DELANEY, Matthew Sylvester, 1927-
DISCRETE EUCLIDEAN UNIVERSES AND ASSOCIATED AUTOMORPHISMS.
The Ohio State University, Ph.D., 1971
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

© 1972
MATTHEW SYLVESTER DELANEY
ALL RIGHTS RESERVED
DISCRETE EUCLIDEAN UNIVERSES
AND ASSOCIATED AUTOMORPHISMS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

Matthew S. Delaney, B.A., S.M.

* * * * *

The Ohio State University
1971

Approved by

Hans Zassenhaus
Adviser
Department of Mathematics
PLEASE NOTE:

Some pages have indistinct print. Filmed as received.

University Microfilms, A Xerox Education Company
ACKNOWLEDGEMENTS

I want to express my complete indebtedness to Professor Hans J. Zassenhaus of the Mathematics Department at the Ohio State University who has spent many hours laboring with me in this work. Without his consistent encouragement and indispensable help this thesis could not have been written.

My thanks also goes to Professor Richard M. Wilson of the Ohio State University Mathematics Department for his help in supplying me with a most useful group.

I wish to express my gratitude and appreciation to Professors Surinder K. Sehgal and John Riedl for their considerable kindness to me during my years at the Ohio State University.

Finally I want to thank Mr. K. C. Wong with whom I have worked for the past four years and to Mr. Frank Adams who has labored much over the typing and setting up of this thesis.
VITA

November 26, 1927 . . . Born - Tipperary, Ireland

1958 . . . . . . . . B.A., Immaculate Heart College, Hollywood, California

1959-1960 . . . . Participant, Academic Year Institute, University Of Notre Dame, Notre Dame, Indiana

1960 . . . . . . . . S.M., University of Notre Dame, Notre Dame, Indiana

1960-1962 . . . . Vice-Principal, Pius X High School, Downey, California

1962-1967 . . . . Assistant Professor, Immaculate Heart College, Hollywood, California

1967-1971 . . . . Teaching Associate, Department of Mathematics, The Ohio State University, Columbus, Ohio

FIELDS OF STUDY

Major Field: Mathematics

Studies in Complex Analysis. Professor Daniel Eustice

Studies in Topology. Professor John Riedl

Studies in Algebra. Professors Hans J. Zassenhaus and Surinder K. Sehgal
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>VITA</td>
<td>iii</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>CHAPTER</td>
<td></td>
</tr>
<tr>
<td>I. SPACE GROUPS</td>
<td>3</td>
</tr>
<tr>
<td>II. DISCRETE EUCLIDEAN UNIVERSE</td>
<td>21</td>
</tr>
<tr>
<td>III. COMBINATORIAL AUTOMORPHISMS IN A DEU</td>
<td>42</td>
</tr>
<tr>
<td>IV. GENERATORS AND RELATIONS IN Aut(DEU)</td>
<td>58</td>
</tr>
<tr>
<td>V. THE GROUP Aut(DEU)</td>
<td>72</td>
</tr>
<tr>
<td>APPENDIX</td>
<td>82</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>94</td>
</tr>
</tbody>
</table>
INTRODUCTION

A discrete universe is a set DU with a 'neighborhood relation' possessing the following properties:
1. No element of DU is a neighbor of itself
2. If X is a neighbor of Y then Y is a neighbor of X
3. Each element of DU possesses only a finite number of neighbors.

The set DU is said to be connected if for any two elements X, Y of DU there is a finite chain \( X = X_0, X_1, \ldots, X_k = Y \) linking X with Y in DU such that \( X_i \) is a neighbor of \( X_{i+1} \) (\( i = 0, 1, \ldots, k-1 \)). The minimum value of k is said to be the combinatorial distance \( d(X, Y) \) from X to Y.

More generally speaking a pointset DEU of the n-dimensional euclidean space \( E_n \) is said to be a crystallographic discrete euclidean universe in the case that
1. DEU is discrete.
2. The Voronoi cell \( V(P) \) of each point P in DEU which is formed by those points of \( E_n \) that are not farther away from P than from any other point of DEU is a convex polytope.
3. The neighbors of the point P in DEU are those points Q of DEU for which the intersection, \( V(P) \cap V(Q) \),
is a common (n-1)-dimensional face (facet) of both V(P) and V(Q).

4. The group of all isometries of $E^n$ mapping DEU onto DEU—the symmetry group of DEU—permutes the points of DEU in a transitive manner.

The symmetry group $G$ of a crystallographic DEU is said to be a euclidean space group in $n$ dimensions. Its action on $E^n$ defines the orbits of $G$. Each orbit of $G$ is a DEU. The euclidean space groups are known for spaces of one, two, and three dimensions and in principle they are known also for 4-dimensional space. The crystallographic DEU's are known only in one and two dimensions.

The purpose of this work is to investigate certain neighborhood preserving combinatorial automorphisms of a DEU. These automorphisms form a group which we designate as Aut(DEU). This group contains a subgroup $\overline{G}$ which is isomorphic to $G$ and the index of $\overline{G}$ in Aut(DEU) is finite. Aut(DEU) contains a finite normal subgroup $N$ such that the factor group Aut(DEU)/$N$ is isomorphic to an abstract space group. A consequence of this result is that it is possible for a member of a discrete euclidean universe who has access only to local information (neighborhood gossip) about his environment to get vital information about the world in which he lives.
CHAPTER I

SPACE GROUPS

The space group concept is connected intrinsically with the process of finding certain extensions of a free abelian group with \( n \) generators. Indeed any group \( G \) which contains a subgroup \( S \) can be considered as an extension of \( S \). However the theory of extensions is generally limited to Schreier extensions, namely those extensions \( G \) such that \( G \) contains a given normal subgroup \( N \) where \( G/N \) is isomorphic to a given group \( H \). There is always one such extension, namely, the direct product of \( N \) and \( H \). We give the following definition:

A group \( G \) is called an extension of a group \( N \) by a group \( H \) if:

(i) \( G \) contains a normal subgroup \( N' \) such that

\[ N' \cong N \]

and

(ii) \( G/N' \cong H \).

This definition is equivalent to saying that the following sequence is exact:

\[ 1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1 \]

i.e., \( \nu \) is a monomorphism and \( \sigma \) is an epimorphism and \( \text{Ker} \sigma = \nu(N) \).
Now let $T$ be a finitely generated free abelian group with $n$ generators. Let $G$ be an extension of $T$ by a finite group $F$. $G$ can be partitioned into cosets $T_g$ of $T$, $(g \in G)$. Let $S$ be a representative system for these cosets and let $S(a)$ denote a representative of the coset corresponding to $a \in F$. Let $1 \rightarrow T \overset{\nu}{\rightarrow} G \overset{\sigma}{\rightarrow} F \rightarrow 1$ be the exact sequence representing the extension in question where $\nu$ is a monomorphism and $\sigma$ an epimorphism. $S$ can be considered as the mapping $S:F \rightarrow G$ such that $\sigma S$ maps $F$ onto itself in an identical manner and $S\sigma$ maps each element of a coset onto a fixed element of that coset, namely the representative. Each element of $G$ can be represented as a product $\nu(t) \cdot S(a)$, $t \in T$, $a \in F$. When $S$ is determined the product is unique. Clearly $S(a)S(b)$ is a member of the coset represented by $S(ab)$.

For an arbitrary extension where $T$ is not abelian we have:

(1) $S(a)S(b) = \nu[C(a,b)] \cdot S(ab)$, $C(a,b) \in T$.

If $S(1_F) = 1_G$ then

(2) $C(1_F,1_F) = C(a,1_F) = C(1_F,a) = 1_G$ for all $a \in F$.

The mapping $C:F \times F \rightarrow T$ is called a factor set of the extension. When $C$ satisfies (2) it is called normalized. Since $\nu(T)$ is a normal subgroup of $G$ then
conjugation in \( G \) with an element \( g \in G \) produces an automorphism \( \gamma_g \) of \( T \) in the manner that

\[
\gamma[g(t)] = g \cdot (\gamma(t)) \cdot g^{-1}, \quad t \in T.
\]

The mapping \( \gamma : G \to \text{Aut}(T) \), the automorphism group of \( T \), is a homomorphism. Indeed if \( a \in F \), then conjugation by \( S(a) \in G \) produces an automorphism \( \phi_a \) of \( T \)

\[
\gamma[\phi_a(t)] = S(a) \cdot \gamma(t) \cdot S(a)^{-1}.
\]

If \( S(1_F) = 1_G \) then \( \phi_{1_F} \) is the identity automorphism of \( T \). Making use of the mapping \( \gamma : F \to \text{Aut}(T) \) the product of two elements in \( G \) can be expressed as

\[
[\gamma(t) \cdot S(a)][\gamma(t') \cdot S(b)] = \gamma[t(\phi_a(t')) \cdot C(a,b)] \cdot S(ab).
\]

If we identify \( \gamma(T) \) with \( T \) in \( G \) this can be expressed more clearly by

\[
[t \cdot S(a)][t' \cdot S(b)] = t \cdot (S(a) \cdot t' \cdot S(a)^{-1}) \cdot (S(a) \cdot S(b))
\]

\[
= t \cdot \phi_a(t') \cdot C(a,b) \cdot S(ab).
\]

From the associativity of multiplication we have a further relation on the factor set:

\[
\phi_a[C(a,b,c)] = C(a,b)C(ab,c)C(a,bc)^{-1}
\]

The mapping \( \phi \) is not always a homomorphism! If we identify \( \gamma(T) \) with \( T \) in \( G \) we have
\[ \varphi_a(\varphi_b(t)) = \varphi_a[S(b) \cdot t \cdot S(b)^{-1}] \]
\[ = S(a) \cdot [S(b) \cdot t \cdot S(b)^{-1}] \cdot S(a)^{-1} \]
\[ = C(a,b) \cdot (S(ab) \cdot t \cdot S(ab)^{-1}) \cdot C(a,b)^{-1} \]
\[ = C(a,b) \cdot \varphi_{ab}(t) \cdot C(a,b)^{-1}. \]

It should also be pointed out that \( \varphi \) depends on the choice of representatives.

Reverting now to our particular situation we have assumed that \( T \) is a free abelian group on \( n \) generators, therefore \( \varphi : F \rightarrow \text{Aut}(T) \) is a homomorphism and does not depend on the choice of representatives for the cosets of \( \mathcal{V}(T) \) in \( G \). Hence we can consider \( F \) as operative through \( \varphi \) on \( T \). In other words \( F \) is a group of operators for \( T \) and \( T \) can be assigned the structure of an \( F \)-module.

We can now state the following theorem:

**Theorem 1.** The mapping \( \varphi : F \rightarrow \text{Aut}(T) \) is a monomorphism if and only if the image \( \mathcal{V}(T) \) of \( T \) in \( G \) is a maximal abelian subgroup of \( G \).

**Proof:** Let the abelian normal subgroup \( \mathcal{V}(T) \) of \( G \) be a maximal abelian subgroup of \( G \). Let \( g \in G \) and let \( \mathcal{V}(T) = T' \). If \( g \cdot t = g \cdot t \) for each \( t \in T' \), then \( g \) also
belongs to $T'$. This is so because the group generated by $g$ and $T'$ is abelian and hence must be contained in $T'$ which is maximal abelian in $G$. We now show that the kernel of $\varphi$, $\text{Ker } \varphi$, consists only of $1_F \in F$. By definition

$$\varphi_a(t) = S(a) \cdot t \cdot S(a)^{-1}, \quad (a \in F, t \in T')$$

and if $a \in \text{Ker } \varphi$ then $S(a) \cdot t \cdot S(a)^{-1} = t$. Thus $\varphi_a$ is the identity automorphism of $T$. We have then $S(a) \in T'$ and $a = 1_F \in F$. Therefore $\varphi$ is a monomorphism.

Conversely, let $\varphi$ be a monomorphism and let $g \in G$ commute with any $t \in T'$. Then

$$t = g \cdot t \cdot g^{-1} = \varphi_{\sigma(g)}(t).$$

But this implies that $\varphi_{\sigma(g)}$ is the identity monomorphism and hence $\sigma(g) = 1_F$. Therefore $g \in T'$ and $T'$ is maximal abelian in $G$. Q.E.D.

We are now in a position to give an abstract definition and several characterizations of the notion of space group.

Definition: Any group $G$ occurring in an extension:

$$1 \rightarrow T \xrightarrow{\varphi} G \xrightarrow{\sigma} F \rightarrow 1$$
where

(i) $T$ is a free abelian group on $n$ generators

(ii) $F$ is finite; $F = G/T$ and

(iii) there exists a mapping $\varphi : F \to \text{Aut}(T)$

which is a monomorphism defined as above

is an abstract $n$-dimensional space group.

For further development of the theory of space groups along the preceding lines we make reference to Ascher and Janner [1].

For our purposes we need a more explicit characterization of space groups. Bieberbach [2] has shown that any $n$-dimensional space group is isomorphic to a subgroup $\overline{G}$ of the group

$$\text{Hol}(\text{GL}(n, \mathbb{Z}), \mathbb{R}^{n \times 1})$$

the holomorph of $\text{GL}(n, \mathbb{Z})$ acting on $\mathbb{R}^{n \times 1}$. This group is formed by matrices of the form

$$X = (H(X), t(X)) = \begin{pmatrix} H(X)_n & t(X) \\ \hline 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $H(X)_n \in \text{GL}(n, \mathbb{Z})$, $t(X) \in \mathbb{Q}^{n \times 1}$. $H(X)_n$ is called the homogeneous part of $X$ and $t(X)$ is called the translative part of $X$. $\overline{G}$ intersects the full translation group

$$\overline{\mathbb{R}^{n \times 1}} = \left\{(I_n, u) \mid u \in \mathbb{R}^{n \times 1}\right\}$$
in the integral translation group

\[ T = \left\{ (I_n, v) \mid v \in \mathbb{Z}^n \times 1 \right\} \]

and the index of \( G \) over \( T \) is finite.

On the other hand Zassenhaus [16] has proved the following:

Let \( T \) be a free abelian group with \( n \) generators \( t_1, t_2, \ldots, t_n \). Let \( G \) be an overgroup of \( T \) in which \( T \) is normal. Let \( G/T = F \). The elements \( c \) of \( F \) are the residue classes of \( G \) modulo \( T \):

\[ c = T \cdot S(c) \]

where the elements \( S(c) \) form a representative system of \( G \) modulo \( T \). If we set up the correspondence

\[ c \leftrightarrow \phi_c \]

where \( \phi_c(t) = S(c) \cdot t \cdot S(c)^{-1} \), we have a one-to-one mapping of \( F \) into \( \text{Aut}(T) \), the automorphism group of \( T \), so that the product of two residue classes modulo \( T \) corresponds to the product of their respective automorphisms in \( \text{Aut}(T) \). The identity residue class \( T \) corresponds to the identity automorphism in \( \text{Aut}(T) \).

If we set

\[ \phi_c(t_k) = S(c) \cdot t_k \cdot S(c)^{-1} = \prod_{i=1}^{n} t_i^{\alpha_{ik}(c)} \]

then we find that the automorphism \( \phi_c \) in \( \text{Aut}(T) \)
corresponding to $c$ can be expressed in the form of an $n \times n$ matrix, $(\alpha_{ik}(c)) = \mathcal{F}_c$. Thus the product of two automorphisms corresponds to the product of their respective representing matrices. All the matrices $\mathcal{F}_c$, $c \in F$, form a group $\mathcal{F}$ which is isomorphic to $F$ and consists of unimodular matrices. The Zassenhaus theorem then states:

If $\mathcal{F}$ (or $F$) is finite then $G$ is isomorphic to a space group in $n$ dimensions.

