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TELEPHONE TRAFFIC IN A HIGH CAPACITY
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DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

Lewis Grant Anderson, B.S., S.M.

* * * * *

The Ohio State University
1972

Approved by

C.E. Warren
Adviser
Department of Electrical Engineering
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ACKNOWLEDGMENT

I wish to express my sincere appreciation to my wife Joyce without whose continued support and encouragement these years of graduate study could not have been completed. To Professor C. E. Warren, my adviser, I offer sincere gratitude for his guidance throughout this project. The support granted by Bell Telephone Laboratories is gratefully acknowledged.

L. G. Anderson
VITA

April 20, 1938 — Born — Sharpsville, Pennsylvania

1960 — — — B.S.E.E., The Pennsylvania State University, University Park, Pennsylvania

1960-1963 — — United States Navy, Engineer Officer, U.S.S. Brister DER 327

1963-1965 — — Research Assistant, Electronic Systems Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts

1965 — — — S.M., Massachusetts Institute of Technology, Cambridge, Massachusetts

1965 — — — Member of Technical Staff, Bell Telephone Laboratories, Columbus, Ohio

FIELDS OF STUDY

Major Field: Electrical Engineering

Studies in Communication Theory. Professor C. E. Warren

Studies in Modern Control Theory. Professor H. Hemami

Studies in Classical Control Theory. Professor John Bacon

Studies in Logic and Coding Theory. Professor R. B. Lackey

Studies in Mathematics and Statistics. Professor Charles Saltzer
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CHAPTER I

INTRODUCTION

1.1 Description of High Capacity Mobile Telephone System

Existing mobile telephone systems serve relatively small numbers of subscribers because of the limited radio frequency spectrum available. One radio frequency channel can serve only one customer at a time in the whole geographical area in which the mobile telephone is permitted to operate. Over the past ten years several papers (references 1-9) have discussed a proposed system which allows several subscribers to simultaneously use the same radio channel. This is accomplished by dividing the allowed operating area into a series of small coordinated cells each with its own transmitting equipment and group of assigned channels. Transmitter power is limited to allow the same group of channels to be assigned to more than one cell in the system. Cells are coordinated to permit subscribers to use their service in any cell and to permit them to move from one cell to another while their mobile telephone is in use. High Capacity Mobile
Telephone System (HCMTS) is the name given to this system. It is conceptually represented in Figure 1.1-1.

A large scale data processing machine will be required to administer this system. When a connection is to be established to a mobile subscriber, this machine will assign a channel for use based upon the cell in which the mobile telephone happens to be located. Location will be established by processing transmission data (e.g., signal strength) from the mobile telephone to several fixed receivers. Channels will be assigned when a mobile telephone moves or is "handed off" from one cell to another.

Systems discussed in references 1 through 7 assume that channels are permanently assigned to cells. References 8 and 9 propose a system using a dynamic channel assignment procedure. Channels are assigned to a cell only when a mobile customer requests service in that cell. Mr. J. S. Engel\(^{(10)}\) has proposed a dynamic channel assignment method to increase the system's call carrying capacity. The idea is to borrow a channel from a nearby cell if all channels are busy. Reference 11 suggests an implementation of an improved version of Engel's proposal.

Several papers have been published while this work was in progress. Reference 12 presents some general ideas about new directions in mobile telephone development. A simulation study of a system composed of a single line of base stations and using various dynamic channel assignment techniques is reported in reference 13.
FIGURE 1.1-1 HIGH CAPACITY MOBILE TELEPHONE SYSTEM
Some aspects of the systems proposed in the references are quite diverse. Thus, at the outset it was decided to limit this work by concentrating on systems with hexagonal cells, to assume that there are 360 channels available for voice use, and further that both mobile and base station radios can tune all 360 channels. Proximity of cells that are using the same channel is restricted by the ratio of signal levels necessary to reduce cochannel interference to acceptable levels. This restriction is expressed as the ratio $D/R$ of distance between cell centers to the maximum radius of a cell. $D/R = 6$ has been used throughout. This value and the number of channels assumed are not claimed to be optimum or the values which would be used in an actual system, but are intended to be realistic enough so that the results will be applicable to a practical system. A similar criterion was used in selecting the cell layout. It is based on the study of the Philadelphia area published in reference 5. Figure 1.1-2 shows the layout and its geographical orientation.

1.2 Problems Considered

A system such as this contains some new and interesting traffic problems. It is to these problems that this work is addressed. As in any other telephone system, subscribers originate and receive calls on a random basis. Statistics of these events have been under study for a long time and specific distributions and parameters are known. However, movement of subscribers from one cell to another is a new statistical problem. It is of interest
both from the standpoint of how many cell changes will be made while

calls are in progress, and how much the telephone traffic in each cell

will vary.

Any channel assignment procedure which does not require a

fixed, static association between cells and channels is called a
dynamic channel assignment procedure. Traffic capacity considerations

associated with the administration of these procedures are also new

and different. They are different because the amount of system

capacity used up when a channel is assigned to a cell is not a con-

stant, but depends on which other cells are using that same channel.

Clearly, the most efficient use of a radio channel results when it is

reused at the minimum D/R ratio. However, dynamic channel assignment

implies that sometimes channels will be reused at D/R ratios greater

than minimum. The whole idea of dynamic channel assignment is to

trade off the more inefficient channel spacing for the increased

system capacity resulting from increased use of the channel. This

tradeoff is made quite complex by its dependence upon the state of

the system and is indeed why the problem is worth studying.

Thus, the traffic problems studied are conveniently divided

into two sections. Those in Chapter II are concerned with channel

assignment procedures, both fixed and dynamic. Chapter III is

devoted to those problems arising from the motion of mobile tele-

phones across cell boundaries. Computer simulation is used as one

means of studying both categories of problems. The simulator

program is quite large and complex. Appendix A is devoted to its
description. Appendix B presents some basic results from telephone traffic theory that are used throughout, and serves to establish a common ground for discussion as well as a review of the material. Appendix C serves the same function for some statistical analysis techniques used in interpreting simulation results, and in verifying simulator operation.
CHAPTER II

CHANNEL ASSIGNMENT

2.1 Fixed

2.1.1 Geometrical Considerations

As mentioned earlier, the hexagonal cell shape is of most interest because of radio coverage considerations. The task of laying out hexagonal arrays and assigning radio channels to them so that interference is held to acceptable levels requires some familiarity with the geometry of hexagonal arrays. This section discusses the required background information.

2.1.1.1 Simplified Derivation of \( N \) and \( \sigma \) Relationships

Proximity of cells using the same channel is restricted by the ratio of signal levels necessary to reduce cochannel interference to acceptable levels. This restriction is expressed as the ratio \( \sigma = D/R \) of the distance between cell centers to the maximum radius of a cell. Once the required \( \sigma \) is specified, then the number of channel sets, \( N \), required to cover the hexagonal array is related to \( \sigma \) by:

\[
N = \frac{1}{3} \sigma^2 \tag{2.1-1}
\]
Possible values of \( N \) must be solutions to the following equation:

\[
N = m^2 + n^2 + mn
\]

where \( n \) and \( m \) are integers. These equations are stated in reference 4 with no explanation. The expression for possible values of \( N \) comes from reference 14 (cited in reference 6) where it is derived from number theoretic considerations. In this section both of these relations are derived from simple geometrical properties of hexagonal arrays.

Consider a rectangular array with row spacings of \( 3/2 \) and column spacings of \( \sqrt{3} \) as in Figure 2.1-1. Let \( m \) denote the row number and \( n \) denote the column. Consider each row column intersection as the only possible cell centers, for example A and B in Figure 2.1-1. In this case the distance between centers must satisfy

\[
D^2 = \left( \frac{3}{2} m \right)^2 + 3n^2. \quad n, m \text{ - integers} \quad (2.1-3)
\]

Now to make a hexagonal array, we deform the array of Figure 2.1-1 by the following rule - shift row \( i \) to the right by an amount equal to \( i \frac{\sqrt{3}}{2} \). The array of Figure 2.1-2 results. Now the distance between any two cell centers must satisfy

\[
D^2 = \left( \frac{3}{2} m \right)^2 + (n \sqrt{3} + m \frac{\sqrt{3}}{2})^2 \quad (2.1-4)
\]

which simplifies to

\[
D^2 = 3(m^2 + n^2 + mn). \quad (2.1-5)
\]
FIGURE 2.1-1 RECTANGULAR ARRAY

FIGURE 2.1-2 HEXAGONAL ARRAY
The array of Figure 2.1-2 has been designed so that hexagonal cells constructed on the possible cell centers will have unit radius. Thus,

\[ \sigma^2 = \left( \frac{D}{R} \right)^2 = 3 \left( m^2 + n^2 + mn \right) \]  \hspace{1cm} (2.1-6)

the desired result.

Now it remains to be seen where \( N = \frac{1}{3} \sigma^2 \) comes from. The following argument is presented as nonrigorous justification of this. We note that \( n \) and \( m \) are coordinates in a hexagonal (nonorthogonal) coordinate system. There are always at least six solutions to

\[ \sigma^2/3 = n^2 + m^2 + nm. \]  \hspace{1cm} (2.1-7)

These points form a hexagonal pattern around the origin. The solutions are:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( m )</th>
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<tr>
<td>( a )</td>
<td>( b )</td>
</tr>
<tr>
<td>(-a)</td>
<td>(-b)</td>
</tr>
<tr>
<td>( a )</td>
<td>(-(a+b))</td>
</tr>
<tr>
<td>(-a)</td>
<td>(a+b)</td>
</tr>
<tr>
<td>( a+b )</td>
<td>(-a)</td>
</tr>
<tr>
<td>(-(a+b))</td>
<td>(a)</td>
</tr>
</tbody>
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**TABLE 2.1-1 - Solutions of Equation (2.1-7)**

Note: If \( a \neq b \), there are six more points corresponding to \( a+b, b+a \), which also form a hexagon of the same size.
Thus, we recognize that if one reuses the same channel at its minimum \( \sigma \), the channel pattern is hexagonal as shown in Figures 2.1-3 and 2.1-4 by the seven cells assigned channel set one. This pattern can be further subdivided into non-overlapping parallelograms with a cell of channel set one at each corner. One of these parallelograms is illustrated in both Figure 2.1-3 and Figure 2.1-4. We claim that the frequency assignments inside this parallelogram must be all different. To convince oneself of this, assign the same channel to two cells inside the parallelogram and then observe that if one attempts to extend this assignment pattern a violation of the minimum \( \sigma \) will occur. Thus, we have established that we must have at least as many channel sets as there are cells inside this parallelogram. At this point it is comforting to note that the area of this parallelogram is a multiple of the area of one hexagonal cell.

\[
\text{Area of Parallelogram } A_p = \frac{\sqrt{3}}{2} \sigma^2
\]

\[
\text{Area of Hexagonal Cell } A_H = \frac{3\sqrt{3}}{2}
\]  

\[
\frac{A_p}{A_H} = \frac{\sigma^2}{3} = m^2 + n^2 + mn --- \text{ an integer since } m, n \text{ are integers.}
\]

That this required number of channel sets is sufficient follows from the observation that this parallelogram can be repeated in all directions without violating \( \sigma \). Hence, we conclude the required number of channel sets is given by the ratio \( \frac{A_p}{A_H} \) computed above.

\[
N = \frac{A_p}{A_H} = m^2 + n^2 + mn
\]  

(2.1-9)
FIGURE 2.1-3 CHANNEL SET ASSIGNMENT

N = 12
σ = 6
FIGURE 2.1-4 CHANNEL SET ASSIGNMENT

$N=7$

$\sigma=4.58$
We summarize by saying that possible channel spacings \( \sigma \) must satisfy

\[
\sigma^2 = 3(m^2 + n^2 + mn)
\]  

(2.1-10)

and given \( \sigma \), the number of channel sets required is given by

\[
N = \frac{\sigma^2}{3}.
\]  

(2.1-11)

2.1.1.2 Some Possible \( N \)s and \( \sigma \)s

Using the relations of Section 2.1.1.1 several possible \( N \), \( \sigma \) combinations are listed in Figure 2.1-5. The specific one considered for this study is \( \sigma = 6 \), \( N = 12 \).

2.1.1.3 Mutual Interference Sets

In this section we introduce the concept of a mutual interference set. It will be used in Section 2.1.2 when we discuss the channel assignment problem. Given an \( N \), \( \sigma \) combination, a mutual interference set is defined as a set of \( N \) cells such that each cell interferes with each of the other \( N - 1 \) cells, that is the center to center distance between each pair of cells divided by a cell radius is less than the prescribed \( \sigma \). That such a set exists and contains \( N \) cells follows from Section 2.1.1.1, for if one could find such a set with more than \( N \) cells, then the \( N \), \( \sigma \) relationship of Section 2.1.1.1 would be violated. It is interesting to note that the \( N \) cells within the parallelograms of Figure 2.1-3 and 2.1-4 do not form mutual interference sets. However, such sets can easily be found by graphical methods. Several examples are shown in Figure 2.1-6 for
### SOLUTIONS TO EQUATIONS 2.1-10 AND 2.1-11

**FIGURE 2.1-5 SOME POSSIBLE N, σ COMBINATIONS**

<table>
<thead>
<tr>
<th>ORIGIN</th>
<th>( N = 1 )</th>
<th>( \frac{1}{\sigma \sqrt{3}} )</th>
<th>( 4 )</th>
<th>( \frac{1}{2\sqrt{3}} )</th>
<th>( 9 )</th>
<th>( \frac{1}{3\sqrt{3}} )</th>
<th>( 16 )</th>
<th>( \frac{1}{4\sqrt{3}} )</th>
<th>( 25 )</th>
<th>( \frac{1}{5\sqrt{3}} )</th>
<th>( 36 )</th>
<th>( \frac{1}{6\sqrt{3}} )</th>
<th>( 49 )</th>
<th>( \frac{1}{7\sqrt{3}} )</th>
<th>( 64 )</th>
<th>( \frac{1}{8\sqrt{3}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma \sqrt{3} )</td>
<td>( 3 )</td>
<td>( 4.58 )</td>
<td>( 7 )</td>
<td>( 13 )</td>
<td>( 21 )</td>
<td>( 31 )</td>
<td>( 43 )</td>
<td>( 57 )</td>
<td>( 75 )</td>
<td>( 12 )</td>
<td>( 19 )</td>
<td>( 28 )</td>
<td>( 39 )</td>
<td>( 52 )</td>
<td>( 61 )</td>
<td>( 76 )</td>
</tr>
<tr>
<td>( \frac{1}{\sigma \sqrt{3}} )</td>
<td>( 6 )</td>
<td>( 7.54 )</td>
<td>( 9 )</td>
<td>( 10.82 )</td>
<td>( 12.49 )</td>
<td>( 14.18 )</td>
<td>( 13.08 )</td>
<td>( 11.35 )</td>
<td>( 10.53 )</td>
<td>( 13.53 )</td>
<td>( 13.75 )</td>
<td>( 13.53 )</td>
<td>( 13.75 )</td>
<td>( 15.1 )</td>
<td>( 15.1 )</td>
<td>( 15.1 )</td>
</tr>
</tbody>
</table>
FIGURE 2.1-6 MUTUAL INTERFERENCE SETS

N=7 \( \sigma = 4.58 \)

N=9 \( \sigma = 5.19 \)

N=12 \( \sigma = 6 \)

N=13 \( \sigma = 6.245 \)

N=16 \( \sigma = 6.92 \)
representative \( N, \sigma \) combinations. A typical system layout would contain many such sets all of which can be found by translation and or rotation of the patterns in Figure 2.1-6.

2.1.1.4 Coordinate System

A useful side result from Section 2.1.1.1 is a hexagonal coordinate system. Cells in an array such as Figures 2.1-2 can be identified by two integer coordinates \( n,m \). Distance, in terms of \( \sigma \), between two cells with coordinates \( n_1,m_1 \) and \( n_2,m_2 \) can be found from application of equation 2.1-6 as

\[
\sigma^2 = 3(m_1 - m_2)^2 + (n_1 - n_2)^2 + (n_1 - n_2)(m_1 - m_2). \tag{2.1-11}
\]

Thus, we have a convenient way of storing cell location information in a computer and of finding the distance between any pair of cells. It is useful in applying the constraints of Section 2.1.1.2 and performing the procedures that will be described in Section 2.1.2.

2.1.2 Channel Assignment Problem

Prior to evaluating dynamic channel assignment procedures, we need to know the maximum traffic capacity of a fixed channel assignment system. If each cell had identical offered load the problem would be trivial. If a cell required \( n \) channels and \( \sigma \) was required to be \( 6 \), then 12 sets of \( n \) channels each would be required. Assuming the offered load grows proportionally in each cell, the system could grow until 12 \( n \) equals the number of channels available. However, actual systems will be more like the one shown in Figure 2.1-7. It is no longer obvious how many channels are required to serve this
REQUIRED NUMBER OF CHANNELS

OFFERED LOAD IN ERLANGS FOR A PROBABILITY OF BLOCKING OF .02 USING THE ERLANG B CURVE.

CHANNEL SET

FIGURE 2.1-7 A TYPICAL SYSTEM LAYOUT
system. If channels are assigned following the regular spacing for \( \sigma = 6 \), it turns out that 158 channels are required. However, if this pattern is distorted slightly by reusing some channels at spacings slightly greater than \( \sigma = 6 \), the required number of channels can be reduced to 151. This distortion has been called fixed borrowing because in a sense channels are borrowed from cells that do not need all the channels available in the set associated with that cell.

In the general case, the minimum number of channels required for a given system layout is a little harder to see. For example, consider the case where instead of one area of high traffic density in the center as in Figure 2.1-7, there are two or more distinct high traffic regions. A procedure has been developed for finding a lower bound on the minimum number of channels when fixed borrowing is allowed. It has not been proven that this lower bound is always sufficient. However, the procedure was tried on several examples and in each case it was possible to make a channel assignment with the derived number of channels. It is felt with a high degree of confidence that the lower bound is in fact the true minimum. The method for finding a lower bound is based on the mutual interference set defined in Section 2.1.1.3. It is argued that the number of channels required for the cells in a mutual interference set is the sum of the individual cell requirements since no channel can be reused within such a set. Thus, the procedure is to enumerate each possible mutual interference set and find the one that contains the maximum number of channels. For example, the controlling set is outlined for Figure 2.1-7 and contains 151 channels.
A computer program has been written which performs the procedure described above. It carries the procedure up to the point of doing the fixed borrowing. This part has not been mechanized because it is easily performed manually since the output from the program can indicate exactly which channels are available for assignment to which cells.

