 + q(x_1-sd_1)^2 - [p(x_1+rd_1) + q(x_1-sd_1)]^2 \\
&= \frac{d_1^2}{2}pq(r + s)^2.
\end{align*}
\]

The variance of the final capital is
\[ V = p^2(L+rd_2+rd_1)^2 + pq(L+rd_2-sd_1)^2 + pq(L-sd_2+rd_1)^2 \\
+ q^2(L-sd_2-sd_1)^2 - [p^2(L+rd_2+rd_1) + pq(L+rd_2-sd_1) \\
+ pq(L-sd_2+rd_1) + q^2(L-sd_2-sd_1)]^2. \]

After much algebra, this simplifies to

\[ V = d_1^2pq(r+s)^2 + d_2^2pq(r+s)^2 = v_1 + v_2. \] QED
APPENDIX C

Convex Programming Solution to the Two-Outcome Multistage Process
APPENDIX C

Convex Programming Solution to the Two-Outcome Multistage Process

In Chapter II it was stated that most multistage decision making problems could be solved either by dynamic programming or by some other optimization technique, only the effort required to determine the optimal policy varying. In this appendix it will be shown that the convex programming solution to the two-outcome, multistage process is identical to the dynamic programming solution. For the situation where \( p_i = p, q_i = q, r_i = r, \) and \( s_i = s \) for all \( i \), the optimal policy for an \( n \) stage process was determined by dynamic programming to be

\[
d = \frac{KL-L}{n(pr-qs)}
\]

with minimum variance

\[
f_n(x_n) = (KL-L)^2 \frac{pq(r+s)^2}{n(pr-qs)^2}.
\]

The identical problem will now be solved by convex programming. The criterion is written as
minimize \( \sum_{i=1}^{n} d_i^2 (r+s)^2 pq \)  

subject to: \( d_i = \frac{KL-x_i}{pr-qs} \)  

\( 0 \leq d_i \leq x_i, \quad i=1,2,\ldots,n. \)

The Lagrange function \( \phi \) is

\[ \phi = \sum_{i=1}^{n} d_i^2 (r+s)^2 pq + \lambda_1 (d_i - \frac{KL-x_i}{pr-qs}) + \lambda_2 (d_i - x_i) . \]

The Kuhn-Tucker conditions are

\[ \frac{\partial \phi}{\partial d_i} = 2d_i (r+s)^2 pq + \lambda_1 + \lambda_2 \geq 0, \quad i=1,2,\ldots,n \]  

\[ d_i \frac{\partial \phi}{\partial d_i} = 0 = d_i [2d_i (r+s)^2 pq + \lambda_1 + \lambda_2], \quad i=1,2,\ldots,n \]  

\[ d_i \geq 0, \quad i=1,2,\ldots,n \]

\[ \frac{\partial \phi}{\partial \lambda_1} = 0 \text{ implies } d_i = \frac{KL-x_i}{pr-qs}, \quad \lambda_1 \text{ unrestricted in sign} \]  

\[ \frac{\partial \phi}{\partial \lambda_2} \leq 0 \text{ implies } d_i - x_i \leq 0, \quad i=1,2,\ldots,n \]

\[ \lambda_2 \frac{\partial \phi}{\partial \lambda_2} = 0 \text{ implies } \lambda_2 (d_i - x_i) = 0, \quad i=1,2,\ldots,n \]

\[ \lambda_2 \geq 0 \]

Assume for all \( i \) that \( d_i \neq 0 \). Equations (5) imply that \( d_i = d \) for all \( i \). From (7)

\[ d_i = \frac{KL-x_i}{pr-qs} . \]
Also,

\[ x_n = L, \]
\[ x_{n-1} = L + d(pr-qs), \]
\[ x_{n-2} = x_{n-1} + d(pr-qs) = L + 2d(pr-qs), \]
\[ x_{n-3} = x_{n-2} + d(pr-qs) = L + 3d(pr-qs), \]
\[ \vdots \]
\[ x_{n-(n-1)} = x_1 = x_2 + d(pr-qs) = L + (n-1)d(pr-qs). \] (10)

Substituting (10) into (7) gives

\[ d_1 = d = \frac{KL-L-(n-1)d(pr-qs)}{pr-qs} \]

\[ d = \frac{KL-L}{n(pr-qs)}. \] (11)

Substituting (11) into (3) gives

\[ (r+s)^2pq \sum_{i=1}^{n} \left[ \frac{KL-L}{n(pr-qs)} \right]^2 \text{ or} \]
\[ (KL-L)^2 \frac{pq(r+s)^2}{n(pr-qs)^2}. \] (12)

Notice that (11) and (12), obtained by convex programming, are identical to (1) and (2) which were obtained by dynamic programming.
APPENDIX D