The theorems of Bieberbach and Zassenhaus lie at the root of the abstract characterization of space groups which is found in the Ascher-Janner paper [1].

The finiteness of the factor group $F$ and its representation by integral matrices as suggested in the Zassenhaus theorem leads us by way of the following theorem to the characterization of space groups as groups of isometries.

Theorem 2: Each finite substitution group $H$ is equivalent to an orthogonal substitution group.

Proof: Suppose the substitution group $H$ is of degree $n$ and that it has real coefficients. Designate the elements of $H$ by $I, h_1, h_2, \ldots, h_t$. If one applies
the elements of \( H \) to the quadratic form
\[ Q = x_1^2 + x_2^2 + \ldots + x_n^2 , \]
one obtains a set of \( t \) positive
definite quadratic forms among which is \( Q \) since \( I \in H \).
Their sum is also a positive definite quadratic form
\[ R = \sum_{i,k=1}^{n} r_{ik} x_i x_k , \quad (r_{ik} = r_{ki}) . \]
Since \( H \) is a group,
\( R \) is invariant under the elements of \( H \). It is well
known that for the positive definite form
\[ R = \sum_{i,k=1}^{n} r_{ik} x_i x_k \]
there exists a substitution \( T \) which
transforms \( R \) into the form \( x_1^2 + x_2^2 + \ldots + x_n^2 \). Now if
one applies \( T^{-1} \) to \( Q \), then \( Q \) is taken into \( R \).
If one then applies \( h \in H \) to \( R \) it leaves \( R \) invariant.
Finally applying \( T \) to \( R \) we obtain \( Q \). Hence \( Q \) is
invariant under the group \( THT^{-1} \). We can conclude then
that \( THT^{-1} \) is the desired orthogonal substitution group
equivalent to \( H \). Q.E.D.

Corollary: An abstract space group \( G \) can, by an affine
transformation, be transformed into a group of isometries.

Proof: The corollary follows from the following:

(i) The translation subgroup \( T(G) \) of \( G \) is
a group of isometries.
(ii) The translation subgroup of the full affine group $A_n$ is a normal subgroup of $A_n$. Thus any conjugate in $A_n$ of $T(G)$ is a translation group.

(iii) Each element $g \in G$ can be expressed in the form $g = \psi(t) \cdot S(a)$ ($t \in T(G)$, $a \in F$).

(iv) The integral representation of $F$ is equivalent to an orthogonal representation.

It can be shown that the problem of finding all groups of integral matrices--integral groups--of degree $n$ can be reduced to the problem of investigating all lattices in $n$-dimensional space which possess particular symmetries (cf. Speiser [12]). In view of this observation and the foregoing discussion it is natural now to introduce the lattice concept.

Let $t_1, t_2, \ldots, t_n$ be $n$ linearly independent vectors. If we consider these $n$ vectors as issuing from a single point $0 \in E^n$, the $n$-dimensional euclidean space, and form all linear combinations

$$x_1 t_1 + x_2 t_2 + \ldots + x_n t_n \quad (x_i \in \mathbb{Z})$$

then we have an $n$-dimensional space lattice. The length of the vectors $t_i$ will be designated by $\sqrt{a_{ii}}$ and

$$\sqrt{a_{ii}} \sqrt{a_{kk}} \cos(t_i, t_k) = a_{ik} = a_{ki}.$$
In this manner the square of the length of the vector

\[ x_1 t_1 + x_2 t_2 + \ldots + x_n t_n \]

will be given by the quadratic form

\[
\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} x_i x_k .
\]

On the other hand we can always associate an n-dimensional lattice with any positive definite quadratic form in n variables. This follows from the fact that when we know the quadratic form we can calculate the length of the fundamental vectors and the angles between them. The lattice associated with a quadratic form is defined up to its position in space and some symmetry operation.

The variables \( x_1, x_2, \ldots, x_n \) are the coordinates of the endpoint of the vector \( x_1 t_1 + x_2 t_2 + \ldots + x_n t_n \) in the coordinate system formed by the vectors \( t_1, t_2, \ldots, t_n \), each considered as a unit vector in its own direction. Since (1) can be written in matrix form we can associate with each quadratic form a matrix \( (a_{ik}) \). The determinant of \( (a_{ik}) \) gives the square of the content of the fundamental parallelepiped of the associated lattice.

Since every n-dimensional space group \( G \) contains \( n \) independent translations (according to Bieberbach's results) we can associate with each such group an
n-dimensional space lattice. Let \( t_1, t_2, \ldots, t_n \) be the \( n \) independent translations contained in a space group \( G \).

By a suitable choice of coordinate systems we can associate with each \( t_i \) a substitute of the form

\[
x'_i = x_{i+1}, \quad x'_k = x_k \quad (k=1) \quad i = 1, 2, \ldots, n.
\]

In the same coordinate system any symmetry of the lattice can be expressed by the equations

\[
x'_k = a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n + a_k \quad (k = 1, 2, \ldots, n).
\]

Thus with each symmetry contained in \( G \) we can associate a linear substitution which has the matrix form

\[
\begin{pmatrix}
H(X)_n & t(X) \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

\( H(X)_n \in GL(n, \mathbb{Z}) \); \( \det(H_n(X)) = \pm 1 \).

The multiplication rule is given by:

\[
\begin{pmatrix}
H_1(X)_n & t_1(X) \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
H_2(X)_n & t_2(X) \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix} = \begin{pmatrix}
K(X)_n & t'(X) \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

where

\[
K(X)_n = H_1(X)_n H_2(X)_n \quad \text{and}
\]

\[
t'(X) = H_1(X)_n t_2(X) + t_1(X).
\]
This group of matrices forms a group $\overline{G}$ isomorphic to $G$. The subgroup $H \subseteq \overline{G}$ composed of matrices of the form

$$
\begin{pmatrix}
H(X)_n & 0 \\
\vdots & \vdots \\
0 & 0
\end{pmatrix}
$$

is called the homogeneous or rotative part of $\overline{G}$ and the subgroup $T \subseteq \overline{G}$ composed of matrices of the form

$$
\begin{pmatrix}
I_n & t(X) \\
0 & 0 \ldots \ldots 0 & 1
\end{pmatrix}, \quad I_n = n \times n \text{ identity matrix}
$$

is the translative group of $\overline{G}$.

It should be pointed out that more than one $n$-dimensional lattice can be associated with a particular $n$-dimensional space group. This depends on the quadratic forms left invariant by the point group of $G$. E.g. In 2 dimensions, the point group $C_{2v}$ has two generators, a half-turn, $C_2$, and a reflection in a vertical axis in the plane, $\sigma_v$, such that

$$C_2^2 = \sigma_v^2 = I, \quad C_2 \sigma_v = \sigma_v C_2.$$
The only integral matrices which possess the same properties and have determinants \( \pm 1 \) are:

\[
M(C_2) = \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix},
\]

\[
M(\sigma_v) = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \text{ or } \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

With the second choice for \( M(\sigma_v) \), the basis vectors of the lattice must be equal in length but may possess arbitrary inclination. With the first choice for \( M(\sigma_v) \) the basis vectors must have inclination \( \frac{\pi}{2} \) but may differ in length. For further treatment of compatibility of lattices and point groups we make reference to Speiser \([13]\) or McWeeny \([10]\).

On the other hand if we are given an \( n \)-dimensional space lattice we can always associate with it a unique space group, namely the group \( G \) of all the symmetries of this lattice. \( G \) contains the full translation group of the lattice together with the set of all operations \( H \)

\[
\left\{ \begin{pmatrix}
H(X)_n \\
\vdots \\
0
\end{pmatrix} \right\} = H
\]

which by reflection or rotation about the origin bring the lattice into self-coincidence. This latter set which
forms a group is called the **point group** or the **holohedry** of the lattice. By combining $H$ with the group of translations $T$ we can find a group isomorphic to $H$ in $G$ for each point of the lattice—point groups exist also for certain points inside the fundamental parallelepiped of the lattice—e.g.: The hexagonal lattice in the plane generated by two linearly independent vectors, $\tau_1$ and $\tau_2$, of equal length and inclined at angle of $\frac{\pi}{3}$. The point group or holohedry of this lattice is $D_6$, the dihedral group of order 12 which is generated by three reflections with the relations

$$R_1^2 = (R_1 R_2)^3 = (R_2 R_3)^6 = (R_3 R_1)^2 = I.$$
In the plane there are five lattices: (1) the general parallelogram lattice, (2) the quadratic, (3) the rectangular, (4) the hexagonal, and (5) the rhombic or face-centered rectangular lattice. With these five lattices we can associate four holohedral groups or full point groups which consist of all symmetries of the lattice which leave the origin fixed. From the example previously given, we know that (3) and (5) have the same holohedral group $C_{2v}$. There are seventeen plane space groups. Of these the dihedral groups of order 8 and 12 are maximal in the sense that they contain the rest of the seventeen as subgroups. For a presentation of these seventeen groups and a table of the subgroup relationships we refer to Coxeter [4].

In 3-dimensional space there are fourteen distinct lattices (Bravais lattices) which belong to seven crystal systems or syngonies. Distinct lattices which have the same holohedry are said to belong to the same crystal system or syngony. We list the seven crystal systems: (1) triclinic, (2) monoclinic, (3) orthorhombic, (4) trigonal or rhombohedral, (5) hexagonal, (6) tetragonal, and (7) cubic. These seven systems are each determined by metrical matrices arising out of certain positive definite quadratic forms. Each crystal
system admits a **primitive lattice** based on **primitive** vectors associated with a certain metrical matrix. E.g., a metrical matrix of the form

\[
\begin{pmatrix}
c & -c/2 & 0 \\
-c/2 & c & 0 \\
0 & 0 & d
\end{pmatrix}
\]

which has

\[
a_{11} = a_{22} ,
\quad a_{21} = a_{12} ,
\quad a_{13} = a_{31} = 0 ,
\quad \text{and} \quad a_{32} = a_{23} = 0
\]

determines a hexagonal lattice with basis vectors \( v_1, v_2, v_3 \) where \( v_1 \) and \( v_2 \) are of equal length, inclined at an angle of \( \frac{2\pi}{3} \) and in a plane perpendicular to \( v_3 \). The holohedry is of course \( D_6 \).

There are 230 3-dimensional space groups, 73 of which are semi-direct products of their rotational (homogeneous) and translational parts. The structure of the remaining 157 space groups is characterized by means of the theory of group extensions. No space group is a direct product of rotations and translations.

We summarize the foregoing discussion by observing that a space group in \( n \)-dimensions appearing in the extension

\[
1 \rightarrow T \xrightarrow{\varphi} G \xrightarrow{\sigma} F \rightarrow 1
\]
can be given a geometric interpretation as follows: The maximal, free-abelian, normal subgroup $T$ of rank $n$ in $G$ can be associated with an $n$-dimensional lattice $L$ with the translations of $T$ as basis vectors. The group $F \cong G/T$, often termed an abstract crystallographic point group, can be associated with some subgroup of the full point group or holohedry of $L$, i.e. the group of \textit{proper} (determinant = 1) and \textit{improper} (determinant = -1) rigid motions of $L$ which leave the origin of $L$ fixed and bring $L$ into self-coincidence. We shall lean very heavily on this interpretation in the ensuing developments.

We conclude this chapter with the yet-unpublished result of Brown, Neubüser, Zassenhaus [3] that there are 710 point groups and approximately 4200 space groups in 4-dimensional space.
CHAPTER II
DISCRETE EUCLIDEAN UNIVERSE

In this chapter we shall be concerned primarily with three central concepts. Firstly we consider discrete groups, secondly we deal with concepts of fundamental set and fundamental region, and thirdly we treat the notion of a finite neighborhood of a point in a discrete space defined by a certain kind of cell which we will call a Voronoi cell. This will eventually lead us to a definition of a Discrete Euclidean Universe.

A topological group is a set $G$ which is at the same time an abstract group and a topological space where $g_n h_n^{-1} \rightarrow g h^{-1}$ whenever $g_n \rightarrow g$ and $h_n \rightarrow h$ ($g_n, h_n, g, h \in G$). This definition is not the most general one but since we are operating in a metric space it is sufficient for our purposes. The products are defined because $G$ is a group and convergence is defined because $G$ is a topological space.

A topological group is called discrete if none of its elements is an accumulation point in the topological space $G$. Since we will be concerned with groups which have matrix representations we define a topology for a matrix group $G$ in the following manner: Let $G = \{ g \}$ be a group of matrices $g = (a_{ij})$, $i, j = 1, 2, \ldots, n$. 

21
We will assume that the matrix coefficients come from a complete metric space. We equip this space of matrices with a metric by defining the distance
\[ d(g_1, g_2) = \max \left\{ \left| a_{ij}^{(1)} - a_{ij}^{(2)} \right| \right\}, \quad i, j = 1, 2, \ldots, n. \quad (g_1, g_2 \in G). \]
It is clear that convergence in \( G \) reduces itself to the elementwise convergence:
\[ g_h \rightarrow g, \quad h \rightarrow \infty, \quad \iff a_{ij}^{(h)} \rightarrow a_{ij}, \quad h \rightarrow \infty \]
for each \( i, j = 1, 2, \ldots, n \).

We shall say that a matrix group is discrete if and only if it contains no convergent sequence of distinct matrices.

Let \( G \) be any group of isometries acting on \( E^n \). Let \( P \) be any point in \( E^n \).

Definition: An orbit of \( P \) under \( G \), denoted \( G(P) \), is a set defined by
\[ G(P) = \left\{ x \in E^n \mid \sigma(x) = P \text{ for some } \sigma \in G \right\}. \]
Thus \( G(P) \) denotes the set of points in \( E^n \) equivalent to \( P \) under \( G \).

Definition: An orbit of \( G \) in \( E^n \) is a pointset \( G(P) \) for some \( P \in E^n \).

Any group \( G \) of one-to-one mappings of the topological space \( E^n \) onto itself will partition that space into mutually disjoint equivalence classes. It is
clear that we have identified these classes with the orbits of $G$.

Definition: A set $F \subseteq \mathbb{E}^n$ which contains one and only one representative from each orbit is called a fundamental set.

No two points of $F$ are equivalent under $G$ and each point of $\mathbb{E}^n$ is equivalent to some point of $F$ under $G$. Fundamental sets exist for all groups of isometries of $\mathbb{E}^n$ but they are not unique. Indeed if $F$ is a fundamental set and $S$ any subset of $F$, then $(F \setminus S) \cup \sigma(S)$, $(\sigma \in G)$, is also a fundamental set. A fundamental set for the full group of euclidean motions consists of just one point. If $F$ is a fundamental set then $\sigma(F)$ is a fundamental set for each $\sigma \in G$.

A fundamental set $F$ cannot be an open subset of $\mathbb{E}^n$ unless $F = \mathbb{E}^n$. For if $F$ is open and $F \not\subseteq \mathbb{E}^n$ then the boundary of $F$ is not empty. Hence $F$ must contain interior points equivalent to points on its own boundary but this is impossible. A more convenient concept than that of fundamental set is that of fundamental region.

Definition: A fundamental region for a group $G$ relative to $\mathbb{E}^n$ is a nonempty open subset $R$ of $\mathbb{E}^n$
containing no distinct \( G \)-equivalent points and every neighborhood of a boundary point of \( R \) contains a point in \( E^n \sim R \) equivalent to a point in \( R \).