Thus, we have progressed to the point that given a system layout and offered load for each cell, we can find the minimum number of channels required to serve that system at a specified level of blocking with fixed channel assignment. However, the problem at hand is to find out how much a given system can grow using fixed channel assignments without channel requirements exceeding the available channels. We propose to solve that problem iteratively by increasing the load then computing channel requirements as above. This approach has been found to work quite nicely on a time shared computer terminal. It is assumed that all cells grow at a uniform rate. For example, it was found that the system of Figure 2.1-7 can grow by a factor of 2.97 until it requires 360 channels. In the next section, it is shown that further growth can be accommodated with the use of dynamic channel assignment techniques. Evaluation of these techniques is measured with respect to the 2.97 growth factor above.

2.2 Dynamic Channel Assignment

As defined in Chapter 1, dynamic channel assignment implies the lack of a fixed relationship between cells and channels. The
above definition allows for almost an infinite variety of dynamic channel assignment procedures. Algorithms studied in this work are restricted to those which have a nominal relation between channels and cells from which deviation is permitted. The nominal channel assignment is used as a first choice group for a particular cell. If it is not possible to serve a particular call with a channel from this group, then the search is extended to other channels. When an idle channel is found as a result of this extended search and the call is assigned to this channel, this channel is said to be borrowed. Thus, a channel is borrowed whenever it is assigned for use in a cell to which it is not nominally assigned. It is assumed that a channel is always radiated by the transmitters of the cell to which it is assigned. In most cases, there is more than one, in fact many, channels which could be borrowed for a given call. It is in the method of selecting one of these channels that makes up most of the variation in the channel assignment algorithms studied.

2.2.1 Algorithms

Three algorithms were studied. For convenience, they are designated algorithm I, II, and III. Algorithm I was proposed in an unpublished memorandum by Mr. J. S. Engel.\(^{(10)}\) Algorithm II is an improved version of I and was a joint effort of Mr. J. S. Engel and Mr. M. M. Peritsky.\(^{(11)}\) Algorithm III was developed by the author after having profited by experience with I and II.
2.2.1.1 Algorithm I

This algorithm assumes a nominal channel assignment as above. When attempting to serve a call, if all nominal channels are busy, an attempt is made to borrow a channel which is nominally assigned to an adjacent cell. This is accomplished by computing the number of channels which are available for borrowing in each adjacent cell. Then a channel is borrowed from the cell which has the largest number available for borrowing. There is also the question of when a borrowed channel should be returned. Two strategies have been tried. The easiest is to let the borrowed channel remain until the call it is serving is terminated. The other returns a borrowed channel as soon as a nominal channel becomes available. This was done by simply maintaining a list of borrowed channels. Whenever a nominal channel becomes idle, a call was transferred from one of the borrowed channels to it. No special effort was made to return channels to the cell most in need, analogous to the procedure for borrowing. The first strategy has been termed normal return, while the second has been called immediate return. Figure 2.2-1 is a flowchart of the borrowing algorithm.

2.2.1.2 Algorithm II

This is an improved version of algorithm I. It is also more complex. We begin with some terminology. Consider a nominal channel assignment as before, then we define:

Nominal Channels - given a cell, those channels nominally assigned to it are its nominal channels.
FIGURE 2.2-1 CHANNEL ASSIGNMENT ALGORITHM I
**Nominal Cells** - given a channel, those cells to which it is nominally assigned are its nominal cells.

**Interferable Cells** - given a cell, any other cell with which it will interfere is one of its interferable cells.

This situation is illustrated in Figure 2.2-2. Three nominal cells are shown for a typical channel. Note that these cells meet the channel spacing requirement $\sigma = 6$ discussed in Section 2.1.2 Also, all of the interferable cells are indicated for a particular cell labeled Borrowing Cell. There can be as many as 36 of these for $\sigma = 6$. In the layout shown in Figure 2.2-2 only 15 of these 36 possible locations are in the system area.

As before, candidates for borrowing are nominal channels assigned to adjacent cells. However, as is shown in Figure 2.2-2, borrowing can have an effect on several nominal cells. It is the intent of Algorithm II to take this fact into consideration and base the decision on which channel to borrow on the state of the worst case nominal cell of all nominal cells affected by the proposed borrowing. Worst case refers to the nominal cell which will have the fewest nominal channels available after the proposed borrowing. Stated in words, the objective of this algorithm is as follows. Choose the candidate channel which maximizes the available nominal channels in the worst case nominal cell which is also an interferable cell. Algorithm I chooses the channel to be borrowed based on only the state of the adjacent cell.
FIGURE 2.2-2 ILLUSTRATION FOR ALGORITHM II
Implementation of this algorithm is illustrated by the flow-chart of Figure 2.2-3. The general procedure is to compute a figure of merit for each candidate channel and choose the one for which this figure of merit is maximum. The figure of merit is the number of nominal channels available for use in the worst case nominal cell which would be affected by the proposed borrowing.

2.2.1.3 Algorithm III

Algorithm III is perhaps the simplest of the three algorithms. As before, we start with a nominal channel assignment. However, the point of view is slightly different. In the previous two algorithms, we assigned nominal channels to cells based on the offered load in each cell. In this case we assign channels to channel sets. Then the nominal channel assignment for each cell is the entire channel set determined by the cell's location as shown in Figure 2.1-7. This results in some cells having many more nominal channels than are needed. Service in these cells is limited by the number of radios provided. Borrowing is attempted when a radio is available, but there are no nominal channels available. The borrowing philosophy is also different. We search for an available channel by searching through channel sets in a prescribed sequence. The first available channel found is then used. Figure 2.2-4 is a flowchart of this algorithm.

2.2.2 Measures of Performance

In this section we examine the various measures of performance that have been selected for use in evaluating these algorithms. With
FIGURE 2.2-3 CHANNEL ASSIGNMENT ALGORITHM II
FIGURE 2.2-4 CHANNEL ASSIGNMENT ALGORITHM III
such a complex system, the number of things one could measure is virtually inexhaustible. The goal has been to find a set of statistics which adequately describe performance without overwhelming us with data.

**Average Blocking** - This is perhaps the statistic of most interest. It is computed by dividing the number of blocked calls by the number of calls received. Thus, it is a weighted average. It gives more weight to the high traffic cells since they receive more calls.

**Service Deviation**\(^1\) - This is a number computed from the fraction blocked, \(B_j\), in each cell according to the following formula:

\[
SD = \left( \frac{\sum_{i=1}^{21} \frac{(B_i - \bar{B})^2}{20}}{21} \right)^{\frac{1}{2}}
\]  

(2.2-1)

where

\[
\bar{B} = \frac{1}{21} \sum_{i=1}^{21} B_i
\]

**High-Cell Blocking** - This is the highest fraction of blocked calls in a particular cell.

**Carried Load Per Channel** - This is the total carried load divided by the number of channels. It serves as a measure of spectrum utilization. It also is very dependent on the particular system layout. Without reusing channels this number could never get much larger than .9 for large numbers of channels and would be smaller for small groups.

---

\(^1\)This statistic often serves as an unbiased estimate of the standard deviation. Its use here is just to indicate the relative amount of variation in grade of service.
Radio Occupancy - This is a measure of how efficiently cell radio equipment is used. It is found by dividing the total carried load by the total number of radios.

Channel Tests Per Call - This provides a rough measure of the relative computing power required for each algorithm. A counter is incremented each time a channel test is made. At the end of the run the number of channel tests is divided by the number of calls received.

Amount of Borrowing - This is the fraction of completed calls that were set up on borrowed channels.

Channel Blocking - This is the fraction of blocked calls that were blocked because a channel was unavailable and not due to lack of radio equipment.

Number of Radios - This is the total number of cell radios (a radio includes both transmitting and receiving equipment) provided in the system.

2.2.3 Algorithm Evaluation

This section describes the effort expended to evaluate the three algorithms presented in Section 2.2.1. The objective is to examine the performance of each algorithm with respect to such things as nominal channel assignment, policy of returning borrowed channels and number of radios. The three algorithms are compared with each other over a range of offered loads. In the 3 sections following, the algorithms are examined separately. Finally, in the fourth section, near optimum configurations of each algorithm are compared to each other as a function of the offered load.
The system simulated is shown in Figure 2.2-5. It requires 360 channels if fixed borrowing is allowed. This system was obtained by allowing the traffic in the layout of Figure 2.1-7 to grow by a factor of 2.97 thus establishing the 360 channel requirement with an offered load of 333.59 erlangs for an average blocking probability of .02 or less. Some cells have spare capacity because of the varied break points on the Erlang B curve. In these cells the offered load is increased until they too have a .02 probability of blocking. This increases the total offered load to 342.579 erlangs. This step was taken to insure that any increased capacity with dynamic channel assignment is due solely to dynamic channel assignment. The resulting system is shown in Figure 2.2-5. It was found experimentally that with dynamic channel assignment the load could be increased by about 25 percent for a system average blocking of .02. Thus, in the next 3 sections, the system of Figure 2.2-5 with a 25 percent increase in offered load is studied.

2.2.3.1 Algorithm I

In Section 2.2.1 we assumed a nominal channel assignment when describing the various algorithms. However, we have not as yet discussed how a nominal channel assignment is made. The first approach was to use the channel assignment in Figure 2.2-5, with the required fixed borrowing for 360 channels, as the nominal channel assignment. Results at 25 percent increase in load are shown in the left column of Table 2.2-1. The average blocking is .0175 with unlimited radios. However, the service deviation at .0099 and high cell blocking of
TOTAL OFFERED LOAD = 342.579 ERLANGS

FIGURE 2.2-5 SYSTEM FOR SIMULATION
### NOMINAL ASSIGNMENT

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Blocking</td>
<td>.0175</td>
<td>.024</td>
<td>.0187</td>
</tr>
<tr>
<td>Service Deviation</td>
<td>.0099</td>
<td>.015</td>
<td>.012</td>
</tr>
<tr>
<td>High Cell Blocking</td>
<td>.0394</td>
<td>.035</td>
<td>.0381</td>
</tr>
<tr>
<td>Carried Load Per Channel</td>
<td>1.16</td>
<td>1.152</td>
<td>1.158</td>
</tr>
<tr>
<td>Radio Occupancy</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Channel Tests Per Call</td>
<td>69.5</td>
<td>77</td>
<td>?</td>
</tr>
<tr>
<td>Amount of Borrowing</td>
<td>.217</td>
<td>.274</td>
<td>.202</td>
</tr>
<tr>
<td>Channel Blocking</td>
<td>All</td>
<td>All</td>
<td>All</td>
</tr>
<tr>
<td>Number of Radios</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Run I.D. Number</td>
<td>35</td>
<td>66</td>
<td>18</td>
</tr>
</tbody>
</table>

### NOMINAL ASSIGNMENT

1. Fixed Borrowing Assignment Used as Nominal
2. B(ACCESS,ACLD) = CONSTANT
3. NOM=ACCESS*OFLD/ACLD

---

**TABLE 2.2-1**

**EFFECT OF NOMINAL CHANNEL ASSIGNMENT WITH UNLIMITED RADIOS, ALGORITHM I**
.03914 seemed excessive in view of the verification run values of Appendix A of .00545 and .0314, respectively. This verification run (11) had the same random number generators and an average blocking of .0162. Considerable effort was spent investigating the service deviation without a great deal of success. Figure 2.2-6 illustrates the situation and shows generally that the highest blocking occurs in the center. An explanation was sought using the techniques of Appendix C. Before describing this effort we introduce some new variables:

Access (cell) = \( \sum_{k} N_{OM}(k) \)

\( k \) (adjacent cells)

= sum of nominal channels in adjacent cells.

i.e., the total number of channels that

this cell has access to for borrowing.

Access Load (cell) = \( \sum_{k} O_{FLD}(k) \)

\( k \) (adjacent cells)

= ACLD (cell) = sum of the offered load

in all cells adjacent to the subject

cell. i.e., the offered load that

subject cell is competing with when

borrowing channels.

Where

\( N_{OM}(k) = \) Nominal channels assigned to cell k.

\( O_{FLD}(k) = \) Offered load to cell k.
360 CHANNELS
LOAD INCREASE FACTOR 1.25
FIRST ALGORITHM
NATURAL RETURN
RUN # 8

FIGURE 2.2-6 FRACTION BLOCKED
Essentially no correlation could be found between the blocking in a cell and any parameter of that cell alone. However, when surrounding cells were considered some correlation was found. Most promising was a correlation coefficient of about .8 between the blocking in a cell and the Erlang B formula using the access and access load as arguments, i.e., B(ACCESS, ACLD). Note that ACCESS(N) and ACLD(N) do not include NOM(N) and OFLD(N). No improvement was made by considering them in the Erlang B formula, i.e., B(ACCESS+NOM, ACLD+OFLD). The next logical step seemed to be to make B(ACCESS, ACLD) constant which would hopefully improve the situation. This was done by solving

\[ B(\text{ACCESS}(N), \text{ACLD}(N)) = \text{CONSTANT} \quad ;N \text{ over all cells} \quad (2.2-2) \]

for ACCESS(N). Then the linear equations implied by the definition of ACCESS,

\[ \sum_{k} \text{NOM}(k) = \text{ACCESS}(N) \quad ;N \text{ over all cells} \quad (2.2-3) \]

\( k \) (adjacent cells)

could then be solved for NOM(N). By trial and error the constant in 2.2-2 was adjusted so that

\[ \sum_{N} \text{NOM}(N) = 360 \quad \text{(High Mutual Interference Set)} \quad (2.2-4) \]

thereby utilizing the available 360 channels. Results in column 2 of Table 2.2-1 were disappointing. Service deviation increased and a check showed that the correlation with the resulting B(ACCESS, ACLD) had disappeared as well.

Next, the same thing was tried with the number of radios limited for .02 radio blocking. Results are shown in Table 2.2-2
### NOMINAL ASSIGNMENT

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Blocking</td>
<td>0.0252</td>
<td>0.0294</td>
<td>0.0244</td>
<td>0.0277</td>
<td>0.0248</td>
<td>0.0219</td>
</tr>
<tr>
<td>Service Deviation</td>
<td>0.0125</td>
<td>0.0204</td>
<td>0.0136</td>
<td>0.0126</td>
<td>0.0118</td>
<td>0.0122</td>
</tr>
<tr>
<td>High Cell Blocking</td>
<td>0.0457</td>
<td>0.0966</td>
<td>0.0521</td>
<td>0.066</td>
<td>0.047</td>
<td>0.047</td>
</tr>
<tr>
<td>Carried Load Per Channel</td>
<td>1.15</td>
<td>1.146</td>
<td>1.152</td>
<td>1.148</td>
<td>1.151</td>
<td>1.155</td>
</tr>
<tr>
<td>Radio Occupancy</td>
<td>0.70</td>
<td>0.70</td>
<td>0.70</td>
<td>0.70</td>
<td>0.70</td>
<td>0.71</td>
</tr>
<tr>
<td>Channel Tests Per Call</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>67.8</td>
<td>61.5</td>
<td>?</td>
</tr>
<tr>
<td>Amount of Borrowing</td>
<td>0.187</td>
<td>0.250</td>
<td>0.193</td>
<td>0.233</td>
<td>0.181</td>
<td>0.138</td>
</tr>
<tr>
<td>Channel Blocking</td>
<td>0.496</td>
<td>0.660</td>
<td>0.616</td>
<td>0.619</td>
<td>0.543</td>
<td>0.619</td>
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<tr>
<td>Number of Radios</td>
<td>589</td>
<td>589</td>
<td>589</td>
<td>589</td>
<td>589</td>
<td>589</td>
</tr>
<tr>
<td>Radio Blocking</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>Run I.D. Number</td>
<td>14</td>
<td>22</td>
<td>23</td>
<td>71</td>
<td>62</td>
<td>19</td>
</tr>
</tbody>
</table>

**NOMINAL ASSIGNMENT**

1. Fixed Borrowing Assignment Used as Nominal
2. \( B(\text{ACCESS,ACLĐ}) = \text{CONSTANT} \)
3. \( B(\text{NOM,OFLĐ}) = \text{CONSTANT} \)
4. \( B(\text{NOM+ACCESS,ACLĐ+OFLĐ}) = \text{CONSTANT} \)
5. \( \text{NOM}=\text{OFLĐ*ACCESS/ACLĐ} \)
6. Similar to 4, but Gave Extra Channels to Lightly Loaded Cells

**TABLE 2.2-2**

**EFFECT OF NOMINAL CHANNEL ASSIGNMENT WITH LIMITED RADIOS, ALGORITHM I**
columns 1 and 2, which were also disappointing. It was then thought that the difficulty might be that \( B[\text{ACCESS, ACLD}] \) considered only surrounding cells. Thus, in Table 2.2-2 column 3, \( B(\text{NOM, OFLD}) \) was set equal to a constant with equally disappointing results. A remaining option would be to consider both the individual and surrounding cells in the nominal channel assignment. That is set

\[
B[\text{ACCESS(N)+NOM(N), ACLD(N)+OFLD(N)}] = \text{CONSTANT}. \tag{2.2-5}
\]

\[
\Sigma \text{NOM(k)+NOM(N)} = \text{TACCESS(N)} \tag{2.2-6}
\]

\( k \) - (adjacent cells)

where

\[
\text{TACCESS(N)} = \text{ACCESS(N)}+\text{NOM(N)} \tag{2.2-7}
\]

as found by 2.2-5, are solved for \( \text{NOM(N)} \). As in 2.2-2 the constant in 2.2-5 was adjusted to satisfy 2.2-4. Results are recorded in column 4 of Table 2.2-2. Some improvement was noted. Next, another approach was tried. It was to set

\[
\text{NOM(N)} = \text{OFLD(N)}*\text{ACCESS(N)}/\text{ACLD(N)} \tag{2.2-8}
\]

where \( \text{ACCESS(N)} \) is found from equation 2.2-2. This makes \( \text{NOM(N)} \) dependent upon the offered load in both the cell \( N \) and the adjacent cells. Again, the results shown in Table 2.2-1 column 3 and Table 2.2-2 column 5 are disappointing with only a slight improvement in column 5. Studying the results of the run in Table 2.2-2 column 5 in more detail indicated that some of the low traffic cells were borrowing channels unnecessarily from high traffic cells. Accordingly, the nominal assignment of column 6 in Table 2.2-2 was made to give the low traffic cells extra channels. This decreased the
blocking slightly but it also slightly increased the service deviation. However, this train of thought led to Algorithm III which did indeed seem to be an improvement.