Solution: Multiple Outcome

Two-Stage Decision Making Model
APPENDIX D

Solution: Multiple Outcome Two-Stage Decision Making Model

The criterion for this process is written as

\[
\text{minimize } \sum_{j=1}^{2} \sum_{i=1}^{m} \sum_{h=1}^{m} p_{ij} p_{hj} (r_{ij} - r_{hj})^2 \quad \text{subject to: } \ 0 \leq d_j \leq x_j, \ j=1,2
\]

\[
x_2 = L
\]

\[
x_j = x_{j+1} + d_{j+1} \sum_{i=1}^{m} p_i, j+1 r_i, j+1, \ j=0,1
\]

\[
x_0 = KL
\]

The transformation equation is

\[
x_1 = L + d_2 \sum_{i=1}^{m} p_i 2 r_{12}.
\]

Stage 1:

\[
f_1(x_1) = \text{minimum } d_1 \sum_{i=1}^{m} \sum_{h=1}^{m} p_{il} p_{hl} (r_{il} - r_{hl})^2 \quad \text{subject to: } \ 0 \leq d_1 \leq x_1
\]

\[
d_1 = \frac{KL-x_1}{\sum_{i=1}^{m} p_{il} r_{il}} \quad (1)
\]

Since \( d_1 = \frac{KL-x_1}{\sum_{i=1}^{m} p_{il} r_{il}} \), no optimization is required at stage 1. The minimum variance is written as
\[ f_1(x_1) = \left\{ \frac{KL-x_1}{\sum_{i=1}^{m} \sum_{h=1}^{m} p_{il} r_{il}} \right\}^2 \sum_{i=1}^{m} \sum_{h=i+1}^{m} p_{il} p_{hl} (r_{il} - r_{hl})^2. \]

Stage 2:

\[ f_2(x_2) = \min_{0 \leq d_2 \leq L} \left[ d_2 \sum_{i=1}^{m} \sum_{h=i+1}^{m} p_{il} p_{hl} (r_{il} - r_{hl})^2 + f_1(x_1) \right] \]

\[ = \min_{0 \leq d_2 \leq L} \left[ d_2 \sum_{i=1}^{m} \sum_{h=i+1}^{m} p_{il} p_{hl} (r_{il} - r_{hl})^2 + \left( \frac{KL-x_1}{\sum_{i=1}^{m} \sum_{h=i+1}^{m} p_{il} r_{il}} \right)^2 \sum_{i=1}^{m} \sum_{h=1}^{m} p_{il} p_{hl} (r_{il} - r_{hl})^2 \right] \]

\[ = \min \left\{ d_2 \sum_{i=1}^{m} \sum_{h=i+1}^{m} p_{il} p_{hl} (r_{il} - r_{hl})^2 + \frac{m}{\sum_{i=1}^{m} \sum_{h=1}^{m} p_{il} r_{il}} \sum_{i=1}^{m} \sum_{h=1}^{m} p_{il} p_{hl} (r_{il} - r_{hl})^2 \right\} \]

\[ + (KL-L-d_2 \sum_{i=1}^{m} p_{il} r_{il})^2 \frac{m}{\sum_{i=1}^{m} p_{il} r_{il}} \]

(2)

Let the term to be minimized be denoted by \( Q_2 \). To find the decision variable \( d_2 \) which minimizes \( Q_2 \), take the derivative of \( Q_2 \) with respect to \( d_2 \), set it equal to zero, and solve for \( d_2 \). The result is
\[ d_2 = (KL-L) \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i1}p_{h1}(r_{i1}-r_{h1})^2 \sum_{i=1}^{m} p_{i2}r_{i2} \]

\[
\left[ \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i1}p_{h1}(r_{i1}-r_{h1})^2 \sum_{i=1}^{m} p_{i2}r_{i2} \right] + \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i2}p_{h2}(r_{i2}-r_{h2})^2 \sum_{i=1}^{m} p_{i1}r_{i1} \]

(3)

It can be shown that \( (d^2Q_2)/(dd_2) \) is positive and therefore (3) is a minimum rather than a maximum. It is necessary to backsolve in order to determine \( d_1 \):

\[
x_1 = L + d_2 \sum_{i=1}^{m} p_{i2}r_{i2}\]

\[
= L + \left[ (KL-L) \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i1}p_{h1}(r_{i1}-r_{h1})^2 \sum_{i=1}^{m} p_{i2}r_{i2} \right]
\]

\[
\left[ \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i1}p_{h1}(r_{i1}-r_{h1})^2 \sum_{i=1}^{m} p_{i2}r_{i2} \right] + \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i2}p_{h2}(r_{i2}-r_{h2})^2 \sum_{i=1}^{m} p_{i1}r_{i1} \]