The connection between a fundamental set and a fundamental region for a group \( G \) relative to \( E^n \) can be explained as follows:

Let \( \bar{R} \) be the closure of \( R \). Suppose \( G(\bar{R}) = \bigcup_{\sigma \in G} \sigma(\bar{R}) \) covers \( E^n \). If we adjoin to \( R \) exactly one point from each equivalence class \( Gx_\alpha \cap \bar{R} \) where \( x_\alpha \) runs through the boundary of \( R \) we obtain a fundamental set \( F \) for \( G \) relative to \( E^n \) and \( R \subset F \subset \bar{R} \).

It can be shown that a maximal nonempty open subset of \( E^n \) containing no distinct \( G \)-equivalent points is a fundamental region for \( G \). (cf. Lehner [8]). This characterization is useful in proving the existence of a fundamental region. For if \( G \) is a group and if there exists a point \( P \in E^n \) such that \( P \) is contained in an \( n \)-dimensional ball \( B_n \) which contains no distinct \( G \)-equivalent points then by taking the nonempty family of open sets which contains \( B_n \) but no distinct \( G \)-equivalent points and applying Zorn's Lemma we can be assured of the existence of a fundamental region for \( G \) in \( E^n \).
Theorem: If \( G \) is a discrete group of isometries and if there exists a point \( P \in \mathbb{E}^n \) such that the stabilizer in \( G \) of \( P \) consists of the identity only then \( G \) possesses a fundamental region in \( \mathbb{E}^n \) with \( P \) as an interior point. Conversely if a group \( G \) of isometries possesses a fundamental region \( R \) then the stabilizer of any interior point \( P \) in \( R \) consists of the identity only.

Proof: Let \( G \) be discrete and \( G_P = 1 \). In view of the remarks at the end of the preceding paragraph all we have to show is that there exists a ball \( B_P = B(P; \varepsilon) \) which contains no distinct \( G \)-equivalent points. Let us assume the contrary. Let \( B_k = B(P; \frac{1}{k}) \). Then for each \( k \) there exists a pair of points \( x_k, y_k \) \((x_k \neq y_k)\) and there exists \( g_k \in G \) such that \( x_k, y_k \in B_k \) and \( g_k(x_k) = y_k \). Thus we have \( x_k \rightarrow P \), \( g_k(x_k) = y_k \rightarrow P \) as \( k \rightarrow \infty \). Let \( B_1 = B(P; 1) \) and \( B = B(P; 3) \). Then for each \( g_k \) we have \( g_k(B_1) \subseteq B \) because \( g_k(x_k) = y_k \), \( x_k, y_k \in B_1 \), and \( g_k \) is an isometry. Let \( g_k \mid_{\overline{B_1}} = g_k^* \) be the restriction of \( g_k \) to \( \overline{B_1} \). The set of isometries \( \{g_k^*\}_{k=1}^{\infty} \) is now a discrete set of functions and hence is closed. We have all the requirements of Ascoli's theorem, hence there exists a subsequence
\left\{ g^i_{k,j} \right\} \infty \text{ of } \left\{ g^i_k \right\} \infty \text{ which converges uniformly to a continuous function } g^0_i \text{ on } \overline{B}_1 \text{ and } g^0_i = g^i_k \text{ for some } k \in \mathbb{Z} \text{ because of closure. We have } g^0_i(p) = p \text{ because } |p - g^0_i(p)| = |p - y_{k_j}| + |g^i_{k,j}(x^i_{k,j}) - g^i_{k,j}(p)| + |g^i_{k,j}(p) - g^0_i(p)| \text{ and each term on the right can be made less than } \frac{\varepsilon}{3} \text{ by choosing } j > M \text{ for some large integer } M. \text{ Hence we can conclude that } g^0_i(p) = I^i_G(p) = P. \text{ Because of discreteness } g^i_{k,j} = g^0_i = I^i_G \text{ for all sufficiently large } j. \text{ Therefore } g^i_{k,j}(x^i_{k,j}) = g^i_{k,j}(x^i_{k,j}) = x^i_{k,j} = y^i_{k,j} \text{ for all sufficiently large } j. \text{ But this contradicts the assumption that } x^i_k \neq y^i_k \text{ for all } k. \text{ Therefore there exists a ball } B_P = B(P; \varepsilon) \text{ such that } B_P \text{ contains no distinct } G\text{-equivalent points and the first part of the theorem is proved.}

The converse is trivial because if } G_P \geq 1 \text{ then each element in } G_P \text{ takes any ball about } P \text{ into itself. Thus every ball with } P \text{ as center contains distinct } G\text{-equivalent points. This would contradict the fact that } R \text{ is a fundamental region.}

We now give the following theorem:
Theorem: Let } G \text{ be an } n\text{-dimensional space group appearing in the extension } 1 \rightarrow T \rightarrow G \rightarrow F \rightarrow 1.
We assert that $G$ has the following properties:

(i) $G$ is a discrete group

(ii) $G$ is a countable group

(iii) $G$ possesses a bounded fundamental region in $E^n$.

Proof: (i) Bieberbach has shown that the free abelian group $T$ is isomorphic to a translation group containing $n$ independent translations. $T$ is clearly discrete. Any finite extension of a discrete group is itself discrete. For if there were a convergent sequence of distinct elements in the finite extension then there would have to be a convergent subsequence of distinct elements in one of the finitely many cosets of the factor group $F$. In that case we could take a fixed representative of that coset, say $\alpha$, and multiply each element of the convergent subsequence by $\alpha^{-1}$. The result would be a convergent sequence of distinct elements all now lying in $T$. This is a contradiction. Therefore the finite extension is discrete and we have proved (i).

(ii) We have shown in Chapter I that each element $g$ of $G$ can be represented as a product $v(t) \cdot S(a)$; $t \in T$, $a \in F$. Since $T$ is countable and $F$ is finite we have property (ii).

(iii) We first show that $G$ possesses a fundamental region. In view of what we have shown already—namely,
that if there exists a point in $E^n$ whose stabilizer in $G$ consists of the identity only, then $G$ possesses a fundamental region—now all we have to show is the existence of such a point. We do this as follows:

$G$ is countable by (ii). Let $g \in G$ and $g \neq I$. Let $F_g$ be the set of points in $E^n$ which are fixed by $g$. Since $G$ is a group of isometries $F_g$ is a closed subset of $E^n$ possessing an empty interior. The family $\mathcal{F} = \{F_g|\ g \in G, g \neq I\}$ is a countable family of nowhere dense subsets of $E^n$. By the Baire Category Theorem we can conclude that

$$\bigcup_{g \in G, g \neq I} F_g \neq E^n.$$ 

Hence there exists a point $P \in E^n$ such that the stabilizer in $G$ of $P$ consists of the identity only. Therefore $G$ possesses a fundamental region $R$.

Next we show that $R$ is bounded. This follows from the fact that $G$ contains $n$ independent translations. For if we first take a fundamental region of the translation subgroup $T$ in $G$ we obtain the interior of a parallelepiped $\bar{W}$ in $E^n$. Every point of $E^n$ is equivalent under $T$ to a point either in the interior of $\bar{W}$ or on the boundary of $\bar{W}$. Now let $F$ be a fundamental set for $G$ with nonempty interior. Every point of $F$ is equivalent under $T$ to some point in $\bar{W}$. Thus by means of the translation subgroup $T$ of $G$ we can build a fundamental set $F'$ for
G which is contained completely in \( \overline{\mathbb{R}} \). Hence \( F' \) is bounded. Since \( \text{Int } F \neq \emptyset \) we can assume that \( \text{Int } F' \neq \emptyset \).

Now we choose as fundamental region for \( G \) the maximal open set \( R \) contained in \( F' \) such that \( R \subset F' \subset \overline{\mathbb{R}} \).

Clearly \( R \) is bounded and we have established property (iii). Q.E.D.

**Definition:** Let \( G \) be a space group in \( n \) dimensions. Let \( P \in \mathbb{E}^n \). We shall call an orbit of \( P \) under \( G \), namely \( G(P) \), a **Discrete Euclidean Universe** and denote it as DEU.

Suppose now we have a DEU and \( P \in \text{DEU} \). We proceed to define a finite neighborhood of \( P \) in DEU. For this purpose we will make use of an extension of the concept of normal polygon (cf. Lehner [9], Fricke-Klein [5]). We will eventually arrive at an \( n \)-dimensional convex cell \( V(P) \) associated with \( P \).

Let \( S \) be any discrete subset of \( \mathbb{E}^n \) and \( P \in S \) define \( V(P,S) \) to be that subset of \( \mathbb{E}^n \) which consists of all points \( X \in \mathbb{E}^n \) such that

\[
d(P,X) \leq d(Q,X) , \quad (Q \in S , \ Q \neq P)
\]

where \( d(P,X) \) denotes the ordinary euclidean distance from \( P \) to \( X \). Thus \( V(P,S) \) consists of those points \( X \in \mathbb{E}^n \) which are closer to \( P \) than they are to any other point of \( S \). \( V(P,S) \) can be defined constructively as follows:
Let \( P, Q \in S \), \( Q \neq P \). Let \( \gamma_{PQ} \) be the \((n-1)\)-dimensional subspace of \( E^n \) such that 
\( Y \in \gamma_{PQ} \iff d(P,Y) = d(Y,Q) \). Thus for each point 
\( Q \in S \setminus P \), \( \gamma_{PQ} \) determines two half-spaces of \( E^n \) one 
of which contains \( P \) and the other \( Q \). Let \( H(P,Q) \) 
be the closed half-space determined by \( \gamma_{PQ} \) which 
contains \( P \). Then

\[
V(P,S) = \cap \{ H(P,Q) \mid Q \in S \& Q \neq P \} .
\]

Definition: A neighborhood of \( P \), denoted by \( \mathcal{N}(P) \), 
in \( S \) is the set

\[
\mathcal{N}(P) = \left\{ Q \in S \setminus P \mid Q \text{ determines an (n-1)-dimensional} \right. 
\left. \text{bounding face of } V(P,S) \right\} .
\]

This definition says, in effect, that \( Q \in \mathcal{N}(P) \) is 
not redundant with respect to the determination of 
\( V(P,S) \). The nature of \( \mathcal{N}(P) \) depends greatly on the 
nature of \( S \). However, if we restrict ourselves to 
the case where \( S \) is a DEU then \( V(P,DEU) \) which we 
shall refer to as the Voronoi cell of \( P \) in DEU--since 
it is analogous to the Voronoi cell in an \( n \)-dimensional 
lattice (cf. Voronoi \([14]\))--will have \textit{finitely} many 
bounding \((n-1)\)-dimensional faces and \( \mathcal{N}(P) \) will be 
finite.

We now discuss some of the properties possessed 
by a Voronoi cell \( V(P,DEU) \), \( P \in DEU \).
(1) \( \text{Int } V(P, \text{DEU}) \) is an open set in \( E^n \). We show that \( \text{Int } V(P, \text{DEU}) \) is not empty. Let \( B_n \) be an \( n \)-dimensional ball about \( P \) of radius \( \frac{\varepsilon}{4} \) where 
\[ \varepsilon = \min\{d(P,Q) \mid Q \in \text{DEU}, Q \neq P\} \]. Now 
\[ d(P, Y_{PQ}) = \frac{1}{2}d(P,Q) \geq \frac{\varepsilon}{2} \]. Hence \( B_n \subseteq V(P, \text{DEU}) \) and \( \text{Int } B_n \subseteq \text{Int } V(P, \text{DEU}) \).

(2) \( V(P, \text{DEU}) \) is convex since each \( H(P,Q) \) is convex and the intersection of the \( H(P,Q) \)'s is convex.

(3) \( V(P, \text{DEU}) \) is connected. This is immediate from (2).

(4) \( V(P, \text{DEU}) \) has finite content since a space group contains \( n \) independent translations.

(5) \( \text{Int } V(P, \text{DEU}) \cap \text{Int } V(Q, \text{DEU}) = \emptyset \) whenever \( P \neq Q \).

Proof: Suppose that \( X \in E^n \) and that \( X \) belongs to the intersection. Since \( X \in V(P, \text{DEU}) \) we have 
\[ d(P,X) = d(Q,X) \] and since \( X \in V(Q, \text{DEU}) \) we have 
\[ d(Q,X) = d(P,X) \]. This statement is contradictory.

(6) \( V(P, \text{DEU}) \) is congruent to \( V(Q, \text{DEU}) \) ; \( V \) \( Q \in \text{DEU} \).

Proof: \( G \) acts transitively on \( \text{DEU} \) hence there is a \( \sigma \in G \) such that \( \sigma(V(P, \text{DEU})) = V(Q, \text{DEU}) \). But \( \sigma \) is an isometry therefore the cells must be congruent.

(7) \( \text{Int } V(P, \text{DEU}) \) contains no distinct \( G \)-equivalent points whenever the stabilizer of \( P \).
in \( G \) consists of the identity alone. For suppose \( X_1, X_2 \in \text{Int} V(P, \text{DEU}) \) and that \( \sigma(X_1) = X_2 \) for some \( \sigma \in G \). Now \( \sigma \) maps \( V(P, \text{DEU}) \) onto some \( V(Q, \text{DEU}) \), \( (P \neq Q) \), and consequently \( X_2 \in \text{Int} V(Q, \text{DEU}) \). This contradicts (5).

(8) \( V(P, \text{DEU}) \) possesses a finite number of bounding \((n-1)\)-dimensional faces.

Proof: Consider all Voronoi cells \( V(Q, \text{DEU}) \), \( Q \in \text{DEU} \), having a point in common with \( V(P, \text{DEU}) \). Let \( \delta \) be the diameter in \( E^n \) of \( V(P, \text{DEU}) \). From (4) we conclude \( \delta < \infty \). The union of all cells which have a point in common with \( V(P, \text{DEU}) \) has diameter of at most \( 3\delta \) and has finite content. Now it follows immediately from (5) and (6) that the number of cells having a point in common with \( V(P, \text{DEU}) \) is finite.

(9) The union, \( \bigcup \{ V(P, \text{DEU}) \mid P \in \text{DEU} \} \), covers \( E^n \) and \( V(P, \text{DEU}) \) intersects \( V(Q, \text{DEU}) \) in at most common boundary points.

(10) \( \text{Int} V(P, \text{DEU}) \) is a fundamental region for \( G \) in \( E^n \) provided that the stabilizer of \( P \) in \( G \) consists of the identity alone. This is a consequence of (5), (7), and (9).

Theorem: The set \( \mathcal{N}(P) \) is finite.

Proof: This is an immediate consequence of property (8).

Remark: \( P \in \mathcal{N}(P) \iff Q \in \mathcal{N}(P) \). Hence the neighborhood relation is symmetric.
It follows now that we can associate with each DEU arising out of a space group \( G \) a Voronoi cell complex which covers \( \mathbb{E}^n \) without overlapping and without gaps. We will now give some examples:

(1) The most fundamental examples of Voronoi cell complexes in \( \mathbb{E}^n \) arise out of \( n \)-dimensional lattices and the cells are the genuine Voronoi cells. In the quadratic lattice the Voronoi cell is a square, in the hexagonal lattice it is a regular hexagon, in the cubic lattice it is a cube, in the body centered cubic lattice it is a cell bounded by six regular hexagonal and four square plane faces. It has been shown by Voronoi [14] that the number of bounding \((n-1)\)-dimensional faces of a cell in an \( n \)-dimensional lattice never exceeds \( 2(2^n-1) \). The interiors of these cells always yield fundamental regions. Figure 1 indicates a hexagonal cell in a general 2-dimensional lattice.