Other effects investigated with Algorithm I are the number of radios (Table 2.2-3) and the strategy for returning borrowed channels (Table 2.2-4). Both of these investigations were made with the original nominal channel assignment. The results of limiting the number of radios is pretty much as expected. However, the effect of the strategy for returning borrowed channels is interesting. Returning them immediately reduces performance in almost all categories. Blocking, service deviation, and high cell blocking are all increased. The amount of borrowing is almost doubled. We defer comment on this until the discussion of Algorithm III.

2.2.3.2 Algorithm II

In general, Algorithm II performs similar too, but slightly better than Algorithm I (See Table 2.2-5). The nominal channel assignment based on the 360 channel system with fixed borrowing was used throughout. Again, the result of limiting the number of cell radios was pretty much as one would expect. Immediate return of borrowed channels caused an increase in blocking and almost doubled the amount of channel borrowing as before. In these runs, the number of channel tests per call was measured also. Immediate return increased these by approximately 90 percent.
<table>
<thead>
<tr>
<th></th>
<th>Run 1</th>
<th>Run 2</th>
<th>Run 3</th>
<th>Run 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Blocking</td>
<td>.0175</td>
<td>.0196</td>
<td>.0207</td>
<td>.0252</td>
</tr>
<tr>
<td>Service Deviation</td>
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<td>.0135</td>
<td>.0129</td>
<td>.0125</td>
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<td>High Cell Blocking</td>
<td>.0394</td>
<td>.0496</td>
<td>.0521</td>
<td>.0475</td>
</tr>
<tr>
<td>Carried Load Per Channel</td>
<td>1.16</td>
<td>1.157</td>
<td>1.156</td>
<td>1.15</td>
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<tr>
<td>Radio Occupancy</td>
<td>—</td>
<td>.64</td>
<td>.666</td>
<td>.70</td>
</tr>
<tr>
<td>Channel Tests Per Call</td>
<td>69.5</td>
<td>66.8</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Amount of Borrowing</td>
<td>.217</td>
<td>.204</td>
<td>.21</td>
<td>.187</td>
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<tr>
<td>Channel Blocking</td>
<td>All</td>
<td>.874</td>
<td>.784</td>
<td>.496</td>
</tr>
<tr>
<td>Number of Radios</td>
<td>651</td>
<td>625</td>
<td>589</td>
<td></td>
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<tr>
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<td>0</td>
<td>.005</td>
<td>.01</td>
<td>.02</td>
</tr>
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<td>Run I.D. Number</td>
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<td>38</td>
<td>16</td>
<td>14</td>
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**TABLE 2.2-3**

EFFECT OF NUMBER OF RADIOS, ALGORITHM I
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Algorithm I</th>
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</thead>
<tbody>
<tr>
<td>Average Blocking</td>
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<td>.0204</td>
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<tr>
<td>Service Deviation</td>
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<td>.0115</td>
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<tr>
<td>High Cell Blocking</td>
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<td>.0415</td>
</tr>
<tr>
<td>Carried Load Per Channel</td>
<td>1.16</td>
<td>1.156</td>
</tr>
<tr>
<td>Radio Occupancy</td>
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<td>-</td>
</tr>
<tr>
<td>Channel Tests Per Call</td>
<td>69.5</td>
<td>?</td>
</tr>
<tr>
<td>Amount of Borrowing</td>
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<td>.392</td>
</tr>
<tr>
<td>Channel Blocking</td>
<td>All</td>
<td>All</td>
</tr>
<tr>
<td>Number of Radios</td>
<td>∞</td>
<td>∞</td>
</tr>
<tr>
<td>Borrowed Channels Returned</td>
<td>Naturally</td>
<td>Immediately</td>
</tr>
<tr>
<td>Run I.D. Number</td>
<td>35</td>
<td>6</td>
</tr>
</tbody>
</table>

**TABLE 2.2-4**

EFFECT OF STRATEGY FOR RETURNING BORROWED CHANNELS, ALGORITHM I
<table>
<thead>
<tr>
<th></th>
<th>0.0173</th>
<th>0.0227</th>
<th>0.020</th>
<th>0.0208</th>
</tr>
</thead>
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<tr>
<td>Average Blocking</td>
<td>0.0117</td>
<td>0.0127</td>
<td>0.00995</td>
<td>0.012</td>
</tr>
<tr>
<td>Service Deviation</td>
<td>0.048</td>
<td>0.0394</td>
<td>0.0322</td>
<td>0.0465</td>
</tr>
<tr>
<td>High Cell Blocking</td>
<td>1.160</td>
<td>1.154</td>
<td>1.157</td>
<td>1.156</td>
</tr>
<tr>
<td>Carried Load Per Channel</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Radio Occupancy</td>
<td>-</td>
<td>-</td>
<td>0.64</td>
<td>0.64</td>
</tr>
<tr>
<td>Channel Tests Per Call</td>
<td>120.4</td>
<td>228.9</td>
<td>119.8</td>
<td>221.8</td>
</tr>
<tr>
<td>Amount of Borrowing</td>
<td>0.207</td>
<td>0.391</td>
<td>0.207</td>
<td>0.378</td>
</tr>
<tr>
<td>Channel Blocking</td>
<td>All</td>
<td>All</td>
<td>0.852</td>
<td>0.943</td>
</tr>
<tr>
<td>Number of Radios</td>
<td></td>
<td></td>
<td>651</td>
<td>651</td>
</tr>
<tr>
<td>Radio Blocking</td>
<td>0</td>
<td>0</td>
<td>0.005</td>
<td>0.005</td>
</tr>
</tbody>
</table>

**TABLE 2.2-5**

**EFFECT OF STRATEGY FOR RETURNING BORROWED CHANNELS AND LIMITED RADIOS, ALGORITHM II**

Borrowed Channels Returned Naturally Immediately Naturally Immediately

Run I.D. Number 33 34 37 63
2.2.3.3 Algorithm III

As previously mentioned, Algorithm III was conceived after observing the performance of Algorithms I and II. The first nominal channel assignment (assignment of channels to channel sets) used was found in the following manner. First, for each channel set listed in Figure 2.2-5 the highest traffic cell having that channel set assigned was identified. Then at the 25 percent increase in offered load, the 360 channels were distributed over these 12 cells so that each would have equal blocking according to the Erlang B formula. Having made a nominal channel assignment, we must next formulate a sequence for searching through these channels to find an available one. Three sequences have been used and they are illustrated in Figure 2.2-7. The regular sequence searches channel sets in a clockwise manner starting with the set immediately to the right of the nominal set. The irregular sequence starts at the same point and sort of jumps around. The idea is to more evenly distribute the locations from which channels are borrowed. Another clockwise regular sequence starting at the channel set to the left was also tried. It is referred to as the alternate starting point sequence. The length of these sequences was also varied. Two lengths were used. One gives every cell access to all 12 sets totaling 360 channels. The other restricts access to only the 6 adjacent channel sets. The first two search sequences and accesses are examined in Table 2.2-6. The regular sequence was always best as was the longest access. It was also observed in these runs that cells in the high mutual interference
REGULAR SEQUENCE - STARTING ON RIGHT

IRREGULAR SEQUENCE STARTING ON RIGHT

REGULAR SEQUENCE - STARTING ON LEFT

FIGURE 2.2-7 SEARCH SEQUENCES
<table>
<thead>
<tr>
<th></th>
<th>0.0224</th>
<th>0.0232</th>
<th>0.0245</th>
<th>0.0256</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Blocking</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Service Deviation</td>
<td>0.0100</td>
<td>0.0102</td>
<td>0.0114</td>
<td>0.0147</td>
</tr>
<tr>
<td>High Cell Blocking</td>
<td>0.035</td>
<td>0.038</td>
<td>0.039</td>
<td>0.0572</td>
</tr>
<tr>
<td>Carried Load Per Channel</td>
<td>1.154</td>
<td>1.153</td>
<td>1.152</td>
<td>1.150</td>
</tr>
<tr>
<td>Radio Occupancy</td>
<td>0.71</td>
<td>0.705</td>
<td>0.704</td>
<td>0.703</td>
</tr>
<tr>
<td>Channel Tests Per Call</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Amount of Borrowing</td>
<td>0.282</td>
<td>0.286</td>
<td>0.272</td>
<td>0.278</td>
</tr>
<tr>
<td>Channel Blocking</td>
<td>0.45</td>
<td>0.398</td>
<td>0.544</td>
<td>0.495</td>
</tr>
<tr>
<td>Number of Radios</td>
<td>589</td>
<td>589</td>
<td>589</td>
<td>589</td>
</tr>
<tr>
<td>Radio Blocking</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>Access</td>
<td>12</td>
<td>12</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>Run I.D. Number</td>
<td>24</td>
<td>25</td>
<td>26</td>
<td>28</td>
</tr>
</tbody>
</table>

**TABLE 2.2-6**

EFFECT OF SEARCH SEQUENCE AND ACCESS, ALGORITHM III
L'et (HMIS) (defined in Figure 2.1-7) experienced both channel and equipment blocking while the other cells experience virtually no channel blocking. This led to the strategy of assigning radios at the .005 level in the HMIS and at .02 in the other cells. This results in substantial improvement in service as seen by comparing column 1 of Table 2.2-7 with column 1 of Table 2.2-6. We will investigate this in more detail later.

Also, in Table 2.2-7 we again see the effect of immediate return of borrowed channels. Blocking is increased, channel tests increase by 71 percent and borrowing almost doubles. This effect has been consistently observed in all algorithms. No real reason is known for this behavior, but it must be related to system time constants. These are discussed for the case of no channel borrowing in Appendix A and are on the order of an average call holding time. Borrowing is caused by a cell that is experiencing a small overload. Because of the system time constant, that overload will persist a while even with Poisson call arrivals. That is, once a cell is carrying an above average number of calls, this increased load will continue for a length of time determined by system time constants.

By not returning the borrowed channel as soon as a nominal one is available, it is reasoned that the system in effect adapts to small overloads. This is just a plausible explanation of the phenomenon. Substantial work would be required to resolve this issue.

As a matter of curiosity two other nominal channel assignments were tried (see Table 2.2-8). First, the channel assignment of
<table>
<thead>
<tr>
<th>Average Blocking</th>
<th>0.0185</th>
<th>0.0236</th>
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</thead>
<tbody>
<tr>
<td>Service Deviation</td>
<td>0.0085</td>
<td>0.0129</td>
</tr>
<tr>
<td>High Cell Blocking</td>
<td>0.0285</td>
<td>0.0502</td>
</tr>
<tr>
<td>Carried Load Per Channel</td>
<td>1.159</td>
<td>1.153</td>
</tr>
<tr>
<td>Radio Occupancy</td>
<td>0.660</td>
<td>0.656</td>
</tr>
<tr>
<td>Channel Tests Per Call</td>
<td>48.9</td>
<td>83.7</td>
</tr>
<tr>
<td>Amount of Borrowing</td>
<td>0.301</td>
<td>0.555</td>
</tr>
<tr>
<td>Channel Blocking</td>
<td>0.775</td>
<td>0.847</td>
</tr>
<tr>
<td>Number of Radios</td>
<td>632</td>
<td>632</td>
</tr>
<tr>
<td>Radio Blocking</td>
<td>0.02/.005</td>
<td>0.02/.005</td>
</tr>
<tr>
<td>Borrowed Channels Returned</td>
<td>Naturally</td>
<td>Immediately</td>
</tr>
<tr>
<td>Run I.D. Number</td>
<td>29</td>
<td>30</td>
</tr>
</tbody>
</table>

**TABLE 2.2-7**

**EFFECT OF RETURN STRATEGY, ALGORITHM III**
<table>
<thead>
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<th></th>
<th>Run 1</th>
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<th>Run 3</th>
</tr>
</thead>
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<tr>
<td>Average Blocking</td>
<td>.0185</td>
<td>.0179</td>
<td>.0175</td>
</tr>
<tr>
<td>Service Deviation</td>
<td>.0085</td>
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<td>.00815</td>
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<tr>
<td>High Cell Blocking</td>
<td>.0285</td>
<td>.0292</td>
<td>.0277</td>
</tr>
<tr>
<td>Carried Load Per Channel</td>
<td>1.159</td>
<td>1.159</td>
<td>1.160</td>
</tr>
<tr>
<td>Radio Occupancy</td>
<td>.660</td>
<td>.66</td>
<td>.661</td>
</tr>
<tr>
<td>Channel Tests Per Call</td>
<td>48.9</td>
<td>47.6</td>
<td>61.5</td>
</tr>
<tr>
<td>Amount of Borrowing</td>
<td>.301</td>
<td>.301</td>
<td>.501</td>
</tr>
<tr>
<td>Channel Blocking</td>
<td>.775</td>
<td>.784</td>
<td>.744</td>
</tr>
<tr>
<td>Number of Radios</td>
<td>632</td>
<td>632</td>
<td>632</td>
</tr>
<tr>
<td>Radio Blocking</td>
<td>.02/.005</td>
<td>.02/.005</td>
<td>.02/.005</td>
</tr>
<tr>
<td>Nominal Assignment</td>
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<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Run I.D. Number</td>
<td>29</td>
<td>31</td>
<td>61</td>
</tr>
</tbody>
</table>

1. Channels assigned to give high cells in each set equal blocking at .02.
2. Channels assigned to give HMIS .02 blocking.
3. Each channel set has 30 channels.

**TABLE 2.2-8**

**EFFECT OF NOMINAL CHANNEL ASSIGNMENT,**

**ALGORITHM III**
algorithm I and II was tried and it was found that average blocking
decreased but service deviation increased. The magnitude was small
in both cases. Similar changes accompanied by a substantial increase
in the amount of borrowing resulted when 30 nominal channels were
assigned to each set. This all suggests that algorithm III is not
too sensitive to its nominal channel assignment.

Table 2.2-9 shows the effect of the starting point on the
regular search sequence. The left starting point is slightly better,
but the results are not very significant and are probably due to the
slightly unsymmetric system layout.

The first two columns of Table 2.2-10 hold a surprise. In
the second column it is seen that when the total number of radios was
reduced, by assigning them at a level of .0086 blocking in the HMIS,
service improved somewhat. Increasing this further to .015 degrades
service with the optimum value being near .01 as shown in the third
column. An explanation is as follows. By restricting the radios in
the HMIS we have reduced the borrowing. This means that more channels
are used at a regular spacing which is more efficient. Thus, there is
some gain in slightly restricting the borrowing.

2.2.3.4 Comparing Algorithms I, II, III

In this section we compare the performance of the algorithms
of the last 3 sections as a function of offered load. Of the many
configurations studied previously, one is selected for each algorithm
for study in this section. For algorithm I it is the one whose results
are tabulated in column 2 of Table 2.2-3 which uses the nominal
<table>
<thead>
<tr>
<th></th>
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<th>Right</th>
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</thead>
<tbody>
<tr>
<td>Average Blocking</td>
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<td>.0179</td>
</tr>
<tr>
<td>Service Deviation</td>
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<td>.00957</td>
</tr>
<tr>
<td>High Cell Blocking</td>
<td>.0287</td>
<td>.0292</td>
</tr>
<tr>
<td>Carried Load Per Channel</td>
<td>1.160</td>
<td>1.159</td>
</tr>
<tr>
<td>Radio Occupancy</td>
<td>.661</td>
<td>.66</td>
</tr>
<tr>
<td>Channel Tests Per Call</td>
<td>46.3</td>
<td>47.6</td>
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<td>Amount of Borrowing</td>
<td>.289</td>
<td>.301</td>
</tr>
<tr>
<td>Channel Blocking</td>
<td>.704</td>
<td>.784</td>
</tr>
<tr>
<td>Number of Radios</td>
<td>632</td>
<td>632</td>
</tr>
<tr>
<td>Radio Blocking</td>
<td>.02/.005</td>
<td>.02/.005</td>
</tr>
<tr>
<td>Starting Point</td>
<td>Left</td>
<td>Right</td>
</tr>
<tr>
<td>Run I.D. Number</td>
<td>58</td>
<td>31</td>
</tr>
</tbody>
</table>

**TABLE 2.2-9**

**EFFECT OF STARTING POINT, ALGORITHM III**
<table>
<thead>
<tr>
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<tr>
<td>EFFECT OF NUMBER OF RADIOS,</td>
</tr>
<tr>
<td>ALGORITHM III</td>
</tr>
<tr>
<td>Average Blocking</td>
</tr>
<tr>
<td>Service Deviation</td>
</tr>
<tr>
<td>High Cell Blocking</td>
</tr>
<tr>
<td>Carried Load Per Channel</td>
</tr>
<tr>
<td>Radio Occupancy</td>
</tr>
<tr>
<td>Channel Tests Per Call</td>
</tr>
<tr>
<td>Amount of Borrowing</td>
</tr>
<tr>
<td>Channel Blocking</td>
</tr>
<tr>
<td>Number of Radios</td>
</tr>
<tr>
<td>Radio Blocking</td>
</tr>
<tr>
<td>Run I.D. Number</td>
</tr>
</tbody>
</table>
channel assignment based on fixed borrowing described at the beginning of Section 2.2.3.1. Cell radios are provided at the level of .005 blocking. Algorithm II uses the same nominal channel assignment and same number of cell radios. Results for 25 percent load increase are in column 3 of Table 2.2-5. Algorithm III also uses the same nominal channel assignment. However, cell radios are provided at the .005 level of blocking only in the HMIS. Other cells are supplied at a blocking level of .02. Results for 25 percent load increase are tabulated in Table 2.2-8, Column 2.

The procedure is to vary the offered load from 15 to 50 percent above the maximum fixed channel assignment case as described in the beginning of Section 2.2.3. All of the measures of performance listed in 2.2.2 are considered.

**Average Blocking** - Figure 2.2-8 shows that to a first order the system with dynamic channel assignment performs as if the high mutual interference set were a full access system of 360 servers. (The Erlang B curve is found by offering the load of the HMIS to 360 servers.) Increased blocking below 25 percent is due to radios being supplied at only the .005 level of blocking. Individually the algorithms rank III, II, I with III being the best by a narrow margin.

**Service Deviation** - The situation in Figure 2.2-9 is about the same, with algorithms performing in order of III, II, I in the 25 percent region for which the average blocking is .02. All of the algorithms show more service deviation than did the verification run.
Figure 2.2-8 Average Blocking vs Load

- Algorithm I
- Algorithm II
- Algorithm III
- Erlang B
- 360 Servers

% Load Increase vs Average Blocking
Figure 2.2-9: Service Deviation vs Load

- Algorithm I
- Algorithm II
- Algorithm III

OBSERVED IN VERIFICATION RUN #11
**High Cell Blocking** - Referring to Figure 2.2-10, the relative performance of the algorithms continues the established trend. However, results are closer to those observed in the verification run than was the case with service deviation.