(4)

Substituting (4) into (1) gives
\[ d_1 = \frac{1}{\sum_{i=1}^{m} p_{il} r_{il}} \left\{ (KL - L) - (KL - L) \right\} \]

\[ = \frac{KL - L}{\sum_{i=1}^{m} p_{il} r_{il}^2} \left\{ \sum_{i=1}^{m} p_{il} r_{il}^2 \left( \sum_{h=1}^{m} p_{l1} r_{il} - r_{h1} \right)^2 \right\} \]

\[ + \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i2} p_{h2} (r_{i2} - r_{h2})^2 \left( \sum_{i=1}^{m} p_{il}^2 r_{il}^2 \right) \]

\[ = (KL - L) \frac{\sum_{i=1}^{m} \sum_{h=1}^{m} p_{i2} p_{h2} (r_{i2} - r_{h2})^2}{\sum_{i=1}^{m} p_{il} r_{il}^2} \frac{\sum_{i=1}^{m} p_{il} r_{il}^2}{\sum_{i=1}^{m} p_{il} r_{il}^2} \]

\[ + \sum_{i=1}^{m} \sum_{h=1}^{m} p_{i2} p_{h2} (r_{i2} - r_{h2})^2 \left( \sum_{i=1}^{m} p_{il}^2 r_{il}^2 \right) \] (5)
Equations (3) and (5) give the optimal policy for this two-stage, multiple outcome process. The minimum variance for this process is found by substituting (3) into (2) and simplifying.

As stated in Chapter III, by applying mathematical induction on the stage number $j$, the optimal allocation can be determined for an $m$ outcome, $n$ stage process.
APPENDIX E

Use of Expected Values in Adaptive Processes
APPENDIX E

Use of Expected Values in Adaptive Processes

In the adaptive processes of Chapters II and III, the prior and posterior expected values of the random variable $p$, the probability of success, were used in order to determine the optimal investment policy. It may be asked why the expected value of $p$ is used rather than some other statistic. It will be shown that using this statistic of $p$, when $p$ is a random variable, gives the same optimal policy as that based on an assumed known value of $p$. The optimal policy for an assumed known value of $p$ is

$$d = \frac{KL-L}{n(pr-qs)}$$

for each stage of an $n$-stage process. The variance of such an $n$-stage, two-outcome process is

$$V = (r + s)^2 pq \sum_{i=1}^{n} d_i^2 .$$

Recall that $p$ is beta distributed with parameters $r_{pr}$ and $n_{pr}$. The prior expected variance of $p$ is
In addition, recall that this optimization problem was constrained by the expected value goal. The expected value of this process is

\[ EV = L + \sum_{i=1}^{n} d_i (pr - qs) \]

and the prior expected value of the mean of \( p \) is

\[
\frac{1}{l} \sum_{p=0}^{l} \left( L + \sum_{i=1}^{n} d_i \left( (r+s)p - s \right) \right) \frac{(n_{pr}-1)!}{(r_{pr}-1)!(n_{pr}-r_{pr}-1)!} \frac{r_{pr}-1}{p_{pr}-1} \frac{n_{pr}-r_{pr}-1}{(1-p) n_{pr}-r_{pr}-1} dp
\]

\[ = \sum_{i=1}^{n} (d_i^2)(r+s)^2 \left[ \frac{r_{pr}}{n_{pr}} - \frac{(r_{pr}+1)(r_{pr})}{(n_{pr}+1)(n_{pr})} \right] . \]

Treating \( p \) as a random variable, the criterion may be stated as

\[
\text{minimize} \sum_{i=1}^{n} d_i^2 (r+s)^2 \left[ \frac{r_{pr}}{n_{pr}} - \frac{(r_{pr}+1)(r_{pr})}{(n_{pr}+1)(n_{pr})} \right] \\
\text{subject to: } L + \sum_{i=1}^{n} d_i \left( \frac{r_{pr}}{n_{pr}} - \frac{n_{pr}-r_{pr}}{n_{pr}} \right) = KL.
\]

The problem may be solved by convex programming with the following optimal solution for each stage:
Recall that \( \frac{r_{\text{pr}}}{n_{\text{pr}}} \) and \( \frac{n_{\text{pr}} - r_{\text{pr}}}{n_{\text{pr}}} \) are the prior expected values of \( p \) and \( q \) respectively. Therefore the use of these expected values of the random variables \( p \) and \( q \) gives the same optimal policy as when assumed known values of \( p \) and \( q \) are used. Similarly it can be shown that the use of the posterior expected value of \( p \) results in the same optimal policy as that based on an assumed known value of \( p \).
BIBLIOGRAPHY