Figure 1
(2) The face centered rectangular lattice generated by two orthogonal translations of unequal length in $E^2$ and a glide reflection in the horizontal lattice lines yields the following configuration:

Figure 2

(3) The full group of symmetries, $C_{6v}$, for the hexagonal lattice together with translations of that lattice yields the following Voronoi cell configuration when $P \in E^2$ is chosen so that its stabilizer in $C_{6v}$ consists of the identity only. The cell determines a fundamental region which is a $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$ triangle.
It has been pointed out that the cell configuration depends not only on the generating group $G$ but also on the choice of $P \in \mathbb{R}^n$. Hence:

(4) As a final example we choose the group $C_6$. This group is a normal subgroup of $C_{6v}$ of index 2. The fundamental cell of the previous example plays an important role in the choice of $P$ determining an orbit for $C_6$. Figure 4 indicates the chosen position of the fundamental cell for $C_{6v}$ in the parallelogram which is the fundamental parallelogram of the hexagonal lattice.

Figure 3

Figure 4
In Figures 5a through 5g: the position of $P$ relative to the fundamental cell of $C_{6v}$ and also to the fundamental parallelogram of the hexagonal lattice is indicated by $\circ$ or $\rightarrow\leftarrow$, the latter symbol meaning that $P$ may take any position along the length of the double-headed arrow and yet the same Voronoi cell will be determined; the bold-faced polygon indicates the Voronoi cell determined by an orbit of $C_6$ with this choice of $P$; the polar coordinates $(r, \alpha)$ indicate the position of $P$ in the fundamental parallelogram of the lattice determined by the hexadic points in the chosen $C_6$ group.

Figure 5a
Figures 5b, 5c, 5d

\[ \alpha = \frac{\pi}{6} \]
\[ r = \frac{\sqrt{3}}{3} \]

\[ \alpha = 0 \]
\[ r = \frac{1}{2} \]

\[ \alpha = 0 \]
\[ 0 < r < \frac{1}{2} \]
\[ r \cos \alpha = \frac{1}{2} \]
\[ 0 < \alpha < \frac{\pi}{6} \]
\[ \frac{1}{2} < r < \frac{\sqrt{3}}{3} \]

Figure 5e

\[ \alpha = \frac{\pi}{6} \]
\[ 0 < r < \frac{\sqrt{3}}{3} \]

Figure 5f

\[ \alpha = \frac{\pi}{18} \]
\[ r = 0.4 \]

Figure 5g
The complex generated by the fundamental cell of Figure 5g is shown in the following diagram:

Figure 6

This example involving the Voronoi cells (Figures 5a through 5g) associated with a DEU generated by $C_6$ deserves the following comments:

1. When the point $P$ is chosen on a fixed symmetry element of $C_6$, then the area of the Voronoi cell increases and it is not a fundamental cell. If the
subgroup of $C_6$ fixing $P$ is of order $\lambda$ then the area of the Voronoi cell is $\lambda$ times the area of a fundamental cell for $C_6$.

2. When the point $P$ is chosen on a fixed symmetry element of the normalizing group, $C_{6v}$, of $C_6$ then the associated DEU can just as well be considered as having been generated by $C_{6v}$. This suggests the existence of a minimal generating group for a DEU as well as a maximal rigid symmetry group for the same DEU. In Figure 5a for instance the minimal generating group is simply the translation group and the maximal symmetry group is $C_{6v}$. The Voronoi cell, namely the regular hexagon, is a fundamental cell for the minimal group.

We conclude this chapter with the observation that when we speak of dimension in a Voronoi cell complex we are using the term in the sense that it has been defined by Hurewicz and Wallman [7]. Consequently, we have the following theorem at our disposal.

*Theorem:* $E^n$ cannot be disconnected by a subset of dimension less than or equal to $n-2$.

The proof of this theorem can be found in [7].
Corollary 1: \( \text{Int } I^n \) cannot be disconnected by a subset of dimension \( n-j \), \( 2 \leq j \leq n \), where \( I^n \) is the closed cube in \( E^n \).

Corollary 2: Any \( n \)-dimensional ball \( B_n \) in \( E^n \) cannot be disconnected by a subset of dimension \( n-j \), \( 2 \leq j \leq n \).

This follows immediately from the theorem and Corollary 1 since \( \text{Int } B_n \) is homeomorphic to \( \text{Int } I^n \).

We have now as a consequence the theorem:

Theorem: Let \( U,V \) be any two points in \( E^n \) and let \( \mathcal{C} \) be an \( n \)-dimensional Voronoi complex associated with some \( \text{DEU} \). There exists a continuous path meeting \( n \)-cells and \( (n-1) \)-cells only which connects \( U \) with \( V \); moreover, this path lies completely in the interior of an \( n \)-dimensional ball \( B_n \) having the segment \( UV \) as its diameter.

Proof: We have only to show that there are finitely many \( (n-j) \)-dimensional cells, \( 2 \leq j \leq n \), contained in \( B_n \). But this is clear since each \( n \)-cell has a fixed content and only a finite number of such cells can intersect \( B_n \). The number of \( (n-j) \)-dimensional cells, \( 2 \leq j \leq n \), each of which belongs to some \( n \)-cell, is therefore finite and the theorem follows.
CHAPTER III
COMBINATORIAL AUTOMORPHISMS IN A DEU

We have defined a neighborhood relation on a DEU in Chapter II. We have seen that each point $P \in \text{DEU}$ possesses a finite number of neighbors. It is clear that all rigid symmetries of a DEU preserve this neighborhood relation. In this chapter we will extend the notion of a rigid automorphism or symmetry of a DEU which preserves the neighborhood relation in question to that of a non-rigid combinatorial automorphism which preserves the same neighborhood relation. In order to achieve this end we fix our attention on the Voronoi cell complex associated with each DEU. In the two dimensional case the Voronoi cell complex is nothing more than an infinite graph on the plane. The permutations on the faces of such a graph which preserve all incidences in the graph come into consideration. We will consider such permutation groups not only for two dimensional complexes but also for higher dimensions. Eventually we will produce a group of automorphisms which preserve the neighborhood relation in a DEU and investigate the relation between this group and an abstract space group.

42
We shall now formalize these notions by introducing the concept of an admissible neighborhood-preserving combinatorial automorphism on the Voronoi cell complex associated with a DEU. This is best done by first introducing a very useful group which will be referred to as Wilson's Group \[15\].

Wilson's Group

Let \( \mathbf{b} \) be a complex associated with some discrete euclidean universe.

Definition: A tower in \( \mathbf{b} \) is an \((n+1)\)-tuple \((s_0, s_1, ..., s_n)\) associated with the closure \( \overline{s_n} \) of the \(n\)-cell \( s_n \) where \( s_0 \) is a 0-cell (vertex), \( s_1 \) a 1-cell (edge), \( s_2 \) a 2-cell (face), etc., and \( s_0 \subset s_1 \subset ... \subset s_n \). Let \( \mathcal{J}(\mathbf{b}) \) be the set of all towers in \( \mathbf{b} \). We will construct a group \( \Sigma \) for which \( \mathcal{J}(\mathbf{b}) \) will be a homogeneous \( \Sigma \)-space. The construction of this group will be carried out explicitly in the 2-dimensional case and its extension to higher dimensions will become clear. We define the following three operations on the set \( \mathcal{J}(\mathbf{b}) \):

1) An involution \( \sigma_0 \) which interchanges two adjacent vertices of an edge \( e \) but leaves both edge and face fixed. More explicitly, let \((v, e, f)\) be a tower in \( \mathbf{b} \), \( v \subset \overline{e} \subset \overline{f} \). Let \( \overline{e} = (v, v') \). Then \( \sigma_0(v, e, f) = (v', e, f) \) and \( \sigma_0^2(v, e, f) = (v, e, f) \).

![Diagram](image.png)
2) An involution $\sigma_1$ which interchanges a pair of edges incident with the vertex $v$ but leaves the vertex and face fixed. If $e$ and $e'$ are incident with $v$, then $\sigma_1(v,e,f) = (v,e',f)$ and $\sigma_1^2(v,e,f) = (v,e,f)$.

3) An involution $\sigma_2$ which leaves the vertex and edge of $T$ fixed but interchanges two faces with the common edge $e$. Again if $f$ and $f'$ are incident with $e$ then $\sigma_2(v,e,f) = (v,e,f')$ and $\sigma_2^2(v,e,f) = (v,e,f)$.

Now let $\Sigma = \langle \sigma_0, \sigma_1, \sigma_2 \rangle$, the group generated by the described operations. We can now define a mapping $\gamma: \Sigma \times J(b) \to J(b)$ by the correspondence $(\sigma,T) \leftrightarrow \sigma(T)$ for any $\sigma \in \Sigma$ and $T \in J(b)$. We have: (1) for any $T \in J(b)$, $1(T) = T$ and (2) for any $T \in J(b)$ and $\sigma, \sigma' \in \Sigma$, $(\sigma \sigma')(X) = \sigma(\sigma'(X))$. Consequently $J(b)$ is a left-space of $\Sigma$. Moreover we can show that $J(b)$ is a homogeneous left $\Sigma$-space, i.e., $\Sigma$ acts transitively.
on $\mathcal{J}(\mathcal{E})$. Let $T_1 = (v_1, e_1, f_1)$ and $T_2 = (v_2, e_2, f_2)$, $v_i \subset e_i \subset f_i$ $(i = 1, 2)$, be any two towers in $\mathcal{J}(\mathcal{E})$. We claim $\exists \sigma \in \Sigma$ such that $\sigma(T_1) = T_2$. The complex $\mathcal{E}$ and its dual are connected. Consequently we can find a path (a minimal path if we so wish) connecting $f_1 \in T_1$ with $f_2 \in T_2$. Let $f_1, h_1, h_2, \ldots, h_n, f_2$ be the chain of faces determined by this path.

By repeated application of $\sigma_0$ and $\sigma_1$ (both of which leave faces fixed) we can take $(v_1, e_1, f_1)$ into $(v_1', e_1', f_1')$ where $v_1' \subset e_1' \subset h_1$. Now an application of $\sigma_2$ to $(v_1', e_1', f_1')$ yields a tower associated with $h_1$, i.e., $\sigma_2(v_1', e_1', f_1') = (v_1', e_1', h_1)$.
If we repeat a similar process we can move this last tower into a tower on $h_2$, and so by an inductive procedure eventually reach $T_2$ associated with $f_2$. The result is a word $\sigma$ in $\Sigma$ such that $\sigma(T_1) = T_2$. Thus $\Sigma$ acts transitively on the towers and $\mathcal{J}(\mathcal{B})$ is a left homogeneous $\Sigma$-space.

In order to obtain all towers associated with a given vertex $v_0$ we consider the triple $(v_0, e, f)$ and allow $\Sigma_0 = \langle \sigma_1, \sigma_2 \rangle$ to operate on it. The orbit of $(v_0, e, f)$ under $\Sigma_0$ is thus the set of all towers with the common vertex $v_0$. In a similar manner we allow $\Sigma_1 = \langle \sigma_0, \sigma_2 \rangle$ to operate on the triple $(v, e_0, f)$ and get all towers associated with $e_0$. $\Sigma_2 = \langle \sigma_0, \sigma_1 \rangle$ applied to $(v, e, f_0)$ gives all towers associated with $f_0$. This all leads to the following correspondences:

\[
\begin{align*}
V(\mathcal{B}) &= \text{Vertices of } \mathcal{B} \quad \xrightarrow{1-1} \quad \text{Orbits of } \Sigma_0 \\
E(\mathcal{B}) &= \text{Edges of } \mathcal{B} \quad \xrightarrow{1-1} \quad \text{Orbits of } \Sigma_1 \\
F(\mathcal{B}) &= \text{Faces of } \mathcal{B} \quad \xrightarrow{1-1} \quad \text{Orbits of } \Sigma_2 
\end{align*}
\]

It is clear that the faces of $\mathcal{B}$ partition the towers in $\mathcal{J}(\mathcal{B})$. Conversely there is a projection from the towers in $\mathcal{J}(\mathcal{B})$ onto the faces. The quadruple $(\mathcal{J}(\mathcal{B}), \sigma_0, \sigma_1, \sigma_2)$ describes the complex $\mathcal{B}$ completely from the abstract combinatorial point of view.
In the three-dimensional case the set of towers will be described as a set of quadruples \((v,e,f,c)\), where \(v\) is a vertex, \(e\) an edge, \(f\) a 2-cell and \(c\) a 3-cell. To extend \(\Sigma\) we simply add another involution \(\sigma_3\) which interchanges 3-cells but leaves the vertex, edge, and 2-cell of a tower fixed. We then have 

\[ \Sigma = \langle \sigma_0, \sigma_1, \sigma_2, \sigma_3 \rangle \]

and the following correspondences:

\[ V(\ell) \xrightarrow{1-1} \text{Orbits of } \Sigma_0 = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \]
\[ E(\ell) \xrightarrow{1-1} \text{Orbits of } \Sigma_1 = \langle \sigma_0, \sigma_2, \sigma_3 \rangle \]
\[ F(\ell) \xrightarrow{1-1} \text{Orbits of } \Sigma_2 = \langle \sigma_0, \sigma_1, \sigma_3 \rangle \]
\[ C(\ell) \xrightarrow{1-1} \text{Orbits of } \Sigma_3 = \langle \sigma_0, \sigma_1, \sigma_2 \rangle \]

It is easy to see that \(J(\ell)\) is a left homogeneous \(\Sigma\)-space in the three dimensional case. The quintuple \((J(\ell), \sigma_0, \sigma_1, \sigma_2, \sigma_3)\) now describes in an abstract combinatorial manner the 3-dimensional Voronoi cell complex associated with a DEU. The extension to higher dimensions should be clear.

Note: It should be observed that when a tower 
\((s_0, s_1, \ldots, s_n)\) is taken into a tower \((s'_0, s'_1, \ldots, s'_n)\) by a word \(\sigma \in \Sigma\) it is not necessary that \(s_i\) and \(s'_i\) be congruent or even have the same number of bounding hyperplanes, only that they be of the same dimension.
Definition: An automorphism of a DEU complex $\mathcal{B}$ is a permutation $\alpha$ of the set of towers $\mathcal{J}(\mathcal{B})$ which commutes with each element of $\Sigma$.

It follows from this definition that $\alpha$ preserves all incidences in the complex $\mathcal{B}$. Hence $\alpha$ preserves the neighborhood relation we defined on the points of DEU. An admissible neighborhood-preserving automorphism of a DEU is one that fulfills the above definition.

Let $A = \text{Aut}(\text{DEU})$ denote the group of all such automorphisms. If $G$ is the generating group of a DEU then each element $g$ of $G$ fulfills the above definition because it permutes the n-cells and hence $g$ is a permutation on the towers of the complex associated with DEU which commutes elementwise with $\Sigma$. Consequently $G \subseteq A$ and since $G$ acts transitively on DEU, so does $A$. We will describe the action of $A$ on DEU in more detail later and give examples of certain DEU's where the generating group $G$ is properly contained in $A$. Before doing so we give the following very important theorem.

Theorem: Let $\mathcal{B}$ be a complex associated with some DEU and let $\mathcal{J}(\mathcal{B})$ be the towers of $\mathcal{B}$. Let $T_1, T_2 \in \mathcal{J}(\mathcal{B})$.

i) There exists at most one admissible automorphism $\alpha$ taking $T_1$ into $T_2$.

ii) There exists one such automorphism if and only if $\Sigma_{T_1} = \Sigma_{T_2}$ where $\Sigma_{T_1}$ denotes the stabilizer in $\Sigma$ of the tower $T_1$. 
Proof:

i) Suppose first that \( \alpha \) fixes \( T_1 \). Then 
\[ T_1^\alpha = T_1 \]. Let \( T \) be any other tower in \( \mathcal{J}(\mathcal{B}) \). Since \( \Sigma \) is transitive on \( \mathcal{J}(\mathcal{B}) \) there exists \( \sigma \in \Sigma \) such that 
\[ T_1^\sigma = T \] but then 
\[ T^\alpha = (T_1^\sigma)^\alpha = T_1^{\sigma \alpha} = T_1^{\alpha \sigma} = (T_1^\alpha)^\sigma = T_1^\sigma = T. \]
Therefore \( \alpha \) leaves every tower fixed and hence is the identity automorphism.