**Carried Load Per Channel** - As shown in Figure 2.2-11 the difference between the three algorithms is indistinguishable. However, this does indicate the effectiveness of the small cell concept in increasing spectrum utilization. In the 25 percent load increase region we achieve about 1.16 erlangs of carried load per channel. This is compared to an upper bound of about 0.9 erlangs per channel in a system with a similar number of channels, but not allowing channels to be reused. This amounts to a gain factor of 1.29. It should be pointed out that the system studied is a small one. The benefits of the small cell concept will have much more impact as the ratio of cells to required channel sets is increased.

**Radio Occupancy** - These data were not plotted since results were in the .60 to .66 range in all cases. This reflects some inefficiency because it is necessary to provide these radios at the .005 blocking level.

**Channel Tests Per Call** - These results plotted in Figure 2.2-12 show significant difference among the three algorithms. The number of channel tests required per call is useful as a rough measure of the computing power required to do dynamic channel assignment relative to the fixed channel assignment case observed in verification run No. 11. The fixed case required 18.5 channel tests per call, while at 25
ALGORITHM I
ALGORITHM II
ALGORITHM III

OBSERVED IN VERIFICATION RUN #11

FIGURE 2.2-10 HIGH CELL BLOCKING VS LOAD
FIGURE 2.2-11  CARRIED LOAD PER CHANNEL VS LOAD
FIGURE 2.2-12  CHANNEL TESTS PER CALL VS LOAD

OBSERVED IN VERIFICATION RUN #11
percent load increase, algorithms I, II, and III required, 66.8, 119.8, and 47.6 channel tests per call respectively. No attempt was made to optimize the implementations with respect to the number of channel tests, but it is felt that the relative magnitudes will not change a great deal with different implementations and are indicative of the relative computing power required.

Channel Borrowing - Figure 2.2-13 plots the fraction of completed calls for which channels were borrowed. Algorithm III requires substantially more borrowing than the other two. It is interesting that in spite of the larger amount of borrowing, algorithm III has less blocking.

Channel Blocking - Figure 2.2-14 indicates the fraction of total blocking that was caused by failure to find a channel as opposed to the unavailability of a cell radio. Dynamic channel assignment will find its most use in systems which are channel limited. At a load increase of 25 percent .8 to .9 of the blocking is due to channels being unavailable. This seems to be a reasonable value.

Number of Radios - Figure 2.2-15 presents an indication of the number of radios required to serve this system.

2.3 Summary and Conclusions on Dynamic Channel Assignment

1) Dynamic channel assignment provides an effective way of increasing traffic handling capacity of systems that are limited by spectrum availability. A first order estimate of this capacity can be obtained by considering the high mutual interference set as a single full access system served by the total number of channels available to the system. This is also an upper bound on improvement.
CHANNEL BORROWING (FRACTION OF COMPLETED CALLS)

ALGORITHM I
ALGORITHM II
ALGORITHM III

FIGURE 2.2-13 CHANNEL BORROWING VS LOAD
Figure 2.2-14: Channel Blocking vs Load

- Algorithm I
- Algorithm II
- Algorithm III

Channel Blocking (Fraction of Total) vs % Load Increase
Figure 2.2-15  Number of Radios vs Load
2) All three algorithms provide similar increases and service characteristics. However, algorithm III has a slight edge over the other two in almost every respect. In particular, blocking and service deviation are slightly lower, while computing effort is significantly reduced. It is recognized that algorithm I and II do not give each cell access to all channels as does III.

3) Dynamic channel assignment causes reduced efficiency of cell radio equipment because it is necessary to have equipment blocking very small in relation to channel blocking to realize the effectiveness of dynamic channel assignment.

4) Service deviation has been of much concern throughout. Within the accuracy of the simulation it does not appear that service deviation is intolerable with dynamic channel assignment although it is substantially worse than with fixed channel assignment. No fundamental reason has been found to indicate that some means cannot be found to improve this situation.

5) Overload performance of dynamic channel assignment systems will be worse than with fixed channel assignment. This is result of the increased efficiency of dynamic channel assignment. Overload performance is roughly equivalent to a full access system with the same total number of channels.
CHAPTER III

HANDOFFS AND THEIR EFFECTS

We now depart from the channel assignment question and consider the second of the two problems proposed in Chapter I, the effect of handoffs on system performance. The approach has been to make some basic assumptions regarding the motion of vehicles. From the resulting model of vehicle motion, the probability distribution of the distance a mobile will travel until it reaches the edge of a cell is computed. Then by assuming a constant speed, or a distribution of speeds, the distribution of the time until a mobile leaves a cell can be found. From this information in conjunction with a call holding time distribution such things as:

1. Probability a particular call will be handed off,
2. Average number of handoffs per call,
3. Probability distribution of the number of handoffs per call, are computed.

A more complex question is the effect handoffs have on probability of blocking and how much the traffic in each cell is modified.
As before a transit time distribution is computed. This enables one to study the problem by computer simulation. In addition to simulation efforts, an analytical model has been formulated based on the well known Erlang B formula for blocking probability. This results in a system of $N$ simultaneous nonlinear equations where $N$ is the number of cells. An efficient iterative procedure for solving these equations has been developed and tested. Solutions are compared to the simulation results.

3.1 Model of Vehicle Motion

First a model of vehicle motion is needed. Some effort was expended in searching vehicular traffic literature to see if any of the information from this area could be applied to our problem. Results of this effort were not very fruitful. In general it appears that the kinds of questions that confront highway designers are not similar enough to be applicable to this problem.

Due to the lack of any data a mathematical model has been created to describe the statistical characteristics of the movement of the mobile subscriber within the system. With this model it is possible to calculate such things as the number of transitions (handoffs) per call, the time between handoffs, etc. The model is defined by the following assumptions:

1. Each point within a cell is equally likely to be the starting point.

2. The mobile subscriber moves in a straight line until he leaves the cell.
3. The direction he travels is evenly distributed from $0^\circ$ to $360^\circ$.

The above assumptions are applicable to the period from the origination of the call until the first handoff. For subsequent handoffs, it is known that the mobile subscriber starts from the cell's edge. Hence, the first and third assumption are modified as follows:

1. Each point on the cell boundary is equally likely as a starting point.

3. The direction he travels is evenly distributed over the $180^\circ$ which leads to the interior of the cell.

Calculations were performed for the two cases above for each of three cell shapes, - hexagon, circle, and square. The hexagon is of most interest because the hexagonal shape is convenient from the radio coverage standpoint. Calculations were performed on the square because some of the systems cited in the literature use square cells. Finally, the circle was included because it is the simplest case and provides an interesting comparison to the other cases.

3.2 Distance Distribution

We begin by calculating the probability distributions of the random variable $D$, the distance from the starting point to the edge of the cell. Specifically we compute

$$P[D < D_0/\theta]$$

and then

$$P[D < D_0] = \int_0^{D_0} P[D < D_0/\theta] \, p(\theta) \, d\theta \quad (3.2-1)$$
where \( \theta \) is the direction of travel. Considering the hexagon, Figure 3.2-1, we draw the locus of all points which are a distance \( D_0 \) from the edge in the direction \( \theta \). Now since each point in the hexagon is equally likely to be the starting point,

\[
P[D < D_0/\theta] = \frac{\text{Cross Hatched Area}}{\text{Total Area of Hexagon}} = \frac{A_{//}}{A_T}. \quad (3.2-2)
\]

A straightforward, but lengthy calculation reveals the following results:

\[
P_1(D_0, \theta) = \frac{4D_0 \cos \theta}{3\sqrt{3}} - \frac{D_0^2}{9} (\cos^2 \theta - 3\sin^2 \theta) ; \quad 0 \leq D_0 \leq D_1 \quad 0 \leq \theta \leq 30^\circ
\]

\[
P_2(D_0, \theta) = -\frac{1}{3} + \left(\frac{2}{\sqrt{3}} \cos \theta + \frac{2}{3} \sin \theta\right) D_0
\]

\[
P[D < D_0/\theta] = \begin{cases} 
  1 & ; \quad D_0 > D_2 \\
  -\left(\frac{2}{3\sqrt{3}} \cos \theta \sin \theta + \frac{2}{9} \cos^2 \theta\right) D_0^2 & ; \quad D_1 \leq D_0 \leq D_2 \\
  \frac{\sqrt{3}}{\cos \theta + \sqrt{3} \sin \theta} & ; \quad 0 \leq \theta \leq 30^\circ
\end{cases}
\]

where

\[
D_1 = \frac{\sqrt{3}}{\cos \theta + \sqrt{3} \sin \theta} \quad \text{and} \quad D_2 = \frac{\sqrt{3}}{\cos \theta} \quad (3.2-3)
\]

\( D_1 \) is a function of \( \theta \) and is the distance at which the shape of the shaded area changes. \( D_2 \), also a function of \( \theta \), is the maximum distance that a mobile may travel within a cell in the direction of \( \theta \).

So far the range of \( \theta \) has been restricted between \( 0^\circ \) and \( 30^\circ \). We claim by symmetry, the above results hold for all \( \theta \). To see this, we note that \( \theta \) is measured from a side of the hexagon to the direction of travel in such a manner that \( \theta \) is inside the hexagon. Now since there is a side of the hexagon oriented every \( 60^\circ \) completely around
FIGURE 3.2-1 LOCUS OF ALL POINTS OF DISTANCE $D_0$, DIRECTION $\theta$

$P[D<D_0/\theta] = P_1(D_0, \theta)$

$P[D<D_0/\theta] = P_2(D_0, \theta)$

FIGURE 3.2-2 REGIONS OF $D_0$, $\theta$ PLANE
360\degree, the direction of travel will always make an angle of 30\degree or less with one side. Thus every possible direction of travel will reduce to the case studied above. Hence we conclude when \( \theta \) is measured in this manner, its density function is as follows:

\[
P_\theta(\theta) = \begin{cases} 
\frac{6}{\pi} ; & 0 \leq \theta \leq 30\degree \text{ or } \frac{\pi}{6} \\
0 ; & \text{elsewhere}
\end{cases}
\]

Thus

\[
P[D < D_0] = \frac{\pi}{6} 
\int_0^{\frac{\pi}{6}} p_\theta(\theta) P[D < D_0|\theta] d\theta. \quad (3.2-4)
\]

When evaluating this integral we must be careful to integrate the correct probability function over its portion of the \( D_0, \theta \) plane. The plot in Figure 3.2-2 aids in this endeavor. Thus

\[
P[D < D_0] = \begin{cases} 
\int_0^{\frac{\pi}{6}} P_1(D_0,\theta) p_\theta(\theta) d\theta ; & 0 \leq D_0 \leq 1 \\
\int_0^{\theta_1} P_1(D_0,\theta) p_\theta(\theta) d\theta + \int_{\theta_1}^{\frac{\pi}{6}} P_2(D_0,\theta) p_\theta(\theta) d\theta ; & 1 \leq D_0 \leq \sqrt{3} \\
\int_0^{\theta_2} P_2(D_0,\theta) p_\theta(\theta) d\theta; & \sqrt{3} \leq D_0 \leq 2 \quad (3.2-5)
\end{cases}
\]
where $\theta_1$ and $\theta_2$ come from

\[
D_1 = \frac{\sqrt{3}}{\cos \theta_1 \sqrt{3} \sin \theta_1} \quad \Rightarrow \quad \theta_1 = \sin^{-1} \frac{\sqrt{3}}{2D_1} - \frac{\pi}{6} \quad (3.2-6)
\]

\[
D_2 = \frac{\sqrt{3}}{\cos \theta_2} \quad \Rightarrow \quad \theta_2 = \cos^{-1} \frac{\sqrt{3}}{D_2} \quad (3.2-7)
\]

Again after a lengthy calculation we find

\[
P[D < D_0] = \begin{cases} 
\frac{\hbar D_0}{\pi \sqrt{3}} + \left[ \frac{1}{9} - \frac{1}{\pi \sqrt{3}} \right] D_0^2 ; & 0 \leq D_0 \leq 1 \\
\frac{1}{3} + \frac{6}{\pi} \left[ \frac{\sqrt{2} D_0^2 - 3}{2\sqrt{3}} + \frac{2}{9} D_0^2 \sin^{-1} \frac{\sqrt{3}}{2D_0} - \frac{1}{3} \cos^{-1} \frac{\sqrt{3}}{2D_0} - \frac{\pi D_0^2}{18} \right] ; & 1 \leq D_0 \leq \sqrt{3} \\
\frac{2\sqrt{3}}{\pi} - \frac{1}{3} + \frac{6}{\pi} \left[ \frac{D_0^2}{6\sqrt{3}} - \frac{\pi D_0^2}{54} - \frac{5\sqrt{2} D_0^2 - 3}{3\sqrt{3}} + \left( \frac{4}{3} + \frac{D_0^2}{9} \right) \cos^{-1} \frac{\sqrt{3}}{D_0} \right] ; & \sqrt{3} \leq D_0 \leq 2. \quad (3.2-8)
\end{cases}
\]

Differentiating we find the density functions

\[
p(D) = \begin{cases} 
\frac{\hbar}{\pi \sqrt{3}} + \left[ \frac{2}{9} - \frac{2}{\pi \sqrt{3}} \right] D ; & 0 \leq D \leq 1 \\
6 \left[ \frac{\sqrt{2} D^2 - 3}{3\sqrt{3} D} + \frac{\hbar D}{9} \sin^{-1} \frac{\sqrt{3}}{2D} - \frac{\pi D}{9} \right] ; & 1 \leq D \leq \sqrt{3} \\
6 \left[ \frac{D}{3\sqrt{3}} - \frac{\sqrt{2} D^2 - 3}{3\sqrt{3} D} + \frac{2D}{9} \cos^{-1} \frac{\sqrt{3}}{D} - \frac{\pi D}{27} \right] ; & \sqrt{3} \leq D \leq 2. \quad (3.2-9)
\end{cases}
\]
The mean and standard deviation can also be found,

\[
\overline{D} = \frac{2}{\pi} - \frac{4}{\sqrt{3}\pi} - \frac{1}{\sqrt{3}\pi} \log (2\sqrt{3} + 3) + \frac{6}{\sqrt{3}\pi} \log (3) = .7699 \quad (3.2-10)
\]

\[
\overline{D^2} = \frac{9}{2\sqrt{3}\pi} + \sigma_D = .48394.
\]

These functions are plotted in Figure 3.2-3.

Calculations for the case of starting at the edge are similar with a length of line segment playing the role of area in the previous calculation. Just the results are given,

\[
P[D < D_0] = \begin{cases} 
\frac{2D_0}{\pi \sqrt{3}} & ; \quad 0 \leq D_0 \leq 1 \\
\frac{1}{3} + \frac{2}{\pi \sqrt{3}} \left[ \sqrt{D_0^2 - 3} - \frac{\sqrt{3}}{3} \cos^{-1} \frac{\sqrt{3}}{D_0} \right] & ; \quad 1 \leq D_0 \leq \sqrt{3} \\
- \frac{1}{3} + \frac{2\sqrt{3}}{\pi} \left[ 1 + \frac{4}{\sqrt{3}} \cos^{-1} \frac{\sqrt{3}}{D_0} - \sqrt{D_0^2 - 3} \right] & ; \quad \sqrt{3} \leq D_0 \leq 2 \quad (3.2-11)
\end{cases}
\]

\[
p(D) = \begin{cases} 
\frac{2}{\pi \sqrt{3}} & ; \quad 0 \leq D \leq 1 \\
\frac{2}{\pi} \frac{\sqrt{D^2 - 3}}{\sqrt{D}} & ; \quad 1 \leq D \leq \sqrt{3} \\
\frac{2\sqrt{3}}{\pi D} \left[ \frac{4 - D^2}{\sqrt{D^2 - 3}} \right] & ; \quad \sqrt{3} \leq D \leq 2 \quad (3.2-12)
\end{cases}
\]

\[
\overline{D} = \frac{3}{\pi} - \frac{2\sqrt{3}}{\pi} - \frac{\sqrt{3}}{2\pi} \log (2\sqrt{3} + 3) + \frac{3\sqrt{3}}{\pi} \log 3 = 1.1549
\]

\[
\overline{D^2} = \frac{3\sqrt{3}}{\pi} + \sigma_D = .56586. \quad (3.2-13)
\]

These functions are plotted in Figure 3.2-4.
FIGURE 3.2-3 HEXAGON-INTERIOR STARTING POINT

FIGURE 3.2-4 HEXAGON-STARTING AT EDGE
Similar calculations were performed for a square and for a circle. For the case of starting from the interior of a square they are:

\[
P[D < D_0] = \begin{cases} 
\frac{2}{\pi} \left[ 2D_0 - \frac{D_0^2}{2} \right] & ; \quad 0 \leq D_0 \leq 1 \\
\frac{2}{\pi} \left[ \frac{\pi}{2} + \cos^{-1} \frac{1}{D_0} - \sin^{-1} \frac{1}{D_0} + 1-2\sqrt{D_0^2-1} + \frac{D_0^2}{2} \right] & ; \quad 1 \leq D_0 \leq \sqrt{3}
\end{cases} \quad (3.2-14)
\]

\[
p(D) = \begin{cases} 
\frac{2}{\pi} (2 - D) & ; \quad 0 \leq D \leq 1 \\
\frac{2}{\pi} \left( D - \frac{2\sqrt{D^2-1}}{D} \right) & ; \quad 1 \leq D \leq \sqrt{2}
\end{cases} \quad (3.2-15)
\]

\[
\bar{D} = \frac{2}{\pi} \left[ \frac{1-\sqrt{2}}{3} + \log (\sqrt{2} + 1) \right] = .4732
\]

\[
\bar{D}^2 = \frac{1}{\pi} + \sigma = .30723.
\]

Starting at the edge, the results are,

\[
P[D < D_0] = \begin{cases} 
\frac{2D_0}{\pi} & ; \quad 0 \leq D_0 \leq 1 \\
\frac{2}{\pi} \left[ \frac{\pi}{2} + \cos^{-1} \frac{1}{D_0} - \sin^{-1} \frac{1}{D_0} + 1 - \sqrt{D_0^2-1} \right] & ; \quad 1 \leq D_0 \leq \sqrt{2}
\end{cases} \quad (3.2-17)
\]
p(D) = \begin{cases} 
\frac{2}{\pi} & ; \quad 0 \leq D \leq 1 \\
\frac{2}{\pi} \left[ \frac{2}{D\sqrt{D^2-1}} - \frac{D}{\sqrt{D^2-1}} \right] & ; \quad 1 \leq D \leq \sqrt{2} 
\end{cases} \quad (3.2-18)

\overline{D} = \frac{1}{\pi} \left[ 1 - \sqrt{2} + 3 \log (\sqrt{2} + 1) \right] = .7098 \quad (3.2-19)

\overline{D^2} = \frac{2}{\pi} + \sigma = .3644.