In general suppose \( T_1^\alpha = T_2 \) for some admissible automorphism \( \alpha \). Let \( T_3 \) be any other tower in \( \mathcal{J}(\mathcal{B}) \). Again because of the transitivity of \( \Sigma \) there exists \( \sigma \in \Sigma \) such that 
\[ T_1^\sigma = T_3 \]. Then 
\[ T_3^\alpha = T_1^{\sigma \alpha} = T_1^{\alpha \sigma} = T_2^\sigma. \]
Therefore the image of \( T_3 \) is determined when the image of \( T_1 \) is determined and this proves the first assertion of the theorem.

ii) Assume there exists an \( \alpha \) such that \( T_1^\alpha = T_2 \). Then 
\[ T_2^{\alpha^{-1}} = T_1 \]. Let \( \sigma \in \Sigma_{T_1} \). Then we have 
\[ T_2^\sigma = T_1^{\alpha \sigma} = T_1^\sigma = T_2. \]
Hence \( \sigma \in \Sigma_{T_2} \). On the other hand suppose \( \tau \in \Sigma_{T_2} \). Then 
\[ T_1^{\tau} = T_2^{\alpha^{-1}} \tau = T_2^{\tau \alpha^{-1}} = T_2^{\alpha^{-1}} = T_1. \]
Hence \( \tau \in \Sigma_{T_1} \).
Therefore \( \Sigma_{T_1} = \Sigma_{T_2} \).
Conversely, assume $\Sigma T_1 = \Sigma T_2$. We claim we can find an $\alpha$ such that $T_1^\alpha = T_2$. Let $H$ be a tower in $\mathcal{J}(\mathcal{B})$. Choose $\sigma \in \Sigma$ such that $T_1^\sigma = H$. Now define $H^\alpha = T_2^\sigma$. We show that $\alpha$ is well-defined (does not depend on $\sigma$). For if $T_1^\sigma = T_1^\sigma' = H$ then $T_1^{\sigma\sigma'^{-1}} = T_1$ and $\sigma\sigma'^{-1} \in \Sigma T_1$. Since $\Sigma T_1 = \Sigma T_2$ then $\sigma\sigma'^{-1} \in \Sigma T_2$. Consequently $T_2^\sigma = T_2^\sigma'$ and $\alpha$ is well-defined. If $H = T_1$ then $T_1^\alpha = T_2$.

Next we show that $\alpha$ commutes elementwise with $\Sigma$. Let $\tau \in \Sigma$. We claim $H^{T\alpha} = H^{\alpha T}$ ($\forall H \in \mathcal{J}(\mathcal{B})$). When defining $H^\alpha$, we considered $\sigma \in \Sigma$ such that $T_1^\sigma = H$ and $H^\alpha = T_2^\sigma$. Hence $H^{\alpha \tau} = T_2^{\sigma\tau}$. To find the image of $H^\tau$ under $\alpha$ pick $\sigma'$ such that $T_1^{\sigma'} = H^\tau$. We simply choose $\sigma' = \sigma \tau$ because $T_1^{\sigma\sigma'^{-1}} = H^\tau$. By definition of $\alpha$ we have $H^{\tau \alpha} = T_2^{\sigma\tau}$. Therefore $H^{\alpha \tau} = H^{\tau \alpha}$.

Finally we must prove that $\alpha$ is a permutation. Assume that $H_1^\alpha = H_2^\alpha$. We choose $\sigma_1$ such that $T_1^{\sigma_1} = H_1$ and $\sigma_2$ such that $T_1^{\sigma_2} = H_2$. By definition $H_1^\alpha = T_2^{\sigma_1}$ and $H_2^\alpha = T_2^{\sigma_2}$ so that $T_2^{\sigma_1} = T_2^{\sigma_2}$ implies $\sigma_1\sigma_2^{-1} \in \Sigma T_2 = \Sigma T_1$. Hence $T_1^{\sigma_1} = T_1^{\sigma_2}$. Therefore $H_1 = H_2$ and $\alpha$ is a one-to-one map on $\mathcal{J}(\mathcal{B})$.

We show that $\alpha$ is an onto map. Suppose that $T^\alpha = K$, $T, K \in \mathcal{J}(\mathcal{B})$. Then $\Sigma T = \Sigma K$ and there exists a mapping $\beta$ such that $K^\beta = T$. Clearly $T^{\alpha \beta} = T$ and $\alpha \beta$ is
the identity on \( \mathcal{J}(L) \). If \( H \) is any tower in \( \mathcal{J}(L) \) then \( H^\alpha \) is the preimage of \( H \) under \( \alpha \). We can conclude then that \( \alpha : \mathcal{J}(L) \rightarrow \mathcal{J}(L) \) is a bijection and hence a permutation. Q.E.D.

We can describe the action of the group \( A \) on DEU in a more precise manner. The direct action of \( A \) is on the towers of the complex \( L \) associated with DEU. However this direct action induces an action by \( A \) on the points of DEU in the following manner:

i) If \( \alpha \in A \) leaves a tower fixed then we know from the theorem it leaves all towers fixed and hence must be the identity action on \( \mathcal{J}(L) \). Thus \( \alpha \) leaves all \( n \)-cells fixed and hence all points in DEU fixed. Therefore \( \alpha \) is the identity automorphism of DEU.

ii) If \( \alpha \in A \) leaves an \( n \)-cell \( V(P) \) containing \( P \) fixed and \( \alpha \neq 1 \) then it must permute the towers of \( V(P) \) among themselves since it preserves incidences. Thus \( \alpha \) is a regular permutation on the towers of \( V(P) \) and the induced action of \( \alpha \) on \( P \) leaves \( P \) fixed. Hence \( \alpha \in A_P \), the stabilizer of \( P \) in \( A \) if and only if \( \alpha \in A_{V(P)} \), the stabilizer of \( V(P) \) in \( A \). Therefore \( A_P \cong A_{V(P)} \).

iii) If \( \alpha \in A \) and \( \alpha \) takes \( V(P) \) into \( V(Q) \) then we say \( \alpha(P) = Q \). Conversely, if we say \( \alpha(P) = Q \)
then $\alpha$ must take a tower of $V(P)$ into a tower of $V(Q)$ and hence the $n$-cell $V(P)$ into the $n$-cell $V(Q)$ since the $n$-cells partition the towers in $\mathcal{B}$. We can say also that the points of DEU partition the towers in $\mathcal{B}$.

Definition: The subgroup $A_p$ of $A$ will be designated as the **inner combinatorial group** of the point $P$ or of the cell $V(P)$.

Definition: The subgroup $\overline{G_p}$ of $A_p$ which consists of rigid motions only will be called the **inner group** of the point $P$ or of its associated cell $V(P)$.

(cf. Sinogowitz [11]).

We now state a theorem of central importance in this whole discussion. It is actually a corollary of the preceding theorem.

Theorem: Let $A$ be the group of admissible automorphisms of any DEU. Let $P \in \text{DEU}$ and $V(P)$ be its associated Voronoi cell. Let $A_p$ be as defined. We assert that $A_p$ is a finite subgroup of $A$.

Proof: If $\alpha \in A_p$ then $\alpha$ is a regular permutation of the towers of $V(P)$. We have shown already that $V(P)$ has a finite number of $(n-1)$-dimensional bounding hyperplanes and hence a finite number of towers associated with it. The regular permutation group of a finite set is finite. Therefore $A_p$ is finite. Q.E.D.
We note that the order of a regular permutation group on \( n \) symbols is \( n! \). Thus if we know the number of towers belonging to \( V(P) \) we get an upper bound for the order of \( A_p \).

Corollary: The order of \( \overline{G}_p \) is finite since \( \overline{G}_p \subseteq A_p \).

The foregoing results allow us now to consider a space group \( G \) consisting of rigid motions as a combinatorial group. Each element of \( G \) is an admissible combinatorial automorphism of any one of its associated DEU's. For every DEU we must consider in general three combinatorial automorphism groups associated with it.

i) The minimal space group required to generate DEU and we will call this the generating group \( G \).

ii) The group \( \overline{G} \) which consists of the rigid symmetries of DEU.

iii) The group \( A = \text{Aut}(\text{DEU}) \) which is maximal in the sense that it contains all possible neighborhood-preserving combinatorial automorphisms of DEU.

Finally we give some examples:

1. In the configuration given in example (1) of Chapter II (Figure 1) we have a tessellation of the plane,
namely \((3,3,3,3,3,3)\). The basic Voronoi cell is the hexagon. This hexagon possesses twelve towers. The configuration is combinatorially equivalent to the regular hexagonal tessellation of the plane which possesses as inner group the dihedral group of order 12. Thus we can conclude that the stabilizers of all twelve towers are equal. The stabilizer of \(P\) in \(\text{Aut}(\text{DEU})\) is thus of order 12 and the rigid stabilizer of \(P\) is not the full dihedral group of order 12. Hence the group \(\text{Aut}(\text{DEU})\) properly contains the rigid symmetry group of the pattern. A similar statement holds for the DEU of Figure 2, Chapter II.

2. In example (4) of the same chapter we examined the cell configuration for the 2-dimensional space group \(G_6\). Figure 6 indicated the tiling of the plane with pentagons. The tessellation is \((6,3,3,3,3)\). It can be shown that the stabilizers of the towers of a pentagon are all distinct. We indicate briefly (referring now to Figure 7) how one goes about accomplishing this. By the composition \(gh\) we mean that one first applies \(h\) and then \(g\). The simplest procedure to follow is to find an element in the stabilizer of one tower of \(V(P)\) which is not contained in the stabilizer of any other
Figure 7

tower on that cell. For instance it is easy to see that the word \((\sigma_2 \sigma_1)^3\) in \(\Sigma\) stabilizes the tower \((v_1,e_1,f)\) but does not stabilize the towers \((v_5,e_1,f)\), \((v_5,e_5,f)\). In fact \((\sigma_2 \sigma_1)^3\) stabilizes the eight towers \((v_1,e_j,f)\), \((i \neq 5)\). We have thus eliminated eighteen pairs of towers. There are forty-five pairs to be considered altogether and a similar process will eliminate the other twenty-seven pairs.
If one numbers the neighbors of $P$ and the points in the environment of $P$ as in Figure 7 then there is another way of determining the non-existence of admissible permutations (other than the identity) of the neighbors of $P$. For instance, if $P_1 \rightarrow P_1$ then either $P_2 \rightarrow P_2$ or $P_2 \rightarrow P_5$. If $P_2 \rightarrow P_2$ then $P_3$ is fixed and we end up with the identity permutation. If $P_2 \rightarrow P_5$ then $P_5 \rightarrow P_2$, $P_3 \rightarrow P_4$, $P_4 \rightarrow P_3$. Now $P_2, P_2$ is next to $P_1$ and $P_2$ therefore must end up next to $P_1$ and $P_5$ which are the images of $P_1$ and $P_5$. But this is impossible since the only neighbor of both $P_1$ and $P_5$ is $P$ which is already fixed. We can summarize the process as follows:

```
P  P_1  P_2  P_3  P_4  P_5  P_2,1  P_2,2
\downarrow  \downarrow  \downarrow  \downarrow  \downarrow  \downarrow  \downarrow  \downarrow  
P  P_1  P_5  P_4  P_3  P_2  P
```

Obviously this is not a permutation of DEU. We can continue this procedure by trying permutations with $P_1 \rightarrow P_2$, $P_1 \rightarrow P_3$, $P_1 \rightarrow P_4$, or $P_1 \rightarrow P_5$. All attempts will end in failure. Thus the stabilizer in $\text{Aut}(\text{DEU})$ of $P$ consists of the identity only. Consequently the space group $C_6$ and the group $\text{Aut}(\text{DEU})$ are identical.
Remark: There is a convenient way of representing pictorially the towers on a 2-cell in a 2-dimensional DEU complex. We simply take the point $P$ in $V(P)$ and connect it to each vertex by a dashed line then connect it with a dotted line to the midpoint of each edge. The "triangles" so formed indicate the towers of $V(P)$.

If we extend this process to the 3-dimensional complexes we will get "tetrahedra" (instead of triangles) representing towers on a cell. This pictorial representation facilitates the calculation of words in the group $\Sigma$. 
CHAPTER IV
GENERATORS AND RELATIONS IN Aut(DEU)

Let $X$ be a group and $S$ an arbitrary subset of $X$. Then $S$ is contained in at least one subgroup of $X$, namely $X$ itself. The intersection $U$ of all subgroups of $X$ containing $S$ is a subgroup of $X$. Indeed $U$ is the smallest subgroup of $X$ containing $S$. We call $U$ the subgroup generated by $S$. In the case that $U = X$ we say that $S$ is a set of generators of $X$ and that $X$ is generated by $S$.

We can characterize the relations on the generators of a group $G$ in the following way:

Let $G$ be a group generated by the set $S = \{x_1, x_2, \ldots, x_n\}$ and let $F$ be the free group generated by the set $T = \{t_1, t_2, \ldots, t_n\}$. Let $\phi: F \rightarrow G$ be the homomorphism given by $\phi(t_i) = x_i$. Let $K = \ker \phi$.

A word in $K$ of the form \( t_{i_1}^{\xi_1} t_{i_2}^{\xi_2} \cdots t_{i_k}^{\xi_k} \) maps onto the word \( x_{i_1}^{\xi_1} x_{i_2}^{\xi_2} \cdots x_{i_k}^{\xi_k} \) (\( \xi_i = \pm 1 \)) in $G$. Since $K = \ker \phi$ we must have

\[ R = x_{i_1}^{\xi_1} x_{i_2}^{\xi_2} \cdots x_{i_k}^{\xi_k} = 1 \text{ in } G. \]
An equation of this form is called a relation in $G$. Let $R$ be a set of generators of the subgroup $K$ of the free group $F$. Since $F$ is completely determined by the set $T$ and the normal subgroup $K$ by the set $R$, then the group $G \cong F/K$ can be defined by exhibiting the set $T$ and the set $R$ whose elements are called the defining relations of $G$. Every relation in $G$ can be expressed as a finite product of elements in $R$.

A system giving the set $S$ and $R$ for a group $G$ is called a presentation of $G$. Such a system is finite if both $S$ and $R$ are finite sets.

Now let $G$ be a space group and let $P \in E^n$. Let $G(P)$ be an orbit of $G$ in $E^n$. Let $b$ be the Voronoi cell complex of this DEU. We proceed to define chains, circuits, and loops in DEU.
Definition: An ordered set of n-cells \( V(P_1), V(P_2), \ldots, V(P_n) \) denoted by \( [P_1, P_2, \ldots, P_n] \), is said to be a chain in DEU if each n-cell is a neighbor of its successor.

The inverse of a chain \( [P_1, P_2, \ldots, P_n] \) is \( [P_n, P_{n-1}, \ldots, P_1] \). The empty chain contains no cells. Two chains can be multiplied if the terminal cell of the first is the initial cell of the second. The multiplication rule amounts to the juxtaposition of the two chains and the elision of duplication on the common cell. If \( K_1 = [P_1, P_2, \ldots, P_j] \) and \( K_2 = [Q_1, Q_2, \ldots, Q_k] \), \( (P_j = Q_1) \) then \( K_1K_2 = [P_1, P_2, \ldots, P_{j-1}, P_j = Q_1, Q_2, \ldots, Q_k] \).

Definition: A circuit is a chain whose initial and terminal cells coincide.
A chain need not be simple, i.e. it may intersect itself.
Definition: A loop is a simple circuit.

If \( C \) is a circuit and \( K \) a chain then \( KCK^{-1} \) is a circuit in the case that \( K \) and \( C \) can be multiplied. If such is the case then we say \( C \) and \( KCK^{-1} \) are equivalent and denote the equivalence class of \( C \) by \([C]\).

Let us now consider the circuit \( C = [P_1, P_2, \ldots, P_t] \), \( (P_t = P_1) \). If \( C \) is not a loop then \( P_j = P_k \) for some \( j \neq k \) \( (1 \leq j, k \leq t) \). There is an \( m \geq j \) and
an n \leq k \text{ such that } L_1 = [P_m, P_{m+1}, \ldots, P_n] \text{ is a loop. We define the chains:}

\[ U = [P_1, P_2, \ldots, P_m] \]

and \[ W = [P_1, P_2, \ldots, P_m, P_{n+1}, \ldots, P_t]. \]

Now \( U \) and \( L_1 \) can be multiplied and since \( P_m = P_n \), \( W \) is a circuit. Therefore

\[ C = UL_1U^{-1}W = [L_1]W. \]

We see that \( W \) is now a shorter circuit than \( C \) so we can apply a similar process to \( W \). After a finite number of steps we can write

\[ C = [L_1][L_2] \cdots [L_s] \]

where each \( L_j \) is a loop. Hence we have the proposition: Every circuit can be written as a product of simple circuits or loops.

Our task now is to relate the chains in a complex associated with a DEU and the elements of both the generating group \( G \) and the group \( A = \text{Aut} (\text{DEU}) \).

Definition: A combinatorial automorphism in \( A \) which carries a point \( P \) into a neighboring point \( Q \) is called a neighborhood operation.

There always exists a rigid neighborhood operation taking a point \( P \) into any one of its neighbors. More precisely, let \( V(P) \), the Voronoi cell of \( P \),
have m neighbors \( \{V(P_1), V(P_2), \ldots, V(P_m)\} \). For each \( P_i \) we choose a rigid neighborhood operation \( g_i \) such that \( g_i(P) = P_i \) where \( g_i \) belongs to \( G \), the generating group of the DEU in question. Thus we can associate with each \( P \in \text{DEU} \) a set of neighborhood operations \( \{g_i\}_{i=1}^m \). As we shall see, this set of neighborhood operations can be so chosen that if \( g_i \) belongs to the set then \( g_i^{-1} \) also belongs.

If we now fix our attention on a particular point \( P \in \text{DEU} \) and its Voronoi cell \( V(P) \) then we can associate with \( V(P) \) its set of towers. We can index these towers in some manner, say \( \{T_1, T_2, \ldots, T_s\} \). We can extend this indexing order to the towers of each cell in DEU because all cells are congruent. If \( g_i(P) = P_i \), then \( g_i \) determines a pairing of the towers on \( P \) with those on \( P_i \). When one pair is determined for a particular \( i \) then all pairs are determined for that \( i \). It is convenient to choose as a representative pair for \( g_i \) the pair which has as first element a tower \((s_0, s_1, \ldots, s_{n-1}, s_n)\) where \( s_{n-1} \) is on the bounding \((n-1)\)-dimensional hyperplane which is common to \( P \) and \( P_i \).

The association of a pair of towers \((T_j, H_k)\) with each neighborhood operation \( g_i \) enables us to describe that rigid neighborhood operation in an exact combinatorial manner.
Let $A_p$ be the stabilizer of $P$ in $A$ and let the order of $A_p$, $|A_p|$, be $\lambda$. Let
\[ M = \{a \in A \mid a(P) = P_i, \quad 1 \leq i \leq m\}. \]

Proposition: $|M| = m\lambda$.

Proof: By definition $\alpha(P) = P, \forall \alpha \in A_p$. Also $g_i\alpha(P) = \alpha g_j(P) = P_t$ for some $j,t \quad 1 \leq j,t \leq m$.

We have $g_i = g_k \iff i = k$ and $g_i\alpha_k = g_i\alpha_1 \iff k = 1$.

Thus with each $g_i$ we can associate a set
\[ \{g_i\alpha_1, g_i\alpha_2, \ldots, g_i\alpha_m\} \]
all of whose elements are distinct.

We get $m$ such sets and they are disjoint since $g_i\alpha_k = g_j\alpha_1 \iff i = j = k = 1$. Therefore we get $m\lambda$ distinct operations $g_i\alpha_j = a_{ij} \quad (1 \leq i \leq m, \quad 1 \leq j \leq \lambda)$ such that $a_{ij}(P) = P_i$ for some $i$.

Therefore $|M| = m\lambda$. Q.E.D.

We will show that these $m\lambda$ operations generate $A$.

Theorem: The set $M$ generates $A$.

Proof: Let $a \in A$ and $V(Q_1)$ be an arbitrary cell in $\mathcal{B}$. Suppose $a$ maps $V(Q_1)$ onto $V(Q_2)$. We claim that $a$ can be expressed as a product of elements in $M$. It is clear that if $g_i$ is a neighborhood operation of $P$ then $g_i^{-1}$ is also a neighborhood operation hence $g_i^{-1} = g_j$ for some $1 \leq j \leq m$.

For if $g_i(P) = P_i$ then $g_i^{-1}(P_i) = P$ and it follows immediately from the neighborhood preserving property
of $\delta_i^{-1}$ that it carries $P$ into some one of its neighbors $P_j$ ($1 \leq j \leq n$). To be more explicit about $a$ let us assume that $a(T) = T'$ where $T$ is a tower of $V(Q_1)$ and $T'$ a tower of $V(Q_2)$. As a consequence of the theorem on connectedness in Chapter II we can find a chain $K$ from $Q_1$ to $Q_2$ passing through $V(P)$ and such that $K$ meets no $(n-j)$-dimensional subspace bounding any Voronoi cell $(j \geq 2)$. Suppose

$$K = [Q_1 = R_1, R_2, \ldots, R_{t-1}, R_t = P, R_{t+1}, \ldots, R_s = Q_2].$$

There exists a $g_k \in M$ such that $g_k(P) = R_{t-1}$. The neighborhood operations of $R_{t-1}$ are the transforms by $g_k$ of the neighborhood operations of $R_t (=P)$. Hence there exists $g_1 \in M$ such that $g_k g_1^{-1}(R_{t-1}) = R_{t-2}$. Then we have the mapping $(g_k g_1^{-1}) g_k : R_t \rightarrow R_{t-2}$, i.e. $g_k g_1$ takes $P$ into $R_{t-2}$. Now the neighborhood operations of $R_{t-2}$ are transforms by $g_k g_1 g_k^{-1}$ of the neighborhood operations of $R_{t-1}$ which in turn are transforms by $g_k$ of those of $R_t$. Hence there exists a $g_j \in M$ determining the mapping

$$[(g_k g_1^{-1}) g_k] g_j [(g_k g_1^{-1}) g_k]^{-1} : R_t \rightarrow R_{t-3}.$$

Reducing this word we get the word $(g_k g_1) g_j (g_k g_1)^{-1}$ taking $P$ into $R_{t-3}$. Proceeding inductively in
In this manner we obtain a word \( \xi_{Q_1} \) in the \( g_i \)'s such that \( \xi_{Q_1}(P) = Q_1 \).

In a similar manner we can compose a word \( \xi_{Q_2} \) in the \( g_i \)'s such that \( \xi_{Q_2}(P) = Q_2 \). Now \( \xi^{-1}_{Q_1} \) takes \( Q_1 \) into \( P \), consequently the word \( \xi = \xi_{Q_2} \xi_{Q_1}^{-1} \) takes \( Q_1 \) into \( Q_2 \). However \( \xi \) may not be the word which takes the tower \( T \) on \( V(Q_1) \) into the tower \( T' \) on \( V(Q_2) \). If not, we can carry out the necessary adjustment by applying a suitable element \( \alpha \) of the point group \( A_P \) on the cell \( V(P) \). In any case we can obtain a word \( \Theta = \xi_{Q_2} \xi_{Q_1}^{-1} \alpha \) such that \( \Theta(Q_1) = Q_2 \) and \( \Theta(T) = T' \).

It has been shown already that once the image of a single tower \( T \) has been determined for a combinatorial automorphism then that automorphism is completely determined. We can conclude then that \( \Theta = a \). Since \( a \) was arbitrary it follows that each element of \( A \) can be expressed as a product of elements of \( M \). Thus \( M \) generates \( A \). Q.E.D.

Corollary: Every relation \( R \) in \( A \) can be expressed as a product \( a_1 a_2 \cdots a_k = 1 \) (\( a_i \in M \)).
**Proposition:** Every relation $R$ in $A$ can be represented by a circuit in DEU.

**Proof:** Let $a_1a_2\cdots a_k = 1$ be a relation in $A$ ($a_i \in M$). We claim that we can get a circuit $C$ beginning and ending at $P$ which represents this relation. We first observe that $a_1a_2\cdots a_k$ can be written in the form

$$[(a_1a_2\cdots a_{k-1})a_k(a_1a_2\cdots a_{k-1})^{-1}]$$

$$[(a_1a_2\cdots a_{k-2})a_{k-1}(a_1a_2\cdots a_{k-2})^{-1}]$$

$$\cdots [(a_1a_2)a_3(a_1a_2)^{-1}][a_1a_2a_1^{-1}]a_1].$$

This form yields $k$ terms in square brackets. We denote them by $\phi_k, \phi_{k-1}, \ldots, \phi_3, \phi_2, \phi_1$. Now notice that $\phi_1$ is a neighborhood operation of $P$; $\phi_2$ is a neighborhood operation of a neighbor of $P$, say $P'$; $\phi_3$ a neighborhood operation of a neighbor of a neighbor of $P$, say $P''$; and so on. Thus $\phi_k \phi_{k-1} \cdots \phi_3 \phi_2 \phi_1(P)$ will determine the circuit

$$C = [P, P', P'', \ldots, P^{(k-1)}, P^{(k)}] \quad (P^{(k)} = P)$$

and the proposition is proved.

It has been shown that any circuit can be written as a product of loops but we can be more specific.

**Definition:** A fundamental loop in DEU is a minimal loop enclosing an $(n-2)$-dimensional bounding subspace of a Voronoi cell.
In two and three dimensional space the fundamental loops are respectively the *vertex circuits* and *edge circuits* of a Voronoi cell. Each cell determines a family $\mathcal{F}$ of inequivalent fundamental loops. If $\mathcal{F}$ and $\mathcal{H}$ are two families associated with $V(P)$ and $V(Q)$ respectively and if $\alpha: V(P) \longrightarrow V(Q)$ then $\alpha \mathcal{F} \alpha^{-1} = \mathcal{H}$.

We now state a theorem which demonstrates the importance of fundamental loops.

**Theorem:** Let $G$ be a space group in $n$ dimensions. Let $V_0$ be a Voronoi cell in a DEU associated with $G$. Assume that $V_0$ is also a fundamental region for $G$. Denote by $\{v_i, i=1,2,\ldots,t\}$ a complete set of inequivalent $(n-2)$-dimensional bounding subspaces of $V_0$ and by $R_i$ the relation determined by the fundamental loop at $v_i$. Then $R_1 = 1$, $R_2 = 1$, $\ldots$, $R_t = 1$ form a basic set of relations for $G$. (cf. Fricke-Klein [6]).

**Note:** It was observed in chapter II that the Voronoi cell $V_0$ associated with a particular orbit of $G$ in $E^n$ was not always a fundamental cell for $G$. When this occurs the inner group $G_{V_0}$ of $V_0$ is larger than 1. In order to get the full set of basic
relations for $G$ we must now insert elements of $G_{V_0}$ into each relation $R_i$ determined by a fundamental loop at $v_i$ and look for new relations. This process is finite since $|G_{V_0}| < \infty$. Eventually we will get a full finite set of basic relations for $G$.

In order to illustrate an application of the previous theorem we will calculate a basic set of relations for the two dimensional space group $G_6$ which has as fundamental Voronoi cell a pentagon with the interior angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ at its hexadic and triadic vertices. The sides enclosing the angle $\frac{\pi}{3}$ are equal as are those enclosing the angle $\frac{2\pi}{3}$. The inner combinatorial group of this cell consists of the identity only.
We define the neighborhood operations as follows:

\[ \sigma_1 : P_0 \rightarrow P_1 \quad i = 1,2,\ldots,5 \]

\( \sigma_1 \) is a rotation about 0 through \( -\frac{\pi}{3} \)

\( \sigma_2 \) is a rotation about the midpoint of the common side of \( P_0 \) & \( P_2 \) through \( \frac{\pi}{2} \)

\( \sigma_3 \) is a rotation about \( Q \) through \( \frac{2\pi}{3} \)

\( \sigma_4 \) is a rotation about \( Q \) through \( -\frac{2\pi}{3} \)

\( \sigma_5 \) is a rotation about 0 through \( \frac{\pi}{3} \).

Clearly we have the following relations between the neighborhood operations: \( \sigma_1 = \sigma_5^{-1} \); \( \sigma_2 = \sigma_2^{-1} \);

\( \sigma_3 = \sigma_4^{-1} \); \( \sigma_4 = \sigma_3^{-1} \); \( \sigma_5 = \sigma_1^{-1} \).

We now derive the vertex circuit relations as follows:

\( 1 \) \( P_0 \rightarrow P_5 \rightarrow P_{2,1} \rightarrow P_{3,1} \rightarrow P_{2,2} \rightarrow P_1 \rightarrow P_0 \):

\( \sigma_5 : P_0 \rightarrow P_5 \) by definition. To take \( P_5 \) into \( P_{2,1} \) we must find the preimage of their common side under \( \sigma_5 \). We see immediately that this preimage is the common side of \( P_0 \) and \( P_5 \). This tells us the neighborhood operation carrying \( P_5 \) to \( P_{2,1} \) must be a transform of \( \sigma_5 \) by \( \sigma_5 \) i.e. \( \sigma_5 \sigma_5 \sigma_5^{-1} = \sigma_5 \).

Thus \( \sigma_5 : P_5 \rightarrow P_{2,1} \). Now to take \( P_{2,2} \) into \( P_{3,1} \) we repeat a similar process and see again that it is a transform of \( \sigma_5 \) that produces the desired
result. Proceeding in this manner we get the relation \( \sigma_5^6 = 1 \) from this vertex circuit.

(2) \( P_0 \rightarrow P_4 \rightarrow P_5 \rightarrow P_0 \):

\( \sigma_4 : P_0 \rightarrow P_4 \). The preimage under \( \sigma_4 \) of the common side of \( P_4 \) and \( P_5 \) is the common side of \( P_0 \) and \( P_2 \). Hence we use a transform of \( \sigma_2 \) by \( \sigma_4 \) to take \( P_4 \) to \( P_5 \) so we have

\( \sigma_4 \sigma_2 \sigma_4^{-1} : P_4 \rightarrow P_5 \). Now the common side of \( P_5 \) and \( P_0 \) has as preimage under \( (\sigma_4 \sigma_2 \sigma_4^{-1}) \sigma_4 \) the common side of \( P_0 \) and \( P_1 \). Hence the transformation taking \( P_5 \) to \( P_0 \) is

\( (\sigma_4 \sigma_2 \sigma_4^{-1} \sigma_4) \sigma_1(\sigma_4 \sigma_2 \sigma_4^{-1} \sigma_4)^{-1} = \sigma_4 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_4^{-1} \).