The calculations were also done for a circular shape,

\begin{align*}
P(D < D_0) &= \frac{D_0}{2\pi} \sqrt{4-D_0^2} + \frac{2}{\pi} \sin^{-1} \left( \frac{D_0}{2} \right) ; \quad 0 \leq D_0 \leq 2 \quad (3.2-20) \\
p(D) &= \frac{\sqrt{4-D^2}}{\pi} ; \quad 0 \leq D \leq 2 \quad (3.2-21) \\
\overline{D} &= \frac{8}{3\pi} = .84883 \quad \overline{D^2} = 1 + \sigma_D = .52867. \quad (3.2-22)
\end{align*}

Starting at the edge

\begin{align*}
P(D < D_0) &= 1 - \frac{2}{\pi} \cos^{-1} \left( \frac{D_0}{2} \right) ; \quad 0 \leq D_0 \leq 2 \quad (3.2-23) \\
p(D) &= \frac{2}{\pi\sqrt{4-D^2}} ; \quad 0 \leq D \leq 2 \quad (3.2-24) \\
\overline{D} &= \frac{4}{\pi} = 1.27324 \quad \overline{D^2} = 2 + \sigma_D = .615515. \quad (3.2-25)
\end{align*}

Distributions for the square and circle are plotted in Figures 3.2-5 through 3.2-8.
FIGURE 3.2-5 SQUARE-INTERIOR STARTING POINT

\[ \bar{D} = 0.47 \]

\[ \sigma = 0.31 \]

FIGURE 3.2-6 SQUARE -STARTING AT EDGE

\[ \bar{D} = 0.71 \]

\[ \sigma = 0.36 \]
FIGURE 3.2-7 CIRCLE-INTERIOR STARTING POINT

FIGURE 3.2-8 CIRCLE-STARTING AT EDGE
3.3 **Handoff Distribution**

In the design of switching equipment to perform the handoffs, one is concerned about how many handoffs occur per call. This information can be computed from the distributions in the previous section. The following notation will be used.

\[ p_t(t_0) \] - density function of \( t \), the time until a mobile leaves the cell, given it started in the interior, or on the edge of the cell.

\[ p_D(D_0) \] - density function of \( D \), the distance from the starting point to the edge of the cell.

\( S \) - speed mobile is traveling. Assumed to be constant.

\( (15, 30, 45, 60 \text{ mph}) \)

\[ p_h(h_0) \] - call holding time density function. Assumed to be exponential with a mean of 3 minutes.

i.e., \( p_h(h_0) = \frac{1}{180} e^{-h_0/180} \) when \( h_0 \) is in seconds.

\( R \) - cell size, 1.25, 2.5, and 5 mile radius.

The specific problem is to compute the probability distribution of the number of handoffs per call for the case of the hexagon.

We will take advantage of the "no memory" property of the exponential holding time distribution. This permits treating the handoff event in each cell as an independent Bernoulli trial, based only on the starting point (edge or interior), vehicle speed, cell size, and average holding time. First we calculate the probability of a
handoff given the call starts in (interior case) or enters (edge case) the specific cell. Referring to Figure 3.3-1,

\[ P[\text{handoff}] = P[h > t] \]

\[ = \int_{0}^{t_m} \int_{t_0}^{\infty} p_t(t_0) p_h(h_0) \, dh_0 \, dt_0 \]

\[ = \int_{0}^{t_m} \int_{t_0}^{\infty} p_t(t_0) p_h(h_0) \, dh_0 \, dt_0. \quad (3.3-1) \]

Recall \( p_h(h_0) = \lambda e^{-\lambda h_0} \)

thus \( P[h > t] = \int_{0}^{t_m} \int_{t_0}^{\infty} p_t(t_0) p_h(h_0) \, dh_0 \, dt_0 \)

\[ = \int_{0}^{t_m} p_t(t_0) \left( \int_{t_0}^{\infty} \lambda e^{-\lambda h_0} \, dh_0 \right) dt_0 = \int_{0}^{t_m} e^{-\lambda t_0} p_t(t_0) \, dt_0. \quad (3.3-2) \]

Now \( p_t(t_0) = \frac{d}{dt_0} P[t < t_0] \)

but \( P[t < t_0] = P[D < D_0] \) since \( D_0 = S t_0 \).

Then \( P[D < D_0] = P[D < S t_0] = \int_{0}^{St_0} P_D(D_0) \, dD_0 \quad (3.3-3) \)
FIGURE 3.3-1 CALL HOLDING TIME, TRANSIT TIME PLANE
\[ p_t(t_o) = \frac{d}{dt_o} P \{ t < t_o \} = S p_D(S t_o). \quad (3.3-4) \]

Substituting back in

\[ P \{ h > t \} = \int_{t_0}^{t_m} e^{-\lambda t_o} S p_D(S t_o) \, dt_o \quad (3.3-5) \]

which with a change of variables gives

\[ P \{ h > t \} = \int_0^{D_m} e^{-\frac{\lambda D_o}{S}} p_D(D_o) \, dD_o. \quad (3.3-6) \]

Now since \( D \) is measured as a fraction of the cell radius and \( \frac{1}{\lambda} \) is the average holding time in seconds, the speed \( S \), must be converted from mph to radii/sec. \( D_m \) is the maximum distance that can be traveled in the cell. For the hexagon it is 2 radii. Since \( p_D(D_o) \) is known, the above integral can easily be evaluated numerically. This was done with care being taken to include the finite area of the pole in \( P_D(D_o) \).

The probability of the first handoff is computed using \( p_D(D_o) \) for a random starting point. This probability is denoted by \( P_1 \).

Similarly the probability of the second and subsequent handoffs is computed using \( P_D(D_o) \) for a starting point on the cell boundary and is denoted by \( P_2 \). Thus, the probability distribution function is:

\[ P \{ k \text{ handoffs} \} = \begin{cases} 
(1 - P_1) & ; \quad k = 0 \\
(1 - P_2) & ; \quad k = 1, 2, 3, \ldots 
\end{cases} \quad (3.3-7) \]
Next we compute the average number of handoffs per call $\bar{k}$.

$$\bar{k} = \sum_{k=0}^{\infty} k P[k] = \sum_{k=1}^{\infty} k P_1 P_2^{(k-1)}(1-P_2)$$

(3.3-8)

Carrying out the algebra and simplifying the resulting geometric series we find

$$\bar{k} = \frac{P_1}{1-P_2}.$$  

(3.3-9)

Some results of this computation are listed in Table 3.3-1.

3.4 Transit Time Distribution

In this section we use an assumed speed distribution to compute a transmit time distribution for use in the next section to examine the effect on traffic capacity. The speed distribution was arbitrarily chosen based on intuitive feeling. It was felt that little would be gained by using a continuous distribution over a discrete one so a discrete one was used to take advantage of the more efficient computations. The distribution chosen is shown in Figure 3.4-1. It is felt that some percentage of mobile telephone calls will be made from parked vehicles. This percentage was chosen to be 30 percent. To avoid the difficulty of an undefined transit time when the speed is zero it is convenient to work with the conditional speed distribution in Figure 3.4-2 when computing the transit time density function.

Let $S_k^r = 15k$ ; $k = 1, 2, 3, 4$ 

(3.4-1)
<table>
<thead>
<tr>
<th>Speed (MPH)</th>
<th>RADIUS (MILES)</th>
<th>1.25</th>
<th>2.5</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>P&lt;sub&gt;1&lt;/sub&gt;</td>
<td>.371</td>
<td>.207</td>
<td>.107</td>
</tr>
<tr>
<td></td>
<td>P&lt;sub&gt;2&lt;/sub&gt;</td>
<td>.233</td>
<td>.112</td>
<td>.055</td>
</tr>
<tr>
<td></td>
<td>( \overline{k} )</td>
<td>.483</td>
<td>.233</td>
<td>.113</td>
</tr>
<tr>
<td>30</td>
<td>P&lt;sub&gt;1&lt;/sub&gt;</td>
<td>.569</td>
<td>.371</td>
<td>.207</td>
</tr>
<tr>
<td></td>
<td>P&lt;sub&gt;2&lt;/sub&gt;</td>
<td>.430</td>
<td>.233</td>
<td>.112</td>
</tr>
<tr>
<td></td>
<td>( \overline{k} )</td>
<td>.998</td>
<td>.483</td>
<td>.233</td>
</tr>
<tr>
<td>45</td>
<td>P&lt;sub&gt;1&lt;/sub&gt;</td>
<td>.675</td>
<td>.487</td>
<td>.295</td>
</tr>
<tr>
<td></td>
<td>P&lt;sub&gt;2&lt;/sub&gt;</td>
<td>.555</td>
<td>.342</td>
<td>.172</td>
</tr>
<tr>
<td></td>
<td>( \overline{k} )</td>
<td>1.52</td>
<td>.739</td>
<td>.357</td>
</tr>
<tr>
<td>60</td>
<td>P&lt;sub&gt;1&lt;/sub&gt;</td>
<td>.740</td>
<td>.569</td>
<td>.371</td>
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<tr>
<td></td>
<td>P&lt;sub&gt;2&lt;/sub&gt;</td>
<td>.637</td>
<td>.425</td>
<td>.233</td>
</tr>
<tr>
<td></td>
<td>( \overline{k} )</td>
<td>2.04</td>
<td>.998</td>
<td>.483</td>
</tr>
</tbody>
</table>

Average Call Holding Time - 3 Min.

**TABLE 3.3-1**

HANDOFF PROBABILITIES AND AVERAGE NUMBER PER CALL
FIGURE 3.4-1 ASSUMED SPEED DISTRIBUTION

$P[SPEED/SPEED^0] = P[SPEED, SPEED^0] / P[SPEED^0]$  

FIGURE 3.4-2 CONDITIONAL SPEED DISTRIBUTION
be the speed in miles per hour. Now for a cell of radius R, the speed in radii/sec is given by

\[ S_k = \frac{15k}{3600R}; \quad k = 1, 2, 3, 4. \]  \hspace{1cm} (3.4-2)

From equation 3.3-4 we can write the conditional transit time density function

\[ P_t (t_o/S_k) = S_k p_D (S_k t_o); \quad k = 1, 2, 3, 4. \]  \hspace{1cm} (3.4-3)

Then the transit time density function conditioned on the event that the speed is greater than zero is

\[ p_t (t_o/S>0) = \sum_{k=1}^{4} S_k p_D (S_k t_o) P[S=S_k/S>0]. \]  \hspace{1cm} (3.4-4)

Assuming a cell radius of five miles, this function is plotted along with the cumulative distribution function in Figure 3.4-3 for both the interior and edge starting cases. Comparing these plots with Figures 3.2-3 and 3.2-4 we see one of the effects of including the speed distribution has been to transform the character of these distributions toward an exponential function. This suggests that one might approximate these functions with exponentials of the same mean. If such an approximation had any validity, then one could apply many of the theoretical results of traffic theory to this problem. A property of the exponential distribution that can be used to check such an approximation is that the mean is equal to the standard deviation. We shall compute the mean and standard deviation and compare their ratio with that for a constant speed.
FIGURE 3.4-3 TRANSIT TIME DISTRIBUTION FOR 5 MILE CELL USING SPEED DISTRIBUTION OF FIG 3.4-2
Now since \( t = \frac{D}{S} \) the mean of \( t \) is given by

\[
\mathbb{E}[t/S_k] = \frac{D}{S_k}
\]  

(3.4-5)

and

\[
\mathbb{E}[t > 0] = \sum_{k=1}^{L} \mathbb{E}[t/S_k] \mathbb{P}[S = S_k] \mathbb{P}[S > 0] = 517 \text{ seconds}
\]

(3.4-6)

using Figures 3.2-3, 3.4-2 and assuming \( R = 5 \) miles for the interior case. Similarly for the exterior case

\[
\mathbb{E}[t/S > 0] = 778 \text{ seconds.}
\]

(3.4-7)

Computing the variance of \( t \) given that \( S > 0 \) we get

\[
\sigma_t^2 = \int_0^{t_m} (t_0 - t)^2 p_t (t_0) \, dt.
\]

(3.4-8)

Substituting from 3.4-4 we get

\[
\sigma_t^2 = \int_0^{t_m} (t-\bar{t})^2 \sum_{k=1}^{L} S_k p_D(S_k t) \mathbb{P}[S = S_k] \mathbb{P}[S > 0] \, dt
\]

\[
= \sum_{k=1}^{L} S_k \mathbb{P}[S = S_k] \int_0^{t_m} (t-\bar{t})^2 p_D(S_k t) \, dt.
\]

(3.4-9)
Now using the same change of variables used in going from 3.3-5 to 3.3-6 we get

\[ \sigma_t^2 = \sum_{k=1}^{4} \frac{P[S=S_k/S>0]}{S_k^2} \sigma_D^2 \]  

(3.4-10)

or \( \sigma_t = 4.55 \) seconds and 5.89 seconds for the interior case and edge case respectively. Now if we compare the ratio of \( \bar{t}/\sigma_t \) we find \( \bar{t}/\sigma_t = 1.59 \) and 2.04 for the constant speed case as opposed to \( \bar{t}/\sigma_t = 1.14 \) and 1.32 for the combined case. This indicates the progression toward an exponential distribution.

3.5 **Effect on Traffic Performance**

In this section we consider the effect that handoffs have on the traffic performance of the HCMTS. Section 3.5.1 develops an analytical handoff model that can be used to solve for the blocking probability in each cell for the fixed channel assignment case. Even though this model must be solved numerically, it is called an analytical model to distinguish from the Monte Carlo simulations reported on throughout this work. Then in Section 3.5.2 a Monte Carlo simulation is performed to verify the results of Section 3.5.1.

3.5.1 **Analytical Handoff Model**

Consider an arbitrary cell \( j \) and all cells adjacent to it. Denote the adjacent cells by \( k = 1, 2, ..., 6 \) as in Figure 3.5-1. Two independent variables serve as parameters to describe this cell.

- \( a_j \) - offered load (erlangs)
- \( C_j \) - assigned channels
\( T \) - AVERAGE TALKING TIME  
\( a_j \) - LOAD OFFERED BY CUSTOMERS IN CELL \( j \)  
\( c_j \) - CHANNELS ASSIGNED TO CELL \( j \)  
\( r_0^* \) - AVERAGE TRANSIT TIME (BEFORE 1\( ^{st} \) HANDOFF)  
\( r_R^* \) - AVERAGE TRANSIT TIME (AFTER 1\( ^{st} \) HANDOFF)  
\( \lambda_{0j} \) - AVERAGE ARRIVAL RATE (ORIGINATING CALLS)  
\( \lambda_{Rj} \) - AVERAGE ARRIVAL RATE FROM SURROUNDING CELLS  
\( \lambda_{Hj} \) - AVERAGE HANDOFF RATE  
\( T_0^* \) - AVERAGE CHANNEL HOLDING TIME (BEFORE 1\( ^{st} \) HANDOFF)  
\( T_R^* \) - AVERAGE CHANNEL HOLDING TIME (AFTER 1\( ^{st} \) HANDOFF)  
\( P_0^* \) - \( P[\text{HANDOFF}] \) (BEFORE 1\( ^{st} \) HANDOFF)  
\( P_R^* \) - \( P[\text{HANDOFF}] \) (AFTER 1\( ^{st} \) HANDOFF)  
\( P_S \) - \( P[\text{SPEED} \neq 0] \) i.e. \( P[\text{VEHICLE IS MOVING}] \)  
\( A_j \) - LOAD OFFERED TO CELL \( j \)  
\( B_j \) - \( P[\text{BLOCKING}] \) FOR CELL \( j \)  

*GIVEN SPEED \( \neq 0 \)

**FIGURE 3.5-1** CELL \( j \) AND SURROUNDING CELLS  
AND DEFINITION OF VARIABLES
When these two parameters are known for each cell in the layout, then
the system is completely specified. In addition, we assume some
average talking time $T$ for all calls in the system and average transit
times $r_0$ and $r_R$ for moving mobiles. Note, $T$ is the average talking
time and is not to be confused with the average channel holding time
denoted by $\tau_0$ and $\tau_R$. Subscripts $0$ and $R$ indicate before or after
first handoff respectively. A reasonable value of $T$ is 3 minutes and
is the value used throughout. The arrival rate $\lambda_{0j}$ of calls that are
originated in cell $j$ is found from

$$\lambda_{0j} = a_j/T.$$  \hspace{1cm} (3.5-1)

These input calls are generated by a Poisson process with parameter
$\lambda_{0j}$. This cell will hand off calls to surrounding cells at some rate
$\lambda_{Hj}$. Similarly it will receive calls from surrounding cells at a
rate of $\lambda_{Rj}$. Adopting the notation that $\lambda_{jk}$ is the rate in calls per
hour that calls are handed off from cell $j$ to cell $k$ we have

$$\lambda_{Hj} = \sum_{k=1}^{6} \lambda_{jk}$$  \hspace{1cm} (3.5-2)

and

$$\lambda_{Rj} = \sum_{k=1}^{6} \lambda_{kj}.$$  \hspace{1cm} (3.5-3)

It is assumed that calls are handed off to all surrounding cells in
equal proportions. Accordingly

$$\lambda_{jk} = \frac{\lambda_{Hj}}{6}.$$  \hspace{1cm} (3.5-4)
For cells located on the edge of the system layout, it is assumed that calls handed off to cells which are not equipped are lost from this system. These assumptions are arbitrary and could be modified as desired.

Based on results of the last section we approximate the cell transit time as an exponentially distributed random variable with mean $r_0$ or $r_R$. For simplicity we assume that the average transit time is the same for all cells, but has two values $r_0$ until the first handoff and $r_R$ after the first handoff. Thus, the following probability density functions describe the vehicle transit time across a cell.

Before the first handoff and

$$p_{r_0}(r) = \frac{1}{r_0} e^{-r/r_0} \quad ; r > 0$$ (3.5-5)

After the first and succeeding handoffs. The talking time is also exponentially distributed with the following density function

$$p_{t_{h}}(h) = \frac{1}{T} e^{-h/T} \quad ; h > 0.$$ (3.5-7)

Next, consider the random variable $t$, the channel holding time. Clearly

$$t = \text{Min} (r,h).$$ (3.5-8)

It is a simple matter to show that $t$ is also exponentially distributed with mean

$$r_0 = \frac{r_0 T}{r_0 + T}$$ (3.5-9)
before the first handoff and

\[ \tau_R = \frac{rR^T}{rR+T} \]  \hspace{1cm} (3.5-10)

after the first and succeeding handoffs. Thus, the channel holding
time is governed by the following density functions.

\[ P_{to}(t) = \frac{1}{\tau_o} e^{-t/\tau_o} ; \ t > 0 \]  \hspace{1cm} (3.5-11)

and

\[ P_{tr}(t) = \frac{1}{\tau_R} e^{-t/\tau_R} ; \ t > 0. \]  \hspace{1cm} (3.5-12)

Using the results from Section 3.4 the handoff probabilities are:

\[ P_o = \frac{T}{T+\tau_o} \]  \hspace{1cm} (3.5-13)

for the first handoff and

\[ P_R = \frac{T}{T+\tau_R} \]  \hspace{1cm} (3.5-14)

for the second and succeeding handoffs.

Next, we denote the probability that a call arriving in a cell,
either a new call or a handoff from another cell, finds no idle channels
and is blocked by B_j. Then the handoff rate is given by

\[ \lambda_{Hj} = [\lambda_{o_j} P_o P_s + \lambda_{R_j} P_R] (1 - B_j) \]  \hspace{1cm} (3.5-15)

where \( P_s = P[\text{speed} \neq 0] \) which is .7 for the example of Section 3.4.

Analogous to the development in Appendix B we can define the offered
load in a cell by

\[ A_j = \lambda_{o_j} P_s \tau_o + \lambda_{o_j} (1 - P_s) T + \lambda_{R_j} \tau_R. \]  \hspace{1cm} (3.5-16)
Now, the only missing item is to find some way of computing $B_j$. We assume that $B_j$ is the well known Erlang B function with arguments $A_j$ and $C_j$, i.e.,

$$B_j = B(C_j, A_j). \quad (3.5-17)$$

This assumption is justified in the succeeding subsections.

Equations 3.5-3, 4, 15, 16, and 17 provide a basis for the formulation of an iterative procedure to solve for $B_j$, $\lambda_{H_j}$, $\lambda_{R_j}$ for $j$ corresponding to all cells. Let the superscript denote the iteration number. For starting values

$$A_j^{(0)} = \lambda_{o_j} P_{s} T + \lambda_{o_j} (1 - P_s) T$$

$$B_j^{(0)} = B(C_j, A_j^{(0)})$$

$$\lambda_{H}^{(0)} = (1 - B_j^{(0)}) P_s \lambda_{o_j} P_o. \quad (3.5-18)$$

Then iterating

$$\lambda_{R_j}^{(n)} = \frac{6}{\Sigma} \frac{\lambda_{H_k}^{(n-1)}}{6}$$

$$A_j^{(n)} = A_j^{(0)} + \lambda_{R_j}^{(n)} \tau_R \quad (3.5-19)$$

$$B_j^{(n)} = B(C_j, A_j^{(n)}).$$

Results of the use of this model on the system layout of Figure 2.2-5 are shown in Table 3.5-1. The effect on offered load is illustrated in Figure 3.5-2. Average transit times are those shown in Figure 3.4-3 for a 5-mile cell. Results for 2.5 and 1.25 mile cells are similar.
<table>
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<th>CELL</th>
<th>a  (Erlangs)</th>
<th>A  (Erlangs)</th>
<th>C</th>
<th>B</th>
<th>( \lambda_o ) (Calls/Hr)</th>
<th>( \lambda_H ) (Calls/Hr)</th>
<th>( \lambda_R ) (Calls/Hr)</th>
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<td>7.7</td>
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<td>1198.7</td>
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</table>

**TABLE 3.5-1**

RESULTS FROM ANALYTICAL MODEL
Figure 3.5-2 Effect of Handoffs on Offered Load
with proportionally more handoffs and resulting more extremes in blocking. Computationally the model is very efficient. Experience with it indicates convergence to 6 significant digits after about 6 iterations. Costs are less than 10 dollars per run which is more than an order of magnitude less than comparable simulation costs.

3.5.1.1 Erlang B Formula With Two Kinds of Traffic

In this section we begin consideration of the validity of equation 3.5-17 for use in the analytical handoff model. First, we show that it holds when there are two kinds of traffic submitted to the system. This situation occurs several times in this system. An example is that handed off calls come at a different rate and have different holding times than do calls that originate in a cell. The situation is this. Some calls arrive at rate \( \lambda_1 \) with average holding time \( \tau_1 \) and some arrive at rate \( \lambda_2 \) with holding time \( \tau_2 \). Now it is necessary to adopt a two dimensional state description, namely \( n_1, n_2 \) the number of calls each of the first and second kinds. We proceed as in Appendix B with the state diagram in Figure 3.5-3. Next, we write a set of algebraic steady state equations based on the fact that the average number of transitions into state \( n_1, n_2 \) must equal those out of it.

\[
P_{n_1, n_2}(\lambda_1 + \lambda_2 + \frac{n_1}{\tau_1} + \frac{n_2}{\tau_2}) = P_{n_1-1, n_2} \lambda_1 + P_{n_1, n_2-1} \lambda_2 + P_{n_1+1, n_2} \frac{n_1+1}{\tau_1} + P_{n_2, n_2+1} \frac{n_2+1}{\tau_2}
\]

(3.5-20)

for \( n_1 \geq 0, n_2 \geq 0 \) and \( n_1 + n_2 < c \).
\[ \lambda_1, \lambda_2 \text{ — ARRIVAL RATES (CALLS/SEC)} \]
\[ \tau_1, \tau_2 \text{ — HOLDING TIMES (CALLS/SEC)} \]

3.5-3 STATE DIAGRAM WITH TWO KINDS OF TRAFFIC
For \( n_1 + n_2 = c \)

\[
P_{n_1,n_2} \left( \frac{n_1}{\tau_1} + \frac{n_2}{\tau_2} \right) = P_{n_1-1,n_2} \lambda_1 + P_{n_1,n_2-1} \lambda_2
\]

for \( n_1 \geq 0, n_2 \geq 0 \). \hfill (3.5-21)

In addition we have some boundary conditions

\[
P_{n_1,n_2} = 0 \text{ for } n_1 < 0, n_2 < 0 \text{ or } n_1 + n_2 > c.
\] \hfill (3.5-22)

These equations were solved by Mr. R. B. Cooper in an unpublished memorandum. The procedure is to first consider the situation when \( C \) is infinite. Then we have two systems which do not interact so the solution is the product of two factors of the form B-2,

\[
P_{n_1,n_2} = \frac{(\lambda_1 \tau_1)^{n_1}}{n_1!} \frac{(\lambda_2 \tau_2)^{n_2}}{n_2!} P_{00}.
\] \hfill (3.5-23)

By substituting equation 3.5-23 into 3.5-20 and 3.5-21 it is seen that 3.5-23 is also a solution of these difference equations. Next, we consider the probability that \( n \) servers are busy in Figure 3.5-3. This probability is found by summing over those state probabilities for which \( n_1 + n_2 = n \). Thus

\[
P_n = \sum_{n_1=0}^{n} \frac{\lambda_1^{n_1}}{n_1!} \frac{\lambda_2^{(n-n_1)}}{(n-n_1)!} \frac{a_2(n-n_1)}{P_{00}}
\] \hfill (3.5-24)

where \( a_1 = \lambda_1 \tau_1, a_2 = \lambda_2 \tau_2 \).

Which by the binomial theorem reduces to
which is exactly the form of equation B-2. Thus, we conclude that the
Erlang B formula B-6 also holds for two kinds of traffic.

3.5.1.2 Interdeparture Time Distribution

In our model, we have assumed that handed off calls arrive at
a cell with an exponential interarrival time. We examine the validity
of that assumption in this section. We will use the following notation:

\( P[t] \) - probability of no departures in \((0, t)\)

\( P_{n_1, n_2}[t] \) - probability of no departures in \((0, t)\) and

system is in state \( n_1, n_2 \) at time \( t \).

Let \( b_1 = \frac{1}{\tau_1} \) and \( b_2 = \frac{1}{\tau_2} \).

Now we can write some differential equations. First, consider \( n_1 + n_2 < C \),

\[
\frac{d}{dt} P_{n_1, n_2}(t) = \lambda_1 \Delta t \left( 1 - b_1 (n_1 - 1) \Delta t \right) P_{n_1-1, n_2}(t) + \lambda_2 \Delta t \left( 1 - b_2 (n_2 - 1) \Delta t \right) P_{n_1, n_2-1}(t) + \left[ 1 - \lambda_1 \Delta t - \lambda_2 \Delta t \right] \left[ 1 - b_1 n_1 \Delta t - b_2 n_2 \Delta t \right] P_{n_1, n_2}(t)
\]

(3.5-26)

for \( n_1 \geq 0, n_2 \geq 0 \).
For $n_1 + n_2 = C$,

$$P_{n_1, n_2}[t+\Delta t] = \lambda_1\Delta t \left(1-b_1(n_1-1)\right)P_{n_1-1, n_2}(t) + \lambda_2\Delta t \left(1-b_2(n_2-1)\right)P_{n_1, n_2-1}(t)$$

1 arrival no departure in $\Delta t$ in $\Delta t$

$$+ [1-b_1(n_1\Delta t-b_2n_2\Delta t)] P_{n_1, n_2}(t)$$

no departure in $\Delta t$

(3.5-27)

for $n_1 > 0$, $n_2 > 0$.

Rearranging terms and letting $\Delta t \to 0$ for $n_1 + n_2 < C$ we get

$$\frac{dP_{n_1, n_2}(t)}{dt} = \lambda_1 P_{n_1-1, n_2}(t) + \lambda_2 P_{n_1, n_2-1}(t)$$

$$- [\lambda_1+\lambda_2 + b_1 n_1 + b_2 n_2] P_{n_1, n_2}(t)$$

(3.5-28)

and for $n_1 + n_2 = C$

$$\frac{dP_{n_1, n_2}(t)}{dt} = \lambda_1 P_{n_1-1, n_2}(t) + \lambda_2 P_{n_1, n_2-1}(t)$$

$$- (n_1 b_1 + n_2 b_2) P_{n_1, n_2}(t).$$

(3.5-29)
The next task is to solve these equations. Consider equation 3.5-28 for \( n_1 = n_2 = 0 \)

\[
\frac{dP_{oo}(t)}{dt} = -[\lambda_1 + \lambda_2] P_{oo}(t) \quad \text{(3.5-30)}
\]

which has as a solution

\[
P_{oo}(t) = Ae^{-(\lambda_1 + \lambda_2)t}. \quad \text{(3.5-31)}
\]

Now we realize that \( A = P_{oo}(0) \) which is the steady state probability denoted by \( P_{oo} \) in the last section. Thus,

\[
P_{oo}(t) = P_{oo}e^{-(\lambda_1 + \lambda_2)t}. \quad \text{(3.5-32)}
\]

In more generality it is easy to verify that equation 3.5-33 is the solution to equation 3.5-28 using equations 3.5-20.

\[
P_{n_1,n_2}(t) = P_{n_1,n_2}e^{-(\lambda_1 + \lambda_2)t} \quad \text{(3.5-33)}
\]

For \( n_1 + n_2 = C \) in equation 3.5-29 the solution is not so simple. The solution is

\[
P_{n_1,n_2}(t) = -\frac{B}{B-\lambda} P_{n_1,n_2}e^{-\lambda t} + \frac{2B-2}{B-\lambda} P_{n_1,n_2}e^{-Bt}
\]

where

\[
B = n_1b_1 + n_2b_2
\]

\[
\lambda = \lambda_1 + \lambda_2. \quad \text{(3.5-34)}
\]
Now using equation 3.5-30 and 3.5-34 we find $P[t]$, the probability of no departures in $(0, t)$, to be given by

$$
P[t] = \sum_{n_1=0}^{c} \sum_{n_2=0}^{c-n_1} P_{n_1,n_2}(t). \tag{3.5-35}
$$

Let $t_D$ be the interdeparture time. Then its cumulative distribution function is

$$
P[t_D \leq t] = 1 - P[t_D > t] = 1 - P[t] \tag{3.5-36}
$$

and the density function is

$$
p(t) = -\frac{dP[t]}{dt} = \sum_{n_1=0}^{c} \sum_{n_2=0}^{c-n_1} \frac{dP_{n_1,n_2}(t)}{dt}. \tag{3.5-37}
$$

From equations 3.5-33 and 3.5-34 the derivatives are

$$
-\frac{dP_{n_1,n_2}(t)}{dt} = P_{n_1,n_2} \lambda e^{-\lambda t} \quad \text{for } n_1+n_2 < c \tag{3.5-38}
$$

and

$$
-\frac{dP_{n_1,n_2}(t)}{dt} = P_{n_1,n_2} \left[ \frac{-\lambda B}{B-\lambda} e^{-\lambda t} + \frac{B(2B-\lambda)}{B-\lambda} e^{-Bt} \right] \tag{3.5-39}
$$

for $n_1 + n_2 = c$.

From equation 3.5-37 it is seen that the summation is over all states in Figure 3.5-3. Furthermore, equation 3.5-38 is used in this summation for all states except where $n_1 + n_2 = c$, i.e., the upper
diagonal row of states. Thus, consider the terms of equation 3.5-37 for which \( n_1 + n_2 < c \).

\[
\begin{align*}
\sum_{n_1=0}^{c-1} \sum_{n_2=0}^{c-1-n_1} \frac{d}{dt} P_{n_1,n_2}(t) &= \lambda e^{-\lambda t} \sum_{n_1=0}^{c-1} \sum_{n_2=0}^{c-1-n_1} P_{n_1,n_2} .
\end{align*}
\] (3.5-40)

Note that the summation in equation 3.5-40 is of the steady state probabilities over all nonblocking states. Now since our systems have an objective blocking of .02, equation 3.5-40 should be approximately

\[
\begin{align*}
\sum_{n_1=0}^{c-1} \sum_{n_2=0}^{c-1-n_1} \frac{d}{dt} P_{n_1,n_2}(t) &\approx 0.98 \lambda e^{-\lambda t} .
\end{align*}
\] (3.5-41)

The remaining terms of equation 3.5-37 have time constants other than \( 1/\lambda \), but since the combined coefficients, that is \( \sum_{n_1=0}^{c} P_{n_1,c-n_1} \), will total approximately .02 it seems reasonable to approximate the inter-departure time distribution as an exponential.

At this point, it is tempting to approximate the distribution with time constant \( 1/\lambda \). However, this means the departure rate is the same as the arrival rate, but intuitively we know it must be

\[
\lambda_D = \lambda [1 - B(c,a)] .
\] (3.5-42)

We can evaluate the mean of the distribution function as

\[
\bar{t} = \frac{1}{\lambda} \left[ 1 - B(c,a) - \sum_{n_1=0}^{c} \frac{B}{B-\lambda} P_{n_1,c-n_1} \right] + \sum_{n_1=0}^{c} \frac{2B-\lambda}{B(\lambda)} P_{n_1,c-n_1} .
\] (3.5-43)
This is not too informative except that it is not \( \frac{1}{\lambda_D} \). This is distressing until we realize that equation 3.5-37 is only a marginal distribution and does not indicate that successive interdeparture times are independent. In fact, it does not seem reasonable that they should be since the terms of equation 3.5-37 are dependent upon the state of the system. Hence, the departure process, aside from not being Poisson, is probably not even a renewal process. However, the results of this section do say that it is reasonable to approximate the departure process as Poisson in our case with the proper rate. Thus, the interdeparture time distribution is approximated by

\[
p(t) = \frac{1}{\lambda_D} e^{-\lambda_D t} \text{ where } \lambda_D = \lambda[1 - B(c, a)].
\]

(3.5-44)

### 3.5.2 Simulator Verification of Model

A sample solution of the analytical handoff model was presented in Table 3.5-1. The same system was simulated using the simulator program described in Appendix A. The speed distribution of Figure 3.4-1 was used. That is 30 percent of the calls come from nonmoving mobiles so they are not handed off. For the 70 percent that come from moving mobiles, handoffs are simulated as follows. First, the call holding time is computed from an exponential distribution with a mean of 180 seconds. Then the transit time is computed from the distribution of Figure 3.4-3 and compared with the call holding time. If the call holding time exceeds the transit time, then the call is handed off to one of the six adjacent cells with equal probability. If the cell is
on the edge of the system and the adjacent cell is outside the system, then it is assumed the call leaves the system. Calls which are handed off to a cell are treated similarly except the 'starting at edge' transit time distribution of Figure 3.4-3 is used to compute a new transit time. This transit time is compared with the remaining call holding time to determine whether there is a second handoff. Cumulative results from two simulation runs are compared with the analytical results in Table 3.5-2. The correlation between the theoretical and simulation results is .96. Similarly Table 3.5-3 lists the simulation results for $\lambda_H$ and $\lambda_R$. Correlation coefficients for these data are .999 and .996 respectively. These results show good agreement between the analytical model and the simulation.

3.6 **Summary and Conclusions on Handoffs and Their Effects**

1) Substantial effort has been expended in developing a mathematical model of the motion of mobile telephone equipped vehicles within a small cell system. Results seem reasonable, but no experimental data is available to check the assumptions.

2) Handoffs will occur in substantial numbers and will have a significant effect on traffic characteristics. The main effect can be viewed as a partial averaging of offered load over the various cells.

3) Good agreement between simulation results and analytical results indicates that, when it is possible to use it, results from the analytical model are as reliable as those obtained by simulation.
<table>
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<th>THEORETICAL BLOCKING</th>
<th>SIMULATOR BLOCKING</th>
</tr>
</thead>
<tbody>
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<td>.0034</td>
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<td>.0703</td>
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Correlation Coefficient = .96

**TABLE 3.5-2**

ANALYTICAL AND SIMULATOR BLOCKING
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<th>$\lambda_R$ (CALLS/HR)</th>
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<tr>
<td>21</td>
<td>26.7</td>
<td>5.3</td>
</tr>
</tbody>
</table>

Correlation Coefficient .999 .996

**TABLE 3.5-3**

SIMULATOR RESULTS TO COMPARE TO TABLE 3.5-1
Advantages of the analytical model are more insight and more efficient computation.