The complete circuit is then described

\( \sigma_4 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_4^{-1} \sigma_4 \sigma_2 \sigma_4^{-1} \sigma_4 \sigma_4 = \sigma_4 \sigma_2 \sigma_1 = 1 \).

(3) \( P_0 \rightarrow P_3 \rightarrow P_4 \rightarrow P_0 \)
yields the relation \( \sigma_3^3 = 1 \).

(4) \( P_0 \rightarrow P_2 \rightarrow P_3 \rightarrow P_0 \)
yields the relation \( \sigma_2 \sigma_1 \sigma_4 = 1 \).

(5) \( P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow P_0 \)
yields the relation \( \sigma_1 \sigma_4 \sigma_2 = 1 \).
If we combine the relations on the neighborhood transformations together with the five vertex circuit relations we can eliminate $\sigma_3$, $\sigma_4$, and $\sigma_5$, obtaining $\sigma_5 = \sigma_1^{-1}$, $\sigma_2 \sigma_1 = \sigma_4^{-1} = \sigma_3$. As a result we get the relations which define $C_6$ namely 

$\sigma_1^6 = 1 \quad (\sigma_2 \sigma_1)^3 = 1 \quad \sigma_2^2 = 1$.

Remark: We calculated all vertex circuits, however, three of the vertices were equivalent under $C_6$ and thus yielded only the one relation $\sigma_2 \sigma_1 = \sigma_4^{-1}$.

We sum up what has been said by pointing out that a presentation of a space group can be derived in a local manner: by this we mean that if the $(n-2)$-dimensional circuits and the inner group of a Voronoi cell in a DEU are known then we get essential knowledge about the associated space group.
CHAPTER V
THE GROUP Aut(DEU)

Our objective now is to show that the group of admissible combinatorial automorphisms of a DEU possesses a representation $\Delta$ by means of affine transformations with a finite kernel $N$.

Let $G$ be the generating group of a DEU and let $A = \text{Aut}(\text{DEU})$ be its associated combinatorial group. We denote by $A_P$ and $G_P$ the stabilizers of $P$ in $A$ and $G$ respectively. We choose a representative system $S$ of $G$ modulo $G_P$. Then

$$G = \bigcup_{g \in S} gG_P$$

(1)

Lemma: For any point $Q \in \text{DEU}$ we can find a $g \in S$ such that $g(P) = Q$.

Proof: Since $G$ acts transitively on $\text{DEU}$ there exists a $g' \in G$ such that $g'(P) = Q$. But $g' = gh$ for some $g \in S$ and $h \in G_P$. Then

$$g'(P) = gh(P) = g(P) = Q$$

and the lemma is proved.
Lemma: \[ A = \bigcup_{g \in S} gA_P \] (2)

Proof: Clearly \[ \bigcup_{g \in S} gA_P \subseteq A. \] We show \[ A \subseteq \bigcup_{g \in S} gA_P. \]

Choose an arbitrary \( a \in A \). The action of \( a \) on \( \text{DEU} \) is completely determined by its action on the towers of \( V(P) \). Suppose \( a(P) = Q \) and that \( a(T) = T' \) where \( T \) and \( T' \) are towers on \( V(P) \) and \( V(Q) \) respectively. By the previous lemma there exists \( g \in S \) such that \( g(P) = Q \). If \( g(T) = T' \) then \( g = a \) and we are finished. If not then there is a tower \( H \) on \( V(P) \) such that \( g(H) = T' \). Since \( a(T) = g(H) = T' \) we can conclude that the stabilizers in \( \Sigma \) of \( T, H, \) and \( T' \) are identical. Hence there exists \( \alpha \in A_P \) such that \( \alpha(T) = H \). Thus \[ g(\alpha(T)) = a(T) = T'. \]

Therefore \( a = g\alpha \) and \( a \in \bigcup_{g \in S} gA_P \) and the lemma is proved.

Theorem: The index of \( G \) in \( A \) is the index of \( G_P \) in \( A_P \).

Proof: We can write
\[ A_P = \bigcup_{i=1}^{A_P} G_{p\alpha_i} (3) \]
where \( a_i \) belongs to a representative system of
\( A_p(\text{mod } G_p) \). Combining (1), (2), and (3) we obtain

\[
A = \bigcup_{g \in S} gA_p = \bigcup_{g \in S} g \left[ \bigcup_{i=1}^{A_p;G_p} G_p a_i \right]
\]

\[
= \bigcup_{g \in S} \left[ \bigcup_{i=1}^{A_p;G_p} gG_p a_i \right]
\]

\[
= \bigcup_{i=1}^{A_p;G_p} \left[ \bigcup_{g \in S} gG_p a_i \right]
\]

\[
= \bigcup_{i=1}^{A_p;G_p} G_a_i.
\]

Now assume \( G_{a_i} \cap G_{a_j} \neq \phi, \ i \neq j \). Then
\( a_i = g' a_j (g' \in G) \). Both \( a_i \) and \( a_j \) leave \( P \) fixed hence

\[
P = a_i(P) = g' a_j(P) = g'(P).
\]

This implies that \( g' \in G_p \) which in turn implies that
\( a_i a_j^{-1} \in C_p \). Hence \( G_p a_i = G_p a_j \). This is true if and only if \( i = j \) but that contradicts the assumption.

Therefore \( A \) is a disjoint union of the \( G_a_i \)'s, \( i = 1, \ldots \), \( A_p;G_p \). We can conclude that

\[
A;G = A_p;G_p \quad \text{Q.E.D.}
\]

The following two lemmas and the next theorem characterize the connection between \( A \) and an abstract space group.
Lemma 1: The commutator group of any group $S$ with a torsion free central subgroup $N$ of finite index is finite.

Proof: For any element (coset) $\sigma$ of the factor group $F = S/N$ choose an element $A_\sigma$ of $\sigma$ such that

$$(1) \quad A_\sigma A_\tau = C_{\sigma,\tau} A_{\sigma \tau} \quad (\sigma, \tau \in F, \ C_{\sigma,\tau} \in N).$$

We embed $N$ into the abelian group $\overline{N}$ formed by the set $\{a^{[F]}^{-1} | a \in N\}$. The operational rules are given by

$$a^{[F]}^{-1}, b^{[F]}^{-1} = (ab)^{[F]}^{-1}$$

$$a^{[F]}^{-1} = b \iff a = b^{[F]} \quad (a, b \in N).$$

Let $\overline{S}$ be the extension of $\overline{N}$ by $F$ with

(i) the factor set $C_{\sigma,\tau}$

(ii) generators $A_\sigma$ ($\sigma \in F$) subject to the defining relations $(1)$ and

(iii) trivial action of $F$ on $N$.

It contains $S$ as a subgroup. The intersection of $\overline{N}$ with $S$ is $N$.

As is shown in finite group theory (cf. Zassenhaus [17]) the $|F|^{\text{th}}$ power of the factor system $C_{\sigma,\tau}$
splits in \( N \). We have

\[ C_{\sigma, \tau} |F| = B_\sigma B_\tau B_{\sigma \tau}^{-1} \quad (B_\sigma, B_\tau, B_{\sigma \tau}^{-1} \in N) \]

Hence \( C_{\sigma, \tau} = B_\sigma |F|^{-1} B_\tau |F|^{-1} B_{\sigma \tau}^{-1} |F|^{-1} \) is in \( \bar{N} \). Thus \( \bar{S} = S_1 \times \bar{N} \) for a certain finite subgroup \( S_1 \) of \( S \). Consequently the commutator group of \( \bar{S} \) is equal to the commutator group of \( S_1 \) which is finite. Since the commutator group of \( S \) is contained in the commutator group of \( \bar{S} \) the lemma is demonstrated.

Lemma 2: Any overgroup \( A \) of finite index over an abstract \( n \)-dimensional space group \( G \) contains a finite normal subgroup with a factor group isomorphic to an abstract \( n \)-dimensional space group.

Proof: The only finite normal subgroup of \( G \) is \( 1 \). Hence the intersection of a finite normal subgroup of \( A \) with \( G \) is \( 1 \).

It follows that the order of any finite normal subgroup of \( A \) is not greater than the index \( A:G \). Therefore there is a largest finite normal subgroup \( N \) of \( A \).

Let \( T \) be the largest abelian normal subgroup of \( G \). \( T \) is free abelian. The index \( G:T \) is finite hence the index of \( T \) in \( A \) is finite. Therefore the
number of conjugates of $T$ under $A$ is finite and each of the conjugates of $T$ in $A$ is of finite index in $A$. The intersection of the conjugates of $T$ under $A$ is a normal subgroup $T_0$ of $A$ and the index of $T_0$ in $A$ is finite. The index of $T_0$ in $T$ is also finite. Hence $T_0$ is a free abelian group of rank equal to that of $T$.

The intersection of $T_0$ with $N$ is 1. The subgroup $T_0N/N$ of $A/N$ is normal free abelian of rank equal to the rank of $T$ and of finite index in $A/N$. Among the normal subgroups of $A$ containing $T_0N$ such that the factor group over $N$ is free abelian and of the same rank as $T$ there is a largest one, say $T_1$.

According to Lemma 1 the commutator subgroup of the centralizer $Z_{A/N}(T_1/N)$ of $T_1/N$ in $A/N$ is finite. Since $T_1/N$ is normal in $A/N$ it follows that $Z_{A/N}(T_1/N)$ is normal in $A/N$. Hence the commutator subgroup of $Z_{A/N}(T_1/N)$ is finite and normal. Because of the maximal property of $N$ one finds that

$$D(Z_{A/N}(T_1/N)) = N/N$$

where $D(Z_{A/N}(T_1/N))$ denotes the commutator subgroup of $Z_{A/N}(T_1/N)$.

In other words $Z_{A/N}(T_1/N)$ is abelian. Again due to the maximal property of $N$ the factor group $A/N$ is
torsion free hence \( Z_{A/N}(T_1/N) \) is torsion free. Since it contains the free abelian group \( T_1/N \) as a subgroup of finite index it is itself free abelian of the same rank as \( T \). Because of the maximal property of \( T \) it follows that

\[
Z_{A/N}(T_1/N) = T_1/N.
\]

Hence \( A/N \) is isomorphic to an \( n \)-dimensional space group. Q.E.D.

From Lemma 1 and Lemma 2 we obtain:

Theorem: The combinatorial group \( A = \text{Aut}(DEU) \) that arises out of an orbit of an \( n \)-dimensional space group \( G \) contains a largest finite normal subgroup \( N \) such that the factor group \( A/N \) is an abstract \( n \)-dimensional space group. The restriction of \( G \) to its action on \( DEU \) defines a transitive subgroup \( \overline{G} \) of \( A \). It is isomorphic to the subgroup \( \overline{G}N/N \) of \( A/N \) of an index dividing the index of \( A/N \) over its maximal abelian normal subgroup.

Proof: Since a space group contains no finite normal subgroup other than the identity subgroup it follows that \( \overline{G} \cap N = 1 \). Thus \( \overline{G} \) is isomorphic to the factor group \( \overline{G}N/N \). We need only prove the last divisibility statement of the theorem.
There is a representation

$$\Delta: A \rightarrow A_n$$

of $A$ by means of affine transformations of $E_n$ with
kernel $\mathbb{N}$ such that $\Delta(A)$ is an affine n-dimensional
space group. This representation is uniquely determined
up to inner automorphisms of $A_n$. Hence it can be chosen
in such a way that the restriction of $\Delta$ to $G$ is the
natural representation:

$$\Delta(\overline{g}) = g$$

where $\overline{g}$ denotes the permutation of DEU obtained by
application of the group element $g$ of $G$.

Let $T_1$ be the translation subgroup of $\Delta(A)$ and
let $T = T_1 \cap G$ be the translation subgroup of $G$. Let
$P \in \text{DEU}$. Then one has

$$x \equiv y \pmod{T} \quad (x, y \in T_1)$$

if and only if

$$y(P) \in T(x(P)).$$

Furthermore, the stabilizer $G_P$ of $P$ under the
action of $G$ intersects $T$ in the identity only. Hence
$T_1:T$ divides the number of disjoint sets $T(Q)$,
$Q \in \text{DEU}$; this number is the quotient of $G:T$ and the
order of $G_P$. Moreover
\[ A: \overline{G}N = \Delta(A):G = \frac{\Delta(A):T_1}{\Delta(G):T} \]

divides both the quotient of \( \Delta(A):T_1 \) and the order of \( G_p \) which in turn divides \( \Delta(A):T \). Q.E.D.

Remark: That it can happen that \( \Delta(A) \) does not send DEU into itself can be seen from the example

\[
G = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rangle \quad n = 1, \quad p = .1.
\]

For this group

\[
\Delta(A) = \langle \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rangle.
\]

We now give a final theorem.

Theorem: If for a DEU the combinatorial group \( A = \text{Aut(DEU)} \) is known then there exists only a finite number of choices for the generating space group \( G \).

Proof: Let \( A \) be given. From the previous theorem we know that \( G \) which is isomorphic to \( \overline{G} \) in \( A \) is therefore isomorphic to \( \overline{G}N/N \) which is contained in \( A/N \).

It has been shown also that the index of \( \overline{G}N/N \) in \( A/N \) divides the index of \( A/N \) over its maximal abelian normal subgroup \( T(A/N) \). Let \( \overline{A} = A/N \). Let \( \overline{A}:T(\overline{A}) = \tau \) and let \( \overline{A}:\overline{G}N/N = \lambda \). Then \( \lambda \) divides \( \tau \).
We have the following:

(i) $\bar{A}^\tau \subseteq \bar{G}N/N (\cong G)$

(ii) $\bar{A}^\tau \subseteq T(\bar{A})$

(iii) $T(\bar{A})^\tau \subseteq T(\bar{A})$

(iv) $T(\bar{A})^\tau \subseteq \bar{A}^\tau$.

$\bar{A}^\tau$ is normal in $\bar{A}$. Also $T(\bar{A})$ is free abelian on $n$ generators and $T(\bar{A}) : T(\bar{A})^\tau = \tau^n$. Thus $\bar{A} : \bar{A}^\tau \leq \bar{A} : T(\bar{A})^\tau = \tau^{n+1}$. Since the isomorphic copy of $G$ in $\bar{A}$ must lie between $\bar{A}^\tau$ and $\bar{A}$ we are left with only a finite number of choices for $G$. 
APPENDIX

We give here two interesting lemmas on distance functions that can be defined on a DEU. The first distance $d$ defines a minimal distance between two points $P, Q$ of DEU. It is invariant under the action of the group $\text{Aut}(\text{DEU})$. The second distance function $d^*$ is a translation invariant distance and is related to the concept of Frechet distance. These two distance functions enable us to define a new distance function $m$, namely, the Minkowski distance between two points $P, Q$ of DEU. The distance function $m$ is of special interest because it turns out to have the property of homogeneity and behaves in a manner very similar to euclidean distance. It has possible applications in packing problems and other areas. I am indebted to Professor Hans J. Zassenhaus of the Ohio State University Mathematics Department for his help in establishing these two lemmas.
Lemma 1: There is a Minkowski distance function \( m \) definable on \( E^n \), the euclidean n-space, that is invariant under \( G \) in the sense that
\[
m(g(U), g(V)) = m(U, V), \quad (U, V \in \text{DEU}, \quad g \in G).
\]

Proof: For any two points \( U, V \in E^n \) we can find a chain \( [X_0, X_1, \ldots, X_\tau] \) such that \( U \in V(X_0) \) and \( V \in V(X_\tau) \), \( X_i \in \text{DEU} \quad (i=0,1,\ldots,\tau) \). We first define a distance function \( d \) by letting \( d(U, V) \) be the minimal number of cells possible on a chain connecting \( U \in V(X_0) \) with \( V \in V(X_\tau) \).