4) This analytical model can be used to evaluate techniques to reduce handoff blocking below originating blocking. This would be done by modifying the state diagram of Figure 3.5-3 then applying those results as was done in the analytical model of Section 3.5.1.

IV. General Summary and Conclusions

Some new telephone traffic problems have been identified in proposed High Capacity Mobile Telephone Systems. These problems have been studied both analytically and by simulation. Where possible, analytical results have been checked by simulation. In other areas simulation has been the only tool.

Dynamic channel assignment has been shown to be an effective means of increasing the traffic capacity of a high capacity mobile telephone system using the small cell concept.

Theoretical descriptions of the handoff phenomenon have been derived and checked by simulation. The effect of handoffs is to modify the system parameters, but the overall system remains unchanged. Thus, handoffs should have no significant effect on the performance of dynamic channel assignment.
APPENDIX A

THE SIMULATOR PROGRAM

1. Introduction

This appendix describes the design of a Monte Carlo simulation program for use in studying the High Capacity Mobile Telephone System. Two versions of the program resulted. One version was used to study dynamic channel assignment algorithms. No handoffs were allowed in this version. The other was used to verify the analytical handoff model. Handoffs were allowed in this version. In general, established simulation techniques have been used. Time is quantized to the level of 1 millisecond and fixed point arithmetic is used. A system clock is used to keep track of simulation time. Event times are computed in advance and a description of the event is stored in a future events list, FEL, in the sequence in which they will occur. Simulation proceeds by executing the next event on the list. Execution of an event consists of four major functions.

1. Advance the system clock to the time of the event.\(^2\)

2. Modify the state of the system in accordance with the event.

\(^2\) Thus the clock is advanced by a variable amount each time. Sometimes this is referred to as variable time increment simulation.
3. Record pertinent statistics about the event for later output.
4. Generate the time and description of any future events that occur as a result of execution of this event.

The future events list is maintained as a singly linked list. This technique was suggested to me by Mr. M. M. Peritsky and is fully described for use in simulation by Knuth\(^{15}\).

2. Random Number Generation

Pseudorandom numbers are generated using the power residue method. The IBM/360 is used which has a 32 bit word, thus it is convenient to do multiplication modulo \(2^{31}\). This results in a generator with a period of \(2^{29}\). The number 65539 is used as a multiplier. Although there is not too much quantitative data available regarding the randomness of generators using this multiplier, it is the one used by IBM's scientific subroutine package and seems to give reasonable results on systems whose performance is known theoretically. To speed up computations, the random numbers are quantized to 1024 integers by using the leftmost 10 bits. Several independent generators are needed. One is used to generate call interarrival times, the second assigns the call to a cell and third generates call holding time. To simulate handoffs, two additional generators are used. One is used to generate cell transit time and the other to select which cell the call will be handed off to. Calls are assumed to come from a Poisson process, so call interarrival times are exponentially distributed random variables and likewise the call holding time is an exponentially distributed random variable. These are generated by computing an
array of $102^4$ logarithms. The ten bit random number is then used to address this array resulting in a discrete random variable which is a close approximation to an exponentially distributed random variable with a mean of 1000 milliseconds. This is then multiplied by the mean in seconds to give the desired exponential random variables. A similar technique is used to distribute the calls to cells. Again an array of $102^4$ elements is constructed. Elements are assigned to cells based on the proportion of offered load in each cell. Thus the random number is used to address this array and the cell assignment is read directly. A similar technique is used for the transit time generator. The cumulative distribution functions of Figure 3.4-3 are used to fill up two $102^4$ element arrays for the interior and edge starting point cases. The following procedure is used.

Let \[ X = P[t < t_0] = f(t_0) \] (A-1)

Then find \( g(x) \) such that

\[ g(f(t_0)) = t_0 \] (A-2)

This is done using an iterative numerical technique in which the density function is used to predict the next point on the curve. The above techniques permit all of the random number calculations to be performed with fixed point arithmetic.

Statistical dependence of the random number generators must be minimized to achieve reasonable results. This is accomplished by using different starting points for the generators. It is easy to
calculate the \(2^n\)th member of the pseudorandom sequence generated by
the power residue method. Thus, the first, and the \(2^{18}\)th through \(2^{21}\)
members are used as starting points for the generators. These numbers
are separated by at least 260,000 members in the sequence. Since only
about 50,000 calls are processed per run, this at least insures non-
overlapping sequences of numbers. Difficulty was experienced when the
same starting point was used. This difficulty disappeared with the
starting points above, thus it appears that this technique reduces
correlation among the generators to acceptable levels.

3. Programming Language

Some effort was spent in selecting a programming language for
the simulation. Because of availability and experience the choices
were quickly narrowed to three, PL/I, Fortran, and GPSS.\(^3\) GPSS was
rejected because estimated run times were too long. These estimates
were based on the author's previous experience in simulation of tele-
phone systems. The choice between PL/I and Fortran is not so easy.
PL/I was chosen because of its list processing and bit string manipu-
lation capabilities. Bit strings provide a convenient and efficient
way of doing logical manipulations.

4. System Representation

System representation is divided into two broad categories the
fixed or static part and the dynamic or transient part. Considering

\(^3\)IBM's General Purpose System Simulator
the fixed part first, each cell is identified by an arbitrarily chosen cell number. Associated with this cell number are the following principal pieces of information:

**Nominal Channel Assignment** - A list of channels nominally assigned to each cell. This list of channels is scanned sequentially for an available channel whenever it is desired to set up a call in the cell.

**Potential Borrowing List** - A list of groups of channels (channel sets or nominal channels) from which a channel may be borrowed in the event all nominal channels are busy. For algorithms I and II this list is composed of adjacent cells whose nominal channels may be borrowed. In the case of algorithm III, this list is a prescribed sequence of channel sets from which channels may be borrowed.

**Interferable Cells** - These are all cells in the system for which the D/R ratio is less than the prescribed value. This information is used in testing a channel to see if it is available for use in a given cell. For a channel to be available for use in a given cell, that channel must not be in use in any of the interferable cells.

**Adjacent Cells** - For the handoff simulation of Section 3.5.2 it is also necessary to maintain a list of adjacent cells to which calls can be handed off.

The dynamic part of the system representation is composed mainly of a large array of variables which is used to maintain the state of the system. There is a variable which corresponds to each radio channel. Numbers of all cells in which this channel is being used are stored in this variable. Every time a call begins or ends, the information in
the channel variable must be changed accordingly. When setting up a call, in a specific cell, the search for an available channel is made by scanning through this array making channel tests on the prescribed channels. A channel test consists of comparing the cells in which the channel is being used with the list of interferable cells. If no conflict exists, then the channel is available.

There is one other piece of dynamic information that is used when it is desired to return borrowed channels as soon as a nominal channel becomes idle. This is a borrowed channel list which is maintained on a per cell basis. It is checked every time a nominal channel goes idle. If the list is not empty, then a borrowed channel is removed from it and the call is transferred to the available nominal channel.

5. Simulator Operation

In its final form the simulator program consists of roughly 800 cards and requires 1 minute of CPU time on an IBM 360/50 to process about 900 calls. Figures A-1 and A-2 are flowcharts of the two versions of the main program. The main program causes the correct tasks to be performed by calling appropriate subroutines. First is the call to the subroutine PRIME which reads the data cards and sets up the system for simulation. It also inserts the first call and the first STAT event in the future events list. (STAT is the name given to the event which causes simulation data to be summarized and printed out.) Control is then returned to the main program. Simulation proceeds by decoding the next event from the FEL. The system clock is
FIGURE A-1 MAIN PROGRAM-WITHOUT HANDOFFS
FIGURE A-2 MAIN PROGRAM-WITH HANDOFFS
updated to the time specified in the event data for its occurrence. Also, the event data specifies the type of event by a job code and pertinent data about the event such as cell number and/or channel number. On the basis of the job code, the path through the main program is selected containing calls to the subroutines that perform the event by actually rearranging the state of the system. Figure A-2 contains the version of the main program modified to include handoffs. Figures A-3 through A-10 are functional flowcharts for the subroutines. These subroutines remain essentially unchanged when used in the two main programs with different channel selection algorithms. Subroutines PRIME and STAT are exceptions and there are separate versions for each main program and channel selection algorithm.

A description of each subroutine follows:

**PRIME** - Performs the 5 main functions illustrated in Figure A-3. First, data is read from data cards. These cards specify each cells' location, offered load and nominal channel assignment. From this data the system representation is generated. The next step, an important one, is to print out the detailed system representation. This is done independent of the system generating step to enable one to verify that the system is actually what is intended. After initializing variables and generating the first events, control is returned to the main program.

**CHAN** - A flowchart is shown in Figure A-4. When CHAN is called it is given a cell number in which a call is to be set up. It first checks to see if radio equipment is available. If so, then a channel search is made using one of the 3 algorithms shown in Figures 2.2-1, 2.2-3,
FIGURE A-3 SUBROUTINE "PRIME"
INCREMENT EQUIPMENT BLOCKING COUNTER

RETURN

PERFORM CHANNEL SELECTION ALGORITHM (I, II, III)

INCREMENT CHANNEL BLOCKING COUNTER

RETURN

IS CHANNEL AVAILABLE?

NO

INCREMENT CHANNEL BLOCKING COUNTER

RETURN

YES

IS CHANNEL AVAILABLE?

NO

INCREMENT CHANNEL BLOCKING COUNTER

RETURN

YES

IS CHANNEL BORROWED?

NO

ADD CHANNEL TO BORROWED CHANNEL LIST

INCREMENT CELL BORROW COUNTER

MAKE CHANNEL BUSY IN THIS CELL

INCREMENT CALLS SERVED COUNTER

GENERATE CALL HOLDING TIME

IS THERE A HANDOFF?

NO

INCREMENT CHANNEL BLOCKING COUNTER

RETURN

YES

GENERATE HANDOFF EVENT

CALL SCHED

RETURN

GENERATE CALL TERMINATION EVENT

FIGURE A-4 SUBROUTINE "CHAN"
or 2.2-4. If a channel is found, then the state of the system is changed to set up a call on that channel. Next, the termination event (call termination or handoff) is generated for this call and inserted in the FEL by a call to subroutine SCHED.

SCHED - Its only function is to add events to FEL. A flowchart is shown in Figure A-5.

GEN - The FEL always contains the next new telephone call. The first one is put there by PRIME. Subsequent ones are generated by a call to subroutine GEN each time the current new call is removed from the FEL. A call to subroutine SCHED inserts it on the FEL. A flowchart is shown in Figure A-6.

TERM - A flowchart is shown in Figure A-7. TERM is used when a call is terminated on a nominal channel. Two versions exist. The simple version simply makes the involved channel and radio idle. The second version will cause a borrowed channel to be made idle if any channels are borrowed by the cell in question. The call on the borrowed channel is transferred to the nominal channel.

BORTERM - It is shown in Figure A-8 and is used whenever a call on a borrowed channel is terminated in that cell.

HANDOFF - Its only function is to determine the cell to which a call is being handed off to. A flowchart is shown in Figure A-9.

STAT - This subroutine shown in Figure A-10 is used to print out all simulation results. It is called by two methods. First, is a STAT event on the FEL which causes it to summarize data and print it out. Also, if the error code has been specified nonzero the complete system
RETURN

SCAN FUTURE EVENTS LIST FOR PROPER POSITION OF NEW EVENT

INSERT NEW EVENT INTO FUTURE EVENTS LIST

RETURN

FIGURE A-5 SUBROUTINE "SCHED"

GENERATE NEXT ARRIVAL TIME

ASSIGN CALL TO CELL

CALL SCHED

RETURN

FIGURE A-6 SUBROUTINE "GEN"
FIGURE A-7 SUBROUTINE "TERM"
Figure A-8 Subroutine "Borterm"
FIGURE A-9  SUBROUTINE "HANDOFF"
SUMMARIZE DATA

PRINT OUTPUT

ERROR CODE = 0 ?

PRINT FUTURE EVENTS LIST

PRINT SYSTEM STATE

PRINT BORROWED CHANNEL LISTS

PRINT DATA ON EVENT (WHEN STAT CALLED)

ERROR CODE = 1 ?

PRINT "ABNORMAL TERMINATION"

STOP SIMULATION

IS SIMULATION OVER?

YES

PRINT "NORMAL TERMINATION"

STOP SIMULATION

NO

GENERATE NEXT STAT EVENT

CALL SCHED

RETURN

NO

YES

FIGURE A-10 SUBROUTINE "STAT"
state at the time of the subroutine call is printed. This is used with the STAT event as an aid in checking out the program. The second way of calling this subroutine is the result of an error in the program. Error conditions that have been identified cause an error message to be printed, set the error code to 1 and initiate a call to STAT. This helps in figuring out what went wrong. Some of these error conditions are indicated in the flowcharts of this section. Like PRIME, there are some differences in this subroutine depending on what system is being simulated. However, the differences are not as extensive.

6. Simulator Verification

Verification of the simulation is accomplished by simulating a system, similar to the one to be analyzed, whose performance is known analytically. Then one considers the question - are the results reasonable? An affirmative answer then strengthens one's confidence in the results. In this case the known system contains the same cell layout as the system in question, but no borrowing of channels is allowed. Hence, we are simulating 21 independent finite serve, loss\(^4\) systems, each of which is engineered for a .02 probability of blocking. This system is illustrated in Figure A-11.

Transient Effects

Many discussions of simulations devote some time to techniques for eliminating transient effects from a simulation run unless that

\(^4\)So named by Riordan.\(^{(16)}\) More commonly called Full Access Blocked Calls Cleared (BCC).
FIGURE A-11 OFFERED LOAD
happens to be the topic under study. No transient effects were observed in any of these simulation runs which seemed a bit suspicious. Since the BCC systems of the verification run are simply birth and death Markov processes, it should be possible to find the dominant time constants in the state transition matrix and thereby estimate how long it will take the state probabilities to reach equilibrium. These systems have been extensively studied and indeed Riordan\(^{(16)}\) gives a recursive relationship for finding the coefficients of the characteristic polynomial of their state transition matrix. The root of the polynomial nearest zero is the reciprocal of the largest time constant of the system. In some limiting cases, this root is known in general. For the specific case of .02 steady-state blocking this root was found by solving the characteristic polynomial. With only two servers, the system time constant is 17\(^{3}\) sec at 20 it is 159 and finally at about 1550 it approaches 90 seconds. The shortest observation interval in any run has been 4200 seconds which is much longer than the dominant system time constant and would explain the absence of any transient effects.

This fact is also useful in the analysis of simulation results. Almost all statistical analysis techniques are based on the assumption of independent samples. It is often difficult to tell how well this assumption is met. However, since observation intervals are much longer than system time constants, this supports the assumption that consecutive observations are independent.
Variance of Statistics

The observation interval of 4000 to 5000 seconds mentioned in the previous section was chosen to meet an intuitive requirement of about 10,000 calls for meaningful results. It is the intent of this section to examine the reasonableness of this intuitive feeling. The approach is similar to that of the last section in that the same verification run is used. A later chapter of Riordan's book \(^{16}\) cites a paper by Kosten \(^ {17}\) et al which deals with the accuracy of blocking probability observations for loss systems. Specifically, he shows that for loss systems

\[
\sigma^2_B \approx \frac{c+a}{c-a} E[B]
\]

where

- \(c\) - number of servers
- \(a\) - total offered load
- \(B\) - number of blocked calls
- \(t\) - time in multiples of average holding time

and

\[
E[B] = at \ P[\text{Blocking}].
\]

Considering the system shown in Figure A-11 which has been engineered for .02 blocking we can compute the mean, variance and standard deviation of \(B\) for each of the 21 cells. Table A-1 lists the mean and standard deviation for a 5250 second time interval. Theoretical values along with sample values from two simulation runs are included. Each run consists of 5-5250 second observation intervals. The runs have
<table>
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<tr>
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<th>SAMPLE VALUES</th>
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</tr>
<tr>
<td>16</td>
<td>2.11</td>
<td>0.800</td>
<td>2.000</td>
<td>1.400</td>
</tr>
<tr>
<td>17</td>
<td>2.11</td>
<td>1.400</td>
<td>0.800</td>
<td>1.100</td>
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<td>18</td>
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<td>1.000</td>
<td>0.800</td>
<td>0.900</td>
</tr>
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<td>19</td>
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<td>3.000</td>
<td>5.600</td>
<td>4.300</td>
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<tr>
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<td>1.200</td>
<td>0.200</td>
<td>0.700</td>
</tr>
<tr>
<td>21</td>
<td>5.25</td>
<td>3.000</td>
<td>6.000</td>
<td>4.500</td>
</tr>
<tr>
<td>TOTAL</td>
<td>199.84</td>
<td>162.600</td>
<td>200.800</td>
<td>181.700</td>
</tr>
<tr>
<td>CELL</td>
<td>THEORETICAL</td>
<td>RUN 11</td>
<td>SAMPLE VALUES</td>
<td>COMBINATION</td>
</tr>
<tr>
<td>------</td>
<td>-------------</td>
<td>--------</td>
<td>---------------</td>
<td>-------------</td>
</tr>
<tr>
<td>1</td>
<td>9.34</td>
<td>10.359</td>
<td>14.856</td>
<td>12.191</td>
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<td>4.58</td>
<td>3.362</td>
<td>2.881</td>
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<td>3.606</td>
<td>3.271</td>
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<td>5.44</td>
<td>3.578</td>
<td>4.879</td>
<td>4.143</td>
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<td>4.336</td>
<td>3.899</td>
<td>4.088</td>
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<td>8.620</td>
<td>9.138</td>
<td>8.470</td>
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<td>15.192</td>
<td>14.543</td>
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<tr>
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<td>7.987</td>
<td>7.765</td>
<td>7.750</td>
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<td>2.236</td>
<td>4.980</td>
<td>3.653</td>
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<tr>
<td>10</td>
<td>16.68</td>
<td>16.888</td>
<td>7.635</td>
<td>12.432</td>
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<tr>
<td>11</td>
<td>22.13</td>
<td>12.402</td>
<td>13.027</td>
<td>15.032</td>
</tr>
<tr>
<td>12</td>
<td>2.37</td>
<td>2.608</td>
<td>3.050</td>
<td>2.677</td>
</tr>
<tr>
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<td>3.97</td>
<td>1.924</td>
<td>2.608</td>
<td>2.319</td>
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<tr>
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<td>8.45</td>
<td>10.977</td>
<td>6.914</td>
<td>9.967</td>
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<td>15</td>
<td>8.45</td>
<td>5.030</td>
<td>3.808</td>
<td>4.572</td>
</tr>
<tr>
<td>16</td>
<td>2.37</td>
<td>1.304</td>
<td>2.345</td>
<td>1.897</td>
</tr>
<tr>
<td>17</td>
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<td>18</td>
<td>2.37</td>
<td>1.414</td>
<td>1.304</td>
<td>1.287</td>
</tr>
<tr>
<td>19</td>
<td>3.97</td>
<td>4.000</td>
<td>3.286</td>
<td>3.713</td>
</tr>
<tr>
<td>20</td>
<td>2.37</td>
<td>1.095</td>
<td>0.447</td>
<td>0.949</td>
</tr>
<tr>
<td>21</td>
<td>4.58</td>
<td>3.674</td>
<td>5.292</td>
<td>4.577</td>
</tr>
</tbody>
</table>

**TOTAL** 39.67 34.177 31.814 34.260

**NOTE:** All results are for a 5250 second observation interval.
different starting values for the random number generators, which should make each run independent. Hence, the combination of these two runs can be considered one long run. Results of this combination run are tabulated in the last column of Tables A-1 and A-2. Since each cell is independent of the other cells, we can examine the total number of blocked calls. The expected value is nearly 200. The sample mean for run 11 is 163 which is within one standard deviation of the expected value. Run 13 and the combination have sample means of 201 and 182, respectively. A more meaningful comparison between expected and observed results can be made using the techniques described in Appendix C. A least squares straight line was fit to the sample values using the theoretical values as independent variables. Table A-2 lists the results of this effort. Considering each run separately, about 85 percent and 65 percent of the square variation was removed from the sample mean and standard deviation respectively. Correlation coefficients are .92 and .81 respectively. In combination, 90 percent and 83 percent of the square variation was removed with correlation coefficients being .95 and .91.