The distance function \( d \) has the following properties:

(a) \( d(U, U) = 0 \); \( d(U, V) \geq 0 \), \( U \neq V \).
(b) \( d(U, V) = d(V, U) \).
(c) \( d(U, V) + d(V, W) \geq d(U, W) \).
(d) \( d(a(U), a(V)) = d(U, V) \); \( a \in A \), \( U, V \in \text{DEU} \).
(e) \( d(U, V) \geq \delta^{-1} \overline{UV} \) where \( \overline{UV} \) represents euclidean distance and \( \delta \) is the diameter of a Voronoi cell.
(f) \( d(U, V) \leq M \overline{UV} \) where \( M \) is a positive constant.

Properties (a), (b), and (c) are obvious.

Property (d) follows from the fact that each \( a \in A \) preserves incidences.
Property (e): Let $d(U,V) = K$. Let $\delta$ be the diameter of a Voronoi cell. The line segment $UV$ is always contained in an $n$-dimensional ball $B_n$ with $U$ as center and $K\delta$ as radius. Hence $\overline{UV} \leq K\delta$. Thus we have established that $d(U,V) \geq \delta^{-1} \overline{UV}$.

Property (f): We showed in Chapter II that there exists a chain connecting $U$ and $V$ which is contained inside a ball $B_n$ of diameter $\overline{UV}$. If we take a ball $B'_n$ of radius $\overline{UV} + \delta$ where $\delta$ is the diameter of a cell then the maximum number of cells in a chain connecting $U$ and $V$ which lie inside $B_n$ is less than or equal to $\frac{\pi(\overline{UV} + \delta)^2}{c}$ where $c$ is the content of a Voronoi cell. Thus

$$d(U,V) \leq \frac{\pi(\overline{UV} + \delta)^2}{c} = M'.$$

If we set $M = \frac{M'}{\overline{UV}}$ we have the result

$$d(U,V) \leq M \overline{UV}.$$

Now define $d^*(U,V) = \max_{U'V' \neq UV} d(U',V')$ ($U,V,U',V' \in E^n$). The distance function $d^*$ has the following properties:

(a') $d^*(U,U) = 0$, $d^*(U,V) > 0$ if $U \neq V$.

(b') $d^*(U,V) = d^*(V,U)$.

(c') $d^*(U,V) + d^*(V,W) \geq d^*(U,W)$.

(d') $d^*(g(U),g(V)) = d^*(U,V)$; $g \in G$, $U,V \in DEU$. 
Property (a'): \( d^*(U,U) = 0 \) is obvious. If \( U \neq V \) then it is always possible to find a vector \( U'V' = UV \) such that the line segment \( U'V' \) intersects a vertex of the Voronoi cell complex in an interior point \( W \) of the segment \( U'V' \). Since all cells are convex then \( U' \) lies in a cell \( V(P) \) and \( V' \) lies in a cell \( V(P') \) where \( V(P) \neq V(P') \). Hence \( d(U',V') \geq 1 \). This implies \( d^*(U,V) \geq 1 \) and the property is established.

Property (b') is clear.

Property (c') follows when we take the maxima across the inequality \( d(U,V) + d(V,W) \geq d(U,W) \).

Property (d') follows from \( G \)'s being a group of isometries.

Property (e') is true because
\[
\frac{d^*(U,V)}{\lambda} \geq \frac{d(U,V)}{\lambda} \leq \delta^{-1} UV.
\]

Property (f') follows from the discussion of property (f) for \( d \).

Finally we define \( m(U,V) = \lim_{\lambda \to +\infty} \frac{d^*(U',V')}{\lambda} \).

The existence of this limit is guaranteed for each pair of points \( U,V \in E^n \) because if \( U'V' = \lambda UV \) then
m has the following properties:

(a" ) \( m(U,U) = 0 \) \( m(U,V) > 0 \) if \( U \neq V \).

(b" ) \( m(U,V) = m(V,U) \).

(c" ) \( m(U,V) + m(V,W) \geq m(U,W) \).

(d" ) \( m(g(U),g(V)) = m(U,V) ; \ g \in G , \ U,V \in DEU \).

(d'" ) \( m(U',V') = \mu m(U,V) ; \ \mu > 0 \) \( \overline{U'V'} = \mu \overline{UV} \).

(e" ) \( m(U,V) \geq \delta^{-1} \overline{UV} \).

(f" ) \( m(U,V) \leq M \overline{UV} \).

Properties (a" ), (e"), and (f" ) follow from (1).

Property (b" ) is clear and property (d'" ) follows from property (d' ) of the distance \( d^* \).

We will now establish the two properties (d'" ) and (c" ).

Property (d'" ) : This is called the homogeneous property of \( m \). We will write \( d^*[\lambda \overline{UV}] \) for \( d^*(U',V') \) when \( \overline{U'V'} = \lambda \overline{UV} \) and \( m[\mu \overline{UV}] \) for \( m(U',V') \) when \( \overline{U'V'} = \mu \overline{UV} \). We have then for a fixed constant \( \mu \)

\[
\frac{1}{\mu} m[\mu \overline{UV}] = \liminf_{\lambda \to +\infty} \frac{d^*[(\lambda \mu) \overline{UV}]}{\lambda \mu} = \liminf_{\lambda \mu \to +\infty} \frac{\lambda \mu}{\lambda \mu} = m[\overline{UV}] .
\]
Hence \( m[\mu \overrightarrow{UV}] = \mu m[\overrightarrow{UV}] \) or \( m(U', V') = \mu m(U, V) \) when \( \overrightarrow{U'V'} = \mu \overrightarrow{UV} \).

This establishes the homogeneity of \( m \).

Property (c''): Let \( \Lambda \) be the set of \( \lambda \)'s on which \( \frac{d^*[\lambda \overrightarrow{UV}]}{\lambda} \) has its least cluster point namely \( m(U, V) \) and no other cluster point. Let \( M \) be the set of \( \mu \)'s on which \( \frac{d^*[\mu \overrightarrow{VW}]}{\mu} \) has its least cluster point \( m(V, W) \) and no other cluster point. For each \( \varepsilon > 0 \) we can find \( \lambda_0 \in \Lambda \) and \( \mu_0 \in M \) such that for all \( \lambda \in \Lambda, \lambda > \lambda_0 \) and for all \( \mu \in M, \mu > \mu_0 \) we have

\[
\frac{d^*[\lambda \mu \overrightarrow{UV}]}{\lambda \mu} < m(U, V) + \varepsilon \tag{1}
\]

\[
\frac{d^*[\mu \lambda \overrightarrow{VW}]}{\mu \lambda} < m(V, W) + \varepsilon \tag{2}
\]

For all \( \lambda, \mu \in \mathbb{R} \) we have by property (c') of \( d^* \)

\[
\frac{d^*[\lambda \mu \overrightarrow{UV}]}{\lambda \mu} + \frac{d^*[\mu \lambda \overrightarrow{VW}]}{\mu \lambda} \geq \frac{d^*[\mu \lambda \overrightarrow{UW}]}{\mu \lambda} \tag{3}
\]

For only finitely many \( \nu \) is

\[
\frac{d^*[\nu \overrightarrow{UW}]}{\nu} \leq m(U, W) - \frac{\varepsilon}{3} \tag{4}
\]

Now letting \( \lambda \) approach +\( \infty \) through \( \Lambda \) and \( \mu \) approach +\( \infty \) through \( M \) we can combine (1), (2), (3), and (4) to obtain
\[ m(U,V) + \frac{\varepsilon}{3} + m(V,W) + \frac{\varepsilon}{3} > m(U,W) - \frac{\varepsilon}{3} \]

or \[ m(U,V) + m(V,W) > m(U,W) - \varepsilon . \]

Since \( \varepsilon > 0 \) can be made arbitrarily small we have established that

\[ m(U,V) + m(V,W) > m(U,W) . \]

**Lemma 2:** If \( U \) and \( V \) are any two distinct points of \( E^n \) then for any \( n \)-dimensional lattice \( L \) in \( E^n \) there is a point \( X \in L \) for which

\[ m(X,U) \neq m(X,V) \]

where \( m(A,B) \) denotes the Minkowski distance from \( A \) to \( B \).

**Proof:** Assume that the lemma is not true. There would be two points \( U, V \), \( U \neq V \), where

\[ m(U,X) = m(V,X) \]

for all \( X \in L \).

Since \( L \) is \( n \)-dimensional we have for each point \( W \in E^n \) a corresponding point \( X(W) \in L \) such that

\[ m\left(W, X(W)\right) \leq \beta \quad \beta \in \mathbb{R} \text{ and } \beta > 0 . \]

Let \( \overrightarrow{W} = \frac{1}{m(U,V)} \overrightarrow{UV} \) ; \( (1) \)

\[ \overrightarrow{VX_i} = i \overrightarrow{UV} \] ; \( (2) \)

\[ Y_i = X(X_i) \] ; \( (3) \)

\[ \overrightarrow{Y_iZ_i} = \overrightarrow{UV} \quad (i = 1, 2, 3, \ldots) . \] \( (4) \)
By assumption \( m(V,Y_i) = m(U,Y_i) \) and by construction

\[ \overrightarrow{UV} = \overrightarrow{Y_iZ_i} \]

\[ m(U,Y_i) = m(V,Z_i) \quad (UZ_iY_iV \text{ is a parallelogram}) \]

Hence

\[ m(V,Y_i) = m(V,Z_i) \quad (i = 1, 2, 3, \ldots) . \quad (5) \]

In addition \( m(V,Y_i) \geq i \, m(U,V) - \beta \quad (i=1,2,3,\ldots) \).

This is so because

\[ m(V,Y_i) + m(Y_i,X_i) \geq m(V,X_i) \]

and \( m(V,X_i) = i \, m(U,V) \) by (2) and the homogeneity of \( m \)

and \( m(Y_i,X_i) \leq \beta \). Hence

\[ m(V,Y_i) \geq i \, m(U,V) - \beta . \]

Consequently

\[ \lim_{i \to \infty} m(V,Y_i) = +\infty . \]

**Note:** If \( Y_i \) is on the line containing \( U \) and \( V \), then

\[ m(U,Y_i) = m(V,Y_i) + m(U,V) \neq m(V,Y_i) \]

(since \( m(U,V) \neq 0 \)), which contradicts our assumption.

For points on a line, the Minkowski triangle of inequality becomes an equality.

Also \( m(V,W) = 1 \) by (1).

Now let

\[ \overrightarrow{VY_i} = \frac{1}{m(V,Y_i)} \overrightarrow{VY_i} \quad (6) \]
\[ V_{z_i} = \frac{1}{m(V, z_i)} V_{z_i}. \] (7)

Hence

\[ m(V, y_i') = 1 = m(V, z_i') \quad \text{and} \quad y_i' \neq z_i'. \]

From (5), (6), and (7) and letting \( m(V, y_i') = Y \) we obtain

\[ \overrightarrow{V_{y_i}} = Y \overrightarrow{V_{y_i}} \]
\[ \overrightarrow{V_{z_i}} = Y \overrightarrow{V_{z_i}}. \]

These two equations together with the homogeneous property of \( m \) yield the following equations:

\[ m(V, y_i) = Y m(V, y_i') \]
\[ m(V, z_i) = Y m(V, z_i') \]
\[ Y \overrightarrow{V_{y_i}Z_i} = \overrightarrow{y_iZ_i}. \]

Consequently \( m(y_i', z_i) = Y m(y_i', z_i') \).

Finally we have

\[ \frac{m(V, y_i)}{m(V, y_i')} = \frac{m(y_i', z_i)}{m(y_i', z_i')} = \left| \frac{y_i'z_i}{y_i'z_i'} \right| = Y \]

which is equivalent to

\[ m(y_i', z_i') = \frac{m(y_i', z_i)}{m(V, y_i)} = \frac{m(U, V)}{m(V, y_i)}. \]
Hence
\[ \lim_{{i \to \infty}} m(Y'_i, Z'_i) = 0. \]

Also since
\[ \frac{|V_{Y'_i}|}{|V_{Y'_j}|} = \frac{|Y'_i Z'_i|}{|Y'_i Z'_j|} = Y \]

we can conclude that the line \( Y'_i Z'_i \) is parallel to the line \( UV \). The limit of the lines \( Y'_i Z'_i \) exists and is the line \( UV \).

We now list the following equations and inequalities:

(a) \[ V_{Y'_i} = \frac{1}{m(V, Y'_i)} \cdot V_{Y'_i} = \frac{1}{m(V, Y'_i)} V_{X'_i} + \frac{1}{m(V, Y'_i)} X'_i Y'_i \]

(b) \[ |m(V, Y'_i) - m(V, X'_i)| \leq \beta \]

(c) \[ m(X'_i, Y'_i) \leq \beta \]

(d) \[ \lim_{{i \to \infty}} m(V, Y'_i) = +\infty \]

(e) \[ \frac{V_{X'_i}}{m(V, Y'_i)} = \frac{V_{X'_i}}{m(V, X'_i)} \cdot \frac{m(V, X'_i)}{m(V, Y'_i)} \]
From these we find that

\[
\lim_{i \to \infty} \overrightarrow{VY_i} = \lim_{i \to \infty} \frac{\overrightarrow{VY_i}}{m(V,Y_i)} = \lim_{i \to \infty} \left[ \frac{\overrightarrow{VX_i} \cdot m(V,X_i)}{m(V,X_i)} + \frac{\overrightarrow{X_iY_i}}{m(V,Y_i)} \cdot m(V,Y_i) \right] = \overrightarrow{WV}.
\]

In a similar manner we can show

\[
\lim_{i \to \infty} \overrightarrow{VZ_i} = \overrightarrow{WV}.
\]

Therefore

\[
\lim_{i \to \infty} Y_i' = W = \lim_{i \to \infty} Z_i' .
\]

We now point out the absurdity of this situation!

Let

\[
B_m = \left\{ x \mid x \in E^n \land m(V,x) \leq 1 \right\}
\]

be the convex ball containing \( V \). The line \( UV \) passes through the inner point \( V \) of \( B_m \). Hence the line \( UV \) has a segment of Minkowski length \( 2 - \varepsilon \) (\( 0 < \varepsilon < 2 \)) in common with \( B_m \). Since the limit of the lines \( Y_i'Z_i' \) as \( i \to \infty \) is the line \( UV \), then for sufficiently large \( i \) the line \( Y_i'Z_i' \) must possess a segment of Minkowski length \( 2 - \varepsilon \) in common with the interior of \( B_m \). But on the other hand \( Y_i' \) and \( Z_i' \) are boundary points of \( B_m \) of Minkowski distance \( m(Y_i', Z_i') \).
from each other. Because of the convexity of $B_m$, the length of any segment of the line $Y_i'Z_i'$ in the interior of $B_m$ cannot be more than $m(Y_i',Z_i')$ which converges to 0 as $i\to\infty$. Thus we are reduced to absurdity in assuming the lemma to be false. Therefore there exists $X \in L$ such that $m(X,U) \neq m(X,V)$. Q.E.D.
REFERENCES


[9] Ibid., p. 147.


[13] Ibid., pp. 76-104.


[15] Wilson, Richard M., Professor of Mathematics at the Ohio State University, Columbus, Ohio. A private communication.