All of this seems to substantiate that the simulator is performing in a reasonable manner.

In addition it is desirable to consider the fraction of blocked calls since this is the measure of performance used by the telephone companies. Table A-3 shows these results for each cell in runs 11 and 13 and the combination of these. Recall that each cell was designed to operate at .02 blocking. In addition Table A-3 contains the service
### TABLE A-2

**LEAST SQUARES ANALYSIS**

<table>
<thead>
<tr>
<th>MEAN</th>
<th>RUN 11</th>
<th>RUN 13</th>
<th>COMBINATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Best Straight Line</td>
<td>( \hat{B} = .8 + .73B )</td>
<td>( \hat{B} = -.57 + 1.1B )</td>
<td>( \hat{B} = .12 + .90B )</td>
</tr>
<tr>
<td>% Variation Removed</td>
<td>87</td>
<td>85</td>
<td>90</td>
</tr>
<tr>
<td>Correlation Coefficient</td>
<td>.93</td>
<td>.92</td>
<td>.95</td>
</tr>
<tr>
<td>F-Ratio</td>
<td>123</td>
<td>104</td>
<td>177</td>
</tr>
</tbody>
</table>

**Standard Deviation**

<table>
<thead>
<tr>
<th>MEAN</th>
<th>RUN 11</th>
<th>RUN 13</th>
<th>COMBINATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Best Straight Line</td>
<td>( \hat{\sigma}_B = .93 + .64 S_B )</td>
<td>( \hat{\sigma}_S = 1.3 + .63 S_B )</td>
<td>( \hat{\sigma}_S = .49 + .79 S_B )</td>
</tr>
<tr>
<td>% Variation Removed</td>
<td>67</td>
<td>65</td>
<td>84</td>
</tr>
<tr>
<td>Correlation Coefficient</td>
<td>.82</td>
<td>.81</td>
<td>.91</td>
</tr>
<tr>
<td>F-Ratio</td>
<td>38</td>
<td>35</td>
<td>97</td>
</tr>
</tbody>
</table>

where

\[
\bar{B} = \frac{1}{n} \sum_{i=1}^{n} B_i
\]

- Sample Mean

\[
S_B^2 = \frac{1}{n-1} \sum_{i=1}^{n} (B_i - \bar{B})^2
\]

- Estimate of Variance

\( B_i = \) Number of blocked calls in a particular cell during interval \( i \).
<table>
<thead>
<tr>
<th>CELL</th>
<th>RUN 11</th>
<th>RUN 13</th>
<th>COMBINATION OF RUN 11 AND 13</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0175</td>
<td>0.0217</td>
<td>0.0196</td>
</tr>
<tr>
<td>2</td>
<td>0.0209</td>
<td>0.0167</td>
<td>0.0188</td>
</tr>
<tr>
<td>3</td>
<td>0.0227</td>
<td>0.0208</td>
<td>0.0217</td>
</tr>
<tr>
<td>4</td>
<td>0.0137</td>
<td>0.0186</td>
<td>0.0161</td>
</tr>
<tr>
<td>5</td>
<td>0.0219</td>
<td>0.0426</td>
<td>0.0324</td>
</tr>
<tr>
<td>6</td>
<td>0.0143</td>
<td>0.0160</td>
<td>0.0145</td>
</tr>
<tr>
<td>7</td>
<td>0.0184</td>
<td>0.0308</td>
<td>0.0247</td>
</tr>
<tr>
<td>8</td>
<td>0.0144</td>
<td>0.0222</td>
<td>0.0183</td>
</tr>
<tr>
<td>9</td>
<td>0.0191</td>
<td>0.0238</td>
<td>0.0214</td>
</tr>
<tr>
<td>10</td>
<td>0.0133</td>
<td>0.0115</td>
<td>0.0124</td>
</tr>
<tr>
<td>11</td>
<td>0.0133</td>
<td>0.0220</td>
<td>0.0176</td>
</tr>
<tr>
<td>12</td>
<td>0.0128</td>
<td>0.0151</td>
<td>0.0140</td>
</tr>
<tr>
<td>13</td>
<td>0.0089</td>
<td>0.0164</td>
<td>0.0127</td>
</tr>
<tr>
<td>14</td>
<td>0.0314</td>
<td>0.0164</td>
<td>0.0240</td>
</tr>
<tr>
<td>15</td>
<td>0.0207</td>
<td>0.0151</td>
<td>0.0179</td>
</tr>
<tr>
<td>16</td>
<td>0.0082</td>
<td>0.0206</td>
<td>0.0144</td>
</tr>
<tr>
<td>17</td>
<td>0.0137</td>
<td>0.0081</td>
<td>0.0109</td>
</tr>
<tr>
<td>18</td>
<td>0.0113</td>
<td>0.0082</td>
<td>0.0097</td>
</tr>
<tr>
<td>19</td>
<td>0.0144</td>
<td>0.0261</td>
<td>0.0203</td>
</tr>
<tr>
<td>20</td>
<td>0.0126</td>
<td>0.0021</td>
<td>0.0074</td>
</tr>
<tr>
<td>21</td>
<td>0.0119</td>
<td>0.0226</td>
<td>0.0174</td>
</tr>
</tbody>
</table>

Weighted Avg. 0.0162 0.0199 0.0181
Arith. Avg. 0.0159 0.0189 0.0174
Service Deviation 0.00545 0.00851 0.00572

TABLE A-3
FRACTION OF CALLS BLOCKED
deviation about the mean. This statistic is defined in Section 2.2.2. Both a weighted average and an arithmetic average are listed on Table A-3. The weighted average is obtained by dividing the total number of blocked calls for a whole run by the total number of calls received. The arithmetic average is simply an average of the numbers recorded in Table A-3. Figure A-12 presents some of this information graphically. Performance both as a function of time and of cell number is indicated.

In Table A-3 it is noted that the fraction of blocked calls is somewhat less than anticipated. The amount of traffic submitted to the system will have a bearing on this. The simulation runs were designed to process 50,000 – 180 second calls in the total time period of 26,250 seconds, for an average traffic intensity of 3.426 erlangs. Run 11 processed 50085 calls with an average holding time of 178.1 seconds, for an offered load of 339.9 erlangs. In run 13, there were 50428 calls with an average holding time of 178.7 seconds for an offered load of 343.4 erlangs. Thus, it would seem that the lower blocking is not caused by insufficient traffic.

In summary, it appears that the simulator functions generally in accordance with known traffic theory. There is no ready explanation for the low fraction of blocking in run 11.
FRACTION BLOCKED

360 CHANNELS — NO BORROWING
RUN #11 6-18-71

CELL NUMBER

SERVICE DEVIATION .00545

FIGURE A-12 CELL BLOCKING
APPENDIX B

BASIC TELEPHONE TRAFFIC THEORY

Consider a simple telephone system comprised of C servers. Customers submit requests for service at an average rate of \( \lambda \) calls per unit time. These requests are distributed according to a Poisson distribution. A request which finds one or more servers idle is assigned a server which it holds for a length of time \( t \). \( t \) is an exponentially distributed random variable with a mean of \( \tau \). Requests that encounter all servers busy are blocked, or denied service. Blocked requests are assumed to leave the system and never return. This assumption is referred to as the "Blocked Calls Cleared" (BCC) assumption. Other blocking assumptions are in common use, but are not used in this work. This system comprises a finite state Markov birth and death process. A state diagram is shown in Figure B-1 which uses the number of busy servers as a state description. Transition probabilities in a small interval \( \Delta t \) are proportional to \( \Delta t \).

We are interested in the statistical equilibrium values of the state probabilities. There are several approaches to this problem. One of the simplest is to realize that in steady state the number of transitions from state \( n \) to \( n+1 \) is equal to those from \( n+1 \) to \( n \). Transitions from \( n \) to \( n+1 \) occur at a rate equal to \( \lambda P_n \), the probability of
\( \lambda \)-ARRIVAL RATE (CALL/SEC)
\( \tau \)-HOLDING TIME (SEC/CALL)

**FIGURE B-1 STATE DIAGRAM FOR A BLOCKED CALLS CLEARED SYSTEM WITH C SERVERS**

\[
\lambda \Delta t \quad \lambda \Delta t \quad \lambda \Delta t \quad \lambda \Delta t \quad \lambda \Delta t \quad \lambda \Delta t
\]

\[
\frac{1}{\tau} \Delta t \quad \frac{2}{\tau} \Delta t \quad \frac{3}{\tau} \Delta t \quad \frac{4}{\tau} \Delta t \quad \frac{5}{\tau} \Delta t \quad \frac{C}{\tau} \Delta t
\]

\[
\text{GROUP OF } C \text{ SERVERS}
\]

\[
\text{OCCUPANCY} = \frac{L}{C}
\]

\[
\text{(BLOCKED CALLS CLEARED)}
\]

\[
\text{OFFERED LOAD} \quad \alpha = \lambda \tau \text{ (ERLANGS)}
\]

\[
\text{CARRIED LOAD} - L
\]

\[
L = \alpha[1 - B(c, \alpha)]
\]

\[
\text{FRACTION BLOCKED} = B(c, \alpha) = \frac{\alpha^c}{c!} \sum_{n=0}^{c} \frac{\alpha^n}{n!} \text{ (ERLANG B FORMULA)}
\]

**FIGURE B-2 BASIC TELEPHONE SYSTEM**
state \( n \), multiplied by the probability of a transition from state \( n \) to \( n+1 \). Hence, we can write the following set of equations

\[
P_n \frac{n}{T} \Delta t = P_{n-1} \lambda \Delta t \quad n=1,2,\ldots,C.
\]  

(S-1)

Solving iteratively for \( P_n \) in terms of \( P \) we have

\[
P_n = \frac{(\lambda \tau)^n P_0}{n!}.
\]  

(S-2)

Then realizing that

\[
\sum_{n=0}^{C} P_n = 1
\]  

(S-3)

we can solve for \( P_0 \)

\[
P_0 = \frac{1}{C} \frac{1}{\sum_{n=0}^{C} \frac{(\lambda \tau)^n}{n!}}
\]  

(S-4)

Substituting in S-2 we have an expression for the state probabilities

\[
P_n = \frac{(\lambda \tau)^n}{C} \frac{1}{\sum_{n=0}^{C} \frac{(\lambda \tau)^n}{n!}}
\]  

(S-5)

Now we are interested in the probability that a random request is blocked. This is simply the probability that a request finds the system in state \( C \) since it is a property of the Poisson arrival process
that the next arrival is independent of all previous events. Hence, the probability of blocking is \( P_c \) denoted by \( B(c,a) \)

\[
B(c,a) = P_c = \frac{\frac{a^c}{C!}}{\sum_{n=0}^{C} \frac{a^n}{n!}}
\] (B-6)

where \( a = \lambda t \).

\( B(c,a) \) is called the Erlang B\(^5\) formula for blocking in a BCC system. Note that \( a \) is a unitless variable equal to the product of the average arrival rate and the average holding time. This variable has been given the name of offered load. An artificial unit called the "Erlang" has been assigned to this variable. Thus, the erlang is a measure of traffic intensity. Another useful concept is the carried load \( L \) which is the number of erlangs of average traffic carried by the system and is given by

\[
L = a[l - B(c,a)].
\] (B-7)

Occupancy is defined as the average fraction of busy servers as

\[
\text{OCC} = \frac{L}{C}.
\] (B-8)

These ideas are summarized in Figure B-2. The ratio \( a/c \), the offered load per server, is also a useful idea even though it has not been

\(^5\)After A. K. Erlang, who first published it in 1917.
given a specific name. In Figure B-3 this ratio a/c has been plotted versus c for B(c,a) equal to a constant. B(c,a) = .02 is a common service objective for a telephone system and is the objective for this system, i.e., on the average we are willing to block two of every 100 requests. Figure B-3 illustrates an important factor in the operation of telephone systems, namely the increased efficiency of larger groups of servers. Indeed this is the whole motivation behind the dynamic channel assignment concept.

So far we have proceeded as if the calling rate $\lambda$ was a constant. This is not the case, and in fact $\lambda$ varies substantially over the period of a day. This problem is solved by the concept of a busy hour. The busy hour is defined as the hour during the day that a switching system experiences its heaviest load. It is then assumed that the arrival rate is constant during this time and all traffic engineering is based upon the busy hour. Hence, a service objective of .02 blocking would apply only during the busy hour. During the rest of the day service should be at least this good or better.

One further comment regarding the Erlang B formula is in order. It was derived assuming an exponential holding time. This is not a necessary requirement. It is easy to see that it also holds for the constant holding time case. In fact, Sevastianov\(^{18}\) has shown that Erlang B holds for an arbitrary holding time distribution. This is not true for the interarrival distribution. It need not be exponential but neither can it be arbitrary. More is said about this in Section 3.5.1.1.
FIGURE B-3 OFFERED LOAD PER SERVER
vs NUMBER OF SERVERS
APPENDIX C

LEAST SQUARES ANALYSIS

This material is adapted from a book by Draper and Smith. (19)

We assume we have a set of two tuples \( \{X_i, Y_i, i = 1, 2, \ldots, n\} \) and we wish to find the straight line which best relates \( Y \) to \( X \) in the square error sense. That is we write

\[
\hat{Y}_i = B_0 + B_1 X_i \quad \text{(C-1)}
\]

and choose \( B_0 \) and \( B_1 \) such that

\[
\text{Error} = \sum_{i=1}^{n} (\hat{Y}_i - Y_i)^2 \quad \text{(C-2)}
\]

is minimum. It is easy to show that the desired \( B_0 \) and \( B_1 \) are given by:

\[
B_1 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n} (X_i - \bar{X})^2} \quad \text{(C-3)}
\]

where

\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{and} \quad \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \quad \text{(C-4)}
\]
then
\[ B_0 = \overline{Y} - B_1 \bar{X}. \]  

Once this has been done, a measure of goodness is needed. One obvious one is derived from the following relation:

\[ \sum_{i=1}^{n} (Y_i - \overline{Y})^2 = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 + \sum_{i=1}^{n} (\hat{Y}_i - Y)^2. \]  

The ratio
\[ R^2 = \frac{\sum_{i=1}^{n} (\hat{Y}_i - \overline{Y})^2}{\sum_{i=1}^{n} (Y_i - \overline{Y})^2} \]

is a measure of the percentage of square deviation explained by the least squares straight line.

So far we have considered the \( X_i \) and \( Y_i \) merely as numbers. Let us assume that the \( Y_i \) are sample values of a random variable \( Y \) which is related to the independent variable \( X \) as follows:

\[ Y = \beta_0 + \beta_1 X + e. \]  

\( \beta_0 \) and \( \beta_1 \) are not random variables, but are in fact constant parameters which are estimated by \( B_0 \) and \( B_1 \). A random error term \( e \) has been added, which is the result of measurement or sampling error. Since the \( Y_i \) are random variables, then all of the previous functions computed from the \( Y_i \) are also random variables. If we assume that \( e \) is
normally distributed with zero mean then the function

\[
F = \frac{B_1^2 \sum_{i=1}^{n} (X_i - \bar{X})^2}{\sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2} \quad (C-9)
\]

has an F distribution with 1 and \(n-2\) degrees of freedom provided that \(B_1 = 0\). If the calculated value of \(F\) exceeds say the .95 point on the \(F\) distribution, then we reject the hypothesis that \(B_1 = 0\) with a .05 probability of error.

If we were to consider both \(X\) and \(Y\) as random variables, the parameter

\[
\sigma_{XY} = \frac{E[(X-\bar{X})(Y-\bar{Y})]}{\sigma_X \sigma_Y} \quad (C-10)
\]

is known as the correlation coefficient between them. From sample values of \(X\) and \(Y\) we compute a sample correlation coefficient \(r_{XY}\) which is an estimate of \(\sigma_{XY}\). This statistic is

\[
r_{XY} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n} (X_i - \bar{X})^2 \sum_{i=1}^{n} (Y_i - \bar{Y})^2}} \quad (C-11)
\]

which is equal to \(R\) whose square was considered earlier.

A time shared computer program was written which calculates the statistics listed above for a set of 21 two tuples \(X_i, Y_i\). Results are then printed out in an Analysis of Variance Table format. This enables rapid evaluation of statistical relationships for the 21 cell
system under scrutiny. As in most cases of statistical analysis it is difficult to know how well the statistical assumptions are met by the data in question. However, this does provide a quantitative way of comparing various intuitive cause and effect relationships. The preceding mathematical development has not been complete, but is only intended to establish a common ground for discussion. For more detail, the reader is referred to a statistics text, an example of which is cited at the beginning of this Appendix.
REFERENCES


